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Polyhedral Lyapunov functions structurally ensure global asymptotic stability of dynamical networks iff the Jacobian is non-singular

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Abstract

For a vast class of dynamical networks, including chemical reaction networks (CRNs) with monotonic reaction rates, the existence of a polyhedral Lyapunov function (PLF) implies structural (*i.e.*, parameter-free) *local* stability. *Global* structural stability is ensured under the additional assumption that each of the variables (chemical species concentrations in CRNs) is subject to a spontaneous infinitesimal dissipation. This paper solves the open problem of *global* structural stability in the absence of the infinitesimal dissipation, showing that the existence of a PLF structurally ensures global convergence if and only if the system Jacobian passes a structural non-singularity test. It is also shown that, if the Jacobian is structurally non-singular, under positivity assumptions for the system partial derivatives, the existence of an equilibrium is guaranteed. For systems subject to positivity constraints, it is shown that, if the system admits a PLF, under structural non-singularity assumptions, global convergence within the positive orthant is structurally ensured, while the existence of an equilibrium can be proven by means of a linear programming test and the computation of a piecewise-linear-in-rate Lyapunov function.

Key words: Dynamical networks; Piecewise-linear Lyapunov functions; Structural stability; Global stability; Chemical reaction networks

1 Introduction

Dynamical networks are composed of dynamic subsystems that interact according to a given interconnection topology. Parameters, often uncertain and time-varying, depend on unpredictable environmental conditions; however, all the dynamical networks in a class characterized by a *structure* (interconnection topology), regardless of parameter values, may have a given property, which is then called *structural* (Nikolov et al. 2007; Blanchini and Franco 2011). Structural properties are often enjoyed by natural systems that perform their task in spite of severe uncertainties and variability, relying on particular structures (motifs, Alon 2006): a structural analysis can then explain their astounding robustness.

For chemical reaction networks (CRNs), structural analysis traces back to the early seventies (Horn and Jackson 1972; Horn 1973a,b). Feinberg (1987, 1995a,b) provided fundamental results upon which a vast literature is built (Craciun and Feinberg 2005, 2006; Chaves 2006; Anderson 2008; Hangos 2010; van der Schaft et al. 2013): the most renowned result, the zero-deficiency theorem, gives a structural sufficient condition (0-deficiency, computable based on the network structure) to ensure that a CRN admits a single positive stable equilibrium. The theorem holds for CRNs with mass action (hence, polynomial) kinetics, although a generalization is possible (Sontag 2001). Structural stability for a vast class of dynamical systems, including CRNs with monotonic reaction rates, is investigated by Blanchini and Giordano (2014). The nonlinear system is absorbed in a linear differential inclusion (LDI) and a structural piecewise-linear (*i.e.*, polyhedral) Lyapunov function (PLF, Blanchini and Miani 2015) is sought by associating the LDI with a suitable discrete difference inclusion: then, an *iterative procedure* generates the unit ball of the PLF (if any). PLFs have been adopted to analyse specific biochemical networks (Blanchini and Franco 2011); piecewise-linear-in-rate Lyapunov functions have been recently proposed by Al-Radhawi and Angeli (2013, 2014, 2016) for the stability analysis of CRNs.

If the iterative procedure mentioned above provides a PLF, then *local* stability is structurally certified, for any choice of monotonic reaction rate functions; *global* stability is structurally ensured if each of the variables (chemical species concentrations in the case of CRNs) is subject to a spontaneous infinitesimal dissipation. Here, we do not assume infinitesimal dissipation and we investigate both *global* structural stability and the existence of an equilibrium, relying on the *BDC*-decomposition (Giordano et al. 2016; Giordano 2016) illustrated in Section 2. We show that:

- the existence of a PLF structurally ensures *global* convergence iff the system Jacobian is structurally non-singular

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(Section 3);

- if the Jacobian is structurally non-singular, assuming positivity of the system partial derivatives, a (unique) equilibrium is guaranteed to exist (Section 4), if the derivatives are upper bounded and bounded away from zero (this excludes, e.g., Michaelis-Menten and Hill functions);
- also for positive systems admitting a PLF, global convergence within the positive orthant is structurally ensured iff the Jacobian is structurally non-singular (Section 5);
- for positive systems the positive-derivative assumption fails on the boundary, but the existence of equilibria can still be established based on linear programming and piecewise-linear-in-rate Lyapunov functions (Al-Radhawi and Angeli 2013, 2014, 2016), which we can compute adopting our iterative procedure (Section 5) under radial unboundedness assumption (hence the result does not hold for Michaelis-Menten and Hill reaction kinetics).

In Section 6, the results are applied to several illustrative examples, including biochemical models from the literature.

2 Dynamical networks and BDC-decomposition

Let $\mathscr{D}$ be a convex subset of $\mathbb{R}^n$ with a non empty interior. Consider the class of *dynamical network* models

$$\dot{x}(t) = Sg(x(t)) + g_0, \tag{1}$$

where the state variables $x \in \mathscr{D} \subseteq \mathbb{R}^n$ are associated with the *nodes* of the dynamical network (and can represent species concentrations, buffer levels, stored energy, *etc.*), while $g(x) \in \mathbb{R}^m$ is a vector of functions associated with the *arcs* (representing reaction rates, flow rates, energy transfer, *etc.*), $g_0 \in \mathbb{R}^n$ is a constant vector, and matrix $S \in \mathbb{R}^{n\times m}$ (typically the stoichiometric matrix $S \in \mathbb{Z}^{n\times m}$ for CRNs) represents the *interconnection topology* of the dynamical network, namely, the system *structure*. Nonnegativity constraints $x_i \geq 0$ will be considered in Section 5, where $\mathscr{D} = \mathbb{R}^n_+$.

Example 1 *The CRN* $A + B \underset{g_c}{\overset{g_{ab}}{\rightleftharpoons}} C$, $A + C \xrightarrow{g_{ac}} \emptyset$, $B \xrightarrow{g_b} \emptyset$, $\emptyset \xrightarrow{a_0} A$, $\emptyset \xrightarrow{b_0} B$, *associated with the graph in Fig. 1, corresponds to the equations*

$$\begin{cases} \dot{a} = a_0 - g_{ab}(a,b) - g_{ac}(a,c) + g_c(c), \\ \dot{b} = b_0 - g_{ab}(a,b) - g_b(b) + g_c(c), \\ \dot{c} = g_{ab}(a,b) - g_{ac}(a,c) - g_c(c), \end{cases}$$

which can be rewritten as system (1) *with* $x = [a \;\; b \;\; c]^\top$,

$$S = \begin{bmatrix} -1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}, \; g(x) = \begin{bmatrix} g_{ab}(a,b) \\ g_c(c) \\ g_{ac}(a,c) \\ g_b(b) \end{bmatrix}, \; g_0 = \begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix}.$$

Assumption 1 *All the component functions of vector* $g(x)$ *are continuously differentiable, with partial derivatives that are positive in the interior of* $\mathscr{D}$.

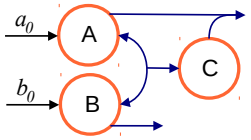

Figure 1. Graph of the dynamical network in Example 1.

Terms of the form $g(\bar{x}_i - x_i)$, with $\bar{x}_i$ constant, can account for decreasing functions (in CRNs, $\bar{x}_i$ is the total concentration of a species that can be either active or inactive).

The local and global *BDC*-decomposition can play a key role when assessing *structural* properties. We begin by defining the *local BDC*-decomposition of the system Jacobian.

