A strain invariant criterion for open hole failure in quasi-isotropic composite laminates

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A strain invariant criterion for open hole failure in quasi-isotropic composite laminates

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Abstract

This master thesis focuses on developing a strain invariant criterion for composite structures with open hole stress concentrations. This could be potentially used in fail safe design of layered composites. The prediction of failure in a laminate with hole becomes more difficult as it depends on factors like layup of laminate, material properties, loading conditions etc. Although modern finite element (FEM) techniques are predominantly used in solving such situations, in case of a steered fibre system it becomes computationally demanding to estimate the stiffness at every single point. Hence, using FEM and other existing failure criteria it is very difficult to qualify a variable stiffness composite laminate as safe to use. In order to design and qualify such a laminate, new system has to be devised which is no longer dependent on orientation of fibre and remains invariant. Thus, the main objective of the thesis work is to analyse the influence of stress concentration on the critical failure envelope which takes care of all fibre angles so that it can be used as a design limit criterion against failure. The process of obtaining a solution for open hole failure involves understanding of the stress field around the hole and requires a good knowledge on the invariant failure envelope. This includes solving for stresses with numerical approximations at intermediate steps. The failure values obtained from envelope are compared against experimental results for validation. The result of this thesis work will not only ease the design process but can also be used as a validation tool, to certify a composite laminate. This writing presents the actions taken in order to achieve the objective of project successfully.
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Nomenclature

List of Acronyms

WWFE  World Wide Failure Exercise
SCF    Stress Concentration Factor
CLT    Classical Laminate Theory
FF     Fibre Failure
CFRP   Carbon Fibre Reinforced Plastic
IFF    Inter-Fibre Fracture
FEM    Finite Element Method
UD     Uni-Directional
DCB    Double Cantilever Beam

List of Symbols

Abbreviations
\[ \mu_{12} \]  Poisson ratio
\[ \sigma_0 \]  Far field stress
\[ \sigma_f, F_{tu} \]  Failure strength of un-notched laminate
\[ \tau_L \]  Longitudinal shear stresses acting on the fracture plane defined
\[ \tau_T \]  Transverse shear stresses acting on the fracture plane defined
\[ \theta_k \]  Fibre orientation angle with respect to laminate axis
\[ \theta_{fp} \]  Angle of the fracture plane
\[ FIF \]  Fibre failure index
\( FI_M \) Matrix failure index
\( G_{ij} \) Strain coefficients
\( I_1 \) Volumetric strain Invariant
\( I_2 \) Maximum shear strain
\( k_\infty^t \) Stress concentration factor at infinity
\( Q_{ij} \) Reduced laminate stiffness components
\( S \) Shear strength
\( S^L \) Longitudinal shear strength
\( S^T \) Transverse shear strength
\( X_c \) Longitudinal strength in Compression
\( X_t \) Longitudinal strength in tension
\( Y_c \) Transverse strength in compression
\( Y_t \) Transverse strength in tension
\( h \) Laminate thickness
\( R \) Normalised radial distance with respect to hole \([R=1 \text{ refers to edge of hole}]\)

- Most of the graphs in this report are in principal planes i.e planes in which we have the shear stress as zero. So, the axis \( \epsilon_I \) and \( \epsilon_{II} \) represents the principal axis strains.

- More terms appear in multiple theories considered and are not listed here, refer to the citation given for better understanding.

- The units of all the material values used in thesis are also presented inside.
1.1 Introduction

In aerospace industry, composites play a very important role because of its inherent high specific stiffness & strength. Development of such advanced composite materials lead to efficient structural design of primary aircraft components. Such transformation of metallic alloys to composites is driven by the necessity to design greener and cleaner means of transportation. In particular, the ability to efficiently use the material properties of composites has been the reason for increased focus on its research and application. The main benefits of using composites are i) high stiffness to weight ratio, ii) better tailor-able anisotropic properties, iii) increased fatigue life. With the main property of anisotropy, it is possible to render and steer the fibres in composite based on functional requirements resulting in an efficient and lighter structure.

Comparing with the conventional metals, designing composite structures against failure becomes difficult as various modes of failures occur in such heterogeneous laminated structures. It is important to have a lighter structures which are critical ensuring the safety of component. In composite design, failure evaluation is one of the cornerstone considerations. Many failure criteria are available in the literature. This enforces the necessity of a criteria with which a failure resistant composite structure can be devised. Many failure criterion are available in literature which can be broadly classified into two classes:

1. Single mode of failure equation (e.g.fibre)
2. Multiple equations one for each failure mode “mode”.

The traditional failure criteria developed by Tsai-Wu [11] and Tsai-Hill [12] belong to the first class, while many of the more successful modern criteria like Hashin [13], Puck [14] and LaRC03 [2] belong to the second. Almost in all the failure criteria, the failure envelopes are expressed in terms of the stresses in fibre direction and normal to fibre direction. The consolidated effect of individual plies as laminate is being determined by taking into the account
of individual ply orientations with the reference axis. This means that the angle between the principal stresses and the fibre directions will always play a role in failure evaluation. Because of that, the evaluation of failure of a laminate has to be done on a ply-by-ply basis and requires explicit knowledge of the stacking sequence.

The method suggested by Tsai and Pagano [15] involving material invariants based on its material properties is an effective tool in formulating the solution. These material invariants are more widely used because they are independent of the plies coordinate systems. The process of analysing failures was mainly characterized by determining the stiffness of the laminate. For a laminate with fibres at defined angles, the stiffness can be estimated using these material invariants. In case of a steered fibre system, even by using classical Finite Element Method (FEM) technique calculating the stiffness is more difficult as the orientation differs at each and every laminate point. So, in order to design & validate such a structure against failure, an invariant system have to be devised which is no longer dependent on fibre angle. In order to achieve this the strain invariant criterion proposed by Tsai and Melo [16] which is based on Tsai-Wu failure can be used. This criterion in strain space is one of the effective way for getting a conservative strain space, where the failure will not occur. Using such formulation, the safe region of the strain space for the laminate can be found which does not depend on the ply angle. The boundary of this curve is referred to as the “omni strain” envelope. This envelope remains invariant that can represent any laminate irrespective of fibre angle but made of same material.

In composite structure for the ease of manufacturing holes are used to join two or more structures and it became more unavoidable. These holes lead to a potential weak zone in the composite due to high stress concentrations. In case of real structures in-plane loads, which are bi-axial in nature, can either increase or decrease the stress concentration effect. Hence it is necessary to study the effect of this stress concentration on omni strain envelope. Therefore, the research objective for this master thesis is formulated as:

"To analyse and propose an analytical solution which characterises the influence of open hole stress concentration, based on omni strain envelope for quasi isotropic fibrous composites"
1.1 Introduction

The report is structured in the following manner: Chapter 2 explains the fundamentals of composite materials and gives a brief description of material invariants. Chapter 3 shows stress concentration in composite, it explains how the orientation of fibres will influence the stress concentration and the advantages of exploring this area. Different methods to solve these complex stress concentration are also discussed and the results are explained. In Chapter 4, the idea of Omni strain envelope is well explained and omni strain envelopes are produced using Tsai-Wu and LaRC03 failure criteria. The basic characterisation of omni strain envelope can be well studied in this chapter. In Chapter 5 an analytical formulation for open hole case is presented and a detailed study is done to validate the formulation by comparing it with closed form Tsai-Wu solution. In Chapter 6, the important characteristics of the solution is presented and is further validated by comparing it with experimental & numerical results. Based on the research objectives and findings from earlier chapters, conclusions and recommendations are made and reported in Chapter 7.
In this chapter, we introduce the basic fundamental works previously done which forms the background of this research work. It begins with a short introduction to fibrous composites Section 2.1 and explains the various methods to predict failure in fibrous composite materials as discussed in Section 2.1.1 & 2.1.3. The basic building blocks of composites from Classical Laminate Theory (CLT), stiffness relations and its dependence on material properties are discussed in section 2.3. The advantages of using formulations in establishing relationships is also briefly explained in this chapter.

2.1 Introduction to Fibrous Composites

Composites are materials which is formed by combining at least two different materials on a macroscopic scale. The main purpose of composite material is to generate a new material with better properties than its constituents. Advanced composites are made up of fibres and matrix. The matrix binds the fibre and helps to hold its position, it also protects fibres from environmental conditions like corrosion, oxidation, etc. The fibres are major component that gives high properties to composites including high strengths and stiffness.

These fibred composites can be classified into the following categories

   (a) Unidirectional fibre composites (fibres in one direction, multidirectional composites can be made by laying unidirectional fibres in other directions).
   (b) Woven fabric composites.
   (c) Random fibre reinforced composites (fibres randomly distributed).

2. Discontinuous fibre-reinforced composites.

In this master thesis work we focus on continuous fibre-reinforced composites specially unidirectional composites.
2.1.1 Failure in composites

Materials tends to fail if its been loaded beyond its capacity, and prior to complete failure of the structure materials will exhibit some loss of property. Likewise composites show loss of stiffness and other properties before it fails completely. Based on loading different forms of failures are prone to exist this failure can be In-plane or out of plane failures. In this Master thesis work we focus on In-plane failures such as tension and compression when a material is subjected to bi-axial loading. These in-plane failures can be due to fibre or matrix failure and it is dependent on the loading involved Fig.2.1.

![Figure 2.1: Different types of in-plane failures in composites](image-url)
2.1.2 Design Failure envelopes

The design failure envelope for composites are widely developed in past and been used in composite industry for design limitations. These failure envelope play a very important role in predicting the failure of structure and can easily be used in design without performing complex experiments for predicting failure. These failure envelopes based on various theories is checked for its correlation to experiments through the World Wide Failure Exercise (WWFE) [17] conducted often and limitations pertaining to each theory is reported Fig.2.2. These design envelopes play a very important role in determining the ability of composite to withstand loads and to predict its applicability for a particular use.

![Figure 2.2: Comparison between the predicted and measured biaxial failure stresses for (0/±45/90) AS4/3501-6 laminates, design failure envelopes from WWFE [1]](image)

It is clear through the results shown and comparisons in WWFE [1] that Tsai quadratic failure criterion for Fibre Failure (FF) gave results close enough to the experimental. The accuracy of Tsai’s criteria increased with increase in the volume fraction of the fibres. Thus its evident that this failure criteria is more dependent on fibres as they are the major strengthening components in a composite. It is not possible to manufacture a composite without having a resin polymer in it as they help to bind the fibres together and keep in place. It is also to protect the fibres from external environmental effects. In 1996, Puck [14] came up with a physically based failure model which is cable of predicting the Inter-Fibre Fracture (IFF). This criteria is found to be more effective as it predicted results which are much closer to experimental than any other criteria existed before through WWFE [1]. In this criteria also individual & mixed modes of fibre and matrix failure were not taken into account but it provided the path of using Mohr-Coulomb method to solve for $\theta_{fp}$. Thus it became necessary to have a interactive failure criteria which takes into account of the matrix failures in composites.
2.1.3 LaRC03 Failure criteria

In 2005, researchers at NASA Langley research centre Davila et.al [2] established a failure criteria which takes account of fibre and matrix failures. It has a set of six phenomenological failure criteria for fiber-reinforced polymer laminates. It can predict the matrix and fibre failure accurately, without using any curve fitting parameters. In case of matrix transverse compression failure the angle of fracture plane ($\theta_{fp}$) is solved by using Mohr-Coulomb method as in Pucks. Fibre misalignment angle is considered while setting the criterion for fibre kinking. Fracture models are also used in calculating the in-situ strengths of the composite laminate. Each mode of failure is found to be predominant in loading scenarios as shown in fig.2.3 where a Uni-Directional (UD) laminate is subjected to biaxial loading.

This failure criteria calculates the in-situ strength for thick plies as a function of UD laminates strength. The transition between thin and thick plies is found to be 0.7mm as stated by Dvorak et. al [18]. Thus considering thick plies and UD laminate properties will give good results for analysis. On adopting this strategy we don’t take into account of the fracture toughness which is experimental data found using Double Cantilever Beam (DCB) experiments.

