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BUCKLING OF SHELLS
WITH GENERAL RANDOM IMPERFECTIONS

by

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ABSTRACT

The paper is concerned with the effect of random axisymmetric and asymmetric imperfections on the buckling of circular cylindrical shells under axial compression. The initial imperfections are considered as random functions of the axial and the circumferential coordinates. This is done by expanding them in terms of the buckling modes of the associated perfect structures, and then treating the Fourier coefficients as random variables. In contrast to earlier works the probabilistic properties (the mean function and the autocorrelation function) are not assumed but the mean vector and the variance-covariance matrix of the Fourier coefficients are calculated from experimental measurements of the shell profiles. The Fourier coefficients needed for the application of the Monte Carlo Method are simulated by a special numerical procedure. Thus a large number of shells is "created". For each shell a deterministic analysis of buckling stress evaluation is carried out. Finally, the reliability function representing the probability (i.e. fraction of an ensemble) of the buckling stress exceeding the specified stress is calculated. The reliability function permits to evaluate the allowable stress for the whole ensemble of shells produced by a given manufacturing process, defined as the stress level for which the required reliability is attained. The paper represents an extension of the approach given in Ref. [1] to asymmetric imperfections.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Abstract</strong></td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. A case capable of exact solution</td>
<td>2</td>
</tr>
<tr>
<td>3. Probabilistic properties of initial imperfections</td>
<td>4</td>
</tr>
<tr>
<td>4. The Monte Carlo Method</td>
<td>6</td>
</tr>
<tr>
<td>5. Reliability of shells with axisymmetric imperfections</td>
<td>8</td>
</tr>
<tr>
<td>6. Reliability of shells with asymmetric imperfections</td>
<td>9</td>
</tr>
<tr>
<td>7. Conclusions</td>
<td>11</td>
</tr>
<tr>
<td>8. References</td>
<td>12</td>
</tr>
<tr>
<td>Tables 1 - 2</td>
<td>14</td>
</tr>
<tr>
<td>Figures 1 - 14</td>
<td>16</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The application of the available theories of imperfection sensitivity to buckling load calculations requires the prior knowledge of the initial imperfections that are present in the structure under consideration. The initial imperfections, however, vary from case to case. This is due to the fact that both shape and magnitude of the initial imperfections derive from the chosen fabrication process, itself subject - by its very nature - to a large number of random variables.

Consider, for example, a cylindrical shell manufactured by electroplating from pure copper, of which a suitable stock may be visualized to be available. Due to uncontrollable random factors involved in the manufacturing process, each realization of the shell will have a different initial shape which cannot be predicted in advance. As illustrated by the two examples shown in Figure 1, the two shells produced by the same manufacturing process \[2\] have totally different imperfection profiles. It is intuitively obvious that even when tested in the same machine these shells would have different buckling loads. This example clearly indicates that for the investigation of the imperfection sensitivity of the buckling load of axially compressed cylindrical shells it makes sense to resort to probabilistic methods.

In this work we treat axisymmetric as well as general nonsymmetric initial imperfections as random functions and solve the resulting stochastic stability problem by the Monte Carlo Method. This work should be considered as the first step in a combined theoretical, experimental and numerical effort towards the development of a probabilistic design code for axially compressed cylindrical shells with general nonsymmetric random initial imperfections. Contrary to earlier investigations the statistical information needed in the analysis will not be assumed but obtained from existing initial imperfection measurements.
2. A CASE CAPABLE OF EXACT SOLUTION

In order to illustrate the stochastic analysis of the initial imperfection sensitivity let us first consider an example which can be solved in "closed" form. In particular we are considering a long isotropic shell with an axisymmetric imperfection in the shape of the classical axisymmetric buckling mode. The nondimensional critical load versus imperfection amplitude relationship, first obtained by Koiter in 1945\[^3\] in his general theory, is shown in Figure 2. Now we shall treat the initial imperfection amplitude $\xi$ as a random variable with the given probability density function shown in Figure 3. The buckling load becomes then also a random variable. We are interested in calculating the reliability of the shell. The reliability is here defined as the probability that the random buckling load $\Lambda$ will be greater than the specified buckling load $\lambda$. That is

