COMMENTS ON THE AERODYNAMICS OF LOW ASPECT RATIO WING-BODY-TAIL COMBINATIONS IN STEADY SUPersonic FLOW

BY

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SUMMARY

During the course of this seminar an attempt is made to illustrate the nature of some of the problems encountered in the design of missiles to operate in the supersonic Mach number range. Some basic theoretical methods applicable to missile configurations, are reviewed and applied to configurations employing rectangular plan form panels arranged in cruciform. Assuming no interference between body and panel, the pressure distribution for a thin, flat, rectangular panel is used to compute the chordwise element of lift, the panel lift coefficient slope, the spanwise center of pressure and panel aileron power. The results obtained by Beskin for a body of revolution in supersonic flow are reviewed and the cruciform wing-body problem considered by discussing the effect of the body on the panels, the effect of the panels on the body, and the effect of one cruciform panel on the other. The wing-tail interference problem is discussed and some results presented. A brief discussion of viscous effects, skin temperatures and elastic deformations is included towards the end of the seminar.
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NOTATION

In general, symbols are defined in the text when introduced.

\( M \) - Mach number.

\( \phi \) - Potential function (throughout the major portion of this work it is the acceleration potential).

\( V \) - Free stream velocity (except where it denotes a volume of integration).

\( u, v, w \) - Free stream perturbation velocity components.

Axes Systems -

(1) During the first part of this work the axes system origin does not particularly affect the work and hence the ordinary co-ordinates \( xyz \) have been used.

(2) During the part of the work dealing with the finite wing problem the axis system origin is important and two reference axes systems have been employed:

(a) Tip axis system (origin at the tip leading edge) and co-ordinates with respect to this system have been denoted by the primed quantities, \( x', y', z' \).

(b) Body axis system (origin at the intersection of the wing leading edge and body center line) and co-ordinates with respect to this system have been denoted by the quantities \( xyz \).

\( r \) - Body radius

\( d \) - Body diameter

\( b \) - Wing span (tip to tip).

\( s \) - Wing semi-span

\( \delta \) - Deflection of the wing panel measured from body center line.

\( \alpha \) - Body angle of attack (or wing incidence, see text).
q - Dynamic pressure.
ρ - Free stream density.
S - Reference area (usually the body cross-section area AB).
c - Wing chord
n - c/r

\( C_L \) - Lift coefficient = Lift/\( q \) \( AB \)
\( C_L \) - Local or section lift coefficient = Section Lift/\( q_c \)
\( C_1 \) - Rolling moment coefficient = Moment/\( q_c A_B \)

\( \frac{\partial \theta}{\partial \delta} \) - Sidewash coefficient.

\( \frac{\partial \phi}{\partial \delta} \) - Downwash coefficient.

\( \frac{\partial C_L}{\partial \delta} \) - The induced rolling moment coefficient due to carry-over of the pressure fields on to the complementary panels, assuming that one planar pair of panels are undeflected whilst the other planar pair are asymmetrically deflected \( +\delta \) and \( -\delta \) from the body center line.

\( \beta = \sqrt{M^2 - 1} \)
INTRODUCTION

As a suitable solution to the problems of performance, stability and control, missile configurations have been adopted which employ low aspect ratio wings and tails arranged in cruciform on rather large diameter bodies. Typical of such configurations is that shown in Figure 1. Because the analyses of the system require close integration of the control, guidance and aerodynamic characteristics, accurate prediction of the configuration aerodynamic properties is a prerequisite to the missile design problem. Consequently many of the interference and intercoupling effects, neglected or approximated at low velocities, must be treated in detail for the high speed case.

Most missile configurations, whether launched from the ground or from a carrier aircraft, are required to perform through the subsonic and transonic speed ranges as well as at supersonic Mach numbers. The field of missile aerodynamics thus embraces the whole field of fluid motion, compressible and incompressible, steady and non-steady. Whilst yielding many useful results in practice, the customary assumptions of linearity and ideal gas flow, made to reduce the equations of motion to a mathematically tractable form, represent only limiting cases for the low aspect ratio configuration. At very low subsonic speeds, where the fluid may be considered as incompressible, the linearized, inviscid theory yields results valid both quantitatively and qualitatively. At high subsonic velocities non-linear effects become important and quantitative agreement no longer exists between the experimental results and linearized, inviscid theory prediction. Flax and Lawrence, reference 1, have presented an excellent treatment of the case of low aspect ratio configurations in high subsonic flow, where non-linear and viscous effects are predominant, and the problem here is essentially that associated with shock wave-boundary layer interaction phenomena. For a treatment of this problem, the reader is referred to the work of Liepmann and Roshko, reference 2. Throughout the supersonic Mach number range, linearized inviscid theory yields results which generally are sufficiently accurate for practical design computations, providing the incidences encountered are not large. The assumptions of zero viscosity break down, of course, when boundary layer effects become large; such as near a wing-body juncture or near the trailing edge of a wing where considerable thickening of the boundary layer due to interaction between the boundary layer and the trailing edge shock wave occurs. The zero-lift drag characteristics are largely dependent on the nature of the boundary layer, and failure to predict accurately boundary layer transition points can result in fairly large
errors in the drag estimates. In addition to failure at high incidences, the assumptions of linearity break down in the case of the wing-body combination where the linear addition of the isolated fields of the wing and the body produce a resultant field which does not satisfy the boundary conditions of the physical system.

In this seminar, we shall confine ourselves to the supersonic regime or, more specifically, to that range of Mach number where the effective panel aspect ratio is greater than unity (that is, to that range of Mach number where the tip leading edge Mach cone is contained inside the cone generated by revolving the panel diagonal around an axis through the tip leading edge point and parallel to the free stream velocity vector). Further, we shall consider only configurations employing rectangular plan-form panels, although the basic methods are not necessarily restricted to this type of configuration. Although some of the most important aspects relating to the automatic control of guided missiles concerns the dynamic characteristics of the vehicle, we shall limit our scope to those aspects relating to steady state flight. For a treatment of the non-steady case, the reader is referred to the work of Garrick and Rubinow, reference 3.
I. THEORY

1.1 General

A body or wing, immersed in a fluid stream, sets up about itself a disturbed flow field. In the case of a free stream velocity somewhat less than the velocity of sound, the field of disturbance about the body can be considered as extending to infinity in all directions. In the case of a supersonic free stream velocity, the field of disturbance is confined to a finite region in general, the region bounded by the envelope of the after Mach cones. If a body is immersed partially or totally in the field of disturbance about another body its aerodynamic characteristics will be modified. If the perturbated velocity of the disturbance field is very small, as in the case of very low subsonic, free-stream velocities, the modification to the aerodynamic characteristics will be correspondingly small. On the other hand in the case of supersonic free stream velocities in which a body is totally immersed in the field of disturbance associated with another body (as is generally the case for almost all low aspect ratio missile configurations) considerable alteration of the aerodynamic characteristics may be found.

The main interference fields associated with a missile configuration are shown in Figure 1. The body disturbance field is confined to the region interior to the conical shock wave originating at the body nose. Under the conditions depicted, both the wing and tail lie completely inside the body disturbance field. (At higher Mach numbers the vertex angle of the conical shock decreases, and eventually the nose cone shock will lie inside the wing leading edge tip, so that only part of the wings will be affected by the body disturbance field.) The tail, in addition to lying within the body disturbance field, also lies within the disturbance field of the wings. The body, besides being immersed in its own field, is partially immersed in the disturbance fields of both the wings and tail.