Pucks failure analysis results [19] are shown in figure for comparison. It is fond that there is good agreement between LaRC03 and Puck in all quadrants except biaxial compression, where LaRC03 predicts an increase of the axial compressive strength with increasing transverse compression.
2.2 Composite laminates and its mechanics

Classical Laminate Theory (CLT) is the basic behind laminate stiffness formulation. It assumes that all layers in an orthotropic or isotropic layup are perfectly attached to each other. CLT also assumes Kirchhoff pure bending assumptions that is in a laminate thickness deformations are zero and strains in out of plane directions are also neglected. By these considerations an approximate state of plane stress prevails. In an orthotropic material loaded along principle axis can be related to laminate strains through the relation:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

(2.1)

The \(Q_{ij}\) reduced stiffness components are given in terms of material modulus in different directions and poisson ratio \((\mu_{12})\) as:

\[
Q_{11} = \frac{E_1}{1 - \mu_{12}\mu_{21}}, Q_{12} = \frac{\mu_{12}E_1}{1 - \mu_{12}\mu_{21}}, Q_{22} = \frac{E_2}{1 - \mu_{12}\mu_{21}}, Q_{66} = G_{12}
\]

(2.2)

Thus for building up a laminate different plies are stacked above each other in various angles with respect to the laminate axis \(\theta_k\) as shown in Figure 2.4

\[\text{Figure 2.4: Composite layup configuration of } N \text{ laminae with orientation } \theta_k, \text{ thickness } t_k \text{ and at a distance } z_k \text{ from mid-plane}\]

By using the CLT assumptions. in-plane stresses on the \(k^{th}\) layer positioned at a thickness location \(z_{k-1} < z < z_k\) is given by:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}_k
\begin{bmatrix}
\epsilon_x + z_k x \\
\epsilon_y + z_k y \\
\gamma_{xy} + z_k xy
\end{bmatrix}
\]

(2.3)

where \(Q_{ij}\)'s are ply stiffness components in laminate co-ordinate system of the \(k^{th}\) layer , given as:
\[
\begin{align*}
Q_{11} &= U_1 + U_2 \cos 2\theta_k + U_3 \cos 4\theta_k \\
Q_{12} &= U_4 - U_3 \cos 4\theta_k \\
Q_{22} &= U_1 - U_2 \cos 2\theta_k + U_3 \cos 4\theta_k \\
Q_{66} &= U_5 - U_3 \cos 4\theta_k \\
Q_{16} &= (U_2 \sin 2\theta_k + 2U_3 \sin 4\theta_k)/2 \\
Q_{26} &= (U_2 \sin 2\theta_k - 2U_3 \sin 4\theta_k)/2
\end{align*}
\]

Thus it is necessary to estimate the stiffness at each layer in order to determine the stresses in that layer and the orientation of fibres in that layer with respect to the laminate axis system plays a vital role in determining the stiffness of that layer.

It is clear from equation 2.3 that through thickness distributions of linear stresses are either constant, when \(k_i\) laminate curvatures are zero. Even if all the layups are made of same material, due to discontinuity of the orientation angle from one layer to another, stresses varies at boundaries and can be seen in figure 2.5. It can be inferred that the spatial distribution of strains (\(\epsilon_i\) and \(k_i\) (from deformation)) in whole, there is no stress like quantity corresponding to these strains. Thus it is difficult to establish stress-strain relationships for entire laminate. A close approximation can be achieved by integrating the layer stresses throughout the laminate thickness(h) shown in equation 2.5. The N’s in are often referred to as stress resultants.

![Figure 2.5: Illustration of linear strain variation and discontinuous stress variation in multi-directional laminate. [3]](image)

\[
\begin{align*}
N_x &= \int_{-h/2}^{h/2} \sigma_x dz \\
N_y &= \int_{-h/2}^{h/2} \sigma_y dz \\
N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} dz
\end{align*}
\]

On substituting the layer stresses from equation 2.3 into equations 2.5, we get the constitutive relations for laminate as:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} +
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
k_x \\
k_y \\
k_{xy}
\end{bmatrix}
\]

where:
The $A$ matrix relates the in-plane stress resultants to the mid-plane strains, and $B$ matrix is called bending-extension coupling matrix as it relates the in-plane stress resultants to the curvatures and moment resultants to the mid-plane strains. This relation can be used in certain applications, however, this relation is typically considered undesirable as it leads to a complex strain rates when curvatures are more. This $B$ matrix can be avoided by making the laminate symmetric with respect to the mid-plane, i.e. having same oriented fibres in top and bottom half of laminate.

2.3 Invariants

Invariants are parameters which signifies the characteristics of a structure for given material properties. In an anisotropic material, properties of laminate differs in each axis direction these parameters can be used to characterise the laminate. In equation 2.4, $U_i$'s are given by the material properties of the $k^{th}$ layer ply, these are invariants with respect to the orientation angle of any particular layer. These invariants $U_i (i = 1, ..., 5)$ are defined in terms of reduced stiffness $Q_{ij}$ matrix components as as:

$$U_1 = \frac{(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})}{8}$$

$$U_2 = \frac{(Q_{11} - Q_{22})}{2}$$

$$U_3 = \frac{(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})}{8}$$

$$U_4 = \frac{(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})}{8}$$

$$U_5 = \frac{(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})}{8}$$

These are the invariants given for the entire laminate through the reduced stiffness matrix components of the composite. Thus, these will remain invariant irrespective of the orientation angle ($\theta$) of the fibre to the laminate axis. These are used to determine the stiffness of each layer given through equation 2.4

2.3.1 Lamination parameters

Lamination parameters provide a compact notation for the description of stiffness properties of a laminate given its lay-up configuration. Material invariants for the composite laminates are also called as “Lamination Parameters”. The main idea of developing material invariants was proposed by Tsai and Pagano [15]. The complete set of stiffness matrices of any laminate can be described as a linear function in terms of only 12 lamination parameters and, for the special case of a symmetric balanced laminate, only four lamination parameters are necessary. They presented stiffness transformation equations for ply rotation in a laminate
as a function of invariants. For a typical composite laminate, in-plane($V_i$) and bending($W_i$)
lamination parameters are defined as

\[
(V_1, V_2, V_3, V_4) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos 2\theta(\bar{z}), \sin 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 4\theta(\bar{z})) d\bar{z}
\]

\[
(W_1, W_2, W_3, W_4) = 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{z}^2 (\cos 2\theta(\bar{z}), \sin 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 4\theta(\bar{z})) d\bar{z}
\]

in which $\bar{z} = \frac{z}{h}$ is the normalized z coordinate of the layers, $h$ is the laminate thickness, and
($\theta$) is the fibre angle at $\bar{z}$ [6]. In a general laminate it has to be noted that the layer orientation
angles typically do not vary continuously and are piece-wise linear, i.e constant in each layer.
Hence, the integrals in equation 2.9 can be replaced by summations. Irrespective of how the
lamination parameters are computed, the classical in-plane and bending laminate stiffness
matrices $A$ and $D$ are then linear functions of their corresponding lamination parameters
which are given by:

\[
A = h(G_0 + G_1 V_1 + G_2 V_2 + G_3 V_3 + G_4 V_4)
\]

\[
D = \frac{h^3}{12} (G_0 + G_1 W_1 + G_2 W_2 + G_3 W_3 + G_4 W_4)
\]

The term $G_i$ and its components are given as:

\[
G_0 = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{bmatrix} \quad G_1 = \begin{bmatrix} U_2 & 0 & 0 \\ 0 & -U_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & U_2 \\ U_2 & U_2 & 0 \end{bmatrix}
\]

\[
G_3 = \begin{bmatrix} U_3 & -U_3 & 0 \\ -U_3 & U_3 & 0 \\ 0 & 0 & -U_3 \end{bmatrix} \quad G_4 = \begin{bmatrix} 0 & 0 & U_3 \\ 0 & 0 & -U_3 \\ U_3 & -U_3 & 0 \end{bmatrix}
\]

The lamination parameters $V_i, W_i$ cannot be arbitrarily prescribed as it is dependent on the
ply angles. Such formulation of constants are widely used to characterize the properties of the
laminate. These are used as design parameters while designing a composite structure against
failure.

### 2.3.2 Reduction of Lamination parameter

Lamination parameters are not independent because trigonometric functions are used
in the equation 2.9. So, using trigonometric dependency the feasible region for the in-plane
lamination parameters is known by using Hammers [20]. This reduction is achieved by using
Krein and Nudelmann [21] method for determining the shape of closed region. The equation
is given as:

\[
2V_1^2(1 - V_3) + 2V_2^2(1 + V_3) + V_3^2 + V_4^2 - 4V_1V_2V_4 \leq 1
\]

\[
V_1^2 + V_2^2 \leq 1 \quad , \quad -1 \leq V_i \leq 1 \quad where \ (i = 1...4)
\]
2.3 Invariants

The method of reduction of lamination parameters for optimal design of constant stiffness can be studied through the works of Miki [22] and Miki and Sugyama [23]. The use of such parameters in the design of variable-stiffness laminates for minimum compliance was explained in the work of Setoodeh and Abdalla [24].

In case of a balanced symmetric laminate only two lamination parameters are needed, namely $V_1$ & $V_3$, using this we can fully model the in-plane stiffness. Thus making $V_2 = V_4 = 0$. This will simplify the equation 2.12 as

\[
\begin{align*}
2V_1^2(1 - V_3) + V_3 &\leq 1 \\
2V_1^2 + 2V_1^2V_3 + V_3^2 &\leq 1 \\
V_3 &\geq 2V_1^2 - 1 \\
-1 &\leq V_i \leq 1 \quad \text{where} \quad (i = 1, 3)
\end{align*}
\]

This relation for in-plane stiffness combined with material invariants can be used to develop an invariant criteria for composite. This type of reduction further helps to reduce the dependence of the system on other variable parameters and will help in establishing a full fledged failure criterion.

In the following chapter effect of open hole and its influence on stresses are studied. This will be useful to identify the best possible method and its solution is discussed.
Chapter 3

Stress Concentration Analysis

The most common way to join two or more structural components is through joints and the most pre-dominant ways are through fasteners such as rivets, bolts. This involves the process of making holes in the structures. These fastened structures when loaded causes the stress flow to concentrate around the holes and hence they are called as stress concentrations. The stress distribution around the hole in a composite is more complex than in isotropic material because composites are anisotropic. The point at or near the maximum stress concentration is normally the initiation location of failure. As composite laminates are piled of individual lamina, in each layer of ply the stress flow & distribution is different so its very hard to predict the stress concentration near the hole. On considering a laminate with different ply angles, it becomes more difficult to effectively predict and solve for stress concentrations. These stress concentrations also contribute to different failure modes depending on the type of loading. The effects of stress concentration in determining the strength of the composite is explained in Section 3.1. Later, in Section 3.2 different effective methods to solve the stress concentration is presented and individual methods are discussed.

3.1 Stress Concentration in Composites

When a isotropic material subjected to uni-axial unit load it is proven that the maximum stress concentration factor is three and failure is prone to happen at the region of maximum stress concentration. But as composites are heterogeneous materials with different properties in all directions its hard to predict the stress concentration. Stress concentration in composites is more complex than metals as stresses in each layer of composites differ and also across their interface. So determining the exact effect of a hole in composite is always difficult and is dependent on loading. In case of a laminate, orientation of fibres to the loading axis direction plays a vital role in determining the effective stress concentration. Apart from this other factors such as type of fastener, force applied on fastener (torquing), have to be considered for better understanding of stress concentration in composites. This makes the it much more complex and critical to study. In this chapter we consider an open hole stress concentration in composite to ease the purpose.
Fibre placement techniques can be effectively used to minimise the effect of stress concentrations [4]. It is evident from the result shown in Figure 3.1b that steered fibre system shows equivalent strength to an un-notched specimen. It has to be noted that the results were based on uni-axial loading for a unidirectional laminate. So, in case of a laminate with multi angled ply layup, these results will not be the same and Stress Concentration Factor (SCF) is largely dependent upon the direction of load transfer. In such situation where fibres in other axis are needed and strength to be maintained a careful study of stress concentration on various fibre angles have to be done.

It is clear from the understanding that fibre placement can be effectively used to reduce the influence of stress concentration. A fail safe criteria is needed to analyse this effect during composite design and to validate a multi angled ply laminate.