Definition 1 *System* (1) *admits a BDC*-decomposition *if, for any* $x \in \mathscr{D}$, *its Jacobian* $J(x) = \partial Sg(x)/\partial x$ *can be written as the positive linear combination of rank-one matrices* R_h:

$$J(x) = \textstyle\sum_{h=1}^q R_h \Delta_h(x) = \sum_{h=1}^q B_h C_h^\top \Delta_h(x),$$

where B_h *and* $C_h^\top$ *are, respectively, column and row vectors (any rank-one matrix can be written as the product of a column vector and a row vector) and are independent of* x, *while* $\Delta_h(x)$, $h = 1,\ldots,q$, *are the positive partial derivatives of the functions* $g_i(x)$.

In a compact form, the Jacobian can be decomposed as

$$J(x) = B\Delta(x)C, \tag{2}$$

where the diagonal matrix $\Delta(x) = \text{diag}\left\{\frac{\partial g_i}{\partial x_j}\right\}$ includes all the non-zero partial derivatives of g, while $B = [B_1 \ldots B_q]$ and $C = [C_1^\top \ldots C_q^\top]^\top$. This expression holds, in particular, for any equilibrium point $\bar{x} \in \mathscr{D}$, such that

$$Sg(\bar{x}) + g_0 = 0. \tag{3}$$

The *BDC*-decomposition is unique up to permutations (in the order of the B_h's and the $C_h^\top$'s) and scaling (dividing B_h and multiplying $C_h^\top$ by the same scalar).

Example 2 *For the dynamical network in Example 1,*

$$\Delta(x) = diag\left\{\tfrac{\partial g_{ab}(a,b)}{\partial a} \; \tfrac{\partial g_{ab}(a,b)}{\partial b} \; \tfrac{\partial g_c(c)}{\partial c} \; \tfrac{\partial g_{ac}(a,c)}{\partial a} \; \tfrac{\partial g_{ac}(a,c)}{\partial c} \; \tfrac{\partial g_b(b)}{\partial b}\right\},$$

$$B = \begin{bmatrix} -1 & -1 & 1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top.$$

The *BDC*-decomposition is also a *global* property of the system. Given a point $\bar{x}$ in the interior of $\mathscr{D}$ and a smooth function $f : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, we have the following identity (which extends that by Khalil (2002), p. 108, Exercise 3.23):

$$f(x) - f(\bar{x}) = \left(\int_0^1 \frac{\partial f}{\partial x}[\sigma(x-\bar{x}) + \bar{x}]d\sigma\right)(x - \bar{x}), \tag{4}$$

for all $x \in \mathscr{D}$. Then, we have the following result.

Proposition 1 *If system* (1) *admits a BDC-decomposition (local) for any $x \in \mathscr{D}$ and has an equilibrium $\bar{x}$ in the interior of $\mathscr{D}$, then it can be equivalently written as*

$$\dot{z}(t) = [BD(z(t))C]z(t) \tag{5}$$

*(*global *BDC*-decomposition*), where $z(t) \doteq x(t) - \bar{x}$.*

Proof If we subtract the equilibrium condition (3) from system (1), and denote $z = x - \bar{x}$, we obtain the shifted system

$$\dot{z} = S[g(z+\bar{x}) - g(\bar{x})]. \tag{6}$$

If $J(x) = B\Delta(x)C$ for all $x \in \mathscr{D}$, an immediate application of the integral formula (4) to the right-hand side of (6) provides

$$\dot{z} = B \operatorname{diag}\left\{ \int_0^1 \frac{\partial g_i(\sigma z+\bar{x})}{\partial x_j} d\sigma \right\} Cz = BD(z)Cz. \tag{7}$$

■

The following result, stemming from the proof of Proposition 1, will be fundamental in the sequel.

Proposition 2 *Let $\mathscr{C} \subset \mathscr{D}$ be a compact set. Then there exist positive numbers ε and μ such that*

$$0 < \varepsilon \leq D_h(x-\bar{x}) = D_h(z) \leq \mu, \;\; h = 1, \ldots, q, \;\; \forall x \in \mathscr{C}. \tag{8}$$

Proof The integral in (7) is computed on a non-zero interval. Hence, functions $D_h(z)$ are strictly positive, continuous and admit a positive maximum and minimum in any closed and bounded domain: $0 < \varepsilon \leq \varepsilon_h \leq D_h(z) \leq \mu_h \leq \mu$. ■

Remark 1 *As shown by Giordano et al. (2016), the class of systems admitting a BDC-decomposition includes all chemical reaction networks (such as* (1)*), hence most biological models, but also flow and compartmental systems. For a survey the reader is referred to Giordano (2016).*

3 Polyhedral Lyapunov functions and global stability

In this section, we assume that $\mathscr{D} = \mathbb{R}^n$ and that system (1) admits an equilibrium $\bar{x}$. We take $\bar{x} = 0$ without restriction and we consider the shifted system

$$\dot{x}(t) = BD(x(t))Cx(t). \tag{9}$$

Definition 2 *System* (9) *is*

- stable *if the origin is Lyapunov stable,* i.e.*, there exists a continuous, strictly increasing and unbounded function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, with $\omega(0) = 0$, such that $\|x(t)\| \leq \omega(\|x(0)\|)$;*
- globally asymptotically stable *if it is stable and, for any initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$.*

Both properties are structural *if they are verified for all $D(x) = diag\{D_1(x), \ldots, D_q(x)\}$ with $D_h(x) > 0$, $h = 1, \ldots, q$.*

Given a matrix $P \in \mathbb{R}^{n\times s}$, whose columns are the vertices p_i of a polytope including the origin in its interior, the function

$$V_P(x) = \min\{\|\alpha\|_1, \;\; \alpha_i \geq 0 \;\; : \;\; P\alpha = x, \; \alpha \in \mathbb{R}^s\}$$

is a polyhedral positive definite function. The dual expression is achieved by a matrix $F \in \mathbb{R}^{s\times n}$

$$V^F(x) = \max_{i\in\{1,\ldots,s\}} f_i^\top x,$$

whose rows $f_i^\top$ are associated with the inequalities $f_i^\top x \leq 1$ that describe the convex polytope.

In the symmetric case, denoting $r = s/2$ (s is even), these functions are both norms and reduce to

$$V_{\tilde{P}}(x) = \min\{\|\alpha\|_1 \;:\; \tilde{P}\alpha = x, \; \alpha \in \mathbb{R}^r\},$$

whose unit ball has vertices $\pm\tilde{p}_i$, where $\tilde{p}_i$ is a column of matrix $\tilde{P} \in \mathbb{R}^{n\times r}$ such that $P = [\tilde{P} \;\; -\tilde{P}]$, and

$$V^{\tilde{F}}(x) = \|\tilde{F}x\|_\infty,$$

whose unit ball has facets on the planes $\pm\tilde{f}_i^\top x = 1$, where $\tilde{f}_i^\top$ is a row of matrix $\tilde{F} \in \mathbb{R}^{r\times n}$, such that $F = [\tilde{F} \;\; -\tilde{F}]^\top$.

Definition 3 *A positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$, locally Lipschitz and radially unbounded ($V(x_k) \to \infty$ for any sequence $x_k \in \mathbb{R}^n$ such that $\|x_k\| \to \infty$) is a* weak Lyapunov function *for system* (9) *if it is non-increasing along all possible system trajectories. A* polyhedral Lyapunov function (PLF) *is a polyhedral function $V_P(x)$ that is a weak Lyapunov function. A (weak) Lyapunov function $V(x)$ is* structural *if it is valid for any choice of the diagonal matrix $D(x) > 0$.*

Definition 4 *Matrix $J = BDC$ is* structurally non-singular *if* $\det(BDC) \neq 0$ *for all diagonal matrices D with positive diagonal entries.*

Definition 5 *System* (9) *is* unitary *if B and C have entries in $\{-1, 0, 1\}$.*

For unitary systems (9), the following result holds.

