A review of theories currently being used to model steady plane flames on flame-holders

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ABSTRACT

The stability of burner flames for arbitrary Lewis number is considered on the basis of large activation energy modelling. Previous leading-order-only approaches are now extended to second order in $\Theta^{-1}$ (the inverse activation energy). The assumption that the unsteady perturbations are small (order $\varepsilon$) means that one must discuss the distinguished limit implicit in the product $\Theta\varepsilon$. It is demonstrated here that different governing equations (and in particular the inner zone equation) are obtained in the two cases, $\Theta, \varepsilon \to 0$ and $\Theta, \varepsilon \to \infty$.

It is shown that there are two important complex frequency relations governing the behaviour of flames near burners and it is readily seen that the now classical free flame dispersion relation can be derived as a special case for flames with infinite stand-off distance, and Lewis number close to unity. However, for finite stand-off distances, the Lewis number can remain as an arbitrary $O(1)$ parameter and the full description of the flame-behaviour for small perturbations will involve both these frequency relations.
Much theoretical analysis has been done in recent months on the stability of pre-mixed laminar flames with and without heat-loss. The book by Buckmaster and Ludford (1982) discusses in detail some of the recent predictions for near equidiffusional flames. For free-flames (i.e. not anchored to a burner) there is a dispersion relation linking complex frequency \( \omega \) and Lewis number \( \lambda_e \) (defined as the ratio of molecular to thermal diffusivities) which must be asymptotically close to unity. The main results of this relation are summarized in a recent review by Margolis and Matkowsky (1982). However, the restriction of all such theories that \( \lambda_e \) is within order \( \Theta^{-1} \) of unity is not unimportant. Though no doubt many mixtures do have such properties it is quite feasible to have values of \( \lambda_e \) down to 0.7 which is difficult to conceive of as anything but an order \( O(1) \) number. It is therefore desirable to develop theories which allow for Lewis numbers away from unity by \( O(1) \) amounts. A recent paper by Durbin (1982) dealing with a complementary problem of the pre-mixed flame in uniform straining counter-flow allows Lewis number to be more realistically treated and yields results for \( \lambda_e \) in a large range. Buckmaster and Ludford (1982, p.47) have also shown that for the one-dimensional flow problem considered here, if time rates of change are considered to be slow, then the restriction on Lewis number is lifted. However, the requirement of slow movement is not always realistic. It is required here to examine unsteady flames where time rates of change are primarily \( O(1) \).

The present work continues previous asymptotic analyses (Clarke and McIntosh 1979, McIntosh and Clarke 1981) to second order in \( \Theta^{-1} \) (inverse activation energy). Heat losses are included by considering the flame to be next to a Hirschfelder type of flame-holder. In the first of the above references it has been shown that the formulae derived for flame stand-off distance and flame-speed agree closely with empirical relationships derived by Kaskan (1957) and Ferguson and Keck (1979). In terms of non-dimensional quantities, the \( O(1) \) stand-off distance is
inextricably connected to the $O(1)$ temperature gradient at the holder. This defines an $O(1)$ localised heat loss which is then proportional to the experimentally measurable heat loss at the surface of the holder. If the heat loss becomes small such that for example, the non dimensional heat loss is $O(\theta_i^{-1})$ then the stand-off distance becomes $O(ln \theta_i)$ and not $O(1)$.

The model proposed here is essentially in agreement with an alternative model proposed by Carrier et al (1978). Briefly, in this alternative approach, the flame holder is represented by a $\delta$-function heat sink situated within the inert pre-heat domain, with stand-off distance defined as the distance between the flame sheet and the heat sink. The two important relationships linking firstly stand-off distance with flame temperature and secondly flame speed with flame temperature are virtually identical with ours. The predictions concerning the order of magnitude of stand-off distance and the order of magnitude of heat loss are identical.

However these results concerning stand-off distance differ from some conclusions recently drawn by Matkowsky and Olagunju (1981). Their model of the flame-holder is based on that proposed by Carrier et al (1978) but differs in some important respects. The notable difference is that $O(\theta_i^{-1})$ heat loss is linked to an $O(1)$ stand-off distance through $O(\theta_i^{-1})$ changes in enthalpy gradients at the holder. In comparing their theoretical results with the afore mentioned empirically derived formulae for flame-speed and stand-off distance, Matkowsky and Olagunju find that agreement is only obtained for a particular value of one of their flame-holder parameters. However, this value violates their original hypothesis about small $O(\theta_i^{-1})$ dimensionless heat loss rates; with this particular parameter value their dimensionless heat loss ceases to be $O(\theta_i^{-1})$ and becomes $O(1)$ and their stand-off distances must be $O(ln \theta_i)$ as in the previous analyses. Our conclusion then is that to match theory and experimental observation $O(1)$ [or $O(ln \theta_i)$] stand-off distances must be linked to $O(1)$[or $O(\theta_i^{-1})$] heat loss terms.
The main advantage of the model used here is that we can proceed with the unsteady analysis, without a priori restrictions on Lewis number $\text{le}$. This ratio $(\text{le})$ is essentially an $O(1)$ parameter at the control of the investigator. The free flame can then be approached in a logical manner from this standpoint.

In this paper it is found that $O(1)$ heat conduction to the holder can have important effects at second order. A second complex frequency relationship is derived which provides correction terms for the leading order estimates as to the unsteady behaviour of flames for given values of Lewis number and stand-off distance. One can in principle extend the analysis to two dimensional disturbances but here only one dimension is considered so that one is restricted to investigation of long wavelength perturbations, that is, pulsating flames.

It should be pointed out that it is important to be clear about the limit processes that are used in analysing the unsteady equations. It is shown that if $\varepsilon$ is the order of the perturbations, the distinguished limit implicit in the product $O,\varepsilon$ affects the analysis. In this work the $\varepsilon \to 0$ limit is taken first with the assumption $O,\varepsilon \to 0$ in that limit. This produces a different inner equation to that obtained usually, when one applies the $O,\varepsilon \to 0$ limit to the perturbed equations. It is noted here that Joulin (1982) has advocated yet another approach to this same problem of flames near burners. His method is based on large $y_{f_1}$ asymptotics (where $y_{f_1}$ is the mass weighted stand-off distance). This method can indeed be useful when $y_{f_1}$ is large but in most practical cases this is not so. Indeed if $y_{f_1}$ is very large, like $O(\Theta_1)$ for example, the burner-flame structure changes (Clarke, 1983). Also Joulin's analysis is for Lewis number unity, and one of the express aims of this work is to make Lewis number arbitrary. Therefore large $\Theta_1$ modelling is adhered to in the present paper, with results that cover a wide range of $y_{f_1}$ and $\text{le}$.
2 FORMULATION

The basic equations used are those governing a simple one-step order 2 irreversible reaction in one-dimensional unsteady flow (McIntosh and Clarke 1981). In co-ordinates fixed in space such that the origin is at the holder, the equations are,

\[ \frac{d^2 C_2}{dt^2} + \frac{dC_2}{dy} - \frac{d^2 C_2}{dy^2} = -L \Theta \Lambda C_2 C_r e^{\Theta \gamma} \quad (1) \]

\[ \frac{dT}{dt} + \frac{dT}{dy} - \frac{1}{\Lambda} \frac{d^2 T}{dy^2} = \frac{L \Theta \Lambda C_2 C_r e^{\Theta \gamma}}{\sigma} \quad (2) \]

The usual non-dimensionalisation has been performed using the characteristic diffusion length and time and the spatial co-ordinate \( y \) is mass weighted to decouple the hydrodynamic effects from the main problem. \( C_2, C_r \) refer to lean and rich species respectively; \( T \) is temperature, \( \Theta \) activation energy (referred to initial burnt temperature), \( \sigma \) the ratio of molecular weights, \( Q \) the reduced heat of reaction and \( \Lambda \), the steady, pre-exponential eigen value which typically for far from stoichiometric conditions is proportional to \( \Theta_0^2 \) (Clarke and McIntosh 1979). The prime concern of this work is to establish the long wavelength dispersion relation so that forcing terms have not been included.

The upstream boundary conditions are consistent with a Hirschfelder type flame-holder (Fig. 1). The inlet mass flux is held constant and the temperature and lean species obey,

\[ T(0,t) = T_0, \text{ constant} \quad (3) \]
\[ C_x(0, t) - \left( \frac{\partial C_x}{\partial y} \right)_{0, t} = \text{constant} \]  \hspace{1cm} (4)

The latter condition is a statement to the effect that mixture strength is held constant at the inlet to the system. Downstream the conditions are simply a statement of chemical equilibrium,

\[ C_x(\infty, t) = 0 \hspace{1cm} , \]  \hspace{1cm} (5)

\[ T(\infty, t) \text{ bounded} \hspace{1cm} . \]  \hspace{1cm} (6)
3 SMALL PERTURBATIONS

The conventional approach in analysing these equations has been to first exploit the limit \( \Theta_i^{-1} \rightarrow 0 \) and match pre-heat and equilibrium expansions using a thin reaction zone, then secondly to assume small (order \( \varepsilon \) ) perturbations to solve the pre-heat equations. Indeed this model has successfully been used to predict the leading order behaviour of planar flames near flame-holders (McIntosh and Clarke 1981). However implied with such models is a restriction on \( \varepsilon \). Since the Arrhenius term contains the exponential,

\[
E = \exp \left[ \Theta_i \left( 1 - \frac{1}{\Theta_i} \right) \right],
\]

then under small perturbations,

\[
T = T_s + \varepsilon T_u,
\]

\( E \) becomes

\[
E = \exp \left[ \Theta_i \left( 1 - \frac{1}{\Theta_i} \right) + \Theta_i \varepsilon T_u \right].
\]

To date, all analyses assume \( \Theta_i \varepsilon \rightarrow \infty \), i.e.,

\[ \varepsilon \gg \Theta_i^{-1} \]

and thus they investigate the limit \( \Theta_i^{-1} \rightarrow 0 \) followed by the limit \( \varepsilon \rightarrow 0 \). If one now takes this approach to second order keeping \( O(\Theta_i^{-1}), O(\varepsilon \Theta_i^{-1}) \) terms in the temperature and lean species expansions but dropping \( O(\varepsilon^2) \) terms, then there is an implication that,

\[ \varepsilon^2 \ll \varepsilon \Theta_i^{-1} \Rightarrow \varepsilon \ll \Theta_i^{-1}. \]
This now contradicts the first condition. To overcome this, $O(\varepsilon^2)$ terms must be included and the method can then only be valid for $\varepsilon$ within a fairly tight band:

$$\Theta^{-2} \ll \varepsilon \ll 1$$

Under these conditions one then has further terms and strictly a further sub-problem involving order $O(\varepsilon^2)$ terms materialises.