### 3.2 Methods to analyse stress concentrations in composites

In literature different theories and methods were proposed to solve the stress concentration in fibrous composites. The most effective ways of solving the stress concentration problem in composites are studied and presented below. The in-plane stress solution for anisotropic plates can be solved using the following methods:

1. Characteristic distance approach (Whitney Nuismer) [25].
   (This is a failure criteria and not a method to solve for stresses)

2. Solving for stress approximations in complex domain
   (a) Savins solution [26].
   (b) Leknitskii solution [27].
3.2 Methods to analyse stress concentrations in composites

3.2.1 Whitney Nuismer Method

This method is very widely used for calculating the stress fields and approximate stress around holes in composite laminates. After Linear Elastic Fracture Models (LEFM) developed by Waddoups et.al [28]. J.M Whitney and Nuismer [25] came up with the method for calculating the stresses around hole by using the characteristic lengths average stress($a_0$) and point stress($d_0$). The development of these failure criteria is based on the study of stress fields around hole for two different hole sizes. The stress distribution of an isotropic plate containing a open circular hole [29] is given as :

\[
\frac{\sigma_y}{\bar{\sigma}} = 1 + \frac{1}{2} \left( \frac{R}{x} \right)^2 + \frac{3}{2} \left( \frac{R}{x} \right)^4
\]  

(3.1)

where $\bar{\sigma}$ is the applied stress parallel to the y-axis at infinity and R is the hole radius.

**Point stress criterion**

It assumes that failure occurs when the stress $\sigma_y$ at some distance $d_0$ away from the hole is greater or equal to the strength of the un-notched laminate. i.e

\[
\sigma_y(x, 0)|_{x=R+d_0} = \sigma_f
\]  

(3.2)

As the SCF at the distance $d_0$ is different for varying hole sizes, this method can predict the hole size effect on the notched strength.

As shown in Figure 3.2 of an infinite plate with a hole, the circumferential stress approximation $\sigma_\theta$ at $\theta = 90$ is given as a function of distance from the centre of hole, its approximation is given by S.C Tan [30] as:

\[
\sigma_x(x = 0, y) = \sigma_0 \left[ 1 + \frac{1}{2} \left( \frac{R}{Y} \right)^2 + \frac{3}{2} \left( \frac{R}{Y} \right)^4 - \left( k_\infty - 3 \right) \left( \frac{5}{2} \frac{R}{Y} \right)^6 - \frac{7}{2} \left( \frac{R}{Y} \right)^8 \right]
\]  

(3.3)

Where R is the radius of hole, Y distance from the hole, $k_\infty$ Stress concentration factor at Infinity. It is noted that when the size of hole is decreased the Maximum Stress concentration...
factor (SCF) remained the same but only a localized decrease in stress redistribution around
the hole.

Whitney and nuismer [31] obtained their point stress criterion by substituting Equation 3.3
into Equation 3.1 and is given by:

\[
\frac{\sigma_N}{\sigma_f} = \frac{2}{2 + \xi_1^3 + 3\xi_1 - (K_T^∞ - 3)(5\xi_1^0 - 7\xi_1^3)}
\]  

(3.4)

where

\[
\xi_1 = \frac{R}{R + d_0}
\]  

(3.5)

Average stress criterion

This method is different from the earlier method that it considers average stress over
a characteristic length \(a_0\). It can be said as failure occurs when average stress is greater or
equal to the strength of the un-notched laminate. This criterion can given as:

\[
\frac{1}{a - 0} \int_R^{R + d_0} \sigma_y(x, 0) dx = \sigma_f
\]  

(3.6)

In case of orthotropic plates with circular hole, the solution is attained by substituting Equa-
tion 3.3 into Equation 3.6:

\[
\frac{\sigma_N}{\sigma_f} = \frac{2(1 - \xi_2)}{2 - \xi_2^2 - \xi_2^4 + (K_T^∞ - 3)(\xi_2^0 - \xi_2^3)}
\]  

(3.7)

where

\[
\xi_2 = \frac{R}{R + a_0}
\]  

(3.8)

Both the point stress criterion and the average stress criterion have two unknowns, the
un-notched strength \(\sigma_f\) and the characteristic length, \(d_0\) or \(a_0\). These unknowns are to be
determined experimentally. The characteristic length can be explained as inelastic, non-
linear material behaviour and imperfections of the hole. This means the theoretical value of
maximum stress concentration will never be achieved just before the laminate fails. It has to
be noted that both these aforementioned methods are only valid for unidirectional composite
laminates. Unlike elasticity theory, a strength criterion with theoretical background can never
have good correlation with experimental results. Although these criteria have been widely
used to predict notched strength of laminates, they should be considered as models rather
than failure criteria as they do not take into account of the complex failure mechanisms [32].
3.2 Methods to analyse stress concentrations in composites

3.2.2 Improved Whitney Nuismer method

The improvement on Whitney Nuismer approach was suggested by Kassapoglou [5]. This aims at determining the analytical solution for the characteristic length $a_0$ in order to determine the average the linear stress solution. This is done by using the formulations of Leknitskii(1968) and Savin(1970) [26, 27]. This formulation makes $a_0$ a laminate constant which is dependent of hole radius($R$) and width($W$). The $a_0$ laminate constant calculated through this model will only be valid for the following condition

$$\frac{W}{2} - R > a_0 \quad (3.9)$$

when Equation 3.9 becomes equal, that is when the material on both sides of the hole is equal to the averaging distance $a_0$ then we get

$$\frac{W}{2R} - 1 = \frac{a_0}{R} \quad (3.10)$$

This can be re-written as

$$\frac{2R}{W} = \frac{1}{1 + (a_0/R)} \quad (3.11)$$

The basic equation 3.3 is modified to suit a finite width plate and expressed it in terms of coefficients $A_6$ and $A_8$

$$\sigma_x(x = 0, y) = \sigma_0 \left[ 1 + \frac{1}{2} \left( \frac{R}{Y} \right)^2 + \frac{3}{2} \left( \frac{R}{Y} \right)^4 + A_6 \left( \frac{R}{Y} \right)^6 + A_8 \left( \frac{R}{Y} \right)^8 \right] \quad (3.12)$$

These coefficients $A_6$ and $A_8$ are solved by using the same relations but for a finite width plate. This is done by (i) recovering the net section stress when integrating from $R$ to $W/2$ and (ii) At $y=R$ on hole edge, adjusted SCF for finite width is recovered. On using this solution method one can obtain the averaging distance $a_0$ for a symmetric balanced laminate. This adaptation involves series of steps and is given in book [5] page 28 & 29.

The final form of the equation is given as:

$$\sigma_0 = \frac{F_{tu}}{s + \left( \frac{1}{2(1+s)} - \frac{1}{2} \left( \frac{1}{(1+s)^2} \right) \right) - \left( \frac{A_6}{5} \right) \left( \frac{1}{(1+s)^2} - 1 \right) - \left( \frac{A_8}{7} \right) \left( \frac{1}{(1+s)^2} - 1 \right)} \quad (3.13)$$

where

$$s = \frac{a_0}{R} \quad (3.14)$$

Thus on using the Equation 3.13 one can obtain the far field stress $\sigma_0$ to cause failure. This gives the ratio of $a_0/R$ as a laminate constant which is only dependent on the layup through
SCF for infinite and finite plates. It is a good improvement as it does not rely on any experimental test results to find the characteristic distance $a_0$. The characteristic distance $a_0$ only depends on $K^\infty_t$ and the width of specimen, the results for orthotropic material are shown in Figure 3.3. In highly orthotropic plates with SCF greater than 6 are neglected, for the range of SCF between 2 to 6 the averaging value of $a_0$ is found to be 3.8mm, which is same as suggested in the experimental approach of whitney-nuismer. More over it does not include the biaxial loading and its interaction with each other. This model presented is not accurate as it does not take account of the transverse direction failure and its associated failure mechanisms. Thus detailed study on ply orientation and mode of failure is very important in obtaining a good analytical model.

### 3.2.3 Savins solution

This is one of the famous and well established way to solve for stresses around a hole. The ideology of using complex variable was first suggested by Muskhelishvili [33] and is used in Savins solution. The solution is obtained by solving the equilibrium equation around the hole and getting complex conjugate pairs of roots. The equilibrium equation around the hole for an anisotropic material having orthogonal axis of symmetry given in-terms of material co-ordinates in tensor notation as

$$S_{1111}^L s^4 + S_{1112}^L s^3 + (2 S_{1122}^L + 4 S_{1212}^L) s^2 - 4 S_{2212}^L + S_{2222}^L = 0$$

(3.15)

where $S_{ijkl}$ is three dimensional laminate Compliance tensor

On solving Equation 3.15 one will get two sets of complex conjugate roots ($s$). Based on the energy considerations of Leknitskii [34] the two roots are taken as the terms with positive imaginary components ($s_1$ and $s_2$). These roots are then transformed into the complex co-ordinates by using

$$Z_1 = x + s_1 y$$

$$Z_2 = x + s_2 y$$

(3.16)
3.2 Methods to analyse stress concentrations in composites

where \(x, y\) are the Cartesian co-ordinates

The in-plane stress solution is given by Saeger [35] as

\[
\begin{align*}
\sigma_{11} &= \sigma_{11}^\infty + 2\text{Re}[s_1^2\Phi_0'(z_1) + s_2^2\Psi_0'(z_2)] \\
\sigma_{22} &= \sigma_{22}^\infty + 2\text{Re}[\Phi_0'(z_1) + \Psi_0'(z_2)] \\
\sigma_{12} &= \sigma_{12}^\infty - 2\text{Re}[s_1\Phi_0'(z_1) + s_2\Psi_0'(z_2)]
\end{align*}
\]

(3.17)

where

\[
\begin{align*}
\sigma_{11}^\infty &= P\cos^2\alpha \\
\sigma_{22}^\infty &= P\sin^2\alpha \\
\sigma_{12}^\infty &= P\sin\alpha\cos\alpha
\end{align*}
\]

(3.18)

\(P\) is the loading and \(\alpha\) is the ply orientation angle and \(\Phi_0, \Psi_0\) are the complex functions for stress around holes.

The solution for the complex functions \(\Phi_0\) and \(\Psi_0\) are derived based on Savins solution and was given by Saeger [35] as:

\[
\Phi_0 = -\frac{iR^2(-is_1 + 1)}{4s_1 - 4s_2}\left(\frac{s_2 \sin (2\alpha) + 2 (\cos (\alpha))^2}{z_1 + \sqrt{z_1^2 - R^2(s_1^2 + 1)}} + \frac{i \left(2s_2(\sin (\alpha))^2 + \sin (2\alpha)\right)}{z_1 + \sqrt{z_1^2 - R^2(s_1^2 + 1)}}\right)
\]

(3.19)

and \(\Psi_0\) is given by

\[
\Psi_0 = \frac{iR^2(-is_2 + 1)}{4s_1 - 4s_2}\left(\frac{s_1 \sin (2\alpha) + 2 (\cos (\alpha))^2}{z_2 + \sqrt{z_2^2 - R^2(s_2^2 + 1)}} + \frac{i \left(2s_1(\sin (\alpha))^2 + \sin (2\alpha)\right)}{z_2 + \sqrt{z_2^2 - R^2(s_2^2 + 1)}}\right)
\]

(3.20)

The formulation of these functional equations \(\Phi_0\) and \(\Psi_0\) are based on conformal mapping and by using Schwarz integral equations. Thus making it one of the most effective ways of formulating the stresses around the hole.

3.2.4 Leknitskii solution

This method of predicting the stresses around stress concentration is well proven for in-plane stress distribution for different types of loadings. The basic formulation principle of Savins [26] and Leknitskii [27] are the same in terms of solving for the forces along edge of the hole. The difference is in the calculation of the roots \(s_1\) and \(s_2\). In Leknitskii method, these are done by using the material parameters \(r\) and \(a\) which does not include the effect of term \(S_{16}\). \(S_{16}\) is the term in the laminate compliance matrix. The Leknitskii method is also based on the theory of Schwarz formula which is well known in the framework of complex variables. The roots are given by:
\[ s_1 = + \sqrt{\frac{r-a}{2}} + i \sqrt{\frac{r+a}{2}} \]
\[ s_2 = - \sqrt{\frac{r-a}{2}} + i \sqrt{\frac{r+a}{2}} \]  
(3.21)

and the terms \( r \) and \( a \) are given as:
\[ r = \sqrt{\frac{S_{22}}{S_{11}}} , \quad a = \frac{1}{2} \left( \frac{2S_{12} + S_{66}}{S_{11}} \right) \]  
(3.22)

This formulation is applicable for symmetric and balanced laminates which doesn’t involve the term \( S_{16} \). The stress function \( \Phi_1(z_1) \) & \( \Phi_2(z_2) \) are attained by using Laurent series as
\[ \Phi_1(z_1) = A_0 + A_1 \ln \zeta_1 + \sum_{n=1}^{\infty} g_1^n \zeta_1^{-n} \]
\[ \Phi_2(z_2) = A_0 + A_2 \ln \zeta_2 + \sum_{n=1}^{\infty} g_2^n \zeta_2^{-n} \]  
(3.23)

The individual terms are explained in book of “Anisotropic Plates” Savin [27]. The stress equations in both savins and leknitskii are the same, only the way of approximating the functions \( \Phi \) are differed.

