Graphene Based Pressure Sensors: Resonant Pressure Transduction Using Atomically Thin Materials

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GRAPHENE BASED PRESSURE SENSORS

RESONANT PRESSURE TRANSDUCTION USING SUSPENDED ATOMICALLY THIN MEMBRANES

by

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Since the introduction of the iPhone and Wii gaming console in 2007 the market for mobile sensor applications has exploded. A continuous effort is made to make these sensors smaller, better and cheaper. This allowed for new applications that caused the market to grow exponentially in recent years. Pressure sensors have recently found implementation in the mobile sensor market, however they require lots of fabrication steps making them relatively expensive. Due to its exceptional properties, graphene can be used as both an electrical and mechanical element reducing the amount of fabrication steps needed to build a pressure sensor. Using this principle, concepts for an absolute (atmospheric) pressure sensor and pressure difference (gage) sensor were developed. Novel analytical solutions were developed and compared to existing measurements of similar devices or to numerical simulations. It is found that these devices will not only be smaller and cheaper, but also have a larger bandwidth and less sensitivity to mechanical disturbances. The thermal noise of these devices is orders of magnitude lower than similar silicon based devices, therefore it is expected that graphene based sensors will meet the demands on accuracy, resolution and signal-to-noise ratio of current high-precision pressure sensors. It is concluded that graphene based resonant pressure sensors have the potential to meet the future mobile market demands.
**Extended Summary**

Micro pressure sensors are found in a wide variety of applications. The demand for these sensors will continue to rise as they become smaller, perform better and become more cost effective. Different principles are used to convert the pressure into an electronic signal, such as piezoresistive element, capacitive readout and resonant transduction. In general these sensors have complex architecture, which means that many processing steps are needed to fabricate them.

Graphene is a sp² bonded monolayer of carbon atoms, which has exceptional electronic and mechanical properties. These properties allow the graphene membrane to be used as both the mechanical element and the electrical element. This simplification of the design reduces the amount of fabrication steps needed to build a pressure sensor, therefore these graphene sensors have the potential to be more cost effective compared to current technologies. This thesis focuses on the analysis of these graphene based sensors which use resonance as the transduction principle.

**Compressed Film Resonant Pressure Transducer**

The compressed film pressure sensor measures the absolute pressure by determining the effort it takes to compress a thin film of gas, in a resonant sensor this can be expressed as a change in the eigenfrequency. In this chapter theory from literature is discussed on the mechanics of membranes, plates and a thin layer of gas. This is used to derive novel equations for a circular membrane or plate over a cavity filled with gas. For a membrane:

\[ \omega_{mn} = \sqrt{\frac{n_0}{\rho h} \left[ \frac{\gamma_{mn}}{a} \right]^2 + \frac{p_a}{n_0 g_0}} \]  

where \( \omega_{mn} \) is the radial eigenfrequency in [rad s⁻¹], \( n_0 \) is the pretension in [N m⁻¹], \( \rho h \) is the mass per unit square in [kg m⁻²], \( \gamma_{mn} \) is a constant corresponding to a mode shape (for the fundamental mode: \( \gamma_{01} = 2.404 \)), \( a \) is the radius in [m], \( p_a \) is the absolute pressure in the cavity in [Pa] and \( g_0 \) is the depth of the cavity in [m].

For a plate a similar equation is derived:

\[ \omega_{mn} = \sqrt{\frac{D}{\rho h} \left[ \frac{\gamma_{mn}}{a} \right]^4 + \frac{p_a}{g_0 D}} \]  

where \( D \) is the bending rigidity in [N m] and \( \gamma_{01} = 3.196 \). These equations are verified by finite element modeling using the Navier-Stokes equations and the existing nonlinear Reynold's equation. These results also allow the effect of damping and gas inertia to be studied.

Based on the results from the modeling it is expected that pressures between 0 and more than \( 10^6 \) Pa can be measured, where it is still unclear where the upper limit is. Notable advantages of these sensors is that the cavity does not have to be hermetically sealed and very clean, which will help to maintain the accuracy during its lifetime. The sensor also shows good linearity at low pressures. It is expected that the same resolution and accuracy can be achieved as existing high-precision pressure sensors for a much lower price.

**Induced Tension Resonant Pressure Transducer**

The induced tension resonant pressure transducer measures a pressure difference over the membrane, the pressure difference changes the potential energy of the membrane and therefore the stiffness. This can be measured by determining the eigenfrequency of the membrane. The eigenfrequency as function of the pressure is determined by first solving:

\[ \Delta p = \frac{4 n_0 \delta}{a^2} + \frac{8 E h \delta^3}{3 a^4 (1 - \nu)} \]  

where \( \Delta p \) is the pressure difference in [Pa], \( \delta \) the deflection of the center of the membrane in [m], \( E \) the bulk Young's modulus in [Pa], \( h \) the thickness of the membrane in [m] and \( \nu \) the Poisson ratio. Solving this
equation for a known pressure difference gives the deflection, this is used in the equation to determine the
eigenfrequency for the fundamental mode:

$$\omega = \frac{2.404}{a} \sqrt{\frac{n_0}{\rho h} + \frac{Eh(2 + \nu)\delta^2}{(1 + \nu^2)a^2\rho h}}$$ (4)

This solution was verified using a finite element model that takes large deflections into account. This model
can also predict more difficult situations such as point loads and distributed loads. Based on the analysis it
is found that this sensor is never linear and its pressure range is limited to approximately $10^6$ Pa due to
delamination. However this sensor can achieve the same cost and size benefits as the compressed film pressure
sensor. Also it is expected that the sensor can compete with current high-precision pressure gages.

**ACOUSTIC CIRCUIT MODELING**

For both devices to function properly the pressure equalization is an important parameter. The compressed
film pressure sensor will only work if the pressure is equalized on both sides, but this may not happen near the
resonance frequency to ensure good compression. The induced pressure sensor may not have any pressure
equalization, therefore in this situation it is also useful to know how to measure pressure equalization.

The pressure equalization is modeled using the existing method of hydraulic circuit modeling, often used
in the field of microfluidics, this makes use of an electrical equivalent model. The voltage is replaced by the
pressure and the current is the volumetric flow rate. The model for the membrane on top of the cavity is mod-
eled as a resistor in series with a capacitor. A capacitor is known as a compliance and it relates the amount
of volume displaced when a certain pressure is applied, in this case this can be derived from the deflection of
the membrane when a uniform pressure field is applied over it. Assuming an actuation frequency far below
the resonance frequency, for a circular membrane the following formula is found:

$$C_h = -\frac{\pi \sqrt{2} a^4}{16n_0}$$ (5)

where $C_h$ is the compliance in m$^3$ Pa$^{-1}$. For a plate:

$$C_h = \frac{0.1812a^6}{D}$$ (6)

A resistance relates the volumetric flow rate to the pressure difference applied over the element. In these
kind of systems the membrane/plate will be permeable and create a resistance, assuming an ideal gas the
following formula is found for the resistance:

$$R_h = \frac{p_\delta h}{\sigma a^2 RT \Phi}$$ (7)

where $\Phi$ is the permeability in mol N$^{-1}$ m$^{-1}$ s$^{-1}$. An important parameter is the acoustic time constant $\tau = \frac{R_h C_h}{\Phi}$. This gives an idea of the time scale needed for pressure equalization, for example if a step response is
applied to the pressure signal the pressure equalization is 63% after $\tau$ seconds and 99% after 4$\tau$ seconds.

This model can also be used the other way around. By measuring the response of the membrane the
acoustic time constant can be determined. By calculating the capacitance the resistance becomes known,
which can be used to calculate the permeability. For the compressed film sensor this model is particularly
useful as it allows to predict or manipulate the bandwidth of the sensor.

**CONCLUSION**

Comparing both sensor types, it is clear that the compressed film sensor is the most promising concept. This
is mainly because of its larger pressure range, linearity at low pressures and the fact that a hermetic cavity is
not needed. Both sensors can achieve a smaller size and higher cost effectiveness. It is expected that both
sensors can compete with current state-of-the-art pressure sensors on the market.
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1.1. ATOMICALLY THIN MATERIALS

It has been presumed that atomically thin layers could not exist in a free-standing state, until the unexpected discovery of graphene a decade ago [2, 3]. This material consists of a sp$^2$ bonded monolayer of carbon atoms, which results in a honeycomb lattice. Graphene can exist as a free standing material because of the strong interatomic bonds which prevents that thermal fluctuations will not lead to the generation of dislocations in the lattice. Also graphene can crumple in the third dimension (Fig. 1.1), this increases the elastic energy but suppresses thermal vibrations which can minimize the total free energy [4, 5].

The high quality of the crystal lattice leads to extraordinary electrical properties of graphene, illustrated alone by the length of the 50-page review paper by Castro Neto et. al. [7] published only five years after the discovery of graphene. This paper shows that a incredible amount of work has been done on the electronic properties of graphene, some noticeable properties which make graphene a material of such high interest are (from [7]):

- The charge carriers which are massless Dirac fermions [3], in agreement with theoretical predictions.
- It is both a semiconductor due to the zero density of states, but also a metal since the band gap is zero.
- The electrons have a very long mean free path.
- The properties can be modified by applying electric or magnetic fields, adding more layers, controlling the geometry and chemical doping.
- It can be probed with relative ease with various scanning probe techniques, such as atomic force microscopy and scanning tunneling microscopy.

Besides these scientific properties, graphene has also the potential for a large number of applications [4], such as chemical sensors and transistors.

Graphene can be produced by mechanical exfoliation of graphite. This is because graphite consists of graphene planes, with only a weak bonding between these planes (Fig. 1.2). The in-plane bonds are very strong, which allows the weak bonds to be broken and to isolate a single-layered plane of graphene. The strong in-plane bonds also result in exceptional mechanical properties as discussed in the following subsection.

1.1.1. MECHANICAL PROPERTIES OF GRAPHENE

More recently, the mechanical properties of graphene have been thoroughly investigated. It was found that few-layered graphene has an exceptionally high Young Modulus of 1050GPa [9], in agreement with the in-plane bulk value of graphite. A monolayer of suspended graphene has an elastic stiffness of about 340N m$^{-1}$ [10], in reasonable agreement with the bulk value of graphite if the interplane distance (0.335nm) is taken as the membrane thickness. These results have all been measured by applying a force using an AFM tip and measuring the deflection [11]. The mathematical description of a circular membrane subjected to a point load is discussed in section 3.1.5.
Another way to measure the mechanical properties of thin films is to apply a pressure to the membrane and measure the deflection of the membrane using a scanning probe [12]. This is possible since graphene is shown to be impermeable to all gases [13]. Mathematical details of this method can be found in section 3.1.4.

An important observation done while employing these methods is that there is always a pretension present, governing the behavior of the membrane at very small deflections. This means that a membrane has a higher resonance frequency than expected for a plate under zero pretension, if there are no external forces in the out-of-plane direction present. Observations by Bunch et. al. [12, 13] suggest that the pretension exist due to the strong van der Waals interaction with the sidewalls. The pretension should be close to the adhesion energy between graphene and the sidewalls (usually SiO$_2$), measured by Koenig and all [14] to be $0.45\, \text{J m}^{-1} = 0.45\, \text{N m}^{-1}$ for monolayer graphene and $0.31\, \text{J m}^{-1} = 0.31\, \text{N m}^{-1}$ for samples with 2 to 5 layers.

### 1.2. Micro Pressure Sensors

Micro pressure sensors are used in a wide variety of applications. Some examples in different industries are [18]:

- Health:
1.2. Micro Pressure Sensors

Figure 1.3: Three applications of micro pressure sensors. Left: a micro pressure sensor used in the health industry to monitor pressure in small blood vessels [15]; middle: packaged micro pressure sensors used as altimeters in the aerospace industry [16]; right: example of a future application that is possible due to the miniaturization of sensors, these smart contact lenses can for example be used to monitor the pressure in the eye which helps patients with glaucoma [17].

- In angioplasty, the pressure is monitored when a balloon is inserted in a blood vessel. About 500,000 units are shipped each year (2008) (Fig. 1.3).
- Infusion pumps, to control the flow of intravenous fluids which allows several drugs to be mixed in one channel, 200,000 units per year (approximately, 2008).
- Disposable blood pressure transducers, sensors with a lifetime of 24-72 hours; 17 million units per year (2008) at a unit price less than $10.
- Intrauterine pressure sensors to monitor the pressure during child delivery, 1 million units per year.

• Aerospace:
  - Pressure sensors can be found in oil, fuel, transmission and hydraulic systems.
  - Measuring air speed in pitot tubes.
  - Measuring the altitude using an altimeter (Fig. 1.3).

• Industrial products:
  - Process pressure transmitters, to monitor manufacturing processes.
  - Monitoring hydraulic systems
  - Refrigeration, heating, ventilation and air conditioning systems.

• Consumer products:
  - Scuba diving watches
  - Fitness gears using hydraulics
  - Digital tire pressure gages
  - Integrated in smartphones [19], recently Samsung has announced it will integrate pressure sensors in all their flagship models.

• Automotive:
  - Control of the manifold for optimal performance
  - Airflow control
  - Monitoring suspension systems
  - Pressure monitoring of:
    - Fuel pump
    - Transmission fluid
    - Engine oil
    - Tire pressure
    - Brake oil
The shear amount of applications show that there is a huge market for micro pressure sensors. This list is far from complete, since the miniaturization of pressure sensors in a ongoing process, new possibilities for application always emerge. An example is the smart contact lens with integrated electronics developed by Google (Fig. 1.3). The smaller these systems become, the better they perform. For example, micro pressure sensors are less sensitive to external vibration due to their low mass and the thermal distortion is very low due to their small size. The small size also results in a lower energy consumption, which makes new applications interesting (in smart phones for example). These sensors are produced using existing integrated circuit production techniques, where these devices are build up from locally adding or removing layers on a silicon wafer. A smaller sensors means that more devices can be produced per wafer, which makes the miniaturization of these devices very cost effective per unit.

In summary, there is a considerable amount of applications for micro pressure sensors and the continuous development allows for new applications to open up. The driving force behind this growing market is making the pressure sensors smaller, improving both the cost-effectiveness and the performance. The next section will present some visions on how this process will develop in the future.
1.3. FUTURE DEMAND FOR MICRO SENSORS

In recent years the demand for microsensors has exploded due to the innovations in the mobile market. Triggered by the emergence of the iPhone and the WII gaming console, the absorptions has gone from 10 million units in 2007 to 3.5 billion in 2012 [20]. In Fig. 1.4 a roadmap is shown for the mobile sensor market in the coming decade. Interesting difference is that the production organizations show a much higher growth than the market research organizations, this is because the production organizations base their prediction on applications that are not yet envisioned by the market researchers. It should be noted is that the mobile sensor explosion in 2007 was also not predicted by the market research organizations. There are several pointers that imply a potential market growth to a trillion units by 2022:

• **Emerging technologies** are defined by the World Economic Forum [21, 22] as:
  - Technologies which arise from new knowledge, or the innovative application of existing knowledge
  - Those that lead to the rapid development of new capabilities
  - Those that are projected to have significant systematic and long-lasting economic, social and political impacts
  - Those that create new opportunities for and challenges to addressing global issues
  - Technologies that have the potential to disrupt or create entire industries

In the Summit on the Global Agenda in 2011 in Abu Dhabi [22, 23] the top 10 of the emerging technologies were compiled:

  - Informatics for adding value to information
  - Synthetic biology and metabolic engineering
  - Green Revolution 2.0 – technologies for increased food and biomass
  - Nanoscale design of materials
  - Systems biology and computational modeling/simulation of chemical and biological systems
  - Utilization of carbon dioxide as a resource
  - Wireless power
  - High energy density power systems
  - Personalized medicine, nutrition and disease prevention
  - Enhanced education technology

These technologies are broadly diverse, but many of them represent or use a smart system (a fusion of sensors, computing and communication). Therefore they can potentially contribute to the continuously rising demand in microsensors.

• **Exponential Technologies** enable growth of supply in excess of the demand for them. Eight exponential technologies that are able to create abundance in just one generation (20 years) are identified [20]:

  - Biotechnologies and bioinformatics
  - Computational systems
  - Networks and sensors
  - Artificial intelligence
  - Robotics
  - Digital manufacturing and infinite computing
  - Medicine
  - Nanomaterials and nanotechnology

Sensors are not only only one of the eight exponential technologies, but are also embedded in the other technologies listed here. It is expected that abundance is achieved when 45 trillion networked sensors are helping to solve global problems such as shortage of food, energy, water, healthcare and education for all people on earth. The roadmap in Fig. 1.4 predicts that this will occur in 2033.
• **Internet of Things** [24] is defined as sensors and actuators embedded in physical objects, often using the Internet Protocol to communicate [25]. The following market segments are expected to deploy the Internet of Things in the (near) future [26]:

  - Smart cities: smart parking, monitoring structural health of buildings, noise urban maps, smartphone detection, electromagnetic field levels, traffic congestion, smart lighting, waste management and smart roads.
  - Smart environment: forest fire detection, air pollution, snow level monitoring, landslide and avalanche prevention and early earthquake detection.
  - Smart water: monitoring quality of tap water, chemical leakage detection, swimming pool remote measurement, pollution levels in the sea, leakage detection in tanks and pipes and water level monitoring in rivers, dams and reservoirs.
  - Smart metering: smart electricity grid, tank level monitoring, photovoltaic installations, water flow in transportation systems and stock calculations for silos.
  - Security and emergencies: perimeter access control, liquid presence, radiation levels and explosive/hazardous gases.
  - Retail: supply chain control, NFC payment, intelligent shopping applications and smart product management.
  - Logistics: quality of shipment control, item location, storage incompatibility detection and fleet tracking.
  - Industrial control: machine diagnostics, indoor air quality, temperature monitoring, ozone detection, indoor location, vehicle auto-diagnostics.
  - Smart agriculture: wine quality enhancement, green houses climate control, golf courses irrigation, meteorological station network, compost production.
  - Smart animal farming: hydroponics, offspring care, animal tracking, toxic gas levels.
  - Domestic automation: energy and water use, remote control appliances, intrusion detection, art and goods preservation.
  - Health: fall detection, medical fridges, sportsman care, patients surveillance, ultraviolet radiation.

As more and more object will have embedded sensors and have the ability to communicate, the resulting information network will create new business model, improve current business processes and reduce cost and risk.

The future for micro pressure sensors looks bright: current exponential technologies, emerging technologies and the Internet of Things will potentially introduce countless new applications. It is expected that these new applications can explode the market up to trillions of sensors. In order for these new applications to break through, the performance of these sensors have to improve while at the same time the price per unit has to drop. This drives the continuous innovation in the micro sensor market.

### 1.3.1. Micro pressure sensors for mobile devices

With Samsung incorporating pressure sensors in all their flagship models recently [19], the market for micro pressure sensor for cellphones is expected to rise to 681 million units in 2016, according to the IHS MEMS & Sensors Service (Fig. 1.5). These sensors are used to determine the altitude of the smartphone, allowing for faster position calculations using the global positioning system and for indoor navigation.

### 1.4. Working principles of micro pressure sensing

This section presents the principles currently used by manufacturers of micro–electromechanical systems (MEMS) to detect a pressure difference, based on an introduction by Tai-Ran Hsu [18]. In general there are two types of pressure sensors: absolute and gage pressure sensors. The absolute pressure sensor measures with respect to vacuum, while the gage pressure sensor measures the difference over the pressure transducer.

In general, the sensing element is a thin silicon die over which the pressure is applied (Fig. 1.6), called the diaphragm. A constraint base made of ceramics or glass supports the silicon dies. By applying a pressure over the die a deformation is induced, which is transduced into an electrical signal. In general, three kinds of transducers are used: piezoresistive, capacitive and resonant transducers.
1.4. WORKING PRINCIPLES OF MICRO PRESSURE SENSING

Figure 1.5: Forecast of shipments of pressure sensors for cellphones, retrieved from [19].

Figure 1.6: Cross section of a typical MEMS micro pressure sensor

1.4.1. PIEZORESISTIVE TRANSUDER

Fig. 1.7 shows a schematic of a pressure sensor using piezoresistive elements to convert the pressure to an electric signal. When a pressure is applied the membrane deforms, this changes the geometry of the resistors and thereby their resistance. A piezoresistive element also changes resistance due to the stresses induced, which is favorable for the sensitivity of the device. Using a Wheatstone bridge also increases sensitivity, the output voltage is expressed as:

\[ V_{\text{out}} = V_{\text{in}} \left( \frac{R_1}{R_1 + R_4} - \frac{R_3}{R_2 + R_3} \right) \]  

(1.1)

In general, the piezoresistive transducer gives a sensor with high gains and a good linearity (with respect to in-plane stresses). The major drawback of piezoresistive elements is the temperature sensitivity.

A example of a real piezoresistive device can be seen in Fig. 1.8. The diaphragm has a complicated design with many layers to enable the electrical connection to the piezoresistors. Polycrystalline silicon is used as a sacrificial layer to obtain a suspended membrane. Because so many layers are needed this design requires many fabrication steps to be build.

1.4.2. CAPACITIVE TRANSUDER

In Fig. 1.9 a pressure sensor is shown which uses capacitance changes for pressure measurements. If the diaphragm deforms the gap will be narrowed which leads to a change in capacitance between the electrodes.
The capacitance $C$ in a parallel-plate capacitor is related to the gap $d$ by the expression:

$$C = \varepsilon_r \varepsilon_0 \frac{A}{d}$$  \hspace{1cm} (1.2)

where $A$ is the area of overlap between the electrodes, $\varepsilon_0 = 8.85\text{pF m}^{-1}$ the permittivity of free space and $\varepsilon_r$ the relative permittivity of the dielectric medium between the plates. The capacitance change is measured using a bridge similar to the Wheatstone bridge, the variable capacitance can be determined using:

$$V_{\text{out}} = \frac{\Delta C}{2(2C + \Delta C)} V_{\text{in}}$$  \hspace{1cm} (1.3)

Micro pressure sensors using capacitance signal transduction are significantly less sensitive to applied pressure as the piezoresistive transducers. However the sensitivity to temperature variations is much lower. Another shortcoming of the capacitive transducer is the nonlinear relation between capacitance change and applied pressure.

In Fig. 1.10 an example of a real capacitive transducer is shown. Since only two electrodes are needed, the design of the diaphragm is much simpler than the piezoresistive transducer. In this case the sensor element is integrated with a CMOS manufactured readout circuit. What can be seen is that the transducer is much larger than the piezoresistive transducer to increase the gain of the sensor.

1.4.3. RESONATING TRANSDUCERS

Fig. 1.11 shows the cross-section of a pressure sensor using a resonating transducer [28]. When the diaphragm deflects tension is induced in the resonating layer. This causes the resonance frequency of this layer to shift. Electrodes between the diaphragm and the resonator layer provide electrostatic actuation and electrical readout of the device.

Advantage of these sensors is the linear relationship between the pressure applied and the frequency shift. Also the sensor has low sensitivity to temperature shifts and a high accuracy over a long lifetime. Disadvantage of these sensors is the cost to fabricate them, because many layers are needed.

1.5. GRAPHENE MEMBRANES FOR RESONANT PRESSURE SENSING

In the previous sections it was shown that atomically thin materials show exceptional mechanical properties and the applications and principles of microscale pressure sensors has been shown. The miniaturization of these devices is an ongoing process, but making silicon membranes very thin is challenging because of the risk of fracture. Resonant transducers show excellent properties regarding the temperature sensitivity, making them favorable over piezoresistive transducers. Also they have low noise characteristics which makes
them perform better than capacitive transducers. However the complexity of these resonant pressure sensors makes them expensive compared to the other transducers. In order to overcome these problems the concept shown in Fig. 1.12 is proposed, here a atomically thin membrane such as graphene is transferred over a cavity. This system is used often in literature as a simple resonator [9, 10, 12, 29–31]. The simplicity of these systems allows them to be made with far less production steps than current technologies, on the condition that a scalable process is found for the transfer of atomically thin membranes on these substrates.

This thesis will focus on gas interacting with these resonators and proposes two principles of pressure sensing. The first principle measures the pressure of the gas in the cavity by compressing it if the membrane vibrates (Fig. 1.13). Due to this compression the gas acts as a spring, increasing the stiffness of the system and thereby the resonance frequency. The second principle will measure a pressure difference over the membrane, the pressure difference causes deflection of the membrane which results in strain (Fig. 1.14). This also increases the stiffness and the resonance frequency. Both concepts will use electrical actuation and readout, as shown in Fig. 1.15.