There is however a much clearer approach which involves taking the $\varepsilon \to 0$ limit first. Temperature, lean species and stand-off distance are written as,

$$T(y,t) = T_s(y) + \varepsilon \tilde{T}_u(y,t) \quad \text{(10)}$$

$$C_l(y,t) = C_{ls}(y) + \varepsilon \tilde{C}_u(y,t) \quad \text{(11)}$$

Noting that $C_l$ is related to $C_{ls}$ through the mixture strength constant $|\Delta_1|$ i.e.,

$$C_{ls} = C_l + |\Delta_1| \quad \text{(12)}$$

and on the strict assumption that $\varepsilon \ll \Theta^{-1}$, the linearised perturbation equations are found to be,

$$\frac{\partial \tilde{T}_u}{\partial t} + \frac{\partial \tilde{T}_u}{\partial y} - \frac{\partial^2 \tilde{T}_u}{\partial y^2} = -\frac{1}{\varepsilon} \Delta_1 \left( \Theta \frac{\tilde{T}_u}{\varepsilon^2} (1 + \Delta_1 + C_{ls}) C_{ls} + (1 + 2 \Delta_1 + 2 C_{ls}) \tilde{C}_u \right) e^{\Theta (1 - \frac{1}{\varepsilon})} \quad \text{(13)}$$

$$\frac{\partial \tilde{C}_u}{\partial t} + \frac{\partial \tilde{C}_u}{\partial y} + \frac{1}{\varepsilon} \frac{\partial \tilde{C}_u}{\partial y} = \frac{1}{\varepsilon} \left( \Theta \frac{\tilde{T}_u}{\varepsilon^2} (1 + \Delta_1 + C_{ls}) C_{ls} + (1 + 2 \Delta_1 + 2 C_{ls}) \tilde{C}_u \right) e^{\Theta (1 - \frac{1}{\varepsilon})} \quad \text{(14)}$$
In that we have assumed a simple order 2 reaction, the reaction term in equations (13, 14) is still relatively simple in form. For a reaction of general order, the chemical term will be substantially more complicated but the following method of solution in principle, will still apply. The boundary conditions at the holder and downstream are then given by,

\[ \tilde{T}_u(0, t) = 0 \quad , \] (15)

\[ \tilde{\phi}_u(0, t) - \left( \frac{\partial \tilde{\phi}_u}{\partial y} \right)_{y=0} = 0 \quad , \] (16)

\[ \tilde{T}_u(\infty, t) \text{ bounded} \quad , \] (17)

\[ \tilde{\phi}_u(\infty, t) = 0 \quad . \] (18)

This completes the first stage of the argument, that is taking the limit \( \varepsilon \to 0 \). In principle one can obtain equations and boundary conditions for each power of \( \varepsilon \) and at each level, \( \Theta_i \) is a parameter in the equations, the largeness of which can then be exploited. But note the order in application of the two limits: first \( \varepsilon \to 0 \) then \( \Theta_i^{-1} \to 0 \).

Since the main concern here is with the stability of pulsations of varying frequency, the solution of (13), (14) is now considered for the particular type of disturbance,

\[ \tilde{T}_u = e^{\omega t} T_u(y) \quad , \] (19)

\[ \tilde{\phi}_u = e^{\omega t} \phi_u(y) \quad . \] (20)
where $\omega$ is a complex frequency term. Substitution of (19), (20) into (13), (14) produces two simultaneous second-order differential equations with non-constant coefficients,

$$
\omega C_{\ell u} + \frac{d}{dy} C_{\ell u} - \frac{d^2}{dy^2} C_{\ell u} = -\omega A_1 (1A_1 + C_{ru}) C_{\ell s} + (1A_1 + 2C_{ru}) C_{\ell u} \exp (\Theta_1 (1 - \frac{1}{\tau}))
$$

(21)

$$
\omega T_u + \frac{d}{dy} T_u - \frac{1}{\lambda_1} \frac{d^2}{dy^2} T_u = \frac{\lambda_1}{\lambda_1} A_1 (1A_1 + C_{ru}) C_{\ell s} + (1A_1 + 2C_{ru}) C_{\ell u} \exp (\Theta_1 (1 - \frac{1}{\tau}))
$$

(22)

with the boundary conditions,

$$
T_u(0) = 0 \quad , \quad T_u(\infty) \text{ bounded} \quad ,
$$

(23a,b)

$$
C_{\ell u}(0) - \frac{d}{dy} C_{\ell u} \bigg|_0 = 0 \quad , \quad C_{\ell u}(\infty) = 0
$$

(24a,b)

In principle for a given steady temperature/lean species profile these two differential equations can be solved by numerical techniques. It is important to point out that the exponential in the reaction term has no dependence on $\Theta_1$, so that the singular behaviour of that term for large $\Theta_1$ only involves the steady temperature $T_s$. It is proposed now to approximate a solution to equations (21), (22) for $T_u$ and $C_{\ell u}$ by making use of the limit $\Theta_1 \to 0$. 
In this section asymptotic series solutions are sought exploiting the smallness of $\Theta_i^{-1}$. Consistent with this approach is the notion that $\omega$ has the asymptotic expansion,

$$\omega = \omega_0 + \omega_1 \Theta_i^{-1} + \ldots$$

(25)

Although Lewis number is mainly an $O(1)$ quantity, the possibility of second order correction terms is allowed for by making the expansion,

$$le = le_0 + l_1 \Theta_i^{-1} + \ldots$$

(26)

However, doubts are raised later as to the usefulness of such an expansion except for the special case when $le_0 = 1$.

At this stage one must also substitute for the steady solution which itself is written as series solutions in three zones.

4.1 Solution of Steady Equations

The main results in the three zones are as follows;

Pre-heat $(0 \leq y < y_{fi})$

$$T_{sp} = T_{sp}^{(1)} + \Theta_i^{-1} T_{sp}^{(2)} + \ldots$$

(27)

$$C'_{lsf} = C'_{lsf}^{(1)} + \Theta_i^{-1} C'_{lsf}^{(2)} + \ldots$$

(28)

where,

$$T_{sp}^{(1)} = 1 - B_i + B_i \exp \left( le_0 (y - y_{fi}) \right)$$

(29)

$$T_{sp}^{(2)} = l_1 B_i y \exp \left( le_0 (y - y_{fi}) \right)$$

(30)
and $B_1$ is the heat release parameter across the flame (Williams 1974) but including the heat loss term. Thus,

$$B_1 \equiv \frac{1 - T_{o1}}{1 - e^{y_{f1} - y_{f2}}}. \quad (33)$$

Note that $y_{f1}$ is the leading order estimate for stand-off distance. It is found (McIntosh 1983) that the second order estimate for stand-off distance ($y_{f2}$) is given by

$$y_{f2} = -\frac{y_{f1}}{T_{o0}}. \quad (34)$$

Equilibrium ($y > y_{f1}$)

$$T_{se} = 1, \quad (35)$$

$$C_{lse} = 0. \quad (36)$$

Reaction ($y$ near $y_{f1}$)

$$T_s = 1 - \Theta_1^{-1} \tau^{(1)}(y) - \Theta_1^{-2} \tau^{(2)}(y) - \ldots, \quad (37)$$

$$C_{ls} = \Theta_1^{-1} \tau^{(1)}(y) + \Theta_1^{-2} \tau^{(2)}(y) + \ldots. \quad (38)$$

$$y \equiv \Theta_1 [y_{f1} - (y_{f1} + \Theta_1^{-1} y_{f2} + \ldots)]. \quad (39)$$
In this zone it is found that $\tau^{(u)}$ and its derivatives satisfy the equations,

\[
\frac{d^2 \tau^{(u)}}{dy^2} = \frac{1}{2} \lambda_0 B^2 f ; \quad f(\tau^{(u)}) = \tau^{(u)} e^{-\tau^{(u)}}, \quad (40a,b)
\]

\[
\frac{d \tau^{(u)}}{dy} = -\lambda_0 B g ; \quad g(\tau^{(u)}) = \sqrt{1 - e^{-\tau^{(u)}}}, \quad (41a,b)
\]

and that,

\[
\mathcal{E}^{(u)} = \frac{\sigma \tau^{(u)}}{\lambda_0 (1+\sigma) Q} . \quad (42)
\]

Restricting investigations to far from stoichiometric conditions (though this is not a vital assumption), the matching of inner reaction zone series solutions with equilibrium and pre-heat solutions, requires the following asymptotic series for $\Lambda_i$:

\[
\Lambda_i = \Theta,^2 \Lambda^{(i)} \left( 1 + \Theta,^{-1} \Lambda^{(2)} + \ldots \right) , \quad (43)
\]

with

\[
\Lambda^{(i)} = \frac{\lambda_0 B^2}{2 |\Lambda_i|} , \quad (44)
\]

\[
\Lambda^{(2)} = 2 \left[ 3 - \frac{2}{B} - \frac{\sigma}{\lambda_0 (1+\sigma) Q |\Lambda_i|} - \frac{1}{B} (1 - \frac{1}{\lambda_0}) + \frac{q_i}{2 \lambda_0} \right] , \quad (45)
\]

\[
I = \int_0^\infty \left[ 1 - g(m) \right] dm = 1.344 \ldots . \quad (46)
\]

These results are simply stated here but explained more fully in McIntosh (1983). The leading order theory is straightforward and has appeared in many publications since first introduced by Bush and Fendell (1970). The second order correction $\Lambda^{(2)}$ for the pre-exponential mass flux term has
also been derived for various cases. Bush and Fendell in the above reference have derived this correction for monopropellant reactions. In a later paper, Bush (1981) derived \( \Lambda^{(u)} \) for the two reactant case of varying order. Equation (45) is essentially the same as that found in Bush (1981) but somewhat more general in that it allows for non-unit values of \( B \), the heat-release parameter. There is also some similarity with that derived in Williams 1974 (Appendix A) but there Lewis number was taken as unity \( (\mathcal{L}_0 = 1, \mathcal{Q} = 0) \). Here Lewis number is kept general and note also the addition of the \( \mathcal{Q} / 2 \mathcal{L}_0 \) term which follows as a direct result of the Lewis number expansion (Equation 26).

Two further results complete the solution of the steady equations in the three zones. In the inner zone \( \tau^{(u)} \) becomes the natural 'independent variable' to the formulation and it is found that the second order inner temperature and species obey the conditions

\[
\frac{d}{d\tau^{(u)}} \left[ g \frac{d}{d\tau^{(u)}} \left( \frac{\tau^{(u)}}{q} \right) \right] = \frac{d}{d\tau^{(u)}} \left[ g \frac{d}{d\tau^{(u)}} \left( \frac{\tau^{(u)}}{q} \right) - \frac{1}{2} f \tau^{(u)} \right] = \frac{1}{2} f \Lambda^{(u)} - \frac{1}{2} f \tau^{(u)} + \frac{\sigma f \tau^{(u)}}{2\mathcal{L}_0 (H_0 Q \Lambda)} + \frac{1}{2B_1} \left( 1 - \frac{1}{\mathcal{L}_0} \right) h e^{-\tau^{(u)}} - \frac{g}{B_1} + \frac{\mathcal{Q}}{2\mathcal{L}_0},
\]

(47)

\[
\frac{\mathcal{L}^{(u)}}{\mathcal{L}^{(u)}} = \frac{\tau^{(u)}}{\tau^{(u)}} + \left( 1 - \frac{1}{\mathcal{L}_0} \right) \frac{h}{B_1 \tau^{(u)}} - \frac{\mathcal{Q}}{\mathcal{L}_0},
\]

(48)

where,

\[
h(\tau^{(u)}) \equiv \int_{\tau}^{\mathcal{L}} \frac{m d_{m}}{g_{m}},
\]

(49)
4.2 The behaviour of $T_u$ near the flame

It is important to see how equations (21), (22) imply the need for $T_u, C_{eu}$ (as well as $T_s, C_{es}$) to be approximated by series expansions in the three zones obtained from the solution of the steady equations. Note that the three zones are fixed relative to the flame-holder. This arises as a direct result of the $\epsilon \rightarrow 0$ limit being taken first and the fact that the perturbations are smaller than $\Theta^{-1}$ (which is also a measure of the flame thickness).

Consider first the equilibrium zone. In this zone $C_{es} = 0$ and $T_s = 1$ from the steady solutions so that equation (21) must then imply that, in the limit $\Theta^{-1} \rightarrow 0$ with $y$ fixed, $C_{eu} = 0$. There are now two routes to follow,

1. $T_u, C_{eu}$ continuous to $O(1)$ across the reaction zone and
2. $T_u, C_{eu}$ discontinuous to $O(1)$ across the reaction zone.

