An analytical approach to the Generalized Tally Game

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An Analytical Approach to the Generalized Tally Game

by

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Dr. C. Kraaijamp, TU Delft

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The Joint Replenishment Problem (JRP) is one of the fundamental problems in inventory theory.\cite{1} This problem arises in a setting with a warehouse and retailers, where multiple products can be shipped together to a retailer. Here, the warehouse ordering cost is fixed, there is a demand for each retailer, and for each retailer there are retailer ordering cost. Our aim is to satisfy all the retailer demand with the costs as low as possible. A retail order of some set of items cannot be placed without the warehouse ordering those items. On the other hand, once the warehouse has ordered a set of items, arbitrary many retailers can use this warehouse order for their retail orders.

To minimize the costs, we need to answer some questions: What is the optimal time between warehouse orders? What is the optimal time between retail orders for each item?

When the warehouse ordering cost is zero, this problem reduces to a straightforward case with independent retailers. When the shipping cost is extremely high, each retailer will have to order together to minimize the cost. In practice, the warehouse ordering cost will be high enough such that we have to combine multiple, but not all, retail orders together in an order. The solution of the JRP consists of a schedule, a set of orders, which satisfies the demand for each item with minimal cost.

In their 2009 paper, Nonner and Souza analyzed an extension of this problem, the Joint Replenishment Problems with Deadlines (JRP-D)\cite{5}. JRP-D is the JRP with the extension that the demand for each item has a deadline until when it has to be satisfied. They also introduced an $5/3$-approximation algorithm for JRP-D, where the output would be at most $5/3$ times the cost of an optimal schedule. They showed that an optimal schedule for JRP-D corresponds to an optimal strategy for a game, which they named the generalized tally game. This means that we could solve the generalized tally game in order to obtain an approximation for JRP-D. These results were then used by Chrobak et al. \cite{4}, where they analyzed the generalized tally game to obtain an improved approximation algorithm for JRP-D.

Most approximations are derived with linear programming with restrictions on the orders and costs. However, in this report we will approach the generalized tally game from a game theory perspective. We aim to solve the generalized tally game with and without these restrictions, First, we will introduce the reader to game theory by looking at two simplified tally games, which analysis will then help us solve the generalized tally game; a finite tally game and an infinite tally game. We will also provide two major theorems that we will need, the minimax theorem and Glicksberg theorem. Next, we will solve the tally game with one draw, and after that we analyze the tally game with 2 draws. Since the number of expected draws in the generalized tally game with optimal is only slightly higher than two, the solution of the two draw tally game should be a good approximation to the solution of the generalized tally game.
Before we introduce the first game, we will introduce some of the terminology used in game theory [2]. First of all, a game consists of the following:

- One or more strategic decision-makers, called players.
- All the possible information states of each player at each decision time.
- The set of possible decisions that each player can choose to make in each of his possible information states. This set is called a strategy set.
- A procedure or formula for determining how the decisions of all the players together determine the possible outcomes of the game.
- Preference of the individual players over these possible outcomes; usually with a payoff or utility function.

The games that we will be looking at are one-stage games, i.e. there will only be one decision time at the start of the game.

A pure strategy for a player in a game is a complete plan describing what decisions from the strategy set that player should make in each of the possible information states. On the other hand, a mixed strategy for a player in a game is a linear combination of the possible pure strategies. That is, a probability distribution $P$ defined over the set $A$ of pure strategies where

$$\sum_{a \in A} P(a) = 1 \tag{1.1}$$

In a one-stage game, the strategy consists only of one decision in the initial information state. Note that each player will simultaneously choose a strategy at the start of one-stage game, without any knowledge of the strategy choices of any other player. A game is said to be finite if the strategy sets for all players are finite.

Let us consider a two-player one-stage game, where each player must choose one of $N$ feasible strategies $S_1, S_2, ..., S_N$. Now, the payoff matrix $V$ for this game is an $N \times N$ matrix that gives the payoff received by each player under each combination of decisions the two players can make. So, each of the rows of $V$ correspond to a feasible strategy of player I and each column of $V$ correspond to a feasible strategy of player II. Each entry is then given by

$$V(i, j) = (S_1(i, j), V_2(i, j)) \tag{1.2}$$

where $V_1(i, j)$ is the payoff to player I and $V_2(i, j)$ is the payoff to player II with player I choosing strategy $S_i$ and player II choosing strategy $S_j$. A game is said to be zero-sum if the sum of the payoffs to all players is 0, regardless of the choices of the players. So, if we denote the strategy of player $i$ by $S_i$ and the payoff function to player $j$ by $f_j(S_1, S_2, ..., S_n)$ a zero-sum game must have

$$\sum_{i=0}^{N} f_i(S_1, S_2, ..., S_n) = 0 \tag{1.3}$$

for all $S_1, S_2, ..., S_n \in S$. Now, for a two-player game, this simply means that if player I has payoff $V(i, j)$ for $S_i, S_j \in S$, then player II must have payoff $-V(i, j)$.

In this thesis, we will mainly focus on one-stage, two-player, zero-sum games. We will later see that for some sets of games can be solved and other sets of games cannot. To demonstrate how to solve a game, we will first introduce the reader to a simplified version of the generalized tally game.
2

INTRODUCTION TALLY GAME

2.1. Finite Tally Game
In this chapter, we will analyze a simplified version of the generalized tally game, the finite tally game. We will focus on the methods of solving these type of games, using some valuable theorems on game theory.

Alice and Bob are going to play a game. Let us write the strategy set \( S = \{1, 2, 3\} \) and let Alice pick a number \( x \in S \) and Bob pick a number \( y \in S \). Next, Bob will have to pay Alice \( V(x, y) \), where

\[
V(x, y) = \begin{cases} 
  x & \text{if } x < y \\
  x - y & \text{if } x \geq y 
\end{cases}
\]  

(2.1)

Then, we can represent the outcome of this game by the following payoff matrix:

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2.1: Payoff matrix of game between Alice and Bob

or, in matrix-form

\[
V = \begin{bmatrix} 
  0 & 1 & 1 \\
  1 & 0 & 2 \\
  2 & 1 & 0 
\end{bmatrix}
\]  

where \( v_{ij} = V(x_i, y_j) \).

(2.2)

In this form, Alice chooses a row, Bob chooses a column and then Bob pays Alice the entry in that row and column. Since this matrix represents both the winnings of Alice and the losing of Bob, we will from now on only refer to Alice’s payoff.

The main objective for Alice is to obtain the highest possible payoff, while Bob wants the lowest possible payoff. Note that both players have strategy set \( S = \{1, 2, 3\} \), so both players have 3 initial strategies. What strategy should each player take? We will first answer this question for Bob; what would be his best response?

Assume Alice picks 1. We see from table 2.1 that Bob should pick 1 to obtain payoff 0. Also, if Alice picks 2 or 3, Bob should pick 2 or 3 respectively to obtain payoff 0. So regardless of what Alice plays, Bob’s best response will always net him a payoff of 0.

Now, what would be the best response for Alice? If Bob picks 1, from 2.1 we see that Alice should pick 3 to obtain payoff 2. If Bob picks 2, then Alice could pick either 1 or 3 to obtain payoff 1 and if Bob picks 3, Alice
should pick 2 to obtain payoff 2. So regardless of what Bob plays, Alice’s best response will give her a payoff of either 1 or 2.

From this we see that Bob can always obtain payoff 0 with optimal play while Alice can always obtain at least payoff 1. So one of the players can always improve their payoff by changing strategies.

Now Alice can improve her strategy by switching between picking 1, 2, and 3. Let Alice draw numbers 1, 2, and 3 with probability \( p_1 \), \( p_2 \) and \( p_3 \) respectively and

\[ p_1 + p_2 + p_3 = 1. \]

To compute the expected payoff for Alice, we first assume Bob picks 1. Then, with probability \( p_1 \) Alice draws 1 and she will receive 0. With probability \( p_2 \) and \( p_3 \) she will draw respectively 2 and 3 and she will receive 1 and 2 respectively. Hence, the expected payoff to Alice when Bob picks 1 is

\[ p_2 + 2p_3. \]

Similarly, when Bob picks 2, Alice obtains the expected payoff

\[ p_1 + p_3 \]

and when Bob picks 3, Alice obtains the expected payoff

\[ p_1 + 2p_2. \]

<table>
<thead>
<tr>
<th>Bob’s number</th>
<th>Payoff Alice</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p_2 + 2p_3 )</td>
</tr>
<tr>
<td>2</td>
<td>( p_1 + p_3 )</td>
</tr>
<tr>
<td>3</td>
<td>( p_1 + 2p_2 )</td>
</tr>
</tbody>
</table>

We could also write this as a matrix product if we write

\[ p = [p_1, p_2, p_3]^T \]

then the payoff to Alice is given by

\[ V^T p = \begin{bmatrix} p_2 + 2p_3 \\ p_1 + p_3 \\ p_1 + 2p_2 \end{bmatrix} \tag{2.3} \]

where \( V \) is given by equation 2.2

Before continuing, it is important to check whether these three strategies are all viable choices for Alice. For example, if picking 2 always gives Alice a larger payoff than picking 3, we say that 3 is being dominated by 2 and therefore should not be picked. To check if there are any dominating strategies for Alice, we refer to the payoff matrix

\[ V = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \tag{2.4} \]

We say that row \( i \) dominates row \( j \) if each payoff in row \( i \) is greater than the corresponding payoff in row \( j \), i.e.

\[ V(x, y_i) \geq V(x, y_j) \text{ for all } x \in S. \]

Since we have 0 on the diagonal of \( V \) and positive numbers on non-diagonal elements, we see that there is no row being dominated. So Alice has indeed three viable strategies.

With this information, how can we calculate what strategy \( p = (p_1, p_2, p_3) \) Alice should be using? We know Bob will always play optimally, which means that Bob will always pick the row that gives the smallest payoff.
Knowing how Bob will respond, Alice wants to find the maximum value of this payoff, i.e. Alice wants to pick \( p \) that maximizes this minimal payoff. The optimal strategy \( p^* \) for Alice can then be obtained with

\[
p^* = \max_p \left\{ \arg \min V^T p \right\}
\]

(2.5)

where \( \arg \min \) returns the points of the domain \( p \) at which \( A^T p \) is minimal. The corresponding payoff is naturally

\[
V = \max \min V^T p
\]

(2.6)

This value is called the lower value of the game. This is the maximum amount that Alice can guarantee regardless of what Bob does.

Before we can solve this game, we have to look at this game from Bob’s perspective when he switches between 1, 2 and 3.