1.2 The Linearized Potential Equation

In this report, as is customary throughout most compressible flow work, it is assumed that the linearized form of the potential equation is valid; so that disturbances to the original stream, produced by the immersed bodies, are sufficiently small that only terms involving their first powers need be retained. Some discussion of the limitations introduced by this assumption will be made later. The potential equation, in its usual form, is then:
where \( \phi \) may be either the velocity potential defined by the relations:

\[
\begin{align*}
\frac{\partial \phi}{\partial x} & = V \frac{\partial u}{\partial x}, \\
\frac{\partial \phi}{\partial y} & = V \frac{\partial v}{\partial y}, \\
\frac{\partial \phi}{\partial z} & = V \frac{\partial w}{\partial z}
\end{align*}
\]  

or the acceleration potential defined by the relations:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} & = V \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial^2 \phi}{\partial y^2} & = V \frac{\partial^2 v}{\partial y^2}, \\
\frac{\partial^2 \phi}{\partial z^2} & = V \frac{\partial^2 w}{\partial z^2}
\end{align*}
\]

or any one of the perturbation velocity components. Following the method of Prandtl (reference 4), we define \( \phi \) as the acceleration potential, and by integration of equations 3 it can be seen:

\[
\begin{align*}
u &= \frac{1}{V} \phi \\
v &= \frac{\partial}{\partial y} \int_{-\infty}^{x} \frac{1}{V} \phi(x, y, z) \, dx \\
w &= \frac{\partial}{\partial z} \int_{-\infty}^{x} \frac{1}{V} \phi(x, y, z) \, dx
\end{align*}
\]
There are numerous mathematical methods available for finding solutions to the linearized potential equation. By the affine transformation:

\[
\begin{align*}
  x &= X \\
  y &= \sqrt{M^2 - 1} \ Y \\
  z &= \gamma M^2 - 1 \ Z
\end{align*}
\]

this equation may be put into an analogous form to the two-dimensional wave equation:

\[
L(\phi) = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial z^2} = 0
\]  

(5)

Some of the most useful generalized mathematical approaches are those presented by Heaslet, Lomax and Jones in references 5 and 6, and Robinson in reference 7. These methods employ Green's theorem to relate the volume integration to the surface integration:

\[
\iiint_Y [\sigma L(\phi) - \phi L(\omega)] \, dV = -\iint_S [\sigma D_n \phi - \phi D_n \omega] \, dS
\]

where \( D_n \) is the directional derivative given by

\[
D_n = n_1 \frac{\partial}{\partial x} - n_2 \frac{\partial}{\partial y} - n_3 \frac{\partial}{\partial z}
\]

\( n_1, n_2, n_3 \), being the direction cosines of the inward vector normal to the surface. By defining the co-normal vector \( \gamma' \) in the form:

\[
\begin{align*}
  \gamma'_1 &= -n_1 \\
  \gamma'_2 &= n_2 \\
  \gamma'_3 &= n_3
\end{align*}
\]

the directional derivative becomes:

\[
D_n = -\left[\gamma'_1 \frac{\partial}{\partial x} + \gamma'_2 \frac{\partial}{\partial y} + \gamma'_3 \frac{\partial}{\partial z}\right] = -\frac{\partial}{\partial \gamma'}
\]
\[ \phi_n = \text{the directional derivative along the co-normal.} \]

If \( \phi \) and \( \sigma \) are two continuous and single valued functions which satisfy the potential equation, then:

\[ \nabla(\phi) = 0 \quad \text{and} \quad \nabla(\sigma) = 0 \]

and Green's theorem reduces to the form:

\[ \iint_S [\sigma D_n \phi - \phi D_n \sigma] \, ds = 0 \]

which, upon using the relations:

\[ D_n \phi = -\frac{\partial \phi}{\partial n} \]
\[ D_n \sigma = -\frac{\partial \sigma}{\partial n} \]

can be written:

\[ \iint_S \left[ \phi \frac{\partial \sigma}{\partial n} - \sigma \frac{\partial \phi}{\partial n} \right] \, ds = 0 \] \tag{6}

If \( \sigma \) is a known particular solution to the wave equation, and the boundary conditions on the surface \( S \) are known, then the integral equation (6), can be solved for the potential function \( \phi \).

In order to illustrate the regions involved in the preceding integrations, let us consider a source of disturbance 0 in a supersonic free stream (Figure 2). This source sets up disturbances at all points interior to the after cone through 0 and in turn makes them sources of disturbances for all points lying within their aftercones. Thus it is seen that the potential at any interior point \( P \) will be due to all sources (or points) lying in the volume contained by the aftercone through the point 0 and the forecone through the point itself. Thus for the conditions depicted in Figure 2, the volume of integration is the volume enclosed by the cones 0-ABCD and P-ABCD, and the surface of integration is the surface bounding \( V \).

The general problem of linearized aerodynamics thus reduces to solving the integral equation (6) for the potential function by finding a particular solution to the wave equation \( \sigma \), valid everywhere within the volume \( V \) (to satisfy equation (5)) and which together with its first derivative, is known on the enclosing surface \( S \). The potential at
any point then can be determined in terms of the boundary potential and potential gradient along the co-normal directional derivative at the boundary.
II. THE GENERAL THIN WING PROBLEM

The integral form of the potential equation is of particular use in the solution of general wing problems, that is in the derivation of general wing characteristics and the flow field about wings. This amenability stems from the interjection of the wing as an additional boundary which exhibits symmetric properties and on which the potential function and its directional derivative are known. In order to clarify this point, consider the conditions depicted in Figure 3, where P is a point lying on the top side of a thin flat plate wing. In this case, since conditions underneath the wing cannot affect the potential at the point P, the surface of integration is made up of \( S_1 \) (the surface of the forecone through P lying above the plane of the wing), \( S_2 \) (the surface of the forecone through P) and \( S_3 \) (the surface of the top side of the wing enclosed by the trace of the forecone). Thus the integral equation (6) may be written:

\[
\iint_{S_1 + S_2 + S_3} \left[ \varphi \frac{\partial \varphi}{\partial y} - \sigma \frac{\partial \varphi}{\partial y} \right] dS = 0 \tag{7}
\]

A similar integral equation may be written for the underside of the wing but the potential at the point P due to this part of the field, designated by \( \varphi^{-} \) must be zero. We may write then:

\[
\iint_{S_1^- + S_2^- + S_3^-} \left[ \varphi^{-} \frac{\partial \varphi^{-}}{\partial y} - \sigma \frac{\partial \varphi^{-}}{\partial y} \right] dS = 0 \tag{8}
\]

Before proceeding further, it is necessary to consider the nature of the \( \sigma \) function. In general, if \( \varphi \) is a solution to the wave equation, then it will possess singular points inside the volume \( V \) which must be excluded from the region of integration, \( XYZ \) itself will be such a point. By the usual process of surrounding the singular points with a surface and taking the limit of the integral as this surface shrinks to zero, the potential at the point can be expressed in terms of the potential and the potential gradient \( \frac{\partial \varphi}{\partial y} \) on the boundary. Determination of the physical conditions on the surface of the fore Mach cone is, generally, not possible. This suggests then, that \( \sigma \) be chosen such that no knowledge of conditions on the Mach forecone is required, i.e., that both \( \varphi \) and \( \frac{\partial \varphi}{\partial y} \) vanish everywhere on the surface \( S_1 \). This consideration led to Volterra's choice of \( \sigma \) as the function:
In this case the singularities are contained along the axis of the forecone, i.e., along the line \((Y - Y_0)^2 + (Z - Z_0)^2 = 0\), this line running from the point XYZ to \(X_i, YZ\), its point of intersection with the envelope of the aftercones through the leading edge. By constructing a cylindrical surface of radius \(\xi\) about this line, and taking the limit of the integral over the surface of this cylinder as \(\xi \to 0\), these singular points can be excluded from the region of integration. Heaslet and Lomax (reference 5) have shown that:

\[
\lim_{\xi \to 0} \int \int \left[ \phi \frac{\partial \phi}{\partial x} - \phi \delta_{x} \phi \right] dS = -2\pi \int_{X_i}^{X} \phi(x_i, y, z) \, dx_i
\]

where \(K\) is the cylindrical surface. Using this result, together with the fact that the integration over the surface of the forecone vanishes, equation (7) may be written:

\[
2\pi \int_{X_i}^{X} \phi(x_i, y, z) \, dx_i = \int \int \left( \phi \frac{\partial \phi}{\partial x} - \phi \delta_{x} \phi \right) \, dS
\]

and differentiated to yield:

\[
\phi(x_i, y, z) = \frac{1}{2\pi} \frac{\partial}{\partial x} \int \int \left( \phi \frac{\partial \phi}{\partial x} - \phi \delta_{x} \phi \right) \, dS
\]