The first route is explained in more detail in the Appendix. To summarize it briefly first write

$$
\begin{align*}
\left\{ \frac{T_u}{C_{eu}} \right\} &= \lim_{y \rightarrow y_{f,-}} \left\{ \frac{T_{up}}{C_{eu,up}} \right\} = \lim_{y \rightarrow y_{f,+}} \left\{ \frac{T_{ue}}{C_{eu,ue}} \right\}, \\
\Theta^{-1} &\ll \delta(\Theta) \ll 1
\end{align*}
$$

One then obtains,

$$
\begin{align*}
\left[ \frac{dC_{eu}}{dy} \right]_{y_{f,-}}^{y_{f,+}} &= \frac{\sigma B_i}{(1+\sigma)Q_i} \frac{\Theta_i}{T_u} \left[ 1 + \ldots \right] + \frac{1}{2} \frac{\Theta_i^2 B_i^2 C_{eu}}{\sigma} \left[ 1 + \ldots \right], \\
\left[ \frac{dT_u}{dy} \right]_{y_{f,-}}^{y_{f,+}} &= -\frac{\Theta_i^3}{(1+\sigma)Q_i} \frac{B_i^2 C_{eu}}{\sigma} \left[ 1 + \ldots \right] - \frac{1}{2} \frac{\Theta_i^3 B_i^2 C_{eu}}{\sigma} \left[ 1 + \ldots \right],
\end{align*}
$$

(51)  

(52)
where $y_{E^-}$, $y_{E^+}$ refer to the pre-heat and equilibrium sides of the flame sheet respectively. These results are analysed further in the Appendix leading to the conclusion that if leading order continuity is assumed across the flame sheet, then one must conclude that $T_u$ and $C_{2u}$ can only be identically zero. Such a trivial result then leads one to question the crucial assumption of continuity. As will now be shown, the correct asymptotic model (in the limit $\Theta_1^{-1} \to 0$) requires $T_u$, $C_{2u}$ to be discontinuous (to $O(1)$) at the flame sheet. This is the second route referred to above.
5 ASYMPTOTIC ANALYSIS WITH DISCONTINUITY IN TEMPERATURE AND SPECIES

With the notion of an $O(1)$ discontinuity in $T_u$ and $C_{eu}$ at the flame, series solutions are now sought for $T_u$ and $C_{eu}$ in the three steady zones already referred to.

5.1 Differential equations in the three steady zones

5.1.1 Pre-heat: $(0 \leq y < y_{f1})$

In this zone, unsteady temperature and species are expanded as

$$T_{up} = T_{up}^{(1)}(y) + \Theta_i^{-1} T_{up}^{(2)}(y) + \ldots $$ \quad (53)

$$C_{eu} = C_{eu}^{(1)}(y) + \Theta_i^{-1} C_{eu}^{(2)}(y) + \ldots $$ \quad (54)

and the outer limiting process

$$\Theta_i^{-1} \to 0 \quad , \; y \text{ fixed}$$ \quad (55)

is used to give the first and second order differential equations in the pre-heat zone. These are,

\begin{align*}
\left\{ \begin{align*}
\omega_0 C_{eu}^{(1)} + \frac{dC_{eu}^{(1)}}{dy} - \frac{d^2 C_{eu}^{(1)}}{dy^2} &= 0 \\
\omega_0 T_{up}^{(1)} + \frac{dT_{up}^{(1)}}{dy} - \frac{1}{\ell_0} \frac{d^2 T_{up}^{(1)}}{dy^2} &= 0
\end{align*} \right. \quad (56)
\end{align*}

\begin{align*}
\left\{ \begin{align*}
\omega_0 C_{eu}^{(2)} + \frac{dC_{eu}^{(2)}}{dy} - \frac{d^2 C_{eu}^{(2)}}{dy^2} &= -\omega_1 C_{eu}^{(1)} \\
\omega_0 T_{up}^{(2)} + \frac{dT_{up}^{(2)}}{dy} - \frac{1}{\ell_0} \frac{d^2 T_{up}^{(2)}}{dy^2} &= -\omega_1 T_{up}^{(1)} - \frac{\rho_1}{\ell_0^2} \frac{d^2 T_{up}^{(1)}}{dy^2}
\end{align*} \right. \quad (57)
\end{align*}
The flame-holder boundary conditions (23a), (24a) become:

\[
\begin{align*}
T_{\text{up}}^{(1)}(0) &= 0, \quad (60) \\
C_{\text{up}}^{(1)}(0) - \frac{dC_{\text{up}}^{(1)}}{dy}\bigg|_0 &= 0, \quad (61)
\end{align*}
\]

\[
\begin{align*}
T_{\text{up}}^{(2)}(0) &= 0, \quad (62) \\
C_{\text{up}}^{(2)}(0) - \frac{dC_{\text{up}}^{(2)}}{dy}\bigg|_0 &= 0. \quad (63)
\end{align*}
\]

Defining

\[
\begin{align*}
T_{\text{up}}^{\text{*}(1)} &= \lim_{y \to y_{f_1}} T_{\text{up}}^{(1)}(y), \quad (64) \\
C_{\text{up}}^{\text{*}(1)} &= (1+\sigma)Q_{1} \lim_{y \to y_{f_1}} C_{\text{up}}^{(1)}(y), \quad (65)
\end{align*}
\]

\[
\begin{align*}
T_{\text{up}}^{\text{*}(2)} &= \lim_{y \to y_{f_1}} T_{\text{up}}^{(2)}(y), \quad (66) \\
C_{\text{up}}^{\text{*}(2)} &= (1+\sigma)Q_{1} \lim_{y \to y_{f_1}} C_{\text{up}}^{(2)}(y), \quad (67)
\end{align*}
\]

the general solution to the pre-heat equations (56-59) is given by,

\[
T_{\text{up}}^{(1)} = T_{\text{up}}^{\text{*}(1)} \exp\left(\frac{1}{2}Le_{0}(y-y_{f_1})\right) \frac{\text{sh}(Le_{0}y)}{\text{sh}(Le_{0}y_{f_1})}, \quad (68)
\]
\[
\frac{(1+\sigma)Q_1 e_{\text{up}}^{(a)}}{\sigma} = e_{\text{up}}^{(a)} \exp\left(\frac{1}{2}(y - y_{v1})\right) \left(\frac{1}{2} \text{sh}(ry) + r\text{ch}(ry)\right) \\
\left(\frac{1}{2} \text{sh}(ry_{v1}) + r\text{ch}(ry_{v1})\right)
\]

\[
T_{\text{up}}^{(a)} = T_{\text{up}}^{(a)} \exp\left(\frac{1}{2} \omega_0(y - y_{v1})\right) \frac{\text{sh}(\omega_0 y)}{\text{sh}(\omega_0 y_{v1})}
\]

\[
+ \left[\frac{1}{2} \omega_0(y - y_{v1}) \left[Y \text{sh}(\omega_0 y_{v1})\left(\frac{1}{2} + s^2\right) \text{ch}(\omega_0 y) + s \text{sh}(\omega_0 y)\right) - Y_1 \text{sh}(\omega_0 y_{v1})\left(\frac{1}{2} + s^2\right) \text{ch}(\omega_0 y_{v1}) + s \text{sh}(\omega_0 y_{v1})\right]\right]
\]

\[
\left[\frac{1}{2} \omega_0(y - y_{v1}) \left[\text{ch}(\omega_0 y) \text{sh}(\omega_0 y_{v1}) - \text{ch}(\omega_0 y_{v1}) \text{sh}(\omega_0 y_{v1})\right]\right]
\]

\[
\frac{(1+\sigma)Q_1 e_{\text{up}}^{(a)}}{\sigma} = e_{\text{up}}^{(a)} \exp\left(\frac{1}{2}(y - y_{v1})\right) \left(\frac{1}{2} \text{sh}(ry) + r\text{ch}(ry)\right) \exp\left(\frac{1}{2}(y - y_{v1})\right)
\]

\[
\left(\frac{1}{2} \text{sh}(ry_{v1}) + r\text{ch}(ry_{v1})\right)
\]

\[
-C_{\text{up}}^{(a)} \omega_0 \exp(ry_{v1}) \left[\frac{1}{2}(y \text{sh}(ry) + r\text{ch}(ry)) \exp\left(\frac{1}{2}(y - y_{v1})\right) - \exp\left(\frac{1}{2}(y - y_{v1})\right)\right]
\]

\[
\frac{C_{\text{up}}^{(a)} \omega_0 \exp\left(\frac{1}{2}(y - y_{v1})\right)}{2 r \left(\frac{1}{2} \text{sh}(ry) + r\text{ch}(ry)\right)^2}
\]

\[
\left(Y \left(\frac{1}{2} \text{ch}(ry) + r\text{sh}(ry)\right)\left(\frac{1}{2} \text{sh}(ry) + r\text{ch}(ry)\right) - Y_1\left(\frac{1}{2} \text{ch}(ry) + r\text{sh}(ry)\right)\left(\frac{1}{2} \text{sh}(ry) + r\text{ch}(ry)\right)\right]
\]

\[
\text{where} \quad \gamma = \sqrt{\omega_0 + \frac{1}{4}} \quad \text{and} \quad \beta = \sqrt{\omega_0 \omega_0 + \frac{1}{4}}
\]

\[
\text{Sh}(\infty) = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \text{Ch}(\infty) = \frac{1}{2}(e^x + e^{-x})
\]

1.2 Equilibrium ($y > y_{v1}$)

Still keeping the same outer limiting process (55), the temperature and species are written as follows,

\[
T_{\text{up}} = T_{\text{up}}^{(a)}(y) + \Theta_{1}^{-1} T_{\text{up}}^{(b)}(y) + \ldots
\]

\[
T_{\text{up}} = T_{\text{up}}^{(a)}(y) + \Theta_{1}^{-1} T_{\text{up}}^{(b)}(y) + \ldots
\]
The latter condition, that $C_{\text{me}}$ is zero is derived from equation (21) with $T_s = 1$ and $C_{2s} = 0$ in this zone. Since $A_1$, is of order $\Theta_i^2$, the only solution for $C_{\text{me}}$ with the outer limiting process (55), whatever the gauge function, is that $C_{\text{me}} = 0$.

Note that in the expansion for $T_{\text{ue}}$, the first coefficient function has superscript $(o)$ so that the $O(\Theta_i^{-1})$ coefficient function still has superscript $(i)$ to maintain consistency with our earlier work. The differential equations obeyed by $T_{\text{ue}}^{(o)}$ and $T_{\text{ue}}^{(i)}$ are,

$$\omega o T_{\text{ue}}^{(o)} + \frac{d T_{\text{ue}}^{(o)}}{d y} - \frac{1}{\lambda o} \frac{d^2 T_{\text{ue}}^{(o)}}{d y^2} = 0 ,$$  \hspace{1cm} (76)$$

$$\omega o T_{\text{ue}}^{(i)} + \frac{d T_{\text{ue}}^{(i)}}{d y} - \frac{1}{\lambda o} \frac{d^2 T_{\text{ue}}^{(i)}}{d y^2} = - \omega i T_{\text{ue}}^{(o)} - \frac{L_i}{\lambda o} \frac{d^2 T_{\text{ue}}^{(o)}}{d y^2} ,$$  \hspace{1cm} (77)$$

with boundedness conditions for downstream;

$$T_{\text{ue}}^{(o)}(\infty) , \quad T_{\text{ue}}^{(i)}(\infty) \quad \text{bounded} . \hspace{1cm} (78a,b)$$

Defining

$$T_{\text{ue}}^{(o)} = \lim_{y \to y_f +} T_{\text{ue}}^{(o)}(y) ,$$  \hspace{1cm} (79)$$

$$T_{\text{ue}}^{(i)} = \lim_{y \to y_f +} T_{\text{ue}}^{(i)}(y) .$$  \hspace{1cm} (80)$$
the general solution to equations (76) and (77) is given by,

\[ T_{ue}^{(o)} = T_{ue}^{(o)} \exp \left[ k_0 \left( \frac{1}{2} - s \right) (y - y_{f1}) \right] , \quad (81) \]

\[ T_{ue}^{(i)} = T_{ue}^{(i)} \exp \left[ k_0 \left( \frac{1}{2} - s \right) (y - y_{f1}) \right] \]

\[ - \frac{T_{ue}^{(o)}}{2s} (\omega + l, (\frac{1}{2} - s))(y - y_{f1}) \exp \left[ k_0 \left( \frac{1}{2} - s \right) (y - y_{f1}) \right] . \quad (82) \]

Note that definitions (64)-(67), (79), (80) underline the approach that is being adopted here, that \( T_{u1} \) is essentially discontinuous. Thus,

\[ T^{(a)}(y_{f1}) \neq T^{(o)}(y_{f1}) \quad ; \quad C^{(a)}(y_{f1}) \neq 0 , \quad (83a,b) \]

\[ T^{(i)}(y_{f1}) \neq T^{(i)}(y_{f1}) \quad ; \quad C^{(b)}(y_{f1}) \neq 0 , \quad (84a,b) \]

and these discontinuities now lead one to consider the inner zone where a profile is found to join the two outer solutions.

