Journal of Computational Finance Volume 16/Number 4, Summer 2013 (3–34)

An efficient pricing algorithm for swing options based on Fourier cosine expansions

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(Received February 4, 2010; revised February 23, 2011; accepted May 3, 2011)

Swing options give contract holders the right to modify amounts of future delivery of certain commodities, such as electricity or gas. We assume that these options can be exercised at any time before the end of the contract, and more than once. However, a recovery time between any two consecutive exercise dates is incorporated as a constraint to avoid continuous exercise. We introduce an efficient way of pricing these swing options, based on the Fourier cosine expansion method, which is especially suitable when the underlying is modeled by a Lévy process.

1 INTRODUCTION

A swing option usually consists of two contract parts: a future part and a swing part. The future contract guarantees that the option seller delivers certain amounts of a commodity (baseload) to the option buyer at certain times, $T_0 < T_1 \leqslant T_2 \cdots \leqslant T_N \leqslant T$, with T the maturity time. The swing part gives the option buyer the right to order extra or deliver back amounts. Usually, the motivation behind the purchase of a swing option is to hedge the uncertainty in the future demand of a commodity. The future part of a swing option can be priced as the discounted expected price of the underlying commodity at the delivery times, whereas the swing part, the focus of the present paper, can vary in contract complexity and is most interesting from a numerical point of view.

In the literature the swing option is often modeled as a Bermudan-style option with swing actions being allowed at the (fixed) delivery times of the baseload, combined with some constraints. Pflug and Broussev (2008) model the bid and ask prices as the least acceptable contract price and the maximal expected profit over demand patterns, respectively, and those prices are determined by stochastic programming. They present an algorithm to find the equilibrium prices from a game-theoretic point of view.

Jaillet *et al* (2003) use a trinomial forest model where a so-called usage level is discretized. Their model is a multiple layer tree which captures the information of the number of exercise rights remaining, the total amount exercised and the price scenario. They move from one tree to another by a swing action. A discrete binomial methodology is also applied by Lari-Lavassani *et al* (2001), where a transition probability matrix is used to calculate the expected profit, to be maximized over different swing actions at each time step.

Carmona and Touzi (2008) view swing options as American-style contingent claims with multiple exercise opportunities and address the problem from the perspective of multiple optimal stopping problems, dealt with by means of Monte Carlo methods and Malliavin calculus. They focus on the Black–Scholes dynamics. Zeghal and Mnif (2006) extend that method to Lévy processes.

Unlike the models in which swing actions are only allowed at discrete times, Dahlgren (2005) proposed a continuous-time model to price the commodity-based swing options. Here the option buyer can exercise the swing option any time before expiry, and more than once, with an upper bound for the maximum amount of additional commodity that can be ordered or delivered back (specified in the contract). After a swing action, the option buyer cannot exercise again unless a recovery time, $\tau_{\mathrm{R}}(D)$, has elapsed, where D represents the amount of commodity and t is the exercise time. This recovery time can be constant, or dependent on the amount of the last swing action. Dahlgren (2005) connected the price of the swing option to a system of discrete variational inequalities of Hamilton–Jacobi–Bellman type, which is solved by means of finite elements and a projected successive over-relaxation (PSOR) algorithm (Cryer 1971). A combination of dynamic programming and a finite difference approximation of the resulting partial integro-differential equation (PIDE) under Lévy jump processes has been presented in Kjaer (2007).

The purpose of the present paper is to develop an efficient alternative solution method for the continuous time model in Dahlgren (2005), which is at least competitive with PIDE solvers or Monte Carlo methods in terms of efficiency, accuracy and flexibility.

Our solution method for the swing option is based on dynamic programming, backward recursion and Fourier cosine expansions, as in Fang and Oosterlee (2008, 2009). For the dynamics of the underlying prices, we employ the Ornstein–Uhlenbeck (OU) mean reverting process, commonly used in commodity derivatives, and the CGMY Lévy jump process (Carr *et al* 2002). The present work can be seen as a generalization, in terms of the financial products, of the work in Fang and Oosterlee (2008, 2009).

The paper is organized as follows. Details of swing options are presented in Section 2. In Section 3, our contribution to pricing swing options is described in detail. We consider both constant and state-dependent recovery times. Numerical results are presented in Section 4. We focus in this paper on the algorithmic description, which is somewhat technical in places. An error analysis is not included here, but it is included in Fang and Oosterlee (2008, 2009) for European and Bermudan options, which are the building blocks of the present swing option algorithm.

2 DETAILS OF THE SWING OPTION

In our discussion, we ignore the future part of the swing option and concentrate on the swing part. Whenever we use the term "swing option", it indicates the swing part of the option.

2.1 Contract details

Our assumptions for the swing option are listed below.

- We adopt the concept of recovery time, denoted by $\tau_{\mathrm{R}}(D)$, which means that if the option buyer has already exercised the swing option with an amount D at time point t, they have to wait $\tau_{\mathrm{R}}(D)$ time before a next swing action can be conducted. Two different models of recovery time will be considered.

  (1) Constant recovery time: if $D \neq 0$, $\tau_{\mathrm{R}}(D) \equiv C$, where C is constant.

  (2) State-dependent recovery time: here the recovery time is assumed to be an increasing function of D, independent of time t, ie, $\tau_{\mathrm{R}}(D) = f(D)$.

  Moreover, $\tau_{\mathrm{R}}(D) = 0$ if and only if $D = 0$, and this holds for both types of recovery time.

- A swing option can be exercised at any time after a recovery time delay until the expiry date T. It implies that we deal with an American-style continuous problem.

- With the constraint of recovery time, a swing option can be exercised more than once before expiry.

- The amount of commodity at each swing action, D, is assumed to be in the range $-L, \dots, -1, 0, 1, \dots, L$, where a negative amount implies back delivery and a positive amount means ordering. The upper bound, L, is necessary as otherwise it may be optimal to order or deliver back an infinite amount of commodity, and thus receive an unrealistic profit.

- The price the option holder has to pay for ordering extra units of the commodity is given by

$$\begin{aligned} S &\quad \text{if } S \leqslant K_{\mathrm{a}}, \\ K_{\mathrm{a}} &\quad \text{if } K_{\mathrm{a}} \leqslant S \leqslant S_{\max}, \\ S - (S_{\max} - K_{\mathrm{a}}) &\quad \text{if } S \geqslant S_{\max}. \end{aligned}$$

  Here S is the price of the underlying commodity, based on a stochastic differential equation for $S(t)$, and the values of the strikes K_{a} and $S_{\max}$ are specified in the contract.

- The price the option holder will receive for delivering back units of the commodity is

$$\begin{aligned} K_{\mathrm{d}} - S_{\min} + S &\quad \text{if } S \leqslant S_{\min}, \\ K_{\mathrm{d}} &\quad \text{if } S_{\min} \leqslant S \leqslant K_{\mathrm{d}}, \\ S &\quad \text{if } S \geqslant K_{\mathrm{d}}, \end{aligned}$$

  where the values of the strikes K_{d} and $S_{\min}$ are also specified in the contract.