(10)

An analogous expression exists for equation (8):

\[
\phi^{-}(x_i, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \int \int \left( \phi^{-} \frac{\partial \phi^{-}}{\partial x} - \phi^{-} \delta_{x} \phi^{-} \right) \, dS = 0
\]

(11)
The integrations of equations (10) and (11) involve integrations over the surfaces of the envelope of the aftercones. Although no general proof exists, it has been found for specific cases (in particular, for thin flat plates satisfying the assumptions of this work) that the integrations over the envelopes of the leading edge Mach cones are symmetric, i.e.:

$$
\iint_{S_2} \left( \phi \frac{\partial \phi}{\partial y} - \sigma \frac{\partial \psi}{\partial y} \right) dS = \iint_{S_3} \left( \phi \frac{\partial \phi}{\partial y} - \sigma \frac{\partial \psi}{\partial y} \right) dS
$$

Hence by adding equations (10) and (11) an expression for the potential function can be derived which involves integration only over the surface of the wing, $S_3$:

$$
\phi(xyz) = \frac{1}{2\pi} \frac{\partial}{\partial x} \iint_{S_3} \left( \phi \frac{\partial \phi}{\partial y} - \sigma \frac{\partial \psi}{\partial y} \right) dS - \frac{1}{2\pi} \iint_{S_3} \left( \phi \frac{\partial \phi}{\partial y} - \sigma \frac{\partial \psi}{\partial y} \right) dS
$$

(12)

If we assume the wing to be a thin flat plate in the $\zeta = 0$ plane, then $\kappa_x = \kappa_y = 0$ and the normal and co-normal vectors become co-linear, and using the relation:

$$
\frac{\partial \phi}{\partial y} = \frac{\partial \phi^*}{\partial y^*}
$$

valid for thin flat plate wings, equation (12) may be written:

$$
\phi(xyz) = \frac{1}{2\pi} \frac{\partial}{\partial x} \iint_{S_3} (\phi - \phi^*) \frac{\partial \sigma}{\partial n} dS
$$

(13)

Differentiating equation (9) along the normal ($\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \zeta}$ in this case) and transforming back to the original spatial co-ordinates $x, y, z$, the potential function becomes (equation 17 of reference 3):

$$
\phi(xyz) = \frac{1}{2\pi} \frac{\partial}{\partial x} \iint_{S_3} \frac{(\phi - \phi^*)(x-x_1)(y-y_1) d\sigma \ d\gamma_1}{[y-y_1]^2 + (g-g_1)^2] \sqrt{(x-x_1)^2 - \beta^2[g-y_1]^2 + (g-g_1)^2]}
$$

(14)
where \( x, y, z \) are the co-ordinates of a point on \( S_3 \).

If it is desired to maintain an analogy with the subsonic case, then the choice of \( \sigma \) will be:

\[
\sigma = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}
\]

which will be recognized as a source function. The problem of obtaining a solution to the integral equation paralleling that obtained for Volterra's choice of \( \sigma \) (which is the integral of the source function), is complicated by the fact that in this case \( \sigma \) is infinite everywhere on the forecone as well as at the point \( XYZ \). It is necessary therefore to resort to Hadamard's method of finite part integration of an infinite integral (reference 8), equations analogous to equations (10) and (11) being obtained by excluding the point \( XYZ \) from the region of integration. Since equation (14) provides the necessary equipment for the future development of this seminar, the finite part method will not be discussed further, other than to quote Hadamard's result that for sources confined to the \( z = 0 \) plane:

\[
\Phi(xyz) = -\frac{\beta^2}{m} \int \frac{\Phi(x'y'z') \, dx'dy'}{\sqrt{[x-x']^2 - \beta^2 [(y-y')^2 + z^2]}}
\]

(15)

where * denotes the finite part of the integral. Ward (reference 9) has integrated this expression and obtained:

\[
\Phi(xyz) = \frac{1}{m} \int \frac{\frac{\partial \Phi}{\partial z} \, dx dy}{\sqrt{[(y-y')^2 + z^2] \left[ (x-x')^2 - \beta^2 [(y-y')^2 + z^2]\right]}}
\]

(16)

It is not meant to convey the impression, by not developing fully the theory for the case when \( \sigma \) represents a source function, that the finite part method is less suitable than the Volterra approach. Indeed the extension of the source concept (even if the compressible source does represent a somewhat physically nebulous quantity) to supersonic flow permits computation of the induced flow fields due to specified distributions of sources and doublets, and in particular is of use in determining the induced flow fields behind wings of finite span. It may be said then, that the finite part method may be applied to more general problems than
those for which Volterra's solution is meaningful. With regard to downwash fields, equations (15) and (16) will be referred to later. For a complete treatment of the basic theory of the finite part method, as well as some applications of it to particular problems, the reader is referred to the work of Robinson (reference 7) and Heaslet and Lomax (reference 6).
III. PARTICULAR RESULTS FOR THIN RECTANGULAR WINGS.

3.1 Solution of the Fundamental Equation.

Equation (14) may be considered as the fundamental equation of thin wing theory. If the potential jump across the wing inside the trace of the Mach forecone on the wing (i.e., region S3) is not constant, that is if:

$$\phi - \phi^- = f(x, y)$$

then the integrations of equation (14) become of the elliptic type and it is not possible to obtain a closed analytical expression for the potential function. However, if the potential jump may be considered constant over the region S3 then the integrations may be carried out. This has been done in reference 5 for the case of a thin, flat, uniformly loaded, rectangular wing, and it has been shown that if the potential jump is represented by $C_0$, for all points lying behind the leading edge wedge, ahead of the trailing edge wedge, and bounded laterally by the inside of the leading edge tip Mach cones:

$$\phi = \pm \frac{C_0}{2}$$

(17)

And that, under the same assumptions, for all points lying within the Mach cone from the leading edge tip, outside of the Mach cone from the trailing edge tip, and forward of the trailing edge wedge:

$$\phi = \frac{C_0}{2\pi} \left( \pm \frac{\pi}{2} - \arctan \frac{x(y - \frac{b}{2})}{\sqrt{\gamma^2 - \beta^2 (y - \frac{b}{2})^2 + z^2}} \right)$$

(18)

The reference axis system for equation (18) is shown in Figure 4, along with the tip reference axis system.

On a finite rectangular wing the loading is not uniform inside the tip Mach cones and consequently the expression for the potential function as given by equation (18) is not valid for points the trace of whose forecones lie within the tip Mach cone trace. Heaslet and Lomax have constructed the loading on the rectangular wing by superposing a series of uniform loads; each uniform load increment is applied over a trapezoidal surface corresponding to the wing with its tips raked. The
amount of each load increment is determined by the increment in rake angle from the preceding load in accordance with conical flow theory. By solving the ensuing integral equation by use of the boundary conditions, the pressure coefficient inside the tip Mach cone is expressed in the form:

\[
\frac{\Delta p}{q} = \frac{8\alpha}{\pi\beta} \sin \sqrt{\frac{\beta}{\alpha}}
\]

(19)

the co-ordinates in this case being with respect to the tip reference axis system, and \( \alpha \) being the panel incidence.

3.2 Lift Distribution Across a Thin Wing Panel.

In Figure 5 a rectangular panel in presence of a body is shown. By putting equation (17) into the standard form for an infinite wing, we may write:

\[
\frac{\Delta p}{q} = \frac{4\alpha}{\beta}
\]

(20)

Equations (19) and (20) then determine uniquely the pressure distribution over the body-panel combination providing no interference effects occur. This distribution is shown in Figure 5.

3.3 Chordwise Element of Lift (Section Lift Coefficient).

When wing panels such as that shown in Figure 5 are immersed in the induced flow fields behind lifting surfaces, both the incidence and local lift coefficient slope vary spanwise across the panel. In order to obtain the total panel lift coefficient it is necessary then, to integrate across the span the product of the local lift coefficient slope and local incidence.