5.1.3 Inner Reaction Zone \((\dagger)\) \((y \text{ near } y_{f1})\)

In this zone, the inner coordinate \( y \) is used,

\[ y = \Theta_i \left[ y - (y_{f1} + \Theta_i^{-1} y_{f2} + ...) \right] , \quad (85) \]

\((\dagger)\) During the preparation of this work it was learnt that Kapila (1982) has independently adopted a similar approach as outlined here to inner zone behaviour.
where from the steady theory, \( y_{\infty} = -l_{x} y_{c} / \rho \) (see Equation (34)). The unsteady temperature \( T_u \) and lean species \( C_{e_u} \) are approximated by the following series expansions,

\[
T_u = -T_u^{(0)}(y) - \Theta_1^{-1} T_u^{(1)}(y) - \Theta_1^{-2} T_u^{(2)}(y) - \ldots, \tag{86}
\]

\[
C_{e_u} = C_{e_u}^{(0)}(y) + \Theta_1^{-1} C_{e_u}^{(1)}(y) + \Theta_1^{-2} C_{e_u}^{(2)}(y) + \ldots. \tag{87}
\]

Note again that superscripts \((\alpha)\) have been used for leading order terms so that \( O(\Theta_1^{-1}) \) terms can remain with superscripts \((\beta)\) to maintain consistency with previous work. Upon application of the inner limiting process,

\[
\Theta_1^{-1} \rightarrow 0, \quad y \text{ fixed}, \tag{88}
\]

the differential equations to \( O(\Theta_1^{2}) \) and \( O(\Theta_1) \) yield,

\[
\frac{1}{\varepsilon_0} \frac{d^2 T_u^{(\alpha)}}{dy^2} = \left( 1 + \alpha \right) \frac{\rho S \Theta_1^2}{2} \left( C_u^{(\alpha)} - C_{e_u}^{(\alpha)} T_u^{(\alpha)} \right) e^{-T_u^{(\alpha)}}, \tag{89}
\]

\[
\frac{d^2 C_{e_u}^{(\alpha)}}{dy^2} = \frac{1}{2} \frac{\rho S \Theta_1^2}{\Theta_1^2 - (1 + \alpha \rho \Delta T)} \left( C_u^{(\alpha)} - C_{e_u}^{(\alpha)} T_u^{(\alpha)} \right) e^{-T_u^{(\alpha)}}, \tag{90}
\]
\[
\begin{aligned}
\left\{-d\frac{\tau^{(0)}}{dy} + \frac{1}{\lambda_0} \frac{d^2\tau^{(0)}}{dy^2} - \frac{l_1}{\lambda_0^2} \frac{d\tau^{(8)}}{dy} - \frac{l_1}{\lambda_0^2} \frac{d^2\tau^{(8)}}{dy^2}\right\} &= \left(1+\sigma\right)Q_1 \lambda_0^2 B^2 e^{-\tau^{(0)}} \left[\frac{\lambda_1}{\lambda_0^2} (\epsilon_u^{(0)} - \epsilon_u^{(1)} \tau_u^{(0)}) + \Lambda^{(0)} \left(\epsilon_u^{(0)} - \epsilon_u^{(1)} \tau_u^{(0)}\right)\right] \\
&\quad - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) + \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) + \frac{\epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) \\
&\quad - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right)
\end{aligned}
\]

(91)

\[
\begin{aligned}
\left\{-d\frac{\epsilon_u^{(0)}}{dy} + \frac{d^2\epsilon_u^{(0)}}{dy^2}\right\} &= \frac{1}{2} \lambda_0^2 B^2 e^{-\tau^{(0)}} \left[\frac{\lambda_1}{\lambda_0^2} (\epsilon_u^{(0)} - \epsilon_u^{(1)} \tau_u^{(0)}) + \Lambda^{(0)} \left(\epsilon_u^{(0)} - \epsilon_u^{(1)} \tau_u^{(0)}\right)\right] \\
&\quad - \frac{2 \epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) + \frac{\epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right) - \frac{\epsilon_u^{(0)} \epsilon_u^{(0)}}{\lambda_1} \left(\tau_u^{(0)} + \tau_u^{(8)}\right)
\end{aligned}
\]

(92)

and a further \(O(1)\) equation will be needed (derived by eliminating the reaction term). This equation is given by,

\[
-\omega_0 \tau^{(0)} - d\frac{\tau^{(0)}}{dy} + \frac{1}{\lambda_0} \frac{d^2\tau^{(0)}}{dy^2} - \frac{l_1}{\lambda_0^2} \frac{d\tau^{(8)}}{dy} - \frac{l_1}{\lambda_0^2} \frac{d^2\tau^{(8)}}{dy^2} + \frac{l_1}{\lambda_0^2} \frac{d^3\tau^{(8)}}{dy^3} + \frac{l_1}{\lambda_0^2} \frac{d^3\tau^{(8)}}{dy^3} + \frac{l_1}{\lambda_0^2} \frac{d^3\tau^{(8)}}{dy^3}
\]

\[
\left(1+\sigma\right)Q_1 \left(\omega_0 \epsilon_u^{(0)} + \frac{d\epsilon_u^{(0)}}{dy} - \frac{d^2\epsilon_u^{(0)}}{dy^2}\right) = 0
\]

(93)

In equations (89) - (92) the leading order estimate for \(\Lambda\) has been substituted using (44) but \(\Lambda^{(0)}\) is explicitly still in the equations - given by (45). We also note that \(\tau^{(8)}\) and \(\epsilon^{(8)}\) (steady inner zone second order coefficient functions) are explicitly in the equations. Relationships for these quantities have already been found (equations (47) and (48)). To integrate these equations we now require the matching conditions satisfied on both sides of the reaction zone.
5.2 Matching

The matching conditions for the inner unsteady coefficient functions are found in a very similar fashion as for their steady counterparts. Thus we simply state them here:

\[
\begin{align*}
\tau_u^{(0)}(-\infty) &= -\tau_u^{(1)}, \\
\xi_u^{(0)}(-\infty) &= \frac{\sigma C_{elup}^{(1)}}{(1+\sigma)Q}, \\
\end{align*}
\]

Upstream Value Matching

\[
\begin{align*}
\tau_u^{(0)}(y \to -\infty) &\sim -\tau_u^{(1)} - (y + y_{fe}) \frac{dT_{up}^{(0)}}{dy} \bigg|_{f_1}, \\
\xi_u^{(0)}(y \to -\infty) &\sim \frac{\sigma C_{elup}^{(1)}}{(1+\sigma)Q} + (y + y_{fe}) \frac{dC_{elup}^{(1)}}{dy} \bigg|_{f_1}, \\
\end{align*}
\]

Downstream Value Matching

\[
\begin{align*}
\tau_u^{(0)}(\infty) &= -\tau_u^{(0)}, \\
\xi_u^{(0)}(\infty) &= 0, \\
\end{align*}
\]

Upstream Gradient Matching

\[
\begin{align*}
\frac{d\tau_u^{(0)}}{dy} \bigg|_{-\infty} &= 0, \\
\frac{d\xi_u^{(0)}}{dy} \bigg|_{-\infty} &= 0, \\
\end{align*}
\]

Downstream Gradient Matching

\[
\begin{align*}
\frac{d\tau_u^{(1)}}{dy} \bigg|_{f_1} &= -\frac{dT_{up}^{(0)}}{dy} \bigg|_{f_1}, \\
\frac{d\xi_u^{(0)}}{dy} \bigg|_{f_1} &= \frac{dC_{elup}^{(0)}}{dy} \bigg|_{f_1}, \\
\frac{d\tau_u^{(2)}}{dy} \bigg|_{y \to -\infty} &\sim -\frac{dT_{up}^{(1)}}{dy} \bigg|_{f_1} - (y + y_{fe}) \frac{d^2T_{up}^{(0)}}{dy^2} \bigg|_{f_1}, \\
\frac{d\xi_u^{(2)}}{dy} \bigg|_{y \to -\infty} &\sim \frac{dC_{elup}^{(1)}}{dy} \bigg|_{f_1} + (y + y_{fe}) \frac{d^2C_{elup}^{(2)}}{dy^2} \bigg|_{f_1}, \\
\end{align*}
\]
The gradient matching is taken up to the $\tau_u^{(z)}$, $\xi_u^{(z)}$ terms for use in the integration of (93). The subscript "$f_1$" simply means "evaluated at $y = y_{f_1}$". These matching conditions are now applied to the integration of equations (89) - (93).

5.3 Integration of inner zone equations

5.3.1 Leading Order

Elimination of the chemical term between (89) and (90) gives the equation,

$$\frac{1}{k e_0} \frac{d^2 \tau_u^{(z)}}{dy^2} = \frac{(1+\sigma)Q_1}{\sigma} \frac{d^2 \xi_u^{(z)}}{dy^2}$$

(104)
Integrating (104) twice using (96a, b) and (101a, b) yields,

\[ L_u^{(0)} = \frac{\sigma}{\kappa_0 (\omega \sigma) Q_i} (\tau_u^{(0)} + T_{ue}^{(0)}) \]  

(105)

Substitution of this equation into (89) with (42) yields the leading order inner equation in \( \tau_u^{(0)} \) alone:

\[ \frac{d^2 \tau_u^{(0)}}{dy^2} = \frac{1}{2} \kappa_0^2 \beta_i^2 \left[ T_{ue}^{(0)} + (1 - \tau_u^{(0)}) \tau_u^{(0)} \right] e^{-\tau_u^{(0)}}. \]  

(106)

As is the case with the steady equations in the inner zone, when integrating the unsteady equations \( \tau_u^{(0)} \) becomes the natural 'independent variable' to the formulation; so that using the identities,

\[ \frac{d}{dy} \equiv \frac{d \tau_u^{(0)}}{dy} \frac{d}{\tau_u^{(0)}} = -\kappa_0 B_i g \frac{d}{\tau_u^{(0)}}, \]  

(107a, b)

\[ \frac{d^2}{dy^2} \equiv \frac{d^2 \tau_u^{(0)}}{dy^2} \frac{d}{\tau_u^{(0)}} + \left( \frac{d \tau_u^{(0)}}{dy} \right)^2 \frac{d^2}{\tau_u^{(0)}}, \]  

(108a)

\[ = \frac{1}{2} \kappa_0^2 B_i^2 f \frac{d}{\tau_u^{(0)}} + \kappa_0^2 B_i^2 g^2 \frac{d^2}{\tau_u^{(0)}}, \]  

(108b)

(where use has been made of (40a, b) and (41a, b)), equation (106) can be written as

\[ q^2 \frac{d^2 \tau_u^{(0)}}{dy^2} + \frac{1}{2} f \frac{d \tau_u^{(0)}}{dy} + \frac{1}{2} (\tau_u^{(0)} - 1) \frac{d^2 \tau_u^{(0)}}{dy^2} \tau_u^{(0)} e^{-\tau_u^{(0)}} = \frac{1}{2} T_{ue}^{(0)} e^{-\tau_u^{(0)}.} \]  

