NON-TYCHONOFF $e$-COMPACTIFIABLE SPACES

K. P. HART AND J. VERMEER

ABSTRACT. We construct a non-Tychonoff space $X$ which is $e$-compactifiable, thus answering a question of S. Hechler. We also answer a question of R. M. Stephenson: whether there exists a Tychonoff space, the largest $e$-compactification of which has a noncompact semiregularization.

1. Introduction. All spaces are Hausdorff. In [He] S. Hechler introduced the class of $e$-compactifiable spaces, i.e. spaces which admit an $e$-compactification. He posed the question whether there exist non-Tychonoff $e$-compactifiable spaces. We show that such spaces exist. In [St] R. M. Stephenson observed that an $e$-compactifiable space has a largest $e$-compactification $eX$, and he asked whether the space $(eX)_\delta$—the semiregularization of $eX$—is always compact. We show that this need not be the case, even if the space $X$ is assumed to be Tychonoff. The example of the space we present is based on an example of J. Chaber.

2. Preliminary definitions and theorems.

DEFINITION 2.1 [He]. Let $D$ be a dense subspace of $X$. $X$ is said to be $e$-compact with respect to $D$ if each open cover of $X$ contains a finite subcollection that covers $D$. If so, $X$ is called an $e$-compactification of $D$ and $D$ is called $e$-compactifiable. Observe that within this terminology the expression “let $X$ be an $e$-compact space” is meaningless. From this definition it readily follows that an $e$-compactification of a space $X$ is an $H$-closed extension. The following theorem shows that the converse need not be true.

THEOREM 2.2 [He]. Let $pX$ be an extension of $X$. Then the following statements are equivalent:

(i) $pX$ is an $e$-compactification of $X$.

(ii) Every ultrafilter on $X$ has an accumulation point in $pX$.

(iii) $pX$ is $H$-closed and $X \cup \{q\}$ is regular, for all $q \in pX$.

It follows that an $e$-compactifiable space is regular. The converse is not the case. From 2.2(iii) we can conclude that each noncompact $H$-closed space (i.e. a regular space which is closed in every regular space in which it is embedded, see [BS]) is an example of a regular non-$e$-compactifiable space. It is clear that every Tychonoff space is $e$-compactifiable, and in [He] the question appeared whether the converse.
holds. In the next section we show that this is not the case. We were unable to characterize the class of e-compactifiable spaces in terms of some separation property.

The following properties of e-compactifiable spaces are known.

**Theorem 2.3** [He]. (i) Let $pX$ be an e-compactification of $X$. Then $\text{cl}_{pX} Y$ is an e-compactification of $Y$, for each $Y \subseteq X$.

(ii) Let $p_i X_i$ be an e-compactification of $X_i$ ($i \in I$). Then $\prod p_i X_i$ is an e-compactification of $\prod X_i$. □

Recall that a subset $U \subseteq X$ is regular-closed if $\text{cl}_{\text{int}} U = U$. The collection of regular-closed subsets of $X$ is a closed base for some topology on $X$. $X$ supplied with this topology is called the semiregularization of $X$, to be denoted by $X_S$. $X$ is called semiregular if $X$ is homeomorphic to $X_S$.

In [St] R. M. Stephenson observed that Theorem 2.3 implies that each e-compactifiable space $X$ has a largest e-compactification $aX$, i.e. if $aX$ is an e-compactification of $X$ then the map $\text{id}: X \to aX$ has a continuous extension over $aX$.

**Theorem 2.4.** (i) [St] Let $X$ be an e-compactifiable space. Then $X$ is an open subspace of $aX$ and $aX - X$ is a closed discrete subspace of $aX$.

(ii) Let $f: X \to Y$ be a continuous map and assume that both $X$ and $Y$ are e-compactifiable. Then there is a continuous extension $ef: aX \to eY$ of $f$.

**Proof.** (ii) According to 2.3(ii) we have that $eX \times eY$ is an e-compactification of $X \times Y$. Define $\tilde{X} = \{(x, f(x)): x \in X\} \subseteq X \times Y$. $\tilde{X}$ is a closed subset of $X \times Y$ and $\Pi X \uparrow \tilde{X}: \tilde{X} \to X$ is a homeomorphism. Since $\text{cl}_{eX \times eY} \tilde{X}$ is an e-compactification of $\tilde{X}$, the map $(\Pi X \uparrow \tilde{X})^{-1}: \tilde{X} \to \tilde{X}$ has an extension $e(\Pi X \uparrow \tilde{X})^{-1}: eX \to \text{cl}_{eX \times eY} \tilde{X}$. Define $ef = \Pi X \uparrow \tilde{X} \circ e(\Pi X \uparrow \tilde{X})^{-1}$. □

As a method to answer the question of S. Hechler, R. M. Stephenson asked the following question.