By definition, the lift coefficient for any surface \( S \) is:

\[
C_L = \frac{1}{qS} \int_S (\rho u_1 - \rho u) \, dS = \frac{L}{qS}
\]
Consider a chordwise strip of constant width $\Delta y$ and length $c$ as shown in Figure 5(b). The lift force per elemental area $\Delta y \, dx$ is:

$$\Delta L = (p_2 - p_1) \Delta y \, dx$$

so that the total force $L$ acting on the strip $\Delta y$ is:

$$L = \int_0^c (p_2 - p_1) \Delta y \, dx = \int_0^{x_1} (p_2 - p_1) \Delta y \, dx + \int_{x_1}^c (p_2 - p_1) \Delta y \, dx'$$

where the subscripts 1 and 2 refer to the pressure differential over the particular area of integration. From this relation we can now form the lift coefficient for the elemental strip $\Delta y \, dx$ based on its own area, depending on whether $y \leq s - \frac{c}{2}$ or $s \leq y > s - \frac{c}{2}$.

For the first condition, that is $y \leq s - \frac{c}{2}$, the section lift coefficient becomes:

$$c_L = \frac{c}{\beta}$$  \hspace{1cm} (21)$$

For the second condition, i.e. $s \leq y > s - \frac{c}{2}$:

$$c_L = \frac{1}{q c \Delta y} \left[ \Delta y \int_0^{x_1} (p_2 - p_1) \, dx' + \Delta y \int_{x_1}^c (p_2 - p_1) \, dx' \right]$$

$$= \frac{1}{c} \left[ \int_0^{\beta y'} \frac{4a}{\beta} \, dx' + \int_{\beta y'}^c \frac{8a \arcsin \sqrt{\frac{\beta y'}{\Delta y}}} {\pi \beta} \, dx' \right]$$

$$= \frac{8q y'}{\pi c} \left[ \arcsin \sqrt{\frac{\beta y'}{\Delta y}} - \cot \arcsin \sqrt{\frac{\beta y'}{\Delta y}} \right]$$  \hspace{1cm} (22)$$
Transforming back to the body axis system by putting \( y' = s - y \) and differentiating to obtain lift coefficient slopes, equations (21) and (22) may be written:

\[
\left( \frac{\partial c_L}{\partial \alpha} \right)_y = \frac{\theta}{\beta}, \quad y = s - \frac{\theta}{\beta}
\]  

(23)

and:

\[
\left( \frac{\partial c_L}{\partial \alpha} \right)_y = \frac{8(k-y)\left[ \tan \sin \sqrt{\frac{\beta(k-y)}{k}} + \cot \sin \sqrt{\frac{\beta(k-y)}{k}} \right]}{\pi \alpha c} \]

(24)

for \( s > y > s - \frac{\theta}{\beta} \)

Equations (23) and (24) define the section lift coefficient at any spanwise position \( y \), on the panel.

3.4 Panel Lift Coefficient Slope

Using equations (23) and (24) the lift coefficient for a thin rectangular panel such as that shown in Figure 5 may be obtained by spanwise integration. Using Figure 6 we may write:

Lift of panel, \( L = L_1 + L_2 \)

\[
L = \mathcal{L}_1 + \mathcal{L}_2
\]

\[
= q \alpha \mathcal{C} \int_{s}^{s - \frac{\theta}{\beta}} (\alpha C_1) \, dy + q \alpha \mathcal{C} \int_{s - \frac{\theta}{\beta}}^{s} (\alpha C_2) \, dy
\]

\[
= q \alpha \mathcal{C} \int_{r}^{s - \frac{\theta}{\beta}} (\text{Eqn. 23}) \, dy + q \alpha \mathcal{C} \int_{s - \frac{\theta}{\beta}}^{s} (\text{Eqn. 24}) \, dy
\]
Using the body cross section area as the reference area, we may write the panel lift coefficient and differentiate to obtain the slope:

\[
(C_L)_{\text{Panel}} = \frac{L_{\text{Panel}}}{\frac{q}{A_B}}
\]

and:

\[
\left(\frac{\partial C_L}{\partial \alpha}\right)_{\text{Panel}} = \frac{C}{A_B} \left[ \int_{-\beta}^{\alpha-\beta} \, dy + \int_{\alpha-\beta}^{\alpha} \, dy \right]
\]

which, when the integrations are carried out becomes:

\[
\left(\frac{\partial C_L}{\partial \alpha}\right)_{\text{Panel}} = \frac{C}{A_B \beta} \left\{ 4 (\alpha - \beta) - \frac{C}{\beta} \right\}
\]

Providing the thin wing concept is valid, equation (25) presents the panel lift coefficient in a form that is very useful for optimization studies, (i.e., for determination of chord length for any given span to yield a maximum lift coefficient).

3.5 Panel Aileron Power

The formulae of the preceding sections may be used to compute the rolling moment due to panel deflection providing no interference effects occur. Under this assumption we shall call the induced moment due to panel deflection the panel aileron power, which is determined by the spanwise integration of the element of rolling moment due to a chordwise element of lift:

\[
L_r = \int_{r}^{\infty} \Delta L_r \, dy = \int_{r}^{\infty} \left(\frac{\partial C_L}{\partial \alpha}\right) \alpha \, y \, q \, c \, dy
\]

The rolling moment coefficient is then:
and integrating yields:

\[
C_l = \frac{\alpha}{A_B \beta} \left\{ 2 \left( \frac{s - c}{\beta} \right)^2 - 2 r^2 - \frac{7c^2}{4\beta^2} + \frac{3ac}{\beta} \right\}
\]

which may be differentiated to yield finally:

\[
\left( \frac{\partial C_l}{\partial \alpha} \right)_{\text{Panel}} = \frac{1}{A_B \beta} \left\{ 2 \left( \frac{s - c}{\beta} \right)^2 - 2 r^2 - \frac{7c^2}{4\beta^2} + \frac{3ac}{\beta} \right\}
\]

Equation (26) is the rolling moment coefficient slope of one wing panel.

3.6 Spanwise Centre of Pressure of a Panel

Knowing the lift coefficient slope and aileron power slope of a panel, equations (25) and (26), the spanwise panel centre of pressure can be derived by using the relation:

\[
y_{c.p.} = \frac{\alpha C_{l,}\alpha}{C_{l,\alpha}} = \frac{\alpha (\text{Eqn 26})}{(\text{Eqn 25})}
\]
IV. BODIES OF REVOLUTION IN SUPERSONIC FLOW

A cylindrical body pitching in a supersonic free stream sets up about itself a circulatory flow field. The flow conditions inside the body disturbance field can be determined by replacing the body by a family of sources distributed along its axis, the distribution strength being determined by the boundary condition (see reference 10). Ferri (reference 10-A) has also calculated the flow field about bodies by direct application of the method of characteristics.

Using linearized theory it has been shown that for a body with a conical head and cylindrical afterbody, the afterbody does not affect the lift of the configuration, the result being obtained that \( C_{L_{\infty}} \approx 2 \) per radian based on body cross section, and the centre of pressure is 66.7% of the cone height aft of the nose. Experimentally, the linearized theory result does not agree with measured values, the normal force coefficient being found about 50% higher than theory predicts and the centre of pressure located considerably further aft (the experimental centre of pressure position corresponds more closely to the 100% cone height point aft of the nose).
V. THE WING-BODY PROBLEM

5.1 Interaction Potential

In the preceding sections we have been concerned with the problems of determining the aerodynamic characteristics of isolated wings and bodies and the flow fields about them. It is now necessary to consider that problem which is fundamental to missile design, that is, the problem of the wing-body combination where due to wing-body interference effects, the aerodynamic characteristics of the combination are considerably different from the characteristics to be expected by linearly superposing the characteristics of the isolated components.