(109)
The left hand side of this equation can be expressed as a total differential giving,

$$\frac{d}{d\tau^{(u)}} \left[ \frac{d}{d\tau^{(u)}} \left( \frac{d\tau^{(u)}}{d\tau^{(u)}} - \frac{1}{2} \tau^{(u)} \right) \right] = \frac{1}{2} T_{ue} \mathbf{i} \cdot e^{\tau^{(u)}} \quad , \tag{110}$$

so that integrating across the reaction zone yields,

$$\left[ \left( 1 - e^{-\tau^{(u)}} \right) \frac{d\tau^{(u)}}{d\tau^{(u)}} - \frac{1}{2} \tau^{(u)} e^{-\tau^{(u)}} \right]_{\tau^{(u)} = 0}^{\infty} = \frac{1}{2} T_{ue} \mathbf{i} \cdot \int_{0}^{\infty} e^{-\tau^{(u)}} d\tau^{(u)} \quad , \tag{111}$$

where we note from matching in the steady theory,

$$\tau^{(u)}(0) = 0 \quad ; \quad \frac{d\tau^{(u)}}{d\tau^{(u)}} \bigg|_{\tau^{(u)} = 0} = 0 \quad , \tag{112a,b}$$

$$\tau^{(u)}(y \to -\infty) \sim -\lambda_{0} B \cdot y \quad ; \quad \frac{d\tau^{(u)}}{d\tau^{(u)}} \bigg|_{y \to -\infty} = -\lambda_{0} B \quad . \tag{113a,b}$$

Equation (111) gives,

$$\frac{d\tau^{(u)}}{d\tau^{(u)}} \bigg|_{\tau^{(u)} = \infty} = \frac{1}{2} T_{ue} \mathbf{i} \cdot \mathbf{j} \quad , \tag{114}$$

which, using (113b) yields,

$$\frac{d\tau^{(u)}}{d\tau^{(u)}} \bigg|_{y \to -\infty} = -\frac{1}{2} \lambda_{0} B \cdot T_{ue} \mathbf{i} \cdot \mathbf{j} \quad . \tag{115}$$

But from the unsteady matching condition (98a),

$$\frac{d\tau^{(u)}}{d\tau^{(u)}} \bigg|_{y \to -\infty} = 0 \quad ,$$

so that (115) implies

$$T_{ue} = 0 \quad , \tag{116}$$
which in turn from (81,82) implies that the equilibrium solutions are simply,

\[ T_{eq}^{(o)} = 0 \]  \hspace{1cm} (117)

\[ T_{eq}^{(u)} = T_{eq}^{(w)} \exp \left[ h_0 \left( \frac{1}{2} - s \right) (y - y_c) \right]. \]  \hspace{1cm} (118)

But one should note that even though the gradient \( \frac{d T_{eq}^{(o)}}{dy} \)

is going to zero at both ends of the inner zone, \( T_{eq}^{(o)} \) itself is not zero. A different inner structure to the traditional approach emerges as follows. Returning to equation (110), one integration yields,

\[ q^2 \frac{d T_{eq}^{(o)}}{d \tau} = -\frac{1}{2} f T_{eq}^{(o)} = \text{constant} \]  \hspace{1cm} (119)

Matching on the equilibrium side (see (96a), (112a) and (116)) shows the constant is zero, so that one can write,

\[ \frac{d T_{eq}^{(o)}}{d \tau} = \frac{1}{2} f T_{eq}^{(o)} \]  \hspace{1cm} (120)

Since \( (q^2)' = f \) (where \( "'" = \frac{d}{d \tau} \)) the solution to (120) using condition (94a) is simply,

\[ T_{eq}^{(o)} = -T_{eq}^{(u)} \exp \left[ q(\tau) \right]. \]  \hspace{1cm} (121)

and represents the leading order estimate as to the inner profile joining the two outer solutions. Now that \( T_{eq}^{(o)} = 0 \), equation (121) satisfies the downstream condition (96a) that
\( \iota_u^{(0)}(y = \infty (i.e. \tau^{(0)} = 0) = 0 \). Note from (105) that \( \iota_u^{(0)} \)

is now given by,

\[
\iota_u^{(0)} = \frac{-\sigma \tau_u^{* (1)} \eta(\tau_u^{(0)})}{\iota_{\infty} (\iota + \sigma) Q},
\]

(122)

so that using (94b) on the pre-heat side, one obtains,

\[
\iota_u^{* (1)} = -\frac{1}{\iota_{\infty}} \tau_u^{* (1)}
\]

(123)

One now considers the next order inner equations (91) and (92) by first forming the equation eliminating the reaction term:

\[
\left(1 - \frac{1}{\iota_{\infty}}\right) \frac{d \tau_u^{(0)}}{dy} = \frac{(1 + \sigma) \eta(\tau_u^{(0)})}{\sigma} \frac{d^2 \tau_u^{(0)}}{dy^2} - \frac{1}{\iota_{\infty}} \frac{d^2 \tau_u^{(0)}}{dy^2} + \frac{b_i}{\iota_{\infty}^2} \frac{d^2 \tau_u^{(0)}}{dy^2}.
\]

(124)

Integration of this equation across the reaction zone and using the matching conditions of \( \S 5.2 \) (with \( \tau_u^{(0)} = 0 \)) yields the condition,

\[
-\left(1 - \frac{1}{\iota_{\infty}}\right) \tau_u^{* (1)} = \frac{(1 + \sigma) \eta(\tau_u^{(0)})}{\sigma} \frac{d \iota_u^{* (0)}}{dy}\bigg|_{f_1} + \frac{1}{\iota_{\infty}} \frac{d \tau_u^{* (1)}}{dy}\bigg|_{f_1}.
\]

(125)

Substituting (68), (69) and (123) into this result yields the leading order frequency relation:

\[
\frac{s^2}{2} - \frac{a}{2} + \frac{R}{2} - R S = 0,
\]

(126)

where

\[
R \equiv \frac{r \text{ch}(ry_{f_2})}{s \text{h}(ry_{f_2})} ; \quad S \equiv \frac{s \text{ch}(\iota_{\infty} y_{f_1})}{s \text{h}(\iota_{\infty} y_{f_1})}.
\]

(127)

Although here derived by a completely different route this relation is exactly the same as that which first appeared in McIntosh and Clarke (1981) and which leads to the neutral
stability curve shown in Fig. 2. This relationship has been discussed in detail in the above reference. We now proceed to find the second order frequency relation.

5.3.2 Second Order

Equation (124) at this stage has only been integrated once. The first integration gives, in general,

$$\frac{(1+\sigma)Q_1}{\sigma} \frac{d\theta_u}{dy} - \frac{1}{\kappa_0} \frac{d\theta_u^{(n)}}{dy} = \left( \frac{1}{\kappa_0} - 1 \right) \tau_u^{(n)} - \frac{l_1}{\kappa_0^2} \frac{d\theta_u^{(n)}}{dy} . \tag{128}$$

Making use of (107b) and (121) and finding the constant of integration by using matching on the equilibrium side, integration of (128) yields,

$$\frac{l_0}{\sigma} \left( 1+\sigma \right) Q_1 E_u^{(n)} = \tau_u^{(n)} + \frac{T_{up}^{(n)}}{B_1} \left( \frac{1}{\kappa_0} - 1 \right) \tau_u^{(n)} + \frac{l_1 T_{up}^{(n)}}{\kappa_0} + T_{ue}^{(n)} . \tag{129}$$

Application of (129) on the pre-heat side of the inner zone and then making use of (34), (95a,b) and (113a) yields an important connection between $T_{up}^{(a)}$, $C_{ep}^{(a)}$, $T_{up}^{(u)}$ and $T_{ue}^{(u)}$:

$$T_{ue}^{(u)} = T_{up}^{(u)} + \frac{l_0 C_{ep}^{(u)}}{B_1} + \frac{l_1 y_{ep} T_{up}^{(a)}}{\kappa_0} \left( \frac{1}{\kappa_0} - 1 \right) - \frac{l_1 T_{up}^{(u)}}{\kappa_0} . \tag{130}$$

This is one of three equations which are to be found connecting $T_{up}^{(a)}$, $C_{ep}^{(a)}$, $T_{up}^{(u)}$ and $T_{ue}^{(u)}$. Equation (130) is the first. A second result is obtained by integrating the full second order inner temperature equation (91). To do this $E_u^{(n)}/\theta_u^{(n)}$ is substituted for, from (129) and (122) along with $E_u^{(n)}/\theta_u^{(n)}$ (inner steady solution) from (48). After substitution of these quantities and making the use of identities (107b), (108b)
the terms can be combined in the following way:

\[
\frac{d}{d\tau^u} \left[ \left( \frac{g^2 d\tau^{*u}}{d\tau^u} \right) - \frac{1}{2} \left( \frac{g^2 d\tau^{wu}}{d\tau^u} \right) \right] = \frac{1}{2} \frac{\tau_{up}^{*u}}{B_i} \left\{ \frac{f}{B_i} - \frac{2 \sigma G f}{\bar{\omega}} \right\} \left( -\tau^u - \tau^{wu} \right) e^{-\tau^u} + \sigma \tau^{wu} e^{-\tau^w} + \frac{1}{B_i} \left( \frac{\varsigma}{\bar{\omega}} \right) f \left( 1 + \frac{\tau^{wu}}{\tau^u} \right) e^{-\tau^w} + \frac{1}{\bar{\omega}} \left( \frac{\varsigma}{\bar{\omega}} \right) f \left( \frac{\varsigma}{\bar{\omega}} \right) e^{-\tau^w} \right\}
\]

Although tedious algebraically, integration of this equation across the reaction zone is possible. The left hand side simply gives \( \frac{1}{\bar{\omega} B_i} \frac{d\tau_{up}^{*u}}{dy} \bigg|_{\chi} \). The right hand side is more difficult; the last term will yield the integral

\[
A = \int_0^\infty q(\tau^u - 2\tau^{wu}) e^{-\tau^w} d\tau^u
\]

\[
A = \left[ \frac{\tau^u}{\bar{\omega}} - \left( \frac{\tau^{wu}}{\bar{\omega}} \right) \right] \left[ q^3 \left( \frac{\tau^{wu}}{\bar{\omega}} \right) \right] d\tau^u
\]

(by using integration by parts)

Since \( q^3 \left( \frac{\tau^{wu}}{\bar{\omega}} \right) \) can be obtained from (47), the integral \( A \) then becomes known. Substituting for \( \Lambda^{(a)} \) from (45), one then finds that nearly all the terms cancel as a result of integrating the right hand side of (131). We do not display the mathematics here but it can be shown that one obtains the simple result that,

\[
\frac{1}{\bar{\omega} B_i} \frac{d\tau_{up}^{*u}}{dy} \bigg|_{\chi} = \frac{1}{B_i} \tau_{up}^{*u} + \frac{1}{2} \tau_{ue}^{*u}
\]
which by substitution of (68) yields,

\[ T_{ue}^{\sigma(1)} = -\frac{2}{B_i} T_{up}^{\sigma(1)} \left( \frac{1}{2} - S \right) \]  

(134)

The last equation linking the second order pre-heat and equilibrium solutions is obtained by integration of (93). Careful use of the matching conditions of §5.2 along with the already derived leading order relationship (126) leads to the following relationship:

\[ -\omega_0 \left( \frac{1}{k_0} - 1 \right) \frac{d}{dy} \left. T_{up}^{\sigma(1)} \right|_{f_i} + T_{up}^{\sigma(2)} - T_{ue}^{\sigma(1)} + C_{eup}^{\sigma(3)} - \frac{1}{k_0} \frac{dT_{up}^{\sigma(4)}}{dy} \bigg|_{f_i} \]

\[ + \frac{1}{k_0} \frac{dT_{ue}^{\sigma(4)}}{dy} \bigg|_{f_i} - \frac{1}{\nu} \frac{d}{dy} \left. C_{eup}^{\sigma(3)} \right|_{f_i} + \frac{B_i}{k_0^2} \frac{dT_{up}^{\sigma(4)}}{dy} \bigg|_{f_i} = 0 \]  

(135)