Assume Bob draws the numbers 1, 2 and 3 with probability \( q_1, q_2 \) and \( q_3 \) such that

\[
q_1 + q_2 + q_3 = 1.
\]

To calculate the expected loss of Bob, we first assume Alice picks 1. Then, with probability \( q_1 \) Bob draws 1 and he must pay 0. With probability \( q_2 \) and \( q_3 \) he will draw respectively 2 and 3 and he must pay 1 and 1 respectively. So, the expected payoff to Alice when Alice picks 1 is

\[
q_2 + q_3.
\]

When Alice picks 2, the expected payoff to Alice is

\[
q_1 + 2q_3
\]

and when Alice picks 3, the expected payoff to Alice is

\[
2q_1 + q_2.
\]

These expected payoffs can be written as the matrix

\[
V q = \begin{bmatrix}
0 & q_2 & q_3 \\
q_1 & 0 & 2q_3 \\
2q_1 & q_2 & 0
\end{bmatrix}
\]

(2.7)

where

\[
q = [q_1, q_2, q_3]^T.
\]

As before, we first want to check if Bob has a pick that dominates another one. To check this, we need to look at the columns of the payoff matrix \( B \). Note that again, since we have 0 on the diagonal and positive numbers for non-diagonal elements, there is no column which dominates another column. Hence all three picks are viable for Bob.

Now, what strategy \( q = (q_1, q_2, q_3) \) should Bob be using? Using the same reasoning as before, Alice will always pick the row that gives her maximum payoff, so Bob should be seeking to minimize this maximum value. The optimal strategy \( q^* \) for Bob is then given by

\[
q^* = \min_q \{ \arg \max V q \}
\]

(2.8)

with corresponding payoff

\[
\nabla \min_q \max V q.
\]

(2.9)
This value is called the upper value of the game, this is the smallest expected loss that Bob can assure himself, regardless of what Alice does.

In order to rewrite these expressions, we note that if Alice chooses a row at random using some strategy \( p^* \) and Bob chooses a number \( j \), then the expected payoff to Alice is given by

\[
\sum_{i=1}^{n} p_i v_{ij} = [p^T v]_j.
\]

On the other hand, if Bob uses some strategy \( q^* \) and Alice picks a number \( i \), then the expected payoff to Alice is given by

\[
\sum_{j=1}^{m} v_{ij} q_j = [V q]_i.
\]

In general, if Alice uses strategy \( p^* \) and Bob uses strategy \( q^* \), then Alice's expected payoff becomes

\[
\sum_{i=1}^{n} \left( \sum_{j=1}^{m} v_{ij} q_j \right) p_i x = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i v_{ij} q_j = \mathbf{p}^T \mathbf{V} \mathbf{q}.
\]

Hence, we can write the lower value of the game as

\[
V = \max_p \min_q \mathbf{p}^T \mathbf{V} \mathbf{q}
\]

and the upper value of the game as

\[
\overline{V} = \min_q \max_p \mathbf{p}^T \mathbf{V} \mathbf{q}.
\]

It is important to note that the lower value is indeed less or equal than the upper value, i.e. \( \overline{V} \leq V \). To see this, assume that there exist a strategy \( p^* \) for Alice such that \( \overline{V} < V \). This guarantees Alice at least \( V \), but Bob is supposed to assure himself losing as much as \( \overline{V} \). This is a clear contradiction. Hence \( \overline{V} \leq V \).

The reverse is not always true. We will later see that \( V \geq \overline{V} \) always holds when the game is finite, but not necessarily when the game is infinite. If there is a difference between the lower value and upper value of a game, we call the difference the duality gap.

We need one more result before we can solve the finite tally game, the equilibrium theorem:

**Theorem 2.1.1** (Equilibrium Theorem). Consider a game with \( n \times m \) payoff matrix \( V \). Let \( p^* = [p_1, p_2, \ldots, p_n]^T \) be an optimal strategy for player I and \( q^* = [q_1, q_2, \ldots, q_m]^T \) be an optimal strategy for player II. Then we have

\[
\sum_{j=1}^{m} v_{ij} q_j = v \text{ for all } i \text{ where } p_i > 0
\]

and

\[
\sum_{i=1}^{n} p_i v_{ij} = v \text{ for all } i \text{ where } q_i > 0
\]

where \( v \) is the value of the game.

**Proof.** Suppose there exists some number \( k \) such that \( p_k > 0 \) and \( \sum_{j=1}^{m} v_{kj} q_j < v \). But then from equation (2.10)

\[
V = \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} v_{ij} q_j \right) < \sum_{i=1}^{n} p_i v = V
\]

which gives us a contradiction. Hence there exists no such \( k \). The latter statement can be proved analogously. \( \square \)
What this theorem tells us is that if there exists an optimal strategy for Alice that gives a positive weight to some row \( i \), then every optimal strategy of Bob if he uses row \( i \) will give Alice the lower value of the game as a payoff. In other words, if there exists an optimal strategy for Alice, this strategy will make Bob indifferent in choosing his (good) strategies. This also means that if there exists an optimal strategy for Bob, this strategy will make Alice indifferent between her strategies and have the upper value of the game as a payoff. For some classes of games, such as the finite tally game, this method is proven to be successful.

To solve the finite tally game for Alice, we need to make Bob indifferent. Using matrix \( 2.2 \) and \( p_1 + p_2 + p_3 = 1 \), we find that Alice’s optimal strategy must satisfy

\[
\sum_{i=1}^{n} p_i v_{ij} = V
\]

or

\[
\begin{aligned}
p_2 + 2p_3 &= V \\
p_1 + p_3 &= V \\
p_1 + 2p_2 &= V \\
p_1 + p_2 + p_3 &= 1
\end{aligned}
\]

Solving this gives us the optimal strategy

\[
p = (1/2, 1/6, 1/3)
\]  \hspace{1cm} (2.14)

with lower value \( V = \frac{5}{6} \). Therefore if Alice picks the strategy \( p = (1/2, 1/6, 1/3) \) she will have an expected payoff of \( \frac{5}{6} \).

We apply the same method to Bob, where we make Alice indifferent. We obtain

\[
\begin{aligned}
q_2 + q_3 &= V \\
q_1 + 2q_3 &= V \\
2q_1 + q_2 &= V \\
q_1 + q_2 + q_3 &= 1
\end{aligned}
\]

Solving this gives us

\[
q = (1/6, 1/2, 1/3)
\]  \hspace{1cm} (2.15)

with \( V = \frac{5}{6} \).

Therefore, the strategy \( q = (1/6, 1/2, 1/3) \) will give him an expected loss of \( \frac{5}{6} \).

From this we see that

\[
V = V = \frac{5}{6}.
\]  \hspace{1cm} (2.16)

Hence, the value \( \nu \) of the game is \( \nu = \frac{5}{6} \) and (2.14) and (2.15) are optimal for Alice and Bob respectively.
2.1.1. Infinite Tally Game

Now we will extend this finite tally game to a more general game. As before, Alice will still choose a strategy $p = (p_1, p_2, p_3)$ for $S = \{1, 2, 3\}$ and then she will draw a number $x$ from this strategy. This strategy is just a distribution and Alice still has a strategy set of 3.

However, Bob now has a semi-infinite strategy $q = (R_1, R_2, \ldots)$ where each $R_i$ is some distribution $R_i = (q_1^i, q_2^i, q_3^i)$ with

$$q_1^i + 2q_2^i + 3q_3^i = 2.$$  

This condition makes sure that for a stochastic $Y \sim R_i$ we have

$$\mathbb{E}[Y] = 2$$

to prevent us from drawing only 1 from this distribution. We can also write this strategy as

$$q = \begin{bmatrix}
q_1^1 & q_2^1 & q_3^1 \\
q_1^2 & q_2^2 & q_3^2 \\
q_1^3 & q_2^3 & q_3^3 \\
\vdots & \vdots & \vdots
\end{bmatrix} \quad (2.17)$$

Note that Bob has an infinite strategy set here. Instead of picking a number as in the previous game, Bob will pick a some pure strategy $R = (q_1, q_2, q_3)$. Next, we will start drawing numbers from this distribution. Bob will keep drawing numbers from his distribution $R$ until the sum of the observed outcomes is greater than Alice’s number.

If we write the $i$-th number drawn as $y_i$, then the stopping criteria of this game is the smallest $n$ for which

$$\sum_{i=1}^{n} y_i > x$$

and the undershoot is then $\sum_{i=1}^{n} y_i$.

Now the game ends and Bob has to pay Alice the difference between the sum of the observed outcomes without the last draw and Alice’s number. This means the payoff $V(x, y)$ to Alice is

$$V(x, y) = x - \sum_{i=1}^{n-1} y_i.$$  

While the first game introduced had only one draw, this game can have up to four draws. To analyze this game, we first let Bob pick a uniform stochastic $U$ such that $U = (1/3, 1/3, 1/3)$. Now, to compute what number Alice should pick, we can construct a tree diagram. For example, if Alice picks 1, the tree diagram is given by figure 2.1
Here, we see that if Alice picks $x = 1$, then with $p = \frac{1}{3}$, Bob will draw 3 and the game ends with payoff $V(x, U) = -1$. When Bob draws 2, we have payoff $V(x, U) = 1$. When Bob draws 1, however, his number is not greater than Alice’s number, so he gets to draw again. No matter the outcome, the payoff will be $V(x, U) = 0$ in this situation. So, the expected payoff to Alice when Bob picks a uniform stochastic $U$ is and Alice picks 1 is

$$V(x, U) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}.$$

We can repeat this process for when Alice picks $x = 2$ and $x = 3$. The results are given below

<table>
<thead>
<tr>
<th>Alice’s number</th>
<th>$V(x, U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 1$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>$\frac{8}{9}$</td>
</tr>
<tr>
<td>$x = 3$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

From this we see that if Bob picks a uniform stochastic, the play from Alice is picking 2. Thus, this game has lower value $V(2, U) = \frac{8}{9}$.