If, in considering a wing-body combination, the potential fields of the body and wings are linearly superposed, the resulting potential distribution will not satisfy the boundary conditions of the physical system. In order to overcome this difficulty a correction potential (the interaction potential of reference 11 and the interference potential of reference 12) has been derived which is a solution to equation (1) and which when superposed linearly onto the combined body and wing fields yields a resultant field which satisfies the physical boundary conditions. The mathematical complexity of the derivation of this correction potential makes the methods of references 11 and 12 extremely laborious when applied to practical configurations. Further since boundary layer effects are neglected, the method yields only approximate answers. For practical design work it has been found that sufficient accuracy is obtained from linearly combining the fields of the wings and bodies, and the amount of labour involved is reduced considerably. In the remainder of this work we shall derive the characteristics by linearly combining fields, leaving to the reader the consulting of references 11 and 12 for more exact treatments.

Considering a pair of rectangular wings arranged in cruciform on a body as shown in Figure 1, the wing body interference problem may be divided into three parts:

1. the effect of the body on the panels,
2. the effect of the panels on the body,
3. the effect of complementary panels on each other.
5.2 The Effect of the Body on The Panels

A pitching body sets up about itself an upwash or downwash field, depending on whether it pitches positively or negatively. For a panel-body combination pitching positively, Beskin (reference 14) under the assumption that the flow field immediately upstream of the wing is cylindrical, has derived the relation:

\[ \frac{\alpha_{\text{local}}}{\alpha_{\text{geometrical}}} = 1 + \left( \frac{r}{y} \right)^2 \]  

(28-1)

where \( y \) is the distance along the span measured from the body centre line, and \( r \) is the radius of the body. Applying strip theory to a rectangular panel, the average incidence over the wing panel can be shown to be:

\[ \alpha_{\text{Av.}} = \alpha_{\text{geom}} \left( \frac{1}{a-r} \right) \int_{r}^{a} \left( 1 + \frac{r^2}{y^2} \right) dy \]

Thus the average incidence across the wing panel is increased by the factor \( 1 + \frac{d}{b} \) and hence for a panel in the presence of a body pitching in the same direction as the wing, the lift coefficient as given by equation (25) must be modified and becomes:

\[ \left( \frac{\partial C_l}{\partial \alpha} \right)_{\text{Panel in presence of positively pitching body}} = \frac{C}{A_b \beta \left( \frac{4(a-r) - \frac{C}{\beta} \right)(1 + \frac{d}{b})} \]

(29)

Equation (29), perhaps requires some clarification. It represents the panel lift coefficient of a pitching wing-body combination, where \( \alpha \) is the angle of pitch. If a deflection of the wing panel \( \delta \) is superposed on the pitch, then the lift coefficient of the panel is given by the relation:

\[ C_l = \frac{\partial C_l}{\partial \alpha} \alpha + \frac{\partial C_l}{\partial \delta} \delta \]

\[ = \text{(Equation 29) } \alpha + \text{(Equation 25) } \delta \]
5.3 Effect of the Panel on the Body

The pressure field developed on the wings is carried over on to the region of the body contained between the root leading and trailing edge Mach helices. These regions are illustrated in Figure 7. A method of computing the magnitude of this carry-over, based on the application of equation (14), becomes apparent if the assumption is made that the flow fields may be linearly superposed. If two-dimensional flow is assumed everywhere over the panel root chord, then equation (14) can be integrated to give a general expression for the potential function, valid everywhere inside the region contained between the root leading edge and trailing edge Mach cones. Under the assumption that the potential remains unaltered by the body surface, the resulting expression would then define the potential distribution over that portion of the body lying inside this region.

Using the linearized form of the pressure coefficient:

\[ C_p = -\frac{2u}{V} \]

and expressing \( u \) in terms of the acceleration potential function (equation 4):

\[ C_p = -\frac{2\phi}{V^2} \]

the pressure coefficient on the body could be determined. This method has not been worked out, and hence no indication of accuracy can be given. Generally speaking, theoretical methods neglecting both boundary layer and tip effects give only fair agreement with experimental observations, and the great amount of labour involved in their application justifies recourse to more "in the large" analyses. Typical of this more approximate type of analysis is that presented by Morikawa in reference 13. In this case the assumption is made that on the shaded portion of the body (see Figure 7) the pressure carried over is constant (at any given Mach number) and equal to the asymptotic pressure at a large distance downstream from the trailing edge. In this case \( \Delta C_{l\alpha_{B\sim W}} \) (the increment in lift slope due to the carry-over from the wing onto the body, based on the exposed area of a pair of wing panels) is:

\[ \Delta C_{l\alpha_{B\sim W}} = \frac{2}{S_w} \int_{S_B} \left( \frac{C_p}{\alpha_0} \right) dS = \frac{2}{\alpha_0} \frac{S_B}{S_w} C_p \alpha = \infty \]
(The above integration is carried out by projecting the shaded area of the body, \( S_B \), onto the x-y plane; \( S_W \) is the wetted area of two wing panels). By putting in the value of the pressure coefficient far downstream of a rectangular wing which has a lift slope based on its wetted area:

\[
C_{L_w} = \frac{4}{\beta} \left( 1 - \frac{1}{4 \alpha_t} \right)
\]

Morikawa derives the relations for the carry-over of lift onto the body:

\[
\frac{\Delta C_{L_B} - \Delta C_{L_w}}{C_{L_w}} = \frac{1}{(1 - \frac{1}{4 \alpha_t})} \left( \frac{d_b}{l - d_b} \right) \left[ 1 - \left( \frac{a_t}{1 - \frac{1}{c_b}} \right) \sin \left( \frac{1 - \frac{d_b}{c_b}}{\frac{a_t}{d_b}} \right) \right]
\]

(31)

for: \( \frac{d}{b} \geq \frac{1}{1 + \frac{\pi}{2} \alpha_t} \)

and:

\[
\frac{\Delta C_{L_w} - \Delta C_{L_B}}{C_{L_w}} = \frac{1}{(1 - \frac{1}{4 \alpha_t})} \left( \frac{d_b}{l - d_b} \right) \left[ 1 - \frac{a_t}{1 - \frac{1}{c_b}} \right]
\]

(32)

for: \( \frac{d}{b} \leq \frac{1}{1 + \frac{\pi}{2} \alpha_t} \)

where

\[
\alpha_t = \beta R_p = \frac{\beta r}{c} \left( \frac{1 - \frac{d}{b}}{\frac{d}{b}} \right), \quad R_p \text{ being}
\]

the aspect ratio of a single wing panel.

It should be observed that equations (31) and (32) are applicable only under the fundamental assumptions of this work as dis-
cussed in the introduction, particular note being made of the fact that the effective panel aspect ratio must be greater than unity.

5.4 Complementary Panel Interference

Generally the nature of the interference effects between two panels at right angles is extremely complex; it is a function of the amount of pitch and yaw of the combination as well as the amount of deflection of the panels, body and wing geometry, and Mach number. In the case of large deflections of a wing panel in the presence of a body, a gap appears at the wing-body juncture and considerable alteration in the flow fields of the body and wing may occur.

The nature of the complementary panel interference effects is such as to alter both the total lift and lift distribution (both spanwise and chordwise), of the panels. Few experimental results describing these effects are available and, because of the complex nature of the problem, the theoretical investigations are rather limited. Regarding theoretical work, it is apparent that equation (14) or equation (16) would be suitable for this type of investigation providing such effects as root-gap flow could be taken into account. Martin (reference 15) has applied the finite part method outlined by Robinson, to non-planar systems and derived the surface pressure distribution and damping in roll characteristics of a family of rectangular fins. At C. A. R. D., the author and M. G. H. Tidy have applied equation (14) to the problem of computing the induced rolling moment (due to carry-over of the pressure field onto the undeflected panels) of a pair of rectangular wings arranged in cruciform on a cylindrical body where one planar pair of wing panels are differentially deflected to produce a rolling moment. The nature of this induced rolling moment, which tends to reduce the aileron power of the wings, is illustrated in Figure 8.