Equations (130), (134) and (135) represent 3 equations linking the four quantities, \( T_{up}^{\sigma(1)}, C_{eup}^{\sigma(2)}, T_{up}^{\sigma(1)} \) and \( T_{ue}^{\sigma(1)} \). In solving these relations one in fact finds (after much calculation) a solvability condition (in a similar way as (126) appeared at leading order). This second order frequency condition is:

\[ l_1 B_i \left[ 1 - \frac{1}{2s^2} \left( \frac{1}{4} + s^2 \right) S + s^2 \right] + 2k_0 \left( \frac{1}{2} - S \right)(s + S') \]

\[ -l_1 B_i y_{fi} \omega_0 \left[ \frac{1}{2s^2(k_0 y_{fi})} + \frac{r^2}{k_0 \left( \frac{1}{2} s h(r_{y_{fi}}) + r c h(r_{y_{fi}}) \right)} \right] \]

\[ + \frac{l_1 B_i \omega_0}{2} \left[ \frac{(r^2 - R^2)}{2r^2(\frac{1}{2} + R)^2} + \frac{\left( \frac{1}{2} R + R \right)}{r^2 \left( \frac{1}{2} + R \right)} - \frac{s}{s^2} \right] \]

\[ + \frac{l_1 B_i \omega_0 y_{fi}}{2} \left[ \frac{\omega_0}{\left( \frac{1}{2} s h(r_{y_{fi}}) + r c h(r_{y_{fi}}) \right)} + \frac{k_0}{s h^2(k_0 y_{fi})} \right] = 0 \]  

(136)

The two results (126) and the above (136) constitute first and second order long wavelength dispersion relations. This latter result (136) has not appeared in the literature to date and constitutes the main result of this analysis. The authors have approached the second order solution of the unsteady equations from other stand points. In particular the more traditional approach where \( \varepsilon > \Theta_i^{-1} \) (but restricted to a
fairly tight band - see § 3) produces the same two results. It should also be noted that in principle these results can be extended to two dimensions upon making suitable assumptions concerning the interaction of the conductive-diffusive terms with the flow field. Here we have restricted the theory to one dimension in order to simplify the complexity of the second order derivation. The two results (126) and (136) are examined in the next chapter.
The leading order relation (126) links complex frequency with stand-off distance \( y_{fc} \). The main result from this relationship is a neutral stability curve (Fig. 2) which has been discussed in detail in McIntosh and Clarke (1981). One then considers the second order complex frequency relation (136) which links estimates of complex frequency \( \omega_0 \) and \( \omega \) to stand-off distance \( y_{fc} \) and Lewis number estimates \( \frac{\lambda_e}{\Theta} \) and \( \lambda \).

In investigating this relationship great care must be exercised in interpreting the results. We discuss under the following headings:

6.1 Ambiguity in second order correction terms

To derive the above second order frequency relation, it has been assumed that

\[
\lambda_e = \lambda_{e_0} + \frac{\lambda_1}{\Theta_1} + \ldots, \quad \omega = \omega_0 + \frac{\omega_1}{\Theta_1} + \ldots, \tag{137a,b}
\]

are valid expansions for Lewis number and frequency. However when one calculates values for \( \omega \) using this relationship, ambiguity becomes apparent. To illustrate, consider a stand-off distance of \( y_{fc} = 4 \) (Fig. 3), and a Lewis number of 1.4. With an activation energy \( \Theta_1 \) of 10, \( \lambda_e \) can be formed in more than one way. We illustrate two,

(a) \( \lambda_{e_0} = 1.2 \); \( \lambda_1 = 2 \)

(b) \( \lambda_{e_0} = 1.4 \); \( \lambda_1 = 0 \)

Case (a) in the first relation (126) yields a complex value for \( \omega_0 \) which is \( 0.06 + 0.25i \). The second order estimate for \( \omega \) from (136) is \( -8.76 - 4.47i \). Thus the actual value of \( \omega \) predicted is \( \omega = -0.82 - 0.20i \) (Note \( B^* \sim 1 - T_0 \), and is approximately unity).

Case (b) yields a complex value for \( \omega_0 \) of \( 0.01 + 0.15i \). However the second order estimate for \( \omega \), (with \( \lambda_1 = 0 \)) is \( -4.17 - 1.72i \).
The actual value for $\omega$ is then $\omega = -0.41 - 0.02i$. As $y_{f_1}$ diminishes, this ambiguity becomes more acute. Fig. 4 displays values of $k_1$ and $Z_1$ for $y_{f_1} = 1$ and $\lambda_e = 1.2$, which are very large (like $\Theta^2$ if $\Theta$ is equal to 10). This sort of behaviour suggests that the asymptotic analysis breaks down for anything other than unrealistically large $\Theta$, values (e.g. $\Theta > 100$), $\lambda_e$ values away from unity and $y_{f_1}$ of modest size. One is tempted to propose that in regions away from $\lambda_e = 1$, $\lambda_e$ should be treated as simply an $O(1)$ parameter ($l = 0$) and we shall in fact do this in much of what follows.

6.2 Areas of validity

Although in most of $(\lambda_e, y_{f_1})$ space the $k_1$ and $Z_1$ values are too large for the theory to be valid, there are some areas where this is not the case. There are two regions marked on Fig. 2 where $|\omega_1| = k_1^2 + Z_1^2$ is less than 10. For both $\lambda_e$ below unity and greater than unity, when $y_{f_1}$ is of modest size (if $y_{f_1}$ is in range 2 to 4) the boundaries show that $\lambda_e$ must be well away from unity for the theory to have validity. In this range of $y_{f_1}$, $\lambda_e$ must be greater than approximately 1.3 or below approximately 0.7. In finding these boundaries (and in all the discussion from here on), $\lambda_e$ is simply treated as an $O(1)$ parameter with $l$ set to zero everywhere. The arguments relating to the magnitude of $\omega_1$ yield (see Fig. 2) the left hand boundaries of validity. However, one should also consider the magnitude of the leading order estimate $\omega_o$ for complex frequency. Although one can rationalise the limit $\omega_o \to 0$ (see later discussion on free-flame limit), in the present context to ensure that time derivatives are still $O(1)$, we choose a bottom limit for $|\omega_o|$ of 0.1 and mark right hand boundaries beyond which $|\omega_o| < 0.1$. Thus two further boundaries emerge marking out the regions where the theory can be validated. (Fig. 2). Note that Figs. 3 and 4 indicate that for $\lambda_e > 1$ the solutions for $\omega_o$ are complex and it is found that the real part of $\omega_o$ (i.e. $\Re \omega_o$) is small. However the $Z_o$ values are large enough to make the modulus of $\omega_o$ still an $O(1)$ quantity.
6.3 Stability

Within the afore-mentioned areas of validity, we now consider whether the solutions indicate stability or instability. For \( \lambda_0 < 1 \), the dominant leading order solution for \( \omega_0 \) is real and positive. Second order corrections (with \( \ell \) set to zero) indicate further amplification from the real part of \( \omega_0 \). Thus for \( \lambda_0 < 1 \) strongly divergent behaviour is still predicted.

For \( \lambda_0 > 1 \), the situation is somewhat different. The leading order theory predicts the neutral stability curve \( (\lambda_0 = \lambda_0^* (y_\ell)) \) as shown in Fig. 2. One now considers the second order corrections \( k_1 \) and \( Z_1 \) (see Figs. 3 and 4 with \( \ell = 0 \)) for \( \lambda_0 \) close to this line. Generally \( k_1 \) is negative, but recalling that the real part of \( \omega \) is represented here by

\[
\kappa = k_0 + \Theta^{-1} k_1
\]

(138)

it can be seen that the apparently modest values of the second order term have a substantial effect on predictions of the neutral stability boundary position in \( (\lambda_0, y_\ell) \) space. This arises because \( k_0 \) is small (typically 0.01 to 0.10) in magnitude, and the second order term is then always larger than \( k_0 \) (unless \( \Theta_1 \) is once again unrealistically large). Thus for typical \( \Theta_1 \) values (\( \Theta_1 \) in the range 10 to 20) and \( \lambda_0 > 1 \), in the area of validity marked on Fig. 2 one can generally predict a stable region (where leading order theory alone predicted instability) for long wavelength disturbances.

6.4 Free-flame limit

As \( y_\ell \) becomes large both the leading order and second order dispersion relations can be approximated. But in doing so the behaviour of the other quantities (notably \( \lambda_0 \) and \( \omega_0 \)) is important. There are two main possibilities.

6.4.1 \( (\lambda_0 - 1) y_\ell \to \infty \) as \( y_\ell \to \infty \)
This condition is true in particular along the neutral stability curve in Fig. 2 for large $y_{fi}$. Under these circumstances $|\omega_0|$ is found to be small and of order $\exp(-\frac{1}{2} y_{fi})$. The leading order dispersion relation (126) can be approximated by,

\[
\frac{\omega_0^2 (1 - \lambda_0)}{\lambda_0} + 2 \omega_0 y_{fi} e^{-\lambda_0 y_{fi}} \approx e^{-\lambda_0 y_{fi}} + \omega_0 e^{-\lambda_0 y_{fi}} + \frac{2 \omega_0 e^{-\lambda_0 y_{fi}}}{\lambda_0} + \omega_0 e^{-\lambda_0 y_{fi}} \tag{139}
\]

which leads to the following approximation for the leading order neutral stability boundary ($\lambda_0 = \lambda_0^*(y_{fi})$, $y_{fi}$ large):

\[
\lambda_0^* (2 y_{fi} - 1) = 2 + \exp \left( \left( \lambda_0^* - 1 \right) y_{fi} \right) \tag{140}
\]

Only at very large stand-off distances ($y_{fi}$) does $\lambda_0^*$ begin to approach unity. Even when $y_{fi} = 50$, $\lambda_0^*$ is only just below 1.1 ($\lambda_0^* = 1.0933$).

This curve has been marked as an asymptote on Fig. 2. Along this curve, the imaginary part of $\omega_0$ (i.e. $\omega_0$) is given by,

\[
\omega_0 = \lambda_0 \sqrt{(\lambda_0 - 1) e^{-\lambda_0 y_{fi}}} \tag{141}
\]

The same approximation can of course be applied to the second order dispersion relation, but little simplification is obtained (if at all) in the form of (136). When second order corrections are applied, the same ambiguity in estimation of Lewis number is found (as described in the first part of this discussion). If $\lambda_1$ is then set to zero the $K_1$ values are of moderate size and negative (typically $-3$ to $-10$). Thus since $|\omega_0|$ is very small, one can generally predict a stable region in these circumstances from second order theory.
6.4.2 \((L_{e_{0}} - 1)y_{f_{1}} \rightarrow O(1)\) as \(y_{f_{1}} \rightarrow \infty\)

This condition will generally hold true below the leading order neutral stability boundary and in the neighbourhood of the line \(L_{e_{0}} = 1\). The complex frequency \(\omega_{e}\) is treated as an \(O(1)\) number and the leading order dispersion relation can then be approximated by,

\[
\frac{\omega_{e} b}{2r}(\frac{1}{2} - r) \approx 2rc\left[(\frac{1}{2} + r)\exp\left[-\frac{b_{y_{f_{1}}}(\omega_{e} + \frac{1}{2})}{r}\right] - (\frac{1}{2} - r)\right],
\]

(142)

where

\[
b \equiv L_{e_{0}} - 1,
\]

(143)

\[
b y_{f_{1}} \sim O(1)\quad \text{(by hypothesis)},
\]

(144)

\[
c \equiv e^{-2ry_{f_{1}}},
\]

(145)

and (see (72a))

\[
r \equiv \sqrt{\omega_{e} + \frac{1}{2}}.
\]

(146)

As indicated on Fig. 2, the leading order theory implies unstable non-oscillatory behaviour for \(L_{e_{0}} < 1\) and unstable oscillatory behaviour for \(L_{e_{0}} > 1\). However, the second order correction \(|\omega_{e}|\) exceeds 10 (typically in the range 10 to 20) so that the theory breaks down unless \(\Theta_{i}\) is unrealistically large. This situation becomes even more acute as \(L_{e_{0}}\) gets even closer to 1. Thus in the special case where,

\[
b y_{f_{1}} \rightarrow 0 \quad \text{as} \quad y_{f_{1}} \rightarrow \infty,
\]

(147)

even though the leading order dispersion relation, simplifies to,

\[
\omega_{e} b = \frac{8r^{3}c}{(\frac{1}{2} - r)}.
\]

(148)
the second order corrections are exceedingly large and the
theory cannot be validated.