Now we want to find the best pure strategy $q = (q_1, q_2, q_3)$ for Bob. Again, we are going to make Alice indifferent. First, assume Alice picks $x = 1$. The tree diagram corresponding to when Alice picks $x = 1$, figure 2.2, is almost identical to figure 2.1, where the probability of drawing 1,2 and 3 is not uniform anymore.
From this tree diagram, we obtain

\[ V(1, q) = q_1 \cdot 0 + q_2 \cdot 1 + q_3 \cdot 1 = q_2 + q_3. \]  

We do the same thing for \( x = 2 \), where the tree diagram is given by figure 2.3

From this tree diagram, we obtain

\[ V(2, q) = q_1(q_1 \cdot 0 + q_2 \cdot 1 + q_3 \cdot 1) + q_2 \cdot 0 + q_3 \cdot 2 \\
= q_1(q_2 + q_3) + 2q_3. \]  

For \( x = 3 \), using the same process, we obtain

\[ V(3, q) = q_1 \cdot [q_1(q_2 + q_3) + 2q_3] + q_2(q_3 + q_2) \]  

From the initial conditions

\[ q_1 + q_2 + q_3 = 1 \]  

and \( q_1 + 2q_2 + 3q_3 = 2 \)

we find that

\[ (q_1, q_2, q_3) = (q_1, 1 - 2q_1, q_1) \text{ for } 0 \leq q_1 \leq 1/2. \]

We can rewrite equations (2.18), (2.19) and (2.20) using \( q = q_1 \):

\[
\begin{align*}
V(1, q) &= 1 - q \\
V(2, q) &= 3q - q^2 \\
V(3, q) &= 1 - 3q + 5q^2 - q^3
\end{align*}
\]
Figure 2.3: Tree diagram when Alice picks 2 and Bob an arbitrary strategy $q$. 

\[ V(2, q) = 0 \]
\[ V(2, q) = 1 \]
\[ V(2, q) = 2 \]

2.1. Finite Tally Game
This gives us the table

<table>
<thead>
<tr>
<th>Alice’s number</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 − q</td>
</tr>
<tr>
<td>2</td>
<td>3q − q^2</td>
</tr>
<tr>
<td>3</td>
<td>1 − 3q + 5q^2 − q^3</td>
</tr>
</tbody>
</table>

Table 2.2: Payoff to Alice when Bob picks an arbitrary strategy q

The optimal strategy \(q^*\) for Bob is now given by

\[ q^* = \operatorname{argmin}\{\max_{1 \leq x \leq 3} V(x, q)\} \quad (2.22) \]

with corresponding upper value

\[ \bar{V} = \min_{0 \leq q \leq 0.5} \max_{1 \leq x \leq 3} V(x, q). \quad (2.23) \]

The payoff functions for each choice of Alice are given by figure 2.4

![Plot of the solution space for an arbitrary strategy](image)

From figure 2.4 we see that the function

\[ \max_{x} V(x, q) = \begin{cases} 
1 - q & \text{for } q \leq 0.27 \\
-q^2 + 3p & \text{for } q > 0.27 
\end{cases} \]

has a minimum on \(2 - \sqrt{3} \approx 0.27\). Therefore

\[ q^* = \operatorname{argmin}_x [\max V(x, q)] \approx 0.27 \]

Thus, the optimal strategy for Bob is for \(q \approx 0.27\), which is the strategy (0.27, 0.44, 0.27) and has value
\[ \bar{V} = 0.7371. \]

We also see that Alice will never choose 3, because the payoff when choosing 3 is being dominated by the payoffs for picking 1 and 3.

Now we want to know the best response from Alice to this strategy from Bob. Since Alice only picks \( y = \{1, 2\} \), in Bob’s optimal strategy, we will consider the strategy \( p = (p, 1-p, 0) \) for Alice. As before, we are going to make Bob indifferent in choosing his strategy \( q \). Note that the restriction \( E[Y] = 2 \) for any \( Y \sim q \) still has to hold. Therefore, we will be using the following mixed strategy \( Q \) for Bob:

- strategy \( \alpha \): \( P(Y = 2) = 1 \) with probability \( q \).
- strategy \( \beta \): \( P(Y = 1) = P(Y = 3) = 1/2 \) with probability \( 1 - q \).

In other words: with probability \( q \), Bob will pick a distribution where he will only draw 2 and with probability \( 1 - q \), Bob will pick a Bernoulli distribution where he will draw 1 with \( p = 0.5 \) and 3 with \( p = 0.5 \).

This is just one of the infinitely many mixed strategies Bob has, but we will see that this strategy already gives us some interesting results.

To calculate the corresponding payoff matrix, we will only compute the payoff \( V(2, \beta) \) here, the payoff when Alice picks 2 and Bob picks the \( \beta \) -distribution. The following table gives all draws and outcomes:

<table>
<thead>
<tr>
<th>draws</th>
<th>probability</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td>1/2</td>
<td>2</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1/4</td>
<td>1</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>1/8</td>
<td>0</td>
</tr>
<tr>
<td>(1,1,3)</td>
<td>1/8</td>
<td>0</td>
</tr>
</tbody>
</table>

which gives us the total payoff 5/4. This leads to the following payoff matrix for Alice

\[
\begin{array}{ccc}
\text{Bob} & 1 & 2 \\
\alpha & 1 & 0 \\
\beta & 1/2 & 5/4 \\
\end{array}
\]

Table 2.3: payoff matrix when Alice picks 1 or 2 and Bob picks \( \alpha \) or \( \beta \).

Now, to make Bob indifferent between strategies \( \alpha \) and \( \beta \), we must solve for \( p \):

\[
p = V \\
1/2p + 5/4(1-p) = V
\]

Solving this system gives

\[
p = \frac{5}{7}, \quad V = \frac{5}{7} = 0.7142.
\]

So setting \( p = \frac{5}{7} \) guarantees Alice a payoff of at least \( V = 0.71 \).

Now, since we already saw that the upper value of this game is \( \bar{V} = 0.7371 \), this means we have a duality gap of

\[
\bar{V} - V = 0.7371 - 0.71 = 0.271.
\]
We see that the duality gap is already quite small, sufficient enough for our analysis. However, if one wants to reduce the duality gap in general we could

- Reduce the upper value of the game by allowing mixed strategies for Bob
- Improve the lower value of the game by checking multiple mixed strategies for Bob

We can now state two major theorems that we will need for the generalized tally game: the minimax theorem and Glicksberg theorem.

### 2.1.2. Minimax Theorem and Glicksberg Theorem

We have solved the finite tally game with the so-called Minimax theorem. Before we formally state this theorem, let us introduce some notation. First, let $V$ be a $m \times n$ matrix representing the payoff matrix for a two-person, zero-sum game. If we denote a strategy pair as $(x, y)$, where $x \in \mathbb{R}^m$ is the strategy for player I and $y \in \mathbb{R}^n$ the strategy for player II, we can define

**Definition 2.1.1.** The payoff for a strategy pair $(x, y)$ is given by

$$V(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

We are interested in the so-called *equilibrium* for this game.

**Definition 2.1.2.** A pair of strategies $(x^*, y^*)$ is said to be an equilibrium point for a two-person, zero-sum game if

$$V(x, y^*) \leq V(x^*, y^*) \leq V(x^*, y)$$

for all $x \in X_m$ and $y \in Y_n$.

Note that this is equivalent to

$$\max_{x \in X_m} \min_{y \in Y_n} V(x, y) = \min_{y \in Y_n} \max_{x \in X_m} V(x, y).$$

Using these definitions we can state the Minimax Theorem

**Theorem 2.1.2** (Minimax Theorem [8]). Every finite, zero-sum, two-person game has an equilibrium point. The *value* $v$ of the game is given by

$$v = \max_{x \in X_m} \min_{y \in Y_n} V(x, y) = \min_{y \in Y_n} \max_{x \in X_m} V(x, y).$$

When a game has an (semi-)infinite strategy set, the minimax theorem does not always hold. Let $X$ and $Y$ now be infinite strategy sets. With the same notation as before, the following holds

$$\max_{x \in X_m} \min_{y \in Y_n} V(x, y) \leq \min_{y \in Y_n} \max_{x \in X_m} V(x, y).$$

The difference is called the *duality gap*. This is equivalent to the duality gap as the difference between the primary and duality solutions when using linear programming. We will give an example of a game with a duality gap, due to Sion and Wolfe [7], which has no value.

### 2.1.3. Games without Minimax Value

Let player I and II choose a number $x$ and $y$ respectively, with $x, y \in [0, 1]$. The payoff to player I is given by

$$V(x, y) = \begin{cases} 
-1 & \text{if } x < y < x + \frac{1}{2} \\
0 & \text{if } x = y \text{ or } y = x + \frac{1}{2} \\
1 & \text{otherwise} 
\end{cases}$$

(2.24)

The payoff to player 2 is $-V(x, y)$, so this is a zero-sum game. If we consider $(x, y)$ as a point on the Cartesian plane, we can plot the payoff to player I.

It is shown in [6] that for this game we have

$$\min_{y \in Y_n} \max_{x \in X_m} V(x, y) = \frac{3}{7},$$
if player I puts only probability weight \( \frac{1}{3} \) on points \( (0, \frac{1}{2}, 1) \). However, we also have

\[
\max_{x \in X} \min_{y \in Y} V(x, y) = \frac{1}{3}
\]

when player II puts probability weight \( \frac{1}{3} \) on points \( (\frac{1}{2}, \frac{1}{2}, 1) \). So if player I knows the strategy from player II, he can guarantee a payoff of at least \( \frac{1}{3} \). But on the other hand, if player II known the strategy from player I, he can guarantee a payoff of \( \frac{1}{3} \). From this we see that there is no equilibrium; there is a duality gap here and this game does not have a minimax value.

However, for a specific class of infinite games, there is a minimax value. This result comes from Glicksberg’s theorem. For this theorem, we will need the following definitions

**Definition 2.1.3.** Consider a function \( f : \mathbb{R} \to \mathbb{R} \) and a point \( x_0 \in \mathbb{R} \). The function \( f \) is **upper semicontinuous** (resp. **lower semicontinuous**) at \( x_0 \) if

\[
f(x_0) \geq \limsup_{x \to x_0} f(x) \quad \text{resp.} \quad f(x_0) \leq \liminf_{x \to x_0} f(x).
\]

Now, we can state Glickberg’s theorem. [3]

**Theorem 2.1.3** (Glicksberg’s theorem). Let \( F \) and \( G \) be compact sets and let the payoff function \( V \) be an upper semicontinuous or lower semicontinuous function on \( F \times G \), then

\[
\sup_{x} \inf_{y} \iint V \, dx \, dy = \inf_{y} \sup_{x} \iint V \, dx \, dy
\]

where \( x \in F \) and \( y \in G \).