Based on the last two assumptions, the payoff function of a swing option is of the form

$$\begin{aligned} g(S, T, D) = D(&\max(S - K_{\mathrm{a}}, 0) - \max(S - S_{\max}, 0) \\ &+ \max(K_{\mathrm{d}} - S, 0) - \max(S_{\min} - S, 0)), \end{aligned} \tag{2.1}$$

with $S = S(T)$. This implies that there can be no profit unless the price of the underlying fluctuates below or above the thresholds K_{d} or K_{a}. The two other thresholds, $S_{\min}$ and $S_{\max}$, are defined to protect an option writer against extreme fluctuations (see Dahlgren 2005). Figure 1 on the facing page shows an example of the payoff for varying S and D.

2.2 Pricing details

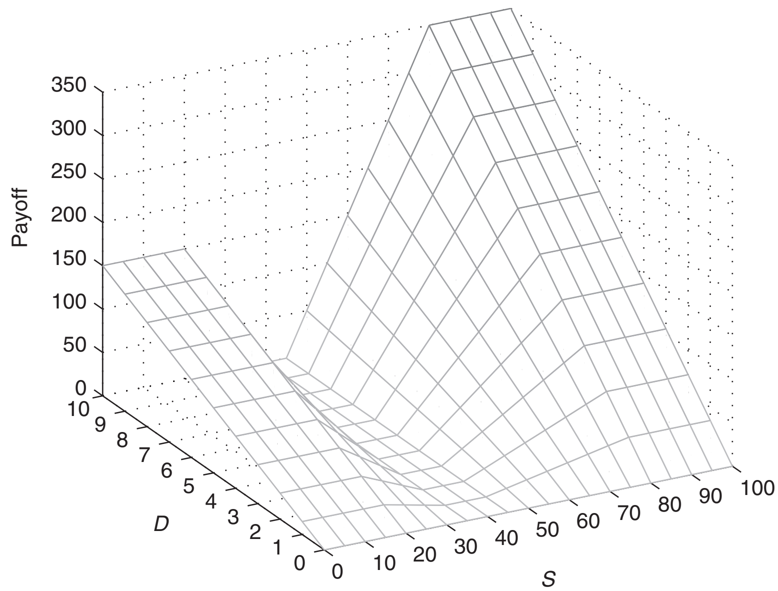
Assume that the first possible time at which a swing action is allowed[1] is T_0: $0 < T_0 < T$. Let

$$n_{\mathrm{s}} := \min\{n \mid n \in \mathbb{N}_+,\ n \geqslant (T - T_0)/\tau_{\mathrm{R}}(1)\}, \tag{2.2}$$

where $\tau_{\mathrm{R}}(1)$ is the recovery time when $D = 1$. Then n_{s} represents the maximum number of swing actions that can be performed in the interval $[T_0, T]$.

[1] If $T_0 > T$ we deal with a futures contract, and with $T_0 = T$ the price of the swing option is just the payoff, $g(S, T, 0)$, if a swing action is not profitable, and $g(S, T, L)$ otherwise.

FIGURE 1 Example of a payoff of a swing option with $S_{\min} = 20$, $K_d = 35$, $K_a = 45$ and $S_{\max} = 80$, and S and D varying.



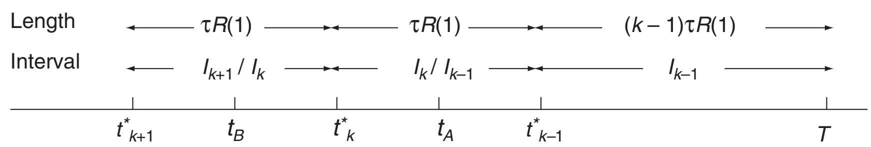
We set $t_n^* := T - n\tau_R(1)$, so that t_n^* is the last point in time for which we can have $n+1$ swing actions, $n = 1, \dots, n_s - 1$. Moreover, let $I_n = (t_n^*, T]$ and $I_{n_s} = [T_0, T]$ be defined as shown in Figure 2 on the next page.[2]

On I_1, there is only one opportunity left for a swing action, which implies that the recovery time has no further influence for the future. Hence, if it is profitable to exercise the swing option during $(t_1^*, T]$ we should exercise the maximum possible amount, L. In this time interval the only issue that needs to be decided is the optimal exercise time. So, the problem is equivalent to an American-style option pricing problem, and the swing option value for any $t \in (t_1^*, T]$ is equal to the value of an American option, starting from t and expiring at T, with payoff $g(S, t', L)$, $t' \in (t, T]$.

At any time $t \in I_{n+1} \setminus I_n$, where $t \neq t_n^*$, $n = 1, \dots, n_s - 1$ (see Figure 2 on the next page), the option holder basically has two options: either exercise the swing option at any time in $[t, t_n^*]$ or do not exercise until t_n^+, the time point immediately after t_n^*.

[2] A division of the time interval into portions $I_{n+1} \setminus I_n$ was first proposed by Dahlgren (2005). Our analysis is based on the appendix in Dahlgren (2005).

FIGURE 2 Division of the time axis and the maximum remaining number of swing rights.



t_A: option holder can exercise maximum k times. t_B: option holder can exercise maximum $k+1$ times.

Note here that the length of interval $I_{n+1} \setminus I_n$ equals $\tau_{\mathrm{R}}(1)$, the recovery time for $D = 1$. It is therefore not possible to exercise more than once within $I_{n+1} \setminus I_n$. In the case of exercise, the problem reduces to the decision of the optimal exercise time within $I_{n+1} \setminus I_n$. So, for each possible amount, D, the problem is equivalent to an American-style option problem, starting at $t \in I_{n+1} \setminus I_n$ and ending at t_n^*, with payoff

$$\bar{g}(S, t', D) = g(S, t', D) + \phi_D^{t'}(S, t'), \quad t' \in [t, t_n^*], \quad t \in I_{n+1} \setminus I_n, \tag{2.3}$$

where

$$\phi_D^{t'}(S, t') = \mathrm{e}^{-r\tau_{\mathrm{R}}(D)} \mathbb{E}_{S,t'}(v(S, t' + \tau_{\mathrm{R}}(D))), \tag{2.4}$$

and $\mathbb{E}_{S,t'}$ represents the conditional expectation of $v(S, t' + \tau_{\mathrm{R}}(D))$ given $S(t')$.

For each possible value of D, ie, $D = -L, \dots, L$, we compute the corresponding value of the swing option at t, assuming that D commodities are bought/sold within $I_{n+1} \setminus I_n$, by an American-style option pricing method. After taking the maximum over all values of D, we obtain the swing option value at $t \in I_{n+1} \setminus I_n$ with $t \neq t_n^*$ if exercise takes place before t_n^+. We denote the corresponding option value by $v_1(S, t)$.

On the other hand, if the option holder decides not to exercise before t_k^+, they have an option worth the discounted expected value:

$$v_2(S, t) = \mathrm{e}^{-r(t_n^+ - t)} \mathbb{E}_{S,t}(v(S, t_n^+)), \quad t \in I_{n+1} \setminus I_n, \tag{2.5}$$

where

$$v(S, t_n^+) = v(S, t_n^* + \delta), \quad 0 < \delta \ll 1.$$

The value $v(S, t_n^+)$ with $t_n^+ \in I_n \setminus I_{n-1}$, has already been obtained at the latest step in the backward recursion. After another, European-type, backward recursion procedure (2.5), value $v_2(S, t)$ is obtained. From the view of a profit maximizing agent, we find that

$$v(S, t) = \max\,(v_1(S, t), v_2(S, t)), \quad t \in I_{n+1} \setminus I_n.$$

Moreover, at each t_n^*, the last time point to perform $n + 1$ swing actions, which is also in $I_{n+1} \setminus I_n$, the option value is the maximum of the payoff $\bar{g}(S, t_n^*, D)$ from (2.3), over all values of D, and the value of $v(S, t_n^+)$.