$$ R (\lambda) = \text{Prob} (\Lambda > \lambda) \quad (1) $$

But from Figure 3 the buckling load $\Lambda$ exceeds the specified value $\lambda$ if the initial imperfection $\xi$ lies between the two values $+\xi_1$ and $-\xi_1$. Thus reliability is represented by the shaded area and it clearly depends on the shape of the probability density curve. That is

$$ R (\lambda) = \int_{-\xi_1}^{\xi_1} f_X (\xi) \, d\xi \quad (2) $$

The reliability function, shown in Figure 4, can be used directly for design purposes. Suppose we demand that the reliability must exceed some specified high level, say 0.95. Now, if we apply to the structure loads less than or equal to the allowable buckling load then the reliability will be greater or equal to 0.95. Consequently the probability of failure will be less than or equal to 0.05.

One can also calculate the mean buckling load, which can be shown to be equal to the area under the reliability curve. However, it appears,
that the "Allowable Buckling Load" associated with the specified "Required Reliability" is a more meaningful design criterion than the mean buckling load.
3. PROBABILISTIC PROPERTIES OF INITIAL IMPERFECTIONS

We have just treated a simple case of a structure described by a single random variable. The question arises: "How does one describe the random initial imperfections in general?". If we have \( N \) realizations of the initial imperfection functions

\[
\overline{W}(x) (m) = \frac{w(x)^{(m)}}{t} = \sum_{i} A_{i}^{(m)} \varphi_{i}(x) \quad (m = 1, 2, \ldots, N)
\]  

(3)

then we can calculate two deterministic functions: the mean imperfection function

\[
m_{\overline{W}}(x) = E \left[ \overline{W}(x) \right]
\]

(4)

and the autocorrelation function

\[
R_{\overline{W}}(x_1, x_2) = E \left\{ \left[ \overline{W}(x_1) - m_{\overline{W}}(x_1) \right] \left[ \overline{W}(x_2) - m_{\overline{W}}(x_2) \right] \right\}
\]

(5)

Without loss of generality we can assume that the mean imperfection function is zero. The autocorrelation function gives information on the probabilistic relations between two sections of the random function, so that if the mean function is zero, at \( x_1 = x_2 \) the autocorrelation function equals the variance. In this case, only the autocorrelation function characterizes the manufacturing process.

If \( \varphi_{j}(x) \) is a complete set on \((0, L)\), where \( L \) is the shell length, then one can represent the autocorrelation function as a double Fourier series\[4\]

\[
R_{\overline{W}}(x_1, x_2) = \sum_{j} \sum_{k} \sigma_{jk} \varphi_{j}(x_1) \varphi_{k}(x_2)
\]

(6)
where, if $\mu_i^2 = \int_0^L \varphi_i^2(x) \, dx$, then

$$
\sigma_{jk} = \frac{1}{\mu_j \mu_k} \frac{L}{2} \frac{L}{2} \int_0^L \int_0^L R_\varphi(x_1, x_2) \varphi_j(x_1) \varphi_k(x_2) \, dx_1 \, dx_2
$$

(7)

is the variance-covariance matrix of the initial imperfections. It can be shown [1], that if $L \to \infty$, i.e. the shell becomes infinite, then instead of the variance-covariance matrix $\sigma_{jk}$ one obtains the generalized spectral density as the double Fourier transform of the autocorrelation function $R_\varphi(x_1, x_2)$. If the initial imperfections are homogeneous in the axial direction, i.e. if $\xi = x_2 - x_1$, then one obtains the conventional spectral density. Amazigo and Budiansky [5,6] have applied the spectral density concept to a infinitely long shell with random axisymmetric imperfections and have derived expressions for the mean buckling load. Our ultimate goal, however, is the reliability function.
4. THE MONTE CARLO METHOD

Since for the finite shell a closed form solution for the reliability function appears unfeasible, we shall use the Monte Carlo Method. The application of this method consists of three steps. First, we simulate the random initial imperfections, i.e. "create" a large number of shells with generally different initial imperfection profiles. Next, for each shell we carry out a buckling load calculation thus making use of the accumulated knowledge about the deterministic imperfection analysis. Finally we calculate the reliability function of the structure.