This theorem ensures the existence of a minimax value when a game has a semicontinuous payoff function. The game of Sion and Wolfe does not have a semicontinuous payoff, so Glicksberg’s theorem cannot be applied. However, if we change the payoff function to be semicontinuous, e.g.

\[
V(x, y) = \begin{cases} 
-1 & \text{if } x \leq y \leq x + \frac{1}{2} \\
1 & \text{otherwise}
\end{cases} \tag{2.25}
\]

then Glicksberg’s theorem states that the game must have a minimax value. Indeed, when we solve the new game, we find that

\[
\min_{y \in Y} \max_{x \in X} V(x, y) = \frac{2}{5} = \max_{x \in X} \min_{y \in Y} V(x, y). \tag{2.26}
\]

From these theorems, we find that if a game is not finite and the payoff function is not semicontinuous, the game might not have a value. We will now get to the main subject of this thesis, the generalized tally game.
In this section, we introduce two games, the tally game and the main subject of this thesis, the generalized tally game [5]. The tally game is a one player, zero-sum game. In this game, we are initially given a number \( x \geq 0 \), after which we choose a strategy which is density function \( g \) of a random variable \( Y \) with

\[
0 \leq Y \leq 1 \text{ and } E[Y] = \frac{1}{2}.
\]  

(3.1)

Then, we draw a sequence of independent samples \((y_n)_{n \in \mathbb{N}}\) from \( g \) until the sum exceeds \( x \), i.e. until

\[
y_1 + y_2 + \cdots + y_n > x.
\]

Let us write

\[
y = \sum_{i=1}^{n-1} y_i.
\]

At the end of the game, we have the following payoff, see also figure (3.1)

\[
V(x, y) = x - y.
\]  

(3.2)

We denote the expected payoff for a density \( g \) and initial value \( x \) by \( E[V(x, g)] \). Note that

\[
V(x, g) = [0, 1) \text{ and } E[V(x, g)] \leq x.
\]

This game can be extended by having another player choose a density \( f \) of a random variable \( X \) with \( 0 \leq X \leq 1 \). This player, which we will refer to as player I, will then draw a number \( x \) from \( f \). This extension is called the generalized tally game. The question then becomes: how should both players choose their densities such that the expected payoff is optimal with respect to their opponents choices? For player I, we need to maximize

\[
\min_{0 \leq y \leq 1} E[V(f, y)],
\]  

(3.3)

and for player II, we need to minimize

\[
\max_{0 \leq x \leq 1} E[V(x, g)].
\]  

(3.4)
The payoff for player II can be written as an integral. The function $x \mapsto \mathbb{E}[V(x, g)] : [0, 1] \to [0, 1]$, satisfies the integral equation

$$
\mathbb{E}[V(x, g)] = \int_0^x g(x - y)\mathbb{E}[V(y, g)]dy + \int_x^1 g(y)dy. \quad (3.5)
$$

Since $V(0, y) = 0$ and $P(V(x, y) = x) \to 1$ as $x \to 0$, we can denote the boundary conditions of this game as

$$
\mathbb{E}[V(0, g)] = 0 \text{ and } \frac{d\mathbb{E}[V(f, 0)]}{dy} = 1. \quad (3.6)
$$

Thus, finding the optimal strategy for player II can be interpreted as an optimization problem: find density $g$ such that the function which satisfies equation (3.5) and boundary conditions (3.6) minimizes equation (3.4).

We cannot explicitly state the expected payoff of player I, because player II could draw multiple times from their strategy $g$. This illustrates the main difficulty in finding the optimal strategy for player I.

### 3.1. Solvability of Generalized Tally Game

We will show that Glicksberg’s theorem applies to the generalized tally game, and thus that it should have a value.

Let $F, G$ be the strategies for respectively player I and player II. Recall that

$$
F : [0, 1] \to \mathbb{R} \text{ and } G : [0, 1] \to \mathbb{R}
$$

and

$$
V : [0, 1] \times [0, 1] \to \mathbb{R}.
$$

To apply Glicksberg’s theorem, we have to show that the following holds:

- the domain $[0, 1]$ is compact
- the payoff function $V$ is uppersemicontinuous

First, to show $[0, 1]$ is compact we give a short proof. To prove compactness, we show that every open cover on $[0, 1]$ has a finite subcover.

Let $\{\mathcal{O}_i\}$ be an arbitrary open covering of $[0, 1]$, i.e.

$$
\bigcup_{i \in I} \mathcal{O}_i = [0, 1].
$$

Consider

$$
A = \{x = [0, 1]||0, x]\text{ can be covered by finitely many of the } \mathcal{O}_i\text{'s}. \quad (3.7)
$$

Since $\mathbb{R}$ is compact, we can take a lowest upper bound of $A$, say $a$.

Suppose $a < 1$. Then $a$ is in some open set $\mathcal{O}_i$, and thus lies in an $\epsilon$-neighborhood lying in $\mathcal{O}_i$.

But now the interval $[0, a - \epsilon/2]$ is covered by finitely many $\mathcal{O}_i$’s and thus also cover $[0, a + \epsilon/2]$. This contradicts the assumption that $a < 1$. Hence $a = 1$ and $[0, 1]$ has a finite subcover.
For upper semicontinuity, fix some $x \in [0,1]$ and let

$$V_x : [0,1] \to \mathbb{R}$$

$$y \mapsto V(x,y)$$

The payoffs $V_x$ for $x = [0.2, 0.5, 0.8]$ are given in figure 3.2. Similarly, if we fix some $y \in [0,1]$ and let

$$V_y : [0,1] \to \mathbb{R}$$

$$x \mapsto V(x,y)$$

the payoffs $V_y$ for $y = [0.2, 0.5, 0.8]$ are given in figure 3.3. From both figures, we can see that there is only one discontinuity, at $x = y$, and the payoff jumps up after the discontinuity. Since all the payoffs only jump up, we have upper continuity. Formally, we have for both $V_x$ and $V_y$ that

$$\text{for any } z \in \mathbb{R}, V^{-1}_x((-\infty, z)) \text{ is open}$$

and

$$\text{for any } z \in \mathbb{R}, V^{-1}_y((-\infty, z)) \text{ is open}.$$ 

From this it follows that $V_x$ and $V_y$ are upper discontinuous.

Hence, Glicksberg’s theorems tells us that the generalized tally game has a minimax value. The objective now is to reduce the duality gap as much as possible to try and approximate the value of the generalized tally game.

Figure 3.2: 3d mesh of the payoff $V(x,y)$

For our analysis of the optimal strategies for the generalized tally, we will first analyze the generalized tally game for just one draw; where player I and player II both draw one number from their respective densities. What we will see is that the result is already a good approximation for the generalized tally game.
3. THE GENERALIZED TALLY GAME

Figure 3.3: 3d mesh of the payoff $V(x,y)$
3.2. **GENERALIZED TALLY GAME WITH ONE DRAW**

In this section, we will take a look at the generalized tally game with just one draw. First, let player I chooses a distribution \( f \) of a random variable \( X \) where \( 0 \leq X \leq 1 \) which is his strategy, and player II chooses a distribution \( G \) of a random variable \( Y \) with \( 0 \leq Y \leq 1 \). Note player II has no restriction on \( \mathbb{E}[Y] \) here. Next, both players draw a number, \( x \sim F \) and \( y \sim G \) respectively, from their chosen distribution. The payoff to player I is given by

\[
V(x, y) = \begin{cases} 
  x & \text{if } x < y \\
  x - y & \text{if } x \geq y
\end{cases}
\]  

(3.10)

and the payoff to player II is \(-V(x, y)\), see figure 3.4

![3d mesh of the payoff \( V(x,y) \)](image)

The objective for player I is to maximize

\[
\min_{0 \leq y \leq 1} \mathbb{E}[V(f, y)]
\]

(3.11)

and for player II to minimize

\[
\max_{0 \leq x \leq 1} \mathbb{E}[V(x, g)]
\]

(3.12)

Here, the expected payoff \( \mathbb{E}[V(f, y)] \) given some \( y \sim G \) for player I can be expressed by

\[
\mathbb{E}[V(f, y)] = \int_0^y xf(x)\,dx + \int_y^1 (x-y)f(x)\,dx.
\]

(3.13)

\[
= \int_0^1 xf(x)\,dx - y \int_y^1 f(x)\,dx.
\]

and for player II the expected payoff \( \mathbb{E}[V(x, g)] \) given \( x \sim F \) can be expressed by

\[
\mathbb{E}[V(x, g)] = \int_0^x (x-y)g(y)\,dy + \int_x^1 xg(y)\,dy.
\]

(3.14)

\[
= x \int_0^1 g(y)\,dy - \int_0^x yg(y)\,dy
\]
From equation 3.13 and figure 3.4, we see that if player I picks 0, \( P(X = 0) = 1 \), then

\[
E[V(0, y)] = 0
\]

and if player I picks 1, the payoff becomes \( V(1, y) = 1 - y \).

Similarly, if player II picks 0 or 1, we get

\[
E[V(x, 0)] = E[V(x, 1)] = x.
\]

From this, we can see that if player I picks \( \epsilon > 0 \) very small, the expected payoff to player I becomes

\[
E[V(\epsilon, y)] = \epsilon \quad \text{for} \quad 0 \leq x \leq 1.
\]

Since player I aims to maximize his expected payoff and picking a small number \( \epsilon \) only gives a small payoff \( \epsilon \), player I should not be picking numbers below a certain threshold \( k > 0 \).

On the other hand, if player II picks \( \epsilon > 0 \) very small, the expected payoff to player I becomes

\[
E[V(x, \epsilon)] = x - \epsilon.
\]

Knowing that player I will only pick numbers above a certain threshold \( k > 0 \), player II should adjust his strategy and only pick numbers above this threshold. Hence, for both players' respective optimal strategies, there must exist some \( k > 0 \) for which

\[
F(k) = 0 \quad \text{and} \quad G(k) = 0 \quad (3.15)
\]

and we can define the strategies \( F, G \) on just the interval \([k, 1]\).

Now, what are the optimal strategies for each player and what is the value of the game? We will first analyze the optimal strategy for player II, since the strategy for player I is generally more difficult to compute.