Theorem 1 *Assume system* (9) *is unitary. Then, it admits a structural weak Lyapunov function if and only if it admits a structural polyhedral weak Lyapunov function.*

Proof The second statement clearly implies the first. Conversely, if $V(x)$ is a structural weak Lyapunov function for (9), then it must be a Lyapunov function for any linear system $\dot{x}(t) = BDCx(t)$, with $D > 0$ diagonal and constant. If $V(x)$ is a common Lyapunov function for all matrices BDC, it is also a common Lyapunov function for the linear parameter-varying (LPV) system $\dot{x}(t) = BD(t)Cx(t)$, which is then marginally stable. The proof follows from (Blanchini and Giordano, 2014, Corollary 3.2), which shows that, for unitary networks, marginal stability implies the existence of a structural polyhedral weak Lyapunov function. ■

For non-unitary networks, the result holds in a weaker form.

Corollary 1 *Given system* (9)*, the existence of a structural weak Lyapunov function is equivalent to the existence of a structural weak Lyapunov function that is a norm.*

Proof The second statement implies the first. The existence of a structural weak Lyapunov function implies marginal stability of the LPV system, hence the existence of a Lyapunov function which is a norm (not necessarily polyhedral; Blanchini 1991; Blanchini and Giordano 2014). ■

Remark 2 *The nature of the result is* structural*: we require the existence of a* single *Lyapunov function valid* for all $D(x)$*. The existence, for any given realization $D(x)$, of a* specific *Lyapunov function, depending on $D(\cdot)$, does not imply the existence of a Lyapunov function that is a norm (a polyhedral function, in the unitary case). Indeed, there are systems that do not admit (rate-independent) piecewise-linear Lyapunov functions, but admit piecewise-linear-in-rate Lyapunov functions (Al-Radhawi and Angeli 2013, 2014, 2016). Polyhedral Lyapunov functions in* reaction coordinates *can be computed with the same iterative procedure adopted to find polyhedral functions in* concentration coordinates*, as shown by Blanchini and Giordano (2015a); such functions become rate-dependent in concentration coordinates.*

Theorem 2 *Assume that system* (9) *admits a structural polyhedral weak Lyapunov function (hence, it is structurally stable). Then, the following statements are equivalent.*

i) *System* (9) *is structurally globally asymptotically stable.*
ii) *Matrix BDC is structurally non-singular.*

Proof i) $\Rightarrow$ ii) is immediate. Indeed, among all possible dynamics, we can consider linear constant systems $\dot{x}(t) = BDCx(t)$. Then, if BDC can be singular, asymptotic stability does not hold. ii) $\Rightarrow$ i) is proven in the next subsection. ■

3.1 Proof of Theorem 2: ii) implies i)

We employ the following shorthand notation for sets: $\{V\ (*)\ \kappa\} = \{x:\ V(x)\ (*)\ \kappa\}$, where $(*)$ is any of the symbols $\leq, =, \geq, <$ and $>$.
Given a polyhedral function $V(x) = V_{\tilde{P}}(x) = V^{\tilde{F}}(x)$, the set $\{V \leq \kappa\}$ is a polytope delimited by planes $f_i^\top x = \kappa$,

$$\{V \leq \kappa\} = \{x\ :\ f_i^\top x \leq \kappa,\ \ i = 1,\dots,s\},$$

also representable by the vertices p_i of its unit ball $\{V \leq 1\}$:

$$\{V \leq \kappa\} = \{x = \textstyle\sum_{i=1}^s\ \alpha_i p_i\ :\ \sum_{i=1}^s\ \alpha_i \leq \kappa,\ \alpha_i \geq 0\}.$$

Since $V(x(t)) = \max_i\ f_i^\top x(t)$ is a convex function, we consider its directional derivative

$$D^+V(x(t)) = \lim_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h} = \max_{i\in\mathscr{A}(x)}\ f_i^\top \dot{x}(t),$$

where $\mathscr{A}(x)$ is the set indexing active planes (namely, the linear functions that achieve the maximum):

$$\mathscr{A}(x) = \{i:\ f_i^\top x = V(x),\ 1 \leq i \leq s\}.$$

A *face* $\mathscr{F}$ of the polytope $\{V \leq \kappa\}$ is a subset of its boundary $\{V = \kappa\}$, in which a certain subset of inequalities, indexed in a set $\mathscr{E}$, are equalities:

$$\mathscr{F} = \{x \in \{V \leq \kappa\} : f_i^\top x = \kappa,\ \ i \in \mathscr{E}\}.$$

All points in the relative interior of a face have the same active set $\mathscr{A}(x)$. We call $\#\mathscr{F}$ the dimension of the face. Formally, taking any point $\tilde{x} \in \mathscr{F}$, $\#\mathscr{F}$ is the dimension of the smallest subspace including all points $\{(x-\tilde{x})\ :\ x \in \mathscr{F}\}$. In three dimensions, *e.g.*, the vertices have $\#\mathscr{F} = 0$, the edges have $\#\mathscr{F} = 1$, the regular 2-dimensional faces have $\#\mathscr{F} = 2$. Any face is the set of all convex combinations of its vertices: if $p_{k_1}\dots p_{k_m}$ are vertices of $\mathscr{F}$, then

$$\mathscr{F} = \{x = \sum_{i=1}^m\ \alpha_i p_{k_i}\ \ :\ \ \sum_{i=1}^m\ \alpha_i = 1,\ \ \alpha_i \geq 0\}.$$

Under the assumptions of Theorem 2, let $x(0)$ be an arbitrary initial condition and $V_{\tilde{P}}(x)$ a structural weak PLF for system (9). Assume that BDC is a non-singular matrix for any positive definite diagonal matrix D.

We denote by $\varphi(t,x_0)$ the solution of (9) with initial condition x_0. In view of Assumption 1, we have the following.

Lemma 1 *The solution $\varphi(t,x_0)$ is a continuous function.*

Lemma 2 *For any x_0, $\lim_{t\to\infty} V(\varphi(t,x_0)) = \kappa_\infty \geq 0$, with $\kappa_\infty = \kappa_\infty(x_0)$, monotonically from above.*

Proof It follows from the fact that $V(x(t)) \geq 0$ is monotonically non-increasing along the system trajectories. ■

Since the system admits the structural weak polyhedral Lyapunov function $V(x)$, the following result holds.