Finally, for $t \in [0, T_0)$, a time interval in which swing actions are not yet allowed, we have

$$v(S,t) = \mathrm{e}^{-r(T_0-t)}\mathbb{E}_{S,t}(v(S,T_0)),$$

which is computed by one step of a European option pricing algorithm.

This concludes the global description of the algorithm for the swing option pricing method.

Summarizing, we can distinguish two major parts of the pricing algorithm.

- For $t \in (t_1^*, T]$, we are faced with an American option pricing problem with payoff $g(S,t,D)$, given by (2.1), which can take five different forms in five different regions of the spot price of the underlying (see Figure 1 on page 7). As mentioned, if it is profitable to exercise the swing option in this time interval, then $D_{\mathrm{opt}} = L$. Hence the swing option price is the maximum of $g(S,t,L)$ and the continuation value.

- For the other time regions, $t \in [T_0, t_1^*)$, we compute the following two quantities and compare them within each time region $I_{n+1} \setminus I_n$:

  - the value of an American option, $v_1(S,t)$, with payoff $\bar{g}(S,t,D) := g(S,t,D) + \phi_D^t(S,t)$, as in (2.3), and ϕ_D^t as in (2.4);
  - the discounted value $v_2(S,t) = \mathbb{E}_{S,t}(v(S,t_n^+))$.

  For the values $v(S,t_n^+)$ we only have to calculate the value of $v_1(S,t_n^+)$, due to the fact that the discounted value of $\mathbb{E}_{S,t}(v(S,t_{n-1}^+))$ equals $\phi_{D=1}^{t_n^+}(S,t_n^+)$ which is less than (or equal to) the payoff with $D = 1$ (since g is nonnegative), and thus less than (or equal to) the corresponding American option value, $v_1(S,t_n^+)$.

2.3 Commodity processes

The commodity underlying for the swing option is modeled by a stochastic differential equation for $x(t) = \ln S(t)$. State variables x and y are defined as the logarithms of the asset price, $S(t)$:

$$x := \ln(S(t_{m-1})) \quad \text{and} \quad y := \ln(S(t_m)),$$

respectively. Consequently, (2.1) can be rewritten (keeping the same notation, g, for the function based on $x(t)$) as

$$\begin{aligned} g(x,t,D) := D(&\max(\mathrm{e}^x - K_{\mathrm{a}}, 0) - \max(\mathrm{e}^x - S_{\max}, 0) \\ &+ \max(K_{\mathrm{d}} - \mathrm{e}^x, 0) - \max(S_{\min} - \mathrm{e}^x, 0)), \end{aligned} \tag{2.6}$$

with $x = x(t)$. Function $\bar{g}$ from (2.3) can be generalized accordingly, also keeping the same notation, $\bar{g}$, for the function based on $x(t)$.

Two underlying processes are considered in this paper, an exponential OU mean reverting process and a CGMY Lévy jump process.

For the exponential OU process, the log-asset process $x(t) = \log(S(t))$ is assumed to be mean reverting:

$$\mathrm{d}x(t) = \kappa(x(t) - \bar{x})\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \tag{2.7}$$

where κ is speed of mean reversion, $\bar{x}$ is long term mean and σ is the volatility. Moreover, under the risk-neutral measure, we should adjust $\bar{x}$ by subtracting a market price of risk parameter λ from $\bar{x}$, as in Dahlgren (2005).

The characteristic function, $\varphi(\omega; x)$, of the conditional probability density function, $f(y \mid x)$, is defined as

$$\varphi(\omega; x) = \mathbb{E}(\mathrm{e}^{\mathrm{i}\omega y} \mid x). \tag{2.8}$$

The well-known characteristic function for the OU process reads

$$\varphi_{\mathrm{OU}}(\omega; x) = \exp\left(x B_x(\omega, \tau) + A(\omega, \tau)\right), \tag{2.9}$$

with

$$\left.\begin{aligned} B_x(\omega, \tau) &= \mathrm{i}\omega \mathrm{e}^{-\kappa\tau}, \\ A(\omega, \tau) &= \frac{1}{4\kappa}(\mathrm{e}^{-2\kappa\tau} - \mathrm{e}^{-\kappa\tau})(\omega^2\sigma^2 + \omega \mathrm{e}^{\kappa\tau}(\omega\sigma^2 - 4i\kappa\bar{x})). \end{aligned}\right\} \tag{2.10}$$

The CGMY process, as defined in Carr *et al* (2002), is a Lévy jump process, a generalization of the variance gamma process, with the characteristic function

$$\varphi_{\mathrm{CGMY}}(\omega; x) = \exp(\mathrm{i}\omega x)\psi_{\mathrm{CGMY}}(\omega, t), \tag{2.11}$$

where

$$\psi_{\mathrm{CGMY}}(\omega, t) = \exp(tC\Gamma(-Y)[(M - \mathrm{i}\omega)^Y - M^Y + (G + \mathrm{i}\omega)^Y - G^Y]). \tag{2.12}$$

It is governed by four parameters. Parameter $Y < 2$ controls whether the process has finite or infinite activity. Parameter $C > 0$ controls the kurtosis of the distribution, and the nonnegative parameters G, M give control over the rate of exponential decay on the right-hand side and the left-hand side of the density, respectively.

Summarizing, we deal here with two characteristic functions of the form

$$\varphi(\omega; x) = \exp(\beta \mathrm{i}\omega x)\psi(\omega, t), \tag{2.13}$$

in which, for the OU process, $\beta = \exp(-\kappa\Delta t)$, whereas for general Lévy processes, we find $\beta = 1$.

3 FOURIER COSINE ALGORITHM FOR SWING OPTIONS

In Section 2, we argued that the price of a swing option can be obtained by a series of Bermudan- and American-style option pricing procedures. In Fang and Oosterlee (2008, 2009) an efficient algorithm, based on the Fourier cosine series expansion (called the COS algorithm), for European and Bermudan early-exercise options was developed. The COS algorithm can be applied to processes for which the characteristic function is available. In this section, we briefly review the COS algorithm, and extend it to pricing swing options.

3.1 Fourier cosine expansions

We depart from the risk-neutral valuation formula

$$v(x, t_0) = \mathrm{e}^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f(y \mid x) \, \mathrm{d}y,$$

where $v(x, t)$ is the option value, $f(y \mid x)$ is the transitional probability density function, x, y can be any increasing functions of the underlying, $S(t)$, at t_0 and T, respectively, and $\Delta t = T - t_0$. We truncate the integration range to $[a, b]$, so that

$$v(x, t_0) \approx \mathrm{e}^{-r\Delta t} \int_{a}^{b} v(y, T) f(y \mid x) \, \mathrm{d}y, \tag{3.1}$$

with

$$\left| \int_{\mathbb{R}} f(y \mid x) \, \mathrm{d}y - \int_{a}^{b} f(y \mid x) \, \mathrm{d}y \right| < \mathrm{TOL},$$

where TOL is the tolerance level, and choose the following integration range, from Fang and Oosterlee (2008):

$$[a, b] := \left[c_1 - 10\sqrt{c_2 + \sqrt{c_4}}, c_1 + 10\sqrt{c_2 + \sqrt{c_4}} \right], \tag{3.2}$$

where c_n denotes the nth cumulant of $\log S$.