**1.6. Indicators for Sensor Performance**

In order to evaluate the performance of these new concepts, they have to be compared to current silicon technologies. For this purpose different indicators are defined. The graphene devices have to at least meet or
exceed all of the following indicators [32]:

- Cost per unit
- Size
- Signal-to-noise ratio
- Accuracy
- Resolution
- Measurable range
- Bandwidth
- Power consumption
- Temperature dependence
- Sensitivity to environmental disturbances
- Lifetime
- Fabrication yield
- Variations between devices after manufacturing
- Cost of packaging, testing and assembly
- Ease of calibration

1.7. CONTENTS OF THE THESIS

This introduction has shown the remarkable properties of atomically thin materials and their possible application in pressure sensing. Chapter 2 goes into the pressure sensor that compresses the gas inside the cavity. This concept poses the problem that the static pressure on both sides of the sensor must be equal. Chapter 4 proposes a modeling approach to solve this problem. When the deflections of membranes become large, the tension induced causes nonlinear behavior. A COMSOL model was built to predict this behavior and is discussed in chapter 3. This modeling is used to predict the behavior of the membrane when a pressure difference is applied, this concept is discussed in section 3.2. The transducers are compared to each other and current silicon technologies in chapter 5, after which the conclusions are drawn.
Figure 1.10: Example of a capacitive pressure transducer manufactured by Toyota [27] integrated with a CMOS readout circuit.
Figure 1.11: Schematic figure of a pressure sensor using a resonant transducer

Figure 1.12: Illustration of a thin membrane (graphene as an example here) suspended over a cavity. The substrate is silicon with an oxidized layer on top. Holes are etched in the oxide to form a cavity. The flakes of thin materials are transferred over these cavity to form suspended membranes.
1.7. CONTENTS OF THE THESIS

Figure 1.13: Illustration of the first concept of resonant pressure transduction using an atomically thin membrane. When the membrane vibrates the gas in the cavity is compressed and expanded which contributes to the stiffness of the membrane. This will cause the resonance frequency to shift. The absolute pressure inside the cavity will determine how much force is exerted during compression and expansion, this means that the change in stiffness and resonance frequency is a measure for the pressure inside the cavity.

Figure 1.14: Illustration of the second concept which uses induced tension by a pressure difference over the membrane. The increase in tension causes the membrane to become stiffer and the resonance frequency shifts, which is used as transduction for the pressure signal.

Figure 1.15: Illustration of the electrical connections for both concepts.
This chapter discusses the first of the proposed two systems, a resonant transducer based on compression of a thin film of gas in the cavity. In order to perform a thorough analysis on the vibration of these systems some theory from the literature is presented. First the harmonic oscillator is introduced to define some basic principles. After this the mechanics of circular membranes, plates and plates with high in-plane tension are analyzed. The mechanics of the thin film of gas is discussed, using the Reynolds equation as a starting point.

Next step is to combine the known theory of the mechanics of the diaphragm and the gas and look into their interaction. In the most simple case only compression of the thin gas film is considered and simple expressions for the resonance frequency are derived. However the validity of this assumption has to be investigated further, for this purpose a system of equations is derived from the Navier-Stokes equations. Solutions from finite element analysis gives insight on effects of damping and inertia of the gas. Finally the transducer is evaluated on the sensor performance indicators from section 1.6.

2.1. Theory

This section governs the basic theory from literature regarding the mechanics of circular membranes and plates and the properties of a thin gas film. The harmonic oscillator is the most basic vibrating system found in engineering and physics and a good starting point for the theory of vibrational mechanics of membranes and plates. In order to analyze the mechanics of the thin gas film the nonlinear Reynolds equation is discussed, this equation is often used to model gas damping in MEMS. A linearization of this equation results in a simple expression only considering the compression of the thin film, which is valid under certain conditions. The theory is closed with a short discussion on the properties of rarefied gas, which is important due to the typically small gap sizes and low pressures.

2.1.1. The Harmonic Oscillator

The harmonic oscillator is probably the most studied system in science, because its behavior is fundamentally related to many concepts in physics and engineering. It can be used to describe vibrations in mechanical systems, the behavior of a coil-capacitor electrical system and light in an optical cavity. The most basic definition of a harmonic oscillator is that the potential energy of the system depends quadratically on the displacement $x$ from the equilibrium position $x = 0$:

$$ V(x) = \frac{1}{2} k x^2 $$

where $k$ is the spring constant of the system. For the undamped system the equation of motion is as follows:

$$ m \ddot{x} + k x = F(t) $$

where $m$ is the mass and $F(t)$ is the forcing function. If free vibration is considered, this equation can be solved to give the eigenvalue:

$$ \omega_0 = \sqrt{\frac{k}{m}} $$
which gives the natural frequency in rad s\(^{-1}\). The harmonic oscillator will vibrate at this frequency when it is displaced from equilibrium and then released to move without any external forces. In the time domain the solution of Eq. 2.2 is:

\[
x(t) = x_0 \cos \omega_0 t
\]  

All practical systems described by a harmonic oscillator experience some sort of damping. The most ideal case is viscous damping, where the damping force is proportional to the velocity of the oscillator:

\[
m \ddot{x} + c \dot{x} + kx = F(t)
\]  

For free vibration of a damped oscillator the following solution for the eigenfrequency is obtained:

\[
\omega_0 = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}
\]  

only in the underdamped case for which if \(c^2 - 4mk < 0\). The solution in the time domain becomes:

\[
x(t) = x_0 e^{-\frac{c}{2m} t} \cos \omega_0 t
\]  

from which it can be seen that the motion of the system has a vibration component (the cosine) and a damping component (the exponential term), see Fig. 2.1. If \(c^2 - 4mk > 0\) the system is overdamped and oscillations are no longer observed.

### 2.1.2. Forced Vibration

The forced vibration of a harmonic oscillator can have many solutions since the forcing function \(F(t)\) can be any arbitrary function. Mostly a harmonic load is considered:

\[
F(t) = F_0 \cos \omega t = \text{Re} \left[ F_0 \exp i\omega t \right]
\]  

where \(F_0\) is the amplitude of the force and \(\text{Re} \left[ \cdots \right]\) is an operator giving the real part of the expression in the brackets. The problem with the damped harmonic oscillator has a solution with a complex amplitude:

\[
\ddot{x}(t) = \frac{F_0}{k} \frac{1}{1 - \left( \frac{\omega}{\omega_0} \right)^2 + i \frac{\omega}{\omega_0} \sqrt{\frac{c^2}{km}}} \cos \omega t
\]  

where the bar over \(\ddot{x}(t)\) indicates that it is a complex quantity. The absolute value \(|\ddot{x}(t)|\) gives the actual displacement in time while the argument \(\angle \ddot{x}(t)\) gives the phase between the displacement and actuation force. A useful tool in the description of the oscillator is the mechanical quality factor \(Q\), defined as:

\[
Q = \frac{\sqrt{km}}{c}
\]
2.1.3. IMPULSE RESPONSE

Any linear system can be described using the impulse response. For the harmonic oscillator, the response to $F(t) = k\delta(t)$ is:

$$h(t) = \sin(o_r t) \exp \left( -\frac{o_r t}{2Q} \right) \sqrt{1 - \frac{1}{4Q^2}} \Phi(t)$$

(2.13)
where $\delta(t)$ is the Dirac delta function and $\Phi(t)$ is the Heaviside step function. For a arbitrary force $F(t)$ one can now calculate the forced response using the convolution integral:

$$x(t) = \int_{0}^{t} h(t-r) \frac{F(r)}{k} \, dr$$  \hspace{1cm} (2.14)

### 2.1.4. Eigenfrequencies of a Membrane

A membrane is a thin piece of (solid) material whose behavior is dominated by tension in the plane and for which the resistance to out-of-plane bending can be ignored. These assumptions are typically valid for very thin plates such as graphene monolayers. Since dynamics are an important part of this thesis, it is a logical step to derive a equation for the eigenfrequencies of the membrane. All derivations regarding the eigenfrequencies of a membrane follow the description by Szilard [33] closely. The equation that governs the vibration of a membrane is the wave equation, in Cartesian coordinates expressed as:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\rho h \partial^2 w}{n_0 \partial t^2} - \frac{p_z}{n_0}$$  \hspace{1cm} (2.15)

where $w(x, y)$ is the deflection in the out-of-plane direction (m), $x$ and $y$ the coordinates (m), $\rho$ the bulk density of the material (kg m$^{-3}$), $h$ the thickness of the membrane (m), $n_0$ the initial tension in the membrane (N m$^{-1}$) and $p_z(x, y)$ the pressure load in the out-of-plane direction (Pa). For a circular membrane it is more convenient to write this equation to cylindrical coordinates:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} = \frac{\rho h \partial^2 w}{n_0 \partial t^2} - \frac{p_z}{n_0}$$  \hspace{1cm} (2.16)

where $\theta$ is the angle (rad) and $r$ the radial coordinate (m). A solution to this equation can be found by using separation of variables:

$$w(r, \theta, t) = R(r)\Phi(\theta)\tau(t)$$  \hspace{1cm} (2.17)

Assuming a harmonic vibration:

$$w = R(r)\Phi(\theta)\sin(\omega t)$$  \hspace{1cm} (2.18)

Substitute Eq. 2.18 into Eq. 2.16 to obtain:

$$\frac{R''(r)}{R(r)} + \frac{1}{r^2} \frac{\Phi''(\theta)}{\Phi(\theta)} + \frac{1}{r} \frac{R'(r)}{R(r)} + \omega^2 \frac{\rho h}{n_0} = 0$$  \hspace{1cm} (2.19)

Now introduce the separation constant $\lambda$ as:

$$\lambda^2 = \omega^2 \frac{\rho h}{n_0}$$  \hspace{1cm} (2.20)

Using this rewrite Eq. 2.19 to:

$$\left[ R''(r) + \frac{1}{r} R'(r) + \lambda^2 R(r) \right] \Phi(\theta) + \frac{R(r)}{r^2} \Phi''(\theta) = 0$$  \hspace{1cm} (2.21)

Using the separation constant $m^2$ this equation can be separated into two ordinary differential equations:

$$\frac{d^2 \Phi(\theta)}{d\theta^2} + m^2 \Phi(\theta) = 0$$  \hspace{1cm} (2.22)

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \left[ \lambda^2 - \frac{m^2}{r^2} \right] R(r) = 0$$  \hspace{1cm} (2.23)

Now $m$ must be chosen such that a harmonic equation in $\theta$ is obtained, the solution of Eq. 2.22 is:

$$\Phi_m(\theta) = C_{1,m} \sin(m\theta) + C_{2,m} \cos(m\theta) \quad m = 0, 1, 2, ...$$  \hspace{1cm} (2.24)

Eq. 2.23 is a Bessel-type differential equation, for a fixed boundary condition ($R(a) = 0$) on the edge the solution is:

$$R(r) = J_m(\lambda a) = 0$$  \hspace{1cm} (2.25)
where \( J_m(\cdot) \) is the Bessel function of the first kind and order \( m \), \( a \) is the radius of the membrane (m). The roots of these equations can be used to evaluate \( \lambda \), first define:

\[
\gamma = a \lambda
\]  

(2.26)

Now a solution for Eq. 2.25 can be found, one will find an infinite number of roots which will be labeled with index \( n \).

For \( m = 0 \) and \( J_0(\gamma) = 0 \):

\[
\gamma_0 = 2.405, 5.520, 8.654, ...
\]

For \( m = 1 \) and \( J_1(\gamma) = 0 \):

\[
\gamma_1 = 3.832, 7.016, 10.173, ...
\]

For \( m = 2 \) and \( J_2(\gamma) = 0 \):

\[
\gamma_2 = 5.135, 8.417, 11.620, ...
\]

The eigenvalues now become:

\[
\omega_{mn} = \frac{\gamma_{mn}}{a} \sqrt{\frac{n_0}{\rho h}}
\]  

(2.27)

which gives the angular frequency of the vibration, the eigenfrequency is:

\[
f_{mn} = \frac{\omega_{mn}}{2\pi a} \sqrt{\frac{n_0}{\rho h}}
\]  

(2.28)

From this equation it can be seen that an increase in radius or mass will reduce the eigenfrequency, an increase in the tension will increase the eigenfrequency. The degenerate (this means at the same frequency) modes of the membrane are given by:

\[
W_{mn}^{(1)}(r, \theta) = R(r)\Phi(\theta) = J_m(\gamma_{mn}r/a)\cos m\theta
\]

(2.29a)

\[
W_{mn}^{(2)}(r, \theta) = J_m(\gamma_{mn}r/a)\sin m\theta
\]

(2.29b)

The first six modes are plotted in Fig. 2.3.

2.1.5. Eigenfrequencies of a Plate

A plate is a thin piece of solid material which has neglectable in-plane tensions, this means that the behavior is dominated by the resistance to bending (known as bending rigidity) in the plate. Since the bending rigidity has become the dominant restoring force, therefore a different formula must be used to calculate the eigenfrequencies. This derivation follows the description by Rao [34] closely. In the case of a free vibrating plate one can write:

\[
D \nabla_r^4 W(r, \theta, t) + \rho h \frac{\partial^2 W(r, \theta, t)}{\partial t^2} = 0
\]  

(2.30)

where \( D(\text{N m}) \) is the bending rigidity, for bulk material defined as:

\[
D = \frac{E h^3}{12(1-\nu^2)}
\]  

(2.31)

where \( E \) is the Young’s modulus (Pa) and \( \nu \) the Poisson ratio (–). The operator \( \nabla_r^2 \) is defined as:

\[
\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]  

(2.32)

After separation of variables of Eq. 2.30, one obtains the relations:

\[
\frac{d^2 \tau(t)}{dt^2} + \omega^2 \tau(t) = 0
\]  

(2.33a)

\[
\nabla^4 W(r, \theta) - \lambda^4 W(r, \theta) = 0
\]  

(2.33b)

where

\[
\lambda^4 = \frac{\rho har^2}{D}
\]  

(2.34)
Figure 2.3: First eigenmodes of a circular membrane. From this figure the physical meaning of the subscript $mn$ becomes clear: $m$ is the number of nodal diameters (straight lines where $w = 0$, crossing the center) and $n$ is the number of nodal circles (circles where $w = 0$, the outer edge included).
Eq. 2.33b equation can be rewritten to:

\[(\nabla_r^2 W + \lambda^2 W)(\nabla_r^2 W - \lambda^2 W) = 0\]  

(2.35)

This can be written can be written as two separate equations:

\[
\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} + \lambda^2 W = 0
\]

(2.36a)

\[
\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} - \lambda^2 W = 0
\]

(2.36b)

Since Eq. 2.33b is a linear differential equation, a solution can be obtained by the substitution of the solutions from Eq. 2.36. Now separate the variables again by using:

\[W(r, \theta) = R(r)\Phi(\theta)\]

(2.37)

Now Eq. 2.36 can be rewritten as:

\[
\frac{r^2}{R(r)} \left[ \frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{d R(r)}{dr} \pm \lambda^2 \right] = -\frac{1}{\Phi(\theta)} \frac{d^2 \Phi}{d\theta^2} = m^2
\]

(2.38)

Where \(m^2\) is constant, now one can write:

\[
\frac{d^2 \Phi}{d\theta^2} + m^2 \Phi = 0
\]

(2.39a)

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{d R}{dr} + \left( \lambda^2 - \frac{m^2}{r^2} \right) R = 0
\]

(2.39b)

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{d R}{dr} + \left( \lambda^2 + \frac{m^2}{r^2} \right) R = 0
\]

(2.39c)

These equations have the following general solutions, in respective order one can write:

\[
\Phi(\theta) = A \cos a\theta + B \sin a\theta \quad a = 0, 1, 2, ...
\]

(2.40a)

\[
R_1(r) = C_1 I_m(\lambda r) + C_2 Y_m(\lambda r) \quad m = 0, 1, 2, ...
\]

(2.40b)

\[
R_2(r) = C_3 I_m(\lambda r) + C_4 K_m(\lambda r) \quad m = 0, 1, 2, ...
\]

(2.40c)

Where \(I_m\) is a Bessel function of the first kind and \(Y_m\) of the second kind. \(I_m\) and \(K_m\) are modified Bessel functions of the first and second kind, respectively. \(m\) is the order of the Bessel function. The general solution of Eq. 2.33b is:

\[
W(r, \theta) = [c_m^{(1)} I_m(\lambda r) + c_m^{(2)} Y_m(\lambda r) + c_m^{(3)} I_m(\lambda r) + c_m^{(4)} K_m(\lambda r)](A_m \cos m\theta + B_m \sin m\theta)
\]

(2.41)

\[
m = 0, 1, 2, ...
\]

The boundary conditions of the circular plate are:

\[W(a, \theta) = 0\]  

(2.42a)

\[\frac{\partial W}{\partial r}(a, \theta) = 0\]  

(2.42b)

The solution of \(W\) must also remain finite, since the functions \(Y_m(\lambda r)\) and \(K_m(\lambda r)\) become infinite at \(r = 0\), Eq. 2.41 reduces to:

\[
W(r, \theta) = [c_m^{(1)} I_m(\lambda r) + c_m^{(2)} I_m(\lambda r)](A_m \cos m\theta + B_m \sin m\theta) \quad m = 0, 1, 2, ...
\]

(2.43)

Eq. 2.42a gives:

\[
c_m^{(3)} = -\frac{I_m(\lambda a)}{I_m(\lambda a)} c_m^{(1)}
\]

(2.44)
Now this constant can be combined with $A_m$ and $B_m$ to give new constants, which results in:

$$W(r, \theta) = \left[ J_m(\lambda r) - \frac{J_m(\lambda a)}{I_m(\lambda a)} I_m(\lambda r) \right] \left( A_m \cos m\theta + B_m \sin m\theta \right) \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (2.45)

The second boundary condition Eq. 2.42b gives the frequency equations:

$$\left[ \frac{d}{dr} J_m(\lambda r) - \frac{J_m(\lambda a)}{I_m(\lambda a)} \frac{d}{dr} I_m(\lambda r) \right] = 0 \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (2.46)

The derivatives on the Bessel functions are known relations:

$$\frac{d}{dr} J_m(\lambda r) = \lambda J_{m-1}(\lambda r) - \frac{m}{r} J_m(\lambda r)$$  \hspace{1cm} (2.47a)

$$\frac{d}{dr} I_m(\lambda r) = \lambda I_{m-1}(\lambda r) - \frac{m}{r} I_m(\lambda r)$$  \hspace{1cm} (2.47b)

This gives the frequency equation:

$$I_m(\lambda a) J_{m-1}(\lambda a) - J_m(\lambda a) I_{m-1}(\lambda a) = 0 \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (2.48)

For each value of $m$ this equation has to be solved to find $n = 1, 2, 3, \ldots$ roots $\lambda_{mn}$. The natural frequencies then follow from Eq. 2.34:

$$\omega_{mn} = \lambda_{mn}^2 \sqrt{\frac{D}{\rho h}}$$  \hspace{1cm} (2.49)

or

$$f_{mn} = \frac{\lambda_{mn}^2}{2\pi} \sqrt{\frac{D}{\rho h}}$$  \hspace{1cm} (2.50)

Some roots of Eq. 2.48 are (35): $\lambda_{01} a = 3.196, \lambda_{11} a = 4.611, \lambda_{21} a = 5.906, \lambda_{02} a = 6.306, \lambda_{31} a = 7.144, \lambda_{12} a = 7.799, \lambda_{22} a = 9.197, \lambda_{03} a = 9.439, \lambda_{13} a = 10.958, \lambda_{23} a = 12.402$. The two degenerate (except for $m = 0$) mode shapes are given by the functions (see also Fig. 2.4):

$$W_{mm}^{(1)}(r, \theta) = [J_m(\lambda_{mn} r) I_m(\lambda_{mn} a) - J_m(\lambda_{mn} a) I_m(\lambda_{mn} r)] \cos m\theta$$  \hspace{1cm} (2.51a)

$$W_{mm}^{(2)}(r, \theta) = [J_m(\lambda_{mn} r) I_m(\lambda_{mn} a) - J_m(\lambda_{mn} a) I_m(\lambda_{mn} r)] \sin m\theta$$  \hspace{1cm} (2.51b)

### 2.1.6. Eigenfrequencies of Plates with In-plane Stresses

In the analysis of atomically thin membranes, one might encounter the situation where the membrane is neither in the membrane (high in-plane tension) nor the plate (large thickness) regime. In this case one must consider both the bending rigidity and the in-plane tension in the equation of motion:

$$\nabla^4 w = -\frac{n_0}{D} \nabla^2 w + \frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} = 0$$  \hspace{1cm} (2.52)

Now assume a solution in the form:

$$w = W(r, \theta) \tau(t)$$  \hspace{1cm} (2.53)

Substitution in Eq. 2.52 and dividing by $W(r, \theta) \tau(t)$ gives:

$$-\frac{\rho h}{D} \frac{\partial^2 \tau(t)}{\partial t^2} = -\frac{1}{W(r, \theta)} \nabla^4 W(r, \theta) - \frac{n_0}{DW(r, \theta)} \nabla^2 W(r, \theta)$$  \hspace{1cm} (2.54)

Recognizing that both sides of this equation must be equal to a constant, chosen as $\lambda^4$, in order to obtain a solution at all times and positions the equations can be separated:

$$\frac{d^2 \tau(t)}{dt^2} + \frac{D}{\rho h} \lambda^4 \tau(t) = 0$$  \hspace{1cm} (2.55a)

$$\nabla^4 W(r, \theta) - \frac{n_0}{D} \nabla^2 W(r, \theta) - \lambda^4 W(r, \theta) = 0$$  \hspace{1cm} (2.55b)
Figure 2.4: First eigenmodes of a circular plate. They are very similar to the eigenmodes of the membrane, except for the added boundary condition on the edge. Like the membrane, \( m \) is the number of nodal diameters (straight lines where \( w = 0 \), crossing the center) and \( n \) is the number of nodal circles (circles where \( w = 0 \), the outer edge not included).
The general solution of Eq. 2.55a is:

$$
\tau(t) = A\cos\left(\lambda^2 \sqrt{\frac{D}{\rho h}} t\right) + B\sin\left(\lambda^2 \sqrt{\frac{D}{\rho h}} t\right)
$$  (2.56)

Eq. 2.55b can be rewritten as follows:

$$
\left(\nabla^4 - \frac{n_0}{D} \nabla^2 - \lambda^4\right) W = 0
$$

(2.57)

$$
(\nabla^2 + \alpha^2)(\nabla^2 + \beta^2) W = 0
$$

$$
\alpha^2 + \beta^2 = -\frac{n_0}{D}
$$

$$
\alpha^2 \beta^2 = -\lambda^4
$$

From the condition that $\Phi$ must have a periodicity of 2, this gives the following system of equations:

$$
\nabla^2 W(r,\theta) + \alpha^2 W(r,\theta) = 0
$$  (2.58a)

$$
\nabla^2 W(r,\theta) - \left(\frac{n_0}{D} + \alpha^2\right) W(r,\theta) = 0
$$  (2.58b)

If the solution of $W$ is assumed in the form $R(r)\Phi(\theta)$ these equations become:

$$
\Phi(\theta) \frac{\partial^2 R(r)}{\partial r^2} + \Phi(\theta) \frac{1}{r} \frac{\partial R(r)}{\partial r} + \frac{R(r)}{r^2} \frac{\partial^2 \Phi(\theta)}{\partial \theta^2} + \alpha^2 R(r)\Phi(\theta) = 0
$$  (2.59a)

$$
\Phi(\theta) \frac{\partial^2 R(r)}{\partial r^2} + \Phi(\theta) \frac{1}{r} \frac{\partial R(r)}{\partial r} + \frac{R(r)}{r^2} \frac{\partial^2 \Phi(\theta)}{\partial \theta^2} - \left(\frac{n_0}{D} + \alpha^2\right) R(r)\Phi(\theta) = 0
$$  (2.59b)