How can we "create" imperfect shells? Briefly the procedure is as follows: Having the $N$ realizations of the initial imperfection functions

$$
\hat{w}(x)^{(m)} = \frac{w(x)^{(m)}}{t} = \sum_{i} A_i^{(m)} \varphi_i(x) \quad (m = 1, 2, \ldots, N)
$$

(3)

we calculate, by taking "ensemble averages", first the estimated mean of the Fourier coefficients $A_i^{(m)}$

$$
\bar{A}_i^{(e)} = \frac{1}{N} \sum_{m=1}^{N} A_i^{(m)}
$$

(8)

and then the estimated variance-covariance matrix

$$
\sigma_{jk}^{(e)} = \frac{1}{N-1} \sum_{m=1}^{N} \sum_{n=1}^{N} \left[ A_j^{(m)} - \bar{A}_j^{(e)} \right] \left[ A_n^{(n)} - \bar{A}_n^{(e)} \right]
$$

(9)

This is a non-negative symmetric matrix and thus can be decomposed as a product of lower and upper triangular matrices

$$
\begin{bmatrix}
\sigma_{jk}^{(e)}
\end{bmatrix} = [c] \triangleright \begin{bmatrix}
0
\end{bmatrix} \triangleleft [c]
$$

(10)
Then the "created" initial imperfection vector can be obtained as follows \[4\]

\[
\{A\} = [C] \begin{bmatrix} \{b\} + \{\tilde{A}^{(e)}\} \end{bmatrix}
\]

(11)

where

\[
\{\tilde{A}^{(e)}\} = \text{estimated "mean" vector}
\]
\[
\{b\} = \text{random vector}
\]

The b's are random numbers generated by the computer. They represent the realizations of a normal random variable with zero mean and unit variance. Taking, for example, 1000 different b's we get 1000 different A's, that is, different shells with the A's as the Fourier coefficients of the initial imperfections. With these "created" initial imperfections one then proceeds to carry out repeated deterministic buckling analysis generating the buckling load histogram of the group of shells under consideration shown in Figure 5. Defining the reliability \(R(\lambda)\) as the probability that the buckling load will exceed a prescribed value (see also Eq. (1)), one can calculate \(R(\lambda)\) from the histogram of the buckling loads by the frequency interpretation (i.e. fraction of an ensemble) yielding the curve shown in Figure 6.

What is the accuracy of this method? For the infinitely long cylindrical shell with cosinusoidal axisymmetric imperfection using Koiter's Eq. (5.2) from Reference [7] as a nonlinear transfer function, the results of the Monte Carlo simulation with 1089 shells practically coincide with the analytical results obtained in terms of error functions \[8\].
5. RELIABILITY OF SHELLS WITH AXISYMMETRIC IMPERFECTIONS

Before continuing further we wish to reemphasize the differences between the approaches used by earlier investigators and the present one. Previous authors have worked with infinite shells and assumed that the random process was stationary and that the ergodicity assumption was valid. We work with finite shells, employ nonstationary processes and replace the ergodicity assumption by the use of ensemble averages. But the most important difference resides in the fact that up to now, with the exception of Makaroff\[9\], all other investigators had to assume some form for the autocorrelation function of the initial imperfections. In our work we make use of the results of experimental imperfection surveys assembled in the Imperfection Data Bank at the Delft University of Technology\[2\] and derive the variance-covariance matrix of the imperfections directly by averaging the ensemble of shells. This matrix derived from the measured data enters as the input to the Monte Carlo Method.