3.2.1. **Strategy player II**

To find the optimal strategy \( G \) for player II, we need to make player I indifferent in picking his strategy. To do this, we consider

\[
x_1 = x \quad \text{and} \quad x_2 = x + \epsilon, \quad x_1, x_2 \in [0, 1]
\]

with \( \epsilon > 0 \) very small. To make player I indifferent between \( x_1 \) and \( x_2 \), we must solve

\[
V(x_1, g) = V(x_2, g).
\]

To compute the expected difference \( \Delta V = V(x_1, y) - V(x_2, y) \) we consider three cases

- \( y < x_1 \) with probability \( G(x_1) \), see figure 3.5

![Figure 3.5: \( \Delta V \) when \( y < x_1 \)](image)

Here, \( V(x_1, y) = x_1 - y \) and \( V(x_2, y) = x_2 - y \). So

\[
\Delta V = x_1 - x_2.
\]

- \( x_1 \geq y < x_2 \) with probability \( (x_2 - x_1)g(y) \), see figure 3.6

Here, \( V(x_1, y) = x_1 \) and \( V(x_2, y) = x_2 - y \). So

\[
\Delta V = x_1 - (x_2 - y)
\]
3.2. Generalized Tally Game with one draw

\[ \Delta V = x_1 - x_2. \]

Adding these equations together holds

\[ \Delta V = (x_1 - (x_2 - y))(x_2 - x_1)g(y) + (x_1 - x_2)(G(x_1 + (1 - G(x_2))) \]
\[ = (e + y)g(y) - e(G(x) + 1 - G(x + e)) \]
\[ = (e + y)g(y) - (G(x) + 1 - G(x + e)). \]

Since we want to make player II indifferent, we must solve

\[ \Delta V = 0. \]

This gives us

\[ (e + y)g(y) = (G(x) + 1 - G(x + e)). \] (3.16)

We take the limit \( e \to 0 \) to obtain

\[ yg(y) = 1 \Rightarrow g(y) = \frac{1}{y}. \]

So, if player II picks \( g \) such that \( g(y) = \frac{1}{y} \) then player I will be indifferent. To find the optimal strategy we already observed that both players should never pick close to 0. So, \( g \) will be given by

\[ g(y) = 1/y \text{ on } [k, 1] \] (3.17)

for some \( k > 0 \). To find \( k \), we simply solve

\[ \int_k^1 \frac{1}{y} dy = 1 \]
\[ \ln 1 - \ln k = 1 \]
\[ k = \frac{1}{e}. \]

So, the optimal strategy of player II is given by

\[ g(y) = \begin{cases} 
1/y & \text{if } \frac{1}{e} \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases} \] (3.18)

see figure 3.8
The optimal distribution is given by

\[ G(y) = \ln(e^y), \quad \frac{1}{e} \leq y \leq 1 \]  

(3.19)

see figure 3.9
This optimal strategy has payoff

\[ V(x, g) = \int_0^x (x - y)g(y) dy + \int_x^1 xg(y) dy \]

\[ = x \int_{1/e}^1 g(y) dy - \int_{1/e}^x yg(y) dy \]

\[ = x - \int_{1/e}^x dy \]

\[ = \frac{1}{e}. \]

So, we have

\[ \min \max_{g} V(x, g) = \frac{1}{e}. \quad (3.20) \]

We will now compute the optimal strategy for player I.

### 3.2.2. Strategy Player I

To find the optimal density \( f \) for player I, we use the same method as for player II. Let us consider

\[ y_1 = y \quad \text{and} \quad y_2 = y + \epsilon, \quad y_1, y_2 \in [0, 1] \quad (3.21) \]

with \( \epsilon > 0 \) really small. To find the expected difference

\[ \Delta V = V(x, y_1) - V(x, y_2) \]

we observe

- \( x < y_1 \) with probability \( F(y_1) \), see figure 3.10

![Figure 3.10: \( \Delta V \) when \( x < y_1 \)](image)

Here, \( V(x, y_1) = x \) and \( V(x, y_2) = x \), So

\[ \Delta V = 0 \]

- \( y_1 < x < y_2 \) with probability \( (y_2 - y_1)f(y_1) \), see figure 3.11

![Figure 3.11: \( \Delta V \) when \( y_1 < x < y_2 \)](image)

Here, \( V(x, y_1) = x \) and \( V(x, y_2) = y_2 - x \). So

\[ \Delta V = -y_2 \]

- \( x > y_2 \) with probability \( 1 - F(y_2) \), see figure 3.12

![Figure 3.12: \( \Delta V \) when \( x \geq y_2 \)](image)

Here, \( V(x, y_1) = y_1 - x \) and \( V(x, y_2) = y_2 - x \). So

\[ \Delta V = y_1 - y_2. \]
So, the expected difference in payoff is

\[
\Delta V = -y_2(y_2 - y_1)f(y_1) + (y_1 - y_2)(1 - F(y_2)) \\
= -(y + \epsilon)f(y) + \epsilon(1 - F(y + \epsilon)) \\
= -(y + \epsilon)f(y) + (1 - F(y + \epsilon)).
\]

To make player II indifferent, we solve for \(\Delta V = 0\). This gives

\[
(y + \epsilon)f(y) = (1 - F(y + \epsilon)).
\] (3.22)

We take the limit \(\epsilon \to 0\) to obtain

\[
yf(y) = 1 - F(y).
\] (3.23)

This is an ordinary differential equation (ODE) that we can solve accordingly. The general solution of this ODE is given by

\[
F(x) = 1 + \frac{c}{x}
\]

where \(c\) is some constant. Since we must have \(k > 0\) for which

\[
F(k) = 0,
\]

the optimal solution is given by

\[
F(x) = 1 + \frac{c}{x}, \quad k \leq x < 1.
\]

Since \(F(x)\) is a distribution function, we need to solve for \(F(k) = 0\) and \(F(1) = 1\). Solving for \(F(k) = 0\) we get

\[
F(x) = 1 - \frac{k}{x} \quad k \leq x < 1.
\] (3.24)

Note that \(\lim_{x \to 1} = 1 - k\). To satisfy \(F(1) = 1\), we must add probability mass \(k\) in 1. This gives us for the optimal density

\[
f(x) = \begin{cases} 
\frac{k}{x^2} & \text{if } k \leq x < 1 \\
k & \text{if } x = 1 \\
0 & \text{otherwise}
\end{cases}
\] (3.25)

and for the optimal distribution

\[
F(x) = \begin{cases} 
1 - \frac{k}{x} & \text{for } k \leq x < 1 \\
1 & \text{for } x = 1
\end{cases}
\] (3.26)

To find the value of \(k\), we again make use of the fact that player II is indifferent in his choice of \(y\) where \(\frac{1}{e} < y < 1\). So let \(y = 1\) to obtain

\[
V(f, 1) = \int_{k}^{1} x f(x) \, dx \\
= \int_{k}^{1} x \frac{k}{x^2} \, dx \\
= k \int_{k}^{1} \frac{1}{x} \, dx \\
= -k \ln k.
\]

Since player II is indifferent, we must have

\[
V(f, y) = -k \ln k \text{ for all } \frac{1}{e} < y < 1.
\] (3.27)
Maximizing this payoff for \( x \) gives us

\[
-k \ln k = 0 \Rightarrow -\ln k - 1 = 0 \Rightarrow k = \frac{1}{e}
\]

So, we find that the optimal strategy for player I is

\[
f(x) = \begin{cases} 
\frac{1}{e^{x/2}} & \text{if } 1/e \leq x < 1 \\
\frac{1}{e} & \text{if } x = 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(3.28)

see figure 3.13.

The optimal distribution for player I is then

\[
F(x) = 1 - \frac{1}{e^{x}} \text{ for } \frac{1}{e} \leq x < 1
\]  

(3.29)

with probability mass \( f(x) = \frac{1}{e} \) in \( x = 1 \), see figure 3.14.

This optimal strategy has payoff
Figure 3.14: optimal distribution for player I

\[ V(f, y) = \int_0^y xf(x) \, dx + \int_y^1 (x - y) f(x) \, dx \]
\[ = \int_0^1 xf(x) \, dx - y(F(1) - F(y)) \]
\[ = \frac{1}{e} \int_{1/e}^1 \frac{d}{dx} x \, dx + \frac{1}{e} - y(1 - (1 - \frac{1}{e})) \]
\[ = \frac{1}{e} \ln 1 - \ln \frac{1}{e} + \frac{1}{e} - \frac{1}{e} \]
\[ = \frac{1}{e}. \]

Hence,

\[ \max_{f} \min_{y \in Y_{\alpha}} V(f, y) = \frac{1}{e} \tag{3.30} \]

Using equation (3.20) and equation (3.30), we see that the generalized tally game with one draw has value

\[ \max_{x \in X_{\alpha}} \min_{y \in Y_{\alpha}} V(x, y) = \min_{y \in Y_{\alpha}} \max_{x \in X_{\alpha}} V(x, y) = \frac{1}{e}. \tag{3.31} \]

where player I has optimal strategy

\[ F(x) = 1 - \frac{1}{e} x, \quad \frac{1}{e} < x < 1 \tag{3.32} \]

with a probability mass of \( \frac{1}{e} \) in 1, see equation (3.29), and player II picks strategy \( G \)

\[ G(y) = \ln x e, \quad \frac{1}{e} < x \leq 1 \tag{3.33} \]

see equation (3.19)
We can use these results later for generalized tally game with two draws. However, we will first show how to solve this particular game with one draw using a numerical approach. This will prove useful for the numerical analysis of the generalized tally game with two draws.

### 3.2.3. NUMERICAL RESULTS

To solve the generalized tally game with one draw numerically, we are going to discretize $x$ and $y$. We do this by introducing a two-dimensional grid of discrete points $x_i$ and $y_j$ with mesh size $h = 1/N$ for some $N \in \mathbb{N}$.


text continues...
The objective for player I is to find a strategy $f$ that maximizes

$$\min_{0 \leq j \leq N} E[V(f, y_j)].$$

If we write

$$\min_{0 \leq j \leq N} E[V(f, y_j)] = K$$

for some $K > 0$, we can write the optimization problem as

maximize $K$
subject to $V^T \cdot f \geq K$
$1 \cdot f = 1$
$f \geq 0$.

Letting

$$f_{opt} = \frac{1}{K} f$$

we can rewrite this as a linear program

minimize $1^T f_{opt}$
subject to $V^T \cdot f_{opt} \leq 1$
$f_{opt} \geq 0$.

(3.35)

The corresponding lower value of the game $V^*$ is then given by

$$V^* = \frac{1}{\sum f_{opt}}.$$

(3.36)

We can solve this linear program numerically with MATLAB using `linprog`. Setting $N = [10, 100, 500]$, the computed lower values are

$$V = [0.3383, 0.3647, 0.3672].$$

(3.37)

Since

$$\frac{1}{e} = 0.3679,$$

we can see that our numerical approximation tends towards the analytical solution. The corresponding optimal strategies are given by figure 3.15. We expected a large mass probability in 1, and no probability mass below $V$, which is what the approximated solutions indicate as well.

For player II, following the same steps, we obtain the following linear program

maximize $1^T g_{opt}$
subject to $V \cdot g_{opt} \geq 1$
$g_{opt} \geq 0$.

(3.38)

where the value for player II $v_{II}$ is given by

$$V = \frac{1}{\sum g_{opt}}.$$

Solving this linear program, again setting $N = [10, 100, 500]$, we obtain

$$V = [0.3383, 0.3647, 0.3672]$$

(3.39)

which is, as expected, equal to equation (3.37). The optimal strategies for player II, shown in figure 3.16, indicate no mass probability below $V$. Also, for $N = 100, 500$ we see that for $x \geq V$, the density is a convex and strictly decreasing function, similar to the analytical solution.
3.2. Generalized Tally Game with One Draw

Figure 3.15: optimal density for player I for $N = 10, 100, 500$

Figure 3.16: optimal density for player II for $N = 10, 100, 500$
3.3. **One Draw with a Fixed Expectation**

When relating the generalized tally game to the joint replenishment problem, most research only analyses the strategy for player II. Let $x$ the choice for player I, $g$ the strategy for player II for some variable $0 \leq Y \leq 1$ and draw $(y_n)_{n \in \mathbb{N}}$ until

$$\sum_{i=1}^{n} y_i > x.$$ 

The main optimization problem can now be stated as

$$\min\{E[Y], 1 - V(x, y)\}$$

where

$$y = \sum_{i=1}^{n-1} y_i.$$

This approach tries to minimize the payoff for player II, while simultaneously increasing the expectation of $E[Y]$. Because of this, we extend the generalized tally game with one draw with a restriction on the expectation for player II, $E[Y] = \mu$. With this analysis we will show the relation between the expectation for player II, $\mu$, and the value of this game.