The conditional density function of the underlying is approximated via the characteristic function by a truncated Fourier cosine expansion, as follows:

$$f(y \mid x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1}{}' \mathrm{Re} \left\{ \varphi\left(\frac{k\pi}{b-a}; x \right) \exp\left(-\mathrm{i} \frac{ak\pi}{b-a} \right) \right\} \cos\left(k\pi \frac{y-a}{b-a} \right), \tag{3.3}$$

where $\mathrm{Re}\{\cdot\}$ denotes taking the real part of the input argument.

The prime at the sum symbol in (3.3) indicates that the first term in the expansion is multiplied by one-half. Replacing $f(y \mid x)$ in (3.1) by its approximation in (3.3) and interchanging integration and summation gives us the COS algorithm to approximate the value of a European option (Fang and Oosterlee 2008):

$$v(x, t_0) = \mathrm{e}^{-r\Delta t} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{\varphi\left(\frac{k\pi}{b-a}; x\right) \exp\left\{-\mathrm{i}k\pi \frac{a}{b-a}\right\}\right\} V_k, \tag{3.4}$$

where

$$V_k = \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y$$

is the Fourier cosine coefficient of $v(y, T)$, which is available in closed form for several European option payoff functions.

Formula (3.4) can be directly applied to calculate the value of a European option, but it also forms the basis for the pricing of Bermudan options.

For a Bermudan option the COS algorithm was generalized in Fang and Oosterlee (2009) as follows. Choose t_m, $m = 1, 2, \ldots, \mathcal{M}$, as the "early-exercise dates". The backward recursion dynamic programming scheme for a Bermudan option with $\mathcal{M}$ exercise dates and $T = t_{\mathcal{M}}$ then reads as follows.

For $m = \mathcal{M}, \mathcal{M} - 1, \ldots, 2$,

$$\left.\begin{aligned} c(x, t_{m-1}) &= \mathrm{e}^{-r\Delta t} \int_{\mathbb{R}} v(y, t_m) f(y \mid x)\, \mathrm{d}y, \\ v(x, t_{m-1}) &= \max(\text{payoff}, c(x, t_{m-1})), \end{aligned}\right\} \tag{3.5}$$

followed by

$$v(x, t_0) = \mathrm{e}^{-r\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y \mid x)\, \mathrm{d}y. \tag{3.6}$$

Functions $v(x, t)$, $c(x, t)$ and "payoff" are the option value, the continuation value and the payoff at time t, respectively.

The Fourier cosine series expansion coefficients, V_k, are now time-dependent and their computation requires an efficient algorithm. The algorithm to compute V_k for swing options is discussed in detail in Sections 3.2 and 3.3.

The value of an American option can be obtained by the backward recursion procedure for discrete Bermudan options, explained above, in combination with a Richardson extrapolation procedure. In particular, a four-point repeated Richardson extrapolation scheme using the prices of Bermudan options for four different numbers of exercise dates, $\mathcal{M}$, $2\mathcal{M}$, $4\mathcal{M}$, $8\mathcal{M}$:

$$\hat{v}_{AM}(\mathcal{M}) = \tfrac{1}{21}(64\hat{v}(8\mathcal{M}) - 56\hat{v}(4\mathcal{M}) + 14\hat{v}(2\mathcal{M}) - \hat{v}(\mathcal{M})), \tag{3.7}$$

has been successfully applied in Chang *et al* (2007) and Fang and Oosterlee (2009). Here, $\hat{v}(\mathcal{M})$ denotes the Bermudan option value, $v(x, t_0)$ from (3.6) with $\mathcal{M}$ exercise dates; $\hat{v}_{AM}(\mathcal{M})$ is the approximation for the American option price with the extrapolation based on $\mathcal{M}$ exercise dates.

The COS algorithm exhibits an exponential convergence rate for European and Bermudan options, for asset processes whose conditional density $f(y \mid x) \in C^{\infty}((a,b) \subset \mathbb{R})$.

In the following subsections we generalize the COS algorithm to pricing swing options.

REMARK 3.1 Subscript n in t_n^*, as well as in t_n^+, decreases, from $n_{\mathrm{s}} - 1$ to 1, if we move forward in time, with t from 0 to T (see Figure 2 on page 8). In contrast, subscript m, denoting the early-exercise dates, in t_m (without $*$) increases and goes from 1 to $\mathcal{M}$ if we move forward in time. Furthermore, there are $N_{\mathrm{R}} = \tau_{\mathrm{R}}(1)/\Delta t \equiv \tau_{\mathrm{R}}(1)\mathcal{M}/T$ early-exercise dates in each time interval $I_{n+1} \setminus I_n$, ie, between time points t_{n+1}^* and t_n^*.

3.2 Algorithm for the final time interval, $t \in I_1$

We start the detailed description of our pricing algorithm for swing options by considering the last time interval, defined as I_1 (see Figure 2 on page 8).

As mentioned in Section 2.2, in I_1, the swing option is equivalent to an American option. We can thus generalize the algorithm based on the Fourier cosine expansions for Bermudan options to the swing option payoff and combine it with a four-point repeated Richardson extrapolation to obtain an approximation of an American option price.

3.2.1 Fourier cosine coefficients

At $t_{\mathcal{M}} = T$, we have for the Fourier cosine coefficients of the swing option value

$$V_k(t_{\mathcal{M}}) = G_k(a, \ln(K_{\mathrm{d}}), D) + G_k(\ln(K_{\mathrm{a}}), b, D),$$

with $D = L$, and a, b as in (3.1). Here

$$G_k(x_1, x_2, D) = \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_{\mathcal{M}}, D) \cos\left(k\pi \frac{x-a}{b-a}\right) \mathrm{d}x \tag{3.8}$$

is the Fourier cosine coefficient of the swing option payoff.

In detail, we find, with $D = L$:

$$\begin{aligned} V_k(t_{\mathcal{M}}) = \frac{2L}{b-a}((&K_{\mathrm{d}} - S_{\min})\psi_k(a, \ln(S_{\min})) \\ &+ K_{\mathrm{d}}\psi_k(\ln(S_{\min}), \ln(K_{\mathrm{d}})) - \chi_k(\ln(S_{\min}), \ln(K_{\mathrm{d}})) \\ &+ \chi_k(\ln(K_{\mathrm{a}}), \ln(S_{\max})) - K_{\mathrm{a}}\psi_k(\ln(K_{\mathrm{a}}), \ln(S_{\max})) \\ &+ (S_{\max} - K_{\mathrm{a}})\psi_k(\ln(S_{\max}), b)), \end{aligned} \tag{3.9}$$

with

$$\chi_k(x_1, x_2) = \frac{1}{1 + (k\pi/(b-a))^2}\left(\cos\left(k\pi\frac{x_2 - a}{b-a}\right)\mathrm{e}^{x_2} - \cos\left(k\pi\frac{x_1 - a}{b-a}\right)\mathrm{e}^{x_1} + \frac{k\pi}{b-a}\left(\sin\left(k\pi\frac{x_2 - a}{b-a}\right)\mathrm{e}^{x_2} - \sin\left(k\pi\frac{x_1 - a}{b-a}\right)\mathrm{e}^{x_1}\right)\right), \tag{3.10}$$

and

$$\psi_k(x_1, x_2) = \left(\sin\left(k\pi\frac{x_2 - a}{b-a}\right) - \sin\left(k\pi\frac{x_1 - a}{b-a}\right)\right)\frac{b-a}{k\pi}, \quad k \neq 0, \tag{3.11}$$

and for $k = 0$, $\psi_k(x_1, x_2) = x_2 - x_1$.

