THEORY OF PROPELLER FORCES
IN A TURBULENT ATMOSPHERE

by

J. B. Barlow

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The topic of this investigation was suggested by Professor B. Etkin. I am indebted to him for sharing with me some of the insights he has gained through an extensive interest in problems related to flight in turbulent air. Of at least equal importance, by his enthusiasm for and energy devoted to his many interests, he sets an example worthy of emulation.

I have received financial support as a National Science Foundation Graduate Fellow, 1967-70 from the National Science Foundation, Washington, D.C. The work has also been supported by the United States Air Force Flight Dynamics Laboratory, Wright Patterson Air Force Base under contract number F33615-C-1055.
ABSTRACT

The problem of determining forces and moments on a propeller operating in a turbulent flow is studied analytically. It is attacked by utilizing the concept of a general aerodynamic transfer function including consideration of time varying parameters as necessary for application to the propeller problem. A quasistatic lifting line model of the propeller response to a nonuniform flow field is given with unsteady effects estimated by applying a Sears function factor based on the rotational speed of the propeller. This response model is used with the general aerodynamic transfer function relation to derive an expression for the aerodynamic transfer function of a propeller.

The transfer function relation is utilized with nonstationary random process theory to obtain expressions for the generalized power spectral density of the forces and moments. Calculations of transfer function coefficients are shown including spatial velocity variations over the plane of the propeller. Power spectral densities of normal force and moment are computed based on a "point" approximation and resulting root mean square values of normal force and moment are found to be about five to ten percent of the thrust value at the specific operating conditions for a specific propeller. The effect of spatial velocity variations is found to be production of peaks in the power spectral densities at multiples of blade frequency.
## CONTENTS

**PART I. INTRODUCTION**

1. Genesis and Statement of the Problem .................................................. 1

2. Related Work
   2.1 Flight in turbulence ................................................................. 2
   2.2 Propeller theory ............................................................................ 4
   2.3 Linear system response and random processes ............................... 5

**PART II. VARIABLE PARAMETER LINEAR SYSTEMS AND AERODYNAMIC TRANSFER FUNCTIONS**

3. Introduction ......................................................................................... 7

4. Linear System Input-Output Relations
   4.1 Purpose of this section ................................................................. 7
   4.2 Input output of variable parameter systems ............................... 7

5. The General Aerodynamic Transfer Function
   5.1 Purpose of this section ................................................................. 11
   5.2 The transfer function concept ...................................................... 11
   5.3 A rigid body in an incompressible nonviscous fluid
      5.3.1 The general class of problems ............................................... 12
      5.3.2 Some functional relations ...................................................... 14
      5.3.3 A reduction to previous results ............................................. 19

**PART III. PROPELLER RESPONSE AND THE PROPELLER TRANSFER FUNCTION**

6. The Propeller in Nonuniform Flow
   6.1 Purpose and scope ........................................................................ 22
   6.2 The model of the propeller
      6.2.1 Representation as rotating lines .......................................... 22
      6.2.2 Expressions for blade element force perturbations ............... 25
      6.2.3 Interpretation as an impulse response .................................. 28

7. The Propeller Transfer Functions
   7.1 Introduction and general relations ............................................... 29
   7.2 Axial motion
      7.2.1 General ................................................................................. 34
      7.2.2 Single blade ............................................................................ 34
      7.2.3 Complete propellers ............................................................... 36
      7.2.4 Reduction to "point approximation" .................................... 37
      7.2.5 A correspondence to angle of attack and comparison with data .................................................. 42
   7.3 Effect of angle of attack ............................................................... 45
| PART IV. PROPELLER RESPONSE TO TURBULENCE AND VARIABLE PARAMETER SYSTEM RESPONSE TO RANDOM INPUTS |
|--------------------------------------------------|---|
| 8. Introduction                                   | 47 |
| 9. Response to Random Processes                   | 48 |
| 9.1 Nonstationary processes                       | 48 |
| 9.2 System response                               | 52 |
| 10. The Propeller Response to Turbulence          | 53 |
| 10.1 The generalized power spectral density       | 53 |
| 10.2 The time averaged power spectral density for the point approximation | 57 |
| 10.3 Discussion of three dimensional effects on the power spectral densities | 74 |

<table>
<thead>
<tr>
<th>PART V. CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. Summary</td>
</tr>
<tr>
<td>12. General Remarks</td>
</tr>
</tbody>
</table>

REFERENCES

APPENDICES

A. Blade Element Force and Moment Perturbations

B. The Dimensionless Coefficients and Variables

C. Algorithm Development for Evaluation of Propeller Transfer Functions
### NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Blade section two dimensional lift curve slopes</td>
</tr>
<tr>
<td>B</td>
<td>Number of blades</td>
</tr>
<tr>
<td>B(p)</td>
<td>Function defined by Eq. (74)</td>
</tr>
<tr>
<td>c</td>
<td>Chord of blade sections</td>
</tr>
<tr>
<td>C_T</td>
<td>Thrust coefficient, see Appendix B</td>
</tr>
<tr>
<td>C_D</td>
<td>Drag coefficient of blade section</td>
</tr>
<tr>
<td>D</td>
<td>Drag</td>
</tr>
<tr>
<td>(\exp{})</td>
<td>Natural logarithm base raised to power of quantity in brackets</td>
</tr>
<tr>
<td>F'</td>
<td>Component of blade element force in plane of rotation</td>
</tr>
<tr>
<td>f</td>
<td>Frequency</td>
</tr>
<tr>
<td>h_{zw}</td>
<td>Element of impulse response matrix relating output to input (w)</td>
</tr>
<tr>
<td>H_{zw}</td>
<td>Element of frequency response matrix relating output (S_z) or (z) (depending on specific form indicated by arguments) to input (S_w)</td>
</tr>
<tr>
<td>(h'_{zw})</td>
<td>Coefficients in the Fourier expansion of the propeller impulse response matrix</td>
</tr>
<tr>
<td>J</td>
<td>Advance ratio</td>
</tr>
<tr>
<td>j</td>
<td>Imaginary unit</td>
</tr>
<tr>
<td>(J_n)</td>
<td>Bessel function of order (n)</td>
</tr>
<tr>
<td>(\hat{i}, \hat{j}, \hat{k})</td>
<td>Unit vectors</td>
</tr>
<tr>
<td>k</td>
<td>Wave number vector</td>
</tr>
<tr>
<td>(k')</td>
<td>Frequency variable defined by Eq. (68)</td>
</tr>
<tr>
<td>(k_{in})</td>
<td>Frequency variable defined by Eq. (61)</td>
</tr>
</tbody>
</table>

vi
L, M, N  
\[ L, M, N \equiv \text{Components of moment perturbation on } oxyz \]

\[ \ell \equiv \text{Summation index in the Fourier expansion of the propeller impulse response matrix} \]

m  
\[ m \equiv \text{Blade index, summation index} \]

\[ o'x'y'z', oxyz \equiv \text{Coordinate frames, Fig. 7} \]

p  
\[ p \equiv \text{Summation index} \]

\[ r', \hat{r}, r \equiv \text{Spatial position vectors} \]

r  
\[ r \equiv \text{Radial distance along propeller blade} \]

R  
\[ R \equiv \text{Propeller radius} \]

\[ R_{ww} \equiv \text{Element of correlation matrix formed from vector } w \]

\[ S_w \equiv \text{Spectrum (Fourier transform) of } w \]

t  
\[ t \equiv \text{time} \]

U  
\[ U \equiv \text{Forward speed, or relative velocity when subscripted} \]

\[ u, v, w, u', v', w' \equiv \text{Fluid velocity components} \]

V  
\[ V \equiv \text{Component of relative velocity perpendicular to the blade axis} \]

W  
\[ W \equiv \text{Component of relative velocity perpendicular to the blade axis and in the plane of rotation} \]

\[ w \equiv \text{Input vector for system relations, the fluid velocity vector} \]

X, Y, Z  
\[ X, Y, Z \equiv \text{Components of force perturbation on } oxyz \]

z  
\[ z \equiv \text{System output vector of force and moment components} \]

\[ \alpha \equiv \text{Angle of attack, perturbation unless subscripted} \]

\[ \beta \equiv \text{Geometric twist of blade sections, angle between section zero lift lines and plane of rotation} \]

\[ \delta \equiv \text{Dirac distribution, i.e. delta function} \]

\[ \theta \equiv \text{Polar coordinate in plane of rotation} \]
\( \lambda \)  
\( \eta \)  
\( \sigma \)  
\( \nu \)  
\( \Theta_k \)  
\( \rho \)  
\( \Lambda \)  
\( \tau \)  
\( \xi \)  
\( \Omega \)  
\( \phi_{ww} \)  

Subscripts  
\( u,v,w \)  
\( i \)  
\( c \)  
\( i \)  
\( zw \)  
\( T \)  

Superscripts  
\( T \)  
\( ' \) (prime)
\( n \) → Functions which are weighted integrals of an \( n \)th order Bessel function

\(*\) → Complex conjugate

Other

[ ] → Matrix

< > → Ensemble average

\( \overline{\text{overbar}} \) → Time average

\( \underline{\text{underbar}} \) → Indicates a vector quantity
PART I. INTRODUCTION

1. Genesis and Statement of the Problem

Within the last decade the intensification of transportation problems in high-density urban areas has focused attention on high-speed systems with potential for operating in areas where little terminal space is available. One of the competitors is a V/STOL aircraft network. There are a number of proposed designs within this category whose relative merits have been discussed by Campbell. The configurations employing propellers or rotors enjoy significant advantages in propulsive efficiency over the jet powered type. It appears that some variation of the rotor-propeller design will reach at least the trial stage for an intercity transport.

Because of the nature of the flight profiles which will be required of these vehicles and the characteristics of the vehicles themselves, they will spend a higher percentage of their flight time in the low altitude regime than is the case for current jet transport aircraft. In this regime, particularly in the planetary boundary layer, unavoidable severe turbulence is expected more frequently than at higher altitudes. Furthermore, the wind characteristics in this lower layer may be strongly influenced by local surface characteristics such as tall buildings, trees, etc. Teunnissen has given a review of the characteristics of turbulence in the planetary boundary layer. It is in this environment that takeoffs, landings, transition from vertical to horizontal flight and vice versa will take place. In any case, the vehicles will be operating at low speeds in which condition they are expected to be more sensitive to gusts of a given velocity. It is important to predict these sensitivities in order that adequate designs and procedures can be evolved. The propeller is, perhaps, the least understood component of these vehicles in a gust environment.

It is known that propellers produce significant side forces along with less significant moments when their axes are inclined to the direction of flight. These forces and moments may be computed by methods of Ribner for steady flight conditions. However, the problem of predicting the characteristics of the aerodynamic forces on a propeller traveling through a continuous turbulent field with known characteristics has not previously been treated except for a recent paper by the present author. A related problem of forces on a stationary propeller in a frozen but nonuniform flow is considered by Morse and Ingard. It is noted that frequency dependence of excitations and responses are important in the analysis of complex dynamic systems.
and therefore methods for the prediction of the spectra of the aerodynamic forces and moments on a propeller are needed.

To be explicit, the objective of the study reported herein was the development of a method for determining the spectra of the aerodynamic forces and moments, particularly those in the propeller plane, on a propeller advancing through a homogeneous turbulent field when the turbulence spectra, geometric form of the propeller, and operating conditions are given.

At present there is no experimental data for comparison with the predicted results except at zero frequency. Mr. T.A.P.S. Apparao at UTIAS is currently engaged in an experimental program which will produce such data.

2. Related Work

2.1 Flight in turbulence:

There exists an extensive literature on the problem of the flight of aircraft through turbulent air. The earliest statistical treatment was given by Liepmann\textsuperscript{6,7} whose work was based on results from the statistical theory of turbulence\textsuperscript{8,9} and the application of similar mathematical models\textsuperscript{10} to problems in communications engineering\textsuperscript{11}. The theory was elaborated by Ribner\textsuperscript{12}, Etkin\textsuperscript{13}, Press and Houbolt\textsuperscript{14}, and Diederich\textsuperscript{15}. The present state of the theory is indicated by references 16-26 and their bibliographies.

In the theory presented in the references cited above, several distinct elements may always be identified as noted by Etkin\textsuperscript{13}. They are: (1) the statistical description of the turbulent field (the input), (2) the determination of an appropriate aerodynamic transfer function or influence function relating elemental gusts to corresponding forces, (3) the determination of vehicular transfer functions or influence functions relating elemental forces to corresponding motions, stresses, or other quantities of interest, and (4) the combination of the transfer functions or influence functions with the input to obtain the resulting outputs of interest. The same sequence of steps is appropriate for computing the effects from propellers on a vehicle flying through turbulence. In fact, the present study consists primarily in performing step (2) for a propeller. There are some significant differences in the present problem and those treated in the above references. The differences are felt to be of potential significance for V/STOL vehicles generally as well as for propellers, so the following discussion is framed in that context.

There are five fundamental assumptions incorporated in previous treatments of flight in turbulent air. They are:
(1) that all required mathematical models are linear within a useful range so that superposition may be freely utilized;
(2) that system parameters are constant; (3) that the turbulent fields encountered are statistically homogeneous; (4) that the time variation of the flow fields is sufficiently slow so that Taylor's hypothesis can be utilized to convert spatial correlations to time correlations; and (5) that the corresponding incident velocities at a given point on an aircraft are statistically stationary functions of time as a result of the vehicle translating rectilinearly through the turbulent field at constant velocity.

A consequence of these assumptions is that all output quantities are statistically stationary functions of time. Another is that if the input probability distribution is normal, or Gaussian, then the output distributions will be likewise. An additional assumption of isotropy is usually made since it simplifies the problems and appears to be a good approximation to turbulence encountered in the atmosphere under most operating conditions; however, it is not fundamental to the theory. The theoretical predictions have been shown to give good agreement with flight data for conventional jet aircraft ranging from large aspect ratio bombers and transports to delta wing fighter types. Applications to V/STOL type vehicles are recent and apparently there have been no published comparisons with flight data. The problems inherent in applying the existing theory to V/STOL vehicles may be indicated by considering the operating regimes of these vehicles in conjunction with the previously stated fundamental assumptions of the theory.

Assumption (1): Linearity may be retained at the expense of more careful definition of perturbation quantities. If an appropriate and useful linear model cannot be established, direct simulation is the only available method of treatment.

Assumption (2): Parameters vary during transition for all types of VTOL craft. Those which have rotors or large propellers have time varying parameters with respect to gust inputs in all operating conditions.

Assumption (3): Gault and Gunter have reported recent experimental data which indicates atmospheric turbulence at low absolute altitudes to be isotropic. The assumption of homogeneity may, therefore, be retained generally although it seems evident that in specific locations local terrain features near landing and takeoff points should be considered. The previously cited review by Teunnissen may be consulted for a discussion of the limitations on this assumption.

Assumption (4): Skelton has concluded that the assumption of frozen gust pattern is reasonable so long as the airspeed is greater than one-third of the mean wind speed. Actually,
this assumption can be removed from the analysis with relatively little difficulty.

Assumption (5): V/STOL vehicles may change speed significantly over distances of the order of the integral scales of the turbulence which will be encountered. A given point on a rotor or propeller performs a general helicoidal motion even when the vehicle travels a rectilinear path. The incident relative velocity at a given point on the vehicle will generally, therefore, be a sample function of a statistically nonstationary random process during passage through a homogeneous turbulent field. This statement is based on the definition of stationary process as given by Lanning and Battin\(^2^8\) or Pugachev\(^2^9\).

The result of the usual theory, that all output quantities are statistically stationary, is upset by either the time variation of parameters or the nonstationarity of the input. The significance of these features in terms of their impact on design or operations is so far undetermined. The problem of determining the forces and moments on a propeller advancing through a homogeneous field which is the primary topic of this study exhibits both of the new features indicated for V/STOL vehicles. It turns out that in certain cases the input transformation due to nonuniform motion of a particular vehicle element can be incorporated as a part of the vehicle characterization. In some cases, therefore, the situation may be represented by a certain system subject to a nonstationary input or by an appropriately modified system subject to a stationary input.

There has been one previous publication of a closely related study which includes some of these features. Gaonkar and Hohenemser\(^2^6\) have studied the flapping response of a single rotor blade to an assumed nonstationary input force which is chosen to simulate a turbulent input. The title of the paper indicates that the analysis is for response to atmospheric turbulence; however, the parameters of the assumed input yield a spectrum which appears much too broad to approximate available atmospheric data. This, however, is only a question of the value of a parameter and the paper presents the only quantitative solution of the response of a time variable system under nonstationary stochastic loading with which I am familiar. Other treatments of nonstationary responses deal with transient behaviour of systems with constant-parameter transfer functions\(^3^0,3^1\).