"Let $X$ be an e-compactifiable space. Is the space $(aX)_S$ always compact?"

Our example of a non-Tychonoff e-compactifiable space provides a negative answer to this question. A partial positive answer to Stephenson's question is the following.

**Theorem 2.5** [St]. Let $X$ be a regular space. If disjoint regular closed sets are contained in disjoint open subsets (in particular, if $X$ is normal), then $X$ is Tychonoff (hence e-compactifiable) and $(aX)_S$ is compact. □

Our second example shows that the answer is negative if $X$ is only assumed to be Tychonoff. The following simple lemma is one of the keys to the construction.

**Lemma 2.6.** Let $X$ be a Tychonoff space. Then $(aX)_S$ is compact iff the map $e(\text{id}): eX \to \beta X$ is injective.

**Proof.** Observe that $X$ is a subspace of $(aX)_S$ and that the map $e(\text{id}): (aX)_S \to \beta X$ is also continuous. Then we have "→", since $(aX)_S$ is a compactification of $X$ and "←" holds because $(aX)_S$ is minimal Hausdorff and the topology of $\beta X$ is weaker than that of $(aX)_S$. □
3. The results. The following theorem is the key to our construction of a non-Tychonoff e-compactifiable space.

**Theorem 3.1.** Perfect preimages of e-compactifiable spaces are e-compactifiable.

**Proof.** Let $X$ be an e-compactifiable space and let $f: Y \to X$ be a perfect map. We construct an e-compactification $aY$ of $Y$ in the following way. The underlying set of $aY$ is $Y \oplus (eX - X)$ and a topology is defined by

(i) $Y$ is open in $aY$;

(ii) For $p \in aY - Y = eX - X$ the collection $\mathcal{U}_p = \{\{p\} \cup f^{-1}(X \cap U): U \text{ open in eX} \& p \in U\}$ is taken as a local base in $p \in aY$.

One readily sees that $aY$ is a Hausdorff extension of $Y$. To see that $aY$ is an e-compactification of $Y$, consider an ultrafilter $\mathcal{F}$ on $Y$. Then $f(\mathcal{F}) = \{f(F): F \in \mathcal{F}\}$ is an ultrafilter on $X$; hence $f(\mathcal{F})$ has an accumulation point $q$ in $eX$. If $q \in X$ then, since $f$ is perfect, $\mathcal{F}$ has an accumulation point in $f^{-1}(q)$. If $q \in eX - X$, then $f(F) \cap U_q \neq \emptyset$ for each open neighborhood $U_q$ of $q$ in $eX$ and $F \in \mathcal{F}$. Since $f(F) \subseteq X$ it follows that $F \cap f^{-1}(U_q \cap X) \neq \emptyset$, i.e. $q$—considered as an element of $aY$—is an accumulation point of $\mathcal{F}$. \quad \square

In [Ch] J. Chaber constructed examples of non-Tychonoff perfect preimages of Tychonoff spaces, and so these examples establish the existence of non-Tychonoff e-compactifiable spaces. From 2.3(i) it follows that subspaces of perfect preimages of Tychonoff spaces are e-compactifiable. We were not able to construct e-compactifiable spaces outside this particular class. Observe that a space $X$ in this class (with $|X| > 1$) admits nonconstant real-valued continuous functions.

**Question 3.2.** Do there exist e-compactifiable spaces on which every real-valued continuous function is constant?

Let us now answer the question of R. M. Stephenson, whether there exist Tychonoff spaces $X$ for which $(eX)_S$ is not compact. Our strategy is as follows. We construct a Tychonoff space $X$, a point $p \in \beta X - X$ and an extension $\alpha X$ of $X$ such that $|\alpha X - X| > 1$ and such that the map $f: \alpha X \to X \cup \{p\} (\subset \beta X)$ defined by $f(x) = x \ (x \in X)$ and $f(\alpha X - X) = p$ is perfect. It then follows that $\alpha X$ is e-compactifiable, and since $e\alpha X$ can be considered as an e-compactification of $X$, we can conclude from the diagram below that the map $e(id): eX \to \beta X$ is not injective. $(e_1$ is the extension of id: $X \to \alpha X \subset e\alpha X$ to $eX$ (see 2.4(iii)). $e_2$ is the extension of id: $X \cup \{p\} \to \beta(X \cup \{p\})$ to $e(X \cup \{p\}).$) Indeed, the diagram shows that $e(id) = e_2 \circ ef \circ e_1$; hence $e(id)^{-1}(p) > 1$.