This procedure was applied to a group of 4 seamless, machined shells (the B-shells) whose Fourier coefficients are shown in Table 1. The "simulated" autocorrelation function from the 100 "created" shells practically coincides with the "measured" autocorrelation function, as can be seen from Figure 7. Interestingly the variance shown in Figure 8 varies with the axial coordinate, so that the measured initial imperfections are not stationary and are not ergodic contrary to the assumptions used by other investigators. Next using the simulation procedure described earlier 100 realizations for the B-shells were "created", which then were used to calculate the histogram of the nondimensional buckling loads (see Figure 9). Finally from the histogram the reliability function shown in Figure 10 was determined. The mean buckling load due to axisymmetric imperfections only came out to be 0.816 which is significantly higher than the experimental mean buckling load of 0.592. This implies that for the correct prediction of the reliability function it is necessary to include also the asymmetric imperfections.
6. RELIABILITY OF SHELLS WITH ASYMMETRIC IMPERFECTIONS

To represent the general imperfections we used the imperfection model shown in Table 2. Here $A_{2,0}$ stands for a half-wave cosine axisymmetric Fourier coefficient, with 2 half-waves in the axial direction (and naturally 0 waves in the circumferential direction). On the other hand $C_{1,10}$ stands for an asymmetric Fourier coefficient with a single half-wave sine in the axial direction and 10 full waves in the circumferential direction. As pointed out by Arbocz and Babcock [10] due to the limited number of meshpoints used (41 x 49) the measured data becomes less reliable for the higher harmonics. Thus it was decided to apply the simulation procedure only to the 8 lower order modes and to use the Donnell-Imbert imperfection model from Reference [10] for finding the amplitudes of the 7 higher order modes. To facilitate the computations of the variance-covariance matrices the following new notation was adopted for the initial imperfection

$$
\tilde{w}(x, y) = \frac{\tilde{w}_0(x, y)}{t} = \sum_{i=1}^{2} A_{i0} \cos \frac{i \pi x}{L} \cos \frac{\pi x}{L} + \sum_{r=1}^{6} C_{r} \sin \frac{k \pi x}{L} \cos \frac{r \pi y}{R} \cos \frac{r \pi y}{R} + D_{r} \sin \frac{k \pi x}{L} \sin \frac{r \pi y}{R} \cos \frac{r \pi y}{R} \cos \frac{r \pi y}{R}
$$

(12)

Then the approximate autocorrelation function becomes

$$
R_{\tilde{w}}(x_1, y_1; x_2, y_2) = E \left\{ [\tilde{w}(x_1, y_1) - \frac{\tilde{w}_0}{t}(x_1, y_1)] \times [\tilde{w}(x_2, y_2) - \frac{\tilde{w}_0}{t}(x_2, y_2)] \right\}
$$

$$
\approx \sum_{i,j=1}^{2} K_{A_{i0}A_{j0}} \cos \frac{i \pi x_1}{L} \cos \frac{j \pi x_2}{L} + \sum_{r,s=1}^{6} K_{C_{r}C_{s}} \sin \frac{k \pi x_1}{L} \sin \frac{k \pi x_2}{L} \cos \frac{r \pi y_1}{R} \cos \frac{s \pi y_2}{R}
$$

(13)
+ \sum_{r,s=1}^{6} K_{D_rD_s} \sin k_r \frac{\pi x_1}{L} \sin k_s \frac{\pi x_2}{L} \sin \frac{y_1}{R} \sin \frac{y_2}{S} \sin \frac{\ell}{R} \sin \frac{\ell}{S}

where the variance-covariance matrices $X_{A D_C S}$, $X_{A D_D S}$, and $C_{D_D S}$ have been neglected. Interestingly, as can be seen from Figure 11 also in this general nonsymmetric case the approximate total variance is not constant. This means that the initial imperfections are not stationary and therefore are not ergodic either. Also, as can be seen from Figure 12 the approximate total variance of the 500 simulated B-shells is almost indistinguishable from the "measured" approximate total variance of the group of 4 B-shells shown in Figure 11.

The deterministic analyses were performed by making use of the so-called Multi-Mode Analysis developed by Arboz and Babcock\textsuperscript{[10]} yielding the histogram of the nondimensional buckling loads shown in Figure 13. The corresponding reliability function is displayed in Figure 14. The inclusion of the asymmetric imperfection has reduced the mean buckling to 0.739, still considerably higher than the experimental mean buckling load of 0.592 or the corresponding deterministic buckling load of 0.663 from Ref. [10], which was based entirely on the Donnell-Imbert imperfection model. It is, however, expected that further refinements in the nonsymmetric random imperfection model will lead to lower mean buckling loads.
7. CONCLUSIONS

In conclusion, one can summarize the results obtained so far as follows:

1. It has been demonstrated that the Monte Carlo Method can be used successfully to obtain reliability functions for shells with axisymmetric as well as asymmetric imperfections.