At the limit \( \eta_\infty = 0 \), the first order dispersion
relation (126) (approximate form (146) above) is satisfied
both by \( \omega_0 = 0 \) (also obtained from case (a) equation
(139) when \( \eta_\infty = \infty \)) and by the more interesting alternative:

\[
b = 0 \quad \text{i.e.} \quad L_0 = 1 \quad , \omega_0 \text{ arbitrary} .
\]  

(149)

This forces one to look at the second order dispersion
relation which degenerates under these circumstances to the
now well known form,

\[
\ell, B_1 = \frac{8r^2}{\frac{3}{2} - r} .
\]  

(150)

Relation (150) has been derived by a number of authors
(Sivashinsky 1977, Joulin and Clavin 1979). But it is
noteworthy that the routes followed by these authors are all
very different from the one used here. Note that in this case
\[ B_1 = 1 - T_01 \]  
is a number very close to unity and (150) is an
expression for long wave disturbances. This result implies
areas of stability/instability which have been discussed by many
authors (see Buckmaster and Ludford 1982, chapter 11). The
important point to note is that only in the limit \( \eta_\infty = \infty \)
is (150) the foremost dispersion relation.

6.5 Lewis number estimation

It is of interest to note that the regions of validity
marked in Fig. 2 include the typical range of \( \eta_\infty \) values
usually found in practice (see Clarke and McIntosh 1979), but
exclude Lewis numbers close to unity. Pelcé and Clavin (1982)
estimate a typical Lewis number (here defined as mass diffusion
divided by thermal diffusion) for propane burning in air of
between 0.58 (fuel-rich) and 1.16 (fuel-lean). The present
theory would then only have valid predictions at the lower end
of such a Lewis number range (where long wave instabilities for
a burner anchored flame would be predicted). When Lewis number is
greater than unity then the theory would not be relevant. The
situation would be little altered for most of the hydrocarbon fuels.
But for hydrogen-air experiments where high binary mass diffusion coefficients can be expected (with only slightly increased thermal diffusion because of the large amount of diluent), one could expect a typical Lewis number to be well in excess of unity and thus the present theory would be quite relevant. Note that Botha and Spalding (1954) found that their propane-air burner flames were stable under fuel-lean conditions (cf the $\text{Le}_\infty$ value of 1.16 quoted above) and instabilities appeared for rich mixtures.

In estimating Lewis number we make some final comments about the free flame limit where (see the earlier part of this discussion) the expansion,

\[ \text{Le} = 1 + \text{Le} \cdot \Theta^{-1} \]  

(151)

can be justified. As mentioned above, Pelcé and Clavin (1982) suggest a practical range for $\text{Le}$ (for a propane-air system) of 0.58 to 1.16 giving for $\Theta \approx 15$ a range for $\text{Le}$ of -6.3 to 2.4. The parameter $\text{Le}$ is then considered an $O(1)$ number. However care must be taken in interpreting such results for, in a completely different problem, where an expansion of the form (151) is not mathematically convenient, Durbin (1982) (in examining premixed flames in straining flow) obtained results for an almost identical $\text{Le}$ range but with $\text{Le}$ treated simply as an $O(1)$ number. Although it is clear that expansions of the form (151) have shed much light on free flame stability one must be clear as to whether it is a true physical constraint or whether it is simply a convenient mathematical tool.

6.6 Time scales

In the light of the foregoing sections, one can now make an assessment of the behaviour of flames over the whole $(\text{Le}, \gamma_f)$ domain.

(a) $|\text{Le} - 1|$ of order unity, $\gamma_f$ of order unity:

In this region (limited by the boundaries mentioned in section 6.2 of this discussion), the response of the flame is predicted to be on an $O(1)$ time scale.
(b) $|\alpha - 1|$ of order unity, $y_{f1}$ large:

As $y_{f1}$ is increased, the $|\omega|$ values become small so that in this region there is a strong indication that time could be rescaled and the 'slowly varying' model (Buckmaster 1977) used to analyse such flames.

(c) $|\alpha - 1|$ small, $y_{f1}$ of order unity:

In this part of the $(\alpha, y_{f1})$ domain, the $|\omega|$ values are becoming exceedingly large indicating again that time should be rescaled, but in this case to take account of fast responses.

(d) $|\alpha - 1|$ small, $y_{f1}$ large:

With $\alpha$ near 1 and $y_{f1}$ large, the $|\omega|$ values become again of order unity and the free flame dispersion relation emerges in the limit as indicated in section 6.4 of this discussion.
CONCLUDING REMARKS

(1) Two complex frequency relationships have been derived for unsteady one-dimensional burner flames for arbitrary Lewis number (i.e. $\text{Le}$ not necessarily close to unity).

These relationships have been derived on the strict assumption that the problem is first linearized for small unsteady perturbations. Only then have activation energy asymptotics been applied to solve the resulting equations to second order in $\Theta_i^{-1}$. It is noted that the inner zone equation differs from the one obtained in the case where activation energy asymptotics are applied first and the small perturbation limit second.

(2) Regions in $(\text{Le}, y_f)$ space have been found where the response of flames to long wave disturbances can be expected to be on different time scales. There are two main areas where the response is expected to be on an $O(1)$ time scale:

   (i) $|\text{Le} - 1|$ of order unity, $y_f$ of order unity

   (ii) $|\text{Le} - 1|$ small, $y_f$ large

(3) It is shown that the free-flame dispersion relation is a special case of the complex frequency relations derived here, and obtained only in the limit of infinite stand-off distance.

(4) For burner anchored flames with Lewis number away from unity there is every indication that Lewis number should be treated as an $O(1)$ parameter.

(5) Ambiguities that arise from writing the Lewis number in the form $\text{Le} = \text{Le}_0 + l_i \Theta_i^{-1}$ lead to a lack of uniqueness in the solutions for complex frequency $\omega$. When $\Theta_i$ is of practical magnitude this lack of uniqueness is no longer tolerable in some $(\text{Le}_0, y_f)$ domains. This is because $|\omega|$ in the expansion $\omega = \omega_0 + \omega_i \Theta_i^{-1}$ is comparable with $\Theta_i$. One must conclude that the asymptotic
theory has broken down under these circumstances.

It is tempting to propose that only leading order theory has validity and that its predictions of numerical values will not be therefore very accurate.

In particular, corrections to the leading order neutral stability boundary are so gross that the true boundary cannot be located by the asymptotic theory.

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Appendix  Temperature and Lean Species in the Reaction Zone

The unsteady perturbations equations are,

\[
\begin{align*}
\omega C_{lu} + \frac{d C_{lu}}{dy} - \frac{d^2 C_{lu}}{dy^2} &= -\frac{\Lambda}{\theta} \left[ \Theta_{tu} (1 + \lambda + C_{0}) C_{0} + C_{lu} (2C_{0} + \lambda_{u}) \right] e^{\Theta(1 - \frac{1}{\Theta})}, \\
\omega T_{u} + \frac{d T_{u}}{dy} - \frac{1}{\theta} \frac{d^2 T_{u}}{dy^2} &= \frac{\lambda(1 + \sigma) \Theta_{tu} \Lambda}{\Theta} \left[ \Theta_{tu} (1 + \lambda + C_{0}) C_{0} + C_{lu} (2C_{0} + \lambda_{u}) \right] e^{\Theta(1 - \frac{1}{\Theta})},
\end{align*}
\]

Initially we assume continuity of \( T_{u}, C_{lu} \) such that

\[
\lim_{y \to y_{f_1}} \left\{ \begin{array}{c} T_{up} \\ C_{up} \end{array} \right\} = \lim_{y \to y_{f_2}} \left\{ \begin{array}{c} T_{ue} \\ C_{ue} \end{array} \right\},
\]

(\text{where subscripts } "p" \text{ and } "e" \text{ refer to pre-heat and equilibrium zones respectively). Note (A3) implies that there are no jumps in relative order of magnitude across the flame.}

We now integrate equations (A1, A2) across the reaction zone over the vanishingly small interval \((y_{f_1} + \delta, y_{f_2} - \delta)\):

\[
\begin{align*}
\omega \int_{y_{f_1} + \delta}^{y_{f_2} - \delta} C_{lu} dy + \left[ C_{lu} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} - \left[ \frac{d C_{lu}}{dy} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} &= -\frac{\Lambda}{\theta} \int_{y_{f_1} + \delta}^{y_{f_2} - \delta} \left( T_{u} \right)_{y_{f_1} + \delta}^{y_{f_2} - \delta} dy - \frac{\lambda(1 + \sigma) \Theta_{tu}}{\Theta} \left[ C_{lu} (2C_{0} + \lambda_{u}) e^{\Theta(1 - \frac{1}{\Theta})} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} \\
\omega \int_{y_{f_1} + \delta}^{y_{f_2} - \delta} T_{u} dy + \left[ T_{u} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} - \frac{1}{\theta} \left[ \frac{d T_{u}}{dy} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} &= \frac{\lambda(1 + \sigma) \Theta_{tu} \Lambda}{\Theta} \left[ C_{lu} (2C_{0} + \lambda_{u}) e^{\Theta(1 - \frac{1}{\Theta})} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta} \equiv \frac{\lambda(1 + \sigma) \Theta_{tu} (I_1 + I_2)}{\Theta} \left[ C_{lu} (2C_{0} + \lambda_{u}) e^{\Theta(1 - \frac{1}{\Theta})} \right]_{y_{f_1} + \delta}^{y_{f_2} - \delta},
\end{align*}
\]

where

\[
\Theta_{i}^{-1} \ll \delta(\Theta_{i}) \ll 1.
\]
The last statement concerning the order of $\delta$ ensures that the interval of integration includes the reaction zone but also that $\delta \to 0$ in the limit $\Theta_1 \to 0$. 

One now considers the two integrals defined by,

$$I_1 = \Lambda_1 \Theta_1 \int_{y_i}^{y_f - \delta} \frac{t_u}{t_s} (r_a + r_s) c_a e^{\Theta_1 (1 - \frac{t_u}{t_s})} dy; \quad I_2 = \Lambda_1 \int_{y_i}^{y_f - \delta} c_a (2c_s + r_a) e^{\Theta_1 (1 - \frac{t_u}{t_s})} dy.$$ \hspace{1cm} (A7a,b)

By use of the Second Mean Value Theorem (Whittaker and Watson 1963, pp. 65, 66) one can write $I_1$ and $I_2$ as,

$$I_1 = \Lambda_1 \Theta_1 \overline{t_u} \int_{y_i}^{y_f - \delta} \frac{t_u}{t_s} (r_a + r_s) c_a e^{\Theta_1 (1 - \frac{t_u}{t_s})} dy; \quad I_2 = \Lambda_1 \overline{c_a} \int_{y_i}^{y_f - \delta} (2c_s + r_a) e^{\Theta_1 (1 - \frac{t_u}{t_s})} dy,$$ \hspace{1cm} (A8a,b)

where $\overline{t_u}$ and $\overline{c_a}$ represent the value of $t_u$ and $c_a$ at suitable $y$ values in the range $(y_i, y_f - \delta, y_f + \delta)$. From the inner zone steady state solutions we have,

$$T_s = 1 - \Theta_1 U''(y) - \cdots; \quad C_k = \frac{\Theta_1^{-1} U''(y)}{\rho_0 (1 + \sigma) \Theta_1} \; \cdots.$$ \hspace{1cm} (A9a,b)

$$y \equiv \Theta_1 (y - y_f - \cdots) \hspace{1cm} \text{(A10)}$$

$$y = y_f + \delta \quad \text{equivalent to} \quad Y = \Theta_1 \delta \hspace{1cm} \text{(A11)}$$

$$y = y_f - \delta \quad \text{equivalent to} \quad Y = -\Theta_1 \delta \hspace{1cm} \text{(A12)}$$

$$\frac{dU''}{dy} = -\rho_0 B, g(U''); \quad g(U'') = \sqrt{1-e^{U''(1+U')}}.$$ \hspace{1cm} (A13a,b)

$$U''(y \to -\infty) \sim -\rho_0 B, y \to +\infty \hspace{1cm} \text{(A14)}$$

$$U''(y \to +\infty) \sim \exp\left(-\frac{\rho_0 B, y}{\sqrt{2}}\right) \to 0.$$ \hspace{1cm} (A15)
These last two results can be obtained by analysis of the inner zone at its two extremities (Williams 1974, Bush 1979) where,

\[
\alpha \equiv \int_0^\infty \left[ \frac{1}{3} - 1 \right] d\tau^{(4)} ; \quad \alpha = 0.935 \ldots \quad \text{(A16)}
\]