Fix $0 < \mu < 1$. Player II has to pick a density $g$ of a random variable $Y$ where

$$E[Y] = \mu, \text{ and } 0 \leq Y \leq 1. \quad (3.40)$$

So, fix $\mu > 0$ and denote $f$ and $g$ as the density of player I respectively player II. Next, they will both draw a number $x$ and $y$ from their respective densities. From before, we already know that

$$\begin{align*}
\mu = 0 \to V(x, y) &= 1 \\
\mu = 1 - \frac{1}{e} \to V(x, y) &= \frac{1}{e}
\end{align*}$$

From this, we could hypothesise that for $0 \leq \mu \leq 1 - \frac{1}{e}$, there holds

$$V(x, y) = 1 - \mu.$$ 

Indeed, for every $0 \leq \mu \leq 1 - \frac{1}{e}$, if player I always picks 1, the payoff from player II becomes

$$V(x, y) = \int_{0}^{1} 1 - yg(y) = \int_{0}^{1} g(y) - \int_{0}^{1} yg(y) = 1 - \mu.$$ 

Picking anything other than 1 will reduce the expected payoff of player I, so this is indeed the optimal strategy for player I. Therefore, for $0 \leq \mu \leq 1 - \frac{1}{e}$, this game has value $v = 1 - \mu$, regardless of the strategy of player II.

In case $1 - \frac{1}{e} \leq \mu \leq 1$, we will need to use that

$$g(y) = \begin{cases} 
 \frac{1}{2} & \text{for } \frac{1}{2} \leq y \leq 1 \\
 0 & \text{otherwise}
\end{cases}$$

is the optimal density for player II for $\mu = 1 - \frac{1}{e}$. Any strategy $g$ for $1 - \frac{1}{e} < \mu \leq 1$ needs to satisfy

$$\int_{0}^{1} g(y) = 1 \quad (3.41)$$

and
3.3. **One draw with a fixed expectation**

\[
\int_0^1 yg(y)\,dy = \mu. \tag{3.42}
\]

To satisfy the second condition, we will have to add some probability mass \( K \) to 1. Then, \( g \) must have the following form

\[
g(y) = \begin{cases} 
\frac{1}{y} & \text{for } c \leq y \leq 1 \\
K & \text{for } y = 1 
\end{cases} \tag{3.43}
\]

for some \( \frac{1}{x} \leq c \leq 1 \). To let \( g \) satisfy the conditions (3.41) and (3.42), we need to solve the following two equations.

\[
\int_c^1 \frac{1}{y} \, dy + K = 1 \tag{3.44}
\]

\[
\int_c^1 dy + K = \mu \tag{3.45}
\]

From equation (3.45), we get

\[
\int_c^1 dy + K = \mu \\
1 - c + K = \mu \\
K = \mu + c - 1
\]

With equation (3.44), we then find that

\[
\int_c^1 \frac{1}{y} \, dy + \mu + c - 1 = 1 - \ln c + \mu + c - 1 = 1 \\
\ln c - c + 2 = \mu. \tag{3.48}
\]

Hence, we obtain an implicit expression for \( c \) and \( \mu \). The optimal strategy for player II is given by

\[
g(y) = \begin{cases} 
\frac{1}{y} & \text{for } c \leq y \leq 1 \\
\mu - 1 + c & \text{for } y = 1 \\
0 & \text{otherwise.} 
\end{cases} \tag{3.46}
\]

which has expected value

\[
\mathbb{E}[V(x, y)] = \int_0^x (x - y)g(y)\,dy + \int_0^1 xg(y)\,dy \\
= x \int_0^1 g(y)\,dy - \int_0^x yg(y)\,dy \\
= x - \int_c^1 dy \\
= c. \tag{3.47}
\]

where

\[
\ln c - c + 2 = \mu. \tag{3.48}
\]

Now we will calculate the optimal strategy for player I. Let \( f(x) \) be the strategy for player I and let player II choose \( g \) where

\[
\mathbb{E}[g] = \mu \tag{3.49}
\]

We assume player II picks the following strategy:

\[
P(Y = y) = p, P(Y = 1) = 1 - p.
\]
To satisfy (3.49), we must have
\[ yp + 1 - p = \mu \]
\[ p = \frac{1 - \mu}{1 - y} \]

The expected payoff for player I is now given by
\[
E[V(f, g)] = p \left[ \int_{0}^{y} f(x)dx + \int_{y}^{1} (x - y)f(x)dx \right] + (1 - p) \int_{0}^{1} xf(x)dx
\]
\[ = -py(1 - F(y)) + \int_{0}^{1} xf(x)dx \]
\[ = -py(1 - F(y)) + E[X]. \] (3.50)

Let us write
\[ \overline{F}(y) = 1 - F(y). \] (3.51)

as the survival function. Since the optimal strategy of player I satisfies
\[
\max_{f} \min_{g} E[V(f, y)]
\]
we first need to calculate
\[
\min_{g} E[V(f, g)].
\]

We will use the principle of indifference; to make player II indifferent, for some \( k > 0 \) there must hold
\[
p y \overline{F}(y) = k
\]
\[ \overline{F}(y) = \frac{k(1 - y)}{y(1 - \mu)}. \] (3.52)

If we write \( c = \frac{k}{1 - \mu} \), we get
\[
\overline{F}(x) = \frac{c}{x} - c.
\]

Note that
\[
\overline{F}(y) = 0 \text{ for } y = 1
\]
\[ \overline{F}(y) = 1 \text{ for } y \leq \frac{c}{c + 1} \]

and
\[
f(x) = \begin{cases} \frac{-c}{x^2} & \text{for } \frac{c}{c + 1} \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

Using this, we calculate
\[
\min_{g} E[V(f, g)] = k + \int_{0}^{1} xf(x)dx
\]
\[ = c(1 - \mu) - c \int_{\frac{c}{c + 1}}^{1} \frac{1}{x}dx
\]
\[ = c(1 - \mu) + c \ln\left(\frac{c}{c + 1}\right) \]
Finally we compute

$$\max_f \min_g \mathbb{E}[V(f, g)]$$

by taking

$$\frac{\partial \min_g \mathbb{E}[V(f, g)]}{\partial c} = 0.$$ 

$$1 - \mu + \ln\left(\frac{c}{c+1}\right) + 1 - \frac{c}{c+1} = 0$$

$$\ln\left(\frac{c}{c+1}\right) - \frac{c}{c+1} + 2 = \mu.$$ 

Setting $v = \frac{c}{c+1}$, we have the optimal strategy

$$F(x) = \frac{v(1-x)}{x(1-v)} \text{ for } v \leq x \leq 1 \quad (3.53)$$

where

$$\ln v - v + 2 = \mu. \quad (3.54)$$

We see that the condition 3.54 is the same as condition 3.48 for player I. Indeed, this strategy has value

$$\max_f \min_g \mathbb{E}[V(f, g)] = -k + \int_0^1 xf(x)dx$$

$$= \frac{v(1-\mu)}{1-v} - \frac{v}{1-v} \int_0^1 \frac{1}{x} dx$$

$$= \frac{v}{1-v} (1 - \ln v + v - 2 \ln \frac{v}{1-v})$$

$$= v \quad (3.55)$$

From (3.55), equation (3.47), we see that this game with one draw and fixed expectation $\mu$ has value

$$\max_{x \in X} \min_{y \in Y} V(x, y) = \min_{y \in Y} \max_{x \in X} V(x, y) = v \quad (3.56)$$

where

$$\ln v - v + 2 = \mu. \quad (3.57)$$
3.3.1. Numerical results

Solving this game numerically, we first need to implement restriction (3.40) for player II, after which we can use the same method as before. So, let us fix $0 \leq \mu \leq 1$, $N \in \mathbb{N}$, $h = 1/N$ and let

$$x = [x_0, x_1, \ldots, x_n]^T \text{ and } y_1 = [y_0, y_1, \ldots, y_h]^T$$

where

$$x_i = i h, \quad y_j = j h \text{ for } i, j = 1, 2, \ldots, N.$$  

To satisfy equation (3.40) we solve

$$py_1 + (1 - p)y_2 = \mu.$$  

(3.58)

for $p$, where we define $y_2$ for $0 < \mu < \frac{1}{2}$ as

$$y_{2j} = \begin{cases} 
\mu & \text{for } y_{1,j} = 0 \\
2\mu - y_{1,j} & \text{for } y_{1,j} < 2\mu, \quad j = 1, 2, \ldots, N \\
0 & \text{for } y_{1,j} \geq 2\mu 
\end{cases}$$

(3.59)

and for $\frac{1}{2} < \mu < 1$

$$y_{2j} = \begin{cases} 
\mu & \text{for } y_{1,j} = 0 \\
1 & \text{for } 2\mu - y_{1,j} > 1, \quad j = 1, 2, \ldots, N. \\
2\mu - y_{1,j} & \text{for } 2\mu - y_{1,j} \leq 1 
\end{cases}$$

(3.60)

The payoff matrix $V$ can then be expressed as

$$V_{i,j} = p_j(x_i \cdot 1_{\{x_i \leq y_{1,j}\}}) + (x_i - y_{1,j}) \cdot 1_{\{x_i > y_{1,j}\}} + (1 - p_j)(x_i \cdot 1_{\{x_i \leq y_{2,j}\}}) + (x_i - y_{2,j}) \cdot 1_{\{x_i > y_{2,j}\}}.$$  

Now, for each $\mu_i = i \cdot h$ for $0 \leq i \leq N$, we can solve the same linear program as in (3.35) and (3.38) to obtain value

$$v = [v_1, v_2, \ldots, v_N].$$

We can plot the solution

![Game value for one draw with fixed expectation $\mu$](image)

Figure 3.17: Plot of the game value for fixed expectation $\mu$

We see that for $\frac{1}{2} < \mu \leq 1$, the approximated value indeed follows the analytical solution

$$\ln v - v + 2 = \mu.$$
We will now numerically compute the optimal strategies for the generalized tally game with two draws. To do this, we approximate the optimal strategy for player I by restricting the strategy set for player II. First, let $F$ be the strategy for player I and $G$ be the strategy for player II. Player I draws $x \sim F$ and player II draws $y_1, y_2 \sim G$ and $y_1 + y_2 = y$. The payoff $V(x, y)$ to player I here is given by

$$V(x, y) = \begin{cases} 
  x & \text{if } x < y_1 \\
  x - y_1 & \text{if } y_1 \leq x < y \\
  x - y & \text{if } x \geq y
\end{cases} \quad (4.1)$$

Note that the probability of drawing some number $y \leq 1$ is the probability that we draw $y$ in one draw plus the probability that we draw $y_1$ and $y_2$ in two draws. We are using this fact to calculate the optimal strategies for player I and player II. We will first calculate the optimal strategy for player II.