At each time step, t_m, $m = \mathcal{M} - 1, \dots, 2$, as in the case of a regular Bermudan option, the log-asset values for which the payoff equals the continuation value are determined by Newton's method. Based on these values we can determine the maximum of the two, as in (3.5). In the case of the swing option, there are two early-exercise points at each time step, as it is profitable to exercise the option when the underlying is less than K_{d} or larger than K_{a}. We denote the lower and upper early-exercise points for time t_m by x_m^{d} and x_m^{a}, respectively. To determine the two early-exercise points by Newton's method, we need the values of $c(x, t_m)$, $g(x, t_m, D)$, $\partial c(x, t_m)/\partial x$ and $\partial g(x, t_m, D)/\partial x$, with the help of the following formulas:

$$c(x, t_m) = \mathrm{e}^{-r\Delta t}\sideset{}{'}\sum_{k=0}^{N-1} \mathrm{Re}\left\{\varphi\left(\frac{k\pi}{b-a}; x\right)\exp\left\{-\mathrm{i}k\pi\frac{a}{b-a}\right\}\right\} V_k(t_{m+1}), \tag{3.12}$$

$$\frac{\partial c(x, t_m)}{\partial x} = \mathrm{e}^{-r\Delta t}\sideset{}{'}\sum_{k=0}^{N-1} \mathrm{Re}\left\{\varphi\left(\frac{k\pi}{b-a}; x\right)\mathrm{i}\beta\frac{k\pi}{b-a} \times \exp\left\{-\mathrm{i}k\pi\frac{a}{b-a}\right\}\right\} V_k(t_{m+1}), \tag{3.13}$$

with $\varphi(\omega; x)$ in (3.12) and (3.13) defined in (2.8). Function g is defined in (2.6) and its derivative is given by the following expression:

$$\frac{\partial g(x, t_m, D)}{\partial x} = \begin{cases} -D\mathrm{e}^x & \text{if } \ln(S_{\min}) \leqslant x \leqslant \ln(K_{\mathrm{d}}), \\ D\mathrm{e}^x & \text{if } \ln(K_{\mathrm{a}}) \leqslant x \leqslant \ln(S_{\max}), \\ 0 & \text{otherwise}. \end{cases} \tag{3.14}$$

Once x_m^{d} and x_m^{a} have been determined, we split the Fourier coefficients V_k into three parts, for $m = \mathcal{M} - 1, \dots, 1$:

$$V_k(t_m) = G_k(a, x_m^{\mathrm{d}}, D) + C_k(x_m^{\mathrm{d}}, x_m^{\mathrm{a}}, t_m) + G_k(x_m^{\mathrm{a}}, b, D),$$

with the Fourier cosine coefficient of the continuation value given by

$$C_k(x_1, x_2, t_m) = \frac{2}{b-a} \int_{x_1}^{x_2} c(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) \mathrm{d}x, \tag{3.15}$$

and $c(x, t_m)$ defined in (3.12), so that the value of $V_k(t_m)$ is obtained from $V_k(t_{m+1})$.

From basic calculus we have that, if $x_m^{\mathrm{d}} < \ln(S_{\min})$,

$$G_k(a, x_m^{\mathrm{d}}, D) = D\frac{2}{b-a}(K_{\mathrm{d}} - S_{\min})\psi_k(a, x_m^{\mathrm{d}}), \tag{3.16}$$

and otherwise

$$\begin{aligned} G_k(a, x_m^{\mathrm{d}}, D) = D\frac{2}{b-a}&((K_{\mathrm{d}} - S_{\min})\psi_k(a, \ln(S_{\min})) \\ &+ K_{\mathrm{d}}\psi_k(\ln(S_{\min}), x_m^{\mathrm{d}}) - \chi_k(\ln(S_{\min}), x_m^{\mathrm{d}})). \end{aligned} \tag{3.17}$$

If $x_m^{\mathrm{a}} > \ln(S_{\max})$, we have

$$G_k(x_m^{\mathrm{a}}, b, D) = D\frac{2}{b-a}(S_{\max} - K_{\mathrm{a}})\psi(x_m^{\mathrm{a}}, b), \tag{3.18}$$

and otherwise

$$\begin{aligned} G_k(x_m^{\mathrm{a}}, b, D) = D\frac{2}{b-a}&(\chi_k(x_m^{\mathrm{a}}, \ln(S_{\max})) - K_{\mathrm{a}}\psi_k(x_m^{\mathrm{a}}, \ln(S_{\max})) \\ &+ (S_{\max} - K_{\mathrm{a}})\psi_k(\ln(S_{\max}), b)), \end{aligned} \tag{3.19}$$

where χ_k and ψ_k are defined by (3.10) and (3.11), respectively.

Next we discuss the computation of $C_k(x_m^{\mathrm{d}}, x_m^{\mathrm{a}}, t_m)$ in (3.15). To determine the value of $C_k(x_1, x_2, t_m)$, we have to compute

$$\begin{aligned} &C_k(x_1, x_2, t_m) \\ &\quad = -\frac{i}{\pi}\mathrm{e}^{-r\Delta t} {\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{\phi\left(\frac{l\pi}{b-a}\right)V_l(t_{m+1})(M_{k,l}^{\mathrm{c}}(x_1, x_2) + M_{k,l}^{\mathrm{s}}(x_1, x_2))\right\}. \end{aligned} \tag{3.20}$$

We can write (3.20) as a matrix–vector product representation, ie,

$$\boldsymbol{C}(x_1, x_2, t_m) = \frac{\mathrm{e}^{-r\Delta t}}{\pi}\{(M_{k,l}^{\mathrm{c}} + M_{k,l}^{\mathrm{s}})\boldsymbol{u}\}, \tag{3.21}$$

where $\{\cdot\}$ denotes taking the imaginary part of the input argument, and

$$\boldsymbol{u} := \{u_l\}_{l=0}^{N-1}, \quad u_l := \varphi\left(\frac{l\pi}{b-a}\right)V_l(t_{m+1}), \quad u_0 = \tfrac{1}{2}\varphi(0)V_0(t_{m+1}). \tag{3.22}$$

Based on the general characteristic function from (2.13), the matrix elements of $M^{\mathrm{c}}_{k,l}(x_1, x_2)$ and $M^{\mathrm{s}}_{k,l}(x_1, x_2)$ are given by

$$M^{\mathrm{c}}_{k,l}(x_1, x_2) = \begin{cases} \dfrac{(x_2 - x_1)\pi}{b-a}, & k = l = 0, \\ \dfrac{1}{(l\beta + k)}\Bigg[\exp\left(\dfrac{((l\beta + k)x_2 - (l+k)a)\pi i}{b-a}\right) \\ \qquad - \exp\left(\dfrac{((l\beta + k)x_1 - (l+k)a)\pi i}{b-a}\right)\Bigg], & \text{otherwise.} \end{cases} \tag{3.23}$$

and

$$M^{\mathrm{s}}_{k,l}(x_1, x_2) = \begin{cases} \dfrac{(x_2 - x_1)\pi i}{b-a}, & k = l = 0, \\ \dfrac{1}{(l\beta - k)}\Bigg[\exp\left(\dfrac{((l\beta - k)x_2 - (l-k)a)\pi i}{b-a}\right) \\ \qquad - \exp\left(\dfrac{((l\beta - k)x_1 - (l-k)a)\pi i}{b-a}\right)\Bigg], & \text{otherwise.} \end{cases} \tag{3.24}$$