Lemma 3 *For any point $x_0 \in \{V = \kappa\}$, there exists $\tau > 0$ such that $V(\varphi(\tau,x_0)) < \kappa$.*

Proof Without restrictions, let $x_0 = x(t_0)$ be on the surface of the unit ball of the norm $V(x)$, hence $\kappa = 1$; this means replacing V by V/κ, which is polyhedral as well.
If $x_0 \in \{V = 1\}$, then either: [RI] x_0 is in the relative interior of some face $\mathscr{F}$, with $\#\mathscr{F} > 0$; or [V] x_0 is a vertex.
Case [V]. If x_0 is a vertex, it must be $BD(x_0)Cx_0 \neq 0$ in view of the non-singularity assumption. Therefore, the solution will leave the vertex and will move either to the interior of $\{V \leq 1\}$ (which concludes the proof), or to its boundary (hence we fall into case [RI]).
Case [RI]. The relative interior of a face $\mathscr{F}$ is the set in which $\mathscr{A}(x)$ is constant, hence

$$x_0 \in \{x \in \{V \leq 1\}:\ \mathscr{A}(x) = \mathscr{A}(x_0)\}.$$

In what follows we show that, at some future time $\tau_1 > 0$, $x(\tau_1) = \varphi(\tau_1,x_0)$ reaches a higher dimensional face $\mathscr{F}'$,

$$x(\tau_1) \in \{x \in \{V \leq 1\}:\ \mathscr{A}(x) \subsetneq \mathscr{A}(x_0)\},$$

namely, $\#\mathscr{F}' > \#\mathscr{F}$, because one of the constraints becomes

inactive (*i.e.*, satisfied as a strict inequality).
In order to show this fact observe that, since the derivative of x is $\dot{x} = BD(x_0)Cx$, the non-positive Lyapunov derivative is

$$D^+V(x_0) = \max_{i \in \mathscr{A}(x_0)} \; f_i^\top BD(x_0)Cx_0 \leq 0.$$

We distinguish two cases:

a) $f_i^\top BD(x_0)Cx_0 < 0$ for some $i \in \mathscr{A}(x_0)$: then there exists τ_1 such that $f_i^\top x(t) < 1$ for $t_0 < t \leq \tau_1$;
b) $f_i^\top BD(x_0)Cx_0 = 0$ for all $i \in \mathscr{A}(x_0)$.

We can show by contradiction that case b) cannot occur. Assume $f_i^\top BD(x_0)Cx_0 = 0$ for all i and consider the system

$$\dot{x}(t) = BD(x_0)Cx \doteq Ax, \tag{10}$$

achieved by "freezing" $D = D(x_0)$. Since (10) belongs to the family of systems, $A = BD(x_0)C$ is non-singular by assumption. Moreover, $V(x)$ is a weak Lyapunov function for system (10) and $\{V \leq 1\}$ is a positively invariant set.

Denote by $x_A(t)$ the solution of (10), and by $p_1, p_2, \ldots, p_m$ the vertices of the polytope $\{V \leq 1\}$ on the face $\mathscr{F}$. We have $x_0 = \sum_{j=1}^m \alpha_j p_j$, hence

$$D^+V(x_0) = \max_{i \in \mathscr{A}(x_0)} \sum_{j=1}^m f_i^\top A \alpha_j p_j = 0.$$

Since x_0 is in the relative interior of the face, all coefficients α_j are to be taken positive, hence it must be

$$f_i^\top A p_j = 0 \tag{11}$$

for all $i \in \mathscr{A}(x_0)$ and all $j \in \{1, \ldots, m\}$.

We now prove that the face $\mathscr{F}$ is positively invariant for $\dot{x} = Ax$ (Blanchini and Miani 2015). We first show that the affine manifold $\mathscr{M}$, which includes the face ($\mathscr{F} = \mathscr{M} \bigcap \{V \leq 1\}$),

$$\mathscr{M} = \{x : \;\; f_i^\top x = 1, \;\; i \in \mathscr{A}(x_0)\},$$

is positively invariant for $\dot{x} = Ax$. The set $\mathscr{M}$ is

$$\mathscr{M} = \{x = x_0 + \sum_{j=1}^m \beta_j (p_j - x_0), \;\; \text{with } \beta_j \in \mathbb{R}\}.$$

For each $f_i^\top$ with $i \in \mathscr{A}(x_0)$ and any $x_0 = x_A(0) \in \mathscr{M}$,

$$\frac{d}{dt} f_i^\top x_A = f_i^\top A x_A = f_i^\top A x_A(0) + \sum_{j=1}^m \beta_j f_i^\top A (p_j - x_A(0)) = 0,$$

in view of the temporary assumption b) and of (11).

Hence $f_i^\top x_A(t) = \text{const} = 1$, and $\mathscr{M}$ is positively invariant. The face $\mathscr{F}$ is the intersection of two invariant sets, $\mathscr{M}$ and $\{V \leq 1\}$ (the polytope is also invariant for the "frozen" linear system): hence, $\mathscr{F}$ is invariant as well. If the bounded set $\mathscr{F}$ is invariant, then there must exist a rest point $\hat{x}$ that is non-zero, because it lies on the surface of $\{V = 1\}$, and for which $A\hat{x} = 0$, against the non-singularity assumption. Then b) cannot occur and the only possibility is case a).

Let us consider case a). By continuity, there exists a right neighborhood $[t_0, \tau_2]$ of t_0 in which the active set cannot grow: $\mathscr{A}(x(t)) \subseteq \mathscr{A}(x_0)$ for $t \in [t_0, \tau_2]$. On the other hand, from condition a), at least one index in $\mathscr{A}(x_0)$ is not in the active set in $(t_0, \tau_1]$. Hence, in the interval $(t_0, \tau_0]$, $\tau_0 = \min\{\tau_1, \tau_2\}$, we have the strict inclusion $\mathscr{A}(x(t)) \subsetneq \mathscr{A}(x_0)$.

Now we just iterate the same reasoning. Consider a new initial time $t_0' > t_0$, for which $\mathscr{A}(x(t_0')) \subsetneq \mathscr{A}(x_0)$. Since case b) is not possible, we are in case a), hence for some $t_0'' > t_0'$ we have $\mathscr{A}(x(t_0'')) \subsetneq \mathscr{A}(x(t_0'))$, and so on. Therefore, at some time τ, all indices that are active in t_0 become inactive (associated with strict inequalities): $f_i^\top \varphi(\tau, x_0) < 1$ for all i in $\mathscr{A}(x_0)$. Hence $V(\varphi(\tau, x_0)) < 1$, and the proof is over. ■

Proof of Theorem 2: ii) $\Rightarrow$ i). Lemma 2 ensures that, for any initial condition x_0, $V(x(t)) \to \kappa_\infty \geq 0$ from above (with κ_∞ depending on x_0). We prove that $\kappa_\infty = 0$, which implies global stability. By contradiction, assume $\kappa_\infty > 0$ and consider $\kappa_\infty = 1$, without loss of generality. In finite time, the solution will be trapped in the compact set $\{V \geq 1\} \cap \{V \leq 2\}$.
Let t_k be a sequence of time instants, $k = 1, 2, \ldots$, such that $t_k < t_{k+1}$ and $t_k \to +\infty$ for $k \to +\infty$. The sequence $x_k = \varphi(t_k, x_0)$ has an accumulation point $\tilde{x}$.
Necessarily, $V(\tilde{x}) = 1$. Indeed, we can extract from $\{t_k\}$ a subsequence $\{t_k'\}$ such that $x(t_k') \to \tilde{x}$. On the other hand, $V(x(t_k')) \to 1$ by assumption. Since V is continuous, it follows that $V(\tilde{x}) = 1$.
In view of Lemma 3, for some $\tau > 0$, $\varphi(\tau, \tilde{x}) \in \{V < 1\}$. On the other hand, due to Lemma 1, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|x - \tilde{x}\| \leq \delta$, then $\|\varphi(\tau, x) - \varphi(\tau, \tilde{x})\| \leq \varepsilon$. Let us take ε such that the ball with radius ε and centre $\varphi(\tau, \tilde{x})$ lies within $\{V < 1\}$.
Since there are points of the solution $x_k = \varphi(t_k, x_0)$ that are arbitrary close to $\tilde{x}$, there are time instants t_k such that $\|\varphi(t_k, x_0) - \tilde{x}\| < \delta$. Then $\|\varphi(t_k + \tau, x_0) - \varphi(\tau, \tilde{x})\| \leq \varepsilon$, which means that $x(t_k + \tau) = \varphi(t_k + \tau, x_0)$ is in $\{V < 1\}$, namely, $V(x(t_k + \tau)) < 1$. However, according to Lemma 2, $V(x(t)) \to \kappa_\infty = 1$ monotonically from above: we fall in a contradiction. Hence, it must be $\kappa_\infty = 0$. ■

Along the same line, the following result can be proven.