$\Phi$ can be separated from both these equations using the separation variable $m^2$, this gives the following system of equations:

$$
\frac{d^2 \Phi(\theta)}{d\theta^2} + m^2 \Phi(\theta) = 0
$$  (2.60a)

$$
\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{d R(r)}{dr} + \alpha^2 R(r) - \frac{m^2}{r^2} R(r) = 0
$$  (2.60b)

$$
\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{d R(r)}{dr} - \left(\frac{n_0}{D} + \alpha^2\right) R(r) - \frac{m^2}{r^2} R(r) = 0
$$  (2.60c)

The solution of Eq. 2.60a is:

$$
\Phi(\theta) = C\cos m\theta + D\sin m\theta
$$  (2.61)

From the condition that $\Phi$ must have a periodicity of $2\pi$, $m$ can only obtain integer values:

$$
\Phi(\theta) = C\cos m\theta + D\sin m\theta \quad m = 0, 1, 2, ...
$$  (2.62)

Eq. 2.60b is a Bessel differential equation of order $m$ whose solution is given by:

$$
R_1(r) = C_m^{(1)} J_m(\alpha r) + C_m^{(2)} Y_m(\alpha r)
$$  (2.63)

Eq. 2.60c is also a Bessel differential equation, but with an imaginary argument, this gives the general solution:

$$
R_2(r) = C_m^{(3)} I_m\left(\sqrt{\frac{n_0}{D} + \alpha^2} r\right) + C_m^{(4)} K_m\left(\sqrt{\frac{n_0}{D} + \alpha^2} r\right)
$$  (2.64)

The condition that $R$ must remain finite at $r = 0$ makes that $C_m^{(2)}$ and $C_m^{(4)}$ must be equal to zero. The general solution for $W(r,\theta)$ becomes:

$$
W(r,\theta) = \left[C_m^{(1)} J_m(\alpha r) + C_m^{(3)} I_m\left(\sqrt{\frac{n_0}{D} + \alpha^2} r\right)\right] [A_m \cos m\theta + B_m \sin m\theta] \quad m = 0, 1, 2, ...
$$  (2.65)
From the boundary condition $W(a, \theta) = 0$ one obtains:

$$C_m^{(3)} = -\frac{J_m(\alpha a)}{I_m\left(a\sqrt{\frac{n_0}{D}} + \alpha^2\right)}C_m^{(1)}$$  \hspace{1cm} (2.66)

Eq. 2.65 now becomes:

$$W(r, \theta) = \left[ J_m(\alpha r) - \frac{J_m(\alpha a)}{I_m\left(a\sqrt{\frac{n_0}{D}} + \alpha^2\right)}I_m\left(r\sqrt{\frac{n_0}{D}} + \alpha^2\right)\right] [A_m \cos m\theta + B_m \sin m\theta]$$  \hspace{1cm} (2.67)

where $A_m$ and $B_m$ are new constants. The clamped plate condition $dW(a, \theta)/dr = 0$ gives the following frequency relation:

$$\alpha I_{m-1}(\alpha a)I_m\left(a\sqrt{\frac{n_0}{D}} + \alpha^2\right) - \sqrt{\frac{n_0}{D} + \alpha^2}I_{m-1}\left(a\sqrt{\frac{n_0}{D}} + \alpha^2\right)J_m(\alpha a) = 0$$  \hspace{1cm} (2.68)

The roots of this equation gives $\alpha$, which can be used to calculate the eigenfrequency. It is more convenient for numerical evaluation to write the frequency relation in the form:

$$\frac{\alpha a}{I_{m-1}\left(a\sqrt{\frac{n_0}{D}} + \alpha^2\right)} = \sqrt{\frac{n_0}{D} + \alpha^2} \frac{J_m(\alpha a)}{J_{m-1}(\alpha a)}$$  \hspace{1cm} (2.69)

It can be seen that this relation is dependent on only two variables, $\alpha^2 \frac{n_0}{D}$ and $\alpha a$. From the system parameters $\alpha^2 \frac{n_0}{D}$ is evaluated to give $\alpha a$, for this purpose a numerical solver or Fig. 2.5 can be used. Dividing by the radius gives the root of the frequency relation. Once $\lambda_{mn}$ is known $\lambda_{mn}$ is evaluated as:

$$\lambda_{mn}^2 = \alpha_{mn}\sqrt{\frac{n_0}{D} + \alpha_{mn}^2}$$  \hspace{1cm} (2.70)

From Eq. 2.56 the eigenfrequency is obtained:

$$\omega_{mn} = \lambda_{mn}\sqrt{\frac{D}{\rho h}} = \frac{\alpha_{mn}\sqrt{\left[\frac{n_0}{D} + \alpha_{mn}^2\right] \frac{D}{\rho h}}}{\sqrt{\alpha_{mn}^2 \frac{n_0}{\rho h} + \alpha_{mn}^4 \frac{D}{\rho h}}}$$  \hspace{1cm} (2.71)

Fig. 2.6 uses this formula to show how a graphene resonator behaves as function of the number of layers. Inspecting the formula shows that for $n_0 >> D$ the formula for the ideal membrane (Eq. 2.27) is obtained, while for the situation where $D >> n_0$ the equation for the plate (Eq. 2.50) is obtained. Also the frequency relation has roots which converge to the membrane situation for $n_0 >> D$ and to the plate situation for $D >> n_0$. Based on Fig. 2.5 a simple test can be used to determine whether the system should be considered a membrane or a plate:

$$a^2 \frac{n_0}{D} < 1 \quad \text{Plate}$$

$$a^2 \frac{n_0}{D} > 10^4 \quad \text{Membrane}$$  \hspace{1cm} (2.72)

for the lowest few modes of the plate. The transition region tends to move to the right as higher modes are considered. As closure of this section, the two degenerate (except for $m = 0$) mode shapes are given by:

$$W_{mn}^{(1)} = J_m(\alpha_{mn} r)I_m\left(a\sqrt{\frac{n_0}{D} + \alpha_{mn}^2}\right) - J_m(\alpha_{mn} a)I_m\left(r\sqrt{\frac{n_0}{D} + \alpha_{mn}^2}\right) \cos m\theta$$  \hspace{1cm} (2.73a)

$$W_{mn}^{(2)} = J_m(\alpha_{mn} r)I_m\left(a\sqrt{\frac{n_0}{D} + \alpha_{mn}^2}\right) - J_m(\alpha_{mn} a)I_m\left(r\sqrt{\frac{n_0}{D} + \alpha_{mn}^2}\right) \sin m\theta$$  \hspace{1cm} (2.73b)
Evaluation of the frequency relation

Figure 2.5: Roots of Eq. 2.69 for the first five orders $m$. The ordering of the solutions $n$ is from low to high. For example lowest four lines correspond to $\omega_{01}$, $\omega_{11}$, $\omega_{21}$ and $\omega_{02}$ in respective order.

Natural frequencies of a graphene resonator as function of thickness

Figure 2.6: Eigenfrequency of a circular graphene resonator with radius 2µm and pretension 0.1N m$^{-1}$. The thickness is assumed 0.335nm per layer and the mass $7.4 \times 10^{-7}$ kg m$^{-2}$ per layer. On the left side it can be seen that the behavior approaches that of an ideal membrane, plotted as a dashed line. On the right side the behavior approached that of an ideal plate (without pretension), plotted as a dotted line.
2.1.7. REYNOLD’S EQUATION

This section derives the Reynolds equation to model the thin film of gas in the cavity, following the explanation by Bao [36] closely. First consider an elemental column with in- and outflow of mass as drawn in Fig. 2.7. The balance of mass flow gives:

\[
(\rho g q_x)_x \, dy - (\rho g q_x)_{x+dx} \, dy + (\rho g q_y)_y \, dx - (\rho g q_y)_{y+dy} \, dx = \frac{\partial \rho g}{\partial t} \, dxdy
\]  
(2.74)

where \( \rho g \) is the density of the gas (kg m\(^{-3}\)), \( q_x \) and \( q_y \) are the specific volumetric flow rates in the respective \( x \) and \( y \) direction (m\(^3\) s\(^{-1}\) m\(^{-1}\)), \( g \) is the gap size of the thin air film (m). Dividing by \( dxdy \) gives the partial differential equation:

\[
\frac{\partial \rho g q_x}{\partial x} + \frac{\partial \rho g q_y}{\partial y} + \frac{\partial \rho g g}{\partial t} = 0
\]  
(2.75)

To find the volume flow components, the velocity distribution in the \( z \)-direction has to be found. For viscous flow between two plates this becomes:

\[
\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}
\]  
(2.76)

For a small gap the pressure \( p(x, y) \) is not a function of \( z \), which means that one can integrate the equation twice to obtain:

\[
u(z) = \frac{1}{2\mu} \frac{\partial p}{\partial x} z^2 + C_1 \frac{1}{\mu} z + C_2
\]  
(2.77)

The no slip boundary condition means that the velocity must be zero at \( z = 0 \) and \( z = g \), therefore:

\[
u(z) = \frac{1}{2\mu} \frac{\partial p}{\partial x} (z-g)
\]  
(2.78)

Now the flow rate per unit width can be calculated by integrating over the thickness of the film:

\[
q_x = \int_0^g u \, dz = -\frac{g^3}{12\mu} \frac{\partial p}{\partial x}
\]  
(2.79a)
\[ q_y = \int_0^g u \, dz = -\frac{g^3}{12\mu} \frac{\partial p}{\partial y} \]  

(2.79b)

Now substituting this into Eq. 2.75 one obtains the nonlinear Reynold's equation:

\[ \frac{\partial}{\partial x} \left( \rho g \frac{g^3}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho g \frac{g^3}{\mu} \frac{\partial p}{\partial y} \right) = 12 \frac{\partial p}{\partial t} \]  

(2.80)

The Reynolds equation is only valid under the conditions that the modified Reynolds number is much smaller than unity, this means that the inertia due to the density of the gas can be neglected:

\[ R_s = \frac{\omega g^2}{\rho g^2 \mu} \ll 1 \]  

(2.81)

For typical silicon microstructures this condition is satisfied, however for atomically thin membranes it is generally not because of the high oscillation frequency. For example, a monolayer graphene membrane with a radius of 2 micron and a pretension of 0.1 N m\(^{-1}\) has a oscillation frequency at \(4.42 \times 10^8\) rad s\(^{-1}\). With a typical air gap thickness of 300nm and properties of rarefied air at atmospheric pressure (see next section) a Reynolds number of \(R_s = 7.1\) is obtained. Around atmospheric conditions a thin membrane therefore requires a more sophisticated analysis, typically solving the much more sophisticated Navier-Stokes equation in its entirety as is done in section 2.3.

### 2.1.8. Linearizing the Reynolds Equation

In the previous section the nonlinear Reynolds equation was derived. This equation is typically hard to solve and requires numerical analysis. For a compressible gas and small deflections around the equilibrium point Eq. 2.80 can be linearized [37]:

\[ p_a \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) - \frac{12\mu}{g_0^2} \frac{\partial p}{\partial t} = \frac{12\mu}{g_0^2} \frac{dg}{dt} \]  

(2.82)

Where \(g_0\) is the gap size when the membrane is undeformed, and \(p_a\) is the average pressure in the gap, equal to the pressure on the other side of the membrane. This equation can be reduced even further, for this purpose the non-dimensional squeeze number is defined:

\[ \sigma = \frac{12\mu \omega l^2}{p_a g_0^2} \]  

(2.83)

Where \(l\) is a typical length, the width or length of a rectangular plate, or the radius of a circular plate. It can be seen that if this squeeze number is very high, the first term in Eq. 2.82 is neglectable and the equation reduces to:

\[ -\frac{g_0}{g_0} \frac{\partial p}{\partial t} = p_a \frac{dg}{dt} \]  

(2.84)

From which a simple relation can be defined:

\[ p_a g + g_0 p = \text{constant} = 2p_a g_0 \]

\[ p_a (g_0 - \bar{w}) + g_0 (p_a + \bar{\rho}) = 2p_a g_0 \]  

(2.85)

where \(\bar{\rho}\) is the variation of the pressure around the average pressure. This is equivalent to Boyle's law, \(pV = \text{constant}\), the gas is fully compressed and fails to escape because the squeeze action is too quick. This derivation will give a simple expression for the eigenfrequency in section 2.2. It seems at this point that a good sensor can be achieved if the squeeze number is very high around the pressure range of interest. Damping effects are not considered using this simple equation, since \(pV = \text{constant}\) holds only for isothermal systems which are reversible.

### 2.1.9. Properties of Rarefied Gas

At the small scale, the thickness of the small air gap (\(g_0\)) might be in the same order as the mean free path length (\(\lambda\)) of the air molecules. The ratio:

\[ Kn = \frac{\lambda}{g_0} \]  

(2.86)
2.2. Compressed Film Effect on the Vibration of Membranes and Plates

The effects of the air in the cavity over which a membrane is vibrating can be quite difficult to comprehend and often needs multiple coupled equations to be described accurately. However if the squeeze number is very high the effects are dominated by compression of the gas, which is the main cause of the frequency shift in the system. From Eq. 2.85 one can write:

$$\tilde{\rho} = \frac{p_w w}{\rho_0}$$

(2.90)

This can be implemented as a load in the equation for a linear membrane (Eq. 2.16), coupling the spring action from the gas to the structural mechanics:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{p_w w}{\rho_0 n_0} = \frac{\rho h \partial^2 w}{n_0}$$

(2.91)

Now an analysis is done similar to the one in section 2.1.4, assumed is that the solution can be found in the form:

$$w(r, \theta, t) = R(r)\Phi(\theta)\tau(t)$$

(2.92)
This gives:
\[
\frac{d^2 R}{dr^2} \Phi_r + \frac{1}{r} \frac{dR}{dr} \Phi_r + \frac{1}{r^2} R \frac{d^2 \Phi}{dr^2} = \frac{\rho h R \Phi}{n_0} \frac{d^2 \tau}{dr^2} + \frac{p_a R \Phi \tau}{n_0 g_0} \tag{2.93}
\]
Dividing by \(R \Phi \tau\) gives:
\[
\frac{d^2 \tau}{dr^2} + \frac{1}{r} \frac{dR}{dr} \frac{d \tau}{dr} + \frac{1}{r^2} \frac{d^2 \Phi}{dr^2} = \frac{\rho h}{n_0} \frac{d^2 \tau}{dr^2} + \frac{p_a}{n_0 g_0} \tag{2.94}
\]
Both sides of the equal sign must be equal to a constant in order to obtain a solution at all times and all positions. Choosing the constant as \(-\lambda^2\) one can write:
\[
\frac{d^2 \tau}{dr^2} + \frac{n_0}{\rho h} \left( \frac{P_a}{n_0 g_0} + \lambda^2 \right) \tau = 0 \tag{2.95a}
\]
\[
\frac{1}{R} \left[ r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right] + \lambda^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{dr^2} \tag{2.95b}
\]
Eq. 2.95b is exactly the same as Eq. 2.21, meaning that the mode shape does not change due to the compression effect. Eq. 2.95a has the following solution:
\[
\tau(t) = A_1 \cos \sqrt{\frac{n_0}{\rho h} \left( \frac{P_a}{n_0 g_0} + \lambda^2 \right)} t + A_2 \sin \sqrt{\frac{n_0}{\rho h} \left( \frac{P_a}{n_0 g_0} \right)} t \tag{2.96}
\]
The frequency relation is the same as Eq. 2.25:
\[
J_m (a \lambda) = 0 \tag{2.97}
\]
With the same roots, which means that the eigenvalues are directly obtained from Eq. 2.96:
\[
\omega_{mn} = \sqrt{\frac{n_0}{\rho h} \left( \frac{\gamma_{mn} a}{a} \right)^2 + \frac{P_a}{n_0 g_0}} \tag{2.98}
\]
where \(\gamma_{mn} = \lambda_{mn} a\) is obtained from Eq. 2.25. The mode shape is not affected by the gas pressure and given by Eq. 2.29. The classical solution for the eigenfrequency of a membrane is obtained for very low pressures, while at very high pressures all the frequencies of each mode converge to the same value (see Fig. 2.9) because the spring action of the gas dominates the behavior.

**Plates**

For a plate the equation of motion with gas pressure becomes:
\[
D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} + \frac{p_a w}{g_0} = 0 \tag{2.99}
\]
Which can be separated in the same way as the membrane, from which one directly obtains the eigenvalues:
\[
\omega_{mn} = \sqrt{\left( \frac{\gamma_{mn}}{a} \right)^4 + \frac{P_a}{g_0 D}} \frac{D}{\rho h} \tag{2.100}
\]
where \(\gamma_{mn}\) is evaluated by solving Eq. 2.48 and the mode shape (Eq. 2.51) remains unaffected by the gas pressure. For a plate with a large initial tension the following equation of motion is derived:
\[
\nabla^4 w - \frac{n_0}{D} \nabla^2 w + \frac{\rho h \nabla^2 w}{D} + \frac{p_a w}{D g_0} = 0 \tag{2.101}
\]
from which the separation also directly gives the eigenvalues:
\[
\omega_{mn} = \sqrt{\alpha_{mn}^2 \frac{n_0}{\rho h} + \alpha_{mn}^4 \frac{D}{\rho h} + \frac{p_a}{g_0 \rho h}} \tag{2.102}
\]
where \(\alpha_{mn}\) is evaluated using Eq. 2.69 and the mode shape by Eq. 2.73. The plate has very similar behavior as the membrane, with the classical solution obtained for very low pressures and all the frequencies converging to the same value for very large pressures.
2.3. NONLINEAR MODELING OF THE GAS FILM

In the previous section simple equations were derived for the frequency change due to the compression of the gas film. This is valid under the assumption that the squeeze number is very high and also the inertia of the gas is neglectable. However these assumptions are not necessarily true and this section will use the Navier-Stokes equations averaged over the thin film to investigate what happens. The first part of this section derives the governing equations and the second part presents the finite element solution.

### 2.3.1. DERIVATION OF GOVERNING EQUATIONS

Consider a membrane or plate suspended on a circular hole. The velocity components in the respective x, y and z-direction are \( u_f, v_f \) and \( w_f \). When assumed that the dynamic viscosity \( \mu \) in the fluid is constant the Navier-Stokes equations can be written as:

\[
\rho_g \left( \frac{\partial u_f}{\partial t} + u_f \frac{\partial u_f}{\partial x} + v_f \frac{\partial u_f}{\partial y} + w_f \frac{\partial u_f}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho_g \mu \left( \frac{\partial^2 u_f}{\partial x^2} + \frac{\partial^2 u_f}{\partial y^2} + \frac{\partial^2 u_f}{\partial z^2} \right) + \rho_g f_x
\]

(2.103a)

\[
\rho_g \left( \frac{\partial v_f}{\partial t} + u_f \frac{\partial v_f}{\partial x} + v_f \frac{\partial v_f}{\partial y} + w_f \frac{\partial v_f}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho_g \mu \left( \frac{\partial^2 v_f}{\partial x^2} + \frac{\partial^2 v_f}{\partial y^2} + \frac{\partial^2 v_f}{\partial z^2} \right) + \rho_g f_y
\]

(2.103b)

\[
\rho_g \left( \frac{\partial w_f}{\partial t} + u_f \frac{\partial w_f}{\partial x} + v_f \frac{\partial w_f}{\partial y} + w_f \frac{\partial w_f}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho_g \mu \left( \frac{\partial^2 w_f}{\partial x^2} + \frac{\partial^2 w_f}{\partial y^2} + \frac{\partial^2 w_f}{\partial z^2} \right) + \rho_g f_z
\]

(2.103c)

where \( f_x, f_y \) and \( f_z \) are body forces (N m\(^{-3}\)) such as gravity, but these will be neglected from here on. Now consider an elemental column \( h dx dy \), as shown in Fig. 2.7. Here \( q_x \) is the volumetric flow rate in the x-direction per unit width in the y-direction and vice versa for \( q_y \). Writing the conservation of mass for this element yields:

\[
(\rho_g q_x)_{+} dy - (\rho_g q_x)_{-} + \rho_g q_y_{+} dx - (\rho_g q_y)_{-} dx dy = \frac{\partial \rho_g}{\partial t} dx dy
\]

(2.104)

which is the same is Eq. 2.75. This can be rewritten into:

\[
\frac{\partial \rho_g q_x}{\partial x} + \frac{\partial \rho_g q_y}{\partial y} + \frac{\partial \rho_g}{\partial t} = 0
\]

(2.105)
Now assume that the dimension of the plate or membrane are much larger than the gap, this means that the components \( u_f \) and \( v_f \) of the velocity are only a function of \( z \). Also the assumption is made that the velocity component \( w \) is neglectable compared to \( u_f \) and \( v_f \). The Navier-Stokes equations are reduced to:

\[
\frac{\partial u_f}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\mu}{\partial z} \frac{\partial^2 u_f}{\partial z^2}
\]

(2.106a)

\[
\frac{\partial v_f}{\partial t} = -\frac{\partial p}{\partial y} + \frac{\mu}{\partial z} \frac{\partial^2 v_f}{\partial z^2}
\]

(2.106b)

Now integrate the continuity equation (Eq. 2.105) from 0 to \( g \) to obtain:

\[
\frac{\partial}{\partial x} \left[ \rho g \int_0^g udz \right] + \frac{\partial}{\partial y} \left[ \rho g \int_0^g vdz \right] + \frac{\partial \rho g}{\partial t} = 0
\]

(2.107)

Since the integrals represent the average velocities multiplied with \( h \), the following equation is obtained:

\[
\frac{\partial \rho g u_f}{\partial t} + \frac{\partial \rho g u_f v_f}{\partial x} + \frac{\partial \rho g v_f}{\partial t} = 0
\]

(2.108)

Eq. 2.106 can be integrated to obtain:

\[
\rho g \frac{\partial}{\partial t} \left[ \int_0^g u_f dz \right] = -\int_0^g \frac{\partial p}{\partial x} dz + \mu \int_0^g \frac{\partial^2 u_f}{\partial z^2} dz
\]

(2.109a)

\[
\rho g \frac{\partial}{\partial t} \left[ \int_0^g v_f dz \right] = -\int_0^g \frac{\partial p}{\partial y} dz + \mu \int_0^g \frac{\partial^2 v_f}{\partial z^2} dz
\]

(2.109b)

which can be rewritten to:

\[
\rho g \frac{\partial u_f}{\partial t} + h \frac{\partial p}{\partial x} + \frac{\partial \rho g u_f}{\partial x} = 0
\]

(2.110a)

\[
\rho g \frac{\partial v_f}{\partial t} + h \frac{\partial p}{\partial y} + \frac{\partial \rho g v_f}{\partial y} = 0
\]

(2.110b)

In order to eliminate \( z \) from the equations, it is assumed that the shape of the velocity profile is not strongly influenced by inertia terms. Using this assumption one can write [40]:

\[
\frac{\partial u_f}{\partial z} \bigg|_{z=0} = -\frac{12\pi f}{g}
\]

(2.111a)

\[
\frac{\partial v_f}{\partial z} \bigg|_{z=0} = -\frac{12\pi f}{g}
\]

(2.111b)

Substitution in Eq. 2.110 gives:

\[
\rho g \frac{\partial u_f}{\partial t} + h \frac{\partial p}{\partial x} + \frac{12\pi f}{g} = 0
\]

(2.112a)

\[
\rho g \frac{\partial v_f}{\partial t} + h \frac{\partial p}{\partial y} + \frac{12\pi f}{g} = 0
\]

(2.112b)

Now by considering that \( \rho g = (P/P_0)\rho_0 \) for an isothermal process, \( g = g_0 - w, p = p_a + \tilde{p} \) and using Eqs. 2.16 and 2.108 a system of equations is derived with four equations and four unknowns:

\[
\rho h \frac{\partial^2 w}{\partial z^2} - n_0 \nabla^2 w = -\tilde{p}
\]

(2.113a)

\[
(g_0 - w) \frac{\partial \tilde{p}}{\partial t} - (p_a + \tilde{p}) \frac{\partial w}{\partial t} + (p_a + \tilde{p}) (g_0 - w) \frac{\partial \tilde{u}}{\partial x} + (p_a + \tilde{p}) (g_0 - w) \frac{\partial \tilde{v}}{\partial y}
\]

(2.113b)

\[
(g_0 - w) \frac{\partial \tilde{v}}{\partial t} + (p_a + \tilde{p}) (g_0 - w) \frac{\partial \tilde{u}}{\partial y} + (g_0 - w) \frac{\partial \tilde{w}}{\partial y}
\]

(2.113c)

\[
\rho_0 \frac{(p_a + \tilde{p})}{P_0} \frac{\partial \tilde{u}}{\partial t} = (g_0 - w) \frac{\partial \tilde{u}}{\partial t} + (g_0 - w) \frac{\partial \tilde{v}}{\partial y} + \frac{12\mu}{(g_0 - w)} = 0
\]

(2.113d)

This system of equations can be solved using the finite element method (section 2.3.2).
2.3.2. Results from Finite Element Modeling

Eq. 2.113 is implemented in COMSOL Multiphysics to obtain a finite element solution to the system of equations. The boundary conditions on the edges are:

- \( w = 0 \) for a clamped membrane.
- \( \vec{V} \cdot \vec{n} = 0 \) where \( \vec{n} \) is a unit vector perpendicular to the boundary, this implements the closed boundary of the cavity.
- \( u = v = 0 \), this enforces the non-slip condition along the edges of the cavity.