The matrixes

$$M_{\mathrm{s}} := \{M^{\mathrm{s}}_{k,l}(x_1, x_2)\}_{k,l=0}^{N-1} \quad \text{and} \quad M_c := \{M^{\mathrm{c}}_{k,l}(x_1, x_2)\}_{k,l=0}^{N-1}$$

have a Toeplitz and a Hankel structure, respectively, if and only if, for all k, l, x_1, x_2,

$$M^{\mathrm{s}}_{k,l}(x_1, x_2) = M^{\mathrm{s}}_{k+1,l+1}(x_1, x_2)$$

and

$$M^{\mathrm{c}}_{k,l}(x_1, x_2) = M^{\mathrm{c}}_{k+1,l-1}(x_1, x_2).$$

In that case, the fast Fourier transform (FFT) can be applied directly for highly efficient matrix–vector multiplication (Fang and Oosterlee 2009), and the resulting computational complexity[3] will be $O(N \log_2 N)$. However, we obtain terms of the form $l\beta - k, l\beta + k$ in the matrix elements in (3.23) and (3.24), in particular for the OU process with $\beta \neq 1$, instead of terms with $l - k, l + k$ as obtained for the Lévy jump processes, with $\beta = 1$ in (2.13). Terms with $\beta \notin \mathbb{N} \cup \{0\}$ lead to computations with $O(N^2)$ complexity.

Since the computation of $G_k(x_1, x_2)$ is linear in N, the overall complexity to determine the V_k-coefficients is dominated by the computation of $C(x_1, x_2, t_m)$, whose complexity is $O(N \log_2 N)$ with the FFT when $\beta = 1$. As a result, the overall

[3] To be precise, we apply forward FFT (FFT) three times, and the inverse FFT (FFT^{-1}) twice.

computational complexity for pricing a Bermudan option with $\mathcal{M}$ exercise dates is $O((\mathcal{M}-1)N\log_2 N)$ in that case, as the work needed for the final step, from t_1 to t_0, is $O(N)$.

Although the algorithm above is only the first step toward solving the pricing problem, it can also be viewed as the complete algorithm for swing options if the option holder is only allowed to conduct a swing action once.

3.3 Algorithm for interval $t \in I_{n_\mathrm{s}} \setminus I_1$

Recall that n_s represents the upper bound for the number of swing rights that can be exercised, as defined in (2.2). In the time interval $I_{n_\mathrm{s}} \setminus I_1$, the option holder has more than one possibility to exercise the swing option. Therefore, apart from the exercise time, the optimal number of commodities to be exercised, D, should also be determined, due to its influence on the recovery time.

REMARK 3.2 In our discussion we deal with the following three functions.

- $c(x, t_m)$, the continuation value, which is typically continuous and differentiable. Moreover, its derivative is usually also continuous.

- $g(x, t_m, D)$, the payoff, which is continuous and piecewise differentiable (see Figure 1 on page 7).

- $v(x, t_m)$, the option value, which is piecewise continuous in time. $v(x, t)$ jumps at t_n^*, where the number of swing rights is decreased by 1.

Note that the equality $v(t_n^*) = v(t_n^+)$ may not hold, since the number of possible exercise times is reduced by 1 from t_n^* to t_n^+. The definition of t_n^+ simply implies that, numerically, we use $t_n^+ = t_n^*$. Therefore, numerically $t_n^* - t = t_n^+ - t$, and $c(x, t_n^*) = c(x, t_n^+)$, but $v(x, t_n^*) \geqslant v(x, t_n^+)$.

Under these assumptions we have that

$$\mathrm{e}^{-r(t_n^*-t)}\mathbb{E}_{x,t}(v(x, t_n^*)) \geqslant \mathrm{e}^{-r(t_n^+-t)}\mathbb{E}_{x,t}(v(x, t_n^+)).$$

3.3.1 Model analysis

By Q and Q_n we denote the continuous interval $\{(x, t) \mid x \geqslant 0, t \in [T_0, t_1^*]\}$ and the discrete set

$$\{(x, t) \mid x \geqslant 0,\ t \in [T_0, t_1^*],\ t \equiv t_n^* := T - n\tau_\mathrm{R}(1),\ n = 1, \dots, n_\mathrm{s} - 1\},$$

respectively.

The swing option value for $(x,t) \in Q \setminus Q_n$ is then given by

$$v(x,t) = \max\Big(\max_D \tilde{v}_{AM}(\bar{g}(x,t,D)), \mathrm{e}^{-r(t_n^+-t)}\mathbb{E}_{x,t}(v(x,t_n^+))\Big), \qquad (x,t) \in Q \setminus Q_n, \tag{3.25}$$

where $\tilde{v}_{AM}(\bar{g}(x,t,D))$ represents the value of an American-style option in any interval $I_{n+1} \setminus I_n$ with payoff

$$\bar{g}(x,t,D) = g(x,t,D) + \phi_D^t(x,t).$$

The quantity $\mathrm{e}^{-r(t_n^+-t)}\mathbb{E}_{x,t}(v(x,t_n^+))$ represents the value of a European option, which cannot be larger than the American option. The term $\mathrm{e}^{-r(t_n^+-t)}\mathbb{E}_{x,t}(v(x,t_n^+))$ is therefore implicitly already included in the first term in (3.25), so that we find, for (3.25),

$$\begin{aligned} v(x,t) &= \max_D \tilde{v}_{AM}(g(x,t,D) + \phi_D^t(x,t)) \\ &= \max_D(\max(g(x,t,D) + \phi_D^t(x,t), c(x,t))) \\ &= \max\Big(\max_D g(x,t,D) + \phi_D^t(x,t), c(x,t)\Big), \quad (x,t) \in Q \setminus Q_n, \end{aligned} \tag{3.26}$$

where $c(x,t)$ is the continuation value. Therefore, the price for $(x,t) \in Q \setminus Q_n$ is reduced to the maximum of American option values over D, ie, $v_1(x,t)$ as defined in Section 2.2.

On the other hand, for $(x,t_n^*) \in Q_n$, the value $v(x,t_n^*)$ is defined by

$$v(x,t_n^*) = \max\Big(\max_D \bar{g}(x,t_n^*,D), v(x,t_n^+)\Big). \tag{3.27}$$

After application of (3.26) to the right-hand side of (3.27), we can rewrite (3.27) as

$$v(x,t_n^*) = \max\Big(\max_D \bar{g}(x,t_n^*,D), \max_D \bar{g}(x,t_n^+,D), c(x,t_n^+)\Big), \tag{3.28}$$

where we assume $c(x,t_n^+) = c(x,t_n^*)$, and $\bar{g}$ is as in (2.3) and (2.4).