Corollary 2 *If a system of the form*

$$\dot{x}(t) = A(x(t))x(t) \tag{12}$$

admits a weak polyhedral Lyapunov function and $A(x)$ is sufficiently regular and non-singular for any x, then system (12) *is globally asymptotically stable.*

Necessity in Theorem 2 comes from the fact that we are looking for structural stability (for any diagonal matrix D in the *BDC*-decomposition), while Corollary 2 provides a condition that is just sufficient for stability. For instance,

matrix $A(x_1,x_2) = \text{diag}\left\{-1\ , -x_2^2\right\}$ is singular if $x_2 = 0$, but the system $\dot{x} = A(x)x$ is globally asymptotically stable, with $V(x) = \max_i\ |x_i|$.

4 Existence of equilibria

A structural result concerning uniqueness of the equilibrium of the nonlinear system can be derived based on the *BDC*-decomposition.

Theorem 3 *Consider system* (1)*, defined on the convex domain $\mathscr{D}$ with non empty interior and admitting a BDC-decomposition. If there are two distinct equilibrium points in $\mathscr{D}$, then the system Jacobian $J(x) = B\Delta(x)C$ cannot be structurally non-singular.*

Proof Given two distinct equilibrium points $\bar{x}$ and $\tilde{x}$, both equilibrium conditions $0 = Sg(\bar{x}) + g_0$ and $0 = Sg(\tilde{x}) + g_0$ must hold. For any x, $S[g(x) - g(\bar{x})] = BD(z)Cz$, with $z = x - \bar{x}$. Choosing $x = \tilde{x}$ provides $BD(z)Cz = S[g(\tilde{x}) - g(\bar{x})] = 0$, in view of the equilibrium conditions. Since $z = \tilde{x} - \bar{x} \neq 0$, being the equilibria distinct by assumption, then $BD(z)C$ (hence, $J(x) = B\Delta(x)C$) must be singular for some x. ■

Hence, if $J = BDC$ is structurally non-singular, the equilibrium (if any) must be unique.

What about the existence of equilibria? First of all, equilibria might not exist if there is no lower bound on the derivative: *e.g.*, the scalar system $\dot{x} = -g(x) + g_0$ admits an equilibrium for any $g_0 \neq 0$ if and only if the monotonic function g is asymptotically unbounded.

We can indeed prove the existence of the equilibrium for any g_0 if the vector function g is radially unbounded.

Definition 6 *Function $g : \mathbb{R}^n \to \mathbb{R}^m$ is* radially unbounded *if $\|g(x_k)\| \to \infty$ for any sequence $x_k \in \mathbb{R}^n$ such that $\|x_k\| \to \infty$.*

Radial unboundedness is structurally ensured by non-singularity along with the following global bound on the derivatives: for some ε and μ,

$$0 < \varepsilon \leq D_h(x) \leq \mu, \quad \forall x \in \mathbb{R}^n. \tag{13}$$

The next result is similar to known results in the literature, see Corollary 3.3 by Zampieri (1992) and the references therein, although we use a tailored proof.

Theorem 4 *If system* (1)*, with $\mathscr{D} = \mathbb{R}^n$, admits a structurally non-singular BDC-decomposition, with the bound* (13)*, then system* (1) *admits an equilibrium for any g_0.*

Proof. The proof requires two steps. First, we show that $BD(x)Cx$ is radially unbounded. Denoting by $\sigma_{min}(M)$ the smallest singular value of a matrix M, we have

$$\|BD(x)Cx\| \geq \sigma_{min}(BD(x)C)\|x\| \geq \min_{\varepsilon \leq D_h \leq \mu} \sigma_{min}(BDC)\|x\|$$

(where $\|\cdot\|$ is the Euclidean norm). Being the set of matrices *BDC* compact, the minimum is achieved and its value is positive: $\min_{\varepsilon \leq D_h \leq \mu} \sigma_{min}(BDC) > 0$. This proves radial unboundedness.

For the second step of the proof, we use a trick borrowed from Zampieri (1992). Denoting $f(x) = BD(x(t))Cx(t) + g_0$, we consider the fictitious system

$$\dot{z}(t) = -J(z(t))^{-1} f(z(t)),$$

which is well defined because $J(z) = \partial f(z)/\partial z = B\Delta(z)C$ is non-singular. Hence, $\sup \|[B\Delta(z)C]^{-1}\| < \infty$.

Consider the Lyapunov-like function $\Psi(z) = f^\top(z)f(z)/2$, which is continuously differentiable and radially unbounded. Any set of the form $\{\Psi \leq \mu\}$ is positively invariant,

$$\dot{\Psi}(z) = f^\top(z)J(z)\dot{z} = -f^\top(z)f(z) \leq 0,$$

and compact, since f is radially unbounded. Therefore, for any initial condition in this set, the solution of the fictitious system is bounded. This immediately implies (Srzednicki 1985; Richeson and Wiseman 2002, 2004) that the fictitious system has an equilibrium $J(\bar{z})^{-1} f(\bar{z}) = 0$, with $J(\bar{z})^{-1}$ non-singular because it has the *BDC* structure. Then $f(\bar{z}) = 0$, hence the original system admits an equilibrium. ■

Remark 3 *The bound* (13) *is a sufficient condition for radial unboundedness, and could be replaced by the weaker assumption of g being radially unbounded. On the other hand, several reaction functions used in biology, such as Michaelis-Menten and Hill-type kinetics, are not radially unbounded; when all reaction functions in g have this form, the existence of an equilibrium is typically guaranteed provided that the input g_0 is not too large.*

5 Positive systems

Important classes of systems, such as chemical and biological networks, must have nonnegative state variables:

$$\dot{x}(t) = Sg(x(t)) + g_0, \quad x(t) \geq 0 \ \text{ componentwise}, \tag{14}$$

i.e., the positive orthant $\mathscr{D} = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ x_i \geq 0\}$ must be positively invariant. The following assumption ensures invariance of $\mathscr{D} = \mathbb{R}^n_+$, because $x_i = 0 \Rightarrow \dot{x}_i \geq 0$.

Assumption 2 *Each component function of vector $g(x)$ is nonnegative in $\mathscr{D}$ and zero if and only if at least one of its arguments is zero. If $s_{ij} < 0$, then g_j must depend on x_i. Moreover, all the partial derivatives of g are positive for $x > 0$ (componentwise).*

We cannot assume positivity of the derivative everywhere: typically, the partial derivatives are positive only in the interior of the positive orthant, but can be zero on the boundary.

We adapt Definition 2 as follows.

Definition 7 *System* (14) *is structurally asymptotically stable if, whenever a positive equilibrium $\bar{x}$ exists, this equilibrium is stable in the sense of Definition 2 i) and convergence to $\bar{x}$ is guaranteed for any $x(0) \geq 0$, independent of the chosen function g, provided that it satisfies Assumption 2.*

Theorem 2 can be extended as follows.