To study the behavior of these systems, two membranes are implemented and discussed in the remainder of this section. The first system is a "small" monolayered graphene membrane with a diameter of 4 \( \mu \)m and a pretension of 0.1N m\(^{-1}\). A gap size of 300 nm is assumed and the properties of air are modeled. This system has a relatively low squeeze number (Eq. 2.83, \( \sigma \approx 43 \)) and high modified Reynolds number (Eq. 2.81, \( R_s \approx 9 \)) in ambient conditions. The second system is a "large" graphene membrane with the same properties as the small one, the diameter is 10 \( \mu \)m and the cavity depth is 100 nm. This system has a higher squeeze number (Eq. 2.83, \( \sigma \approx 106 \)) in a large pressure range and a lower Reynolds number (Eq. 2.81, \( R_s \approx 1 \)), but not smaller than one.

**Small Monolayered Graphene Membrane**

The first system that is implemented is a graphene monolayer with a diameter of 4\( \mu \)m and a pretension of 0.1N m\(^{-1}\). A gap size of 300 nm is assumed and the properties of air are implemented as discussed in previous sections. The output of the model is a number of complex eigenvalues \( \bar{\omega} \) for each pressure. The quality factor of oscillation is obtained by:

\[
Q = \frac{\text{Re}(\bar{\omega})}{2\text{Im}(\bar{\omega})} \tag{2.114}
\]

In Fig. 2.10 the eigenvalues for a small graphene membrane are shown. Also the squeeze number and modified Reynolds number are plotted. From this figure it becomes clear that there are 2 groups of eigenvalues to which the solver can converge: 1) a group with low damping and a \( Q > 7 \) and 2) a group with \( Q < 7 \). This is explained by the convective inertia of the gas that is implemented in these equations, the second group will disappear if this is ignored (this will be shown later in this section). The second group experiences a lot of damping and can be ignored as the resonance peaks will be small compared to group 1. In Fig. 2.11 the model output from group 1 are shown. The eigenvalues are compared with Eq. 2.98 and show good agreement.

The eigenvalues from group 1 are affected by more damping as the pressure increases, however the resonance frequency is still accurately predicted by Eq. 2.98. Group 2 is not predicted well by this equation and also the mode shape is affected (Fig. 2.12). The most striking difference is that the deflection has a gradient of zero near the edges, this was a boundary condition that was only implemented for the pressure field but not for the deflection. It is suspected that this is caused by the gas that is resonating in the cavity, the membrane passively follows the pressure field.

This becomes clear when the pressure field corresponding to both groups is plotted (Fig. 2.13) for the fundamental mode. From this it can be seen that the pressure has the same sign on each position for the high quality factor (group 1) mode. This means that the gas is fully compressed when the membrane resonates and it also explains why Eq. 2.98 is in such good agreement with these modes. The mode shape with the low quality factor (group 2) has a sign change in the pressure field, this means that gas is exchanged along the cavity during the vibration and the inertia of the gas is the cause of these modes. It also explains why the frequency corresponding to these modes becomes lower when the pressure increases, a high pressure means a higher gas density, adding mass to the system. The deflection of the membrane is following the pressure field variations due to this gas exchange. More damping is to be expected from such a mode since the gas has to flow further and due to the viscosity of the gas a resistance is experienced. Due to this the damping will be more significant than for the compressed case, causing the difference in quality factor for both cases.

**Large Monolayered Graphene Membrane**

In Fig. 2.14 the modeling results for the 10 \( \mu \)m membrane are shown. It can be seen that the behavior is entirely dominated by the compression of the film, the resonance frequency is in excellent agreement with Eq. 2.98, see Fig. 2.15. The effect of pretension is present in a much smaller pressure range, combined with the low damping effect this gives a large pressure range where the frequency is rising as function of the pressure. The damping shows interesting behavior:
Figure 2.10: Top: real and imaginary part of the eigenvalue $\omega$, the real part is the undamped natural frequency and the imaginary part proportional to the damping; middle left: the absolute value of $\omega$ which is the actual damped resonance frequency; middle right: the quality factor of the oscillation; bottom left: squeeze number as function of pressure, the higher the value the more the behavior is dominated by compression of the film; bottom right: modified Reynolds number as function of pressure, a number higher than one means that inertia from the gas should be considered in the modeling.
Figure 2.11: The same plot as in Fig. 2.10, but the modes with a low quality factor are removed. Top: real and imaginary part of the eigenvalue $\omega$, the real part is the undamped natural frequency and the imaginary part proportional to the damping; bottom left: the absolute value of $\omega$ which is the actual damped resonance frequency; bottom right: the quality factor of the oscillation. The real part and absolute value are compared with Eq. 2.98 plotted as lines.
Figure 2.12: Left: eigenmode corresponding to a resonance with a high quality factor; right: eigenmode corresponding to a resonance with a low quality factor. Important difference is the gradient of the deflection near the edges, since this is zero for resonances with a low quality factor, it is suspected that these modes are a result from the inertia of the gas. The gradient of zero near the edges would mean that the membrane is passively following the pressure gradient due to this resonance.

Figure 2.13: Left: eigenmode of the pressure corresponding to a resonance with a high quality factor; right: eigenmode of the pressure corresponding to a resonance with a low quality factor. Arrows are drawn representing $u$ and $v$, with the same scale factor in both figures. It can be seen that the pressure in the left figure has the same sign over the entire geometry, while on the right the sign in the middle is different that that near the edges. The mode presented on the left will therefore fully compress the gas in the cavity, while on the right the gas is exchanged between the middle and the edges. From this it is clear that the left mode will have a higher quality factor while the right mode will have a lower one, as viscous flow causes more damping.
• At very low pressures the rarefied gas causes very little damping and the quality factor is high.

• As the pressure increases, so does the effective viscosity (see Fig. 2.8). This causes the gas to exert viscous damping forces on the membrane and the quality factor decreases.

• Above a certain pressure, the spring constant from the gas starts to play a significant role. Due to the high viscosity and vibration frequency the gas has no time to displace laterally and the compression effect is dominant, this is described by the squeeze number (Eq. 2.83). The increase in resonance frequency causes the compression effect to become even more dominant, because at higher frequency the lateral flow is even further restricted. These effects cause the quality factor to increase significantly as the pressure increases.

The above is illustrated in Fig. 2.16.

**Comparison With the Nonlinear Reynolds Equation**

The results from the previous analysis are compared with simulations using the nonlinear Reynolds equation as implementation. The results for a 4 µm diameter graphene membrane are shown in Fig. 2.17. In comparison to the Navier-Stokes implementation the following differences are found:

• The eigenmodes with a low quality factor are not found, this is a result of the convective inertia of the gas that is ignored in the nonlinear Reynolds equation.

• The absolute part of $\omega$ is also well approximated with Eq. 2.98, but above a certain pressure a clear deviation from this solution can be seen (black lines in Fig. 2.17). This is an effect of the decrease in the squeeze number, which means that viscous forces start to become important around this pressure for this specific system.

• This transition was not seen in the Navier-Stokes implementation, this is because the solver simply did not converge to the modes that describe this effect. This does not necessarily mean that any of the models is wrong. It shows however that the nonlinear Reynolds equation is useful since it is able to find these modes, with a small error due to the inertia that is ignored.

• The quality factors show that the Reynolds equation predicts less damping, this is because of the inertia effects that are ignored. The inertia forces are governed by the change in pressure gradient over time, which results in a damping force.

The 10 µm membrane (Fig. 2.18) shows results which are in good agreement for the resonance frequency, because these are dominated by compression effects. The imaginary part of $\omega$ and the quality factor show some slight differences, this is because inertia effects are ignored in the Reynolds equation.

## 2.4. Notes on the Modeling

In section 2.1.6 a formula was derived for the eigenfrequencies and modes of a plate with a high in-plane tension. Thein Wah has already solved this problem in 1962 [41]. However the analysis in this thesis uses easier constants to separate the equations, this gives the same results in terms of eigenfrequency and eigenmode. In this thesis the result is rewritten to physical parameters with only one constant, this made the interpretation easier as it showed that there is a smooth transition between plate-like behavior and membrane-like behavior. Since only one unknown parameter has to be determined, the analysis is greatly simplified. The downside of the analysis in this thesis is that it cannot be used for compressive in-plane tensions.

Modeling the squeeze film effects in the cavity has been extensively studied in MEMS devices, however the atomically thin membrane resonators have typically much higher eigenfrequencies. This means that under certain conditions extensive modeling is needed. Two numerical models were used to analyse the effect of the gas film on the vibration of the membrane: the Navier Stokes equation averaged over the thin film and the nonlinear Reynolds equation. These models show that strange things happen due to damping if the squeeze number no longer very high. A pressure sensor will not operate properly under such high damping, also the resonance frequency should always rise when the pressure increases in the measurement range. It is found that systems with a very high squeeze number and low Reynolds number show much lower damping and a large measurement range, in general these systems have a high radius-cavity depth ratio.

A simple expression is derived for determining the resonant frequency as function of the absolute pressure. This expression was in good agreement with both numerical models if the squeeze number is very high. A striking result is that the eigenfrequencies converge to the same value if the pressure if very high.
Figure 2.14: Results from numerical simulation using the averaged Navier-Stokes implementation. Top: real and imaginary part of the eigenvalue $\omega$, the real part is the undamped natural frequency and the imaginary part proportional to the damping; middle left: the absolute value of $\omega$ which is the actual damped resonance frequency; middle right: the quality factor of the oscillation; bottom left: squeeze number for each oscillation; bottom right: modified Reynolds number for each oscillation.
2.5. SENSOR PERFORMANCE

In order to say something about the performance of this transducer in an actual MEMS device, the indicators in section 1.6 are used as guidance. This section will discuss whether this sensor can meet or exceed these specifications compared to an example of an existing silicon device with a similar working principle.

2.5.1. SILICON DEVICE FOR COMPARISON

In Fig. 2.19 a device is shown that was characterized by Lalit Kumar [42]. This device can be used as a compressed film pressure transducer since the gap size is small (950nm) compared to the radius (between 160 – 170µm). The hole is present for pressure equalization, but is expected to be small enough to not have any effect when the diaphragm is at its resonance frequency. This system is chosen because of the availability of measurement data, which will be useful to compare to the analytical result. Downside of this system is that some parameters such as thickness are not precisely known.

COMPARISON TO MODELING

This device can be used to compare some analytical results in this chapter to measurements. The eigenfrequency as function of the pressure is one of the novel equations that can be used to compare to the experiments. Modeling the diaphragm as a circular plate made out of silicon germanium, the eigenfrequency is evaluated using Eq. 2.100, repeated here:

$$\omega_{mn} = \sqrt{\left(\frac{\gamma_{mn}}{a}\right)^4 + \frac{p_a}{g_0 D} \frac{D}{\rho h}}$$

The following parameters were found to provide a very good fit to the measured data, as seen in Fig. 2.20:

- $g_0 = 950$nm, which was known.
- $\rho = 2350$kg m$^{-3}$, approximate bulk value for SiGe.
- $h = 10$µm
- $D = 2.24$N m
Figure 2.16: Illustration of the cause of damping and spring action of the gas. Top: low pressures, the gas molecules barely interact with each other and a small gas damping term comes from collisions of gas molecules with the membrane. For thin membranes the damping is very likely dominated by intrinsic mechanisms. Middle: at intermediate pressure the gas molecules start to interact and flow through the gap as the membrane deflects, the viscous friction forces cause significant damping on the membrane. Bottom: as the pressure increases further, so does the effective viscosity (see Fig. 2.8), the gas tries to displace along the gap but has no time due to the increased resistance, this results in the gas being compressed and less damping.
Figure 2.17: Eigenvalues calculated using the nonlinear Reynolds equation. Top: real and imaginary part of the eigenvalue $\omega$, the real part is the undamped natural frequency and the imaginary part proportional to the damping; bottom left: the absolute value of $\omega$ which is the actual damped resonance frequency; bottom right: the quality factor of the oscillation. The black lines in the real part and absolute part of $\omega$ are solution of Eq. 2.98.
Figure 2.18: Eigenvalues calculated using the nonlinear Reynolds equation. Top: real and imaginary part of the eigenvalue $\omega$, the real part is the undamped natural frequency and the imaginary part proportional to the damping; bottom left: the absolute value of $\omega$ which is the actual damped resonance frequency; bottom right: the quality factor of the oscillation.
2.5. SENSOR PERFORMANCE

2.5.2. SENSITIVITY OF IDEAL GRAPHENE DEVICES

The sensitivity tells how much effect the pressure has on the eigenfrequency:

\[ S_f = \frac{df_0}{dp_a} \]  

(2.115)

In Fig. 2.21 the sensitivity of two systems with different diameters are shown, assumed that only compression is important. It is clear that the devices are more sensitive for pressure changes when the absolute pressure is small. This was not directly clear from Fig. 2.9 due to the logarithmic axis. The sensitivity can be interpreted as the resolution of the resonance frequency measurement needed to achieve a pressure sensing resolution of 1 Pa. It can be seen that for small absolute pressures the device will have a linear relationship between pressure and frequency, since the curve is flat. This will be a very good property if the device is to be used in vacuum applications. The linear range can be enlarged by reducing the size of the sensor, but this will be at the cost of the sensitivity at low pressures.

2.5.3. SENSOR NOISE

The sensitivity of the device gives some information about how the frequency changes when the pressure changes, however this is not an indication on how well the pressure can be measured. There is a limitation on the accuracy at which the frequency can be measured, which means that this also limits the resolution of the pressure measurement. The spread in the frequency measurement of an oscillator is limited by the Brownian motion of the resonator, the standard deviation is given by [42]:

\[ \sigma[\omega(t) > t_r] = \frac{\omega_0}{2Q} \sqrt{\frac{k_B T}{\tau P_{sig}}} \]  

(2.116)
Comparison between measurement and analytical results for compressive film sensors

![Graph](image)

Figure 2.20: Comparison of the results from the analytical equation with the measurement data on the SiGe diaphragms by Lalit Kumar.

where \( \sigma(\omega(t)) \) is the standard deviation of the frequency, \( k_B \) the Boltzmann constant, \( T \) the temperature (K), \( \tau \) the measurement time (s) and \( P_{\text{sig}} \) the power dissipated in the transducer. For a mass-spring-damper system, the power is given by \( P_{\text{sig}} = \frac{1}{2} k x^2_{\text{max}} \) where \( x_{\text{max}} \) is usually chosen as the maximum deflection where the system is still linear. This standard deviation can be converted to a pressure using Eq. 2.115. This is done for the systems in Fig. 2.21 and plotted in Fig. 2.22. For this a maximum deflection of 10nm is assumed, a temperature of 300K and a quality factor of 50 (which is a typical value for monolayered circular resonators) are assumed. It can be seen that the limit on the resolution lies somewhere between 0.4 and 100\( \mu \)Pa. For comparison, the MEMS device in Fig. 2.19 is limited to 10mPa around 1 bar and with the same measurement time. Due to the low thermal noise it is expected that an actual sensor will be limited by noise from the electronics and therefore should at least meet the specifications of current silicon based technologies.

### 2.5.4. Cost

By checking several suppliers [www.alibaba.com] it can be seen that a typical absolute pressure sensor for automotive applications can cost as low as $1. For graphene sensors to be applied in the practice the price per unit has to be comparable. Since the substrate is a quite simple design, it is expected that this can be produced for less than $0.04 for a 1 by 1 mm substrate. This is a coarse estimate based on examples in the article by Lawes [43]. The size is much bigger than the transducer itself, this allows for bondpads to be produced and leaves room for dicing lines.

Graphene can currently be produced for 8000$/m^2$ which is 0.8 cents per mm$^2$ [32]. Per device this means 0.8 cents for graphene costs, assuming that the transfer of graphene can be done for $0.03 per chip, the total cost so far is about $0.08 per device. Since the substrate is quite simple design, it is expected that this can be produced for less than $0.04 for a 1 by 1 mm substrate. This is a coarse estimate based on examples in the article by Lawes [43]. The size is much bigger than the transducer itself, this allows for bondpads to be produced and leaves room for dicing lines. Packaging will cost approximately 80% of the total price to manufacture the system [43]. This means that the estimated total production cost for the transducer will be around $0.40.

It should be noted that this estimate is very coarse and depends on the wafer size, production volumes, feedstock prices, etc. Also a simple estimate was made, but the price can probably be reduced if the production process is optimized. Electronic readout should be done somewhere outside the device with these assumptions, meaning that some added costs can be expected. However IC-based frequency counters are available for retail prices of \( \approx $0.60 \), which is most likely the most sophisticated part of the readout circuit. It is expected that significant cost reduction is achieved if the transducer and electronic readout is all integrated on one circuit, it should be possible to achieve a 1 dollar device.
Figure 2.21: Sensitivity of two graphene monolayers suspended over a cavity with 300 nm and a pretension of 0.1 N m$^{-1}$.

Figure 2.22: The limit on the resolution due to sensor noise for two monolayered graphene devices. A 300nm cavity depth and a pretension of 0.1N m$^{-1}$ is assumed.
2.5.5. Size
The size of the transducer can be made exceptionally small. In the mathematical analyses of section 2.3.2 it was found that a good transducer is made if the diameter is \(20\mu m\) in terms of damping and frequency shift. This is exceptionally small compared to the system in Fig. 2.19 which has a diameter of \(320\mu m\). The piezoresistive transducer in Fig. 1.8 has a diaphragm of \(100\mu m\) and the capacitive transducer \(2000 \times 2000\)mm, which shows that graphene transducers can be made significantly smaller than current technologies.

However in practical systems bond pads (typically \(100 \times 100\mu m\)) are needed which limits the minimal size one can achieve. The packaging costs also limit the minimal size one can achieve, since a smaller die is more difficult to handle. This is a good argument to implement this transducer in an integrated circuit. The die size will very likely be dominated by the size of the electronic readout circuit (\(20 \times 20\mu m\) is a very small spot in a \(2 \times 2\)mm dice, for example). The increase in size will partially be compensated by the reduced packaging costs.

2.5.6. Measurable range
In the compressed film transducer the pressure is equal on both sides, this means that there are no large tensions induced on the membrane. From a mechanical point of view, there is no limit on the pressure range that one can measure. From the nonlinear modeling of the devices using the nonlinear Reynolds equation, it could be seen that devices with a small diameter – cavity depth ratio are expected to be limited by damping. For devices with a larger ratio, it is not clear if the fluid dynamics will limit the pressure range from the simulations. However it was observed that the squeeze number is continuously decreasing in value as function of pressure (Fig. 2.14), so it is expected that the effects of damping will dominate at a certain point. Improved simulations are needed in order to predict the limit on the pressure range.

2.5.7. Bandwidth
The bandwidth of a compressed film resonant pressure transducers is limited by the pressure equalization. The pressures should be equal on both sides of the membrane in a low frequency range, but there should be no pressure equalization on the high frequency range to ensure good compression near the resonance frequency. Chapter 4 develops a model to predict the behavior of the pressure equalization for different frequencies.

2.5.8. Mechanical disturbance
The influence of mechanical disturbances on the performance of a graphene device will be very small due to its low mass. For the inertial forces one can write:

\[
p_{\text{inert}} = \rho h \frac{\partial^2 w}{\partial t^2}
\]

which gives the force per unit square for an acceleration component in the out of plane direction. The acceleration from gravity will introduce a force of \(7.3 \times 10^{-6}\)N m\(^{-1}\), this is barely above the pressure resolution (Fig. 2.22). Even when the acceleration is 100 times the gravitational acceleration it will not introduce any significant force \((7.3 \times 10^{-3}\)N m\(^{-1}\)) on the membrane. This means not only that the mechanical disturbance is very low, if the device is exposed to large accelerations (from dropping the device for example) the silicon substrate is far more likely to break than the graphene membrane.

2.6. Conclusion
Formulas have been developed to predict the eigenfrequency of a membrane or plate as function of the absolute gas pressure inside the cavity. Using models from literature the properties of the gas inside the cavity can be predicted within the molecular range. The Navier-Stokes equations were implemented in COMSOL to verify the effect of the gas inside the cavity on the vibration of the membrane. It was found that in general a good pressure range for measurement is obtained if the non-dimensional squeeze number is much larger than one, these systems also experience low damping. The Navier-Stokes solutions are compared with the nonlinear Reynolds equation commonly used in MEMS, it was found that there is a slight disagreement in damping of the system, due to the fact that gas inertia is ignored in the Reynolds equation.

Pressure sensors using a compressed gas film have a good potential for application in real MEMS. It is expected that these transducers can beat current silicon based transducers by reduced costs per unit, reduced size and reliability.
This chapter goes into the concept which uses the change in frequency of a diaphragm if a pressure difference is applied over it. In these systems the deflections become very large, which means that the geometric nonlinear effects must be taken into account. In order to model this behavior a finite element model was built in COMSOL and verified using some analytical solutions or approximations.

The first section discusses the nonlinear model for membranes and plates. The governing equations for two different models are derived and several solutions from literature are compared to the finite element solution. The second section goes into the vibration of the membrane when a static pressure difference is applied. It was found that these are not in agreement with formula’s from literature, but a better equation is derived for the vibration of the fundamental mode as function of pressure. The chapter closes with a discussion on this pressure transducer and the conclusions.

### 3.1. Modeling of the Nonlinear Behavior of Atomically Thin Membranes

This section goes into the theory of nonlinear membranes. A membrane becomes nonlinear if the tensions induced by external forces are comparable to, or larger than the pretension in a membrane. Also the tensions in a plate can become so large that is starts to behave as a nonlinear membrane. This system only provides analytical solutions if the applied loads and boundary conditions are relatively simple, but even these analytical solutions can become quite complicated. Therefore a model was developed in COMSOL that combines these effects and which can solve complex situations in an effective manner.

The first section derives the governing differential equations used to build the COMSOL model. After this, the analytical solution for the deflection of a pressurized membrane is derived and compared to the model. The third section goes into the analytical derivation of a membrane subjected to a point load in the center. After this the model is used to analyze the induced tension transducer. The concept will be evaluated on the sensor performance indicators from section 1.6 and the chapter will close with the conclusions.

#### 3.1.1. Derivation of Governing Differential Equations for Membranes

The equations for the membrane forces are based on methods and free body diagrams as shown by Szilard [33]. In order to derive the governing differential equation the force equilibrium of a membrane element with size $dx \; dy$ is considered, while assuming small angles in the membrane (Fig. 3.1). The in-plane membrane forces per unit length, denoted by a small letter, are: $n_x$, $n_y$ and $n_{xy} = n_{yx}$. For the equilibrium on the $X$-axis one can write:

$$\left( n_x + \frac{\partial n_{x}}{\partial x} \right) dy - n_x \; dy + \left( n_{yx} + \frac{\partial n_{yx}}{\partial y} \right) dx - n_{yx} \; dx = 0 \quad (3.1)$$

Simplification of Eq. 3.1 can be done by dividing by $dx \; dy$ and disregarding the higher order terms:
\begin{equation}
\frac{\partial n_x}{\partial x} + \frac{\partial n_{yx}}{\partial y} = 0 \tag{3.2}
\end{equation}

In a similar manner the equilibrium in the Y-direction is obtained:

\begin{equation}
\frac{\partial n_{yx}}{\partial x} + \frac{\partial n_y}{\partial y} = 0 \tag{3.3}
\end{equation}

For the summation of the forces in the Z-direction a deformed element must be drawn, Fig. 3.2, where for simplicity the two far sides are fixed and in the XY-plane. The summation of the forces in the Z-direction yields:

\begin{equation}
\Sigma F_z = \left( n_x + \frac{\partial n_x}{\partial x} \right) dy \frac{\partial^2 w}{\partial x^2} dx + \left( n_{yx} + \frac{\partial n_{yx}}{\partial y} \right) dx \frac{\partial^2 w}{\partial y^2} dy + \left( n_{xy} + \frac{\partial n_{xy}}{\partial x} \right) dy \frac{\partial^2 w}{\partial x \partial y} dx + \left( n_y + \frac{\partial n_y}{\partial y} \right) dy \frac{\partial^2 w}{\partial x \partial y} dx \frac{\partial^2 w}{\partial x \partial y} dy - P_z = 0 \tag{3.4}
\end{equation}

For the propagation of heat in the membrane one can write the equation:

\begin{equation}
\nabla^2 kT = Q \tag{3.5}
\end{equation}
3.1.2. Stress Strain Relations

The stress strain relations shown below are derived in the book by Landau [44], his methods are followed closely.