If $(x, t_n^* + \tau_{\mathrm{R}}(D)) \in Q \setminus Q_n$, with the number of exercise possibilities the same for $t_n^* + \tau_{\mathrm{R}}(D)$ and $t_n^+ + \tau_{\mathrm{R}}(D)$, we have

$$v(x, t_n^* + \tau_{\mathrm{R}}(D)) = v(x, t_n^+ + \tau_{\mathrm{R}}(D)).$$

If $t_n^* + \tau_{\mathrm{R}}(D) \in Q_n$, we have

$$v(x, t_n^* + \tau_{\mathrm{R}}(D)) \geqslant v(x, t_n^+ + \tau_{\mathrm{R}}(D)).$$

So, $v(x, t_n^* + \tau_R(D)) \geqslant v(x, t_n^+ + \tau_R(D))$ for any x, thus from (2.4) we have

$$\phi_D^{t_n^*}(x, t_n^*) \geqslant \phi_D^{t_n^+}(x, t_n^+).$$

Equation (3.28) is now given by

$$v(x, t_n^*) = \max\Big(\max_D g(x, t_n^*, D) + \phi_D^{t_n^*}(x, t_n^*), c(x, t_n^*)\Big). \tag{3.29}$$

As a result, from (3.26) and (3.29), we find that for all $t \in [T_0, t_1^*]$:

$$v(x, t) = \max\Big(\max_D g(x, t, D) + \phi_D^{t}(x, t), c(x, t)\Big). \tag{3.30}$$

Equation (3.30) tells us that the swing option is an American-style option with recovery time and multiple exercise opportunities. Its pricing algorithm is therefore different from a standard American option. Instead of taking the maximum of the payoff and the continuation value, we take the maximum over the resulting payoff for all possible values of D, and the continuation value from the previous time step. Another difference is that for any amount, D, the payoff also includes the term $\phi_D^t(x, t)$ from an earlier time step.

It is easy to determine the value of $g(x, t, D)$ for any x, t, D according to (2.6). We therefore focus on the values $\phi_D^t(x, t)$ and $c(x, t)$, which are both obtained in the recursion of Fourier cosine coefficients V_k. To calculate $c(x, t_m)$, one only needs the values of $V_k(t_{m+1})$, like in the case of a Bermudan option. However, to compute the value of $\phi_D^t(x, t)$ we need the coefficients $V_k(t + \tau_R(D))$, which depend on the function for the recovery time.

REMARK 3.3 In time interval $t \in [0, T_0]$ swing actions are not yet allowed. Therefore, we have

$$v(t, x) = \mathrm{e}^{-r(T_0 - t)} {\sum_{k=0}^{N-1}}' \operatorname{Re}\left\{\varphi\left(\frac{k\pi}{b-a}; x\right) \exp\left\{-\mathrm{i}k\pi \frac{a}{b-a}\right\}\right\} V_k(T_0),$$

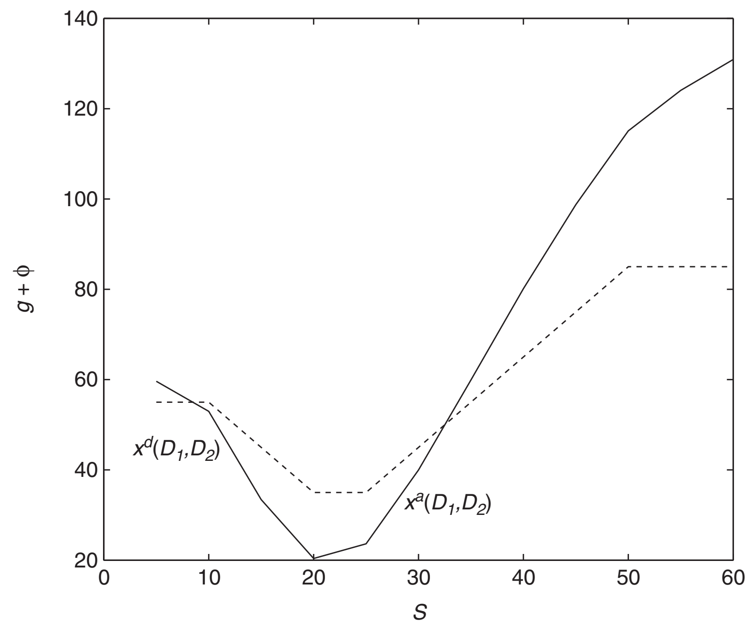
where $V_k(T_0)$ is obtained by a backward recursion procedure.

3.4 The early-exercise points

In this section we consider the state-dependent recovery time, $\tau_R(D)$, which is assumed to be an increasing function of D.

The option value is obtained by means of a backward recursion on $V_k(t_m), m = \mathcal{M} - 1, \dots, 1$. At each time step, as shown in Section 3.3.1, the payoff, $\bar{g}(x, t_m, D)$, for all possible values of D and the continuation value, $c(x, t_m)$, are compared. The largest value represents the swing option value at t_m. We therefore need to identify the following regions in our pricing domain.

FIGURE 3 Payoff function $g + \phi$ for two different D.



Solid line: D_1. Dashed line: D_2.

- A_D, $D = 1, \dots, L$: the regions in which exercising the swing option with D commodity units will result in the highest profit $g(x, t_m, D) + \phi_D^{t_m}(x, t_m)$.
- A_c: the region in which $c(x,t)$ is the maximum. In other words, with the commodity price in A_c, it is profitable not to exercise the swing option.

With these regions determined, the Fourier cosine coefficients, $V_k(t_m)$, for the swing option can be determined with a splitting, as follows:

$$V_k(t_m) = \frac{2}{b-a}\biggl(\int_{A_c} c(x, t_{m+1}) \cos\left(\frac{k\pi(x-a)}{b-a}\right) \mathrm{d}x + \sum_{D=1}^{L} \int_{A_D} g(x, t_m, D) \cos\left(\frac{k\pi(x-a)}{b-a}\right) \mathrm{d}x\biggr) \tag{3.31}$$

We now describe the procedure to determine the different regions A_c and A_D, $D = 1, \dots, L$. As an example, let us first look at the payoff functions for two values $D = D_1$ and $D = D_2$, where $D_1 > D_2$, shown in Figure 3.

Points $x^{\mathrm{d}}(D_1, D_2)$ and $x^{\mathrm{a}}(D_1, D_2)$ denote the "early-exercise points", where the strategy of exercising D_1 or D_2 units results in the same $\bar{g}$-values. Between $x^{\mathrm{d}}(D_1, D_2)$ and $x^{\mathrm{a}}(D_1, D_2)$, the value for D_2 is largest, in other words, it is profitable to exercise a smaller amount of commodity. Beyond $x^{\mathrm{d}}(D_1, D_2)$ and $x^{\mathrm{a}}(D_1, D_2)$, it is profitable to exercise the larger amount D_1.

REMARK 3.4 An explanation of the behavior of the two payoff functions in Figure 3 on the facing page is as follows. The payoff, $\bar{g}(x, t, D)$, is the sum of $g(x, t, D)$ and $\phi_D^t(x, t)$. For D increasing, the true payoff $g(x, t, D)$ increases, but the quantity $\phi_D^t(x, t)$ decreases because of the time penalty due to the longer recovery.

For all $0 < D_2 < D_1$, we have

$$\bar{g}(x, t, D_1) - \bar{g}(x, t, D_2) = g(x, t, D_1) - g(x, t, D_2) - (\phi_{D_2}^t(x, t) - \phi_{D_1}^t(x, t)). \tag{3.32}$$

With the underlying between K_{d} and K_{a}, we have $g(x, t, D_1) = 0$, $g(x, t, D_2) = 0$ and $\phi_{D_2}^t(x, t) > \phi_{D_1}^t(x, t)$. Therefore, $\bar{g}(x, t, D_1) < \bar{g}(x, t, D_2)$ and it is more profitable to exercise the smaller amount, D_2.