Theorem 5 *Assume that system* (14) *admits a positive equilibrium* $\bar{x} > 0$ *and a BDC representation*

$$\dot{z}(t) = BD(z(t))Cz(t), \tag{15}$$

with $z = x(t) - \bar{x}$. *If the system admits a structural weak PLF* $V(z)$, *the following conditions are equivalent.*

i) *System* (15) *is structurally globally asymptotically stable.*
ii) *BDC is* structurally *non-singular.*

Proof. *i)* $\Rightarrow$ *ii)* The argument adopted in Theorem 2 cannot be used because $\dot{z} = BDCz$ with constant D is not necessarily a positive system. We proceed by contradiction, assuming that i) holds but the Jacobian BDC is not structurally non-singular. Then there are two cases.

a) The Jacobian is not identically singular.
b) The Jacobian is identically singular.

Case a). It is not possible, because the PLF could not exist. Indeed, $\det(-BDC)$ is a multilinear function of the diagonal entries D_h of D, hence it is linear with respect to each of the D_h's. If it is null at some point $(D_1, D_2, \ldots, D_q) > 0$ (but not identically), then it must assume both positive and negative values: therefore, for some D, $\det(-BDC) < 0$. However, $\det(-BDC)$ is the 0-degree term of the characteristic polynomial $p(s) = \det(sI - BDC)$. If $\det(-BDC) < 0$, then BDC has eigenvalues with positive real part, hence it is unstable and no Lyapunov function can exist.
Case b). If for some function g^* an equilibrium $\bar{x} > 0$ exists, such that $Sg^*(\bar{x}) + g_0 = 0$, then there exists another function g (the details on its construction are in Appendix A) that is associated with the same equilibrium $\bar{x}$ and affine in a neighborhood $\mathscr{N}_V$ of $\bar{x}$, namely,

$$\dot{x} = Sg(x) + g_0 = BDCx + g_0, \text{ for } x \in \mathscr{N}_V,$$

for some constant D such that $\det(BDC) = 0$. In such a neighborhood $\mathscr{N}_V$, $z(t) = x(t) - \bar{x}$ satisfies the equation

$$\dot{z}(t) = BDCz(t).$$

Since BDC is singular, we cannot have asymptotic stability.

ii) $\Rightarrow$ *i).* Consider an arbitrary $\kappa > 0$ and a (nonnegative) initial condition $x(0)$ in the convex and compact set

$$\mathscr{D}_\kappa \doteq \{V(x - \bar{x}) \leq \kappa\} \bigcap \mathbb{R}^n_+$$

For any point in this set, the integral in (7) is strictly positive, because it is carried out on a segment in the *interior* of the positive orthant, where all the partial derivatives are positive. By continuity, $D(z) = D(x - \bar{x})$ has a positive global minimum in $\mathscr{D}_\kappa$, and the result follows from Theorem 2. ■

Remark 4 ***(Stoichiometric compatibility class.)*** *For a chemical reaction network, any equilibrium* $\bar{x}$ *is associated with the stoichiometric compatibility class*

$$\mathscr{S}(\bar{x}) = \{x = \bar{x} + Ra(S)\},$$

where $Ra(S)$ *is the range of matrix S. Only if* $x(0) \in \mathscr{S}(\bar{x})$, *we can have convergence to* $\bar{x}$. *If* $rank(S) < n$, *then the affine variety* $\mathscr{S}(\bar{x})$ *has a dimension smaller than n and BDC is structurally singular. Is the system globally stable for all* positive *initial conditions inside the variety? If a Lyapunov function exists, then we can define a new weak Lyapunov function* $V_{\mathscr{S}}$ *inside* $\mathscr{S}(\bar{x})$, *having the polytope* $\{V_{\mathscr{S}} \leq 1\} = \{V \leq 1\} \bigcap \mathscr{S}(\bar{x})$ *as a unit ball. We can repeat exactly the same reasoning in the "restricted state space" and claim that, if BDC has full relative rank with respect to the restriction to* $\mathscr{S}(\bar{x})$, *then we have global stability for all positive initial conditions inside* $\mathscr{S}(\bar{x})$. *In practice, the test is done by taking an orthonormal basis Q of* $Ra(S)$, *and checking whether* $Q^\top BDCQ$ *is structurally non-singular.*

Existence of equilibria. Under positivity constraints, Theorem 4 cannot be extended: in fact, the positivity of the derivative is not guaranteed on the boundary of the positive orthant (i.e., if some state components are zero). An alternative criterion can be provided based on a piecewise-linear-in-rate Lyapunov function.

Since $g(\bar{x}) > 0$ for $\bar{x} > 0$, an equilibrium can exist only if the linear programming problem

$$Sr + g_0 = 0, \quad r > 0 \tag{16}$$

admits a positive solution $\bar{r} > 0$. Unfortunately, this is a necessary condition only. Even when $\bar{r} > 0$ exists, indeed, if $m > n$ there is generally no $\bar{x} \in \mathbb{R}^n_+$ such that $g(\bar{x}) = \bar{r}$.

Denote by $y = g(x) - \bar{r}$ the rate function. Then

$$\dot{y} = \frac{\partial g}{\partial x}\dot{x} = \frac{\partial g}{\partial x}[Sg(x) + g_0] = \frac{\partial g}{\partial x}S[g(x) - \bar{r}] = \frac{\partial g}{\partial x}Sy.$$

This equation admits a BDC-decomposition

$$\dot{y} = [ED(y)F]y, \tag{17}$$

where E and F are constant matrices (different from B and C for the original system) and D is a diagonal matrix of partial derivatives. Matrix EDF has the same structure as BDC. Hence, the procedure proposed by Blanchini and Giordano (2014) for the generation of a PLF can be adopted: if it converges, then (17) admits a weak (non-increasing) piecewise-linear Lyapunov function $U(y) = \|Hy\|_\infty$. Hence, if g is radially unbounded, an equilibrium point exists.

Assumption 3 *Function* $g(x)$ *is radially unbounded in* $\mathbb{R}^n_+$: *for any nonnegative sequence* $x_k \geq 0$, $\|x_k\| \to \infty$ *implies* $\|g(x_k)\| \to \infty$.

Theorem 6 *Under Assumption 3, assume also that* (16) *admits a positive solution* $\bar{r} > 0$ *for* $g_0 \geq 0$ *and that* (17) *admits a weak polyhedral Lyapunov function* $U(y)$. *Then, there exists an equilibrium* $\bar{x} \geq 0$.

Proof Consider any initial condition $x_0 \geq 0$ in a η-ball, $\|x_0\| \leq \eta$. Denoting $y_0 \doteq g(x_0) - \bar{r}$, let $y(t) = g(x(t)) - \bar{r}$ be the corresponding solution in terms of rate variables, which

satisfies (17). Due to the existence of the PLF $U(y)$, $y(t)$ is bounded. In view of the radial unboundedness assumption, all the solutions $x(t)$ starting in the ball are bounded as well. This ensures that a stationary point exists (Srzednicki 1985; Richeson and Wiseman 2002, 2004). ■

Without radial unboundedness, the rate function $g(x)$ can remain bounded, and still $x(t)$ can be unbounded. This can be the case of networks with saturated Michaelis-Menten or Hill reaction kinetics. Note that, even in the case of mass action kinetics networks, radial unboundedness may fail (e.g., $\kappa x_1 x_2$ is not radially unbounded, see Section 6).

Remark 5 *If problem* (16) *is feasible, equilibrium points on the boundary can be excluded as follows. For any variable x_i, eliminate from the linear programming problem* (16) *all reaction variables r_j for which the corresponding reactions include x_i by setting $r_j = 0$, while keeping $r_h \geq 0$ for the other variables. If these linear programming problems become unfeasible, then there cannot be non-positive equilibria (however, feasibility of the reduced linear programs does not imply that boundary equilibria exist).*

6 Examples

We present here simple examples that emphasise relevant aspects of the proposed results, along with examples taken from the biochemical literature. PLFs have been generated using the iterative procedure (Blanchini and Giordano 2014).