2. It was found that for finite shells nonstationary statistics must be used (thus, ergodicity is not applicable).

3. Using the simulation procedure developed by Elishakoff,[4] the measured initial imperfections have been used directly to generate input data for the Monte Carlo Method.

It is hoped that these preliminary results will encourage many investigators all over the world to compile extensive experimental information on initial imperfections, classified according to the manufacturing processes. The existence of these Initial Imperfection Data Banks will make it possible to associate statistical measures with the different methods of fabrication. As outlined in this paper, the variance-covariance matrices and the mean vectors can be used effectively to generate input for the Monte Carlo Method, which in turn yields the reliability functions associated with the different manufacturing processes. It is felt that by this means the imperfection sensitivity concept can be finally introduced routinely into the design procedures.
8. REFERENCES


<table>
<thead>
<tr>
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<th>B-1</th>
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Table 1: First fifteen Fourier coefficients \( a_i^{(m)} \) of the initial imperfections expanded as

\[
w_0^{(m)}(x) = t \sum_{i=0}^{14} a_i^{(m)} \cos \frac{i \pi x}{L}.
\]

for the group of B-shells

\( R = 101.6 \text{ mm}, t = 0.2007 \text{ mm}, L = 134.30 \text{ mm}, E = 1.065 \times 10^5 \text{ N/mm}^2, \nu = 0.3, i_{cL} = 17 \)

Note: Here \( w_0^{(m)}(x) \) is positive outward!
Table 2: The modified 15-modes imperfection model used for the Monte-Carlo simulation
a. Shell A-7 [2]

b. Shell A-8 [2]

Fig.1 Three-dimensional plots of measured initial imperfections
Fig. 2  Nonlinear transfer function $\lambda$ vs $\bar{\xi}$ by Koiter (1945)\textsuperscript{[3]}
a. Critical buckling load $\lambda$ vs imperfection amplitude $\bar{\xi}$

b. Probability density of the initial imperfection amplitude

Fig. 3 Illustration of the reliability calculation
Fig. 4 Plot of the reliability function $R(\lambda)$
Fig. 5  Histogram of nondimensional buckling loads $\lambda$

$R(\lambda) = \text{Prob}(\Lambda > \lambda)$

$R(\lambda) = \int_{-\xi_{l}}^{\xi_{l}} f_{\xi}(\xi) d\xi$

Monte Carlo simulation

Fig. 6  Comparison of analytical reliability function with results of the Monte Carlo method
a. Estimated autocorrelation function
   (Group of B-shells$^{21}$)

b. Simulated autocorrelation function
   (Group of 100 equivalent B-shells)

Fig. 7 Comparison of the estimated and the simulated autocorrelation functions
\[ \frac{\sigma^2(x)}{0.202} = \frac{R_{yy}(x,x)}{0.202} \]

\[ \begin{align*}
\sigma^2(x) & = \frac{R_{yy}(x,x)}{0.202} \\
\end{align*} \]

Fig. 8  Variance of the measured initial imperfections
(Group of B-shells \(^{[2]}\))
Fig. 9  Histogram of the nondimensional buckling loads $\lambda$
(Group of B-shells$^{[2]}$)

Fig. 10  Calculated reliability function for the axisymmetric
imperfection model (Group of B-shells$^{[2]}$)
Fig. 11Estimated approximate variance
(Group of $B$-shells$^{[2]}$)

Fig. 12Simulated approximate variance
(Group of 500 equivalent $B$-shells)
Fig. 13  Histogram of the nondimensional buckling loads $\lambda$
( Group of $B$-shells$^{[2]}$ - Asymmetric )

$R(\lambda) = \text{Prob}(\Lambda \geq \lambda)$

Fig. 14  Calculated reliability function for the asymmetric
imperfection model ( Group of $B$-shells$^{[2]}$ )