In the derivation of the governing equations an important kinematic assumption is made, namely that the forces in the $x$ and $y$-direction remain in the $xy$-plane after deformations, this is known as the Kirchhoff hypothesis. The plate does not change thickness, normally this would be a simplification, but for the atomically thin layer this will actually be true. These assumptions imply that the displacement field can be written as:

$$u(x, y, z) = -z \frac{\partial w}{\partial x}$$

$$v(x, y, z) = -z \frac{\partial w}{\partial y}$$

$$w(x, y, z)$$

The components of the three dimensional Lagrange-Green strain tensor are defined as:

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \frac{\partial w}{\partial x_j} \right)$$

Substituting Eq. 3.6 into Eq. 3.7 and enforcing that the thickness does not change gives the following result for the components of the Lagrange-Green strain tensor:

$$\epsilon_x = -z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left[ z^2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + z^2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]$$

$$\epsilon_y = -z \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left[ z^2 \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + z^2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]$$
\( \epsilon_z = 0 \) \hspace{1cm} (3.8c)

\( \epsilon_{xy} = -z \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2} \left[ z^2 \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) + \frac{1}{2} \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial y \partial x} \right) + \frac{\partial w \partial \partial w}{\partial y} \right] \) \hspace{1cm} (3.8d)

\( \epsilon_{xz} = 0 \) \hspace{1cm} (3.8e)

\( \epsilon_{yz} = 0 \) \hspace{1cm} (3.8f)

Now the stress-strain relationship is assumed linear and defined by the generalized Hooke's Law, which in Matrix form writes as:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} = -\frac{E}{(1-v^2)} \begin{bmatrix}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & 1-v
\end{bmatrix} \begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_{xy}
\end{bmatrix} - \frac{E}{2(1-v)} \begin{bmatrix}
T \\
T \\
0
\end{bmatrix} + \begin{bmatrix}
\sigma_0 \\
\sigma_0 \\
0
\end{bmatrix}
\]

Since the stress resultants can be expressed as: \( n_{ab} = \int_{-h/2}^{h/2} \sigma_{ab} \, dz \) one can write:

\[ n_x = \frac{Eh}{2(1-v^2)} \left( \left( \frac{\partial w}{\partial x} \right)^2 + v \left( \frac{\partial w}{\partial y} \right)^2 \right) - EhaT + n_0 \]

\[ n_y = \frac{Eh}{2(1-v^2)} \left( \left( \frac{\partial w}{\partial y} \right)^2 + v \left( \frac{\partial w}{\partial x} \right)^2 \right) - EhaT + n_0 \]

\[ n_{xy} = \frac{Eh}{2(1+v)} \frac{\partial w \partial w}{\partial x \partial y} \]

Note that the force equilibrium in the \( z \)-direction (Eq. 3.4) can be written in the compact form:

\[ \nabla \cdot \left( \begin{bmatrix} n_x \\ n_{xy} \\ n_y \end{bmatrix} \otimes \nabla w \right) = P_z \] \hspace{1cm} (3.11)

Using D'Alembert’s principle the inertia term can be treated as a force, since the equations are two-dimensional the inertial term becomes:

\[ \rho h \frac{\partial^2 w}{\partial t^2} \]

Adding this term to Eq. 3.11 gives the total system of equations:

\[ \rho h \frac{\partial^2 w}{\partial t^2} - \nabla \cdot \left( \begin{bmatrix} n_x \\ n_{xy} \\ n_y \end{bmatrix} \otimes \nabla w \right) = -P_z \] \hspace{1cm} (3.13a)

\[ n_x = \frac{Eh}{2(1-v^2)} \left( \left( \frac{\partial w}{\partial x} \right)^2 + v \left( \frac{\partial w}{\partial y} \right)^2 \right) - EhaT + n_0 \] \hspace{1cm} (3.13b)

\[ n_y = \frac{Eh}{2(1-v^2)} \left( \left( \frac{\partial w}{\partial y} \right)^2 + v \left( \frac{\partial w}{\partial x} \right)^2 \right) - EhaT + n_0 \] \hspace{1cm} (3.13c)

\[ n_{xy} = \frac{Eh}{2(1+v)} \frac{\partial w \partial w}{\partial x \partial y} \]

\[ \nabla^2 kT = Q \] \hspace{1cm} (3.13e)

### 3.1.3 1D Axisymmetric Model

Another way to describe the mechanics of the membrane is to make the assumption that the model is axisymmetric, this reduces the dimension to 1 (the radial coordinate \( r \)). This has been done by Timoshenko et. al. and Yen et. al. [45, 46], this section follows their methods closely but also includes thermal expansion.

The strains for very large deflections [47] are defined as:

\[ \epsilon_r = \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 + \frac{1}{2} \left( \frac{du}{dr} \right)^2 \]

\[ \epsilon_\theta = \frac{u}{r} \]

\[ \epsilon_z = 0 \] \hspace{1cm} (3.14)

\[ \epsilon_{r\theta} = 0 \] \hspace{1cm} (3.15)
3.1. MODELING OF THE NONLINEAR BEHAVIOR OF ATOMICALLY THIN MEMBRANES

Figure 3.3: Forces acting on a membrane element taken from a circular membrane.

Application of Hooke’s Law and adding terms for the pretension and thermal stress gives:

\[
n_r = \frac{Eh}{1-\nu^2} (\epsilon_r + \nu \epsilon_\theta) + n_0 + \alpha T = \frac{Eh}{1-\nu^2} \left[ \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 + \frac{1}{2} \left( \frac{du}{dr} \right)^2 + \frac{v}{r} \frac{u}{r} \right] + n_0 + \alpha T \tag{3.16a}
\]

\[
n_\theta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_r) + n_0 + \alpha T = \frac{Eh}{1-\nu^2} \left[ \frac{du}{dr} \frac{v}{2} \left( \frac{dw}{dr} \right)^2 + \frac{v}{2} \left( \frac{du}{dr} \right)^2 + \frac{u}{r} \right] + n_0 + \alpha T \tag{3.16b}
\]

Assuming these are the only forces present in the system, one can now write the equilibrium of forces based on Fig. 3.3.

The sum of the projections in the radial direction yields:

\[
\frac{dn_r}{dr} - n_\theta + n_r = P_r \tag{3.17}
\]

and in the out-of-plane direction:

\[
-\nabla n_r \frac{dw}{dr} = P_z \tag{3.18}
\]

The dynamic analogy of these equations are:

\[
\rho h \frac{\partial^2 w}{\partial t^2} - \nabla n_r \frac{\partial w}{\partial r} = P_z \tag{3.19a}
\]

\[
\rho h \frac{\partial^2 u}{\partial t^2} + r \frac{\partial n_r}{\partial r} - n_\theta + n_r = P_r \tag{3.19b}
\]

Now substitute Eq. 3.16 into Eq. 3.19 to obtain the following system of equations:

\[
\nabla \left[ \frac{\partial u}{\partial r} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{v}{r} \frac{u}{r} + \frac{1-\nu^2}{Eh} (\alpha T + n_0) \right] \nabla w = -\frac{1-\nu^2}{Eh} P_z \tag{3.20a}
\]

\[
-\rho h \left( \frac{1-\nu^2}{Eh} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{2} \frac{\partial u}{\partial r} \frac{\partial^2 w}{\partial r^2} + \frac{1-\nu^2}{Eh} \frac{\partial w}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{2r} \frac{\partial u}{\partial r} \frac{\partial^2 w}{\partial r^2} + \frac{1-\nu}{2r} \frac{\partial^2 u}{\partial r^2} \right) + \frac{1-\nu}{2r} \frac{\partial^2 u}{\partial r^2} + a \frac{\partial T}{\partial r} = \frac{1-\nu^2}{Eh} P_r \tag{3.20b}
\]
EXTENDING TO MULTILAYER MEMBRANES
Depending on the pretension and the size of the membrane, when more layers are added the resistance to bending becomes more important for the mechanical behavior. When approximately 5 layers are added, the bending rigidity should already be taken into account (see Fig. 2.6).

For the 2D case, the governing differential equation for a plate is the well known Kirchhoff’s plate equation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P(x,y)}{D} \tag{3.21}$$

Where $D$ is the bending rigidity, in the continuum case defined as:

$$D = \frac{E h^3}{12(1-\nu^2)} \tag{3.22}$$

In the case of few layer material, the bending rigidity should be measured because of quantummechanical effects. Now using the nabla operator Eq. 3.21 can be written into a more condensed form:

$$D \nabla^4 w = P_z \tag{3.23}$$

Since the bending rigidity is entirely caused by moments in the plate and has no contribution to the normal stresses, the principle of superposition can be used to determine the effect of the load on the deflection, basically:

**Inertia term + Bending Term + Stretching (membrane) Term = Total load.**

This means that for the 2D model one can write:

$$\rho h \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w - \nabla \cdot \left[ \left( \begin{array}{cc} n_x & n_{xy} \\ n_{xy} & n_y \end{array} \right) \otimes \nabla w \right] = -p_z \tag{3.24a}$$

$$n_x = \frac{E h}{2(1-\nu^2)} \left( \frac{\partial w}{\partial x} \right)^2 + \nu \left( \frac{\partial w}{\partial y} \right)^2 - E h a T + n_0 \tag{3.24b}$$

$$n_y = \frac{E h}{2(1-\nu^2)} \left( \frac{\partial w}{\partial y} \right)^2 + \nu \left( \frac{\partial w}{\partial x} \right)^2 - E h a T + n_0 \tag{3.24c}$$

$$n_{xy} = \frac{E h}{2(1+\nu)} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \tag{3.24d}$$

$$\nabla^2 k T = Q \tag{3.24e}$$

For the axisymmetric model the bending rigidity will also project in the out-of-plane direction. So for the 1D case one can write:

$$\frac{1-v^2}{E h} \left[ -\rho h \frac{\partial^2 w}{\partial t^2} - D \nabla^4 w \right] + \nabla \left[ \frac{\partial u}{\partial r} + \frac{1}{2} \frac{(\partial w)^2}{\partial r} + \frac{1}{2} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{v}{r} \frac{\partial u}{\partial r} \right] + \frac{1-v^2}{E h} \left( \alpha T + n_0 \right) \nabla w = \frac{1-v^2}{E h} P_z \tag{3.25a}$$

$$-\rho h \frac{1-v^2}{E h} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial w \partial^2 w}{\partial r^2} + \frac{1-v}{2r} \left( \frac{\partial w}{\partial r} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1-v}{2r} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{\partial T}{\partial r} = \frac{1-v^2}{E h} P_r \tag{3.25b}$$

3.1.4. PRESSURE-INDUCED DEFLECTION
The bulge test is a method used to derive the mechanical properties of thin films. The derivation of the deflection versus the pressure difference of a membrane is described in detail in the PhD thesis of Scott Bunch [12] and this section will follow the derivation closely.

A pressure difference $\Delta P$ is applied to a circular membrane suspended over a hole with radius $a$ (Fig. 3.4). The membrane takes the shape of a uniform curvature $R$ with a maximum deflection $\delta$. The tension $n$ in the membrane is assumed biaxial and uniform across the membrane. The pressure force is balanced by the tension in the membrane:

$$\Delta p\pi R^2 = 2\pi n R \tag{3.26a}$$
3.1. Modeling of the Nonlinear Behavior of Atomically Thin Membranes

\[ n = \frac{\Delta p R}{2} \]  (3.26b)

If the deflections are small the radius of curvature can be approximated with:

\[ R \approx \frac{a^2}{2\delta} \]  (3.27)

Now the tension becomes:

\[ n = \frac{\Delta p a^2}{4\delta} \]  (3.28)

The strain can be calculated using the arc length [48]:

\[ \epsilon = \frac{R\theta - a}{a} \approx \frac{a^2}{6R^2} \]  (3.29a)

Now plug in Eq. 3.27 to obtain:

\[ \epsilon \approx \frac{2\delta^2}{3a^2} \]  (3.29b)

Hooke’s Law states that:

\[ n = \frac{Eh}{1-\nu} \epsilon \]  (3.30)

Now combining Eq. 3.29b with Hooke’s Law one obtains for the concomitant tension:

\[ n_p = \frac{2Eh\delta^2}{3a^2(1-\nu)} \]  (3.31)

The total tension in the membrane should include the pretension, which is often significant for atomically thin membranes:

\[ n = n_0 + n_p = n_0 + \frac{2Eh\delta^2}{3a^2(1-\nu)} = \frac{\Delta p a^2}{4\delta} \]  (3.32)

Rewriting this gives the following relation for a circular membrane:

\[ \Delta p = \frac{4n_0\delta}{a^2} + \frac{8Eh\delta^3}{3a^4(1-\nu)} \]  (3.33)

Comparison to Modeling

This situation can be modeled in COMSOL by applying a uniform pressure. The results are in good agreement with the analytical formulas (Fig. 3.5). The 1D model has been corrected for the fact that the force-direction changes when the deflections are large, because the pressure load will always be perpendicular to the surface. The 2D model has not been corrected for that fact, this leads to a small error when the applied pressure is large.
3.1.5. CENTRAL POINT LOADED MEMBRANE

The central point loaded membrane is a difficult subject for which a solution in the membrane limit is currently not available. If it is assumed that a thin membrane such as graphene is a very thin plate, a good prediction of the membrane limit is obtained. The response of a membrane has two components: a contribution due to pretension and one from the elastic response, this results in the following equation ([10, 49, 50], see appendix B for the derivation):

\[ F = \frac{2\pi D}{a^2} g_s(\beta_0) \delta + \frac{Eh}{(f(\nu))^3 a^2} \delta^3 \]  

(3.34)

This is a cubic polynomial which can be used to fit a curve to measured results. \( \delta \) is the deflection at the center, \( \beta_0 \) is defined as:

\[ \beta_0 = \sqrt{\frac{n_0 a^2}{D}} \]  

(3.35)

Assuming that all physical parameters are known, except \( n_0 \) and \( E \), this formula can be used to derive these properties from a force-displacement measurement. A minimizing algorithm commonly used in engineering optimization problems can be used to find the parameters closest to the measured results.

The functions are evaluated as follows, for \( f(\nu) \) [50]:

\[ f(\nu) \approx 1.0491 - 0.1462\nu - 0.15827\nu^2 \]  

(3.36)

And for \( g_s(\beta_0) \) [49]:

\[ g_s(\beta_0) = \frac{1}{\beta_0^2} \left( \frac{1 - \beta_0 K_1(\beta_0)}{\beta_0 I_1(\beta_0)} \right) \left[ 1 - I_0(\beta_0) \right] - \log \frac{2}{\beta_0} + \gamma + K_0(\beta_0) \]  

(3.37)

where \( \gamma \) is the Euler-Mascheroni constant (\( \gamma = 0.577216 \)).

COMPARISON TO MODELING

Modeling the membrane subjected to a point force can be quite complicated. Some problems that were found is that implementing a point force in the 1D model (Eq. 3.20) using COMSOL will result in unrealistic results. Implementing an uniform pressure with a very small area in the center solves the problem. It can be seen that if the size of the applied force is reduced, the force-displacement curve appears to converge (Fig. 3.6). The 2D model (Eq. 3.13) gives reasonable results compared to real point force measurements [10, 12, 29, 31, 51–53].

The equation derived by Wan [49], Eq. 3.34, is quite complicated. It is counterintuitive that this equation is still dependent on the bending rigidity, while this is not modeled. Lee et. al. suggest that this equation can be simplified to:

\[ F = n_0 \pi \delta + Eh (a \delta^3) \left( \frac{\delta}{a} \right) \]  

(3.38)
for a point force. However this simplification cannot be defended properly, it is impossible to determine the asymptote of the thickness going to zero, since there is a singularity in Eq. 3.34 when \( h = 0 \). The asymptote is however still there and can be approximated by using a very small bending rigidity \( D = 1 \times 10^{-19} \text{N m} \), this gives a force displacement curve which is very close to the modeling results as can be seen in a modeled force displacement curve of a graphene membrane, Fig. 3.7.

### 3.2. Differential Pressure Sensor

Now that a model has been developed for the nonlinear behavior of the membrane, the eigenfrequency as function of pressure can be evaluated. Analytical solutions are available for this situation, but not in agreement with the finite element models from the previous section. This section will therefore also derive an improved equation for the eigenfrequency as function of pressure.

#### 3.2.1. Linearized Vibration of Pressurized Membranes

The previous section showed that if a membrane is subjected to a load in the out-of-plane direction, there are tensions induced. When these tensions become larger than the pretension the membrane is in the elastic regime. This added tension means that the simple equations for determining the eigenfrequency are no longer valid, since the tension is no longer constant. However one could linearize with the tension induced from a external load, this can be done in COMSOL by first evaluating the static solution and finding the eigenfrequencies around this solution, this is known as a "prestressed analysis". First some analytical derivations are done to verify this modeling approach.

From Eq. 3.32 one obtains the total tension in a membrane. According to Scott Bunch et. al. [12, 13], if the geometry does not change significantly (the deflections are small, but the tensions not), a substitution in
Figure 3.7: Force deflection curves as modeled by the 1D axisymmetric model, the 2D model, the equation by Wan et. al. (Eq. B.32) [49] and the approximation by Lee et. al. (Eq. 3.38) [10]. It is clear that the modeling results are in agreement with the formula's derived by Wan, while the equation from Lee is just a approximation, not valid for determining the pretension in the membrane. A larger pretension will increase the error. In this case a 4 µm diameter monolayer graphene membrane was modeled with a pretenion of 0.23 N m$^{-1}$.

Eq. 2.28 can be done to obtain:

$$f_{mn} = \frac{\gamma_{mn}}{2\pi a} \sqrt{\frac{n_0 + n_p}{\rho h}}$$

(3.39)

Where $n_p$ is determined by first calculating $\delta$ using Eq. 3.33, this can be substituted in Eq. 3.31, repeated here:

$$n_p = \frac{2Eh\delta^2}{3a^2(1-\nu)}$$

$$\Delta p = \frac{4n_0\delta}{a^2} + \frac{8Eh\delta^3}{3a^4(1-\nu)}$$

3.2.2. COMPARISON WITH MODELING

The 2D model is used to first find a static solution and linearizing around this point gives the eigenfrequency of a pressurized membrane. In section 3.1.4 it was already shown that this model is accurate regarding the static deflection of the membrane, with only a small error due to the fact that a dead load is used to implement the pressure instead of a live load. The linearization around this static solution gives the result in Fig. 3.8, compared with the analytical solution (Eq. 3.39). It can be seen that the calculated frequencies are in perfect agreement for very low pressure differences, but at higher pressure differences the results are no longer in agreement. Next step is to use the 1D model to do a prestressed analysis. This model uses a live load and in section 3.1.4 it was shown that this causes the numerical solution to be in perfect agreement with the analytical solution, regarding the static deflection. In Fig. 3.9 the result of this analysis can be seen. From this analysis a similar disagreement can be observed, this excludes the possibility that the difference between a live and a dead load is the cause of the problem.
3.2. Differential Pressure Sensor

3.2.3. Eigenfrequency of Fundamental Mode

In the previous section it was concluded that adding induced tension as additional stiffness in the eigenfrequency equation gives an answer that does not agree with the results from numerical modeling. Assuming that the numerical modeling is correct, means that the stiffness must be evaluated in a different manner. First step is to write the potential energy for a membrane subject to large deflections, from [45, 46]:

\[
V = \pi \int_0^a \left( n_0 e_1 + \frac{Eh}{2(1-\nu)} [e_1^2 - 2(1-\nu) e_2] \right) r \, dr \tag{3.40}
\]

where \( e_1 \) and \( e_2 \) are the first and second strain invariant, defined as:

\[
e_1 = \varepsilon_r + \varepsilon_\theta \tag{3.41a}
\]
\[
e_2 = \varepsilon_r \varepsilon_\theta \tag{3.41b}
\]

To approximate the behavior of the first mode, the assumption is made that this made takes on the shape of an inflated membrane (section 3.1.4). This reduces the system to one generalized coordinate \( \delta \). The uniform tensions result in uniform biaxial strain (Eq. 3.29b), therefore the strain invariants become:

\[
e_1 = \frac{4\delta^2}{3a^2} \tag{3.42a}
\]
\[
e_2 = \frac{4\delta^4}{9a^4} \tag{3.42b}
\]

These strains are not dependent on \( r \) or \( \theta \), thus substitution in Eq. 3.40 gives:

\[
V = \frac{4\pi n_0 \delta^2}{3} + \frac{\pi Eh}{(1-\nu^2)} \left( \frac{4\delta^4}{9a^2} - 2(1-\nu) \frac{\delta^4}{9a^2} \right) \tag{3.43}
\]
Figure 3.9: Comparison between the 1D model and the analytical equation for calculating the eigenfrequency as function of a pressure load for the first 3 axisymmetric mode shapes of oscillation. The membrane modeled is a monolayer of graphene with a radius of 2 microns and a pretension of 0.1N m$^{-1}$. The numerical results are almost identical to the numerical results from the 2D model (except for a small disagreement due to the live load – dead load difference). The disagreement is the same as the one observed in the 2D model.

Around a static solution from inflating the membrane, $\delta_0$, a Taylor expansion can be used approximate a quadratic function around this minimum. This will result in the equation for the harmonic oscillator:

$$ V \approx \frac{1}{2} (\delta - \delta_0)^2 \frac{d^2 V(\delta_0)}{d\delta^2} $$

(3.44)

A coordinate transform will give a new variable $x = \delta - \delta_0$ around the potential energy minimum:

$$ V \approx \frac{1}{2} \frac{d^2 V(\delta_0)}{d\delta^2} x^2 $$

(3.45)

Comparing this to the standard form for the harmonic oscillator: $V = kx^2/2$ it becomes clear that the effective stiffness is equal to the second derivative of the potential energy, evaluated at the static solution:

$$ k_{\text{eff}} = \frac{d^2 V(\delta_0)}{d\delta^2} = \pi n_0 \frac{4}{3} \frac{\pi E h}{(1 - \nu^2)} \left[ \frac{16\delta_0^2}{3a^2} - 2(1 - \nu) \frac{4\delta_0^2}{3a^2} \right] $$

(3.46)

By using the equation for the eigenfrequency of the fundamental mode (Eq. 2.27) at zero pressure difference, the equivalent modal mass for this generalized coordinate becomes:

$$ m_{\text{eff}} = \frac{4\pi a^2}{3 \cdot 2.404} \rho h $$

(3.47)

Now the eigenfrequency of the fundamental mode becomes:

$$ \omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \frac{2.404}{a} \sqrt{\frac{n_0}{\rho h} + \frac{E h (2 + \nu) \delta_0^2}{(1 - \nu^2) a^2 \rho h}} $$

(3.48)
Since $\delta_0$ can be determined from Eq. 3.33, a direct solution for the eigenfrequency as function of pressure has been found. In Fig. 3.10 the results from the 1D model are again compared to this new analytical solution. It is found that they are in good agreement. Therefore it is concluded that calculating the eigenfrequencies with Eq. 3.39 is too simple. The correct approach is to linearize the potential energy of the membrane around the static solution when it is pressurized.

![Graph showing fundamental eigenfrequency as function of pressure load](image)

Figure 3.10: Comparison between the 1D numerical model and the improved equation for calculating the eigenfrequency as function of a pressure load for the fundamental mode (Eq. 3.48). The membrane modeled is a monolayer of graphene with a radius of 2 microns and a pretension of 0.1N m$^{-1}$. The results are in good agreement with each other.