2.2 Propeller theory:

Historically most of the work on propeller theory has naturally been concentrated on the problem of accurate prediction of thrust and torque in the axial operating condition. Even in recent years when the power of computers has been brought to bear on rotor and propeller problems, the emphasis has been on the calculation of static (zero advance) performance with
comparatively little effort expended on side force and moment prediction in nonaxial operation. The additional complexity of the propeller geometry as compared to wings has resulted in the use of less sophisticated aerodynamic techniques for propeller analysis. Fortunately, useful results have been obtainable with comparatively simple techniques as, for instance, a combination of blade element theory, momentum theory, and the Prandtl tip loss analysis as detailed by McCormick. More important to the present study, Ribner's work on the problem of propellers in yaw yields reasonable agreement with experimental data without a detailed consideration of unsteady aerodynamic effects. The success of that theory is the basis of the approach taken in the present study wherein a technique is required for determining the forces and moments resulting from any arbitrarily given perturbation flow field.

Other related work is that of DeYoung who has developed relations for propeller characteristics at high angles of attack by curve fitting using Ribner's results and various propeller data. Crigler and Gilman have given some results which might be used for estimating unsteady aerodynamic and compressibility effects. The most complete set of data on a propeller at angle of attack with which I am familiar is that of Yaggy and Rogallo which shows that normal force and yawing moment on a propeller are nearly linear with thrust axis angle of attack up to thirty degrees for a wide range of operating conditions.

There has been some related work on the problem of marine propellers operating in the nonuniform flow of a ship's wake. These studies include an application of unsteady lifting surface theory. The cited references show comparisons between various theoretical methods which demonstrate large differences between the three-dimensional and strip theories for low-aspect-ratio marine propellers. For the typically higher aspect ratios of air propellers it is expected that there would be much less discrepancy.

The method used in the present work is based on the assumption that the perturbation force on a particular point on a blade is a linear function of the local incident velocity perturbation with the unsteady aerodynamic effect given by applying Sears function evaluated at a reduced frequency determined only by the rotational speed of the propeller.

2.3 Random process theory and linear system response:

There are many books and papers which treat this topic, including some of the references cited under section 2.1. Only a few, however, treat nonstationary processes. The primary references for the methods used in this study are Bendat and Piersol, Papoulis, and Sveshnikov. The primary result from the latter is a theorem to the effect that the outputs of
linear systems having constant or variable parameters, subject to Gaussian inputs are also Gaussian. The importance of this theorem is that the outputs, even though nonstationary, are still completely specified in the sense that probability densities of any order are in principle calculable, when the correlations or spectral densities, means, and mean squares are known. This is significant for the problems of estimating level crossing rates, probable lifetimes, etc.

Most of the necessary nonstationary input output relations are developed by Bendat and Piersol\textsuperscript{19}; however more convenient forms are developed in the present work, and an effort is made to interpret the various functions. Furthermore, since there are six outputs (3 force components, 3 moment components) and three inputs (3 fluid perturbation velocity components), the relations are given in matrix form. The inputs are functions of three or four scalar variables depending on whether or not the flow field is an explicit function of time as well as of the three space coordinates. It has been found convenient to utilize the generalized definition of "transfer function" as given by Ribner\textsuperscript{12} which simply means the forces generated upon passage through a sinusoidal shear wave given as a function of the wave number vector.
PART II. RESPONSES AND AERODYNAMIC TRANSFER FUNCTIONS OF VARIABLE PARAMETER SYSTEMS

3. Introduction

This part contains a discussion of some general linear system relations with time as the independent variable and their extension to the aerodynamic case when there may be as many as four independent variables. The concept of the "transfer function" of an aerodynamic surface as defined by Liepmann, Ribner, and Etkin is discussed with the additional consideration of parameter variations. The general expressions are reduced, by introducing appropriate assumptions, to specific forms for a propeller. A perturbation blade element analysis of a propeller is given and utilized to carry out numerical evaluations of some propeller transfer functions. Certain limiting cases are compared to previous results and experimental data for propellers in yaw or at angle of attack.

4. Linear System Input-Output Relations

4.1 Purpose of this section:

The objectives of this section are threefold. The first is the provision of a summary of some relations for variable parameter linear systems with arbitrary inputs along with some remarks on interpretation which I believe are new and helpful in the application of the results. The second objective is the explicit definition of certain terms, whose use has not become completely uniform, so that their meaning in the present context may be unambiguous.

4.2 Input-output of variable-parameter systems:

Consider a linear system which has time dependent characteristics and for which the inputs and outputs are functions of time only. These relations will later be generalized to more independent variables. Since the systems are linear, the response to any given, but arbitrary, input can be found by summation or integration if the responses to any elemental inputs capable of being superposed to form the given, but arbitrary, input are known. The two most commonly used elemental responses are the responses to an impulse, or Dirac distribution, and the unit amplitude sinusoid. The properties of, and relationships between, these responses are widely used by engineers and scientists engaged in analyses of the dynamics of constant parameter systems. The properties of these elemental responses of variable parameter systems are less widely used.

Fig. 1 indicates schematically some input-output relations for a linear variable parameter system. Here the underbar indicates a vector quantity. The system functions are generally
matrices. It is assumed throughout that the systems are absolutely stable and that the relevant integrals converge.

The definitions of the quantities symbolized by Fig. 1 will be phrased for single input, single output cases; but they carry over directly to the multidimensional case.

First, note that the input and output specifications form Fourier transform pairs. The Fourier transform of a function of time as in Eq. (2) and Eq. (4) will be called the "spectrum" of that function. It is frequently helpful to consider the system functions of Fig. 1 to be functions of the first argument with the second argument representing a parameter.
\[ w(t) = \int_{-\infty}^{\infty} S_w(f) \exp\{j2\pi ft\} df \]  
(1)

\[ S_w(f) = \int_{-\infty}^{\infty} w(t) \exp\{-j2\pi ft\} dt \]  
(2)

\[ z(t) = \int_{-\infty}^{\infty} S_z(f) \exp\{j2\pi ft\} df \]  
(3)

\[ S_z(f) = \int_{-\infty}^{\infty} z(t) \exp\{-j2\pi ft\} dt \]  
(4)

Impulse response: \( h(t, \tau) \) is defined to be the system output at time \( t \) resulting from an impulsive input a time increment \( \tau \) earlier. This is a system function relating the input in the time domain to the output in the time domain. The impulse responses for all physical systems have the property that \( h(t, \tau) = 0 \) for \( \tau < 0 \). The output for arbitrary input at any time is given by Eq. (5).

\[ z(t) = \int_{0}^{\infty} [h_{zw}(t, \tau)] w(t-\tau) d\tau \]  
(5)

Time varying frequency response: \( H_{zw}(t, f_i) \) is the complex amplitude of the output at time \( t \) when the input is a unit amplitude sinusoid of frequency \( f_i \). Thus, if the input is \( \exp\{j2\pi f_i t\} \), the output is \( H(t, f_i) \exp\{j2\pi f_i t\} \). This is a system function relating the input in the frequency domain to the output in the time domain. Some of its properties are discussed by Gibson\(^4\). The output for arbitrary input is related to the input as indicated schematically by Fig. 1b and mathematically by Eq. (6).

\[ z(t) = \int_{-\infty}^{\infty} [H_{zw}(t, f_i)] S_w(f_i) exp\{j2\pi f_i t\} df_i \]  
(6)

The time varying frequency response is related to the impulse response by Eq. (7) and its inverse, i.e. they form a Fourier transform pair.

\[ [H_{zw}(t, f_i)] = \int_{-\infty}^{\infty} [h_{zw}(t, \tau)] \exp\{-j2\pi f_i \tau\} d\tau \]  
(7)
Impulse response spectrum: This system function relates the input in the time domain to the output in the frequency domain. Since it appears to be a somewhat novel concept, a brief derivation will be given.

Consider the impulse response in the form \([h_{zw}(\hat{t}, t)]\) representing the response at time \(\hat{t}\) to an impulse at time \(t\). Then:

\[ z(\hat{t}) = \int_{-\infty}^{\infty} [h_{zw}(\hat{t}, t)]w(t)dt \]

Utilizing Eq. (4):

\[ S_z(f_0) = \int_{-\infty}^{\infty} [h_{zw}(\hat{t}, t)]w(t)dt \exp\{-j2\pi f_0 \hat{t}\} \hat{t} \]

Change the order of integration and define:

\[ [h'_{zw}(f_0, t)] = \int_{-\infty}^{\infty} [h_{zw}(\hat{t}, t)] \exp\{-j2\pi f_0 \hat{t}\} d\hat{t} \]

Now make a change of variable \(\hat{t} = t+\tau\) so that \([h_{zw}(\tau, t)]\) is the response at time \(t+\tau\) to an impulse at time \(t\). Now

\[ [h'_{zw}(f_0, t)] = \exp\{-j2\pi f_0 \tau\} \int_{-\infty}^{\infty} [h_{zw}(\tau, t)] \exp\{-j2\pi f_0 \tau\} d\tau \]

\[ = [h_{zw}(f_0, t)] \exp\{-j2\pi f_0 t\} \quad (8) \]

The "impulse response spectrum" is defined to be \([h_{zw}(f_0, t)]\) to correspond to the definition of the time varying frequency response. From Eq. (8) it is seen to be the complex amplitude of the response spectrum when the input is an impulse at time \(t\). The input-output relation indicated schematically by Fig. 1c is given in equation form by Eq. (9).

\[ S_z(f_0) = \int_{-\infty}^{\infty} [h_{zw}(f_0, t)] \exp\{-j2\pi f_0 t\} w(t) dt \quad (9) \]

Thus if the input is \(\delta(t-a)\), the spectrum of the output is \([h_{zw}(f_0, a)] \exp\{-j2\pi f_0 a\}\).

When the system has constant parameters, the impulse response spectrum and the time varying frequency response reduce to the same forms.
Frequency response: The frequency response of time varying systems is defined to be the spectrum of the output as a function of the frequency variable $f_0$ when the input is a unit amplitude sinusoid of frequency $f_i$. The input and output spectra are related by:

$$S_2(f_0) = \int_{-\infty}^{\infty} [H_{zw}(f_0,f_i)]S_w(f_i)df_i$$

(10)

The frequency response is related to the time varying frequency response and the impulse response spectrum by Eq. (11) and Eq. (12) respectively.

$$[H_{zw}(f_0,f_i)] = \int_{-\infty}^{\infty} [H_{zw}(t,f_i)]\exp{j2\pi(f_i-f_0)t}dt$$

(11)

$$[H_{zw}(f_0,f_i)] = \int_{-\infty}^{\infty} [h_{zw}(f_0,t)]\exp{j2\pi(f_i-f_0)t}dt$$

(12)

For constant parameter systems, the usual relations follow since if $[H_{zw}(t,f_i)]$ and $[h_{zw}(f_0,t)]$ are independent of $t$, Eqs. (11) and (12) give:

$$[H_{zw}(f_0,f_i)] = [H_{zw}(f_i)]\delta(f_i-f_0) = [h_{zw}(f_0)]\delta(f_i-f_0)$$

and Eq. (10) then yields

$$S_2(f_0) = [H_{zw}(f_0)]S_w(f_0) = [h_{zw}(f_0)]S_w(f_0).$$

5. The General Aerodynamic Transfer Function

5.1 Purpose of this section:

The purpose of this section is to define a class of problems which contains the propeller problem as well as the wing problems which have been treated previously$^6,7,12,13$ and to show the generalization required to include the propeller problem. The discussion is physically based and leans heavily on the concept of the aerodynamic transfer function$^{12,13,22}$. It could also be stated in a purely mathematical form as a boundary value problem.

5.2 The transfer function concept:

The concept of a transfer function is widely used in dynamic analyses. Most applications are to systems whose states are
specified as a function of a single independent variable, time, and which have constant parameters so that the transfer functions are dependent on a single transform variable. If the concept is extended to variable parameter systems as in section 4, the transfer function or frequency response becomes a function of two variables, i.e. Fig. 1.

The application of the transfer function concept to aerodynamic problems\textsuperscript{7,12,13,22} involves an increase in dimensionality. The system input is now considered to be a function of the three space coordinates as well as time. The forms given by Liepmann\textsuperscript{7}, Ribner\textsuperscript{12}, and Etkin\textsuperscript{13,22} assume the input not to be an explicit function of time, i.e. the spatial velocity patterns are assumed frozen. More generally, the transfer function in the aerodynamic case is a function of four transform variables if the system has constant parameters. The variables are identified as the three components of a wave number vector and an ordinary frequency. The associated input is a sinusoidal shear wave traveling in the direction of the wave number vector. Examples of aerodynamic systems with constant parameters which have been treated by this concept are wings\textsuperscript{6,7,12,13,23,24} traveling at constant speed. The classical special case is the treatment by Sears\textsuperscript{43} of an airfoil moving through a shear wave with the direction of motion parallel to the wave number vector and the fluid velocity perpendicular to the plane of the airfoil.

If an aerodynamic system has variable parameters, the transfer function is dependent on five scalar variables, i.e. the three components of the wave number vector plus two frequency variables as in section 4. A propeller represents such a system. Another simpler case is the problem of Sears\textsuperscript{43} generalized by allowing the airfoil velocity to be a function of time. Generally, any rigid body which translates through a fluid with constant velocity may be represented as a constant parameter system since a given gust pattern will produce forces on the body which may be given as functions of the time from entry with no dependence on the absolute time of entry. Rigid bodies which translate with varying speed or rotate and for which the motion is given as a function of time may generally be considered to be variable parameter systems. In this case, the forces produced by a given gust pattern will depend on the absolute time of entry as well as the time elapsed from entry into the gust pattern.

5.3 A rigid body in an incompressible nonviscous fluid:

5.3.1 The general class of problems

Consider the class of problems defined by the following specifications. Let a coordinate frame \( o'x'y'z' \) immersed in an incompressible nonviscous fluid of large extent be defined
such that the space and time averages of the fluid velocity relative to it are zero. Let the local fluid velocity components on this frame be designated by \((u', v', w')\), Fig. 2.

Fig. 2. Coordinate Frames and Fluid Velocities

It is assumed that \(o'x'y'z'\) is an inertial reference. Let a rigid body be specified by a surface \(B'(x', y', z', t) = 0\) and assume its motion relative to \(o'x'y'z'\) consists of a translation at constant speed, \(U\), of some point, \(o\), of the body along \(o'x'\) plus a rotation at constant angular velocity, \(\Omega\), about an axis through \(o\). A coordinate system \(oxyz\) is defined as in Fig. 2 which translates (but does not rotate) with the body and which has \(ox\) along the angular velocity vector. The angular velocity vector may be taken, as shown, in the \(o'x'z'\) plane without loss of generality. The body can be specified by a surface given by the functional form \(B(x, y\cos\Omega t + z\sin\Omega t, z\cos\Omega t - y\sin\Omega t) = 0\). It is desired to determine the forces and moments exerted on the body when \((U, \Omega)\) are given and \((u', v', w')\) are given, but arbitrary, functions of \((x', y', z', t)\). Since the fluid is assumed nonviscous, an equivalent problem is the determination of the pressure distribution over the body surface under the same conditions. The pressure at a given point may be considered to be a sum of two parts. First, there is a contribution determined by the body shape and motion with \(u' = v' = w' = 0\) and second a contribution attributable to the velocity field \((u', v', w')\). It is assumed here that the body shape, its motion, and the
velocity field are such that the second contribution can be sufficiently approximated by linear functionals of the \((u',v',w')\) field. The sufficiency of the approximation is ultimately to be determined by comparison with experimental results. Linearity is necessary if the previously discussed transfer function ideas are to be useful in the treatment of arbitrary velocity fields. In the present work on the propeller, only the contribution of \((u',v',w')\) is considered.

5.3.2 Some functional relations

The forces and moments will be considered in terms of components on the oxyz reference frame. In the subsequent discussion the force component \(Z\) will be taken as representative. \([Z\) is the perturbation due to \((u',v',w')].\) Generally if the pressure distribution is known at any time, \(t\), the various force and moment components may be obtained by appropriate integrations. If pressure impulse functions [say \(h_{pu}(r,t;\hat{r},\tau)\), \(h_{pv}(r,t;\hat{r},\tau)\), \(h_{pw}(r,t;\hat{r},\tau)\) where \(h_{pu}(r,t;\hat{r},\tau)\) is defined as the pressure increment at \((r,t)\) resulting from an impulse in \(u\) at \((\hat{r},t-\tau)\), i.e. \(u = \delta(r-\hat{r})\delta(t-\tau)\)] could be found, then the pressure distributions at any time, \(t\), and consequently the forces and moments could in principle be determined. In the case of a propeller, at this time the rigorous determination of such functions appears impractical or perhaps impossible. Note that for a wing translating at uniform speed these pressure impulse functions would not depend on \(t\).