Making use of (A9-A15) and (43,44) in the main text (for \( \Lambda_i \)), one can write,

\[
\mathcal{I}_i = \frac{\Theta_i \overline{\tau}_u \sigma B_i}{2 \kappa_0 (1 + \sigma) Q_i} \int_0^\infty \frac{\tau^{(4)} e^{-\tau^{(4)}} d\tau^{(4)}}{q(\tau^{(4)})} \left[ 1 + \mathcal{O}(\Theta^{(-1)}) \right], \quad \text{(A17)}
\]

\[
\mathcal{I}_i = \frac{\Theta_i \overline{\tau}_u \sigma B_i}{\kappa_0 (1 + \sigma) Q_i} \left[ 1 + \mathcal{O}(\Theta^{(-1)}) \right]. \quad \text{(A18)}
\]

The integral \( \mathcal{I}_i \) can be considered in a similar manner, but the limit \( \Theta_i \to 0 \) must be considered carefully. One can write first,

\[
\mathcal{I}_2 = \lim_{\Theta_i \to 0} \left\{ \frac{\Theta_i \overline{C}_u B_i}{2} \int_{\tau^{(4)}(y=\Theta,\delta)} \frac{e^{-\tau^{(4)} d(\tau^{(4)})}}{q(\tau^{(4)})} \left[ 1 + \mathcal{O}(\Theta^{(-1)}) \right] \right\}. \quad \text{(A19)}
\]

From (A14,A15) in the limit \( \Theta_i \to 0 \), one can write,

\[
\tau^{(4)}(y = -\Theta,\delta) \sim \kappa_0 B_i \Theta, \delta \to \infty, \quad \text{(A20)}
\]

\[
\tau^{(4)}(y = \Theta,\delta) \sim a \exp \left( -\frac{\kappa_0 B_i \Theta, \delta}{\sqrt{2}} \right) \to 0. \quad \text{(A21)}
\]
In the limit $\Theta_i^{-1} = 0$, the integral in (A19) is unbounded due to the behaviour of $g(\tau^\nu)$ near $\tau^\nu = 0$. Using (A20, A21), one can write,

$$I_2 = \lim_{\Theta_i^{-1} \to 0} \left\{ \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} \left[ 1 + O(\Theta_i^{-1}) \right] \right\}.$$

(A22)

The integral can be split up as,

$$I_2 = \lim_{\Theta_i^{-1} \to 0} \left\{ \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} + \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} \right\}.$$

(A23)

where $b$ is in the range,

$$\exp\left(-\frac{\Theta_i B_i \delta}{\sqrt{2}}\right) < b \ll 1.$$

(A24)

Now since,

$$g(\tau^\nu) \to 0 \approx \frac{1}{\sqrt{2}} \tau^\nu + \ldots,$$

(A25a)

$$\frac{e^{-\tau^\nu}}{q(\tau^\nu)} \approx \frac{\sqrt{2}}{\tau^\nu} + \ldots,$$

(A25b)

we can write

$$I_2 = \lim_{\Theta_i^{-1} \to 0} \left\{ \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} + \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} \right\}.$$

(A26)

i.e.

$$I_2 = \lim_{\Theta_i^{-1} \to 0} \left\{ \frac{\Theta_i C_{\alpha\beta} B_i}{2} \int_{\tau^\nu = \nu}^{\nu} \frac{e^{-\tau^\nu} d\tau^\nu}{q(\tau^\nu)} + \frac{\Theta_i C_{\alpha\beta} B_i}{2} \left[ \frac{\log(\tau^\nu)}{\alpha} + \Theta_i B_i \delta \right] \right\}.$$

(A27)
The first two terms will always be smaller than the last term as $\Theta \rightarrow 0$. Thus $I_2$ can be expressed as,

$$I_2 = \frac{1}{2} L \Theta B (\delta \Theta^2) \left[ 1 + O(\Theta^{-1}) \right]. \quad (A28)$$

On the assumption of the continuity of $C_{eu}$ and $T_u$, as expressed by (A3) we have,

$$\lim_{\delta \to 0} \left( \frac{C_{eu}}{y_{x+d}} \right) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \left( \frac{T_u}{y_{x+d}} \right) = 0, \quad (A29)$$

$$\lim_{\delta \to 0} \left( \frac{C_{eu}}{y_{x+d}} \right) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \left( \frac{T_u}{y_{x+d}} \right) = 0, \quad (A30)$$

so that, since $\frac{dC_{eu}}{dy}$, $\frac{dT_u}{dy}$ are $O(1)$, by hypothesis, equations (A3,A4) yield,

$$\begin{bmatrix}
\frac{dC_{eu}}{dy} \\
\frac{dT_u}{dy}
\end{bmatrix}_{y_{x+}} = \frac{\sigma B_1 T_u}{(1+\sigma)Q} \left[ 1 + \ldots \right] + \frac{1}{2} L \Theta B (\delta \Theta^2) C_{eu} \left[ 1 + \ldots \right], \quad (A31)$$

$$\begin{bmatrix}
\frac{dC_{eu}}{dy} \\
\frac{dT_u}{dy}
\end{bmatrix}_{y_{x+}} = -\frac{L \Theta B (\delta \Theta^2) C_{eu}}{2 \sigma} \left[ 1 + \ldots \right]. \quad (A32)$$

Now $T_u$ and $C_{eu}$ are mean values within the reaction zone proposed firmly on the basis of continuity according to (A3). Thus by hypothesis,

$$\begin{bmatrix}
\frac{T_u}{C_{eu}}
\end{bmatrix}_{y \to y_{x-}} = \lim_{y \to y_{x-}} \begin{bmatrix}
\frac{T_u}{C_{eu}}
\end{bmatrix}_{y \to y_{x+}} = \lim_{y \to y_{x+}} \begin{bmatrix}
\frac{T_u}{C_{eu}}
\end{bmatrix}_{y \to y_{x+}} = \lim_{y \to y_{x+}} \begin{bmatrix}
\frac{T_u}{C_{eu}}
\end{bmatrix}_{y \to y_{x+}} \quad (A33)$$
The immediate conclusion then from equations (A31, A32) is that \( T_u^\prime \) and \( C_{eu} \) cannot be \( O(1) \) but that both are certainly \( o(1) \). One then proceeds to solve equations (A1, A2) in a pre-heat zone \( 0 \leq y < y_{f_1} \) where

\[
T_u = T_{up}^{(\prime)}(y) + \Theta^{-1} T_{up}^{(2)}(y) + \ldots \quad (0 \leq y < y_{f_1}),
\]

\[
C_{eu} = C_{eu}^{(\prime)}(y) + \Theta^{-1} C_{eu}^{(2)}(y) + \ldots \quad (0 \leq y < y_{f_1}),
\]

The coefficient functions \( T_{up}^{(\prime)} \) , \( C_{eu}^{(\prime)} \) satisfy,

\[
\frac{d}{dy} C_{eu}^{(\prime)} + \frac{d}{dy} C_{eu}^{(2)} = 0,
\]

\[
\frac{d}{dy} T_{up}^{(\prime)} + \frac{1}{Le} \frac{d^2}{dy^2} T_{up}^{(\prime)} = 0,
\]

with holder conditions (see equations (23a, 24a) in main text):

\[
T_{up}^{(\prime)}(0) = 0 \quad ; \quad C_{eu}^{(\prime)}(0) - \frac{d}{dy} C_{eu}^{(\prime)} \bigg|_0 = 0.
\]

Since \( T_u \) and \( C_{eu} \) are \( o(1) \), result (A33) implies

\[
T_{up}^{(\prime)}(y_{f_1}) = 0 \quad ; \quad C_{eu}^{(\prime)}(y_{f_1}) = 0
\]

and the only solutions for \( T_{up}^{(\prime)} \), \( C_{eu}^{(\prime)} \) are,

\[
T_{up}^{(\prime)}(y) \equiv 0 \quad ; \quad C_{eu}^{(\prime)}(y) \equiv 0.
\]

If in the equilibrium zone \( (y > y_{f_1}) \) one writes,

\[
T_u = T_{ue}^{(\prime)}(y) + \Theta^{-1} T_{ue}^{(2)}(y) + \ldots
\]

because of downstream boundary conditions, one obtains the same result,

\[
T_{ue}^{(\prime)}(y) \equiv 0.
\]
If one now proceeds to seek lower order solutions but still maintaining continuity, as expressed in (A33), one writes,

\[ T_{up}, \quad C_{ew}^{up} \sim O(\mathcal{O}_{t}^{-1}) \]  
\[ T_{u}, \quad C_{ew} \sim O(\mathcal{O}_{t}^{-1}) \]  
\[ T_{ue} \sim O(\mathcal{O}_{t}^{-1}) \]  

But now it is evident one is in exactly the same position as at leading order. All the terms in (A31,A32) are just one order less in magnitude and again the pre-heat equations will imply

\[ T_{up}^{(\alpha)}(y) \equiv 0 \equiv T_{ue}^{(\alpha)}(y); \quad C_{ew}^{(\alpha)}(y) \equiv 0 \]  

Similar results will be obtained successively at each order. We thus conclude that if one insists on continuity such that (A33) is true, the only possible solution is that \( T_{u} \) and \( C_{ew} \) are identically zero to all orders. This outcome is in violation of the basic hypothesis that there is an \( O(1) \) unsteady disturbance. One is in fact driven to the conclusion that \( T_{u} \) and \( C_{ew} \) are in fact \textit{discontinuous} to \( O(1) \) at the flame sheet. Thus,

\[ \lim_{y \to y_{i}^{-}} \left\{ \begin{array}{c} T_{up} \\ C_{ew}^{up} \end{array} \right\} \neq \lim_{y \to y_{i}^{+}} \left\{ \begin{array}{c} T_{ue} \\ C_{ew}^{ue} \end{array} \right\} \]  

The method used in this appendix will then no longer apply since (A29,A30) will no longer hold true. One proceeds to match solutions in three zones as discussed in §5 in the main text.
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**Fig 1** SCHEMATIC OF ONE-DIMENSIONAL PRE-MIXED FLAME WITH FLAME-HOLDER.
**FIG 2** STABILITY BOUNDARY PREDICTED BY LEADING ORDER THEORY IN 
(Le, yf1) SPACE. SUPERIMPOSED ARE BOUNDARIES OF VALIDITY, FOUND 
AT SECOND ORDER AND REFERRED TO IN THE MAIN TEXT.
Fig. 3. Second order corrections for complex frequency:
The main curve is an Argand diagram for leading order estimate $\omega_0$ of complex frequency for stand off distance $y_{st} = 4$ and varying leading order estimate for Lewis number $L_e$. At four positions are listed the second order corrections for complex frequency $\omega = R_i + z_i$, for different values of $R_i$ (second order estimate for Lewis number). Note $\delta_i^* = 1 - T_{0i}$. 
Fig. 4. Second order corrections for complex frequency: notation as for Fig. 3. Stand off distances $y_{f1} = 1$ and $y_{f1} = 2$. Note scales are different to those used in Fig. 3.