### 3.2.4. Notes on the Modeling

It was shown that for the fundamental mode an analytical solution can be obtained for the eigenfrequency as function of pressure by linearizing around the static deflection. In principle this analysis can be expanded to mode shapes other than the fundamental one. The analysis can be done as follows, for the strains [54]:

$$
\varepsilon_r = \frac{1}{2} \left( \frac{\partial w(r, \theta, t)}{\partial r} \right)^2
$$

$$
\varepsilon_\theta = \frac{1}{2r^2} \left( \frac{\partial w(r, \theta, t)}{\partial \theta} \right)^2
$$

Where $\delta_0$ is determined from Eq. 3.33 and for each separate mode the deflection $w(r, \theta, t)$ is assumed to be of the form:

$$
w(r, \theta, t) = W_{mn}(r, \theta) \eta(t)
$$

Now the potential energy can be obtained using Eq. 3.40 and linearized around $\eta = 0$. However evaluating this using Eq. 3.40 gives an integral with bessel functions up to the fourth power, for which an analytical solution is not known at this point. It is recommended to use a final element solution for this situation, since it is proven that the linearization from this method works for the fundamental mode it is safe to assume that it will also work for the higher order modes.

An important note on this sensor is that if a small cavity is used, the resonance frequency can also be affected by the compressed film effect. This can be avoided by using vacuum in the cavity as reference pressure, or by making the cavity very deep. It should be considered during the design of the device that an electrostatic back gate is also needed, which means that some kind of cavity has to be present in the device.
3.3. SENSOR PERFORMANCE

This section will be on sensor performance, for which the remarks on costs, size and mechanical disturbance are the same as for the compressed film transducer (see section 2.5).

3.3.1. SENSITIVITY OF IDEAL DEVICES

The improved equation for determining the frequency change as function of pressure (Eq. 3.48) can be used to calculate the sensitivity of the resonance frequency to a pressure change. This is shown in Fig. 3.11 as function of pressure. It can be seen that the graph is never constant over a pressure range, meaning that the device never has a linear relationship between pressure and resonance frequency.

3.3.2. SENSOR NOISE

In order to have some idea of the pressure resolution that can be achieved by these transducers, the limitation due to sensor noise is studied. The spread in the frequency measurement is given by Eq. 2.116, repeated here:

\[ \sigma[\omega(t)]_{t,\tau} = \frac{\omega_0}{2Q} \sqrt{\frac{k_B T}{\tau P_{\text{sig}}}} \]

this is directly converted to a pressure giving the resolution that can be achieved for a pressure measurement (Fig. 3.12). The resolution changes a lot as function of pressure, it is in the range between 0.25\(\mu\)Pa and 0.2mPa. The resolution can be considered very high compared to the pressure differences that are being measured. For example, if the 4\(\mu\)m device is used to measure a pressure difference of 1 \(\times\) 10^5 Pa the resolution is limited to approximately 5\(\mu\)Pa. If only limited by sensor noise, one can measure a pressure difference with 10 digit accuracy.

3.3.3. MEASURABLE RANGE

The graphene diaphragm is fixed by adhesion to the surface, this is the limiting factor to the maximum pressure difference that can be measured using these devices. Based on measurements from literature [55], delamination will occur at approximately 1MPa for a 4\(\mu\)m device. This would mean that this device can be used to measure tire pressures in cars (pressure difference typically 0.2MPa), however for the tire pressure in bicycles the sensor would be working on its limit since the pressure difference can be as high as 0.9MPa. In vacuum systems the sensor could work fine, since the pressure difference will never exceed 0.1MPa.
3.3.4. BANDWIDTH
This device has no limit on the bandwidth due to pressure equalization. The bandwidth is limited by the measurement time used to measure the frequency (Eq. 2.116). However a larger bandwidth means a less accurate measurement, which should be considered during the design of these systems.

3.3.5. COMPARING TO CURRENT TECHNOLOGIES
Currently pressure gages are considered to be high-precision if the accuracy is 0.001% [42, 56, 57]. Looking at the limitations of the thermal noise (0.00000001% accuracy), it is expected that this transducer will be limited by the noise from the electronics. This means that the same specifications can be met. Also the price of high precision silicon sensors lies in the hundreds of dollars, it is expected that these graphene devices can bring these costs down to only a couple of dollars (section 2.5.4). These numbers suggest that even with added electronics to correct for nonlinearities and other electronics the device will be much cheaper while obtaining the same accuracy and resolution.

3.4. CONCLUSION
Systems of partial differential equations have been derived which were implemented in COMSOL multiphysics to predict the nonlinear behavior of membranes in different situations. An equation has been derived that relates the pressure difference to the resonance frequency of a circular membrane, it was found that this equation was an improvement to current models and in agreement with finite element modeling.

Induced tension transducers are promising because of their high accuracy and low costs compared to current technologies. Downside is their limited measurable pressure range and the nonlinear relationship between pressure and resonance frequency.
In the previous chapters two concepts for resonant pressure transduction have been discussed. In the compressed film sensor the pressure in the cavity determines the behavior, this means that pressure equalization to the volume of interest is important. However this pressure equalization must be much slower the resonance frequency, since fast equalization reduces the compression effect. The resonance transduction by induced tension has the opposite problem, since here pressure equalization would mean that the reference pressure is no longer defined resulting in poor measurements. These problems are investigated using a model commonly used in complex fluidic systems: a hydraulic circuit model. Since this thesis deals with gas instead of liquids, the term acoustic circuit modeling will be used for this method.

This chapter starts with theory from literature, first discussing the electrical resistance-capacitance circuit. This systems shows many analogies with the acoustic circuit considered in this chapter. The second part of this theory introduces hydraulic circuit modeling as used in (micro-)fluidic systems. One trick for pressure equalization is to use a permeable membrane [58–61], the second section discusses how the permeability can be related to the acoustic resistance of the system. The third section derives equations for the acoustic equivalent of the capacitance, this needs an extension of the mechanics discussed in chapter 2 by deriving the forced vibration of membranes and plates. For verification of this model it is necessary to know how the acoustic circuit couples to the structural mechanics, which will be discussed in the fourth section.

4.1. Theory

This section goes into the theory from literature to support the mathematical derivations of this chapter. First the theory of the electrical resistor-capacitor circuit is presented. The second part goes into hydraulic circuit modeling, often used in microfluidics to model flow through devices.

4.1.1. Resistor-Capacitor Circuit

![Image of a simple electrical circuit with a resistor and capacitor in series.]

A resistor-capacitor (RC) circuit is an electrical system which also has analogies with other physical systems. Other than the electrical system it can also be used to describe gas flow between the cavity and the outside, as will be shown in this chapter. The simplest circuit one can consider is a capacitor and resistor in
series (Fig. 4.1). The current through a capacitor is given by:

\[ I = C \frac{dV_C}{dt} \]  

(4.1)

where \( I \) is the current (A), \( V_C \) the voltage across the capacitor (V) and \( C \) the capacitance (F). In the frequency domain this becomes:

\[ I(\omega) = C i \omega V \]  

(4.2)

The current through the resistor is given by Ohm’s law:

\[ I = \frac{V_R}{R} \]  

(4.3)

where \( V_R \) is the voltage across the resistor and \( R \) is the resistance (Ω). Note that for a constant frequency one can rewrite Eq. 4.2 to:

\[ Z(\omega) = \frac{V}{I} = \frac{1}{i \omega C} \]  

(4.4)

where \( Z \) is an equivalent resistance for a certain frequency, known as the impedance. Using this, the system shown in Fig. 4.1 can be regarded as a voltage divider. In the frequency domain, the voltage across the capacitor becomes:

\[ V_C(\omega) = \frac{1/(i \omega C)}{R + 1/(i \omega C)} = \frac{1}{1 + i \omega RC} V_{in}(\omega) \]  

(4.5)

and across the resistor:

\[ V_R(\omega) = \frac{R}{1 + 1/(i \omega C)} = \frac{i \omega RC}{1 + i \omega RC} V_{in}(\omega) \]  

(4.6)

Both these responses are plotted in Fig. 4.2. The total complex impedance of the system is obtained by adding the resistance to the complex impedance of the capacitor:

\[ Z = \frac{V}{I} = R + \frac{1}{i \omega C} \]  

(4.7)
4.1. Theory

**Step Response**

One way to measure the properties of a RC system is to look at the response to a Heaviside step function (with amplitude 1). These can be obtained by using the inverse Fourier transforms of Eqs. 4.5 – 4.6

\[ V_C(t) = 1 - e^{-\frac{t}{RC}} \]  \hspace{1cm} (4.8)

and for the resistor:

\[ V_R(t) = e^{-\frac{t}{RC}} \]  \hspace{1cm} (4.9)

An factor that is often used is the time factor \( \tau = RC \). When the voltage in the step response is equal to 63% of the final value, the time is equal to the time constant.

**4.1.2. Hydraulic Circuit Modeling**

This section discusses the Hagen-Poiseuille law, which is used to approximately describe the fluid flow in complex rigid networks of fluid channels. This linearized approximation can also be used to predict unsteady flow in flexible channels. A detailed description of this theory can be found in the book by Kirby [62].

For the steady flow of a Newtonian fluid through a channel with uniform circular cross section the velocity along the channel is given by:

\[ u_z = -\frac{1}{4\eta} \frac{\partial p}{\partial z} (R^2 - r^2) \]  \hspace{1cm} (4.10)

where \( \eta \) is the kinematic viscosity (m\(^2\) s\(^{-1}\)), \( \partial p/\partial z \) the pressure gradient along the channel (Pa m\(^{-1}\)) and \( R \) the outer radius of the tube. The volumetric flow rate is now given by:

\[ Q = -\frac{\pi R^4}{8\eta} \frac{\partial p}{\partial z} \]  \hspace{1cm} (4.11)

where \( Q \) is the volumetric flow rate (m\(^3\) s\(^{-1}\)). Strictly speaking, this only holds for a straight tube of infinite length, however one could approximate the behavior of a tube with finite length by substituting \(-\partial p/\partial z\) with \(\Delta p/L\). This approximation results in:

\[ Q = \frac{\pi R^4}{8\eta L} \Delta p \]  \hspace{1cm} (4.12)

Now define the hydraulic resistance as:

\[ R_h = \frac{8\eta L}{\pi R^4} \]  \hspace{1cm} (4.13)

This results in the Hagen-Poiseuille law:

\[ Q = \frac{\Delta p}{R_h} \]  \hspace{1cm} (4.14)

Figure 4.3: Step response of a RC circuit with a Heaviside step function at \( t = 0 \) as input voltage. Left: the voltage over the capacitor, right: the voltage over the resistor.
Note that this law is equivalent to Ohm’s law if the pressure is considered the voltage and the volumetric flow rate is considered the current. This allows modeling of complex systems like multiple fluid channels in parallel or series, but this is not necessary for the systems considered in this thesis.

A channel is not the only cause of resistance. Another relevant cause of acoustic resistance is the permeability of atomically thin membranes. This might be due to etching processes or a property of the membrane itself (atomically thin MoS$_2$ for example is permeable). Expressions for this resistance will be derived in section 4.2.

**Hydraulic Compliance**
The Hagen-Poiseuille law can be extended by adding a equivalence of a capacitor, known as the hydraulic compliance, this describes the amount of (effective) volume $V$ ($\text{m}^3$) that can be stored if the system is pressurized. The definition is given by:

$$C_h = \frac{dV}{dp} \tag{4.15}$$

The fluid can be stored by an increase of volume accessible to the fluid, this would be caused by the deflection of the membrane for example. Another cause is the compression of the fluid inside the cavity. For ideal gases this compliance becomes:

$$\frac{dV}{dp} = -\frac{nRT}{p^2} \tag{4.16}$$

where $p$ is the pressure inside the cavity, $n$ is the amount of gas molecules (mol), $R$ is the universal gas constant ($\text{J K}^{-1} \text{ mol}^{-1}$) and $T$ is the absolute temperature ($\text{K}$). In this thesis the compression effect on the compliance of the system is ignored, because the only interest lies in the pressure equalization of the system. This thesis does however take into account the influence of the compression on the mechanics of the system. The compression is only important if the squeeze number is very high, which is usually only the case at very high frequencies. The consequence is that the compliance model is only valid when the squeeze number is low, which is true for low frequencies.

The volume stored will be governed by the volume the membrane encloses when it deflects from its equilibrium position. Therefore first the deflection must be derived, using the forced response. This will be done in section 4.3.1 for the membrane and in 4.3.3 for the plate. The deflection can be integrated to give the enclosed volume. Using Eq. 4.15 an expression for the compliance is obtained.

**Hydraulic Impedance**
The total response of the system can now be determined by combining the resistance and compliance in an complex impedance. For a harmonic excitation the impedance becomes:

$$Z_h = R_h + \frac{1}{i\omega C_h} \tag{4.17}$$

The complex form of the Hagen-Poiseuille law becomes:

$$\Delta p = Q Z_h \tag{4.18}$$

The complex form means that the volume flow has a certain amplitude and phase shift with respect to the excitation.

Since the general form of this theory focuses on fluids, the terms hydraulic resistance and compliance are used. However this thesis mostly deals with gases, therefore from now on the term acoustic will be used instead of hydraulic. The relevant acoustic circuit that is used to model the pressure equalization behavior is a resistor and compliance in series, as shown in Fig. 4.4.

### 4.2. Permeability as a Source of Acoustic Resistance

The most simple case of an acoustic resistor has already been introduced, however there are other possibilities to obtain a resistance from the cavity to the outside world. Suggested by Scott Bunch [12] is the diffusion of air molecules through the lattice of silicon dioxide, which would give a very high resistance. Other resistances could be a permeable membrane such as MoS$_2$ or a UV etched graphene membrane, the latter has attracted scientific interest because it can be used for gas separation [58–61]. MoS$_2$ has shown to be a good material for resonators in recent research [30].
4.2. PERMEABILITY AS A SOURCE OF ACOUSTIC RESISTANCE

Figure 4.4: The series acoustic circuit used to model the pressure equalization of the compressed film transducer. The equivalent electrical circuit is shown on the bottom. $P_0$ is the force per unit square used to actuate the membrane; $P_1$ is the gas pressure inside the cavity (assumed uniform); $P_2$ is the pressure of the surroundings, usually assumed zero to simplify the analysis.

Since the permeable membrane shows great potential for various future applications of atomically thin materials, it is useful to have a simple model regarding its behavior. Since the flux through the membrane is proportional to the pressure difference over the membrane, one can define the permeability:

$$J = -\frac{\mathcal{P}(p_2 - p_1)}{h}$$ (4.19)

where $\mathcal{P}$ (mol N$^{-1}$ m$^{-1}$ s$^{-1}$) is the permeability of the membrane, $(p_2 - p_1)$ the pressure difference over the membrane and $h$ the thickness. The flux $J$ is measured in mol m$^{-2}$ s$^{-1}$, the ideal gas law can be used to calculate the corresponding volumetric flow rate:

$$V = \frac{nRT}{p_a}$$ (4.20)

which is the amount of gas volume for $n$ moles of gas. Now one can rewrite Eq. 4.19 to:

$$J \frac{\pi a^2 RT}{p_a} = Q = -\frac{\pi a^2 RT \mathcal{P}(p_2 - p_1)}{p_a h} = \frac{p_2 - p_1}{R_h}$$ (4.21a)

$$R_h = \frac{p_a h}{\pi a^2 RT \mathcal{P}}$$ (4.21b)

$$\mathcal{P} = \frac{p_a h}{\pi a^2 RT R_h}$$ (4.21c)

This means that by measuring the hydraulic resistance of a membrane over a sealed cavity, the permeability can be measured. One way to do this is looking at the damping present in the system. For this purpose it is convenient to know the pressure difference across the membrane:

$$p_2 - p_1 = R_h Q$$ (4.22)
Since \( Q \) can be a complex number, it will give information on whether the system is acting like a spring or a viscous damper.

This model does not work if a mixture of gases with different partial pressures is considered and the pore sizes are in the order of the kinetic diameter of the gas molecules, because the permeability is different for each component in the gas. In this case the partial pressure difference of each gas should be considered. The difference in resistance in this case could also be exploited for building a gas sensor.

### 4.3. Acoustic Compliance

In this section the acoustic compliance of circular membranes and plates are calculated. This is done by calculating the volume that the membrane or plate stores when it deflects, compression is not considered. In order to calculate the volume enclosed, the deflection must be known, for this the forced response of both membranes and plates are calculated.

#### 4.3.1. Forced Vibration of Circular Membranes

The eigenfrequencies and mode shapes in the previous section gives the response of the system if it is freely vibrating without an external force. However sometimes it can be useful to know the response of a system when a certain force is applied. In this section the forced response of a membrane subjected to a uniform pressure load is derived following the derivation from Rao [34], this can be used to calculate the acoustic compliance.

Consider a harmonically varying uniform pressure load acting on the membrane:

\[
\Delta p(r, \theta, t) = p_0 \cos(\Omega t), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi
\]  

(4.23)

The equation of motion is given by Eq. 2.15:

\[
m_0 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) + \Delta p = \rho_h \frac{\partial^2 w}{\partial t^2}
\]

(4.24)

Now use modal analysis to find a solution in the form:

\[
w(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} W_{mn}(r, \theta) \eta_{mn}(t)
\]

(4.25)

Where \( W_{mn} \) are the natural vibration modes and \( \eta_{mn} \) are the generalized coordinates corresponding to this mode, represented by a harmonic oscillator. Eq. 2.40 gives two mode shapes which are 90° apart, written as \( W_{mn}^{(1)} \) and \( W_{mn}^{(2)} \), except when \( m = 0 \), this means that Eq. 4.25 can be rewritten to:

\[
w(r, \theta, t) = \sum_{n=1}^{\infty} W_{0n}^{(1)}(r, \theta) \eta_{0n}^{(1)}(t) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} W_{mn}^{(1)}(r, \theta) \eta_{mn}^{(1)}(t) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} W_{mn}^{(2)}(r, \theta) \eta_{mn}^{(2)}(t)
\]

(4.26)

The modes can be normalized using the following formula:

\[
\int_A \rho_h C_{1mn}^2 W_{mn}^2 dA = 1
\]

(4.27)

Evaluating this equation gives:

\[
W_{0n}^{(1)} = \frac{\sqrt{2}}{\sqrt{\pi} \rho_h a \sqrt{J_0^2(\gamma_{0n}) + J_1^2(\gamma_{0n})}} J_0(\gamma_{0n} r / a)
\]

(4.28a)

\[
W_{mn}^{(1)} = \frac{\sqrt{2}}{\sqrt{\pi} \rho_h a \sqrt{J_m^2(\gamma_{mn}) + J_{m+1}^2(\gamma_{mn})}} J_m(\gamma_{mn} r / a) \cos m\theta
\]

(4.28b)

\[
W_{mn}^{(2)} = \frac{\sqrt{2}}{\sqrt{\pi} \rho_h a \sqrt{J_m^2(\gamma_{mn}) + J_{m+1}^2(\gamma_{mn})}} J_m(\gamma_{mn} r / a) \sin m\theta
\]

(4.28c)
4.3. ACOUSTIC COMPLIANCE

Using the frequency equation (Eq. 2.25) means that \( I_m(\gamma_{mn}) = 0 \), this simplifies the result to:

\[
W_{0n}^{(1)} = \frac{\sqrt{2}}{\sqrt{\pi \rho \bar{h} J_1(\gamma_{0n})}} J_0(\gamma_{0n} r/a)
\] (4.29a)

\[
W_{mn}^{(1)} = \frac{\sqrt{2}}{\sqrt{\pi \rho \bar{h} J_{m+1}(\gamma_{mn})}} J_m(\gamma_{mn} r/a) \cos m\theta
\] (4.29b)

\[
W_{mn}^{(2)} = \frac{\sqrt{2}}{\sqrt{\pi \rho \bar{h} J_{m+1}(\gamma_{mn})}} J_m(\gamma_{mn} r/a) \sin m\theta
\] (4.29c)

When Eq. 4.25 is used, one can obtain from Eq. 2.15:

\[
\ddot{\eta}_{mn}(t) + \omega_{mn}^2 \eta_{mn}(t) = N_{mn}(t)
\] (4.30)

Where \( N_{mn}(t) \) is the generalized force given by:

\[
N_{mn}(t) = \int_0^{2\pi} \int_0^{2\pi} W_{mn}(r, \theta) \Delta p(t) r dr d\theta
\] (4.31)

Since modal analysis breaks the system into a set of harmonic oscillators representing each mode, the impulse response is given by \( h(t) = \sin \omega_{mn} t \) (Eq. 2.13). The convolution integral between the generalized force and the impulse response gives the forced response (Eq. 2.14), in this case equal to the generalized coordinate since this represents the harmonic oscillator:

\[
\eta_{0n}(t) = \frac{1}{\omega_{0n}} \int_0^t N_{0n}^{(1)}(\tau) \sin \omega_{0n}(t - \tau) d\tau
\] (4.32a)

\[
\eta_{mn}^{(1)}(t) = \frac{1}{\omega_{mn}} \int_0^t N_{0n}^{(1)}(\tau) \sin \omega_{mn}(t - \tau) d\tau
\] (4.32b)

\[
\eta_{mn}^{(2)}(t) = \frac{1}{\omega_{mn}} \int_0^t N_{2n}^{(2)}(\tau) \sin \omega_{mn}(t - \tau) d\tau
\] (4.32c)

Using Eqs. 4.28 – 4.31 the generalized forces can be evaluated:

\[
N_{0n}^{(1)}(t) = \int_0^{2\pi} \int_0^{2\pi} W_{0n}^{(1)}(r, \theta) \Delta p(t) r dr d\theta = \int_0^{2\pi} \int_0^{2\pi} W_{0n}^{(1)} \cos \Omega t r dr d\theta = \int_0^{2\pi} \int_0^{2\pi} W_{0n}^{(1)} \cos \Omega t r dr d\theta
\]

\[
N_{mn}^{(1)}(t) = \int_0^{2\pi} \int_0^{2\pi} W_{mn}^{(1)}(r, \theta) \Delta p(t) r dr d\theta
\]

\[
N_{mn}^{(2)}(t) = \int_0^{2\pi} \int_0^{2\pi} W_{mn}^{(2)}(r, \theta) \Delta p(t) r dr d\theta
\]

Note that the uniform pressure will only actuate the axisymmetric modes, since the waveform of Eq. 4.23 is orthogonal to all other modes. Using Eq. 4.32, the generalized coordinates become:

\[
\dot{\eta}_{0n}^{(1)}(t) = \frac{2 \pi p_0}{\sqrt{\pi \rho \bar{h} \Omega_{0n}} \gamma_{0n}} \int_0^t \cos \Omega t \sin \omega_{0n}(t - \tau) d\tau
\]

\[
= \frac{2 \pi p_0 (\cos \omega_{0n} t - \cos \Omega t)}{\sqrt{\pi \rho \bar{h} \gamma_{0n}} (\Omega^2 - \omega_{0n}^2)}
\] (4.34a)
Figure 4.5: The magnitude and phase shift of the deflection of a circular membrane, evaluated at \( r = 0 \). The properties of a monolayer of graphene are assumed with a radius of 4\( \mu \)m and a pretension of 0.1N m\(^{-1}\). Damping is added to the system by assuming that \( Q = 100 \) for each resonance. Eq. 4.36 is approximated by using only the first twelve terms in the equation.

\[
\eta_{nm}^{(1)}(t) = 0
\]
\[
\eta_{nm}^{(2)}(t) = 0
\]

Now the response of the membrane can be expressed as:

\[
w(r, \theta, t) = \frac{2\sqrt{2}p_0}{\rho h} \sum_{n=1}^{\infty} \frac{J_0(\gamma_0 r/a)}{J_1(\gamma_0 a)} (\cos \omega_{0n} t - \cos \Omega t)
\]

The response is now expressed as a sum of harmonic oscillators, compare with Eq. 2.11. Recognizing that these equations are equivalent, the damped response becomes:

\[
w(r, \theta, t) = \frac{2\sqrt{2}p_0}{\rho h} \sum_{n=1}^{\infty} \frac{J_0(\gamma_0 r/a)}{J_1(\gamma_0 a)} \left( \cos \omega_{0n} t - \cos \Omega t + i \frac{\omega_{0n}}{\sqrt{Q}} \right)
\]

This equation can be approximated by using the first couple of terms, since the dependence on the eigenfrequency \( \omega_{0n}^2 \) the higher eigenfrequencies have a smaller contribution. An example of a forced response can be seen in Fig. 4.5.