The relation between the pressure impulse functions and an impulse function for the \(Z\) force component is given by Eq. (13). Here \(n_z(r,t)\) is the local direction cosine of the normal to the body surface with respect to the oz axis and the integration is over the body surface. The result of the integration, \(h_{zw}(t;\hat{r},\tau)\), is the increment in the \(Z\) force component at time \(t\) resulting from an impulse in \(w\) at \((\hat{r},t-\tau)\). This is a generalization of the impulse response represented by Fig. 1a and is analogous to the impulse response functions, \(h(\tau,y)\), considered by Liepmann\(^7\) and Ribner\(^{12}\). In the present case there is an additional dependence on \(t\). This is the mathematical statement of the time variable character of the system as stated in section 5.2. An explicit representation of the situation is given by Fig. 3 which indicates a wing and a propeller entering two dimensional impulsive gusts and the corresponding \(Z\) force generation. The difference in the character of the response is readily seen.
It is also seen that any rigid body performing the motion specified in section 5.3.1 is, in the present context, a representative of a periodic system with period at most equal to the period of revolution. This characteristic is important in the detail development of the propeller transfer function.

The expression for the Z force component resulting from the w component of the velocity field and which is analogous to Eq. (5) is given by Eq. (14).
\[ Z_w(t) = \int \int_{\mathbb{B}} h_{zw}(t, \hat{r}, \tau) w(\hat{r}, t-\tau) \, d\tau \, dS \]  

(14)

(The symbol \( Z_w \) will be used later to signify \( \frac{\partial Z}{\partial w} \), but no confusion will arise if the discussion is followed through.)

Starting with Eq. (14) as the basic relation, a set of input-output relations for the aerodynamic system can be developed which is analogous to the set of Eqs. (5) to (12). In this case, however, the input functions depend on four scalar variables; and the system functions depend on five scalar variables.

The system function \( h_{zw}(t, \hat{r}, \tau) \), as stated earlier, is the \( Z \) force at time \( t \) resulting from an impulse in \( w \) at \( (\hat{r}, t-\tau) \), i.e. 
\[ w = \delta(\hat{r}-\hat{r}) \delta(t-\tau) \]. Such an impulsive gust is difficult to visualize and impossible to generate physically although it entails no mathematical difficulties in principle. The alternative, analogous to the transfer function of ordinary system analysis, is to represent the spatial velocity pattern as a superposition of traveling sinusoidal shear waves. This is a slight generalization of the representation implied by Liepmann\(^7\), and stated explicitly by Ribner\(^12\), and Etkin\(^13\) which eliminates the need for assuming a frozen gust pattern. Some intermediate representations have also been utilized\(^15,16\). A set of these input-output relations will now be given functionally and schematically in the manner of Fig. 1 and Eqs. (5) to (12).

First, Fig. 4 corresponds to Eq. (14) where the limits on the space integration may be extended to infinity by reasons analogous to those allowing extension of the lower limit on the \( \tau \) integration. The function \( h_{zw}(t, \hat{r}, \tau) \) is nonzero only on the body surface.

\[ Z_w(t) = \int \int \int_{\mathbb{B}} h_{zw}(t, \hat{r}, \tau) w(\hat{r}, t-\tau) \, d\tau \, d\hat{r} \]

Fig. 4. Representation of Response as Superposition of Responses to Impulsive Gusts

Next an intermediate transformation is considered. Let the gust field be represented as a superposition of sinusoidal waves traveling in the direction of the \( ox' \) axis but with amplitude a function of \( y' \) and \( z' \). These are not three dimensional
shear waves, but simply sinusoidal velocity variations along lines \( y' = c, z' = c \). This representation is indicated mathematically by the two dimensional Fourier transform pair of Eqs. (15) and (16). There are limitations, of course, on the velocity variations which are imposed by the fluid dynamic equations. These restrictions are assumed to be satisfied by all the subject velocity fields.

\[
w(r, t) = \iint S_{wx}(k_x, y', z', f) \exp\{j2\pi(k_x r + ft)\} dk_x df
\]  

(15)

\[
S_{wx}(k_x, y', z', f) = \iint w(r, t) \exp\{-j2\pi(k_x r + ft)\} dx'dt
\]  

(16)

This leads to the input-output relation symbolized by Fig. 5. In this representation the system function, \( H_{zw}(t, k_x, y', z', f_i) \exp(2\pi f_i t) \) is the contribution to the \( Z \) force of a unit amplitude sinusoid of wave number \( k_x \) and frequency \( f_i \) applied to an elemental surface at \((y', z')\). This is the time variable version of the system functions used by the panel methods of treating response to turbulence\(^{16,18}\). The relation between the system functions of Fig. 4 and Fig. 5 is given by Eqs. (17) and (18). The stationary form of this system function

\[
H_{zw}(t, k_x, y', z', f_i) = \iiint h_{zw}(t, z', \tau) \exp\{j2\pi(k_x r - f_i \tau)\} dk_x df_i
\]  

(17)

\[
h_{zw}(t, z', \tau) = \iiint H_{zw}(t, k_x, y', z', f_i) \exp(-j2\pi(k_x r - f_i \tau)) dk_x df_i
\]  

(18)

is useful when two dimensional theory is applied stripwise to wings. It will not be used in the propeller analysis, but is included here to indicate connections to other analyses which are in the literature. Comparison of Fig. 5 and Eq. (17) with Eqs. (6), (10), and (11) indicates the procedure for obtaining the output force in the frequency domain.
There are other intermediate transforms which could be discussed with physical interpretation. However, for present purposes of illustration the above example seems sufficient. Fig. 6 symbolizes the input-output relations with the input represented by the previously mentioned superposition of traveling sinusoidal shear waves. Here $f_i$ is the frequency of the velocity variation which would be measured by a stationary observer, i.e. an observer with zero velocity relative to the $o'x'y'z'$ frame. Fig. 6a indicates the output in the time domain while Fig. 6b indicates the output in the frequency domain.

$$S_w(k, f_i) \rightarrow H_{zw}(t, k, f_i) \rightarrow Z(t)$$

$$Z(t) = \iint_{-\infty}^{\infty} H_{zw}(t, k, f_i) \exp\{j2\pi f_i t\} S_w(k, f_i) \, dk \, df_i$$

Fig. 6a. Representation of Response as a Superposition of Responses to Sinusoidal Shear Waves

$$S_w(k, f_i) \rightarrow H_{zw}(f_0, k, f_i) \rightarrow S_z(f_0)$$

$$S_z(f_0) = \iint_{-\infty}^{\infty} H_{zw}(f_0, k, f_i) S_w(k, f_i) \, dk \, df_i$$

Fig. 6b. Representation of Output Spectrum as Superposition of Frequency Responses to Sinusoidal Shear Waves

The representation of the input indicated on Fig. 6 is related to the gust space-time function by the four dimensional Fourier transform pair of Eqs. (19) and (20).

$$w(\hat{r}, t) = \iiint_{-\infty}^{\infty} S_w(k, f) \exp\{j2\pi (k \cdot \hat{r} + ft)\} \, dk \, df$$

(19)

$$S_w(k, f) = \iiint_{-\infty}^{\infty} w(\hat{r}, t) \exp\{-j2\pi (k \cdot \hat{r} + ft)\} \, d\hat{r} \, dt$$

(20)
The system functions or frequency responses indicated on Fig. 6 are related to the system function or impulse response of Fig. 4 by the relations indicated by Eqs. (21) to (24).

\[
H_{zw}(t,k_f_i) = \iint_{-\infty}^{\infty} h_{zw}(t,\hat{r},\tau)e^{j2\pi(k\cdot\hat{r}-f_i\tau)}d\hat{r}d\tau \tag{21}
\]

\[
h_{zw}(t,\hat{r},\tau) = \iint_{-\infty}^{\infty} H_{zw}(t,k_f_i)\exp{-j2\pi(k\cdot\hat{r}-f_i\tau)}dkdf_i \tag{22}
\]

\[
H_{zw}(f_0,k_f_i) = \int_{-\infty}^{\infty} H_{zw}(t,k_f_i)\exp{j2\pi(f_i-f_0)t}dt \tag{23}
\]

\[
H_{zw}(t,k_f_i) = \int_{-\infty}^{\infty} H_{zw}(f_0,k_f_i)\exp{-j2\pi(f_i-f_0)t}df_0 \tag{24}
\]

These relations are analogous to Eqs. (7) and (11). The system function \(H_{zw}(t,k_f_i)\) is the complex amplitude at frequency \(f_i\) of the \(Z\) force resulting on passage through a sinusoidal shear wave of unit amplitude with wave vector \(k\) traveling with speed \((f_i/k)\). This is a generalized time varying frequency response. The system function \(H_{zw}(f_0,k_f_i)\) is the spectrum of the \(Z\) force time history on passage through such a traveling shear wave.

5.3.3 A reduction to previous results

For illustrative purposes, a reduction to a form of Liepmann will be performed. This is the case of a rectangular wing lying in the \(x-y\) plane. First, Eqs. (21) and (23) are combined to form Eq. (25). Note that Eq. (25) is not restricted to the planar case.

\[
H_{zw}(f_0,k_f_i) = \iiint_{-\infty}^{\infty} h_{zw}(t,\hat{r},\tau)e^{j2\pi(k\cdot\hat{r}-f_i\tau)}\exp{j2\pi(f_i-f_0)t}d\hat{r}d\tau \tag{25}
\]

It is assumed that a stripwise application of two dimensional theory is permissible, that the wing is sensitive only to gusts at \(z = 0\), and that the airfoil response is independent of \(y\). (i.e., Two dimensional airfoil theory is used.) This yields an impulse response of the form of Eq. (26).
\[ h_{zw}(t, \hat{\tau}) = h_{zw}(t, x', \tau) \delta(z') ; \quad -b \leq y' \leq b \tag{26} \]

where \( 2b \) is the span of the wing.

In \( \text{oxyz} \) coordinates:

\[ \hat{\tau}(t-\tau) = (U(t-\tau)+x)\hat{i}+y\hat{j}+zk \tag{27} \]

where \( o \) has been assumed to be at the center of the wing. This transformation with Eq. (26) and the known stationary or time invariant characteristics yields

\[ h_{zw}(t, \hat{\tau}, \tau) = h_{zw}(x, \tau) \delta(z) ; \quad -b \leq y \leq b \tag{28} \]

Eqs. (27) and (28) are substituted into Eq. (25) with \( f_1 = 0 \) to correspond to Liepmann's case. The result may be written:

\[ H_{zw}(f_0, k, f_1) = \int \int \int h_{zw}(x, \tau) \exp\{j2\pi k_x(x-U\tau)\} dx d\tau \]

\[ \times \int_{-\infty}^{\infty} \exp\{j2\pi k_y y\} dy \int_{-\infty}^{\infty} \exp\{j2\pi k_z z\} dz \]

\[ \times \int_{-\infty}^{\infty} \exp\{j2\pi(k_x U-f_0) t\} dt \]

The integrations over \( y, z, \) and \( t \) may be evaluated immediately. By comparison of the geometry of the problem with that of Sears' \(^3\) and the definition of \( h_{zw}(x, \tau) \), the result of the integration over \( x \) and \( \tau \) may be seen to be a Sears function \( \pi \rho cUS(\pi k_x c) \)

where \( c \) is the airfoil chord. See Sears Eqs. (9), (10), and (22). Thus

\[ H_{zw}(f_0, k) = \pi \rho cUS(\pi k_x c)(1/\pi k_y) \sin(2\pi k_y b) \delta(k_x U-f_0) \tag{29} \]

where the span of the wing is \( 2b \).

Eq. (29) corresponds to Liepmann's \(^7\) Eq. (14a). Here the fact of the single frequency output (which may in the time invariant case be obtained by inspection) has been recovered formally. In more difficult time varying cases the formal approach appears indispensable. Note also that the result of Eq. (29) is independent of \( k_z \) so that the integration over \( k_z \) indicated on Fig. 6b can in principle be carried out independently of the
system function and a two dimensional spectrum obtained before transition to the spectral density form where this step is usually indicated\textsuperscript{12},\textsuperscript{13}. This observation is not of practical import in analysis of response to turbulence, but offers a further piece in the overall puzzle.

Note that in general a set of transfer relations from each fluid velocity component to each force and moment component is required for dynamic analyses.
PART III. THE PROPELLER RESPONSE AND TRANSFER FUNCTIONS

6. The Propeller in Nonuniform Flow

6.1 Purpose and scope:

This section and section 7 contain the primary results of this study. It was for this application that the general ideas and specific forms of the preceding two sections were considered. Approximate relations are developed which are utilized to compute the transfer functions of a specific propeller. These results will be applied in Part IV to compute response to turbulence.

The approach taken is to consider an idealization of a propeller as a set of rotating lifting lines which behave locally in two dimensional quasistatic fashion. The justification for this approach rests on the success of techniques given by Ribner for computing side forces and pitching moments on propellers in yaw which do not include detailed consideration of the unsteady aerodynamics. The local blade elements of a propeller in pitch moving through a uniform stream experience periodic inflow with the period determined by the shaft angular speed. In the present application to flight in atmospheric turbulence, the time variation of the inflow at a blade element will be determined principally by shaft rate since the turbulent flow field variation will be slow compared to the angular speed of the propeller. It is, therefore, concluded that a quasistatic blade element analysis applied to the turbulence case will have substantially the same accuracy as similar techniques applied to propellers in yaw in a uniform stream. A plausible correction by a Sears function factor is included in the analysis.

The only available experimental check is a comparison of results for the case of a propeller at angle of attack. The most complete such data appears to be that of Yaggy and Rogallo\textsuperscript{36} and it is this condition which has prompted the transfer function calculations to be performed for their particular propeller.

6.2 The model of the propeller:

6.2.1 Representation as rotating lines

As mentioned above, the propeller is represented by a set of rotating lines. Fig. 7 corresponds to Fig. 2 for this problem. A three blade propeller is illustrated. The blade position $\theta_m$ is given by:

$$\theta_m = \Omega t + \frac{2\pi}{B}(m-1)$$  \hspace{1cm} (30)
where \( B \) is the number of blades and \( m \) is the blade index.

\[
\begin{align*}
    r_m' &= \left( Ut + r \sin \theta_m \cos \alpha_T \right) \hat{i} + \left( r \cos \theta_m \right) \hat{j} + \left( r \sin \theta_m \cos \alpha_T \right) \hat{k} \\
    \end{align*}
\]

Here \( 0 \leq r \leq R \). Eq. (31) for fixed \( r \) defines the path along which a particular blade element samples the velocity field. If \( \alpha_T = 0 \), it is, of course, a helix. The input velocity field is assumed to be given in terms of the components on the \( oxyz \) frame. These are indicated by \((u,v,w)\) on Fig. 7. The blade element geometry and relative velocity are shown by Fig. 8.

The primes on force expressions do not refer to a coordinate system but indicate association with a blade element or equivalently a partial derivative of the unprimed quantity with respect to the radial coordinate. On Fig. 8a are shown the relative velocity components and forces when \( u=v=w=0 \). All the force and velocity components are considered positive in the senses shown on the figures. The contributions to the relative velocity indicated by \( U_i \) and \( W_i \) are the result of induction effects associated with the propeller forces with \( u=v=w=0 \). The contributions of the blade element to these forces are indicated by \( X_i'(r) \) and \( F_i'(r) \). For present purposes these forces and associated flow field are assumed to be known or specified quantities. Methods given by Ribner\(^3\) may be used to calculate
Fig. 8a. Blade Element Geometry with Relative Velocities and Force Components for \( u' = v' = w' = 0 \)

Them for \( \alpha_T \) small and light loading.