In deriving an expression for the induced roll moment coefficient of the configuration depicted in Figure 8 two approaches were used. In the first, it was assumed that the body may be replaced by two flat plates equal in semi-span to the body radius and arranged in cruciform at zero incidence to the free stream. The pressure field from the deflected panel was then assumed to act on the area of the undeflected panel normally enclosed by the body as well as on the enclosed area of the exposed panel. The contribution to the induced moment of the added flat plate section lying in the plane of the differentially deflected panels was assumed to be zero, this plate being considered to act only as a reflection plane preventing any alteration of the pressure di-
tribution over the inboard portion of the deflected panel due to circulatory 
flow around the root. In this approach then, the limit of integration with 
respect to \( \gamma \) (see equation 37-1) is zero, and a rather simple expres-
sion for the induced rolling moment may be derived. In the second ap-
proach employed, the area of the undeflected panel normally enclosed by 
the body is excluded from the region of integration, and a somewhat more 
complex expression for the induced moment coefficient results.

Because of the nature of the flow \( (M > 1) \), it is possible to 
employ a reflection plane technique, and hence the flow field need be con-
sidered only in one quadrant. This will be true providing that the diagonal 
joining the root leading edge point of the deflected panel to the tip trailing 
edge point of the undeflected panel lies ahead of the root leading edge Mach 
cone. If this condition is not satisfied, then the reflection plane technique 
must be abandoned and a lift cancellation method employed. The assumed 
wave field in one quadrant is shown in Figure 9. In addition to those dis-
cussed in the introduction, the following assumptions are inherent in this 
work:

(i) root-gap-flow effects are negligible;
(ii) the body is at zero incidence;
(iii) wing deflection angles are sufficiently small that the 
wing panels may be considered to lie in the \( \gamma = 0 \) plane;
(iv) the wing panels are thin, flat plates;
(v) the configuration has zero rate of roll.

The significance of this last assumption, aside from the fact that it re-
resents a condition readily duplicated in the supersonic wind tunnel, 
will appear later. Using the linearized relation for the pressure co-
efficient as given by equation (30), the induced rolling moment co-
efficient (based on body cross section area and wing chord), may be 
written as:

\[
C_{l} = \frac{1}{A_{b}c} \iint_{\Gamma_{xz}} C_{p} \gamma dS = -\frac{2}{V_{b}^{2} A_{b}c} \iint_{\Gamma_{xz}} \phi(x \circ y) \gamma dS
\]

where, as shown in Figure 9, \( \Gamma_{xz} \) is the appropriate area en-
closed, beneath the trace of the root leading edge Mach cone from the 
deflected panel, on the vertical panel.
The acceleration potential function \( \phi(x, y, z) \) is given by equation (14) where \( \gamma = y = 0 \) and \( S_3 \) is the region enclosed on the deflected panel by the trace of the forecone through the point \( (x, 0, z) \) as shown in Figure 9. The equation of the Mach forecone through \( (x, 0, z) \) is:

\[
(x-x_1)^2 - \beta^2 \left[ y_1^2 + (z - y_1)^2 \right] = 0
\]

and the equation of the intersection of this cone with the \( z = 0 \) plane is:

\[
(x-x_1)^2 - \beta^2 (y_1^2 + z^2) = 0
\]  

(34)

The region of integration \( S_3 \) is then the region bounded by the root and leading panel edges and the hyperbola defined by equation (34). Hence in making the integrations of equation (14), the \( y_1 \) integration is made over the interval:

\[
r \to \frac{1}{\beta} \sqrt{(x-x_1)^2 - \beta^2 y_1^2}
\]

and the integration with respect to \( x_1 \) is made over the interval:

\[
o \to x - \beta \sqrt{r^2 + z^2}
\]

The integral expression for the potential function on the \( y = 0 \) plane may then be written in definite integral form:

\[
\phi(x, 0, z) = \frac{\Phi - \Phi^{-}}{2\pi} \int_0^x \frac{\left( x-\beta \sqrt{r^2 + z^2} \right) \left( x-\beta \sqrt{(x-x_1)^2 - \beta^2 y_1^2} \right)}{(y_1^2 + z^2)(x-x_1)^2 - \beta^2 (y_1^2 + z^2)} dy_1
\]

and integrated to yield:

\[
\phi(x, 0, z) = \frac{\Phi - \Phi^{-}}{2\pi} \left( \frac{\pi}{2} - \arctan \frac{r x}{\sqrt{x^2 - \beta^2 (y_1^2 + z^2)}} \right)
\]

and integrated to yield:

\[
\phi(x, 0, z) = \frac{\Phi - \Phi^{-}}{2\pi} \left( \frac{\pi}{2} - \arctan \frac{r x}{\sqrt{x^2 - \beta^2 (y_1^2 + z^2)}} \right)
\]
Under the assumptions of linearized theory:

\[ \Phi - \Phi' = \frac{1}{\rho} (p_l - p_u) \]

\[ = \frac{V^2}{2} \frac{p_l - p_u}{p_a} = \frac{V^2}{2} \frac{\Delta p}{\rho} \]

and for effective panel aspect ratios greater than unity,

\[ \frac{\Delta p}{\rho} = \frac{\pi}{\beta} \delta \]

where \( \delta \) is the panel deflection (see Figure 8). Thus:

\[ \Phi - \Phi' = \frac{2V^2\delta}{\beta} \]

and the potential function on the \( y = 0 \) plane may be written:

\[ \phi(x, y) = \frac{V^2\delta}{\pi\beta} \left[ \frac{\pi}{2} - \arctan \frac{rx}{\sqrt{r^2 - \beta^2(y^2 + z^2)}} \right] \tag{35} \]

Finally equation (35) may be substituted into equation (33) and the rolling moment coefficient written as:

\[ C_l = \frac{-2\delta}{\beta A_{ab} \pi \alpha} \iint_{x,y} \left[ \frac{\pi}{2} - \arctan \frac{rx}{\sqrt{r^2 - \beta^2(y^2 + z^2)}} \right] z \, dx \, dy \tag{36} \]

The integrations of equation (36) can be performed, once the area is known.
First Approach:

As previously discussed, in this case is the region enclosed by the trace of the root leading edge Mach cone on the undeflected panel and that portion of the undeflected panel enclosed by the body. The equation of the root Mach cone is:

$$\chi^2 - \beta^2 \left[(y-r)^2 + z^2 \right] = 0$$

and the equation of the trace on the vertical fin is:

$$\chi^2 - \beta^2 \left[r^2 + z^2 \right] = 0$$

Thus the integration with respect to $\chi$ is made over the interval from $0$ to $r \sqrt{\frac{\chi^2}{\beta^2} - 1}$ where $\chi = \alpha / r$ and the integration with respect to $\xi$ is made over the interval from $\beta \sqrt{r^2 + z^2}$ to $nr$. Putting in these limits the induced rolling moment coefficient (equation 36) may be written in the definite integral form:

$$C' = -\frac{2}{\pi \beta A_B \Delta} \int_0^{nr} g \, d\xi \int_0^{\frac{\pi}{2}} \left[ \frac{\frac{\pi}{2} - \tan^{-1} \left( \frac{\frac{\pi}{2} - \tan^{-1} \left( \frac{nr}{\beta \sqrt{r^2 + z^2}} \right)}{\beta \sqrt{r^2 + z^2}} \right)}{\beta \sqrt{r^2 + z^2}} \right] \, d\chi \quad (37-1)$$

which, when integrated, yields:

$$C' = -\frac{r^3 \delta}{6 \beta^2 A_B \Delta} \left[ \beta^2 - 9n^2 + \frac{5n^3}{\beta} + 3n\beta \right] \quad (37)$$

Equation (37) represents the contribution of one quadrant only and must be multiplied by four to give the final result for wings in cruciform as depicted in Figure 8. Thus by differentiating equation (37) and multiplying by 4, the total roll moment coefficient derivative due to complementary panel carry-over (designated by the subscript $w = w$) may be written:

$$\left( \frac{\partial C'_c}{\partial \delta} \right)_{w = w} = -\frac{4r^3}{6 \beta^2 A_B \Delta} \left( \beta^2 - 9n^2 + \frac{5n^3}{\beta} + 3n\beta \right) \quad (38)$$
Second Approach