### 3.3 Stress concentration analysis in composites

The above analysis reveals, the method of solving for stresses in complex domain to be more effective as it is dependent on the layup and the orthotropic property of the laminate. The complex domain method allows to solve for stress inside the domain of composite so by doing this one can establish the relation to predict the failure for bi-axial loads.

It is inferred through the analysis that at the hole edge the stress solution is always unidirectional \( \sigma_\theta \) it does not depend upon the layup or the loading condition. i.e at the hole edge the critical stress is always \( \sigma_\theta \) which causes the stress to accumulate and fail. This can be seen in Figure 3.4a for a symmetric laminate. The red line denotes the variation of \( \sigma_\theta \) around the hole and light blue \( \sigma_r \) & green lines \( \sigma_r \) over laps each other and stays zero around the hole.

### Table 3.1: Laminate configuration and its SCF for unit load applied along x-direction

<table>
<thead>
<tr>
<th>Laminate Code</th>
<th>Layup</th>
<th>Max SCF</th>
<th>Min SCF</th>
<th>Layup Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[30/45/0/0/45/30]</td>
<td>4.004</td>
<td>-0.7948</td>
<td>Symmetric</td>
</tr>
<tr>
<td>B</td>
<td>[0/90/0/90/0/90]</td>
<td>5.3811</td>
<td>-0.8402</td>
<td>Cross-ply laminate</td>
</tr>
<tr>
<td>C</td>
<td>[45/-45/-30/30]</td>
<td>2.7920</td>
<td>-0.7049</td>
<td>Angle ply laminate</td>
</tr>
<tr>
<td>D</td>
<td>[45/-30/30/45]</td>
<td>2.7920</td>
<td>-0.7049</td>
<td>Anti-Symmetric laminate</td>
</tr>
<tr>
<td>E</td>
<td>[30/60/-45/-30/-60/45]</td>
<td>2.5664</td>
<td>-1</td>
<td>Balanced laminate</td>
</tr>
<tr>
<td>F</td>
<td>[0 -45 90 90 -45 45 0]</td>
<td>3</td>
<td>-1</td>
<td>Quasi-isotropic laminate</td>
</tr>
</tbody>
</table>
3.3 Stress concentration analysis in composites

(a) Stress solution around the hole for a Symmetric laminate A

(b) SCF at hole edge $\sigma_\theta$ for cross-ply laminate B(red) and Q.I laminate F(blue) table 3.1

Figure 3.4: Stress distribution at hole edge for unit load along x axis ($P_x = 1$, $P_y = 0$) loaded at infinity

Figure 3.5: SCF at hole edge $\sigma_\theta$ for Quasi-isotropic laminate presented in table 3.1 with loading $P_x = 1$ and $P_y = 2$

The Figure 3.4a also reveals the solution to be symmetric around $180^\circ$ as Uni-Directional (UD) straight fibres are placed around the hole which carries the load. Thus in case of UD fibre laminates its enough to analyse for stresses in $180^\circ$ and this region can still reduce for a quasi-isotropic or a cross ply laminate. It is shown through the Figure 3.4b that the SCF depends upon the layup and the fibre orientations of plies in the laminate. In case of a quasi-isotropic
system with hole the stress solution resembles the solution for an isotropic plate getting a maximum value of 3 and minimum value of -1. The Figure 3.4b above represents the stress variation in a laminate when subjected to uni-axial loading at infinity. The same case can be studied for multi axial loading by using the formulation of Savins solution. In case of a biaxial loading as shown in Figure 3.5 the stress solution remains unidirectional $\sigma_\theta$ around the hole and thus it is shown that irrespective of laminate layup or loading the major stress at the hole junction is $\sigma_\theta$ which leads to failure.

The variation of $\sigma_\theta$ for various laminate layups table 3.1, around the hole is shown in Figure 3.6. The maximum value of the SCF can be much higher than the typical value of three for isotropic metals and it can be increased by increasing the number of $0^\circ$ plies. The results show that for laminate B with only $0^\circ$ and $90^\circ$ plies has the maximum stress concentration factor of 5.3 and as the number of plies in the other direction increases the peak SCF decreases this can be seen in the laminates C, D and E. In case of C and D laminates the stress concentration effect is less compared to isotropic material. The laminates C and D produced the same results, proving that SCF is independent of the stacking sequence.

![Graph](image)

**Figure 3.6:** SCF at hole edge $\sigma_\theta$ for orthotropic laminates presented in table 3.1 for unit load along x axis $P_x = 1$ and keeping $P_y = 0$

The method of solving using complex variables proves to be efficient when biaxial loads are considered. Proven the ability to predict the stress concentrations under biaxial loading the Savins solution can be used in the omni strain criterion. The idea of omni strain criteria and the procedure to build it is described in the next chapter.
The idea of invariant based design envelope where the laminate will not fail is proposed and known as omni strain envelope. Composite laminates are made up of multiple fibre plies with fibres oriented at different angles to the loading direction. When loaded each layer of the laminate gets stressed differently and failure is possible, this is fully based on the orientation of lamina in laminate. A single layer fibre failure can provide to be the weak spot for the structure to fail when subjected to fatigue loading. Thus through omni strain envelope one can determine the safe region within which the failure will be completely avoided. It is important to understand the failures in composite when it is subjected to bi-axial loads. The failures can be due to fibre, matrix or mixed modes of failure. The fibre orientation ($\phi$) to the load paths plays a vital role in determining the failure. In case of a laminate with multiple layers of fibres oriented in different directions ($\phi$) Figure 4.1 subjected to uniform loading each layer can fail in different modes of failure at various load levels. The individual lamina failures are mostly in-plane and those areas tends to be weak zones in a laminate. The critical strain envelope referred to as “Omni strain envelope” by Tsai and Melo [16] is discussed in this chapter. The closed form solution Using Tsai-Wu Failure criterion is presented and the adaptability using stress concentration factor is also discussed. In section 4.1 and section 4.2 omni strain envelope using Tsai-Wu failure criteria and LaRC03 are discussed.

Figure 4.1: Laminate and ply axis systems [6]
4.1 Formation of Omni strain envelope using Tsai-Wu

Among the existing failure theories for fibre in composites Tsai-Wu gives the prediction for first ply failure and has the closed form expression given as:

\[ F_{11}\sigma_{1}^{2} + F_{22}\sigma_{2}^{2} + F_{66}\tau_{12}^{2} + F_{1}\sigma_{1} + F_{2}\sigma_{2} + 2F_{12}\sigma_{1}\sigma_{2} = 1 \] (4.1)

where \( F_{i} \) and \( F_{ij} \) represents the second and fourth order tensors. The terms in equation 4.1 are given as

\[
F_{11} = \frac{1}{X_{t}X_{c}} \quad F_{22} = \frac{1}{Y_{t}Y_{c}} \quad F_{1} = \frac{1}{X_{t}} - \frac{1}{X_{c}} \\
F_{2} = \frac{1}{Y_{t}} - \frac{1}{Y_{c}} \quad F_{12} = -\frac{1}{2\sqrt{X_{t}X_{c}Y_{t}Y_{c}}} \quad F_{66} = \frac{1}{S_{2}}
\]

The material stresses are related to material strains through the reduced stiffness matrix (Q). The failure criterion is thus given as

\[ G_{11}\epsilon_{1}^{2} + G_{22}\epsilon_{2}^{2} + G_{66}\epsilon_{12}^{2} + G_{1}\epsilon_{1} + G_{2}\epsilon_{2} + 2G_{12}\epsilon_{1}\epsilon_{2} = 1 \] (4.2)

The strain coefficients \( G_{ij} \) are given in Appendix A. The material strains \((\epsilon_{1},\epsilon_{2},\epsilon_{12})\) are related to laminate strains \((\epsilon_{x},\epsilon_{y},\epsilon_{xy})\) using the following transformation [3]

\[
T = \begin{bmatrix}
c^2 & s^2 & 2cs \\
s^2 & c^2 & -2cs \\
-cs & cs & c^2 - s^2
\end{bmatrix}
\]

(4.3)

where \( c = \cos(\varphi) \) and \( s = \sin(\varphi) \)

substituting the transformed strains using (4.3) into equation (4.2) we get the failure envelope in terms of laminate strains and ply angle in the form

\[ F(\epsilon_{x},\epsilon_{y},\epsilon_{xy},s,c) = 0 \] (4.4)

The main idea is to construct a safe design region within which failure does not occur and independent of ply orientation angle \((\varphi)\). Therefore in order to get it we construct a geometric envelope which is characterized by the failure surfaces 4.4 using the respective ply orientations \((\varphi)\) . Thus the equation for the envelope is given by

\[ \frac{dF}{d\varphi} = 0 \] (4.5)

On expanding equation (4.5) using chain rule we get

\[ \frac{dF}{d\varphi} = c\frac{\partial F}{\partial s} - s\frac{\partial F}{\partial c} = 0 \] (4.6)

The function \( F \) is polynomial function of Cosine and sine. Because sine and cosine are dependent upon the ply angle \((\varphi)\) and have to satisfy the trigonometric condition
4.1 Formation of Omni strain envelope using Tsai-Wu

\[ s^2 + c^2 - 1 = 0 \]  \hspace{1cm} (4.7)

The final form of the failure envelope is obtained by eliminating the sine and cosine from the equations (4.4)(4.6)(4.7) using Dixon’s resultant. This reduction was done using Mathematica and the resulting equation is as shown below [7,36].

\[ 4u_6^2I_2^2 - 4u_6u_1I_2^2 + 4(1 - u_2I_1 - u_3I_1^2)(u_1 - u_6) + (u_4 + u_5I_1)^2 = 0 \]  \hspace{1cm} (4.8)

\[ u_4^2I_2^4 - I_2^2(u_4 + u_5I_1)^2 - 2u_1I_2^2(1 - u_2I_1 - u_3I_1^2) + (1 - u_2I_1 - u_3I_1^2)^2 = 0 \]  \hspace{1cm} (4.9)

The material invariants \((u_i)\) \(i=1,2,...6\) are given in the Appendix A. The equation (4.8) and (4.9) represents surface traced out by the Tsai-Wu failure criteria for all ply orientations. Where \(I_1\) is volumetric strain invariant and \(I_2\) is the maximum shear strain given as

\[ I_1 = \epsilon_x + \epsilon_y \]
\[ I_2 = \sqrt{(\epsilon_x - \epsilon_y)^2 + \epsilon_{xy}^2} \]  \hspace{1cm} (4.10)

The feasible design space, given through the equations (4.8) and (4.9) is material dependent as \(u_i\) is function of \(G_{ij}\) which is again a function of the reduced stiffness matrix \(Q\) and material strength coefficients \(F_i,F_{ij}\). The given equations represent the conservative form of Tsai-Wu failure criterion in terms of strain invariants.