Fig. 8b indicates the effect of flow field nonuniformity. Here it is assumed that the induction associated with the perturbation forces is negligible so that the total relative velocity is sufficiently approximated as shown on Fig. 8b. It is assumed also that the perturbations, \( (X'(r), F'(r)) \), of the force on the blade element are linearly related to the instantaneous local flow field values \( (u, v, w) \).
6.2.2 Expressions for blade element force perturbations

Fig. 9 shows the lift and drag for the reference \( u=v=w=0 \) and perturbed conditions. The perturbation force components, \([X'(r), F'(r)]\), shown on Fig. 8b are composed of the corresponding components of the perturbations \( dL' \) and \( dD' \) shown on Fig. 9.
The detail development of expressions for blade element contributions to force and moment components is given in dimensional form in Appendix A and corresponding nondimensional relations are defined in Appendix B. The results are summarized here.

\[ \lambda = \frac{U}{\Omega R}, \quad \eta = \frac{r}{R}, \quad \sigma = \frac{C}{R}, \quad \nu_0 = \frac{W_0}{\Omega R}, \quad \nu_i = \frac{W_i}{\Omega R}, \]

\[ u_i = \frac{U_i}{\Omega R} \]  \hspace{1cm} (32)

\[ \nu_0 = \eta + \lambda \cos \theta \sin \alpha_T - \nu_i \]  \hspace{1cm} (33)

\[ \lambda_0 = \lambda \cos \alpha_T + u_i \]  \hspace{1cm} (34)
It is cautioned that \((u,v,w)\) are used as both dimensional and dimensionless variables. The expressions in which they appear are always such that the intent is evident. The force and moment derivative expressions are listed below.

\[
F'_{cu}(\theta, \eta) = \sigma[\alpha_0 v_0^2 + \alpha_0^2 v_0 + 2\alpha_0^2 v_0^2] - c_{D_0} \lambda_0 v_0] / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
F'_{cw}(\theta, \eta) = \sigma[\alpha_0^2 + \alpha_0^2 \lambda_0 v_0] + c_{D_0} [\lambda_0^2 + 2v_0^2]] \sin \theta / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
F'_{cv}(\theta, \eta) = -\sigma[\alpha_0^2 + \alpha_0^2 \lambda_0 v_0] + c_{D_0} [\lambda_0^2 + 2v_0^2]] \cos \theta / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
x'_{cu}(\theta, \eta) = \sigma[\alpha_0 v_0 + \alpha_0 \lambda_0 v_0^2] + c_{D_0} [\lambda_0^2 + 2v_0^2]] \sin \theta / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
x'_{cw}(\theta, \eta) = -\sigma[\alpha_0 v_0 + \alpha_0 \lambda_0 v_0^2] - c_{D_0} \lambda_0 v_0] \cos \theta / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
x'_{cv}(\theta, \eta) = \sigma[\alpha_0 v_0 + \alpha_0 \lambda_0^2 + 2\alpha_0 v_0^2] - c_{D_0} \lambda_0 v_0] \sin \theta / (\lambda_0^2 + v_0^2)^{1/2}
\]

\[
y'_{cu}(\theta, \eta) = F'_{cu}(\theta, \eta) \sin \theta
\]

\[
y'_{cw}(\theta, \eta) = -F'_{cw}(\theta, \eta) \cos \theta
\]

\[
y'_{cv}(\theta, \eta) = -F'_{cv}(\theta, \eta) \cos \theta
\]

\[
z'_{cu}(\theta, \eta) = -F'_{cu}(\theta, \eta) \cos \theta
\]

\[
z'_{cw}(\theta, \eta) = -F'_{cw}(\theta, \eta) \cos \theta
\]

\[
z'_{cv}(\theta, \eta) = -F'_{cv}(\theta, \eta) \cos \theta
\]

\[
l'_{cu}(\theta, \eta) = -\eta F'_{cu}(\theta, \eta)
\]

\[
l'_{cv}(\theta, \eta) = -\eta F'_{cv}(\theta, \eta)
\]

\[
l'_{cw}(\theta, \eta) = -\eta F'_{cw}(\theta, \eta)
\]
\[ M_{cu}(\theta, \eta) = nX_{cu}(\theta, \eta) \sin \theta \] (50)

\[ M_{cv}(\theta, \eta) = nX_{cv}(\theta, \eta) \sin \theta \] (51)

\[ M_{cw}(\theta, \eta) = nX_{cw}(\theta, \eta) \sin \theta \] (52)

\[ N_{cu}(\theta, \eta) = -nX_{cu}(\theta, \eta) \cos \theta \] (53)

\[ N_{cv}(\theta, \eta) = -nX_{cv}(\theta, \eta) \cos \theta \] (54)

\[ N_{cw}(\theta, \eta) = -nX_{cw}(\theta, \eta) \cos \eta \] (55)

### 6.2.3 Interpretation as an impulse response

The quantities defined by Eqs. (38) through (55) may be interpreted as the elements of an impulse response matrix in the sense of Fig. 4. As indicated previously the use of quasi-static aerodynamics implies the \( \tau \) dependence to consist of an impulse function \( \delta(\tau) \) as a factor so that \([h_{zw}(t, r, \tau)] = h_{zw}(t, r) \delta(\tau)\). Furthermore, according to the lifting line model, the propeller is sensitive only to the fluid velocity instantaneously along the lines \((x=0; \theta = \theta_m, 1 \leq m \leq B; 0 \leq r \leq R)\) where a cylindrical polar coordinate system on the oxyz frame is implied. This information is incorporated in the mathematical relation as follows.

\[ [h_{zw}(t, r) \delta(\tau)] = [h_{zw}(r, \theta)] \delta(\tau) \delta(x) \delta(\theta-\theta_m); 1 \leq m \leq B, 0 \leq r \leq R \]

\[ = 0 \quad r > R \] (56)

The expression corresponding to Fig. 4 now becomes:

\[ Z(t) = \int_0^B \int_0^R \int_{-\infty}^{\infty} [h_{zw}(r, \theta)] \delta(\tau) \delta(x) \delta(\theta-\theta_m) w(x, r, \theta, t-\tau) d\tau dx d\theta \]

\[ = \sum_{m=1}^{B} \int_0^R [h_{zw}(r, \theta_m)] w(r, \theta_m, t) dr \] (57)
Here $\theta_m$ implicitly contains the time variable. See Eq. (30).

If $w$ is known in $o'x'y'z'$ coordinates, the proper result for use in Eq. (57) may be obtained simply by writing $w(r'_m)$ in place of $w(r, \theta_m, t)$ where $r'_m$ is given by Eq. (31). Eq. (57), which gives force and moment components on $oxyz$ when the elements of $[h_{zw}(r, \theta_m)]$ are defined by Eqs. (38) through (55) and $w$ is given by components on $oxyz$, can be simply transformed to yield the same result with input $w'$ in terms of components on $o'x'y'z'$. In terms of dimensionless variables with $\theta_1$ as the time variable:

$$z_c(\theta_1) = \sum_{m=1}^{B} \int_0^1 [h'_zw(\theta_m, \eta)]w'(\theta_m, \eta, \theta_1)d\eta$$

where

$$[h'_zw(\theta_m, \eta)] = [h_{zw}(\theta_m, \eta)] \begin{bmatrix} \cos \alpha_T & 0 & -\sin \alpha_T \\ 0 & 1 & 0 \\ \sin \alpha_T & 0 & \cos \alpha_T \end{bmatrix}$$

With this interpretation, the general relations of section 5 can be used to derive various corresponding forms of the propeller transfer functions.

7. The Propeller Transfer Functions

7.1 Introduction and general relations:

In this section the procedures are given for detailed evaluation of the propeller transfer functions on the basis of the propeller model of section 6 and the definition of Eq. (25). The transfer functions will be developed in terms of wave number components on the $o'x'y'z'$ system with the input sinusoidal velocity components considered to be on the $o'x'y'z'$ frame but the output force and moment components on the $oxyz$ frame.

The special case of Eq. (25) defining the propeller transfer function can now be written. It may be helpful to consider the impulse response first in the form given as the integrand of Eq. (57). In dimensional form:
This is the lifting line, quasistatic propeller transfer function or generalized frequency response. It yields the spectra of the various force and moment components as functions of \( f_0 \) on passage through a travelling sinusoidal wave with wave number \( k \) and temporal frequency \( f_1 \).

The dimensionless form of Eq. (59) utilizing the variables introduced in section 6 and Appendix B is given by Eq. (60) where no distinction is made by notation for the coefficient form of the transfer function or the dimensionless wave number components and frequency variables.

\[
[H_{zw}(f_0,k,f_1)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{1} \left[ h'_{zw}(r,\theta_m) \right]
\times \exp\left\{ j2\pi[ k_x(U+rsin\alpha_T+sin\theta_m)+k_yrcos\theta_m \right.
\left. +k_zrcos\theta_m+\left(f_1-f_0\right)t] \right\} dt dr 
\]

In Eq. (60), the frequency variables \((f_1,f_0)\) are in units of shaft frequency and the wave number components are in units of wave lengths per propeller radius.

A somewhat simpler form can be obtained by introducing the following variables.

\[
k_1 = 2\pi k_x\lambda+ f_1- f_0
\]

\[
k_\theta = \left[ (k_y^2+(k_x\sin\alpha_T+k_z\cos\alpha_T)^2) \right]^{1/2}
\]

\[
\theta_k = \tan^{-1}\left[ \frac{(k_x\sin\alpha_T+k_z\cos\alpha_T)}{k_y} \right]
\]

It is seen that \( \theta_k \) is the component of the wave number vector in the plane of rotation, \( yz \), and \( \theta_k \) is the angle which this component makes with the \( oy \) axis. In terms of these variables, Eq. (60) becomes:

\[
[H_{zw}(f_0,k,f_1)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{1} \left[ h'_{zw}(r,\theta_m) \right]
\times \exp\left\{ j2\pi[ k_x(U+rsin\alpha_T+sin\theta_m)+k_yrcos\theta_m \right.
\left. +k_zrcos\theta_m+\left(f_1-f_0\right)t] \right\} dt dr 
\]
\[ [H_{zw}(f_0, k, f_1)] = \frac{1}{2\pi} \int_{m=1}^{B} \int_{0}^{\infty} [h^t_{zw}(\eta, \theta_m)] \times \exp\{j[k_1\theta_1 + 2\pi k_\theta \eta \cos(\theta_m - \theta_k)]\} d\theta_1 d\eta \quad (62) \]

The elements of the impulse response matrix \([h^t_{zw}(\eta, \theta)]\), as in section 6, are generally periodic functions of \(\theta\) with period \(2\pi\). They may, therefore, be expanded in Fourier series as in Eq. (63).

\[ [h^t_{zw}(\eta, \theta)] = \sum_{l=-\infty}^{\infty} [h^l_{zw}(\eta)] \exp{-jl\theta} \quad (63) \]

Evaluation on the mth blade gives:

\[ [h^l_{zw}(\eta, \theta_m)] = \sum_{l=-\infty}^{\infty} [h^l_{zw}(\eta)] \exp{-j\theta_m} \quad (64) \]

where, recalling Eq. (30):

\[ \theta_m = \theta_1 + \frac{2\pi}{B}(m-1) \quad (65) \]

The factor \(\exp\{j2\pi k_\theta \eta \cos(\theta_m - \theta_k)\}\) of the integrand of Eq. (62) can be written as an infinite series in the form:

\[ \exp\{j2\pi k_\theta \eta \cos(\theta_m - \theta_k)\} = \sum_{n=-\infty}^{\infty} j^n J_n(2\pi k_\theta \eta) \cos(\theta_m - \theta_k) \quad (66) \]

or as Eq. (67) if the cosine is put in exponential form as will later prove convenient.

\[ \exp\{j2\pi k_\theta \eta \cos(\theta_m - \theta_k)\} = \frac{1}{2} \sum_{n=-\infty}^{\infty} j^n J_n(2\pi k_\theta \eta)(\exp\{jn(\theta_m - \theta_k)\}) + \exp\{-jn(\theta_m - \theta_k)\} \quad (67) \]

where \(J_n\) is the Bessel function of order \(n\).

Define a variable:

\[ k_{\theta, n} = k_\theta + \ell \pm n \quad (68) \]

Then utilizing the preceding expressions in the integrand of Eq. (62):
\[ [H_{zw}(f_0, k, f_i)] = \frac{1}{2\pi} \sum_{m=1}^{B} \int \int_{0}^{\infty} \sum_{k=-\infty}^{\infty} [h_{zw}(\eta)] \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} j^{n} j_{n}(2\pi k \theta n) \]

\times \left\{ (\exp{jk_{+n} \theta_1}) (\exp{-jn \theta_k}) (\exp{j(n-\ell)\frac{2\pi}{B}(m-1)}) \right\}

\[ + \left( \exp{jk_{-n} \theta_1} \right) (\exp{jn \theta_k}) (\exp{-j(n+\ell)\frac{2\pi}{B}(m-1) \right\}} \]

\times d\theta_1 d\eta \quad (69)\]

The integration over \( \theta_1 \) yields a set of delta functions the arguments of which define a set of discrete frequencies for the output. The form is now:

\[ [H_{zw}(f_0, k, f_i)] = \frac{1}{2\pi} \sum_{m=1}^{B} \int \int_{0}^{\infty} \sum_{k=-\infty}^{\infty} [h_{zw}(\eta)] \sum_{n=-\infty}^{\infty} j^{n} j_{n}(2\pi k \theta n) \]

\times \left\{ (\exp{-jn \theta_k}) (\exp{j(n-\ell)\frac{2\pi}{B}(m-1)}) \delta(k^{+\ell \theta_1}) \right\}

\[ + \left( \exp{jn \theta_k} \right) (\exp{-j(n+\ell)\frac{2\pi}{B}(m-1)}) \delta(k^{-\ell \theta_1}) \right\} d\eta \quad (70)\]

From Eqs. (61), (68), and (70), it is seen that the output frequencies are given by:

\[ f_0 = 2\pi k_x \lambda + f_i \pm p \; ; \; p \text{ an integer or zero, } 0 \leq p \quad (71)\]

The amplitudes at the various frequencies depend on propeller geometry as will be developed. Many of the frequencies will have zero amplitude in certain cases.

It is observed that the dependence on blade index is contained in the factors,

\[ \exp{j(\pm n-\ell)\frac{2\pi}{B}(m-1)} \]

By changing the order of the summations, interior factors of the form of Eq. (72) can be obtained.
\[ \sum_{m=1}^{B} \exp\{jpB^{2\pi}(m-1)\} \quad p \text{ an integer or zero} \quad (72) \]

Letting \( x = \exp\{jpB^{2\pi}\} \), this becomes:

\[ \sum_{m=1}^{B} x^{m-1} = \]

\[ \begin{cases} B & \text{if } x = 1 \\ \frac{1-x^n}{1-x} & \text{if } x \neq 1 \end{cases} \quad (73) \]

Now \( x^B = 1 \) for all \( B \) and \( p \) of physical significance; and \( x = 1 \) if \( \frac{p}{B} \) is an integer or zero. Therefore define:

\[ B(p) = \begin{cases} B & \text{if } \frac{p}{B} \text{ an integer or zero} \\ 0 & \text{otherwise} \end{cases} \quad (74) \]

Utilizing this result, Eq. (70) is rewritten.

\[ [H_{zw}(f_0, k, f_i)] = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} j^n [h_{zw}(n)] J_n(2\pi k_\theta n)dn \]

\[ \times \{ \exp(-jn\theta)^k \delta(k+n) B(n-l) + \exp(jn\theta)^k \delta(k-n) B(n+l) \} \quad (75) \]

Equation (75) appears to be reduced to as simple a form as possible in general terms. Though not particularly simple, it provides a relatively straightforward procedure for computing propeller frequency response.

Several general features of the equation are worthy of note. First, the frequencies at which output may appear are determined by \( k_x \) and \( f_i \). Second, the amplitudes of the outputs at those frequencies are determined by \( k_\theta \). Third, the spacing of permissible frequencies increases with blade number, which is reasonable. Permissible frequencies are, in fact, the sums and differences of the effective input frequency with multiples of the blade frequency.
7.2 Axial motion:

7.2.1 General

In the case of axial motion, there are a number of simplifications. From considerations of symmetry it follows that the elements of the impulse response matrix as given by Eqs. (35) through (55) have parameters \( v_0, \lambda_0, \alpha_0, \) and \( c_D \) independent of \( \theta \). Also the transformation matrix of Eq. (58) becomes the identity matrix. The net result is that the \( \theta \) dependence is explicitly given by the trigonometric factors appearing in Eqs. (35) through (55). From this it follows that the Fourier series representation of Eq. (63) will contain only terms with \(-2 \leq l \leq 2\).