In the second approach to solving this problem, the body is excluded from the region of integration, so that the expression for the induced rolling moment coefficient is:

\[
C_l = \frac{(\varphi - \varphi')}{\pi V^2 A_B c} \int_{\beta}^{\frac{\beta^2}{\beta^2 - 1}} \int_{\beta}^{\frac{n^2}{\beta^2}} \left[ \frac{\beta}{\sqrt{\beta^2 + \eta^2}} \right] d\eta \, d\beta
\]

which when integrated (substituting in the expression for \( \varphi - \varphi' \) previously derived), yields:

\[
\left( \frac{\partial C_l}{\partial \delta} \right) \bigg|_{w=w_{\text{avg}}} = \frac{-n^3}{2\pi^2 A_B c} \left[ \pi \left( \frac{5}{3} \frac{n^2}{\beta^2} - \frac{3n}{\beta} + \frac{\beta}{3n} \right) \right.
\]

\[
+ \left( \frac{6n}{\beta} - \frac{4\beta}{6n} \right) \ln \frac{\beta}{\beta^2} + 2 \tan^{-1} \frac{n}{\sqrt{n^2 - 2\beta^2}}
\]

\[
+ \left( \frac{n^2}{\beta^2} - 1 \right) \tan^{-1} \frac{\sqrt{n^2 - 2\beta^2}}{n} - \frac{4}{6n} \sqrt{n^2 - 2\beta^2}
\]

\[
- \frac{4}{3} \frac{\beta^2}{n} \tan^{-1} \frac{\sqrt{n^2 - 2\beta^2}}{\beta} - \left( \frac{n^2}{\beta^2} + 1 \right) \tan^{-1} \frac{\beta^4}{n^2} \frac{n^2 - 2\beta^2}{n^2 - 3\beta^2}
\]

\[
- \frac{4}{3} \frac{n^2}{\beta^2} \tan^{-1} \frac{\beta^2}{n^2 - 2\beta^2}
\]

\[
(39)
\]

Under the restrictions of this derivation (\( \beta R_p > 1 \)), the values of the induced rolling moment coefficient as given by equations (38) and (39) agree rather closely in magnitude, although their percentage difference can be great in some cases.
The above work deals with a configuration with zero rate of roll, and is of particular value in determining the roll characteristics of a roll stabilized configuration. The results should hold for configurations with very small roll rates. In the case of configurations with rather large rolling rates, computation of the induced rolling moment due to complementary panel carry-over becomes more difficult because of the rolling of the undeflected panels into the root Mach cone and the spanwise variation in local lift coefficient along the deflected panel due to the roll velocity. The first effect results in a reduction of the area enclosed by the trace of the root leading edge Mach cone of the deflected panel on the undeflected panel and may be calculated readily. The second effect, the spanwise variation in local lift coefficient, presents a serious difficulty, because the potential jump within the trace of the forecone on the deflected panel, \( \phi - \phi' \), becomes a function of \( \chi \) and \( \gamma \) so that the integrations of equation (14) become of the elliptic type, and hence a closed, analytic form of solution is not obtainable. Several methods of circumventing the difficulty are being investigated presently at C.A.R.D.E., but as yet no satisfactory answer has been found.
VI. THE WING-BODY-TAIL PROBLEM

In addition to the wing-body problem already discussed, the general wing-body-tail problem reduces to consideration of the wing-tail interference problem which involves determining the induced flow field at the tail due to the wings, including any alteration in the velocity distribution caused by the body. Because of the large spanwise velocity gradients usually encountered across a tail surface lying downstream of finite lifting wings, the total lift coefficient of the tail must be computed by spanwise integration of the product of local lift coefficient and local incidence. The local lift coefficient of a rectangular tail panel with no flow induced on it by the body, is defined by equations (23) and (24). In the case of a pitching body, the variation in local incidence across the tail span due to body upwash must be superposed onto the induced incidence due to the wing wash fields.

In general, determination of the downwash and sidewash fields behind low aspect ratio wings is considerably complicated by the large amount of distortion associated with the trailing vortex pattern. Additional complication is introduced by the cruciform arrangement of missile wings as well as the rather large body diameter-to-wing span ratios usually employed. The standard concepts of line vortex theory have been extended from the subsonic case to cover compressible flow conditions by Mirels and Haefeli (reference 16). Spreiter and Sacks (reference 17) have considered the general problem of the rolling up of the trailing vortex sheet, as well as leap-frogging of the vortices at high bank angles. Lagerstrom and Graham (reference 18) under the assumption of no rolling up of the vortices, have applied Busemann's conical flow method (reference 19) to the problem of determining induced flow fields behind semi-infinite wings at supersonic speeds. This method consists of expressing the downwash and sidewash associated with the planar semi-infinite quadrant in a closed analytical form and then obtaining the induced field due to a semi-infinite rectangular wing by superposing onto the field along the line of the trailing edge, an infinity of negatively lifting conical semi-infinite wings in such a fashion as to decrease to zero the perturbation velocity (lifting pressure) in the plane of the wing outside of the wing plan form. The summation of these wings cannot, in general, be expressed in a closed analytical form except for the case of calculations made in the Trefftz plane, that is, the plane infinitely far downstream from the wings. In this case Lagerstrom and Graham have obtained the expressions for downwash and sidewash (referred to the axis system shown in Figure 10) in the form:
\[ \frac{\partial \phi}{\partial \delta} = -\frac{2}{\pi} \left[ \frac{(\frac{\beta z'}{c} + 1)^2 + (\frac{\beta y'}{c})^2}{(\frac{\beta y'}{c})^2 + (\frac{\beta y'}{c})^2} \right]^{\frac{1}{4}} \sin \psi \]  \tag{40}

\[ \frac{\partial \psi}{\partial \delta} = \frac{2}{\pi} \left\{ 1 - \left[ \frac{(\frac{\beta y'}{c} + 1)^2 + (\frac{\beta z'}{c})^2}{(\frac{\beta y'}{c})^2 + (\frac{\beta y'}{c})^2} \right]^{\frac{1}{4}} \cos \psi \right\} \]  \tag{41}

\[ \psi = \frac{1}{2} \left[ \tan^{-1} \left( \frac{\beta z'}{\beta y' + 1} \right) - \tan^{-1} \left( \frac{\beta z'}{\beta y'} \right) \right] \]  \tag{42}

(\delta \text{ is the angle of attack of the wing}).

For reasonable tail span-to-body diameter ratios, experiments have shown that the Trefftz plane calculations are reasonably accurate for tails located more than two wing chords downstream from the wing trailing edge.

Recently Leslie (reference 20) has obtained a general expression for the downwash behind wings by integrating by parts Ward's result (equation 16) and differentiating to obtain:

\[ w(xyz) = \frac{\partial \phi}{\partial z} = -\frac{1}{\pi} \int \int \frac{\partial^2 \phi}{\partial x^2} K(xz) \, dx \, dy \]  \tag{43}

where:
and the integration is made over the part of the wing lying inside the
forecone through \( x, y, z \). Equation (43) proves extremely useful. By
assuming conical flow Leslie shows that it can be integrated to yield
the formula of Lagerstrom and Graham, whilst a similar formula to
that of Mirels and Haefeli can be obtained by assuming the wing to be
replaced by a lifting line along the line \( \chi_L = 0 \).