Figure 4.2: Tsai-Wu closed form expressions 2nd and 4th order equations for Carbon/PEEK(AS4)

In Figure 4.2 the second order equations form the inner closed region. Hence, for this material AS4 the critical solution of strains will be the strains that satisfy the second order equation.
Table 4.1: Material Properties [3, 9]

<table>
<thead>
<tr>
<th>Properties</th>
<th>Carbon/PEEK (AS4)</th>
<th>Carbon/epoxy (IM6)</th>
<th>Boron/epoxy (B5.6)</th>
<th>Carbon/PEKK (Cytec-AS4D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal modulus (E₁,GPa)</td>
<td>142.0</td>
<td>177.0</td>
<td>201.0</td>
<td>139</td>
</tr>
<tr>
<td>Transverse modulus (E₂,GPa)</td>
<td>10.3</td>
<td>10.8</td>
<td>21.7</td>
<td>10.3</td>
</tr>
<tr>
<td>Shear modulus (G₁₂,GPa)</td>
<td>7.2</td>
<td>7.6</td>
<td>5.4</td>
<td>5.2</td>
</tr>
<tr>
<td>Poisson’s ratio (μ₁₂)</td>
<td>0.27</td>
<td>0.27</td>
<td>0.17</td>
<td>0.30</td>
</tr>
<tr>
<td>Longitudinal tensile strength (Xₜ,MPa)</td>
<td>2280.0</td>
<td>2860.0</td>
<td>1380.0</td>
<td>2463</td>
</tr>
<tr>
<td>Longitudinal compressive strength (Xₗ,MPa)</td>
<td>1440.0</td>
<td>1875.0</td>
<td>1600.0</td>
<td>1493</td>
</tr>
<tr>
<td>Transverse tensile strength (Yₜ,MPa)</td>
<td>57.0</td>
<td>49.0</td>
<td>56.6</td>
<td>61</td>
</tr>
<tr>
<td>Transverse compressive strength (Yₗ,MPa)</td>
<td>228.0</td>
<td>246.0</td>
<td>125.0</td>
<td>254</td>
</tr>
<tr>
<td>Shear strength (S,MPa)</td>
<td>71.0</td>
<td>83.0</td>
<td>62.6</td>
<td>100</td>
</tr>
</tbody>
</table>

Figure 4.3: Tsai-Wu critical envelope Carbon/PEEK(AS4)

Figure 4.3 represent the critical region alone for AS4. This closed form second order equation contains only the material values and properties and is not dependent on the ply orientation.
4.1 Formation of Omni strain envelope using Tsai-Wu

(a) Tsai-Wu closed form expressions 2nd and 4th order equations for Carbon/Epoxy(IM6)

(b) Tsai-Wu critical envelope Carbon/Epoxy(IM6)

**Figure 4.4:** Tsai-Wu omni strain regions based on 2nd and 4th order equation solutions for Carbon/Epoxy(IM6)

The critical region traced by the second order equation is shown in Figure 4.4b. This strain envelope is similar to the one traced for AS4 as their ratio’s of stiffness $E_1/E_2$ are close to each other. For material IM6 the ratio of stiffness is bit higher than AS4 leading to some intersection regions of the 4th order equation itself. But the second order equation in itself forms the inner region.

The Figure 4.3 and Figure 4.4b represent the second order equation as it forms in the closed inner intersecting region. In Figure 4.5b and Figure 4.6b fourth-order equation describes the inner envelope, it is factor-able into the product of two equations leading to a self-intersecting non-smooth envelope.

It is evident that the feasible design region lies with in the closed form equations (4.8), (4.9). The safe region is the common intersecting region of Tsai-Wu failure envelope for all the angles. The omni strain region entirely dependent on the material under consideration. For better understanding of this feasible design region envelopes, different materials are considered Table:[4.1] and their design critical envelopes are plotted. The materials are chosen in such a way that they have different stiffness ratios. $E_1/E_2$ are considered ranging from 9 to 17. In all these cases $\epsilon_{xy}$ has been set to zero. It is evident from the figures that either one of the equations second (4.8) or Fourth (4.9) order equations prescribe the inner envelope and is independent of the fibre orientation. It is clear from the observations that critical envelope is symmetric around an imaginary $45^\circ$ line. This is mainly because when the signs of tensions and compression are reversed, it yields the equivalent strains with only change in direction.
Figure 4.5: Tsai-Wu omni strain regions based on 2nd and 4th order equation solutions for Boron/Epoxy(B56)

(a) Tsai-Wu closed form expressions 2nd and 4th order equations for Boron/Epoxy(B56)

(b) Tsai-Wu critical envelope Boron/Epoxy(B56)

Figure 4.6: Tsai-Wu omni strain regions based on 2nd and 4th order equation solutions for Carbon/PEEK(AS4D)

(a) Tsai-Wu closed form expressions 2nd and 4th order equations for Carbon/PEEK(AS4D)

(b) Tsai-Wu critical envelope Carbon/PEEK(AS4D)
The fibrous composite material is composed of both fibres and matrix. In this mixed built up with wide variations of fibre orientations different failure modes cease to exist. The failure criteria should include all the mixed modes of failure in order to predict and represent the exact behaviour of the laminate. To assist this Nasa Langely Research Centre with its renowned scientists came up with the criteria which can predict and include multiple modes of laminate failure LaRC03 [2]. This criteria is well established that it predicts the failure of system closest to the experimental results (WWFE) [37]. LaRC03 criteria was based on plane stress assumptions which includes special relations for matrix tension and Compression failures. A new fibre kinking failure criterion for fibre compression is also established by using the matrix failure criteria to the configuration of kink. This results in six sets of failure criteria and is denoted as LaRC03. Thus this criterion is more valid in all the loading regions as it correctly predicts the failure of composites in all the failure modes. It has to be noted that this theory takes assumptions of inbuilt material fractures and hence the failure is not exactly the first ply failure.

In order to develop the Omni strain envelope using LaRC03, all the possible fibre combinations have to be checked for failure in all the possible failure modes. This makes the algorithm bit more complex as multiple failures have to be checked. Continuously checking for the failure against all the failure modes for a bi-axial loading will result in-plane failure envelope for each ply in the laminate. This can further be interpreted to get the omni strain envelope.

Figure 4.7: LaRC03 ply failure envelopes for Carbon/PEEK(AS4)

Quasi-isotropic laminates are considered for predicting the omni strain region using LaRC03 failure criteria. As they have almost same material properties in all in-plane directions analysing 0, 45 & 90 degree plies would be enough to determine the omni strain region. The Figure 4.7 shows the traced failure regions of three angled plies. In Figure 4.7 the failure regions of individual fibres are traced for a quasi-isotropic laminate.
The equations for the system of failure is given in LaRC03 is as follows:

<table>
<thead>
<tr>
<th>Failure mode</th>
<th>LaRC03 Mode</th>
<th>Equation FI (Failure Index)</th>
<th>Failure &amp; validity Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix Compression ($\sigma_{22} &lt; 0$)</td>
<td>LaRC03#1</td>
<td>$FI_m = \left( \frac{\tau_{eff}}{S_{eff}} \right)^2 + \left( \frac{\sigma_{22}}{S_{ts}} \right)^2$</td>
<td>$FI \leq 1$ &amp; $\sigma_{11} \geq Y_c$</td>
</tr>
<tr>
<td>Matrix Tension ($\sigma_{22} \geq 0$)</td>
<td>LaRC03#2</td>
<td>$FI_m = (1 - g)\left( \frac{\sigma_{22}}{Y_{ts}} \right)^2 + g\left( \frac{\tau_{eff}}{S_{eff}} \right)^2$</td>
<td>$FI \leq 1$</td>
</tr>
<tr>
<td>Fibre Tension ($\sigma_{11} \geq 0$)</td>
<td>LaRC03#3</td>
<td>$FI_f = \frac{\epsilon_{11}}{\epsilon_T}$</td>
<td>$FI \leq 1$</td>
</tr>
<tr>
<td>Fibre Compression ($\sigma_{11} &lt; 0$)</td>
<td>LaRC03#4</td>
<td>$FI_f = \left( \frac{\sigma_{22}}{Y_{ts}} \right)^2$</td>
<td>$FI \leq 1$ &amp; $\sigma_{22}^m &lt; 0$</td>
</tr>
<tr>
<td>Fibre Kinking ($\sigma_{11} &lt; 0$)</td>
<td>LaRC03#5</td>
<td>$FI_f = (1 - g)\left( \frac{\sigma_{22}}{Y_{ts}} \right)^2 + g\left( \frac{\tau_{m22}}{S_{22}} \right)^2$</td>
<td>$FI \leq 1$ &amp; $\sigma_{22}^m \geq 0$</td>
</tr>
<tr>
<td>Bi-axial Matrix compression (Matrix Kinking) ($\sigma_{22} &lt; 0$)</td>
<td>LaRC03#6</td>
<td>$FI_m = \left( \frac{\tau_{mT}}{S_{mT}} \right)^2 + \left( \frac{\tau_{mL}}{S_{mL}} \right)^2$</td>
<td>$FI \leq 1$ &amp; $\sigma_{11} &lt; Y_c$</td>
</tr>
</tbody>
</table>

![Figure 4.8: LaRC03 omni strain envelope for Carbon/PEEK(AS4)](image-url)
Algorithm 1 LaRC03 Omni strain Algorithm

1: procedure LaRC03 Algorithm
2:   for i=1 to number of layers in laminate (l) do
3:     for θ=0 to 360 do //Variable θ is used to simulate the effect of Bi-axial loading
4:       The strains inputs given for solving the scaling variable λ are given as
5:       $e_i = \cos(\theta)$ and $e_{ii} = \sin(\theta)$
6:       Mixed failure modes of LaRC03 are implemented which returns a value
7:       New F.I=0 at failure.
8:       //The New F.I =F.I-1 where F.I=1 allocated for all modes of failure
9:       The values of λ and corresponding strains $e_i$, $e_{ii}$ are stored for each layup
10:   end for
11: end for
12: end procedure LaRC03 ALGORITHM

13: procedure LaRC03 Omni-Strain Algorithm
14:   // Getting the omni strain envelope involves
15:   for j=1 to number of layers in laminate (l)-1 do
16:     The common region of the envelopes is got by comparing one fixed with others and
17:     finding the common region of points.
18:     This is achieve by using Intersection command Polybool(’intersection’) in Matlab
19:     This step is repeated for all layers and the common region points are found
20:   end for
21:   Plotting common region points and its strain values will give Omni strain Envelope
22: end procedure

Figure 4.9: LaRC03 omni strain regions based on material properties of Carbon/Epoxy(IM6)

The Table 4.2 illustrates the conditions for different modes of failures to happen in a composite laminate. All these conditions have to be checked for at each load step in order to check for
failure of any lamina in the laminate. The failure is characterized by the failure index which attains a value of one at failure point. This failure index is dependent on the stresses in the layer and depending on each type of failure the validity conditions were also set.

Figure 4.10: LaRC03 omni strain regions based on material properties of Boron/Epoxy(B5.6)

Figure 4.11: LaRC03 omni strain regions based on material properties of Carbon/PEEK(AS4D)

It is inferred through the Figures 4.8, 4.9b, 4.10b & 4.11b that the omni strain region traced using LaRC03 resembles the omni strain envelopes of Tsai-Wu.
4.3 Analysis of Tsai-Wu and LaRC03 omni strain regions

It is evident through Figure 4.12a & 4.12b that for bi-axial loading Tsai-Wu over predicts the failure in the compression-compression region. Apart from this region in the other quadrants Tsai-Wu tends is more conservative and predicts failure well below the actual failure. This is also well proven in the (WWFE) [37] that in biaxial compression region Tsai-Wu doesn’t match with the experimental results. The failure mode in the compression-compression region can be characterized by matrix properties which are not considered in Tsai or any local insitu properties of strain. It can be seen that for all material cases both the omni strain remain to be symmetric around a imaginary 45° line implying almost similar failure in laminates which are perpendicular to each other.
Chapter 5

Omni strain envelope with open hole

The analysis of omni strain region under the influence of open hole is critical, as hole can further deteriorate the strength of composite. The omni strain envelope tends to change with the influence of stress concentration. The effect of stress concentration on the critical envelope is crucial to study as it forms the weak zone for failure. The understandings of the stress concentration solution discussed in chapter 3 with the construction procedure of omni strain region discussed in chapter 4 will be helpful in this chapter to form a new design region. An analytic solution for quasi-isotropic laminates which can predict the omni strain envelope without analysis of stresses is also presented in section 5.2. The chapter ends with detailing the region of validity of the solution.