The expressions for \( k_\theta \) and \( \theta_k \) of Eq. (61) become:

\[
\begin{align*}
    k_\theta &= (k^2 + k_z^2)^{1/2} \\
    \theta_k &= \tan^{-1}\left(\frac{k_z}{k_y}\right)
\end{align*}
\]  

(76)

7.2.2 Single blade

The elements of the transfer function matrix which pertain to the normal force coefficient \( Z_c \) and the yawing moment coefficient \( N_c \) will now be considered for a single blade. From Eqs. (35)-(40), (44)-(46), and (53)-(55):

\[
\begin{align*}
    Z'_c (\theta, \eta) &= -\sigma [a(-\alpha_0 \lambda_0 \nu_0 - 2\alpha_0 \lambda_0^2 - c_D \lambda_0 \nu_0)] \cos \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2} \\
    Z'_cv (\theta, \eta) &= -\sigma [a(\lambda_0^2 + \alpha_0 \lambda_0 \nu_0) + c_D \lambda_0 (\lambda_0^2 + 2\nu_0^2)] \sin \theta \cos \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2} \\
    Z'_cw (\theta, \eta) &= \sigma [a(\lambda_0^2 + \alpha_0 \lambda_0 \nu_0) + c_D \lambda_0 (\lambda_0^2 + 2\nu_0^2)] \cos^2 \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2} \\
    N'_c (\theta, \eta) &= -\pi \sigma [a(\lambda_0 \nu_0 - \alpha_0 \lambda_0 \nu_0) + c_D (2\lambda_0^2 + \nu_0^2)] \cos \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2} \\
    N'_cv (\theta, \eta) &= -\pi \sigma [a(\lambda_0 \nu_0 + \alpha_0 \lambda_0 \nu_0 + 2\alpha_0 \nu_0^2) - c_D \lambda_0 \nu_0] \sin \theta \cos \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2} \\
    N'_cw (\theta, \eta) &= \pi \sigma [a(\lambda_0 \nu_0 + \alpha_0 \lambda_0 \nu_0 + 2\alpha_0 \nu_0^2) - c_D \lambda_0 \nu_0] \cos^2 \theta / (\lambda_0^2 + \nu_0^2) \frac{1}{2}
\end{align*}
\]
By comparison define $F_1'(\eta)$, $F_2'(\eta)$, $X_1'(\eta)$, and $X_2'(\eta)$ as:

$$Z_{cu}'(\theta, \eta) = -F_1'(\eta) \cos \theta$$
$$N_{cu}'(\theta, \eta) = -\eta X_1'(\eta) \cos \theta$$

$$Z_{cv}'(\theta, \eta) = -F_2'(\eta) \sin \theta \cos \theta$$
$$N_{cv}'(\theta, \eta) = -\eta X_2'(\eta) \sin \theta \cos \theta$$

$$Z_{cw}'(\theta, \eta) = F_2'(\eta) \cos^2 \theta$$
$$N_{cw}'(\theta, \eta) = \eta X_2'(\eta) \cos^2 \theta$$

The Fourier expansions of Eq. (63) for the normal force expressions are:

$$Z_{cu}'(\theta, \eta) = -F_1'(\eta) \left(\frac{1}{2} \exp\{-j(-2)\theta\} + \frac{1}{2} \exp\{-j(+2)\theta\}\right)$$

$$Z_{cv}'(\theta, \eta) = -F_2'(\eta) \left(-\frac{1}{4} j \exp\{-j(-1)\theta\} + \frac{1}{4} j \exp\{-j(+1)\theta\}\right)$$

$$Z_{cw}'(\theta, \eta) = +F_2'(\eta) \left(-\frac{1}{4} \exp\{-j(-1)\theta\} + \frac{1}{4} \exp\{-j(+1)\theta\}\right)$$

and similarly for the yawing moment expressions. For these elements there will be only two or three terms in the summation over $\lambda$ of Eq. (75). Also the integrals to be evaluated vary only with $\eta$ since the $\eta$ dependence in Eq. (84) does not vary from term to term.

Define:

$$F_1^n(k_\theta) \equiv \int_0^1 F_1'(\eta) J_n(2\pi k_\theta \eta) d\eta$$

$$F_2^n(k_\theta) \equiv \int_0^1 F_2'(\eta) J_n(2\pi k_\theta \eta) d\eta$$

$$X_1^n(k_\theta) \equiv \int_0^1 \eta X_1'(\eta) J_n(2\pi k_\theta \eta) d\eta$$

$$X_2^n(k_\theta) \equiv \int_0^1 \eta X_2'(\eta) J_n(2\pi k_\theta \eta) d\eta$$

Eq. (75) with (84)-(86) yield Eqs. (89)-(91) for the normal force transfer functions.
The expressions for the yawing moment transfer functions are the same as (89)-(91) with $X_1^n(k\theta)$, $X_2^n(k\theta)$ replacing $F_1^n(k\theta)$ and $F_2^n(k\theta)$. The number of significant terms in the summations above is determined by the relative sizes of $F_1^n(k\theta)$ for the maximum value of $k$ to be considered. The maximum value of this parameter must be a compromise between the desire to treat short wave lengths and the expected deterioration of the quasistatic aerodynamic model as the wave length decreases.

7.2.3 Complete propellers

The transfer functions for the complete propeller of $B$ blades have the same forms as those for a single blade except for the factors $B(n-\ell)$ and $B(n+\ell)$ as shown in Eq. (75). The effect of these factors is to remove frequencies for which $\frac{n+\ell}{B}$ is not an integer or zero and to multiply the remaining terms by $B$. The principal computation required for a complete propeller remains that of evaluating the functions defined by Eqs. (85)-(88).
The calculations indicated by Eqs. (86) and (88) have been carried out for several cases using the data of propeller number 1 from NASA TND-318 and a method detailed in Appendix C. The functions \( F_n^2(k_\theta) \) and \( X_n^2(k_\theta) \) are shown on Figures 10-13 for \( \beta = 25^\circ, J = 1.0, \) and \( \beta = 30^\circ, J = 1.4. \) The curves have been normalized by the value of the respective functions with \( n = 0 \) and \( k_\theta = 0 \) since this represents a value for uniform flow as will be shown in the next paragraphs. It is seen that the \( n = 0 \) terms are dominant for \( k_\theta \leq 0.2. \) This is true for other values of blade setting and advance ratios also. Eqs. (89)-(91) indicate that the \( n = 0 \) term contributes to sinusoidal outputs at frequencies with \( f_0 - 2k_\theta J - f_i = \pm \ell \) where \( \ell = 0,1, \) and 2. The \( n = 1 \) term contributes to sinusoids similarly with \( f_0 - 2k_\theta J - f_i = \pm 1 \pm \ell, \ell = 0,1, \) and 2 so that in addition to reinforcing the frequencies of the \( n = 0 \) terms, an additional frequency is added. This is a feature of each term in succession as \( n \) increases. Discussion of the importance of the terms for \( n \neq 0 \) and the range of \( k_\theta \) will be taken up in Part IV on response to turbulence since it is in conjunction with the input characteristics that these points become clarified.

7.2.4 Reduction to "point approximation"

Problems of flight in turbulence have frequently been treated by a "point approximation" wherein the turbulent velocity is assumed uniform at any instant over the extent of the vehicle but varies with time. In the present notation this is equivalent to considering only those wave lengths for which \( k_\theta = 0. \) When this condition holds Eqs. (85)-(88) yield nonzero results only for \( n = 0 \) since \( J_0(0) = 1; J_n(0) = 0, n \neq 0. \) Equations (89)-(91) in this case have only the \( n = 0 \) term. The frequency content of the output for various blade numbers then reduces to that given by the author in reference 4. Namely, a single blade will produce normal force at frequencies:

\[
f_0 = 2\pi k_x \lambda + f_i ; 2\pi k_x \lambda + f_i + 1 ; \text{ and } 2\pi k_x \lambda + f_i + 2 \quad (92)
\]

A two bladed propeller will produce normal force at frequencies:

\[
f_0 = 2\pi k_x \lambda + f_i ; 2\pi k_x \lambda + f_i + 2 \quad (93)
\]

A propeller with three or more blades produces normal forces at frequencies:

\[
f_0 = 2\pi k_x \lambda + f_i \quad (94)
\]
Figure 10. Transfer Function Coefficient; $\frac{F_2^n(k_\theta)}{F_2^0(0)}$

$\beta = 25^\circ$, $J = 1.0$, $0 \leq n \leq B$

38
Figure 11. Transfer Function Coefficient; \( \frac{F^n(\theta)}{F_2^0(0)} \)

\( \beta = 30^\circ, J = 1.4, 0 \leq n \leq B \)
Figure 12. Transfer Function Coefficient; $\frac{x_2^n(k_\theta)}{x_2^0(0)}$

$\beta = 25^\circ$, $J = 1.0$, $0 \leq n \leq 5$
Figure 13. Transfer Function Coefficient; $\frac{x_2^n(k_\theta)}{x_2^0(0)}$

$\beta = 30^\circ, J = 1.4, 0 \leq n \leq B$
These statements hold for the yawing moment frequencies also. It is emphasized that the results stated in Eqs. (92) to (94) are valid only if the perturbation velocities are uniform over the entire propeller at any instant in time.

7.2.5 A correspondence to angle of attack and comparison with data

The case of \( k = f_i = 0 \) corresponds to a propeller operating in steady flow with small deviations from a reference condition. For example, a propeller operating at small \( \alpha_T \) in a uniform stream may be considered equivalent to a propeller in axial motion with a small zero frequency \( w \) perturbation. The transfer function expression for this limiting case will be compared with available data. This is the only case, so far as I know, for which experimental data exists. Even in this case, only the zero frequency or time average of the output is available for suitable propellers.

For this case Eq. (91) and the similar equation for yawing moment become:

\[
H_{Zw}(0) = F_2^0(0)\left(\frac{B}{2}\right) \tag{94}
\]

\[
H_{Nw}(0) = X_2^0(0)\left(\frac{B}{2}\right) \tag{95}
\]

These will be the same as \( H_{YV} \) and \( H_{Mv} \) for similar conditions with the propeller yawed.

For the case of uniform velocity over the propeller to which Eqs. (94) and (95) apply force and moment derivatives for the entire propeller with respect to the various gust components can be defined. In coefficient form for the \( Z \) force and yawing moment with respect to \( w \):

\[
Z_{cw} \equiv \frac{\partial \left( \frac{1}{2} \rho \omega^2 R^* \right)}{\partial (w/\omega R)} = H_{Zw}(0) \tag{96}
\]

\[
N_{cw} \equiv \frac{\partial \left( \frac{1}{2} \Omega^2 S^* \right)}{\partial (w/\omega R)} = H_{Nw}(0) \tag{97}
\]

Calculations for these quantities have been carried out by the method detailed in Appendix C for a propeller designated as propeller number 1 in NASA TND-318. The calculated results and experimental data from TND-318 are shown on Figures 14 and 15. Results of calculations by Ribner's method are also shown. Both techniques depend on previous availability.
Fig. 14. Normal Force Derivative as a Function of Advance Ratio and Blade Setting, Experimental Points from NASA TN-318, Propeller Number 1
Fig. 15. Yawing Moment Derivative as a Function of Advance Ratio and Blade Setting, Experimental Points from NASA TND-318, Propeller Number 1
of thrust data as a function of advance ratio and blade setting either from experiment or theoretical calculations. Based on the Z force calculations, the relative importance of the parameters \(a, \alpha_0\), and \(C_{D_0}\) appearing in Eqs. (77) to (82) shifts with advance ratio. Near the zero thrust condition, the effective lift curve slope is the dominant parameter. As the advance ratio decreases, the reference effective angle of attack through its effect on lift appears to become significant, and at still lower advance ratios the drag rise due to stall becomes dominant.

The trend of the curves computed by the present methods to agree at low advance ratios is accomplished by including a drag rise due to estimated stall conditions. Ribner's method could be modified to incorporate such a feature.

The significance of the variations in the two methods of computations and the experimental data is difficult to assess since the data on which the experimental curves are based was stated to have a scatter of \(\pm 9\) per cent from the faired curves. This variation overlaps the calculated curves for side force near the zero thrust condition.

The moment derivative calculations are much less in agreement with the data although the best agreement is near the zero thrust condition as expected. It seems reasonable that the moments are more strongly affected by the details of the flow pattern near the propeller blade tips. It is in these regions that the present methods fail since they do not incorporate three dimensional tip effects.

The accuracy appears adequate for evaluation of turbulence effects on lightly loaded propellers. Figs. 14 and 15 define the end points (for \(k = 0, n = 0\)) of families of transfer function coefficients such as those shown by Figs. 11 and 13.

### 7.3 Effect of angle of attack:

When a propeller operates with \(\alpha_T \neq 0\), normal forces and moments are generated. The time averages may be found from data such as Figs. 14 and 15 for \(\alpha_T\) small. In this case the blade loads vary with \(\delta\) even in the uniform flow case. The loads on the entire propeller, however, are constant if \(B \geq 3\).

The effect of \(\alpha_T \neq 0\) on the propeller transfer function may be seen by considering the forms of the elements of the impulse response matrix along with Eqs. (63) and (75). As already discussed, for axial motion the expansion of Eq. (63)
has terms only for $-2 \leq \ell \leq +2$ because the elements are exactly sinusoidal in this case. For nonaxial motion, there will be additional terms in the expansion. This means that the $n = 0$ term of Eq. (75) will now contribute to output at additional frequencies. The additional frequencies are all at higher multiples of blade frequency so it may be said that the output band width is increased when $\alpha_T \neq 0$. For $\alpha_T$ small, the elements of the impulse response matrix will still be nearly sinusoidal so that terms of Eq. (63) for $|\ell| > 2$ will have relatively small coefficients and, consequently, the additional terms occurring in Eq. (75) will be small.
8. Introduction

The time and space dependences of the atmospheric velocity field which are important to the prediction of stability and control effects and stress levels in aircraft are generally known only in a statistical sense. It follows that the resulting output quantities can be specified only in a similar manner. The problem in such a case is to predict a set of statistical parameters describing the output of interest from a corresponding set describing the input. In the case of flight in atmospheric turbulence, the input is generally given by the spectral densities or correlations of the velocity field along with the probability distribution of the intensities. Discussions of such descriptions are given by Etkin and Houbolt, Steiner and Pratt. The assumption is generally made that the probability distribution is normal or Gaussian although the supporting data is only tentative. This assumption is of major importance in the treatment of linear system response, since according to a theorem given by Sveshnikov, the output of any linear system, constant or variable parameter, will be normally distributed if the input is normally distributed. With this assumption, therefore, the inputs and outputs are completely described in a statistical sense when their means and correlations or spectral densities are given.

The problem to be considered in this part, therefore, is the determination of the correlations or spectral densities of the forces on a propeller from the correlations or spectral densities of the atmospheric turbulence.

There are well known input output relations for constant parameter linear systems subject to stationary random inputs. In the present instance, as detailed in Part III, the system under consideration has variable parameters. In such a case, the output is generally statistically nonstationary regardless of whether or not the input is stationary. Bendat and Piersol give input output relations for the one dimensional, single independent variable, nonstationary case. The multidimensional, single independent variable, relations have been given by the present author in a form considered to be somewhat more convenient for interpretation than the direct extension of the relations of Bendat and Piersol. In the following section similar relations are given for the case with inputs which are functions of four independent variables.

The nonstationary processes do not have averages over the independent variable equal to corresponding ensemble averages.
They are evidently not ergodic which leads to some difficulties in the interpretation of some quantities appearing in the analysis. The question of whether or not various quantities are directly measurable, in a physical sense, is related to the time average properties of nonstationary processes. An important theorem regarding this question has been given by Papoulis and will be quoted at the necessary point in the following development.

Although the turbulence input to be used in the problem of the propeller is assumed homogeneous, the relations will be developed to include nonstationary inputs. It frequently occurs that observation of the more general relations cause the less general cases to appear less difficult.

9. Response to Random Processes

9.1 Nonstationary processes:

The correlation matrix of the input, \( w(\mathbf{r}, t) \), is defined by Eq. (98) or (99) where the \( v \) brackets denote an ensemble average.

\[
[R_{ww}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2)] = \langle w(\mathbf{r}_1, t_1) w^T(\mathbf{r}_2, t_2) \rangle
\]  

or:

\[
[R_{ww}(\mathbf{r}, t, \xi, \tau)] = \langle w(\mathbf{r}, t) w^T(\mathbf{r}-\xi, t-\tau) \rangle
\]  

In the stationary and homogeneous case, the form given by Eqs. (99) is independent of \( \mathbf{r} \) and \( t \).

Now consider the spectral representation in the form of Eq. (19) and Eq. (100) valid for \( \tilde{w}(\mathbf{r}, t) \) real

\[
\tilde{w}(\mathbf{r}, t) = \iint_{-\infty}^{\infty} s_w^*(k, f) \exp\{-j2\pi(k \cdot \mathbf{r} + ft)\} dk df
\]  

The integrals are assumed to exist although for application to fields with infinite energy a rigorous statement would require that some of the integrands be of Stieltjes form. The end result is the same and the present procedure has intuitive advantages. Furthermore, it can be argued that only cases with finite energy may be treated experimentally. Using Eqs. (19) and (100) with Eq. (99), the correlation matrix can be written as:
The spectral density matrix is defined by
\[
[R_{ww}(\tau, t, \tilde{\xi}, \tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle S^*(k_1, f_1) S^T(k_2, f_2) \rangle \nonumber
\]
\[\times \exp\{-j2\pi(k_1 \cdot \tau + f_1 t)\} dk_1 df_1 \nonumber
\]
\[\times \exp\{j2\pi[k_2 \cdot (\tau - \tilde{\xi}) + f_2 (t - \tau)]\} dk_2 df_2 \tag{101}\]

The inverse relation to Eq. (103) is: (where "\(g\)" indicates the number of integrals)
\[
[\Phi_{ww}(k_1, f_1, k_2, f_2)] = \int_{-\infty}^{\infty} [R_{ww}(\tau, t, \tilde{\xi}, \tau)] \nonumber
\]
\[\times \exp\{-j2\pi[k_1 \cdot (k_1 - k_2) + t(f_1 - f_2) + k_2 \cdot \tilde{\xi} + f_2 \tau]\} dk_1 dk_2 df_1 df_2 \tag{103}\]

It is seen from Eq. (104) that in the event the correlation function is independent of \((\tau, t)\), the integrations over \(\tau\) and \(t\) give:
\[
[\Phi_{ww}(k_1, k_2, f_1, f_2)] = [\Phi_{ww}(k_2, f_2)] \delta(k_1 - k_2) \delta(f_1 - f_2) \tag{105}\]
If required, these relations could be used in a development of expressions for the response of flight vehicles to nonhomogeneous turbulence. However, because in the present case the treatment will be for homogeneous stationary inputs and the relations are cumbersome to manipulate, further consideration of nonstationary processes will be restricted to time dependence only. This will be sufficient for the problem under study.