The effect of the body on the wing downwash and sidewash
fields at the tail is twofold in nature and becomes of particular impor-
tance when large body-diameter to span ratios are employed. First,
the presence of the body causes the vortex spacing to be increased,
experimental results indicating a vortex spacing ratio (i.e. the ratio
of the distance between the tip trailing vortices and the tip-to-tip wing
span) in the region \( .94 \) to \( 1 \) for body diameter to wing-span ratios in
the range \( .1 \) to \( .3 \). The second effect is the blanketting by the body of
part of the tail panel from the effect of the tip vortex on the opposite
side of the body. A method of accounting for this blanketting effect,
based on the assumption of conical flow and assuming that any portion
of the tail panel optically shielded is totally shielded from any induced
flow effect, has been derived at C.A.R.D.E., and when applied to
specific configurations has yielded good agreement with experimental
results. For convenience of illustration a configuration with its tail
panel out of line with respect to the wing panels, will be considered
(Figure 11). The origin of the conical flow system is assumed to lie
on the panel tip chord located chordwise aft of the leading edge at the
centre of pressure. Further, it is assumed that the tail panel lies
well inside the after Mach cone originating at \( 0 \). It is apparent then,
that there will be a family of rays emanating from \( 0 \) bounded by the
rays \( OAB \) and \( OCD \) (Figure 11) which will intersect the tail panel,
their trace on the tail panel being given by the line \( BD \). In this
method it is assumed that any ray lying below the family of tangent
rays will be completely reflected and consequently that portion of the tail enclosed between the root chord and the line BD will have no induced field due to the wing panel on the opposite side of the body. (In practice, refraction and reflection of the rays at the boundary-layer would be expected, but this effect would be small). For any geometrical arrangement the line BD may readily be found.

In computing the resultant lift of the tail panel due to the wing panel on the opposite side of the body the blanketted portion of the panel is excluded completely and the product of the section lift coefficient and local incidence integrated from BD to the panel tip. For configurations where the blanketted area is a large percentage of the panel area, it was found necessary to apply this correction. The theoretical basis of this method is very much lacking in rigour, but the agreement with experiment (perhaps fortuitous) was good in the cases investigated. It is envisaged that a more sophisticated method can be developed, based on this approach.
VII. THE EFFECT OF THE BOUNDARY LAYER

For missile configurations such as those considered here, the viscous problem is more that of the shock-wave boundary-layer interaction type than of the boundary-layer alone. Studies of the characteristics of compressible boundary layers have been made by Van Driest (reference 21) and Lees (reference 22). The shock-wave boundary layer interaction problem has been studied by Liepmän, Barry et al in reference 2. The effects of surface roughness and skin temperature gradients encountered on most practical missiles introduces appreciable errors into the theoretical predictions of transition point and boundary layer thickness.

Generally in missile work, the ultimate effect of the boundary layer must be obtained from experiment. For the class of configuration discussed here, it has been found that the effect of the boundary layer on the aerodynamic characteristics (except drag) is small in the case of moderate pitch angles, but can be appreciable when rectangular panels are deflected in the presence of cylindrical bodies (due largely to the effect of the boundary layer on the flow around the root gap). The effect of the boundary layer on the zero lift characteristics is, in almost all cases, fairly large. Attempts at estimating the friction drag of this type of configuration by using the predicted transition point and compressible boundary layer theory have yielded results which are not too satisfactory. A semi-empirical approach, adequate for most initial design estimates, has evolved. It consists in applying the incompressible turbulent data, increased by a factor of 1.5, to the supersonic case. A typical formula which may be used to estimate the drag of a configuration at supersonic Mach numbers is Prandtl's subsonic formula (see reference 23) increased by a factor 1.5:

\[
C_{DF} = \frac{0.683}{(\log_{10} R_e)^{2.58}}
\]  

(44)

based on the wetted area \(A_s\), and \(R_e\) is the Reynolds number of the component, the characteristic length being taken parallel to the free stream. It has been found that reasonable estimates of configuration drag can be obtained by applying equation (44) and linearly superposing the component drags.
The boundary layer affects the transfer of energy to the surface of the body, and hence the skin temperature. Recently (Feb. 1952) an excellent report discussing the factors affecting skin temperatures has been published (reference 24); in this report a rather complete set of solutions to the skin temperature problems are presented. The authors present an empirical law for temperature rise which because of its simplicity and surprising practical accuracy is quoted here:

$$\Delta T = \left(\frac{V}{100}\right)^2 \text{ degrees Centigrade} \quad (45)$$

where $\Delta T$ is the ultimate rise in body temperature in degrees centigrade and $V$ is the velocity in miles per hour. For a more complete treatment of this problem, the above referenced work is recommended.
VIII. CONCLUSIONS AND DISCUSSION

During the course of the seminar we have attempted to illustrate the nature of some of the problems encountered in the design of missiles to operate in the supersonic Mach number range. We have reviewed some basic theoretical methods applicable to missile configurations and by applying them have derived expressions for specific characteristics in a form somewhat different than usually encountered. These characteristics expressed in the form given here, have proven particularly useful in determining the detailed aerodynamic characteristics of low aspect ratio wing-body-tail combinations and have been found to be reasonably accurate. Because of time considerations the application of these fundamental methods has been restricted to the case of configurations which have rectangular wing panels. However, they may equally well be applied to configurations employing delta and other type wing panels.

In this work the components of the configuration were considered to be perfectly rigid bodies. In practice the deformation of the components due to loading may introduce appreciable alteration in the aerodynamic characteristics. Hence in computing the performance, stability and control characteristics of missile configurations it is essential to include the interweaving between the aerodynamics and elastic properties of the configuration components. Generally then, the computations of the resultant characteristics of a configuration involve the simultaneous solution of a set of equations, one group of which represents the elastic characteristics of the components, the other group representing the aerodynamic properties.

The assumption of attached flow conditions has tacitly been made throughout this report. However, in many cases (moderate incidences at low Mach numbers) separation occurs and the aerodynamic characteristics are altered by the resulting mixed subsonic and supersonic flow fields. An experimental study of the effect on the aerodynamic characteristics of transition from an attached to a detached bow shock wave at the leading edge of a finite span lifting wedge is presented in reference 25. For the isolated wedge it is concluded that no radical changes occur in the characteristics at separation, rather the change is a gradual one. The theory developed in the referenced literature of reference 25 can be applied to determine the characteristics of an isolated body, but for a complex wing-body-tail combination it is doubtful if any satisfactory results can be obtained from theoretical predictions because of the inherently large boundary layer effects.
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Fig. 1: TYPICAL MISSILE CONFIGURATION AND ASSOCIATED FLOW FIELDS
FIG. 2.
SOURCE DISTRIBUTION DUE TO O, AFFECTING P

FIG. 3.
SURFACE OF INTEGRATION FOR P
FIGURE 4

REFERENCE AXIS SYSTEMS
EQUATION (18) ... (x, y, z)
TIP SYSTEM ...... (x' y' z')
FIG. 5.
PRESSURE DISTRIBUTION OVER A RECTANGULAR PANEL IN THE PRESENCE OF A BODY
FIGURE 6

REGIONS OF APPLICATION OF EQUATIONS
23 AND 24
FIG. 7.

REGIONS OF PRESSURE FIELD CARRY-OVER ---WINGS

TO BODY
Trace of root leading edge Mach Cone on undeflected wing panel

Negative pressure carried over.

Positive pressure carried over.

Deflected panel $-\delta$

Deflected panel $+\delta$

Aileron Moment $+2L_w l_w$

Induced Moment $-2L_{w-w} w_w$

FIG. 8

INDUCED ROLLING MOMENT FROM COMPLEMENTARY PANEL INTERFERENCE
Fore Mach Cone through general pt. \((x,0,z)\) on the complementary (undeflected) fin.

Trace of root L.E. Mach Cone on undeflected panel

\[ x^2 - \beta^2 (z^2 + r^2) = 0 \]

\[ (c,0,r) \]

\[ \sqrt{\frac{n}{\beta^2 - 1}} \]

Trace of fore cone in the plane of the wing

\[ (x-x_1)^2 - \beta^2 (y_1^2 + z^2) = 0 \]

\( S_3 \)

Deflected panel

\((0,r,0)\)

\((c,s,0)\)

\((0,s,0)\)

FIG. 9

ASSUMED WAVE FIELD IN ONE QUADRANT
FIGURE 10

SEMI INFINITE RECTANGULAR WING WITH TIP AXIS SYSTEM USED IN EQUATIONS 40, 41 AND 42.
FIG. 11.
BLANKETTING OF TAIL BY BODY