5.1 Stress concentration envelope using Tsai-Wu

The Tsai-Wu closed form equations 4.8 & 4.9 which represent the omni strain region for any composite material can be used to predict the influence of omni strain envelope. The Tsai-Wu envelope equation can be scaled by using strength constraint as an objective function, for optimization, the equations of design envelope can be reformulated in terms of the safety factor \( \lambda \), which is defined as:

\[
\lambda = \frac{b}{a}
\]  

(5.1)

where \( a \) is the distance between the origin and an arbitrary point \( P \) in design space, and \( b \) is the length of the vector from origin through point \( P \) to the point on the envelope boundary \( P^* \), as shown in Figure 5.1.

In general, \( \lambda \) is a scaling factor that, when applied to the values of \( \epsilon \) at any point \( P \) representing applied strains, gives the values of \( \epsilon^* \) at the corresponding point on the boundary \( P^* \). Therefore, the strain invariants \( I_1 \) and \( I_2 \) can be related to those at the boundary of the failure envelopes by substituting \( I_1^* = \lambda I_1 \) and \( I_2^* = \lambda I_2 \) into the failure envelope Equations 4.8 and 4.9, yielding two polynomial equations in terms of \( \lambda \).
Algorithm 2 Tsai-Wu omni strain solution with hole Implementation
1: procedure Tsai-Wu omni strain solution with hole Implementation
2:   for i=1 to 360 do
3:     // The Bi-axial loading is characterised as \( N_x = \cos(i) \) \( N_y = \sin(i) \)
4:     The equations for for max and min are given in 5.10 and 5.9
5:     Strains found using inplane stiffness and unidirectional property of stress vector
6:     // As the stresses are varying the max and min have to be reassigned in the loop
7:     if sigtemp(1,1) > sigtemp(2,1) then
8:       sig1=sigtemp(1,1)
9:       sig2=sigtemp(2,1)
10:      else
11:        sig1=sigtemp(2,1)
12:        sig2=sigtemp(1,1)
13:     end if
14:     // The scaling factors found using Tsai-wu is multiplied with analytic stresses
15:     factorcompression=abs(r+*(sig2))
16:     factortension=(r+ * (sig1))
17:     // The critical of tension and compression is found using
18:     rana=max(factorcompression,factortension)
19:     // The original varying strains based on loading is found using
20:     strainana=inv(A)*[N_x;N_y;0]
21:     strainx=strainana(1,1)/rana
22:     strany=strainana(2,1)/rana
23:     // These strains represent the scaled strain based on Analytic expression
24:   end for
25: end procedure

Figure 5.1: Scaling factor \( \lambda \) with respect to arbitrary point \( P \) with in design envelope and the corresponding point \( P^* \), on the boundary [7]
5.1 Stress concentration envelope using Tsai-Wu

\[ f_1(\lambda) = a_{12}\lambda^2 + a_{11}\lambda + a_{10} \]  

(5.2)

\[ f_2(\lambda) = a_{24}\lambda^4 + a_{23}\lambda^3 + a_{22}\lambda^2 + a_{21}\lambda + a_{20} \]  

(5.3)

where the coefficients \( a_{ij} \) are functions of \( I_1 \) and \( I_2 \) and are listed in Appendix A. Solving for \( \lambda \) yields up to six roots; the equation with the smallest positive real root represents the active envelope, because the smallest safety factor is critical. The active envelope is not independent of the strains. It is also noted that the fourth-order envelope self-intersects and can therefore be thought of as two smooth curves, as can be seen in Figure 4.5b. Corresponding to each of these two smooth curves is a positive root \( \lambda \), which is a continuous function of strains.

It is shown in chapter 3 that irrespective of the laminate configuration, the stress at hole edge is found to be unidirectional \( \sigma_\theta \). In most cases the hole edge becomes the most critical region for failure as stresses are more concentrated in that region. Because of this property at the hole edge, the strains can be given in the form:

\[ \epsilon = A^{-1}\sigma_\theta \]  

(5.4)

Given the laminate layup and other properties, we can get the \( \sigma_\theta \) using the results discussed in section 3.2.3. This Stress Concentration Factor (SCF) is taken for the whole laminate and not for a single ply as the material critical envelope is not dependent on any layup.

The scaling parameter for open hole can be obtained by substituting the \( \epsilon \) from equation 5.4 into equations 5.2 & 5.3 and solving for the scaling parameter \( \lambda \) & \( r \). The most critical value of \( r \) is found for varying load cases and is used to scale the corresponding strains produced due to loading. A short algorithm is presented which will be useful for better understanding.

In case of a quasi-isotropic laminate of AS4 the omni strain envelope with stress concentration is as shown in figure 5.2. It can be seen that the elliptical omni strain material envelope forms a parallelepiped envelope region under the influence of stress concentration. It is seen that the region formed is not a simple linear scaled version of the material failure region as the scaling changes with respect to loading in all regions of the quadrant. A similar behaviour of Tsai-Wu failure over prediction in the compression-compression region can be seen in new omni strain region as well , this comparison is in turn with the other loading regions formed.
5.2 Analytical solution for stress concentration envelope

A simplified analytic solution will be easier to apply in order to estimate the design safe limit of a composite structure. Getting an analytic solution which is not dependent on the stress state will prove to be more useful. The formulation of an analytical expression to estimate the failure envelope region is presented below.

For an isotropic plate with uniaxial loading the analytic solution for the stresses around hole [38, 39] is given by

\[
\sigma_{rr}(r, \theta) = \frac{\sigma}{2} \left[ \left( 1 - \frac{a^2}{r^2} \right) + \left( 1 + 3 \frac{a^4}{r^4} - 4 \frac{a^2}{r^2} \right) \cos(2\theta) \right] \tag{5.5}
\]

\[
\sigma_{\theta\theta}(r, \theta) = \frac{\sigma}{2} \left[ 1 + \left( \frac{a^2}{r^2} \right) - \left( 1 + 3 \frac{a^4}{r^4} \right) \cos(2\theta) \right] \tag{5.6}
\]

\[
\sigma_{r\theta}(r, \theta) = -\frac{\sigma}{2} \left[ \left( 1 - 3 \frac{a^4}{r^4} + 2 \frac{a^2}{r^2} \right) + \sin(2\theta) \right] \tag{5.7}
\]

At \( a=r \) (i.e) when stresses are measured at the hole edge, \( \sigma_{rr} \) and \( \sigma_{r\theta} \) becomes zero only \( \sigma_{\theta\theta} \) remains non zero. So, for biaxial loading \( \sigma_{\theta\theta} \) can be written as
\[ \sigma_{\theta\theta} = N_x \left( 1 - 2\cos(2\theta) \right) + N_y \left( 1 - 2\cos(2\varphi) \right) \]

For loading in Y direction \( N_y \) we can write it in terms of \( \theta \) as \( \varphi + \frac{\pi}{2} = \theta \)

\[ = N_x \left( 1 - 2\cos(2\theta) \right) + N_y \left( 1 - 2\cos(2\theta - \pi) \right) \]

\[ = N_x \left( 1 - 2\cos(2\theta) \right) + N_y \left( 1 - 2\cos(2\theta) + 2\cos(\pi) \right) \]

\[ = N_x \left( 1 - 2\cos(2\theta) \right) + N_y \left( 1 - 2\cos(2\theta) - 2 \right) \]

\[ = N_x \left( 1 - 2\cos(2\theta) \right) + N_y \left( -1 - 2\cos(2\theta) \right) \]

\[ = N_x \left( 1 - 2\cos(2\theta) \right) - N_y \left( 1 + 2\cos(2\theta) \right) \]

Expanding the equation we get

\[ = N_x - N_y - 2 \left( N_x + N_y \right) \cos(2\theta) \]

Thus based on the \( \cos(2\theta) \) variation we get

\[ \sigma_{\theta\theta} = N_x - N_y \pm 2 \left( N_x + N_y \right) \]

\[ (5.8) \]

Thus we get the minimum and maximum equations as

\[ \text{Min } \sigma_{\theta\theta} = (3N_y - N_x) \]

\[ (5.9) \]

\[ \text{Max } \sigma_{\theta\theta} = (3N_x - N_y) \]

\[ (5.10) \]

The equations 5.9 & 5.10 represents the minimum and maximum possible unidirectional stress \( \sigma_{\theta} \) at the hole junction. These equations are independent of the stresses caused due to hole and are only dependent on the loading thus making it easier to evaluate stresses without solving for the complex stress solution around the hole in composite.
The analytic solution procedure is implemented for biaxially loaded composite and the result is shown in the Figure 5.3. It is clear from the solution that it is symmetric around an imaginary $45^\circ$ line. It can also be seen that compared with all the areas four quadrants the region in compression-compression is bigger, this is because of the combined effect of loading in this region and the failure prediction capabilities of Tsai-Wu failure model.

5.3 Correlation of Analytical solution with the Tsai-Wu failure based model

A comparison is made between the proposed analytical solution discussed in the previous section with the reduced material envelope of Tsai-Wu in the presence of the open hole. This study is vital to establish that the analytical solution can predict the same behavior of the reduction without solving for the complex stress functions around the hole.
The Figure 5.4 shows that both the design failure envelopes traced by Tsai-Wu omni strain and the analytical expression correlate with each other predicting the same failure at the boundary of the hole. These results were checked for multiple quasi-isotropic material types and found to be consistent in predicting the same failure behaviour. The analytic solution is proven for quasi-isotropic laminate because the in-plane stress distribution around hole is similar to isotropic laminate whereas in other laminates because of anisotropy the stress distributions are different.

The closed figure forms a parallelogram shaped region to predict the failure. It can be said from the inference that a closed form analytic solution which is only dependent on loads can characterize the behaviour of a open hole stress concentration failure in laminate in biaxial loading. i.e. only knowing the failure loads from Open Hole Tension (OHT) and Open Hole Compression (OHC) tests of a quasi-isotropic laminate one can develop the design failure region for biaxial loading.
5.4 Evaluating Failure at a distance away from hole

In case of combined loading it is possible for failure not to happen at the load, in this case evaluating failure outside the hole is critical. In practice it is done using Whitney-Nuismer approach of using the characteristic distance was discussed in section 3.2.1. This method can be used to determine the failure inside the domain i.e. evaluating failure at a point away from the hole. So this method can be potentially used to determine the failure at any point near to hole. It is critical and important to analyse the validity of analytic solution away from the hole. The following strategies were adopted to analyse the applicability of solution.

5.4.1 Failure analysis at far field

The failure has to be analysed inside the built up domain of composite, in order to determine and analyse the applicability of solution. To analyse this Failure envelopes are plotted at radius distances of $R=1$ to 5 with increments of 0.5, where radius $R=1$ represents the hole edge. The material omni strain region traced based on Tsai-Wu failure is taken as the critical envelope for scaling. At very large distances the critical region in presence of hole should re-traced the Tsai-Wu material omni strain envelope because of the fact that the effect of stress concentration will be less at far field. It is interesting to note that the scattering of failure points based on radius is not linear and varies on the loading biaxial schemes.

![Failure envelopes plotted at intervals of R=1 to 5 with increments of 0.5 (increasing envelopes of blue dots). Inner green box denotes the omni strain region at hole edge(R=1). The envelopes are traced using Tsai-Wu omni strain failure region for AS4.](image)

**Figure 5.5:** Failure envelopes plotted at intervals of $R=1$ to 5 with increments of 0.5 (increasing envelopes of blue dots). Inner green box denotes the omni strain region at hole edge($R=1$). The envelopes are traced using Tsai-Wu omni strain failure region for AS4.
5.4 Evaluating Failure at a distance away from hole

Figure 5.6: Failure envelopes plotted at intervals of R=1 to 5 with increments of 0.5 (increasing envelopes of blue dots). Inner green box denotes the omni strain region at hole edge (R=1). The envelopes are traced using Tsai-Wu omni strain failure region for B56.

The Figure 5.6 represents possible failures at far field of the hole for material B5.6 traced using Tsai-Wu failure condition. Through these results presented above we get to know that the failure envelopes do not intersect each other at greater radius of measurement from the hole. This result is proven for all the second and fourth order based omni strain critical regions based on Tsai-Wu as its verified for multiple materials.

5.4.2 Failure analysis around close vicinity of hole

It is important to establish the validity of solution closer to the hole as well. The regions closer to the hole is considered for this analysis of solution, the increment is made such small so that we could analyse the validity at small radius. The failures are calculated from R=1 (hole edge) to R=1.5 with increments of 0.1. The results of this analysis are shown in the graphs below 5.7 and 5.8.