In this regard the output force notation will be used such that:

\[ [R_{zz}(t,\tau)] = \langle z(t)z^T(t-\tau) \rangle \]  

(108)

Analogous to Eqs. (106) and (107), there are:

\[ [R_{zz}(t,\tau)] = \iint_{-\infty}^{\infty} [\Phi_{zz}(\tilde{\vec{f}},\vec{f})\exp{-j2\pi(\tilde{\vec{f}}+\vec{f})}]d\tilde{\vec{f}}d\vec{f} \]  

(109)

\[ [\Phi_{zz}(\tilde{\vec{f}},\vec{f})] = \iiint_{-\infty}^{\infty} [R_{zz}(t,\tau)]\exp{j2\pi(\tilde{\vec{f}}+\vec{f})}d\tilde{\vec{f}}d\vec{f}d\tau \]  

(110)

Two intermediate transforms which may be said to lie between Eqs. (109) and (110) are:

\[ [\Phi_{zz}(t,\vec{f})] = \iint_{-\infty}^{\infty} [R_{zz}(t,\tau)]\exp{j2\pi\vec{f}\tau}d\tau \]  

(111)

\[ [R_{zz}(\tilde{\vec{f}},\vec{f})] = \iint_{-\infty}^{\infty} [\Phi_{zz}(\tilde{\vec{f}},\vec{f})]\exp{-j2\pi\tilde{\vec{f}}t}d\tilde{\vec{f}} \]  

(112)
The one dimensional relation corresponding to Eq. (111) and in slightly different form has been called an "instantaneous power spectrum".\textsuperscript{39} From its inverse relation:

\[ [R_{zz}(t,\tau)] = \int_{-\infty}^{\infty} [\Phi_{zz}(t,f)] \exp(-j2\pi f \tau) df \] (113)

it is seen that its integral over \( f \) yields the matrix of time varying mean products. Evidently when the mean products do not depend on time, \([\Phi_{zz}(t,f)]\) is independent of time, and its elements become the ordinary power and cross spectral densities.

Next the time average of \([\Phi_{zz}(t,f)]\) is considered primarily because it is a quantity which can be measured by techniques used in the measurement of stationary signals\textsuperscript{39}. Forming the time average:

\[ \bar{\Phi}_{zz}(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\Phi_{zz}(t,f)] dt \] (114)

From Eqs. (111) and (114) there is:

\[ \bar{\Phi}_{zz}(f) = \int_{-\infty}^{\infty} [\bar{R}(\tau)] \exp(j2\pi f \tau) d\tau \] (115)

and from the second part of Eq. (111):

\[ \bar{\Phi}_{zz}(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} [\Phi_{zz}(\tilde{f},f)] \exp(-j2\pi \tilde{f} \tau) d\tilde{f} d\tau \] (116)

Following Papouli\textsuperscript{40}, assume \([\Phi_{zz}(\tilde{f},f)]\) finite except at a finite number of finite values of \( \tilde{f} \) such that it may be expressed as \([\Phi_{zz}(\tilde{f},f)] = \sum_{i} [\Phi_{zz}(f)] \delta(\tilde{f}-a_i)\), where the first term is finite and continuous everywhere. It is seen that the only contribution to \([\bar{R}_{zz}(f)]\) will be that resulting from the second term for \( a_i = 0 \) if such exists. This is a different and apparently more straightforward demonstration of a result that has been called\textsuperscript{26} Papouli\textsuperscript{26} theorem. Restating for conciseness:

If \([\Phi_{zz}(\tilde{f},f)]\) contains real valued singular or line masses along the line \( \tilde{f} = 0 \), i.e. it contains a contribution \([\Phi_{zz}(f)] \delta(\tilde{f})\), then a real valued time averaged spectral density matrix exists and is given by:

\[ \bar{\Phi}_{zz}(f) = [\Phi_{zz}(f)]_o \] (117)
Now return briefly to Eq. (112). The quantity defined there, \( r_{zz}(f, \tau) \), when evaluated at \( \tau = 0 \) represents the spectrum of the time variation of the mean products \( R_{zz}(t, 0) \).

It is noted that the characteristics given for nonstationary properties in the time and frequency domains have corollaries for nonhomogeneous turbulence in the space and wave number domains.

9.2 System response:

There are a number of forms in which the system input output relations can be cast. A number of them are given by Bendat and Piersol\(^4\) and other forms by the present author. In this section, a form suitable for use with the previously developed propeller transfer function, Eq. (75), will be given.

The appropriate deterministic relation is given on Fig. 6b and is repeated for convenience as Eq. (118) in vector form.

\[
S_z(f_0) = \int \int [-\infty, \infty] [H_{zw}(f_0, k, f_1)] S_z(k, f_1) dk df_1
\]  

(118)

Forming the spectral density matrix as in Eq. (102) and making the appropriate manipulations:

\[
[\phi_{zz}(f_0, f_0)] = \int \int [-\infty, \infty] [H_{zw}^*(f_0, k, f_1)]
\]

\[
\times \left[ \phi_{ww}(k_2, f_{i_2}) \right] \delta(k_1 - k_2) \delta(f_{i_1} - f_{i_2})
\]

\[
\times [H_{zw}(f_0, k_2, f_{i_2})]^T dk_1 dk_2 df_{i_1} df_{i_2}
\]

(119)

where the form given by Eq. (105) has been used corresponding to the assumption of homogeneous turbulence. The integrations over \( k_1 \) and \( f_{i_1} \) may be executed by inspection and a transformation \( \tilde{f}_0 = f_0 - f_0 \) is made corresponding to that leading from Eq. (104) to Eqs. (106) and (107). The result is:

\[
[\phi_{zz}(\tilde{f}_0, f_0)] = \int \int [-\infty, \infty] [H_{zw}^*(f_0 + \tilde{f}_0, k, f_1)] [\phi_{ww}(k, f_1)]
\]

\[
\times [H_{zw}(f_0, k, f_1)]^T dk df_1
\]

(120)
This is the relation which may be used with the propeller transfer function, Eq. (75), and the spectral density matrix of the turbulence, \( \phi_{\text{ww}}(k,f_i) \), to obtain the double frequency spectral density matrix, \( \phi_{zz}(\tilde{f}_0,f_0) \), characterizing the propeller forces and moments.

10. The Propeller Response to Turbulence

10.1 The generalized power spectral density

The elements of the double frequency spectral density matrix defined by Eq. (120) will be called the generalized power and cross spectral densities. As an example, the generalized power spectral density of the Z force will be considered. The contributions from the cross spectral density elements of \( \phi_{\text{ww}}(k,f_i) \) are assumed small so that the appropriate element from Eq. (120) is given by Eq. (121).

\[
\phi_{zz}(\tilde{f}_0,f_0) = \iint \{ H^*_{zu}(f_0+\tilde{f}_0,k,f_i) \phi_{uu}(k,f_i) H_{zu}(f_0,k,f_i) \\
+ H^*_{wv}(f_0+\tilde{f}_0,k,f_i) \phi_{wv}(k,f_i) H_{wv}(f_0,k,f_i) \\
+ H^*_{zw}(f_0+\tilde{f}_0,k,f_i) \phi_{zw}(k,f_i) H_{zw}(f_0,k,f_i) \} dkdf_i
\]

The transfer functions are given by Eq. (75). The expression for \( H_{zu} \) is typical.

\[
H_{zu}(f_0,k,f_i) = \sum_{\ell=\infty}^{\infty} \sum_{n=\infty}^{\infty} h^\ell_{zu}(k_0) \{ \exp\{jn(\frac{\pi}{2} - \theta_k)\} \delta(k^\ell_{+n})B(n-\ell) \\
+ \exp\{+jn(\frac{\pi}{2} + \theta_k)\} \delta(k^\ell_{-n})B(n+\ell) \}
\]

where:

\[
h^\ell_{zu}(k_0) = \int h^\ell_{zu}(\eta) J_n(2\pi k_0 \eta) d\eta
\]

\[
k^{\ell}_{\pm n} = 2\pi k_{\pm x} + f_i - f_0 - \ell \pm n
\]

Expressions for the spectral density matrices of frozen turbulence are given by Batchelor\(^9\). For homogeneous turbulence the form is:
\[ \phi_{ij} = b^2 \delta_{ij} \left( \frac{k_ik_j}{k^2} \right) + a_ia_j \left( 1 - \frac{b^2}{a^2} \right) \] 

where \( \delta_{ij} \) is the Kronecker delta; \( a \) and \( b \) are vector functions of \( k \). When the turbulence is isotropic, this relation reduces to

\[ \phi_{ij}(k) = \frac{E(k)}{8\pi^2k^4} \left( k^2 \delta_{ij} - k_ik_j \right) \]

where \( E(k) \) is an energy spectrum dependent on the magnitude of \( k \) only.

The difficulty in expanding Eq. (121) with the restriction to the form of Eq. (126) is sufficient to discourage the attempt to incorporate Eq. (125). Furthermore, only the axial motion case will be considered in the remainder of the development. The diagonal terms from Eq. (126) in terms of wave vector components \( k_x, k_\theta \), corresponding to the same variables in the propeller transfer function for axial motion are:

\[ \phi_{uu}(k) = \frac{E(k)}{8\pi^2k^4} k_x^2 \]

\[ \phi_{vv}(k) = \frac{E(k)}{8\pi^2k^4} \left( k_x^2 + k_\theta^2 \sin^2 \theta_k \right) \]

\[ \phi_{ww}(k) = \frac{E(k)}{8\pi^2k^4} \left( k_x^2 + k_\theta^2 \cos^2 \theta_k \right) \]

where \( k^2 = k_x^2 + k_\theta^2 \). As already noted, these forms are for frozen turbulence. Since they do not depend on \( f_1 \) and the transfer function dependence on \( f_1 \) occurs only in the argument of the distributions, the integration over \( f_1 \) of Eq. (121) simply implies the deletion of \( f_1 \) from the delta function arguments.

The contributions from the three terms of the integrand of Eq. (121) will be denoted by \( G_1, G_2, \) and \( G_3 \) respectively. Furthermore, the single blade form of Eq. (122) will be considered since the multiple blade case simply means deleting those terms for which the frequencies are not multiples of
blade frequency and multiplying the remaining terms by B.

The contribution from the first term is:

\[ G_{1}(\tilde{f}_{0}, f_{0}) = \int_{-\infty}^{\infty} \int_{0}^{\infty} 2\pi \frac{E(k)}{8\pi^{2}k^{4}} \left\{ \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h_{0}^{n}(k_{\theta})_{n} \times \{ \exp\{-jn(\frac{\pi}{2} - \Theta_{k})\} \delta(2\pi k_{x} - \tilde{f}_{0} - f_{0} - n) \right. \]

\[ + \exp\{-jn(\frac{\pi}{2} + \Theta_{k})\} \delta(2\pi k_{x} - \tilde{f}_{0} - f_{0} - n) \left\} \times \{ \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h_{0}^{m}(k_{\theta})_{p} \right. \]

\[ \times \left. \{ \exp\{jp(\frac{\pi}{2} - \Theta_{k})\} \delta(2\pi k_{x} - f_{0} - m) \right. \}

\[ \left. + \exp\{jp(\frac{\pi}{2} + \Theta_{k})\} \delta(2\pi k_{x} - f_{0} - m) \right\} d\theta_{k} dk_{\theta} dk_{x} \]

(130)

The properties of the distributions are such that products of the form

\[ \delta(2\pi k_{x} - \tilde{f}_{0} - f_{0} - n) \delta(2\pi k_{x} - f_{0} - m) \]

are nonzero only if \( \tilde{f}_{0} \) is an integer or zero. Utilizing the properties of the distributions along with

\[ \int_{0}^{2\pi} \exp\{\pm jn\theta\} d\theta = 0 \quad n \neq 0 \]

\[ = 2\pi \quad n = 0 \]

(131)

and

\[ J_{-n}(z) = (-1)^{n} J_{n}(z) \]

(132)

Equation (130) can be written (after several pages of manipulations) as:

\[ G_{1}(\tilde{f}_{0}, f_{0}) = 8\pi \sum_{\tilde{f}_{0}=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{1}(\frac{f_{0} + \tilde{f}_{0} + n}{2\pi \lambda}) \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{0} + \tilde{f}_{0} + \lambda - n}{2\pi \lambda} \]

\[ ; \tilde{f}_{0} \text{ an integer or zero} \]

\[ = 0 \quad ; \text{otherwise} \]

(133)
where:

\[
I_1(k_x')_{\text{np}}^{l_m} = \int_0^{\infty} \frac{E(k_x')}{8\pi^2 k_x'} k_x' h_x^l Z_u(k_\theta) n Z_u(k_\theta) p dk_\theta
\] (134)

Similarly, the contributions from the last two terms of the integrand of Eq. (121) may be found to be:

zero; if \( f_0 \) is not an integer or zero

if \( f_0 \) an integer or zero:

\[
G_2(\tilde{f}_0, f_0) = 8\pi \sum_{\ell = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} I_{2x}\left(\frac{f_0 + \tilde{f}_0 + \ell - n}{2\pi\lambda}\right)_{ nn}
\]

\[
+ 4\pi \sum_{\ell = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} I_{2\theta}\left(\frac{f_0 + \tilde{f}_0 + \ell - n}{2\pi\lambda}\right)_{ n,2}
\]

where:

\[
I_{2x}(k_x')_{\text{np}}^{l_m} = \int_0^{\infty} \frac{E(k_x')}{8\pi^2 k_x'} k_x' h_x^l Z_u(k_\theta) n Z_u(k_\theta) p dk_\theta
\] (136)

\[
I_{2\theta}(k_x')_{\text{np}}^{l_m} = \int_0^{\infty} \frac{E(k_x')}{8\pi^2 k_x'} k_x' h_x^l Z_u(k_\theta) n Z_u(k_\theta) p dk_\theta
\] (137)

\[
G_3(\tilde{f}_0, f_0) = 8\pi \sum_{\ell = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} I_{3x}\left(\frac{f_0 + \tilde{f}_0 + \ell - n}{2\pi\lambda}\right)_{ nn}
\]

\[
+ 4\pi \sum_{\ell = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} I_{3\theta}\left(\frac{f_0 + \tilde{f}_0 + \ell - n}{2\pi\lambda}\right)_{ n,2}
\]

\[
+ 4\pi \sum_{\ell = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} I_{3\theta}\left(\frac{f_0 + \tilde{f}_0 + \ell - n}{2\pi\lambda}\right)_{ n,2}
\] (138)
The generalized power spectral density of the side force is given by:

\[ \phi_{zz}(f_0, f_0) = G_1(f_0, f_0) + G_2(f_0, f_0) + G_3(f_0, f_0) \]  

(141)

Similar developments can be carried out for the generalized power spectral densities of the other force and moment components. These functions have not been evaluated for a specific case, but some general properties will be discussed after the results of calculations for an approximation are given.

10.2 The time averaged power spectral density for the point approximation:

The time averaged power spectral density of the Z force and N moment will be discussed and results of calculations presented. In this case, as discussed in section 7.2.4, the transfer functions are given by Eqs. (89)-(91) and similar expressions with \( k_\theta + 0 \). The summation over \( n \) reduces to a single term with \( n = 0 \), all others being zero. The generalized power spectral density of the Z force is computed by the corresponding special case of Eq. (121) and other similar equations for other force and moment components. According to Eq. (117) and its development, the measurable time averaged power spectral density will be given by the generalized power spectral density evaluated for \( f_0 = 0 \).

Carrying out the substitutions results in Eq. (142)

\[ \phi_{zz}(0, f_0) = \overline{\phi}_{zz}(f_0) = B^2 \left\{ F_0^2 \frac{1}{16} \left[ \phi_{VV}(\frac{f_0-2}{2\pi \lambda}) + \phi_{WW}(\frac{f_0-2}{2\pi \lambda}) \right] + \frac{1}{4} F_0^0 \left[ \phi_{uu}(\frac{f_0-1}{2\pi \lambda}) + \phi_{uu}(\frac{f_0+1}{2\pi \lambda}) \right] B(1) + \frac{1}{16} F_0^0 \left[ \phi_{VV}(\frac{f_0+2}{2\pi \lambda}) + \phi_{WW}(\frac{f_0+2}{2\pi \lambda}) \right] B(2) + \frac{1}{4} F_0^0 \phi_{WW}(\frac{f_0}{2\pi \lambda}) \right\} \]  

(142)
For isotropic turbulence to which this equation is restricted, \( \phi_{ww} = \phi_{vv} \). Also recall that the advance coefficient \( \lambda \) is related to the more familiar advance ratio by \( \lambda = J/\pi \).