Why piecewise-linear? Consider the system corresponding to the graph in Fig. 2:

$$\begin{cases} \dot{a} = -g_a(a) + g_b(b) - g_{ab}(a,b) + a_0 \\ \dot{b} = g_a(a) - g_b(b) - g_{ab}(a,b) + b_0 \end{cases} \tag{18}$$

Let $\alpha = \frac{\partial g_{ab}}{\partial a}$, $\beta = \frac{\partial g_a}{\partial a}$, $\gamma = \frac{\partial g_{ab}}{\partial b}$, $\delta = \frac{\partial g_b}{\partial b}$ be all positive.

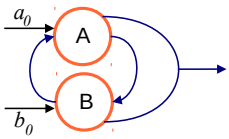

Figure 2. Graph corresponding to system (18).

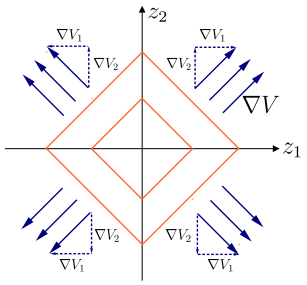

Figure 3. Unit ball of the PLF for system (18).

As shown by Blanchini and Giordano (2015b), (18) does not admit structural quadratic Lyapunov functions, but admits a structural PLF (not surprisingly: quadratic functions are known to be conservative, Molchanov and Pyatnitskiy 1989). We show now that the 1-norm $V(z_1,z_2) = |z_1| + |z_2|$ is the only possible rate-independent structural Lyapunov function for system $\dot{z} = Jz$: any Lyapunov function must have sublevel surfaces shaped as diamonds. We have that

$$J = BDC = \begin{bmatrix} -(\alpha+\beta) & \delta-\gamma \\ \beta-\alpha & -(\gamma+\delta) \end{bmatrix}.$$

Consider the gradient $\nabla V = [\nabla V_1 \;\; \nabla V_2]$ in the positive quadrant $z_1, z_2 > 0$. We must have that $\dot{V}(z) = \nabla V J z \leq 0$, namely

$$\begin{aligned} \dot{V}(z) = &[\beta(\nabla V_2 - \nabla V_1) - \alpha(\nabla V_2 + \nabla V_1)]z_1 \\ &+ [\delta(\nabla V_1 - \nabla V_2) - \gamma(\nabla V_1 + \nabla V_2)]z_2 \leq 0, \end{aligned}$$

for any choice of $\alpha, \beta, \gamma, \delta > 0$. In turn, this implies that for $z_1, z_2 > 0$, the inequality *structurally* holds iff $\nabla V_2 = \nabla V_1 > 0$. In the second quadrant $z_1 < 0$, $z_2 > 0$, we get in the same way $\nabla V_2 = -\nabla V_1 > 0$. By considering all quadrants, we conclude that the essential condition is that $|\nabla V_1| = |\nabla V_2|$ and $\text{sign}[\nabla V_1 \;\; \nabla V_2] = \text{sign}[z_1 \;\; z_2]$. Taking into account continuity, this condition implies that all the sublevel surfaces are diamonds (cf. Fig. 3), hence the only possible structural Lyapunov function is polyhedral.

Let us now remove the term $g_b(b)$ from the equation of $\dot{a}$ and the term $g_a(a)$ from the equation of $\dot{b}$. A solution $[\bar{r}_1, \bar{r}_2, \bar{r}_3]^\top > 0$ of (16) exists for any $a_0 > 0$ and $b_0 > 0$. The matrix in equation (17) is

$$EDF = \begin{bmatrix} -\beta & 0 & -\beta \\ 0 & -\delta & -\delta \\ -\alpha & -\gamma & -(\alpha+\gamma) \end{bmatrix}.$$

This system admits a PLF $U(y) = \|y\|_\infty$, but it is only marginally stable. Indeed, all the points $v = \lambda[1 \;\; 1 \;\; -1]^\top$, $\lambda \in \mathbb{R}$, are equilibria: $EDFv = 0$. Yet, the existence of $U(y)$ is fundamental to establish boundedness, hence existence of the equilibrium $\bar{x}$. The *BDC*-decomposition matrix is

$$J = BDC = \begin{bmatrix} -(\alpha+\beta) & -\gamma \\ -\alpha & -(\gamma+\delta) \end{bmatrix}.$$

The corresponding system (in "concentration" coordinates) admits the Lyapunov function $V(x-\bar{x}) = \|x-\bar{x}\|_1$. Matrix *BDC* is robustly non-singular and this, in turn, ensures global asymptotic stability of the original system.

Let us consider a new variation of system (18), with $g_a(a) = g_b(b) \equiv 0$. The corresponding rate system admits a polyhedral Lyapunov function. However, if $g_{ab}(a,b) = \kappa ab$, we do not have radial unboundedness: since $g_a = g_b = 0$, $\|g\| = g_{ab}$ and, if we take the norm-diverging sequence $a_k = k$ and $b_k = 1/k$, function g_{ab} remains constant. In fact, if $b_0 > a_0 > 0$, then $d/dt(b-a) = b_0 - a_0 = \text{const} > 0$, hence all trajectories diverge and there is no equilibrium.

A fourth order system. For the system

$$\begin{cases} \dot{a} = -g_a(a) - g_{ad}(a,d) + a_0, \\ \dot{b} = \ g_a(a) - g_b(b), \\ \dot{c} = \ g_a(a) - g_c(c) + c_0, \\ \dot{d} = \ g_c(c) - g_{ad}(a,d) - g_d(d), \end{cases}$$

the procedure yields a PLF $V(x)$ whose unit ball has 14 vertices. *BDC* is structurally non-singular. This means that, if an equilibrium exists, then it is globally asymptotically stable. To establish the existence of an equilibrium, consider the linear programming problem $Sr + g_0 = 0$, $r > 0$, with

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 \end{bmatrix}, \quad g_0 = \begin{bmatrix} a_0 \\ 0 \\ c_0 \\ 0 \end{bmatrix}.$$

Via standard Gaussian elimination, we can see that this system is equivalent to $\tilde{S}r = -\tilde{g}_0$, with

$$\tilde{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & +\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}, \quad -\tilde{g}_0 = \begin{bmatrix} \frac{a_0}{2} - \frac{c_0}{2} \\ \frac{a_0}{2} + \frac{c_0}{2} \\ \frac{a_0}{2} - \frac{c_0}{2} \\ \frac{a_0}{2} + \frac{c_0}{2} \end{bmatrix}.$$

Then we have $r_1 = (r_5 + a_0 - c_0)/2$, $r_2 = (-r_5 + a_0 + c_0)/2$, $r_3 = (r_5 + a_0 - c_0)/2$ and $r_4 = (r_5 + a_0 + c_0)/2$, along with $r_i > 0$, $i = 1, \ldots, 5$. Hence, if $a_0 - c_0 > 0$ and if we choose $0 < r_5 < a_0 + c_0$, a positive $\bar{r} > 0$ exists. By computing the *EDF* decomposition for the system in reaction coordinates and applying the procedure, we find a PLF $U(y)$ with 16 vertices (or 16 facets). In view of Theorem 6 we have boundedness, in rate and concentration coordinates, hence there is a globally asymptotically stable (unique) equilibrium.