The finding through these analysis revels that at some point closer to the vicinity of hole there exists a region of higher stress (at higher radius) compared to the stress at the same angle at smaller radius. This is mainly due to the loading cycle and the corresponding stresses fluctuating stresses produced. It is to be noted that outside the hole edge the stress state is not uni-axial anymore i.e not only dependent on $\sigma_\theta$. These stress effects cause the failure regions to intersect and serves as a limiting point of validity. The results are checked for multiple materials and limiting region is well discussed in the sections coming after.
Figure 5.7: Failure envelopes plotted at intervals of $R=1$ (hole edge) to 1.5 with increments of 0.1 (increasing envelopes of blue dots). Inner green box denotes the omni strain region at hole edge. The envelopes are traced using Tsai-Wu failure based omni strain region for AS4.

Figure 5.8: Failure envelopes plotted at intervals of $R=1$ (hole edge) to 1.5 with increments of 0.1 (increasing envelopes of blue dots). Inner green box denotes the omni strain region at hole edge. The envelopes are traced using Tsai-Wu failure based omni strain region for B5.6
5.4 Evaluating Failure at a distance away from hole

5.4.3 Failure Analysis using Whitney Nuismer characteristic Distance

In process to determine the region of validity the Whitney-Nuismer solution is used, based on Whitney-Nuismer model of point stress criterion \(d_0\) which is based on experiments the characteristic distance value \(d_0\) is 0.04\[^{[25]}\]. This distance predicted by Whitney & Nuismer is purely based on certain experimental results on multiple materials and through data analysis this characteristic distance was found. Hence in order to relate it with the hole size, the measurement region \(R\) is taken as:

\[
R = 1 + \frac{0.04}{Holesize}\quad (5.11)
\]

The hole size is varied from 0.05 to 0.5 with increments of 0.05.

\textbf{Figure 5.9:} Failure envelopes for varying hole sizes at whitney-Nuismer characteristic distance \(d_0\) for as increasing radius (blue dot envelopes). Red line denotes the material omni strain region without hole for AS4.
It is clear from the results that in combined loading cases of tension and compression there is an interference of the curves with the failure envelopes traced at lower radii. This proves that the whitney Nuismer characteristic distance $d_0$ cannot be applied to predict the failure in all possible load combinations. It is clear from the procedure implemented that this approximation distance becomes problem if the hole size is decreased beyond a certain point. It can be inferred that as the hole becomes smaller, then the failure is not because of the hole. So, in order to determine the valid region of solution, the scaling factor ($r$) have to be checked this is done in the next section.

**Figure 5.10:** Failure envelopes for varying hole sizes at whitney-Nuismer characteristic distance $d_0$ for as increasing radius (blue dot envelopes). Red line denotes the omni strain region at far field for B5.6
It is important to analyse and evaluate the valid region of the solution. This can be done by using the values of scaling factor \((r)\) along the radius. The scaling factor based on Tsai-Wu defines the amount to be scaled to get the solution at some radius. If the scaling factor is plotted along \(\sigma_\theta\) then the local minimum point of radius will serve as the limit of solution. The schematic of the physical meaning and the procedure for determination of the limiting point can be found in Figure 5.11.

Figure 5.11: Determination of distance of validity of solution based on Tsai-Wu failure scaling\((r)\) for AS4

It can be seen that on going inside the domain, away from the hole edge the stress is not unidirectional any more and the effect of all the three stresses \(\sigma_\theta\), \(\sigma_r\) and \(\sigma_{r\theta}\) have to be taken into account while calculating the scaling factor based on Tsai-Wu closed form expression. The variation of scaling factor\((r)\) at various radius for a particular loading in \(x\) and \(y\) corresponding sinusoidal loading to \(135^\circ\) can be seen in Figure 5.11b. It is clear from the Figure 5.11b that the angle measurements over laps each other and the top curve is not formed for a particular angle. so multiple angles have to be computed and supper imposed to get the limiting curve.
Figure 5.12: Determination of the distance (R1) valid region of solution
5.5 Evaluating the valid region of solution

The Figure 5.12b represents the characteristic curve which denotes the maximum possible variation of scaling factor (r) for all the angles as blue line, with the points of variation of radius in x-axis. Through this analysis it can be seen that there exists a local maxima after getting a local minima in the curve hence, the corresponding value of the scaling factor (r) in the local maxima will be the used to determine the limiting radius (R1). Extending the point of local maxima to get a point in the curve before the local minima is produced (point denoted in red colour in Figure 5.12b) which is used to determine the limiting distance (R1).

It is found that the valid region of the envelope is dependent on the radius of measurement, the loading conditions and the stresses induced near hole. This helps to calculate the failure based on Tsai-Wu closed form equations to form limiting the omni strain envelope. For each of the material cases discussed so far, there exists a previously found overlapping region in second and fourth quadrant of biaxial loads, the limiting radius is found using the scaling factor (r) for the same load case and the validity of solution is proven.

The analysis is conducted for material cases AS4, IM6 and B5.6 with material properties in Table 4.1 and the maximum valid region of solution based on Tsai-Wu failure criterion can be seen in figures 5.13, 5.14 and 5.15. The limiting radius distances (R1) calculated are shown in table below.
### Table 5.1: Solution validity region based on Tsai-Wu failure for various materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Limiting radius (R1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon/PEEK(AS4)</td>
<td>1.20</td>
</tr>
<tr>
<td>Carbon/Epoxy(IM6)</td>
<td>1.24</td>
</tr>
<tr>
<td>Boron/Epoxy(B5.6)</td>
<td>1.26</td>
</tr>
</tbody>
</table>

The limiting values from table 5.1 is not always constant and varies with the material. This is because, the limiting radius(R1) is dependent on the stresses at particular location on a laminate. The figure 5.15 & 5.14 shows the valid region of Tsai-Wu failure based omni strain envelope for an open hole.
Figure 5.15: Valid region of solution inside the domain (blue dot envelopes showing the increasing omni strain envelopes with increasing radius) Red region is the omni strain envelope for material without hole (B5.6)
Omni strain envelope with open hole
Chapter 6

Validation of the Solution

The analytic solution presented in chapter 5 is only dependent on the loadings in $N_x$ and $N_y$ and is valid around the hole. Applying this for a laminate with many angled plies, it is possible to estimate the design safe region beyond which the laminate is prone to failure. To evaluate the applicability of the analytical solution it has to be compared against real time experimental results. However, in the available experimental results from literature, the data corresponding to first ply failure is not reported, rather the final failure loads are reported. The failure values attained through experiments are usually final failures of the laminate. In this chapter the results from analytical model is compared with the experimental results in section 6.1 and a brief discussion is presented on the use of solution. This solution approach is also validated by comparing it with results of numerical models and is discussed in section 6.3.

Figure 6.1: Schematic shows biaxial loads being applied on a composite laminate, the loads can either be in tension or in compression
6.1 Experimental results

The experimental setup needed to conduct biaxial tests are complex and requires special tooling the setup has to be devised in such a way that it can provide loads in all different load regimes of tension-tension, tension-compression and compression-compression. Proper evaluation of failures have to be defined as multiple mixed modes of failure exists in composite laminates. The calibration of measuring devices also plays a vital role in determining the accuracy of the solution. An experimental study using biaxial loads was recently conducted by Wilhelmus et.al [9] for determining the strength of notched thermoplastic composite and reported. Their study was primarily done to study the strength values of notched and un-notched composite laminate and focussed on a particular material they used. Their test took Uni-Directional (UD),Carbon Fibre Reinforced Plastic (CFRP) quasi- isotropic laminate[45/0/ − 45/90]s of AS4D material properties presented in Table 4.1 with a ply thickness of 0.14mm. The experimental set-up was made on the basis of the ASTM test system for crack propagation studies under biaxial loading [8]. The principle of operation of the test set-up and the plan of the grip to apply loads into specimen is given in [8].

![Diagram of experimental setup](image)

**Figure 6.2:** Method of applying biaxial load to curiform specimen with two actuators; the specimen passes through the horizontal centre frame(reproduced from [8])
6.1 Experimental results

The Figure 6.2 shows the schematic view of the test setup, in order to conduct a study the set-up will also have lot of actuators and measuring devices attached to the frame. In their study not all variations of load $N_x/N_y$ is studied as it is difficult to implement and requires highly sophisticated tools to automate the process. A special anti-buckling guide (ABG) was designed using DIC strain measurements to predict the failure in biaxial tension-compression and compression-compression load cases. The load cases they considered are presented in the Table 6.1.

<table>
<thead>
<tr>
<th>Bi-axial test code</th>
<th>Description</th>
<th>Expected stress/strain in section</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>uni-axial tension validation test in 1-direction</td>
<td>$\sigma_1 &gt; 0; \sigma_2 = \sim 0; \epsilon_1 &gt; 0; \epsilon_2 &lt; 0$</td>
</tr>
<tr>
<td>C2</td>
<td>uni-axial tension validation test in 2-direction</td>
<td>$\sigma_2 &lt; 0; \sigma_1 = \sim 0; \epsilon_1 &gt; 0; \epsilon_2 &lt; 0$</td>
</tr>
<tr>
<td>T1/T2</td>
<td>biaxial tension/tension test</td>
<td>$\sigma_1 = \sigma_2 &gt; 0; \epsilon_1 &gt; 0; \epsilon_2 &gt; 0$</td>
</tr>
<tr>
<td>2T1/T2</td>
<td>biaxial tension/tension test</td>
<td>$\sigma_1 = 2(\sigma_2) &gt; 0; \epsilon_1 &gt; 0; \epsilon_2 &gt; 0$</td>
</tr>
<tr>
<td>T1/C2</td>
<td>biaxial tension/compression test</td>
<td>$\sigma_1 = -\sigma_2 &gt; 0; \epsilon_1 &gt; 0; \epsilon_2 &lt; 0$</td>
</tr>
<tr>
<td>C1/C2</td>
<td>biaxial compression/compression test</td>
<td>$\sigma_1 = \sigma_2 &lt; 0; \epsilon_1 &lt; 0; \epsilon_2 &lt; 0$</td>
</tr>
</tbody>
</table>

The failure stresses found for different load cases are plotted in form of a failure envelope, which represents the combinations of biaxial loading. The Figure 6.3 shows the experimental failure stress as red points where the laminate fails and the black points denote the failure initiation point. A characteristic curve is also drawn using Whitney-Nuismmer point stress method (blue line) and is found to be in good agreement with the experimental results. The test failure initiation data points show closer correlation with the theoretically predicted envelope, and considered close enough to first ply failure.

![Figure 6.3: Failure envelopes for open-hole specimens of AS4D quasi- isotropic laminate (reproduced from [9])]
6.2 Comparison of solution with test data

The solutions reported in various experimental literature should be compared in order to establish the correctness and accuracy of the method. In this process, it has to be taken care that relevant quantities should be matched with each other, so that the study is more valid without limitations. In the case of omni-strain solution, verification with experimental results is always difficult as it represents the region where failure will never occur in any lamina. All the real time experimental results & solutions presented are observed failure of the laminate and not based on lamina.

6.2.1 Comparison with experimental test data

The experimental results shown in section 6.3 is compared with the proposed solution procedure for Tsai-Wu failure and LaRC03 failure. The same material properties of experimental test results are used in getting the analytical solution. The experimental result shows the final failure of the composite laminate considered. A special anti buckling guide was designed in their experimental set-up and used in tension-compression and compression-compression regions. These might be the limitations to be considered while comparing the omni-strain envelope with the experimental results. For getting a good fit with the experimental results they used puck failure combined with Whitney-nuismer point stress method as claimed in the literature.

![Comparison of omni-strain envelopes based on LaRC03 failure with the open hole against experimental results](image)

**Figure 6.4:** Comparison of omni-strain envelopes based on LaRC03 failure with the open hole against experimental results

The LaRC03 omni strain solution presented is extended to open hole failure and is compared with experimental results are shown in Figure 6.4. It can be inferred that the failure solution
6.2 Comparison of solution with test data

presented for hole (R=1) is well inside the original failure thus confirming the solution to be invariant and conservative in all possible regimes of biaxial loading. Further, the distance approach was used in order to predict the final failure using omni-strain envelope. This resulted in getting a good fit when failure is evaluated at a radius of 1.4. It can be seen that at this radius all the peak points in second, third and fourth quadrants lie outside the experimental results proving the claim made by the author. He claimed buckling as the major failure phenomenon in majority of his mixed loading cases of tension-compression and compression-compression loading. This out-of plane failure behaviour is well characterised by the reduction in strengths seen in these quadrants. It can be seen that LaRC03 omni-strain implementation along with determination of its characteristic distance can predict the failure of a laminate to the closest.