### 4.3.2. Acoustic Compliance of Membranes

The volume stored can now be evaluated by calculating the volume taken by the deflected membrane:

\[
V = \int_0^{2\pi} \int_0^r w(r, \theta, t) dr d\theta
\]
Using the forced response (Eq. 4.35) this equation gives:

\[
\gamma = \frac{4\pi\sqrt{2}\rho_0}{\rho h} \sum_{n=1}^{\infty} \left( \cos \omega_{0n} t - \cos \Omega t \right) \int_{0}^{a} \frac{r}{Y_{0n}} J_{0}(y_{0n} r/a) dr
\]

\[
= \frac{2\pi\sqrt{2}\rho_0}{\rho h} \sum_{n=1}^{\infty} \left( \cos \omega_{0n} t - \cos \Omega t \right) \int_{0}^{a} \frac{r}{Y_{0n}} J_{1}(y_{0n} r/a) \left| r = 0 \right. \]

\[
= \frac{2\pi\sqrt{2}\rho_0}{\rho h} \sum_{n=1}^{\infty} 1 \left( \cos \omega_{0n} t - \cos \Omega t \right)
\]

(4.38)

Taking the derivative with respect to the pressure gives the acoustic compliance:

\[
C_h = \frac{dV}{dp} = \frac{2\pi\sqrt{2}a^2}{\rho h} \sum_{n=1}^{\infty} \frac{1}{\gamma_{0n}^2 (\Omega^2 - \omega_{0n}^2)}
\]

(4.39)

This can be used to calculate the pressure equalization over a wide pressure range. However in most cases the pressure equalization is slow with respect to the resonance frequency (\(\Omega << \omega_0\)), in this case one can write:

\[
C_h = -\frac{2\pi\sqrt{2}a^4}{\rho h} \sum_{n=1}^{\infty} \frac{1}{\gamma_{0n}^2 \omega_{0n}^2}
\]

(4.40)

Substituting Eq. 2.28, this becomes:

\[
C_h = -\frac{2\pi\sqrt{2}a^4}{n_0} \sum_{n=1}^{\infty} \frac{1}{4n^4}
\]

(4.41)

For the infinite series of zeros of Bessel functions the solutions are known [63]:

\[
\sum_{n=1}^{\infty} \frac{1}{\gamma_{v,n}^4} = \frac{1}{16(v+1)^2(v+2)}
\]

(4.42)

This means that the infinite series is converging to:

\[
\sum_{n=1}^{\infty} \frac{1}{\gamma_{v,n}^4} = \frac{1}{32}
\]

(4.43)

Now for \(\Omega << \omega_0\) the following solution is obtained:

\[
C_h = -\frac{\pi\sqrt{2}a^4}{16n_0}
\]

(4.44)

### 4.3.3. Forced Vibration and Acoustic Compliance of Circular Plates

The solution of the forced vibration of a circular clamped plate is typically not found in textbooks. However the modal analysis, as used for the forced response of a membrane, should also apply for the plate. This section finds the solution for the forced response of a plate, which will be used to calculate the acoustic compliance. Consider a harmonically varying uniform pressure load acting on the plate:

\[
\Delta p(r, \theta, t) = p_0 \cos(\Omega t), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi
\]

(4.45)

The equation of motion is given by:

\[
D V^2 \gamma^2 \gamma \frac{w(r, \theta, t)}{\rho h} + \frac{\partial^2 w(r, \theta, t)}{\partial t^2} = 0
\]

(4.46)

Only axisymmetrical deformations are considered, like the case of the membrane the non-axisymmetric modes will not appear in the solution. Now use modal analysis to find a solution in the form:

\[
w(r, \theta, t) = \sum_{n=0}^{\infty} W_{0n}(r) \eta_{0n}(t)
\]

(4.47)

Where \(W_{0n}\) are the axisymmetric natural mode shapes given by:

\[
W(r) = l_0(\lambda_{0n} r) J_0(\lambda_{0n} r) - J_0(\lambda_{0n} r) l_0(\lambda_{0n} r)
\]

(4.48)
The normalized mode shapes are evaluated as:

\[ \int_0^a \int_0^{2\pi} C_{0n}^2 \rho h |J_0(\lambda_{0n}a)J_0(\lambda_{0n}r) - J_0(\lambda_{0n}a)J_0(\lambda_{0n}r)|^2 r \, dr \, d\theta \]

\[ = 2\pi C_{0n}^2 \rho h \int_0^a \left( I_0^2(\lambda_{0n}a) f_0^2(\lambda_{0n}r) - 2 J_0(\lambda_{0n}a) I_0(\lambda_{0n}r) J_0(\lambda_{0n}a) J_0(\lambda_{0n}r) + J_0^2(\lambda_{0n}a) I_0^2(\lambda_{0n}r) \right) r \, dr \]

\[ = 2\pi C_{0n}^2 \rho h I_0^2(\lambda_{0n}a) \int_0^a f_0^2(\lambda_{0n}r) r \, dr - 4\pi C_{0n}^2 \rho h J_0(\lambda_{0n}a) I_0(\lambda_{0n}a) \int_0^a J_0(\lambda_{0n}r) J_0(\lambda_{0n}r) r \, dr \]

\[ + 2\pi C_{0n}^2 \rho h f_0^2(\lambda_{0n}a) \int_0^a I_0^2(\lambda_{0n}r) r \, dr = 1 \] (4.49)

These three integrals can be evaluated as follows:

\[ \int_0^a r f_0^2(\lambda_{0n}r) r \, dr = \frac{1}{2} a^2 \left( I_0^2(\lambda_{0n}a) + f_0^2(\lambda_{0n}a) \right) \] (4.50a)

\[ \int_0^a r I_0^2(\lambda_{0n}r) r \, dr = \frac{1}{2} a^2 \left( I_0^2(\lambda_{0n}a) - I_0^2(\lambda_{0n}a) \right) \] (4.50b)

\[ \int_0^a r J_0(\lambda_{0n}r) I_0(\lambda_{0n}r) r \, dr = \frac{a}{2\lambda_{0n}} (I_1(\lambda_{0n}a)J_0(\lambda_{0n}a) + I_0(\lambda_{0n}a)J_1(\lambda_{0n}a)) \] (4.50c)

From Eq. 4.49 can now be rewritten to give:

\[ C_{0n}^2 \pi \rho h \left[ \frac{a}{\lambda_{0n}} \left( aI_0^2(\lambda_{0n}a)I_{0n}(f_0^2(\lambda_{0n}a) + f_0^2(\lambda_{0n}a)) \right) - 2 J_0(\lambda_{0n}a) I_0(\lambda_{0n}a) \left( I_1(\lambda_{0n}a)J_0(\lambda_{0n}a) + I_0(\lambda_{0n}a)J_1(\lambda_{0n}a) \right) + f_0^2(\lambda_{0n}a) a \lambda_{0n} (I_0^2(\lambda_{0n}a) - I_0^2(\lambda_{0n}a)) \right] = 1 \] (4.51)

From the frequency relation (Eq. 2.48) one obtains the relation:

\[ I_0(\lambda_{0n}a)J_1(\lambda_{0n}a) = - J_0(\lambda_{0n}a)I_1(\lambda a) \] (4.52)

This simplifies the normalization quite significantly to:

\[ 2\pi C_{0n}^2 \rho h a^2 I_0^2(\lambda_{0n}a) f_0^2(\lambda_{0n}a) = 1 \] (4.53a)

\[ C_{0n} = \frac{1}{\sqrt{2\pi \rho h a I_0(\lambda_{0n}a) J_0(\lambda_{0n}a)}} \] (4.53b)

This can be evaluated to give the normalized mode shape:

\[ W_{0n} = \frac{I_0(\lambda_{0n}a)J_0(\lambda_{0n}r) - J_0(\lambda_{0n}a)I_0(\lambda_{0n}r)}{\sqrt{2\pi \rho h a I_0(\lambda_{0n}a) J_0(\lambda_{0n}a)}} \] (4.54)
The generalized force can now be expressed as:

\[
N_{0n}(t) = \int_0^a \int_0^\theta \frac{2\pi a}{2\pi p\omega_0} J_0(\lambda_{0n} r) J_0(\lambda_{0n} a) p_0 \cos(\Omega t) r dr d\theta
\]

\[
= 2\pi p_0 \cos(\Omega t) \int_0^a J_0(\lambda_{0n} r) r dr - \frac{2\pi p_0 \cos(\Omega t)}{\lambda_{0n} \sqrt{2\pi p\omega_0}} \int_0^a J_0(\lambda_{0n} a) J_0(\lambda_{0n} r) r dr
\]

\[
= \frac{\pi}{\lambda_{0n} \sqrt{2\pi p\omega_0}} \frac{I_0(\lambda_{0n} a) I_1(\lambda_{0n} a)}{I_0(\lambda_{0n} a)} p_0 \cos(\Omega t)
\]

\[
\eta_{0n}(t) = \frac{1}{\omega_0} \int_0^t N_{0n}(r) \sin(\omega_0(t-r)) dr = \frac{2\pi p_0 I_1(\lambda_{0n} a)}{\omega_0 \lambda_{0n} \sqrt{2\pi p\omega_0}} \int_0^t \cos(\Omega t) \sin(\omega_0(t-r)) dr
\]

\[
= -\frac{\pi p_0}{\lambda_{0n} \sqrt{2\pi p\omega_0}} \frac{I_1(\lambda_{0n} a) \cos(\omega_0 t - \cos(\Omega t))}{I_0(\lambda_{0n} a)}
\]

Eq. 4.47 can be evaluated to give:

\[
w(r, t) = \sum_{n=0}^{\infty} W_{0n}(r) \eta_{0n}(t)
\]

\[
= \sum_{n=0}^{\infty} -\frac{I_0(\lambda_{0n} a) J_0(\lambda_{0n} r) - J_0(\lambda_{0n} a) I_0(\lambda_{0n} r)}{\lambda_{0n} \sqrt{2\pi p\omega_0} I_0(\lambda_{0n} a)} \frac{\pi p_0}{\lambda_{0n} \sqrt{2\pi p\omega_0}} \frac{I_1(\lambda_{0n} a) \cos(\omega_0 t - \cos(\Omega t))}{I_0(\lambda_{0n} a)}
\]

\[
= \sum_{n=0}^{\infty} -\frac{p_0[I_0(\lambda_{0n} a) J_0(\lambda_{0n} r) - J_0(\lambda_{0n} a) I_0(\lambda_{0n} r)] I_1(\lambda_{0n} a) \cos(\omega_0 t - \cos(\Omega t))}{2\rho h a \lambda_{0n} I_0^2(\lambda_{0n} a) I_0(\lambda_{0n} a) (\Omega^2 - \omega_0^2)}
\]

The damped response is:

\[
w(r, t) = \sum_{n=0}^{\infty} -\frac{p_0[I_0(\lambda_{0n} a) J_0(\lambda_{0n} r) - J_0(\lambda_{0n} a) I_0(\lambda_{0n} r)] I_1(\lambda_{0n} a) \cos(\omega_0 t - \cos(\Omega t))}{2\rho h a \lambda_{0n} I_0^2(\lambda_{0n} a) I_0(\lambda_{0n} a) (\Omega^2 - \omega_0^2 + \frac{\tau^2 \omega_0^2}{\Omega^2})}
\]

An example of a circular plate in MEMS systems is a microphone, for which the forced response can be seen in Fig. 4.6.

Now the acoustic compliance can be evaluated similar as in section 4.3.2, giving:

\[
C_h = \frac{8\pi a^2}{\rho h} \sum_{n=1}^{\infty} \frac{1}{\lambda_{0n}^2 (\Omega^2 - \omega_0^2) \left(I_1(\lambda_{0n} a) / I_0(\lambda_{0n} a) \right)^2}
\]

(4.59)

For \(\Omega \ll \omega_0\) and using \(\gamma_{0n} = \lambda_{0n} a\) this becomes:

\[
C_h = \frac{8\pi a^2}{\rho h} \sum_{n=1}^{\infty} \frac{1}{\gamma_{0n}} \left(I_1(\gamma_{0n}) / I_0(\gamma_{0n}) \right)^2
\]

(4.60)
Figure 4.6: The magnitude and phase shift of the deflection of a circular plate, evaluated at $r = 0$. The system used is a MEMS microphone, consisting of a SiGe plate with radius 175µm and thickness 3µm. A quality factor of 200 is assumed for each resonance. Eq. 4.57 is approximated using the first ten terms in the equation.
Using numerical evaluation, it is found that the infinite series converges to:

$$
\sum_{n=1}^{\infty} \frac{1}{\gamma_0^4} \left( \frac{I_1(\gamma_0\omega_n)}{I_0(\gamma_0\omega_n)} \right)^2 \approx 0.007211 \quad (4.61)
$$

the estimated error by using the first ten terms is 0.013 %. The acoustic compliance for \( \Omega << \omega_0 \) becomes:

$$
C_h = \frac{0.1812a^6}{D} \quad (4.62)
$$

### 4.4. COUPLING TO THE STRUCTURAL MECHANICS

The analogy between the RC circuit and the acoustic behavior is the key to performing measurements to this system. However at the small scale of atomically thin membranes it is impossible to place a flow sensor in the system. Therefore it is necessary to look at the interaction between the gas flow and the mechanics of the membrane. Recognizing that potential difference in a RC circuit is equivalent to pressure difference in the system, it becomes clear that the pressure difference over the membrane is equivalent to voltage over a resistor, therefore:

$$
\Delta p_r = \frac{i\Omega R_h C_h}{1 + i\Omega R_h C_h} p_0(\Omega) \quad (4.63)
$$

Looking at the transfer function of this system, Fig. 4.7, it can be seen that below \( \frac{1}{\tau_h} \), the acoustic cutoff frequency, \( p_r \), is a damping force while above the acoustic cutoff it is a spring force.

Recognizing that there is a force added due to the acoustic resistance, the forced response can be corrected. The approach used here is to write Eq. 4.30 in the frequency domain and add the force from the resistance (Eq. 4.63). This gives in the frequency domain:

$$
\eta_{0n}(\Omega) = \frac{2\pi p_0(\Omega)}{\sqrt{\pi \rho h_0 a_0}} \left( \frac{i\Omega R_h C_h}{1 + i\Omega R_h C_h} \right) \frac{2\pi p_0(\Omega)}{\omega_0^2 - \Omega^2} \quad (4.64)
$$

Rewriting and multiplying by the mode shape (Eq. 4.25) gives the solution for the deflection in the frequency domain:

$$
\omega(\Omega) = \frac{2\sqrt{2} I_0(\gamma_0 r/a)}{\rho h_0 (1 + i\Omega R_h C_h)(\omega_0^2 - \Omega^2) a I_1(\gamma_0) p_0(\Omega)} \quad (4.65)
$$

This has been plotted in Fig. 4.8 along with the compressed film effect. The compressed film effect only plays a role at high frequencies, the cutoff is approximately at \( \sigma = 10.1342 \) [37]. If this cutoff is sufficiently far away from the acoustic cutoff, the frequency response can be used to find the force from the acoustic circuit that is acting on the membrane. This should give a result similar to Fig. 4.7.

### 4.4.1. STEP RESPONSE

Like the electrical equivalent of this system, the step response can be used as a tool to characterize the behavior of the system. If the acoustic cut-off frequency is much smaller than the compression cut-off frequency,
Amplitude and phase shift of the center deflection of a membrane with an acoustic resistance and capacitance.

Figure 4.8: Amplitude and phase of the center deflection of a circular monolayer graphene membrane of 4 µm diameter and a pretension of $n_0 = 0.1$ N m$^{-1}$. The cavity is 100nm deep and the pressure is atmospheric, the properties of air are used. The acoustic compliance at low frequencies for this system is $C_h = -1.2151 \times 10^{-27}$ m$^3$ Pa$^{-1}$ and the resistance is tuned to $R_h = 2.9117 \times 10^{24}$ Pa s m$^{-3}$. This gives an acoustic cutoff frequency of $f_{ac} = 45$ Hz. There is a second cutoff due to the compressed film effect, increasing the stiffness above the cutoff frequency. Using the theory by Bao [37] the cutoff is at $f_{sq} = 1.3042 \times 10^7$ Hz, where $\sigma = 10.1342$. Each cutoff results in an increase of stiffness which results in a step on the amplitude around this cutoff. There is a 'bump' in the phase near the acoustic cutoff, this is due to the transition from a damping force to a spring force. The squeeze film cutoff does not show an effect in the phase because all damping in the squeeze film is ignored.
4.5. **OTHER APPLICATIONS OF THE ACOUSTIC CIRCUIT MODEL**

The main goal of acoustic circuit modeling was to predict the pressure equalization behavior of the compressed film transducer. However this model shows more opportunities for application:

- It can be used to measure permeability. This was previously only possible using the blister test which is on a very slow time scale [13]. Now it is possible to measure the permeability on a timescale of milliseconds or even microseconds, dependent on how many times per second the deflection can be measured. It is possible to measure the permeability of single-layered MoS$_2$ for the first time and to characterize the UV oxidative etching process which makes graphene membranes permeable.

- It can be used to build a partial pressure sensor. Since the resistance can be dependent on the properties of a gas, the pressure equalization will be different for each gas. Measuring where the cut-off frequency lies for each gas and the change in stiffness means that one can use it to measure the partial pressure of the gas.
4.6. CONCLUSION
A model has been developed to predict the pressure equalization behavior of the compressed film sensor, using a simple resistor capacitor circuit. This system has a cut-off frequency: at lower frequencies there is pressure equalization and above this frequency there is no pressure equalization. By careful design the resistance and capacitance can be optimized for good compressive behavior and high bandwidth. It was found that the model can be used for different applications, like measuring permeability and creating a partial pressure sensor.
DISCUSSION AND CONCLUSION

In this chapter the compressed film transducer, the induced tension transducer and silicon transducers will be compared to each other. After this the conclusions will be drawn and recommendations are made for further research.

5.1. Comparison between Graphene Based Transducers

The applications of both transducer types are different, the compressed film transducer can only be used as an absolute pressure sensor and the induced tension transducer as a pressure difference sensor. The induced tension transducer can be used as an absolute pressure sensor as well, if the pressure in the cavity is very well known (vacuum for example).

It is not sure what the limit is on the pressure range that the compressed film transducer can measure; finding a finite element solution at high pressures is difficult due to the fact that the eigenfrequencies come close together. It is expected that the inertia effect from the gas will limit the measurable pressure range. The induced tension will have a far lower pressure range, since it is limited by the pressure difference the graphene membrane can handle without delamination from the substrate.

The membrane of the compressed film transducer has to be made permeable, such that the specific time of the acoustic circuit has the desired value. This means that an extra UV-oxidation step of the graphene membrane is needed. On the other end, the induced tension transducer will need hermetically sealed membranes. Currently it is hard to obtain wafer-scale graphene with such quality. Also gas molecules can diffuse through the silicon dioxide substrate [12, 13], equalizing the pressure difference. This could be solved by using another type of substrate.

The compressed film transducer can be made into a linear sensor for low absolute pressures, this pressure range can be increased at the cost of some sensitivity. The linearity makes the sensor suitable for application in vacuum systems, for measurement of higher pressures additional measures might be necessary. The induced tension transducer never has a linear relationship between resonance frequency and pressure.

The bandwidth of the compressed film sensor is limited by the time it takes for the pressure inside the cavity to become equal to the outside. If a pressure equalization channel or a permeable membrane is used the model in chapter 4 can be used to evaluate this behavior. The induced tension transducer’s bandwidth will be limited by the measurement time.

The costs and size of both transducers are comparable, since similar designs are being used. The absolute limit on the resolution is the sensor noise, see Fig. 5.1. It can be seen that both transducers can achieve a very high resolution when limited by thermal noise, therefore it is expected that noise from the electronic circuit will limit the resolution of both transducers.

A summary of this discussion can be seen in table 5.1.

5.2. Comparison between Graphene Devices and Current Silicon Technologies

The big question of this thesis is whether graphene devices can exceed the performance of current silicon technologies. Several sections in this thesis have compared a performance specification of a graphene de-
Table 5.1: Comparison of different characteristics between the compressed film pressure transducer and the induced tension transducer.

<table>
<thead>
<tr>
<th></th>
<th>Compressed Film</th>
<th>Induced Tension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Application</td>
<td>Absolute Pressure Sensor</td>
<td>Pressure Difference Sensor</td>
</tr>
<tr>
<td>Pressure range</td>
<td>0 to $&gt; 10^6$ Pa</td>
<td>0 to $10^6$ Pa</td>
</tr>
<tr>
<td>Membrane</td>
<td>Permeable</td>
<td>Hermetically sealed</td>
</tr>
<tr>
<td>Linearity</td>
<td>Low Absolute Pressures</td>
<td>Never</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>Limited by pressure equalization or measurement time</td>
<td>Limited by measurement time</td>
</tr>
<tr>
<td>Costs</td>
<td>$\approx 0.40$</td>
<td>$\approx 0.40$</td>
</tr>
<tr>
<td>Size</td>
<td>Diaphragm $&lt; 20\mu$m possible</td>
<td>Diaphragm $&lt; 20\mu$m possible</td>
</tr>
<tr>
<td>Resolution</td>
<td>If limited by sensor noise: see Fig. 5.1</td>
<td>If limited by sensor noise: see Fig. 5.1</td>
</tr>
</tbody>
</table>

Figure 5.1: Comparison between the resolution of 4 different graphene resonant transducers limited by sensor noise. Note that the compressed film transducer measures absolute pressures while the induced tension transducer measures pressure differences. This means that for this comparison the induced tension transducer has vacuum as reference pressure and the pressure on the horizontal axis is then the absolute pressure. For comparison, the device from section 2.5.1 has a limit of $1 \times 10^{-2}$ Pa around atmospheric conditions.
vice to current silicon technologies. Whether the graphene devices are expected to meet or exceed current technologies is summarized in table 5.2. It can be seen that on almost all the performance specifications discussed in this thesis the graphene devices are expected to meet or exceed the performance of current silicon technologies. The measurable range is classified as meeting the current performance of silicon devices, this is because the limit on the maximum pressure for the compressed film device is not known and the induced tension device has a limited measurable range compared to existing technologies.

For some performance specifications it is not yet known whether graphene devices will meet or exceed current technologies. These answers will be provided when

- A complete design is made of the electrical readout circuit (Power consumption).
- The temperature dependence is either measured or simulated using finite element modeling.
- A suitable technique for transferring graphene on wafer scale is found, then the fabrication yield and variations between devices can be evaluated.
- Cost of packaging, assembly, test and the ease of calibration will become clear in a later stage of the design process. Currently the devices that have been discussed are on concept-level.
- Lifetime and reliability has to be evaluated. Several classical MEMS failure mechanisms can be the cause of reliability issues such as stiction of the membrane to the back surface of the cavity and dielectric charging and breakdown [66]. Particle contamination is expected to be a major problem in these kind of sensors, as the membranes are very thin small particles will have a significant effect on the mass and electric properties [67].

5.3. Conclusion

Two concepts have been developed for pressure sensors using graphene as a resonant transducer, the compressed film sensor can measure absolute pressures and the induced tension transducer can measure pressure differences. Both concepts have the potential to be cheaper per unit, have a smaller size and achieve much higher accuracies and resolutions than current silicon-based technologies.

The compressed film transducer is considered a more promising concept over the induced tension transducer. This is because the compressed film transducer is linear in a certain pressure range, while the induced tension transducer is always nonlinear. Also the hermetic sealing needed for the induced tension transducer can be problematic, while compressed film transducers need pressure equalization which makes them much easier to realize. Also the compressed film transducer can measure a much larger pressure range, since there are no large tensions induced in the graphene membrane.