First, let us fix some $x \in [0, 1]$. With probability $1 - G(x)$, we have $y_1 \geq x$ with payoff

$$V(x, y) = x. \quad (4.2)$$

With probability $G(x)$ we have $y_1 < x$. Now the payoff is

$$V(x, y) = x - y_1 \quad (4.3)$$

when $y_2 < x - y_1$ with probability $G(x - y_1)$ and

$$V(x_1, y) = x_1 \quad (4.4)$$

when $y_2 > x - y_1$ with probability $1 - G(x - y_1)$.

The corresponding tree diagram is given in figure 4.1.
The expected payoff for player II is

$$E[V(x, g)] = \int_0^x g(y) \cdot E[V(x - y, g)] dy + \int_x^1 xg(y)dy.$$ 

See also equation (3.5). We could also rewrite this as

$$E[V(x, g)] = \int_0^x g(y) \cdot E[V(x - y, g)] dy + \int_x^1 xg(y)dy.$$ 

We will use this last expression to numerically compute the expected payoff for player II later on.

To compute the optimal strategy for player II, we again need to make player I indifferent in his strategy. The result that we obtain is very similar to the solution obtained by Chrobak at al. [4]. The main difference is our optimal strategy for player II focussed purely on the optimal solution for the Generalized Tally Game, while the solution by Chrobak optimized the generalized tally game while simultaneously putting constraints on the expectation.

To find the optimal strategy for player I, we use a variational method. We first put a restriction on the strategy of player II, after which we make him invariant under changes of this strategy.
4.1. STRATEGY PLAYER II

We will first find the optimal strategy for player II. Let $F, G$ be the distributions for player I and player II respectively. As before, both players will only draw numbers above some threshold $k > 0$, so

$$G(k) = F(k) = 0.$$  

We are going to make player I indifferent in choosing between $x_1 = x$ and $x_2 = x + \epsilon$ for some $x \in [0, 1]$ and $\epsilon > 0$ very small. The difference in expected payoff is given by

$$\Delta E[V(x, g)] = E[V(x + \epsilon, g) - E[V(x, g)].$$

We can now compute $\Delta E[V(x, g)]$ as we did for the one-draw tally game

$$\Delta V(x, g) = \begin{cases} 
\epsilon & \text{with } G(x) \\
-y_2 + \epsilon & \text{with } \epsilon g(x) \\
\epsilon & \text{with } 1 - G(x + \epsilon)
\end{cases}$$

This gives us

$$\Delta E[V(x, g)] = \epsilon (G(x) + 1 - G(x + \epsilon) - y g(x)). \tag{4.5}$$

We must have $\Delta E[V(x, y)] = 0$ for player I to be indifferent. By letting $\epsilon \to 0$, we obtain

$$y g(y) = 1. \tag{4.6}$$

To calculate $g$, fix $0 \leq x \leq 1$, draw $y_1, y_2 \sim g$ and write $y = y_1 + y_2$. We now partition the $[0, 1]$-interval for some $\frac{1}{4} < k < \frac{1}{3}$:

$$P = (0, k, 2k, 1) \tag{4.7}$$

To determine the optimal density, we calculate the optimal density on each interval, by making player I indifferent on each interval.

On $[0, k]$ we will have $g(y) = 0$, since we assumed $G(k) = 0$. We write

$$g^k_0(y) = 0. \tag{4.8}$$

On $[k, 2k]$, we need to solve

$$y g(y) = 1 \quad \text{with } k \leq y \leq 2k. \tag{4.9}$$

Since we cannot draw $z \in [0, k]$, we must have picked $y \in [k, 2k]$ in one draw. Thus the optimal strategy on $[k, 2k]$ will be the same as the optimal strategy for the tally game with one draw

$$g^k_{2k}(y) = \frac{1}{y}. \tag{4.10}$$

For $y \in [2k, 1]$, we again solve

$$y g(y) = 1 \quad \text{where } 2k \leq y \leq 1. \tag{4.11}$$

There are two possible situations:

- We draw $y \in [2k, 1]$ in one draw with probability $g^1_{2k}(y)$; this is the density we want to obtain.
• We first draw some \( z \in [k, y-k] \) and then \( y-z \in [k, y-k] \). This requires us to draw twice from \( g^1_k(z) : \ k \leq z \leq y-k \).

Hence, the probability that we end up in \( y \) with drawing \( z \) and \( y-z \) is

\[
g^1_k(z) g^1_k(y-z) \text{ for } k \leq z \leq y-k
\]  

(4.12)

So, combining this with equation (3.16) and using

\[
\frac{1}{a-b} = \frac{1}{a} \left( \frac{1}{a-b} + \frac{1}{b} \right)
\]  

(4.13)

we can compute \( g^1_{2k}(y) \)

\[
1 = y \left( g^1_{2k}(y) + \frac{1}{2} \int_{k}^{y-k} g^2_k(y-z) g^1_k(z) dz \right)
\]

\[
y g^1_{2k}(y) = 1 - \frac{y}{2} \int_{k}^{y-k} \frac{1}{y-z} dz
\]

\[
g^1_{2k}(y) = \frac{1}{y} \left( \int_{k}^{y-k} \frac{1}{y-z} + \frac{1}{z} dz \right)
\]

\[
g^1_{2k}(y) = \frac{1}{y} \left[ \ln(y-z) + \ln(z) \right]_{z=k}^{z=y-k}
\]

\[
g^1_{2k}(y) = \frac{1}{y} \left( 2 \ln(y-k) - 2 \ln(k) \right)
\]

\[
g^1_{2k}(y) = \frac{1}{y} \left( \ln\left( \frac{y-k}{k} \right) \right)
\]

This gives us

\[
g^1_{2k} = \frac{1}{y} \left( \ln\left( \frac{y-k}{k} \right) \right).
\]  

(4.14)

Note, that

\[
g^1_{2k} = g^1_{2k}.
\]  

(4.15)

Finally, we have to make sure that \( G(1) = 1 \). This can be achieved by defining the probability of picking 1 as

\[
1 - \int_k^1 g(y) dy \text{ for } y = 1.
\]  

(4.16)

The optimal strategy of player II using \( P \) can be written as

\[
g(y) = \begin{cases} 
0 & \text{for } y \in [0,k] \\
\frac{1}{2} & \text{for } y \in [k,2k] \\
\frac{1}{y} - \frac{1}{y} \ln\left( \frac{y-k}{k} \right) & \text{for } y \in [2k,1]
\end{cases}
\]  

(4.17)

with probability mass \( 1 - \int_k^1 g(y) dy \) in \( y = 1 \).

Now, since player I is indifferent under this strategy, we must have

\[
E[V(k,g)] = E[V(1,g)].
\]

Solving this for \( k \) gives us

\[
k = 0.36456.
\]  

(4.18)

Note that indeed \( \frac{1}{3} < k < \frac{1}{4} \).
4.2. **Strategy Player I**

To compute the optimal strategy for player I, we first need to restrict the strategy $G$ for player II. Let $y_1, y_2 \in [0, 1]$ such that $y_1 + y_2 = 1$. Player II still has an infinite strategy set, so we define the following two strategies $g_1(y)$

- With probability $p = \frac{1}{2}$, player II draws $y_1$ first, then $y_2$.
- With probability $p = \frac{1}{2}$, player II draws $y_2$ first, then $y_1$.

and $g_2(y)$

- With probability $p = \frac{1}{2}$, player II draws $y_1 + \epsilon$, then $y_2 - \epsilon$.
- With probability $p = \frac{1}{2}$, player II draws $y_2 - \epsilon$, then $y_1 + \epsilon$.

Now, instead of making player II indifferent between an infinite number of strategies, we make player II indifferent between just these strategies, thus

$$\Delta E[V(x, g)] = E[V(x, g_2)] - E[V(x, g_1)] = 0.$$

We make a distinction between

- $x < y_1$ with probability $F(y_1)$, we have
  $$\Delta E[V(x, g)] = 0$$

- $y_1 < x \leq y_1 + \epsilon$ with probability $f(y_1)\epsilon$, we have
  $$\Delta E[V(x, g)] = \frac{1}{2}(x - y_1) - \frac{1}{2}x$$
  $$= \frac{1}{2}y_1.$$

- $y_1 + \epsilon < x \leq y_2 - \epsilon$ with probability $F(y_2) - F(y_1)$, we have
  $$\Delta E[V(x, g)] = \frac{1}{2}(x - y_1) - \frac{1}{2}(x - y_1 - \epsilon)$$
  $$= -\frac{1}{2}\epsilon.$$

- $y_2 - \epsilon \leq x < y_2$ with probability $f(y_2)\epsilon$, we have
  $$\Delta E[V(x, g)] = \frac{1}{2}(x - y_2) - \frac{1}{2}x$$
  $$= -\frac{1}{2}y_2.$$

- $y_2 \leq x < y_1 + y_2$ with probability $F(y_1 + y_2) - F(y_2)$, we have
  $$\Delta E[V(x, g)] = \frac{1}{2}\epsilon - \frac{1}{2}\epsilon$$
  $$= 0.$$
Thus, we must have

\[ \Delta \mathbb{E}[V(x, g)] = 0 \]

\[ \frac{1}{2} \epsilon (y_1 f(y_1) = y_2 f(y_2) - y_2 F(y_2) + y_1 F(y_1)) \]

\[ y_1 f(y_1) + F(y_1) = y_2 f(y_2) + F(y_2). \] (4.19)

To solve equation (4.19), we need to solve the ODE

\[ x f(x) + F(x) = c \] (4.20)

with \( c \) some constant. The general solution of (4.20) is given by

\[ F(x) = c - \frac{d}{x}. \] (4.21)