Equation (142) can therefore be written as:

\[
\overline{\phi}''_{ZZ}(f_0) = B^2 F_2^0 \frac{2}{4} \left[ \phi_{ww} \left( \frac{f_0-2}{2J} \right) + \phi_{ww} \left( \frac{f_0+2}{2J} \right) \right] B(2) + B^2 F_1^0 \frac{2}{4} \phi_{uu} \left( \frac{f_0-1}{2J} \right)
+ \phi_{uu} \left( \frac{f_0+1}{2J} \right) B(1) + B^2 F_2^0 \frac{2}{4} \phi_{ww} \left( \frac{f_0}{2J} \right)
\]

(143)

Here \( B F_2^0 \) and \( B F_2^0 \) are simply the static derivative of \( Z \) force with respect to \( w \) and \( u \) or \( Y \) force with respect to \( v \) and \( u \).

Equation (143) and its counterpart for yawing moment are rewritten in normalized form for presentation. They are normalized with respect to

\[
\frac{B^2 F_2^0}{4} \phi_{ww}(0)
\]

Therefore define:

\[
\overline{\phi}''_{ZZ}(f_0) \equiv \overline{\phi}''_{ZZ}(f_0) / \left( \frac{B^2 F_2^0}{4} \phi_{ww}(0) \right)
\]

(144)

and

\[
\phi_{ww}(f) \equiv \phi_{ww}(f) / \phi_{ww}(0)
\]

(145)

\[
\phi_{uu}(f) \equiv \phi_{uu}(f) / \phi_{ww}(0)
\]

Now:

\[
\overline{\phi}''_{ZZ}(f_0) = \frac{1}{2} \left[ \phi_{ww} \left( \frac{f_0-2}{2J} \right) + \phi_{ww} \left( \frac{f_0+2}{2J} \right) \right] B(2)
+ \phi_{uu} \left( \frac{f_0+1}{2J} \right) B(1) + \phi_{ww} \left( \frac{f_0}{2J} \right)
\]

(146)

The yawing moment expression is:

\[
\frac{\phi''_{NN}(f_0)}{X_2^0} = \frac{1}{2} \left[ \phi_{ww} \left( \frac{f_0-2}{2J} \right) + \phi_{ww} \left( \frac{f_0+2}{2J} \right) \right] B(2)
+ \phi_{uu} \left( \frac{f_0+1}{2J} \right) B(1) + \phi_{ww} \left( \frac{f_0}{2J} \right)
\]

(147)
where:

\[ \Phi_{\text{NN}}'(f_0) \equiv \Phi_{\text{NN}}(f_0) / \left( \frac{B^2}{4} x_2^2 \phi_{ww}(0) \right) \]  

\[ (148) \]

The normalized expressions differ only in the coefficients of the second term for a single blade. For complete propellers, the normalized expressions are identical.

The von Karman expressions for turbulence spectra were used in computations. Figures 16-21 show the results of calculations for the example propeller for a single blade for blade settings of 25° and 30° at advance ratios of 1.2 and 1.4 respectively. Each case is shown for turbulence scales of 10, 20, and 50 propeller radii.

The principle features of Figures 16-21 are the peaks at \( f_0 = 1 \) and \( f_0 = 2 \). They resemble resonance peaks of system responses and are general features of responses of periodic systems to random inputs. Figures 22-24 show the result of calculations for two blade propellers. These normalized curves are general in that they depend parametrically on \( L/J \) only, not on design parameters. This is also true of the spectral density curves of Figs. 25-29 for three or more blades. These last curves exhibit no peaks. They are simply the turbulence spectra with the frequency scale shifted depending on the parameter \( L/J \).

The time averaged mean square for the \( Z \) force and yawing moment are given by Eqs. (149) and (150).

\[ \bar{Z}_C^2 = \left( \frac{B}{2} \right)^2 (f_2')^2 \phi_{ww}(0) \int_{-\infty}^{\infty} \Phi_{ZZ}'(f_0) df_0 \]  

\[ (149) \]

\[ \bar{N}_C^2 = \left( \frac{B}{2} \right)^2 (x_2')^2 \phi_{ww}(0) \int_{-\infty}^{\infty} \Phi_{\text{NN}}'(f_0) df_0 \]  

\[ (150) \]

Several values have been calculated for the example propeller. They are presented in Tables 1 and 2.
Fig. 16. Time Averaged Power Spectral Density of Z Force;
\[ \beta = 25, \, J = 1.2, \, L = 10, \, \frac{F_1}{F_2} = 1.19 \]
\[ B = 1 \]
Fig. 17. Time Averaged Power Spectral Density of Z Force;
\( \beta = 25, \ J = 1.2, \ L = 20, \ F_1/F_2 = 1.19 \)
\( B = 1 \)
Fig. 18. Time Averaged Power Spectral Density of Z Force;
\[ \beta = 25, J = 1.2, L = 50, \frac{F_1}{F_2} = 1.19 \]
\[ B = 1 \]
Fig. 19. Time Averaged Power Spectral Density of Z Force;

$\beta = 30$, $J = 1.4$, $L = 10$, $F_1/F_2 = .95$

$B = 1$
Fig. 20. Time Averaged Power Spectral Density of $Z$ Force;
$\beta = 30$, $J = 1.4$, $L = 20$, $F_1/F_2 = .95$

$B = 1$

64
Fig. 21. Time Averaged Power Spectral Density of Z Force; 
\( \beta = 30; J = 1.4, L = 50, F_1/F_2 = .95, B = 1 \)
Fig. 22. Time Averaged Power Spectral Density of Z Force; \( L/J = 10 \quad B = 2 \)
Fig. 23. Time Averaged Power Spectral Density of Z Force; 
$L/J = 20, B = 2$
Fig. 24. Time Averaged Power Spectral Density of Z Force; L/J = 30, B = 2
Fig. 25. Time Averaged Power Spectral Density of Z Force; \( L/J = 10, B = 3 \)
Fig. 26. Time Averaged Power Spectral Density of Z Force; 
\( L/J = 20, B = 3 \)
Fig. 27. Time Averaged Power Spectral Density of Z Force; \[ L/J = 30, \ B = 3 \]
Fig. 28. Time Averaged Power Spectral Density of Z Force; 
L/J = 50, B = 3
Fig. 29. Time Averaged Power Spectral Density of Z Force; $J = 1, L = 100, B = 3$
Table 1. Root Mean Square Normal Force Coefficients, $\sqrt{\frac{Z}{C}}$
RMS Turbulence = .03$\Omega R$

<table>
<thead>
<tr>
<th>$J = .6$</th>
<th>$\beta = 12$</th>
<th>scale = 10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>20</td>
<td>.0054</td>
<td>.0024</td>
</tr>
<tr>
<td>1.4</td>
<td>30</td>
<td>.0090</td>
<td>.0040</td>
</tr>
</tbody>
</table>

Table 2. Root Mean Square Yawing Moments Coefficients, $\sqrt{\frac{N}{C}}$
RMS Turbulence = .03$\Omega R$

<table>
<thead>
<tr>
<th>$J = .6$</th>
<th>$\beta = 12$</th>
<th>scale = 10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>20</td>
<td>.0041</td>
<td>.0018</td>
</tr>
<tr>
<td>1.4</td>
<td>30</td>
<td>.0065</td>
<td>.0029</td>
</tr>
</tbody>
</table>

These values are five to ten per cent of the thrust values at the given operating conditions. They are also found to be nearly the same for the given blade design for a single blade, two blades, or three blades. The value would go up for more blades. For a single blade, the normalized spectrum has three peaks, each containing roughly the same area for a wide range of operating conditions. For two blades, there are two such peaks, and for three blades or more, only the one at zero frequency.

10.3 Discussion of three dimensional effects on the power spectral densities:

The effect of turbulent velocity variation over the disk can be estimated by considering the forms of the transfer function coefficients of Eq. (75) along with the results of the preceding section on the point approximation. The additional terms which occur in Eq. (75) when $k_\theta \neq 0$ give rise to outputs at multiples of blade frequency regardless
of the number of blades on the propeller. The power spectral density in that case will have peaks at multiples of blade frequency. These features are evident from the equations (133), (135), and (138) if it is realized that the functions $I_1$, $I_2$, and $I_3$ are weighted integrals of the turbulence three dimensional power spectra over one of the wave number components. The result will still exhibit peaks for values of $f_0$ for which the argument goes to zero. The amplitudes of the peaks are determined by the weighted integrals of Eqs. (134), (136), (137), (139), and (140). Based on the form of the transfer function coefficients as shown by Figs. 10-13, it appears that the spatial variation should produce relatively small effects on the mean square forces and moments provided that the input spectral density is down by two or more orders of magnitude for $k = 0.2$. For broader input spectral densities it may be necessary to proceed with the evaluation of the relations of section 10.1.
PART V. CONCLUSION

11. Summary

Appropriate variable parameter linear system input output relations have been given in forms useful for application to the problem of propeller response to turbulence. The general concept of the aerodynamic transfer function has been discussed at some length including consideration of time varying parameter systems which lead to concepts which are helpful in the treatment of the propeller problem.

A quasistatic lifting line model of the propeller in nonuniform flow has been given with unsteady flow effects estimated by applying a Sears function factor based on the propeller shaft speed. This model has then been introduced into the previously developed general aerodynamic transfer function relation to derive an expression for the aerodynamic transfer function of a propeller.

In preparation for treating the propeller response to turbulence, some useful forms of input-output relations for time varying systems subject to random inputs have been given. These are then utilized with the propeller transfer function to obtain expressions for the generalized spectral density of the side force on a propeller in turbulence. Evaluations of the time averaged power spectral density of the normal force and yawing moment have been carried out using von Karman relations for the turbulence spectral densities and a point approximation. It was calculated that the root mean square force and moment components in the plane of the propeller may be about five to ten percent of the thrust values at a specific set of operating conditions which are near the points of maximum efficiency for given blade settings and advance ratios. The effect of turbulent velocity variation over the disk has been found to be production of peaks in the output spectral density at multiples of blade frequency.

12. General Remarks

The theoretical development contained in this report points the way toward considerable additional work on the subject problem. A modest amount of computation has been carried out to evaluate trends but relations are available to compute many other output characteristics. It is an area, however, which will require a careful preview of probable important effects because of the multiplicity of parameters of the problem.
In addition, the statistical parameters of level crossings and probabilities of exceeding given loads may be studied. Methods for treating these problems are available in the literature\textsuperscript{16, 39, 40}. 
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APPENDIX A

BLADE ELEMENT FORCE AND MOMENT PERTURBATIONS

This appendix contains a development in dimensional form of the contributions of a blade element to the various components of the force and moment perturbations. The relevant geometry is defined by Fig. 8, 9, and A1. Note the correspondence of $U_0$ and $W_0$ on Fig. A1 to the variables on Fig. 8.

\[ U_0 \equiv U \cos \alpha_T + U_i \]

\[ W_0 \equiv \Omega r + U \cos \theta \sin \alpha_T - W_i \]

---

Fig. A1. Blade Element with Reference and Perturbed Relative Velocities and Forces
The lift and drag are assumed to be given by expressions of the form of Eqs. (A1) and (A2).

\[ L' = \frac{C}{2} \rho V^2 \alpha (\alpha + \alpha_0) \quad (A1) \]

\[ D' = \frac{C}{2} \rho V^2 (C_D) \quad (A2) \]

For the unperturbed state, these are:

\[ L_0' = \frac{C}{2} \rho V_0^2 \alpha_0 \quad (A3) \]

\[ D_0' = \frac{C}{2} \rho V_0^2 C_D \quad (A4) \]

Note that the drag coefficient magnitude is assumed unchanged as would be closely satisfied if the section is operating near its design condition.

The perturbations \( X' \) and \( F' \) are composed of the respective components of \( dL' \) and \( dD' \). They may be written as:

\[ X' = L' \cos \Lambda - L_0' \cos \Lambda_0 + D_0' \sin \Lambda_0 - D' \sin \Lambda \quad (A5) \]

\[ F' = L' \sin \Lambda - L_0' \sin \Lambda_0 + D' \cos \Lambda - D_0' \cos \Lambda_0 \quad (A6) \]

From Fig. AI follow the relations of Eqs. (A7) to (A13) where the approximations are for small \( \alpha \), and small \( (u,v) \).

\[ V^2 = (U_0 - u)^2 + (W_0 + v)^2 \]

\[ = U_0^2 + W_0^2 + 2(vW_0 - uU_0) + u^2 + v^2 \]

\[ = V_0^2 + 2(vW_0 - uU_0) \quad (A7) \]
\[
\cos \Lambda = \cos (\Lambda_0 - \alpha) = \cos \Lambda_0 \cos \alpha + \sin \alpha \sin \Lambda_0
\]

\[
= \cos \Lambda_0 + \alpha \sin \Lambda_0 \tag{A8}
\]

\[
\sin \Lambda = \sin (\Lambda_0 - \alpha) = \sin \Lambda_0 \cos \alpha - \sin \alpha \cos \Lambda_0
\]

\[
= \sin \Lambda_0 - \alpha \cos \Lambda_0 \tag{A9}
\]

\[
\tan \alpha = \tan (\Lambda_0 - \Lambda) = \frac{\tan \Lambda_0 - \tan \Lambda}{1 + \tan \Lambda_0 \tan \Lambda} \tag{A10}
\]

\[
\tan \Lambda_0 = \frac{U_0}{W_0} \quad \tan \Lambda = \frac{U_0 - u}{W_0 + v} \tag{A11}
\]

\[
\tan \alpha = \frac{\nu U_0 + u W_0}{W_0^2 + U_0^2 + \nu W_0^2 - u U_0} \tag{A12}
\]

\[
\alpha = \frac{u}{W_0} \cos^2 \Lambda_0 + \frac{\nu}{U_0} \sin^2 \Lambda_0 \tag{A13}
\]

Equations (A1) to (A13) can be combined to obtain expressions for \(X'\) and \(F'\) in terms of the reference conditions and \((u, v, w)\) or \((u, \nu)\). The resulting expressions to the first order in \(u\) and \(v\), assuming that the change in drag coefficient magnitude is second order, are given by Eqs. (A14) and (A15).

\[
X' = \frac{C}{2} \rho \left\{ [a (W_0^2 - 2 U_0^2 a v) + c D_0 (2 U_0^2 + W_0^2)]u \\
+ [a (U_0 W_0 + U_0^2 + 2 a U_0 W_0^2) - c D_0 v v_0 U_0 v] (U_0^2 + W_0^2)^{-1/2} \right\} \tag{A14}
\]

\[
F' = \frac{C}{2} \rho \left\{ [a (-2 a W_0^2 + U_0 W_0 - 2 a U_0^2) - c D_0 U_0 W_0]u \\
+ [a (U_0^2 + 2 a U_0 W_0) + c D_0 (U_0^2 + 2 W_0^2)]v] (U_0^2 + W_0^2)^{-1/2} \right\} \tag{A15}
\]

Noting that \(\frac{\partial v}{\partial v} = \sin \theta\) and \(\frac{\partial v}{\partial w} = -\cos \theta\), the elements of the matrix of force and moment derivatives corresponding to Eq. (A14) may be written. They are given in terms of the components on \(oxyz\). First note:
\[ X'_u = \frac{C}{2} \rho [a (W_0^2 - \alpha_0 U_0 W_0) + c_D (2U_0 + W_0^2)] \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A16)

\[ X'_v = \frac{C}{2} \rho [a (U_0 W_0 + \alpha_0 U_0^2 + 2\alpha_0 W_0^2) - c_D W_0 U_0] \sin \theta \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A17)

\[ X'_w = -\frac{C}{2} \rho [a (U_0 W_0 + \alpha_0 U_0^2 + 2\alpha_0 W_0^2) - c_D W_0 U_0] \cos \theta \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A18)

\[ F'_u = \frac{C}{2} \rho [a (-\alpha_0 W_0^2 + U_0 W_0 - 2\alpha_0 U_0^2) - c_D U_0 W_0] \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A19)

\[ F'_v = \frac{C}{2} \rho [a (U_0^2 + \alpha_0 W_0 U_0) + c_D (U_0^2 + 2W_0^2)] \sin \theta \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A20)

\[ F'_w = -\frac{C}{2} \rho [a (U_0^2 + \alpha_0 W_0 U_0) + c_D (U_0^2 + 2W_0^2)] \cos \theta \left/ \sqrt{(U_0^2 + W_0^2)} \right. \] (A21)

where the subscript indicates partial differentiation. The form of the matrix whose elements are being evaluated is shown by (A22). The compact notation for the matrix, \([Z'_w]\), is intended to recall the relations of section 4. It is seen that Eqs. (A16), (A17), and (A18) are expressions for the first row of \([Z'_w]\).