5.4. Recommendations

In order to complete table 5.2:

- More research is needed to find out how the origin of damping and the pretension in the membrane. Currently monolayered membranes are limited to a quality factor of approximately 50. The accuracy and resolution of the devices can be improved significantly if the quality factor can be improved. Knowing how the pretension and quality factor are affected or can be improved will help to design a good wafer-scale transfer process for graphene. Variations between the quality factor and the pretension are expected to be the major causes of variation between devices after fabrication, controlling these factors will be essential to meet the performance of current silicon technologies.
- A temperature-dependent model of the substrate and the graphene membrane has to be made in order to see if the temperature sensitivity is lower than current silicon technologies.
- The electrical readout circuit has to be designed carefully in order to obtain a device that has low power consumption and low noise levels.
Table 5.2: Expected performance of graphene devices compared to current silicon technologies

<table>
<thead>
<tr>
<th></th>
<th>Meet/exceed/unknown</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost per unit</td>
<td>Exceed</td>
<td>Section 2.5.4</td>
</tr>
<tr>
<td>Size</td>
<td>Exceed</td>
<td>Section 2.5.5</td>
</tr>
<tr>
<td>Signal-to-noise ratio</td>
<td>Meet</td>
<td>Fig. 5.1, section 2.5.3 and 3.3.2</td>
</tr>
<tr>
<td>Accuracy and resolution</td>
<td>Meet</td>
<td>Fig. 5.1, section 2.5.3 and 3.3.2</td>
</tr>
<tr>
<td>Measurable range</td>
<td>Meet</td>
<td>Section 2.5.6 and 3.3.3</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>Exceed</td>
<td>Section 2.5.7 and 3.3.4, chapter 4, [57]</td>
</tr>
<tr>
<td>Power consumption</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Temperature dependence</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Sensitivity to mechanical disturbances</td>
<td>Exceed</td>
<td>Section 2.5.8</td>
</tr>
<tr>
<td>Sensitivity to other disturbances</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Lifetime and reliability</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Fabrication yield</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Variations between devices after manufacturing</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Cost of packaging, assembly, test</td>
<td>Unknown</td>
<td>–</td>
</tr>
<tr>
<td>Ease of Calibration</td>
<td>Unknown</td>
<td>–</td>
</tr>
</tbody>
</table>
In this chapter the modeling of the membrane in COMSOL multiphysics is discussed. A special feature in the software allows the user to implement their own equations, which is discussed in the first section. As closure of this chapter, the model will be verified by comparing to analytical equations for certain situations.

A.1. COEFFICIENT FORM PDE MODELING

COMSOL allows the user to define their own equations in a physics package called Coefficient Form PDE (PDE stands for partial differential equation). The coefficient form PDE has the following form:

\[
e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial^2 u}{\partial t^2} + \nabla \cdot (-c \otimes \nabla u + a u + \gamma) + \beta \cdot \nabla u + a u = f
\]  

(A.1)

The nomenclature for this equation is:

- \(e_a\) Mass coefficient
- \(d_a\) Damping coefficient
- \(c\) Diffusion coefficient
- \(a\) Conservative Flux Convection Coefficient
- \(\gamma\) Conservative Flux Source
- \(\beta\) Convection Coefficient
- \(a\) Absorption Coefficient
- \(f\) Source Term, this can also be defined on only a part of the domain by the source node

\(\nabla\) is a vector with first-order derivative operators

\(u\) is a vector with all the dependent variables

By matching the coefficients one can implement any equation. Suppose when the 2D model without temperature effects (Eqs. 3.13 – 3.13d) is to be implemented, then coefficients are:

- \(e_a = \rho \text{sqrt}\)
- \(d_a = 0\) if damping is omitted, or replaced with a damping coefficient.
- \(c\) is a 4 by 4 by 2 by 2 tensor with \(c(1, 1, m, n) = \left[ \begin{array}{cc} n_x & n_y \\ n_y & n_x \end{array} \right] \) and zero anywhere else.
- \(a\) is a 4 by 4 matrix with \(a(2, 2) = a(3, 3) = a(4, 4) = 1\) and zero anywhere else.
- \(f\) is a 4 by 1 vector where the right-hand sides of Eqs. 3.13 – 3.13d are substituted.
• All other coefficients are put to zero.

This implementation can be done for any equation, even for a fourth order one such as Eq. 3.24a. This can be achieved by splitting the fourth order equation into two second order equations. Suppose one wants to model the simple plate equation:

\[ D \nabla^4 w = P \]  

(A.2)

This can be split into a form which is easily implemented in the coefficient form PDE:

\[ \nabla^2 w = \Omega \]
\[ D \nabla^2 \Omega = P \]

(A.3)

### A.1.1. Axisymmetric Implementation

The axisymmetric 1D model (Eqs. 3.20a – 3.20b) can be implemented in a similar manner. However, one must be careful with implementing radial-symmetric equations in COMSOL, since the definition of the second order gradient is different from the mathematical definition. In COMSOL the second order gradient is defined as:

\[ \nabla^2 w = \frac{\partial^2 w}{\partial r^2} \]

(A.4)

while the mathematical definition is:

\[ \nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \]

(A.5)

This problem is easily corrected, for example if one wants to model a linear axisymmetric circular membrane:

\[ n_0 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) = P \]

(A.6)

One can use the following coefficients to obtain the correct results:

• \( c = n_0 \)
• \( \beta = -\left(\frac{1}{r}\right) n_0 \)
• \( f = P \)

### A.2. Additional Modeling Results

This section presents additional results from the models presented in section 3.1.

#### A.2.1. AFM Profiles

This simulation is most relevant to measurements performed by Bunch et. al. [13]. In this paper, graphene membranes are pressurized and measured using atomic force microscopy (AFM). Interesting observation is that the graphene membranes appear to adhere to the sidewalls, it is stated that the cause of the adherence is the strong van der Waals interaction between the sidewalls and the membrane. Also it is mentioned that a 17nm dip along the edges is observed.

One of the downsides of AFM measurements is that a force is always applied to the membrane, meaning that the measurement will give a different deflection profile than the pressurized membrane shape. This is demonstrated in Fig. A.1, this particular simulation shows that the measurement in the center is approximately 30nm off and a 5nm dip is observed near the edges. These effects are a result from performing the measurement and are not present in the actual pressurized deflection profile. This disagreement leads to the following recommendations:

• During measurements, different forces must be applied. As shown in Fig. A.2 it can clearly be seen that the edge effects are force-dependent, measurement have been performed showing very similar behavior (Fig. A.3). This gives some idea of the measurement error and whether the dip along the edges is due to adhesion or a result of the measurement.

• The force that the AFM tip exerts on the membrane should be well known, this allows for comparison between the measurement and finite element modeling. This gives an idea of the measurement error and the adhesion effects that are present.
Figure A.1: Result from the 2D model regarding AFM measurements. If the AFM is measuring with along a line through the center, one would obtain the profile in the black line. However the actual deflections from the finite element modeling (colored lines) show that the profile that is obtained never represents the actual deflection of the membrane. A 4µm diameter monolayered graphene membrane was implemented with a pretension of 0.23N m\(^{-1}\). A pressure difference of 10000Pa is applied in the upward direction and the AFM tip presses down with a 10nN force.

Figure A.2: Same system as in Fig. A.1, but different forces are applied. The dependence of the edge effects on the force gives a better idea of the real deflection shape.
A.2.2. Force-deflection curves
Mechanical properties of a membrane or plate can be extracted by applying a point force in the center [10, 12, 29, 31, 51–53], the response to such a force has been discussed in section 3.1.5. One simulation that was performed is to see how these force deflection curves change when a pressure difference is applied, as plotted in Fig. A.4. Above certain pressure differences the force deflection curves become no longer symmetric. When nonsymmetric curves are found in measurements, one should exclude the possibility that a pressure difference is the cause. Another cause of nonsymmetric curves is a water meniscus between the tip and the membrane [31].

A.2.3. Eigenfrequency as function of electrostatic force
If a voltage is applied between a graphene membrane and the back gate, the graphene membrane will be attracted to the gate. The pressure from the electrostatic force can be expressed as:

\[ p_V = \frac{\varepsilon_0}{2} E^2 \]  

(A.7)

where \( p_V \) is the electrostatic pressure, \( \varepsilon_0 \) the permittivity of free space and \( E \) the electric field. When the membrane deflects the electric field changed locally, however the voltage on the conductors remain constant as function of position. This means that one can write for the electrostatic pressure:

\[ p_V(x, y) = \frac{\varepsilon_0}{2} \frac{V^2}{(g_0 - w(x, y))^2} \]  

(A.8)

This can be implemented as a force in the finite element model. Note that this force is dependent on position and therefore it behaves as a nonlinear spring. One experiment that can be performed is increasing the DC voltage and see how the deflection and eigenfrequency change.

A.3. Discussion
Figure A.4: Force deflection curves for pressurized membranes. It can be seen that with increasing pressures the force deflection curves become less symmetric.

Figure A.5: Deflection of the center as function of applied voltage for monolayered graphene membranes of different diameters. A pretension of 0.23 N m$^{-1}$ and a gap size of 300nm are assumed. Solutions below approximately 200nm deflection are not found, because here the pull-in voltage is exceeded.
Figure A.6: Change in eigenfrequency as function of pressure. Also plotted is the line that represent spring hardening by the geometric nonlinearity, calculated by excluding the electrostatic force in the linearization. The difference between the spring hardening and the actual behavior gives the red line. This represents the spring softening due to the electrostatic force. It can be seen that in the next voltage step the spring softening would be larger than the spring hardening, this means that the pull-in voltage is exceeded.
RESPONSE OF A MEMBRANE TO A CENTRAL POINT LOAD

Finding an analytical solution to the situation where a circular membrane is subjected to a central point force is quite complicated. Using the assumption that the bending rigidity can be neglected will lead to non-singularities in the solution for example. This section will use a slightly different formulation of the Föppl Von Kàrmàn equations and follows the derivations by Wan et. al. [49] closely.

PRETENSION

It is assumed that the plate is circular and isotropic with flexural rigidity \( D = Eh^3/12(1 - v^2) \) and under a pretension \( n_0 \). A load in the form of a pressure distribution \( q(r) \) is acting on the membrane. The equations are written in the form [45, 49]:

\[
-D \left( \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} + n_r \frac{dw}{dr} \right) = -q(r) \quad (B.1a)
\]

\[
\frac{d}{dr} (n_r + n_t) + \frac{Eh}{2r} \left( \frac{dw}{dr} \right)^2 = 0 \quad (B.1b)
\]

\[
n_t = \frac{d}{dr} (rn_t) \quad (B.1c)
\]

Here \( q(r) \) is related to the load function \( \Phi(r) \) as:

\[
\Phi(r) = \frac{1}{r} \frac{d}{dr} (rq(r)) \quad (B.1d)
\]

The following boundary conditions are used:

- \( w = 0 \) at \( r = a \)
- \( \frac{dw}{dr} = 0 \) at \( r = 0 \) and \( r = a \)
- \( n_t - vn_r = 0 \) at \( r = a \)

It is assumed that the membrane stress is constant over the whole membrane, \( n_r = n_t \) and that the gradients of the deflection are small. The first assumption makes that Eq. B.1c is automatically satisfied. Since \( \frac{dw}{dr} \approx 0 \) Eq. B.1b is also satisfied, leaving Eq. B.1a as the only governing equation. However since there is still a nonlinearity in the equations which cannot be entirely satisfied, therefore the following equation is written for the concomitant stress \( n_m \):

\[
n_m = \frac{Eh}{2a^2(1 - v^2)} \int_0^a \left( \frac{dw}{dr} \right)^2 r dr \quad (B.2)
\]
Now the following dimensionless quantities are defined for the analysis:

\[
\beta = \sqrt{\frac{na^2}{D}} \quad \text{(B.3a)}
\]
\[
\xi = \frac{r}{a} \quad \text{(B.3b)}
\]
\[
W = \frac{w}{h} \quad \text{(B.3c)}
\]
\[
\theta = \frac{a \frac{dw}{dr}}{h} \quad \text{(B.3d)}
\]
\[
Q = \frac{a^3}{Dh} q(r) \quad \text{(B.3e)}
\]
\[
\varphi = \frac{Fa^2}{2\pi Dh} \quad \text{(B.3f)}
\]
\[
\zeta = \frac{c}{a} \quad \text{(B.3g)}
\]

which results in the following set of equations:

\[
\xi^2 \frac{d^2 \theta}{d\xi^2} + \xi \frac{d\theta}{d\xi} - (1 + \beta^2 \xi^2) \theta = \xi^2 Q(\xi) \quad \text{(B.4a)}
\]
\[
\beta^2_m = 6 \int_0^1 \theta^2 \xi d\xi \quad \text{(B.4b)}
\]

Where \( \beta = \beta^2_m + \beta^2_0 \). A solution is sought in the form of a homogeneous (not dependent on the load function) and a particular solution (dependent on the load function).

The homogeneous solution is given in the form:

\[
\theta_h = C_0 I_1(\beta \xi) + C'_0 K_1(\beta \xi) \quad \text{(B.5)}
\]

where \( I_1 \) is modified Bessel function of the first kind and \( K_1 \) of the second kind. Now an external load \( F \) is applied via an distributed ring of radius \( c \). The load function now becomes:

\[
\Phi(r) = \begin{cases} 
0 & \text{for } r < c \\
F \delta(r) & \text{for } c \leq r \leq a 
\end{cases} \quad \text{(B.6)}
\]

Where \( \delta(r) \) is the Dirac delta function, this means that:

\[
Q(\xi) = \begin{cases} 
0 & \text{for } r < c \\
\frac{\varphi}{\xi} & \text{for } c \leq r \leq a 
\end{cases} \quad \text{(B.7)}
\]

Now substitute this into Eq. B.4a gives:

\[
\theta = \begin{cases} 
C_3 I_1(\beta \xi) + C_4 K_1(\beta \xi) & \text{for } \xi < \zeta \\
C_5 I_1(\beta \xi) + C_6 K_1(\beta \xi) - \frac{\varphi}{\beta^2 \xi} & \text{for } \zeta \leq \xi \leq 1 
\end{cases} \quad \text{(B.8)}
\]

The boundary conditions are:

- \( \theta = 0 \) for \( \xi = 0 \)
- \( \theta = 0 \) for \( \xi = 1 \)
- \( W = 0 \) for \( \xi = 1 \)
- \( \theta \) and \( \frac{d\theta}{d\xi} \) are continuous at \( \xi = \zeta \)
These boundary conditions give the following equations for the constants:

\[
C_3 = -\frac{1}{\beta^2} I_1(\beta) \times \\
\frac{-I_1(\beta)\{K_0(\beta) + K_2(\beta)\} - K_1(\beta)\{I_0(\beta) + I_2(\beta)\} + \frac{2}{\beta^2} I_1(\beta)K_1(\beta) - I_1(\beta)K_1(\beta)}{I_1(\beta)\{K_0(\beta) + K_2(\beta)\} + K_1(\beta)\{I_0(\beta) + I_2(\beta)\}}
\]

\(\text{Eq. B.9a}\)

\[
C_4 = 0
\]

\(\text{Eq. B.9b}\)

\[
C_5 = -\frac{1}{\beta^2} I_1(\beta) - \frac{1}{\beta^2} I_1(\beta) \times \\
\frac{K_1(\beta)\{I_0(\beta) + I_2(\beta)\} + \frac{2}{\beta^2} I_1(\beta)K_1(\beta)}{I_1(\beta)\{K_0(\beta) + K_2(\beta)\} + K_1(\beta)\{I_0(\beta) + I_2(\beta)\}}
\]

\(\text{Eq. B.9c}\)

\[
C_6 = -\frac{1}{\beta^2} I_1(\beta) \times \\
\frac{I_0(\beta) + I_2(\beta)}{I_1(\beta)\{K_0(\beta) + K_2(\beta)\} + K_1(\beta)\{I_0(\beta) + I_2(\beta)\}}
\]

\(\text{Eq. B.9d}\)

The blister profile is given by:

\[
\theta = \begin{cases} 
C_3 \frac{\phi}{\beta} [I_0(\beta) - I_0(\beta)] + W_\xi & \text{for } \xi < \xi \\
\frac{\varphi}{\beta} \left[ C_5[I_0(\beta) - I_0(\beta)] - C_6[K_0(\beta) - K_0(\beta)] - \frac{\log \xi}{\beta} \right] & \text{for } \xi \leq \xi \leq 1
\end{cases}
\]

\(\text{Eq. B.10}\)

The deflection at \(\xi = \xi\) is given by:

\[
W_\xi = \frac{\varphi}{\beta} \left[ C_3[I_0(\beta) - I_0(\beta)] - C_6[K_0(\beta) - K_0(\beta)] - \frac{\log \xi}{\beta} \right]
\]

\(\text{Eq. B.11}\)

The deflection of the center at \(\xi = 0\):

\[
W_0 = C_3 \frac{\varphi}{\beta} [I_0(\beta) - I_0(\beta)] + W_\xi
\]

\(\text{Eq. B.12}\)

Substituting Eq. B.8 into Eq. B.4b gives for the applied load:

\[
\varphi = \frac{\beta m}{\sqrt{6}} \left[ \frac{1}{\frac{f_{p1}(\beta \xi) - f_{p1}(0)}{[f_{p2}(\beta) - f_{p2}(\beta \xi)]}} \right]
\]

\(\text{Eq. B.13}\)

Where the functions can be evaluated as:

\[
f_{p1}(x) = C_3^2 \frac{x^2}{\beta^2} \left[ \left( 1 + \frac{1}{x^2} \right) I_1(x)^2 - \left( I_0(x) - \frac{I_1(x)}{x} \right)^2 \right]
\]

\(\text{Eq. B.14a}\)

\[
f_{p2}(x) = C_6^2 \frac{x^2}{\beta^2} \left[ \left( 1 + \frac{1}{x^2} \right) I_1(x)^2 - \left( I_0(x) - \frac{I_1(x)}{x} \right)^2 \right] + \frac{2C_5C_6}{\beta^2} \left[ K_0(x) - K_0(\beta) \right] + \frac{1}{\beta^2} \log x - \frac{2}{\beta^2} [C_5I_0(x) - C_6K_0(x)]
\]

\(\text{Eq. B.14b}\)

Now the equations have to be modified by taking the limit \(\xi \to 0\). For this limit the following profile can be found from Eq. B.10:

\[
W = \frac{\varphi}{\beta^2} \left[ \frac{1 - \beta K_1(\beta)}{\beta I_1(\beta)} \right] [I_0(\beta \xi) - I_0(\beta)] - [K_0(\beta \xi) - K_0(\beta)] - \log \xi
\]

\(\text{Eq. B.15}\)
with the deflection of the center given by:

\[ W_0 = \frac{q}{\beta^2} \left( \frac{1 - 6K_1(\beta)}{\beta I_1(\beta)} \right) \left[ 1 - \frac{I_0(\beta)}{\beta^2} \right] - \log \frac{2}{\beta} + \gamma + K_0(\beta) \]  

(B.16)

where \( \gamma \) is the Euler-Mascheroni constant (\( \gamma \approx 0.577216 \)). The load applied to the membrane can be evaluated as:

\[ q = \sqrt{\frac{F_{mn}}{6(f_{cp}(\beta) - f_{st}(\beta))}} \]  

(B.17)

where the functions are evaluated with:

\[ f_{cp}(x) = \frac{C_7^2 x^2}{\beta^2} + \left[ 1 + \frac{1}{x^2} \right] I_1(x)^2 - \left( I_0 - \frac{I_1}{x} \right)^2 \] + \[ \frac{C_7^2}{\beta^2} + \left( \frac{K_1(\beta)}{I_1(\beta)} - \frac{K_1(\beta)}{I_1(\beta)} \right) \left( K_0(x)^2 - \frac{2K_0(x)K_1(x)}{x} \right) \] + \[ \frac{1}{\beta^4} \log x - \frac{2}{\beta^3} [C_7I_0(x) - C_8K_0(x)] \] + \[ \frac{2C_7C_8}{\beta^2} \left[ x^2 \left( I_0(x)K_0(x) + I_1(x)K_1(x) - xI_1(x)K_0(x) \right) \right] \]  

(B.18a)

\[ f_{st}(x) = \frac{2}{\beta^2} \left( \frac{1 - \gamma}{2} + \frac{\log 2}{2} - \frac{1 - xK_1(x)}{xI_1(x)} \right) \]  

(B.18b)

And the constants are defined as:

\[ C_7 = \frac{1}{\beta^2} \left[ \frac{K_1(\beta\xi) - \frac{K_1(\beta)}{\xi}}{I_1(\beta)K_1(\beta\xi) - I_1(\beta\xi)K_1(\beta)} \right] \]  

(B.19a)

\[ C_8 = \frac{1}{\beta^2} \left[ \frac{I_1(\beta\xi) - \frac{I_1(\beta)}{\xi}}{K_1(\beta)I_1(\beta\xi) - K_1(\beta\xi)I_1(\beta)} \right] \]  

(B.19b)

When \( W_0 \) is small Eq. B.17 approaches asymptotically to:

\[ q = \frac{1}{g_s(\beta_0)} W_0 \]  

(B.20)

with:

\[ g_s(x) = \frac{1}{\beta_0^2} \left( \frac{1 - \beta_0K_1(\beta_0)}{\beta_0I_1(\beta_0)} \right) \left[ 1 - \frac{I_0(\beta_0)}{\beta_0^2} \right] - \log \frac{2}{\beta_0} + \gamma + K_0(\beta_0) \]  

(B.21)

Eq. B.20 becomes \( q/\beta_0 = 8 \) when the prestress is zero, this is the classical solution for a pure bending plate under a central point load. To find the asymptote for the limit \( D \to 0 \) is difficult, because of the singularities in the functions. Therefore in this thesis \( D \) is approximated by a equivalent thickness, to obtain a very small value. Eqs. B.20 – B.21 can be used to approximate the linear part of a force-displacement curve.

**ELASTIC RESPONSE**

In this section the elastic response of a membrane subjected to a point load in the center is derived. The derivation of this formula is done by Schwerin [68] by introducing the following variables:

\[ V = \frac{A}{2} \left( \frac{32\pi^2 E^2 a^2}{r^2} \right) \]  

(B.22a)

\[ \xi = \frac{r^2}{a^2} \]  

(B.22b)

This results in a differential equation of the simple form:

\[ \frac{d^2V}{d\xi^2} = \frac{1}{V^2} \]  

(B.23)
By integrating this equation one obtains the following solution for $\xi$:

$$\xi = C_1(\phi - \sin \phi) + C_2$$  \hspace{1cm} (B.24)

where:

$$\phi = 2 \arccot \sqrt{-\frac{2\sqrt{C_1^2}}{v - 1}}$$  \hspace{1cm} (B.25)

For the special case where $\nu = 1/3$ and no prestress the following particular solution is obtained:

$$V = -\frac{9}{2} \xi^2$$  \hspace{1cm} (B.26)

In terms of the physical parameters this can be written to [50]:

$$\frac{w_0}{a} = \sqrt{\frac{3F}{\pi Eh}}$$  \hspace{1cm} (B.27)

Komaragiri et al. [50] propose an approximate solution for this equation by using a correction factor dependent on the Poisson ratio:

$$\frac{w_0}{a} = f(\nu) \sqrt[3]{\frac{F}{Eah}}$$  \hspace{1cm} (B.28)

Where the correction factor $f(\nu)$ is defined by the approximate formula:

$$f(\nu) \approx 1.0491 - 0.1462 \nu - 0.15827 \nu^2$$  \hspace{1cm} (B.29)

**Total response**

Since these restoring forces arise independent of each other, one can use the principle of superpositions to add the restoring forces. The stiffness response is usually characterized by a force-displacement curve, for this purpose Eq. B.20 is rewritten to:

$$F = \frac{2\pi D}{a^2 g_s(\beta_0)} w_0$$  \hspace{1cm} (B.30)

where $g_s$ is defined by Eq. B.20. Eq. B.28 is rewritten to:

$$F = \frac{Eh}{(f(\nu))^3 a^2} w_0^3$$  \hspace{1cm} (B.31)

The total response to a point force applied in the center of a circular membrane can now be calculated as:

$$F = \frac{2\pi D}{a^2 g_s(\beta_0)} w_0 + \frac{Eh}{(f(\nu))^3 a^2} w_0^3$$  \hspace{1cm} (B.32)

This is a cubic polynomial which can be used to fit a curve to measured results. Assuming that all physical parameters are known, except $n_0$ and $E$, this formula can be used to derive these properties from a force-displacement measurement. A minimizing algorithm commonly used in engineering optimization problems can be used to find the parameters closest to the measured results.


96 BIBLIOGRAPHY