![Experimental fitted results](attachment:image)

**Figure 6.5:** Comparison of omni-strain envelopes based on Tsai-Wu failure with the open hole against experimental results

In the second analysis Tsai-Wu omni strain envelope for open hole is drawn for the material considered in their experimental study and its results are shown in Figure 6.5. It can be seen that the solution at the hole edge is more conservative and lies well inside the experimental failure data points. The Tsai-Wu is extended till its limiting distance, for this case it is found to be R=1.3 from the hole edge (R=1). It is seen that the maximum possible region of Tsai-Wu without intersection of curves lies well inside the material experimental failure region. This is due to the main fact that the experimental predicted failure region with open hole lies outside the omni strain envelope of Tsai-Wu without hole Figure 6.12, thus limiting the possibility of Tsai-Wu to trace beyond the material failure point. Whereas in case of LaRC03 it is possible because the material curve without hole lies outside the experimental curve. This phenomenon is mainly due to the considerations included in the failure theories and it can be seen from analysis of [40] that LaRC03 is better in predicting results closer to experimental test data.
6.3 Numerical results

In case of multi layered composites failure prediction using Finite Element Method (FEM) is proven to be always time consuming process. The method to predict the failure in a particular layer is always a problem and progressive damage modelling has to be done in order to get good results. A numerical model was developed by Pranav and Melo et.al [10] to calculate the failure of open hole composite laminates. In their model a standard square plate with dimensions shown in Figure 6.6 is taken for analysis. FEM was chosen for numerical modelling and standard commercial software ABAQUS is used in their model. In order to calculate the mesh size required for computation, Open hole tension (OHT) and Open hole compression (OHC) values are compared to against standard reference values. Progressive damage model was used using UMAT subroutine implemented in ABAQUS by this method the material properties of the failed elements are degraded once specified failure criterion is satisfied. They used Tsai-Wu quadratic failure as damage initiation criteria. The degradation factors used were estimated from the validation methods for OHT and OHC strengths against known values. Quasi static displacement controlled loading was used to conduct the analysis. The tests were conducted on two quasi-isotropic 4mm thick [45/90/−45/90]_4s and [0/±60]_5s laminates, these are chosen as they did not show noticeable buckling under compressive load. The material considered in their analysis is IM7/8552 composite and its properties are given in Table 6.2.

Table 6.2: Lamina properties of IM7/8552 composite [10]

<table>
<thead>
<tr>
<th></th>
<th>$E_{11}$</th>
<th>$E_{22}$</th>
<th>$G_{12}$</th>
<th>$\mu_{12}$</th>
<th>$X_t/X_c$</th>
<th>$Y_t/Y_c$</th>
<th>$S$</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>150</td>
<td>11</td>
<td>4.6</td>
<td>0.3</td>
<td>2400/1690</td>
<td>111/250</td>
<td>120</td>
<td>0.125</td>
</tr>
</tbody>
</table>

Figure 6.6: (a) Dimensions of the laminate taken for analysis (b) Structured meshing scheme adopted using S4R elements (reproduced from [10])
The Figure 6.7 represents the damaged zones of failure due to an equal biaxial load, thus in a quasi-isotropic laminate failure behaviours on 1 and 2 directions are the same.

![Figure 6.7: (a) Initial damage on $\{45/90\} - 45/90\}$, (b) total damage in the model at the end of loading (reproduced from [10])]}

It can be seen through Figure 6.8 that the shape of ultimate failure envelope at points A and B deviates from the box shape formed by the points of uni-axial tension and compression strengths. It shows a larger strengths are possible at the corners than initially predicted uni-axial tension and compression strengths. At point B with reference to uni-axial tension and compression, the damage propagates perpendicular to the fibres thus causing fibre failure. Therefore, this perpendicular direction stress would inhibit damage propagation and increase in strength. This results in resulting values for tension-tension be higher than in uni-axial tension. Similarly, compression-compression stress state results in higher strength than uni-axial compression.

### 6.3.1 Comparison with numerical analysis test data

The numerical solution presented in the Section 6.3, is compared to the analytical solution. The numerical solution for ultimate failure from Figure 6.8, is converted to strain space and is shown in Figure 6.9. This conversion is done by using the in-plane stiffness calculated for the given material properties. This envelope closely resembles the box shape formed by the analytical solution. As these corresponds to the ultimate failures and not the first ply failures, However, the analytical solution at the far field for the same material can be compared with this solution for better understanding.
Validation of the Solution

Figure 6.8: Ultimate failure envelopes for notched $[45/90/ -45/90]_4s$ composites (units Mpa) (reproduced from [10])

Figure 6.9: Ultimate Failure strains predicted numerically for $[45/90/ -45/90]_4s$ composites [10]
Figure 6.10 shows the numerical result data from literature with the omni-strain curve developed based on LaRC03 failure. It shows that the omni-strain region at the hole(R=1) is within the limits of numerical data in first, second and fourth quadrants. It over predicts failure in the compression-compression quadrant, this is because of the fact that they assumed Tsai-Wu failure with degradation parameters in their model. These considerations led to some lower values in compression-compression than in tension (not in-line with regular Tsai-Wu failure which over predicts in compression). Assuming this limitation in their study, it can be said that at hole(R=1) the failure regions is within the final failure predicted through numerical analysis. The LaRC03 failure is evaluated at a distance away from the hole and the best fit to the predicted numerical result was found at a radius of R=1.10. This proves the ability of the LaRC03 approach to predict the final failure.
Figure 6.11 shows the comparison between the Omni-strain envelope (with hole) predicted using Tsai-Wu failure and the numerical model results. We can say that the Omni-strain envelope at hole (R=1) is well within the final failure limits compared with the numerical model results. In their numerical study they assumed Tsai-Wu as damage initiation criterion. On extending our Tsai-Wu failure model to greater radius till the limiting region of validity (to avoid interference of envelopes) we attain the failure envelopes shown as blue dots. Thus at a distance R=1.3 (limiting distance attained for Tsai-wu) predicted failure envelope closer to the numerically predicted failure. It is close enough in compression than in tension mainly because of their numerical model considerations, a similar reason stated while comparing LaRC03 solution. Their numerical study was not conducted at all load combinations so it will not cover the limiting distance of our study. Thus it can be well said that the Omni-strain model for open hole failures resembles the predicted values through numerical analysis.
A plot is made to analyse the various failure models with the presence and absence of hole in the material and shown in Figure 6.12. Through this comparative study it is found that the proposed omni strain analytical solution for open hole is well within the omni-strain material of Tsai-Wu & LaRC03 and experimental results of final failure(with open hole). The experimental results shows similar pattern in failure leading to a box shaped region, but lacks in analysing buckling conditions for failure. It is found that experimentally predicted failure region is well above Tsai-Wu material envelope (without hole) in three quadrants proving the failures attained are of mixed failures between fibre and matrix systems. The above conclusion well suits the experimental results as it lies within LaRC03 material failure envelope(without hole). In the analysis of LaRC03 with experimental results 6.2.1 also proves its applicability to predict the final failure. It is found through this validation that the omni-strain failure region for open hole(both Tsai & LaRC03) lies within the region of all final failures reported. Thus, supporting the claim that it is ply based failure criteria and no failure will occur if stayed within this region.
Chapter 7

Conclusion and Recommendation

7.1 Conclusion

In this section we try to summarize the thesis by mentioning the main conclusions and observations from our study. This chapter refocuses on the research objective made earlier "To analyse and propose an analytical solution which characterises the influence of open hole stress concentration, based on omni strain envelope for quasi isotropic fibrous composites". The main points that are concluded from this research.

- The effect of bi-axial loading on a composite with a hole was studied through analysis in Chapter 3 and based on analysis conclusions were drawn on how laminate layup can affect the SCF. The advantages of using fibre steering is also explained earlier in that chapter, these changes in the fibre angle orientation will make the existing failure criteria unusable and requires a system which does not depend on fibre angle. This really enforces the need of a criteria which remains invariant to the fibre angles in the composite.

- The omni strain envelope can be used to solve the stress concentration problem with arbitrary fibre angles and the principles behind omni strain envelope were well presented in Chapter 4. It is found that material properties determines the omni strain envelope and for other changes in the laminate configuration, the omni strain envelope remained invariant. As omni strain assumes failure definitions, Tsai-Wu and LaRC03 failures were considered and a comparative study between these theories was carried out. It is concluded that Tsai-Wu resulted in a more conservative omni strain envelope except in bi-axial compression.

- The omni strain envelope was combined with the effect of SCF and a suitable method was proposed such that it predicts the behaviour of failure at the hole. An analytical solution was also developed for a quasi-isotropic laminate which can predict the failure region at the hole. The results of analytical and proposed omni strain envelope were found to be in consistent with each other, resulting in the same failure region.
• Further, an analysis had been carried out to extend the solution inside the composite domain to understand the applicability and limitations of the solution to predict the final failure. In this process, Whitney-Nuismer point stress method of using characteristic distance was used and found to be non-applicable, as the characteristic distance predicted interfering regions of failure envelopes at certain loading conditions.

• In the last section of Chapter 5, a detailed study is made to predict the valid region of the proposed solution based on Tsai-Wu failure model. The limiting conditions was well established and valid regions of solution were replotted and found to be without interference. It is found that the regions of validity are dependent on the material properties.

• A validation study against experimental results had been carried out in Chapter 6 and it revealed that omni strain failure envelope showed similar characteristic behaviour of the failure regions traced from experimental data points. The omni strain was extended to predict the final behaviour, i.e to its limiting distances and the results showed good concurrence.

To summarise, the proposed analytical solution has the desired characteristics of being independent of fibre orientations and has the ability to predict the region where failure will not occur. One can determine the failure region just by having two data from experiments i.e failure loads from of Open Hole compression (OHC) and Open Hole Tension (OHT) tests. Further, it is also proven that it has the ability to predict the final failure by determining its valid region of applicability.

7.2 Recommendations

The potential research possible in composites is wide open, lot of improvements can be made in terms of predicting the behaviour of composite for various conditions. But based on this work, I see the following areas more appealing to carry out future research. They are as follows:

• This research work can be extended to more reliable failure criteria (LaRC0x,..) which are being developed. By doing this, the accuracy of the predicted solution increases to match the real time experimental results.

• In this thesis, analysis is done for an infinite plate with a open hole. This can be extended to a finite width plate considering the stress redistribution effects.

• Extending the proposed analytical solution by adding the damage degradation parameters, in order to predict the final failure envelope.

• In this work, a composite laminate with a open hole was considered. The formulation can also be extended to deal with closed hole systems.


Material Invariants

The material coefficients for the strain equivalent Tsai-Wu failure criterion is given using the reduced stiffness matrix \(Q\) as:

\[
G_{11} = Q_{11}^2 F_{11} + Q_{12}^2 F_{22} + 2F_{12} Q_{11} Q_{12}
\]

\[
G_{22} = Q_{12}^2 F_{11} + Q_{22}^2 F_{22} + 2F_{12} Q_{12} Q_{22}
\]

\[
G_1 = (Q_{11} F_1 + Q_{12} F_2)
\]

\[
G_2 = (Q_{12} F_1 + F_2 Q_{22})
\]

\[
G_{12} = Q_{11} Q_{12} F_{11} + Q_{12} Q_{22} F_{22} + F_{12} Q_{12}^2 + F_{12} Q_{12} Q_{22}
\]

\[
G_{66} = 4Q_{66}^2 F_{66}.
\]

The following material invariants are used while solving the feasible region and are given as:

\[
u_1 = G_{11} + G_{22} - 2G_{12}
\]

\[
u_2 = (G_1 + G_2)/2
\]

\[
u_3 = (G_{11} + G_{22} + 2G_{12})/4
\]

\[
u_4 = G_1 - G_2
\]

\[
u_5 = G_{11} - G_{22}
\]

\[
u_6 = G_{66}
\]