\[
\begin{bmatrix}
X'_u & X'_v & X'_w \\
Y'_u & Y'_v & Y'_w \\
Z'_u & Z'_v & Z'_w \\
L'_u & L'_v & L'_w \\
M'_u & M'_v & M'_w \\
N'_u & N'_v & N'_w
\end{bmatrix}
\]

Expressions for the remaining elements are listed below by Eqs. (A23) to (A27).
\[ y'_u = F'_usin\theta \quad y'_v = F'_vsin\theta \quad y'_w = F'_wsin\theta \]  
\[ z'_u = -F'_ucos\theta \quad z'_v = -F'_vcos\theta \quad z'_w = -F'_wcos\theta \]  
\[ l'_u = -rF'_u \quad l'_v = -rF'_v \quad l'_w = -rF'_w \]  
\[ m'_u = X'_ursin\theta \quad m'_v = X'_vsin\theta \quad m'_w = X'_wsin\theta \]  
\[ n'_u = -X'_urcos\theta \quad n'_v = -X'_vcos\theta \quad n'_w = -X'_wrcos\theta \]  

(A23)  
(A24)  
(A25)  
(A26)  
(A27)
APPENDIX B

THE DIMENSIONLESS COEFFICIENTS AND VARIABLES

The geometry and dimensional variables are defined by Figs. 7, 8, 9, and A1 along with the equations developed in Appendix A. Typical force and moment coefficients are defined here by Eq. (B1).

\[
\begin{align*}
X'_u & = \frac{1}{2} \rho (\Omega R) RX'_c u \\
L'_u & = \frac{1}{2} \rho (\Omega R) R^2 L'_c u \\
X'_u & = \frac{1}{2} \rho (\Omega R) R^2 X'_c u \\
L'_u & = \frac{1}{2} \rho (\Omega R) R^3 L'_c u \\
X & = \frac{1}{2} \rho (\Omega R)^2 R^2 X_c \\
L & = \frac{1}{2} \rho (\Omega R)^2 R^3 L_C \\
\end{align*}
\]

(B1)

Basic dimensionless variables are defined by Eq. (B2).

\[
\lambda \equiv \frac{U}{\Omega R} , \quad \eta \equiv \frac{r}{R} , \quad \sigma \equiv \frac{c}{R} \quad (B2)
\]

Subscripts will be used with these as with the dimensional variables. All velocities are nondimensionalized with respect to \(\Omega R\). For example, in Appendix A:

\[
W_0 \equiv \Omega R + U \cos \theta \sin \alpha - W_i \quad (B3)
\]

Now there results

\[
v_0 \equiv \eta + \lambda \cos \theta \sin \alpha - v_i \quad (B4)
\]

where

\[
v_0 \equiv \frac{W_0}{\Omega R} , \quad v_i = \frac{W_i}{\Omega R} .
\]

To illustrate resulting expressions, Eqs. (A16) to (A21) are given in dimensionless form.

\[
X'_c u = \sigma [a(\nu_0^2 - \alpha_0 \lambda_0 \nu_0) + c_{D_0}(2\lambda_0^2 + \nu_0^2)] / (\lambda_0^2 + \nu_0^2)^{1/2} \quad (B5)
\]

\[
X'_c v = \sigma [a(\lambda_0 \nu_0 + \alpha_0 \lambda_0^2 + 2\alpha_0 \nu_0^2) - c_{D_0}\lambda_0 \nu_0] \sin \theta / (\lambda_0^2 + \nu_0^2)^{1/2} \quad (B6)
\]

\[
X'_c w = -\sigma [a(\lambda_0 \nu_0 + \alpha_0 \nu_0^2 + 2\alpha_0 \nu_0^2) - c_{D_0}\lambda_0 \nu_0] \cos \theta / (\lambda_0^2 + \nu_0^2)^{1/2} \quad (B7)
\]
Reiterating definitions: $a$ is the section lift curve slope, 

$\sigma \equiv \frac{C}{R}$, $\nu_0$ is defined by Eq. (B4), $\alpha_0$ is defined by Fig. 8, 
$c_{D_0}$ is the section drag coefficient for conditions shown on 
Fig. 8a, and $\lambda_0 \equiv \lambda \cos \alpha_T + u_i$ where $u_i$ is nondimensionalized with respect to $\Omega R$.

There are several forms of propeller coefficients which are used frequently. The one adopted here offers no unique advantages as is the case with the others. However, the presentation of derivatives with respect to perturbation velocities instead of angle of attack, etc. is more natural for the study of turbulence response, or so it seems to the present writer. Also, this set yields a straightforward set of equations without unnecessary constant multipliers. The most common form of propeller data is the presentation of $C_p$ and $C_T$ as a function of advance ratio $J$ where these are defined by the following:

$$C_T \equiv \frac{T}{\rho n^2 D^4} \quad C_P \equiv \frac{P}{\rho n^2 D^5} \quad J \equiv \frac{U}{nD}$$

where: $C_T \equiv$ thrust coefficient

$C_P \equiv$ power coefficient

$J \equiv$ advance ratio

$T \equiv$ thrust

$P \equiv$ power

$U \equiv$ forward speed

$D \equiv$ propeller diameter

$n \equiv$ propeller rotation rate in rev/sec

$\rho \equiv$ air density
These are related to the corresponding variables in the present notation as follows:

\[ J = \pi \lambda \]

\[ C_T = \frac{\pi^2}{8} \rho \lambda \]

\[ C_P = \frac{\pi^3}{8} \rho \lambda \]

(B12)

The moment coefficients, for example the pitching moment, are related as:

\[ C_M = \frac{\pi^2}{16} \rho \lambda \]

(B13)

where \( C_M \equiv \frac{M}{\rho n^2 D^5} \)

The other relations which are used in comparison of data of propellers at angle of attack or yaw are the normal force and moment derivatives with respect to angle of attack.

\[ C_{N\alpha} \equiv \frac{\partial C_N}{\partial \alpha} \quad C_N \equiv \frac{N}{\rho n^2 D^5} \]

(B14)

\[ C_{Z\alpha} \equiv \frac{\partial C_Z}{\partial \alpha} \quad C_Z \equiv \frac{Z}{\rho n^2 D^4} \]

These are related to the system used in this report by

\[ C_{Z\alpha} = \lambda \frac{\pi^2}{8} C_{cw} \]

\[ C_{N\alpha} = \lambda \frac{\pi^2}{16} C_{cw} \]

(B15)
APPENDIX C

ALGORITHM DEVELOPMENT FOR EVALUATION OF PROPELLER TRANSFER FUNCTIONS

C1. General

The process of calculating propeller transfer functions as defined in Part III of the text primarily consists of the evaluation of a certain set of integrals over a blade. They are of the form:

\[ G_n^\theta (k_\theta) = \int_{\eta_0}^{1} g_n^\theta (\eta) J_n \left( 2\pi k_\theta \eta \right) d\eta \]  

(C1)

The g's are determined parametrically by the propeller geometry and operating conditions.

The forms of g_n^\theta (\eta) will now be considered. First define:

\[ F'_1 (n, \theta) = \frac{\sigma [a(\lambda_0 v_0 - \alpha_0 \nu_0^2 - 2\alpha_0 \lambda_0^2) - \gamma_0 \lambda_0 v_0]}{\left( \lambda_0^2 + \nu_0^2 \right)^{1/2}} \]

(C2)

\[ F'_2 (n, \theta) = \frac{\sigma [a(\lambda_0^2 + \alpha_0 \lambda_0 \nu_0 + \gamma_0 \lambda_0)]}{\left( \lambda_0^2 + \nu_0^2 \right)^{1/2}} \]

(C3)

\[ X'_1 (n, \theta) = \frac{\sigma [a(\nu_0^2 - \alpha_0 \lambda_0 v_0 + \gamma_0 \lambda_0)]}{\left( \lambda_0^2 + \nu_0^2 \right)^{1/2}} \]

(C4)

\[ X'_2 (n, \theta) = \frac{\sigma [a(\lambda_0 v_0 + \alpha_0 \lambda_0^2 + 2\alpha_0 \nu_0^2 - \gamma_0 \lambda_0 v_0)]}{\left( \lambda_0^2 + \nu_0^2 \right)^{1/2}} \]

(C5)

The various g(\eta)'s are the coefficients in the Fourier expansions with respect to \theta of the following forms [see Eq. (63)] which are a restatement of Eqs. (38)-(55) with the trigonometric factors shown explicitly.

\[ X'_c (n, \theta) = X'_1 (n, \theta) \]  

(C6)

\[ X'_v (n, \theta) = X'_2 (n, \theta) \sin \theta \]  

(C7)

\[ X'_w (n, \theta) = -X'_2 (n, \theta) \cos \theta \]  

(C8)

\[ Y'_c (n, \theta) = F'_1 (n, \theta) \sin \theta \]  

(C9)

\[ Y'_v (n, \theta) = F'_2 (n, \theta) \sin^2 \theta \]  

(C10)

\[ Y'_w (n, \theta) = -F'_2 (n, \theta) \sin \theta \cos \theta \]  

(C11)

\[ Z'_c (n, \theta) = -F'_2 (n, \theta) \cos \theta \]  

(C12)
The variables appearing in Eqs. (C2)-(C5) are defined in Appendices A and B. The operating condition is specified by \( \lambda \) and \( \alpha_T \). The chord distribution \( \sigma(\eta) \) is a direct geometric parameter, but the remaining variables, \( \alpha, \lambda_0, \nu_0, \alpha_0, \) and \( c_{D_0} \) generally depend on operating conditions as well as geometry.

Some appropriate propeller theory is necessary to find these parameters from the complete propeller geometry and operating condition. Propeller geometry is usually specified by giving as a function of radius the chord distribution, \( \sigma(\eta) \), the blade twist, \( \beta(\eta) \), and the section parameters of profile shape, thickness, and design lift coefficient. From the section specifications, the two dimensional lift curve slope, \( a(\eta) \), and the drag properties are determined. It has been assumed in the calculations carried out for this study that the reference drag coefficient is given by:

\[
c_{D_0} = \kappa_1 (\alpha_0 - c_{\rho_d})^2 + \kappa_2 (\alpha_0 - a_{ST})^p
\]

where \( \kappa_1, c_{\rho_d}, \) and \( \alpha_0 \) are determined from section data; \( a_{ST} \) is an estimated stall angle of attack; \( \kappa_2 = 0 \) for \( \alpha_0 < a_{ST} \) and is about 1. with \( p = 1.33 \) for the propeller for which calculations have been carried out. The inclusion of a drag computation
such as Eq. (C24) extends the range of agreement with available data. (See Figs. 12 and 13) It has no effect on the results near the zero thrust condition.

Consideration of the time dependence of the flow at a blade section of a slightly inclined propeller, or the equivalent case of a propeller with small uniform side flow, reveals that the blade experiences an approximately sinusoidal angle of attack variation along with an approximately sinusoidal pulsation about the mean relative velocity. The effective reduced frequency in the Sears sense is given by:

\[ k = \frac{\Omega C}{2(U^2 + \Omega^2 R^2)^{1/2}} = \frac{\sigma}{2(\lambda^2 + \eta^2)^{1/2}} \]  

(C25)

To show the range of this parameter, let \( \lambda = \frac{2}{\pi} \), \( \eta = 0.2 \) which gives \( \frac{k}{\sigma} = 2.38 \) and \( \lambda = \frac{2}{\pi}, \eta = 1 \) which gives

\[ k = 0.42. \]  

If \( \sigma \approx 1 \) as for the example propeller, this gives a range of \( 0.04 \leq k \leq 0.24 \) which corresponds to \( 0.72 \leq S(k) \leq 0.9 \) where \( S(k) \) is the Sears function. This gives the right order of reduction in the effective lift curve slope although the flow field is curved and quite different from the two dimensional potential flow for which the unsteady lift function was developed.

Ribner\(^3\) has treated the unsteady effects indirectly by momentum considerations. However, this requires certain assumptions of the form of the induced inflow over the propeller disk which can not be made for a generally nonuniform incident flow field. As shown by Figs. 13 and 15, the direct application of Eq. (C25) with Sears result to obtain an effective lift curve slope yields results comparable to the method of Ribner in the static angle of attack case.

In summary, the general procedure is as follows. From the given characteristics of a propeller; \( \sigma(\eta), \beta(\eta) \), and section specifications; and the operating condition; \( \lambda, \alpha_0 \); the parameters \( \lambda_0, \alpha_0, \) and \( c_{D_0} \) are to be determined by propeller theory. In general they are functions of \( \eta \) as well as of \( \eta \). The lift curve slope is to be modified according to Eq. (C25). The resulting expressions from Eqs. (C2)-(C5) are used with Eqs. (C6)-(C23) to form the elements of the impulse response matrix which are then to be Fourier analyzed to obtain the \( g^l(\eta)'s \) of Eq. (C1). Eq. (C1) will then yield the \( C^l_{\theta}(k) \)'s which define the amplitudes of the output at the various output frequencies according to Eq. (75).
C2. Axial Motion with Experimental Thrust Data

The procedure will be described which was used to generate the data shown by Figs. 14 and 15. It is based on the existence of experimental thrust data as a function of angle of attack and an assumption that the blade twist is not drastically different from most propeller designs of the day.

From the geometry of the problem the angle of attack at a blade section is given by:

\[ \alpha_0(\eta) = \beta(\eta) - \Lambda_0(\eta) \]  \hfill (C26)

where:

\[ \tan \Lambda_0(\eta) = \frac{\lambda + u_1(\eta)}{\eta - v_1(\eta)} \]  \hfill (C27)

If the inflow is assumed uniform over the propeller plane, momentum theory gives:

\[ \lambda_0 = \frac{\lambda}{2} \left[ 1 + \left(1 + \frac{\delta C_T}{\pi J^2}\right)^{1/2} \right] \]  \hfill (C28)

where the tangential induced flow has been neglected and the expression is given in terms of \( C_T \) and \( J \) because most thrust data is given in terms of these parameters. It follows from the stated assumptions that:

\[ \tan \Lambda_0 = \frac{\lambda_0}{\eta} \]  \hfill (C29)

It appears that the twist distributions of many propellers can be closely approximated by the form:

\[ \beta(\eta) = \beta_{.75} + \beta_0 + \tan^{-1} \frac{\lambda_0}{\eta} \]  \hfill (C30)

where \( \beta_{.75} \) is the usual blade setting and \( \beta_0, \lambda_0 \) are constants to be determined from the given twist distribution and, if necessary, the set of values of \( (C_T, J, \beta_{.75}) \) which yield zero thrust. The parameter \( \lambda_0 \) is determined completely by the twist distribution. If the values of \( \beta \) at \( \eta = 0.2 \) and \( 1 \) are used for the fit, it is found that

\[ \lambda_0 = \frac{1}{2} \left\{ \frac{4}{5 \tan \Lambda_0} + \left[ \left( \frac{4}{5 \tan \Lambda_0} \right)^2 - \frac{4}{5} \right]^{1/2} \right\} \]  \hfill (C31)
Fig. C1. Propellers # 1 and 2 TND-318 Twist Distributions
where $\Delta \beta$ is $\beta(2) - \beta(1)$. Two such fits are shown on Fig. C1 for the propellers designated as number 1 and number 2 of NASA TN-318. It has been found that the expression Eq. (C31) evaluated by slide rule may easily result in an overall error of four or five degrees. Calculations to five significant figures are necessary. To complete the fit it is assumed that the condition of zero thrust corresponds to zero angle of attack at $\eta = .75$. This is accomplished by the relation:

$$\alpha_0(\eta) = \beta_{.75} + \beta_0 + \tan^{-1}\left\{\frac{\eta(\lambda^0 - \lambda_0)}{\eta^2 + \lambda^0 \lambda_0}\right\}$$

(C32)

from which $\beta_0$ may be computed with the previously stated assumptions. Once $\beta_0$ and $\lambda^0$ have been obtained, Eq. (C32) is assumed to give the reference condition angle of attack distribution. With section data assumed known and the fact that $v_0 - \eta$ with the neglect of $v_1$, all the necessary information for evaluating Eqs. (C2)-(C5) is now available.
The problem of determining forces and moments on a propeller operating in a turbulent flow is studied analytically. It is attacked by utilizing the concept of a general aerodynamic transfer function including consideration of time varying parameters as necessary for application to the propeller problem. A quasi-steady lifting line model of the propeller response to a nonuniform flow field is given with unsteady effects estimated by applying a Sears function factor based on the rotational speed of the propeller. This response model is used with the general aerodynamic transfer function relating to derive an expression for the aerodynamic transfer function of a propeller. The transfer function relation is utilized with nonstationary random process theory to obtain expressions for the generalized power spectral density of the forces and moments. Calculations of transfer function coefficients are shown including spatial velocity variations over the plane of the propeller. Power spectral densities of root mean square values of normal force and moment are found to be about five to ten percent of the thrust value at the specific operating conditions for a specific propeller. The effect of spatial velocity variations is found to be production of peaks in the power spectral densities at multiples of blade frequency.

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