![Diagram](image-url)
The example we present is almost identical to the one constructed by J. Chaber. The only difference lies in the fact that we want the point \( p \) to lie in the Čech-Stone remainder of \( X \). For the reader’s convenience we give the construction in detail.

**Example 3.3.** Put \( T = (\omega_1 + 1) \times (\omega_1 + 1) - \{ (\omega_1, \omega_1) \} \). The set of pairs of the form \( (\alpha, \omega_1) \in T \) will be called the left edge of \( T \). The set of pairs of the form \( (\omega_1, \alpha) \in T \) will be called the right edge of \( T \). Define the space \( T^n \), for \( n \in \mathbb{N} \), as the space obtained by identification in the sum \( \bigoplus_{i=1}^{n} T(i) \) where \( T(i) = T \times \{ i \} \), of the right edge of \( T(i) \) with the left edge of \( T(i + 1) \). Let \( \varphi_n : \bigoplus_{i=1}^{n} T(i) \to T^n \) be the corresponding identification-map. For each \( 0 \leq k \leq n \) we define an open subset \( U^n_k \subset T^n \) by

\[
U^n_k = \begin{cases} 
\text{int} \varphi_n(T(1)) & (k = 0), \\
\text{int} \varphi_n(T(k) \cup T(k + 1)) & (k = 1, \ldots, n - 1), \\
\text{int} \varphi_n(T(n)) & (k = n).
\end{cases}
\]

Finally we define \( X = \bigoplus_{n=1}^{\infty} T^n \).

It is well known that \( |\beta T^n - T^n| = 1 \), for each \( n \in \mathbb{N} \). For \( \alpha < \omega_1 \) put \( Z_\alpha = [\alpha, \omega_1] \times [\alpha, \omega_1] - \{ (\omega_1, \omega_1) \} \). Then \( \{ Z_\alpha : \alpha < \omega_1 \} \) is a base for the unique nonfixed \( \omega \)-ultrafilter on \( T \). If we define, for \( n \in \mathbb{N} \), \( Z^n_\alpha = \varphi_n(\bigoplus_{i=1}^{n} (Z_\alpha \times \{ i \}) \) then \( \{ Z^n_\alpha : \alpha < \omega_1 \} \) is a base for the unique nonfixed \( \omega \)-ultrafilter \( Z^n \) on \( T^n \).

Next we define a point \( p \in \beta X - X \). Let \( \mathcal{G} \) be a nonfixed ultrafilter on \( \mathbb{N} \). For \( G \in \mathcal{G} \) and \( \alpha < \omega_1 \) put \( Z(G, \alpha) = [\alpha, \omega_1] \times [\alpha, \omega_1] - \{ (\omega_1, \omega_1) \} \). Then \( \{ Z(G, \alpha) : G \in \mathcal{G}, \alpha < \omega_1 \} \) is a base for a nonfixed \( \omega \)-ultrafilter \( \mathcal{Z} \) on \( X \). Let \( p \in \beta X - X \) be the point in \( \beta X \) corresponding to \( \mathcal{G} \), i.e. \( \{ p \} = \bigcap \{ \text{cl}_{\beta X} F : F \in \mathcal{G} \} \). In the space \( X \cup \{ p \} \) we have the following: If \( U \) is open in \( X \) then \( U \cup \{ p \} \) is a neighborhood of \( p \) in \( X \cup \{ p \} \) iff \( \exists G \in \mathcal{G} \exists \alpha < \omega_1 \) such that \( Z(G, \alpha) \subset U \). (*)

(This is not completely trivial, since \( X \) is not normal. However, it follows easily by considering the space \( \tilde{X} = \bigcap_{n=1}^{\infty} \text{cl}_{\beta X} T^n \subset \beta X \), which is \( \sigma \)-compact (hence normal).

We omit the details.)

Let us now introduce a topology on the set \( X \cup [0, 1] \) ([0, 1] is the unit interval) in the following way. For \( t \in [0, 1] \) let \( \{ V(t) \}_{t=1}^{\infty} \) be a countable local base at \( t \). For \( t \in [0, 1], l \in \mathbb{N}, G \in \mathcal{G}, \alpha < \omega_1 \), define

\[
U(t, l, G, \alpha) = \bigcup_{n \in G} \bigcup_{s \in V(t)} (U^n_{[n,s]} \cap Z^n_\alpha) \cup V(t).
\]

(Here \([n,s]\) denotes the greatest integer not greater than \( n.s.\)) And next we put:

\( X \) is open in \( X \cup [0, 1] \).

For \( t \in [0, 1] \) the collection \( \{ U(t, l, G, \alpha) : l \in \mathbb{N}, G \in \mathcal{G}, \alpha < \omega_1 \} \) is defined to be a local base of \( t \) in \( X \cup [0, 1] \).