Solving for \( F(k) = 0 \) gives us

\[ F(x) = c(1 - \frac{k}{x}). \] (4.22)

with a probability mass of \( 1 - c(1 - k) \) in \( x = 1 \). There must hold \( 0 \leq c \leq \frac{1}{1 - k} \). Now, to find \( c \) we will maximize the expected payoff

\[ \mathbb{E}[V(f, g)] = \frac{1}{2} \int_k^y x f(x) dx + \int_{y_1}^y (x - y_1) f(x) dx + \int_y^1 (x - y) f(x) dx \]

\[ + \frac{1}{2} \int_k^{y_2} x f(x) dx + \int_{y_2}^y (x - y_2) f(x) dx + \int_y^1 (x - y) f(x) dx \]

\[ = \int_k^1 x f(x) dx - \frac{1}{2} y_1 \int_{y_1}^y f(x) dx - \frac{1}{2} y_2 \int_{y_2}^y f(x) dx - y \int_y^1 f(x) dx \]

\[ = -ck \ln k - \frac{1}{2} y_1 (F(y) - F(y_1)) - \frac{1}{2} y_2 (F(y) - F(y_2)) - 11 y(1 - F(y)). \]

\[ = -ck \ln k + \frac{1}{2} \left( y_1 F(y_1) + \frac{1}{2} y_2 F(y_2) + yF(y) \right) - y \]

\[ = c(y - k \ln k - \frac{3}{2}) - y. \]

From this we see that \( \mathbb{E}[V(f, g)] \) is linear in \( c \). Since \( 0 \leq c \leq \frac{1}{1 - k} \), the expected payoff \( \mathbb{E}[V(f, g)] \) is maximal for

\[ c = \frac{1}{1 - k}. \]

Hence,

\[ F(x) = \frac{x - k}{x(1 - k)} \text{ for } k \leq x \leq 1 \] (4.23)

and

\[ f(x) = \frac{k}{1 - k} \frac{1}{x^2} \text{ for } k \leq x \leq 1. \] (4.24)
4.3. Numerical Results

In this section, we will provide a lower bound for the lower value of the generalized tally game with two draw. We will solve a slightly different game for player I, where we change the strategy for player II. Let player I choose a strategy \( F \) and draw \( x \sim F \). Now player II chooses densities \( g_1, g_2 \) and draws two numbers \( y_1 \sim g_1 \) and \( y_2 \sim g_2 \). So player II gets to choose a different density for the second draw. Although this strategy will be a better strategy for player II, since he can adjust the density for the second draw, this modified game has the advantage that we can solve it with linear programming, similarly to what we have done before. Therefore, the solution will be underestimating the real solution, since player II gets to improve his strategy in this game compared to the generalized tally game.

Let us fix \( N, M \in \mathbb{N} \), \( h_1 = 1/N \), \( h_2 = 1/M \) and let

\[
x = [x_0, x_1, \ldots, x_N]^T, \quad y_1 = [y_0^1, y_1^1, \ldots, y_M^1] \quad \text{and} \quad y_2 = [y_0^2, y_1^2, \ldots, y_M^2]^T
\]

where

\[
x_i = ih_1 \quad \text{for} \quad i = 0, 1, \ldots, N
\]

and

\[
y_j^1 = y_j^2 = jh_2 \quad \text{for} \quad j = 0, 1, \ldots, M.
\]

Now, we want to calculate the payoff for each \( x_i : i = 0, 1, \ldots, N \) and each pair \((y_{1,j}, y_{2,k}) : j, k = 0, 1, \ldots, M\). This will give us the \((N + 1) \times (M + 1)^2\) payoff matrix \( V \) with

\[
V = \begin{bmatrix}
V(x_0, (y_0^1, y_0^2)) & V(x_0, (y_0^1, y_1^2)) & \ldots & V(x_0, (y_0^1, y_M^2)) & V(x_0, (y_1^1, y_0^2)) & \ldots & V(x_0, (y_1^1, y_M^2)) \\
V(x_1, (y_0^1, y_0^2)) & V(x_1, (y_0^1, y_1^2)) & \ldots & V(x_1, (y_0^1, y_M^2)) & V(x_1, (y_1^1, y_0^2)) & \ldots & V(x_1, (y_1^1, y_M^2)) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
V(x_N, (y_0^1, y_0^2)) & V(x_N, (y_0^1, y_1^2)) & \ldots & V(x_N, (y_0^1, y_M^2)) & V(x_N, (y_1^1, y_0^2)) & \ldots & V(x_N, (y_1^1, y_M^2))
\end{bmatrix}.
\]

Here, for each pair \((y_j^1, y_k^2)\) we can write

\[
y_l = (y_j^1, y_k^2)
\]

where

\[
l = k + j \cdot (N - 1).
\]

Using these expressions, we get the payoff

\[
V(i, l) = x_i \cdot \mathbb{I}_{x_i < y_j^1} + (x_i - y_j^1) \cdot \mathbb{I}_{y_j^1 \leq x_i < y_j^2} + (x_i - y_j^2) \cdot \mathbb{I}_{x_i \geq y_j^2}.
\]

Now, we can define a strategy \( f \in \mathbb{R}^{N+1} \) for player I as

\[
f = [f_0, f_1, \ldots, f_N]^T \quad \text{where} \quad f_i = P(X = x_i) \quad \text{for} \quad i = 0, 1, \ldots, N
\]

and writing \( M_1 = (M + 1)^2 - 1 \) the strategy \( g \in \mathbb{R}^{M_1} \) for player II as

\[
g = [g_0, g_1, \ldots, g_{M_1}]^T \quad \text{where} \quad g_j = P(X = y_j) \quad \text{for} \quad l = 0, 1, \ldots, M_1.
\]

If we denote the \( k \)-th row of \( V \) as

\[
V_k = [V(x_k, y_0), V(x_k, y_1), \ldots, V(x_k, y_M)]
\]

then we can write the expected payoff of player I as
and the expected payoff of player II as

$$\mathbb{E}[V(x_i, g)] = V \cdot g$$ for \(i = 0, 1, \ldots, N\).

As seen before, the objective for player I is finding a strategy \(f\) that maximizes

$$\min_{0 \leq j \leq N} \mathbb{E}[V(f, y_j)].$$

If we write

$$\min_{0 \leq i \leq N} \mathbb{E}[V(f, y_j)] = K$$

we can write the optimization problem as

$$\begin{align*}
\text{maximize} & \quad K \\
\text{subject to} & \quad V^T \cdot f \geq K \\
& \quad \sum f = 1 \\
& \quad f \geq 0
\end{align*}$$

Letting

$$f_{\text{opt}} = \frac{1}{K} \cdot f$$

we can rewrite this linear program as

$$\begin{align*}
\text{minimize} & \quad \|f\|_1 \\
\text{subject to} & \quad V^T \cdot f_{\text{opt}} \leq 1 \\
& \quad f_{\text{opt}} \geq 0
\end{align*}$$

(4.27)

Here, the value for player I \(V\) is then given by

$$V = \frac{1}{\sum f_{\text{opt}}}. $$

Now, with \(N = M = 100\) we obtain lower value

$$V = 0.21054.$$

and the strategy from figure (4.2) There is a probability mass in 1, and the density is 0 whenever \(x < V\), just as we would expect. More interesting is the fact that there is another jump close to \(x = 0.6\). This jump indicates that player I should pick high numbers more often relative to then tally game with one draw.
Figure 4.2: strategy player I for $N, M = 100$. 
This thesis used the principle of indifference to analyze the generalized tally game. We first solved the finite tally game, a simplified tally game, with a finite strategy set for both players. Next, we extended this to the infinite tally game, where player II had a semi-infinite strategy set. This game illustrated the difficulties to obtain an optimal strategy, especially for player I.

Next, we provided two main theorems, the minimax theorem and Glicksberg theorem. We introduced the generalized tally game and proved that Glicksberg theorem indicated the generalized tally game has a minimax value.

Then we analyzed and solved the generalized tally game with one draw. We found an optimal strategy for both players and the value of the game. This analysis lays the foundation for the game two draws, and potentially the generalized tally game. We also stated and solved an extension of this game by restricting the strategy of player II. Other research on the Joint Replenishment Problem aim to maximize the payoff for player II while simultaneously minimize the expectation (of a draw) for player II. This analysis may prove useful for future work on the Joint Replenishment Problem, since we have found an analytical solution that is a good approximation for the real solution.

Next, we analyzed the tally game with two draws where we found an optimal strategy for player II by partitioning the interval and using the principle of indifference on each interval. To get an analytical solution for player I, we restricted the strategy of player II, similar to the extension for the one draw game. The numerical analysis of this game proved us with a lower value of the two draw game.

For the generalized tally game, the value of the game should be close to $\frac{1}{7}$. This means that we should realistically expect 2, at most 3 draws from player II. Hence, any solutions to the tally game with two draw can be considered as good approximations for the actual solution to the generalized tally game.
BIBLIOGRAPHY


MATLAB PROGRAMS

A.1. Generalized Tally Game with One Draw

```matlab
A = clear
% This program computes the value and optimal strategies
% for the tally game with 1 guess without a restriction
% on the expected value of a guess. Value = 1/e.
N = 100; Dx = 1/N;
x = [0:Dx:1]; y = x; f = ones(1, N+1); nul = zeros(N+1, 1); V = zeros(N+1, N+1);
A = x' * f;
B = f' * y;
V = max(A - B) + triu(A, 1);
z = linprog(-f, V, [], [], nul, []);
value = 1 / (f * z);
w = linprog(f, -V', -f, [], [], nul, []);
dualvalue = 1 / (f * w);
```

A.2. Generalized Tally Game One Draw Fixed Expectation

```matlab
A = clear all;
N = 100; Dx = 1/N;
x = [0:Dx:1]; y1 = x; f = ones(1, N+1); nul = zeros(N+1, 1);
Value = ones(N+1, 1); dualvalue = ones(N+1, 1);
Value2 = zeros(N+1, 2);
s1 = zeros(N+1, 1);
y2 = zeros(1, N+1);
p = zeros(1, N+1);
for i = 2:N+1
    if (mu(i) <= 0.5)
        mu(i);
        y2(1:2*i-1) = [mu(i) 2*mu(i)-Dx-Dx:0];
        e = mu(i) * ones(1, N+3-2*i) ./ ((2*mu(i):Dx:1) .* ones(1, N+3-2*i));
    
```
clear
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %%
N=100; Dx=1/N;
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %%
%This program computes the value and optimal strategies for the tally game with 2 draws.
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %%
x=[0:Dx:1]; y1=x; y2=x; f1=ones(1,N+1); f2=ones(1,(N+1)^2); nul1=zeros(1,N+1); nul2=zeros(1,(N+1)^2);
V=zeros(N+1,N+1);
for i = 1:N+1
  for j=1:(N+1)*(N+1)
    int1 = floor(j/(N+2))+1;
    int2 = mod(j-1,N+1)+1;
    V(i,j) = (x(i)<y1(int1))+(x(i)>=y1(int1) & x(i)<y1(int1)+y2(int2))+(x(i)-y1(int1))+(x(i)-y1(int1)+y2(int2))*(x(i)-y1(int1)+y2(int2));
  end
end
z=linprog(-f2,V,f1,[],[],nul2,[]);
value=1/(f2*z);
w=linprog(f1,-V,f1,[],[],nul1,[]);
dualvalue=1/(f1*w);