Phosphorylation-dephosphorylation reversible cycle (Golbeter and Koshland 1981). The chemical reactions $A + B \underset{g_c(c)}{\overset{g_{ab}(a,b)}{\rightleftharpoons}} C \xrightarrow{g_c^*(c)} D + B$ and $D + E \underset{g_f(f)}{\overset{g_{de}(d,e)}{\rightleftharpoons}} F \xrightarrow{g_f^*(f)} A + E$, where B and E are the enzymes kinase and phosphatase, A is the protein and D the phosphorylated protein, yield the positive system

$$\begin{cases} \dot{a} = -g_{ab}(a,b) + g_c(c) + g_f^*(f), \\ \dot{b} = -\dot{c} = -g_{ab}(a,b) + g_c(c) + g_c^*(c), \\ \dot{d} = -g_{de}(d,e) + g_f(f) + g_c^*(c), \\ \dot{e} = -\dot{f} = -g_{de}(d,e) + g_f(f) + g_f^*(f), \end{cases}$$

along with the conservation laws $b + e = K$, $e + f = H$, $e + b - a - d = M$, where we assume $K + H \neq M$ to exclude equilibria on the boundary. The existence of an equilibrium is ensured by the boundedness of the trajectories. After replacing $c = K - b$, $e = M + a + d - b$ and $f = H - M - a - d + b$, we consider the reduced order system in the variables a, b and d, whose Jacobian

$$J = \begin{bmatrix} -(\alpha+\theta) & -\beta-\gamma+\theta & -\theta \\ -\alpha & -(\beta+\gamma+\delta) & 0 \\ -(\zeta+\eta) & \zeta+\eta-\delta & -(\varepsilon+\zeta+\eta) \end{bmatrix}$$

is structurally nonsingular: $\det(-J) > 0$ structurally. Hence, convergence of the procedure (providing a polyhedral Lyapunov function whose unit ball has 12 vertices, or 12 facets in the dual case) guarantees *global asymptotic stability of the equilibrium in the stoichiometric compatibility class*.

Phosphotransfer (Del Vecchio and Murray 2014). Given the inactive proteins A and D, along with their phosphorylated versions E and B, consider the phosphotransfer reaction $A + B \underset{g_c(c)}{\overset{g_{ab}(a,b)}{\rightleftharpoons}} C \underset{g_{de}(d,e)}{\overset{g_c^*(c)}{\rightleftharpoons}} D + E$, the dephosphorylation reaction $E \xrightarrow{g_e(e)} A$ and the phosphorylation reaction $D \xrightarrow{g_d(d)} B$, along with the conservation laws $b + c + d = K$ and $a + c + e = H$. The corresponding positive system

$$\begin{cases} \dot{a} = -g_{ab}(a,b) + g_c(c) + g_e(e), \\ \dot{b} = -g_{ab}(a,b) + g_c(c) + g_d(d), \\ \dot{c} = g_{ab}(a,b) + g_{de}(d,e) - g_c(c) - g_c^*(c), \\ \dot{d} = -g_{de}(d,e) + g_c^*(c) - g_d(d), \\ \dot{e} = -g_{de}(d,e) + g_c^*(c) - g_e(e), \end{cases}$$

does not admit equilibria on the boundary, but an equilibrium exists in view of boundedness. The reduced order system in the variables a, b and c has Jacobian

$$J = \begin{bmatrix} -(\alpha+\varepsilon) & -\beta & \gamma-\varepsilon \\ -\alpha & -(\beta+\delta) & \gamma-\delta \\ \alpha-\eta & \beta-\zeta & -(\gamma+\zeta+\eta+\theta) \end{bmatrix},$$

which is structurally nonsingular: $\det(-J) > 0$ structurally. The procedure provides a polyhedral Lyapunov function whose unit ball has 14 vertices (or 12 facets in the dual case), thus ensuring *global asymptotic stability of the equilibrium in the stoichiometric compatibility class*.

7 Conclusions

We have investigated structural global stability and existence of an equilibrium for a class of dynamical networks, including CRNs with monotonic reaction rates. For these systems, structural stability can be certified by PLFs, which can be found based on our iterative procedure; however, our previous results guaranteed *local* stability, or global stability in the presence of an infinitesimal auto-degradation affect-

ing each of the variables. Here we have shown that, if the system admits a PLF, *global* stability (of the unperturbed system) is ensured if and only if the Jacobian is structurally non-singular. Structural non-singularity of the Jacobian also ensures the existence of an equilibrium, under suitable assumptions on the system partial derivatives. If the system trajectories are constrained in the positive orthant (*e.g.*, for chemical and biological systems) and a PLF exists, structural non-singularity ensures global convergence to a nonnegative equilibrium point, while linear programming and piecewise-linear-in-rate Lyapunov functions can be employed to guarantee that an equilibrium exists.

It is worth comparing the analysis based on PLFs (in concentration coordinates) with that based on piecewise-linear-in-rate (PWLR) Lyapunov functions, a class of functions proposed by Al-Radhawi and Angeli (2013, 2014, 2016) for CRNs. A vast campaign of numerical tests (Giordano 2016) suggests that, in general, our generating iterative procedure is more likely to converge for PWLR Lyapunov functions when the number of species n is greater than the number of reactions m, and for PLFs if $m > n$. If $n = m$, we have convergence in rates iff we have convergence in concentrations (Blanchini and Giordano 2015a). Therefore, we believe that the two types of functions are complementary and an interesting future research direction aims at generating mixed piecewise-linear functions in concentrations and rates.

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A Construction of function g.

Assume that $Sg^*(\bar{x}) + g_0 = 0$ for some $\bar{x} > 0$. Then, inside the positive orthant, there exist a neighborhood $\mathcal{N}_\nu = \{x > 0 : \ |x_i - \bar{x}_i| \leq \nu\}$ where another function g is affine and admits the same equilibrium.

Let g_i be the ith component of g. We consider for simplicity the case of 3 variables, but the argument works for any number of variables. The value $g_i^*(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is fixed by the equilibrium. Take g so that $g_i = a_i(x_1, x_2, x_3)x_1 + b_i(x_1, x_2, x_3)x_2 + c_i(x_1, x_2, x_3)x_3$.

Due to Assumption 2, such a choice is admissible if $a_i(x_1, 0, x_3)$, $a_i(x_1, x_2, 0)$, $b_i(x_1, x_2, 0)$, $b_i(0, x_2, x_3)$, $c_i(x_1, 0, x_3)$ and $c_i(0, x_2, x_3)$ are zero for all i. Take the smooth and non-decreasing functions $\Phi_1(x_1)$, $\Phi_2(x_2)$ and $\Phi_3(x_3)$, such that $\forall j \in \{1,2,3\}$ $\Phi_j(0) = 0$ and $\Phi_j(x_j)$ is constant inside $\mathcal{N}_\nu$: $\Phi_j(x_j) = 1$ for $|x_j - \bar{x}_j| \leq \nu$, $\Phi_j(x_j) > 1$ for $x_j > \bar{x}_j + \nu$. Define

$$
\begin{aligned}
a_i(x_1, x_2, x_3) &= \alpha_i \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3), \\
b_i(x_1, x_2, x_3) &= \beta_i \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3), \\
c_i(x_1, x_2, x_3) &= \gamma_i \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3).
\end{aligned}
$$

To preserve the equilibrium, we can select α_i, β_i and γ_i so that, at the rest point $\bar{x}$, $g_i^*(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \alpha_i \bar{x}_1 + \beta_i \bar{x}_2 + \gamma_i \bar{x}_3$. With this choice of the components of g, in $\mathcal{N}_\nu$ we have $Sg(x) + g_0 = BDCx + g_0$, where D is a constant, diagonal matrix including the coefficients α_i, β_i and γ_i.