Observe that \([0, 1]\) is embedded in \( X \cup [0, 1] \). It is easy to check that \( X \cup [0, 1] \) is a Hausdorff space. In fact our topology has more open sets than Chaber’s.

**Claim.** Let \( U \) be a subset of \( X \cup [0, 1] \) which contains \([0, 1]\). Then \( U \) is neighborhood of \([0, 1]\) in \( X \cup [0, 1] \) iff \( \exists G \in \mathcal{G} \exists \alpha < \omega_1 \) such that \( Z(G, \alpha) \subset U \).
PROOF. Assume \( Z(G, \alpha) \subseteq U \). Then, for \( t \in [0, 1] \), \( t \in U(t, l, G, \alpha) \subseteq Z(G, \alpha) \).
Hence \([0, 1] \subseteq \text{int} \, U \). On the other hand, assume \([0, 1] \subseteq \text{int} \, U \). Then, \( \forall t \in [0, 1] \exists \ell(t) \in \mathbb{N} \exists g(t) \in \mathcal{G} \exists \alpha(t) < \omega_1 \) such that
\[
t \in U(t, l(t), G(t), \alpha(t)) \subseteq U.
\]
Since \([0, 1] \) is compact, \([0, 1] \) can be covered by finitely many of these sets. Say \([0, 1] \subseteq \bigcup_{k=1}^{K} U(t_k, l(t_k), G(t_k), \alpha(t_k)) (\subseteq U) \). Put \( G = \bigcap_{k=1}^{K} G(t_k) (\in \mathcal{G}) \) and \( \alpha = \sup \{\alpha(t_k) : i < k\} (\prec \omega_1) \). We claim that \( Z(G, \alpha) \subseteq \bigcup_{k=1}^{K} U(t_k, l(t_k), G(t_k), \alpha(t_k)) (\subseteq U) \). Choose \( p \in Z(G, \alpha) \), say \( p \in Z_{n,p} \) for some \( n \in \mathbb{N} \). Since \( T^n = \bigcup_{k=0}^{n} U_k^n \), there exists \( k \leq n \) such that \( p \in U_k^n \). Choose \( s \in [0, 1] \) such that \((n.s) = k \). If \( s \in U(t_k, l(t_k), G(t_k), \alpha(t_k)) \) then, since \( G \subseteq G(t_k) \) and \( Z_n \subseteq Z_{\alpha(t_k)} \), we conclude that \( p \in Z(\alpha(t_k)) \cap U_{[n,s]}^n \) for “some” \( n \in G(t_k) \), i.e. \( p \in U(t_k, l(t_k), G(t_k), \alpha(t_k)) \). The claim follows.

From the claim and from (*) we conclude that the space obtained from \( X \cup [0, 1] \) by identifying \([0, 1] \) to a point is homeomorphic to \( X \cup \{p\} \). Obviously the map \( f: X \cup [0, 1] \to X \cup \{p\} \) defined by \( f(x) = x (x \in X) \) and \( f[0, 1] = p \) is a perfect map. Hence, all the required properties are satisfied.

REMARK 3.4. It is well known that each space \( T^n \), as defined in 3.3, has a unique (nontrivial) regular extension, namely \( \beta T^n \). It follows that \( \text{cl} \, e_X T^n = \beta T^n \), for all \( n \in \mathbb{N} \). Consider the space \( \tilde{X} = \bigoplus_{n=1}^{\infty} \beta T^n \). Then \( X \subseteq \tilde{X} \subseteq eX \). \( \tilde{X} \) is a \( \sigma \)-compact, hence normal, and according to 2.5 this implies that \( (eX)_S = \tilde{X} = \beta X \). Since \( (eX)_S \neq \beta X \), we conclude that the map \( \text{id}: \tilde{X} \to eX \) cannot be extended continuously to \( eX \). At first glance this may seem a contradiction, but it is not. One cannot use 2.4(ii) to ensure that such an extension should exist since \( eX \) is not \( e \)-compactifiable \((eX \) is not even semiregular), nor the fact that \( eX \) is the largest \( e \)-compactification, since \( eX \) is not an \( e \)-compactification of \( \tilde{X} \). (A nonfixed ultrafilter on \( \tilde{X} - X \) does not have an accumulation point in \( eX \).

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SUBFACULTEIT WISKUNDE EN INFORMATICA, VRIJE UNIVERSITEIT, DE BOELELAAN 1081, 1081 HV AMSTERDAM, THE NETHERLANDS (Current address of K. P. Hart)

Current address (J. Vermeer): Department of Mathematics, University of Kansas, Lawrence, Kansas 66045
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