From (2.6) we find for the derivative:

$$\begin{aligned}\frac{\partial g}{\partial D} &= (\max(\mathrm{e}^x - K_{\mathrm{a}}, 0) - \max(\mathrm{e}^x - S_{\max}, 0) \\ &\qquad\qquad + \max(K_{\mathrm{d}} - \mathrm{e}^x, 0) - \max(S_{\min} - \mathrm{e}^x, 0)) \\ &\equiv g(x, t, 1),\end{aligned}$$

which increases as $S = \mathrm{e}^x$ decreases/increases beyond K_{d} or K_{a}, until payoff $g(x, t, 1)$ reaches its upper bound (see Figure 1 on page 7). Therefore, if, before x reaches $\log(S_{\min})$ or $\log(S_{\max})$, we have, for some x,

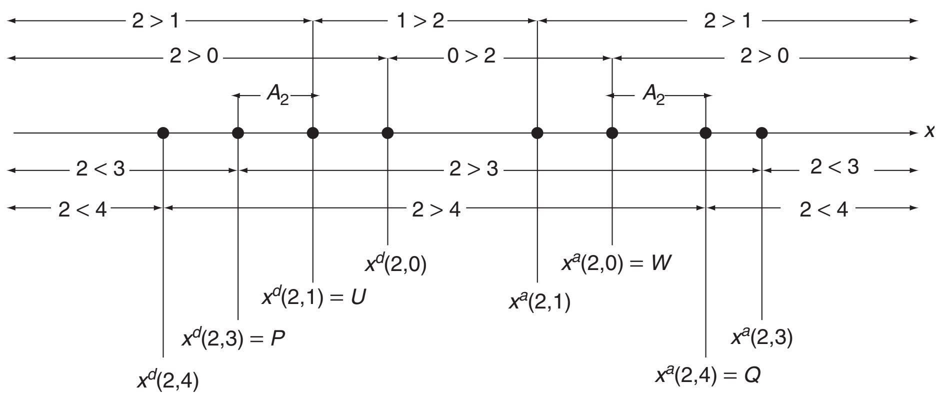
$$\frac{\partial g(x, t, D)}{\partial D} > \frac{\partial \phi_D^t(x, t)}{\partial D},$$

this implies that the payoff function $g(x, t, D)$ is more sensitive with respect to variation in D than function $\phi_D^t(x, t)$, and it is thus more profitable to exercise at the larger amount D_1.

Based on the insight in Remark 3.4, let us look at a second example with $L = 4$ and we will determine A_2, ie, the region where it is most profitable to exercise two units. The example is detailed in Figure 4 on the next page, where the relation between the payoffs for any two different amounts of commodity is graphically sketched. Figure 4 on the next page is a one-dimensional picture with only the x axis, which consists of different sections, where purchasing two different amounts of commodity is compared on each horizontal line in Figure 4.

In Figure 4 on the next page, "0" denotes the continuation value $c(x, t)$. Point sets $x^{\mathrm{d}}(2, D_j)$ and $x^{\mathrm{a}}(2, D_j)$, $j = 0, 1, 3, 4$, are the two sets of points for which $D = D_j$

FIGURE 4 An example to illustrate the exercise region A_2 with $L = 4$.



gives the same payoff value as $D = 2$. In order to determine the region A_2, we need to find the subregions in which $D = 2$ gives the largest payoff compared with the other D-values.

The value $D = 2$ returns a larger value than $c(x,t)$, if $x < x^{\mathrm{d}}(2,0)$ or $x > x^{\mathrm{a}}(2,0)$; Similarly, $D = 2$ returns a larger value than $D = 1$, if $x < x^{\mathrm{d}}(2,1)$ or $x > x^{\mathrm{a}}(2,1)$. So, $D = 2$ returns larger values than both $c(x,t)$ and $D = 1$, if x is either smaller than both $x^{\mathrm{d}}(2,0)$ and $x^{\mathrm{d}}(2,1)$, or larger than both $x^{\mathrm{a}}(2,0)$ and $x^{\mathrm{a}}(2,1)$. To determine these regions we compute the following early-exercise points (see again Figure 4 for the values of U and W for this example):

- $U := \min(x^{\mathrm{d}}(2,0), x^{\mathrm{d}}(2,1)) \equiv x^{\mathrm{d}}(2,1)$; and
- $W := \max(x^{\mathrm{a}}(2,0), x^{\mathrm{a}}(2,1)) \equiv x^{\mathrm{a}}(2,0)$.

$D = 2$ now returns a larger value for $x < U$ or $x > W$.

We proceed in the same spirit. To make sure that $D = 2$ returns larger values than $D = 3$ and $D = 4$, x should be larger than both $x^{\mathrm{d}}(2,3)$ and $x^{\mathrm{d}}(2,4)$, or smaller than both $x^{\mathrm{a}}(2,3)$ and $x^{\mathrm{a}}(2,4)$. This is again related to the global behavior of the payoff functions with $D_1 > D_2$, as in Figure 3 on page 20. Therefore, we calculate

- $P := \max(x^{\mathrm{d}}(2,3), x^{\mathrm{d}}(2,4)) \equiv x^{\mathrm{d}}(2,3)$ and
- $Q := \min(x^{\mathrm{a}}(2,3), x^{\mathrm{a}}(2,4)) \equiv x^{\mathrm{a}}(2,4)$.

Now $D = 2$ returns a larger value than $D = 3$ and $D = 4$ for $x > P$ or $x < Q$.

So, $D = 2$ returns the largest value, if $P < x < U$ or $W < x < Q$. Therefore, $A_2 = [P, U] \cup [W, Q]$, as shown in Figure 4.

More generally, for each $D = 1, \dots, L$, we determine

$$P_D = \max_{j>D} x^{\mathrm{d}}(D, j), \qquad Q_D = \min_{j>D} x^{\mathrm{a}}(D, j),$$
$$U_D = \min_{j<D} x^{\mathrm{d}}(D, j), \qquad W_D = \max_{j<D} x^{\mathrm{a}}(D, j),$$

and set $A_D = [P_D, U_D] \cup [W_D, Q_D]$. Here P_D, Q_D represent the early-exercise interval boundaries, within which exercising D units of commodity returns a larger payoff than exercising more units. U_D, W_D are the left and the right boundary, respectively, beyond which exercising D units returns a larger value than when fewer or no units are exercised. Similarly, we have

$$A_L = \Big[a, \min_{j<L} x^{\mathrm{d}}(L, j)\Big] \cup \Big[\max_{j<L} x^{\mathrm{a}}(L, j), b\Big],$$
$$A_c = \Big[\max_{j>0} x^{\mathrm{d}}(0, j), \min_{j>0} x^{\mathrm{a}}(0, j)\Big].$$

All early-exercise points, $x^{\mathrm{d}}(D, j), x^{\mathrm{a}}(D, j),\ j = 0, \dots, L$, are computed by Newton's method.

With the regions A_c and A_D, $D = 1, \dots, L$ fixed, (3.31) can be rewritten as

$$\begin{aligned} V_k(t_m) = {} & C_k\Big(\max_{j=1,\dots,L} x^{\mathrm{d}}(0, j), \min_{j=1,\dots,L} x^{\mathrm{a}}(0, j), t_m\Big) + \sum_{D=1}^{L} G_k(P_D, U_D, D) \\ & + \sum_{D=1}^{L} G_k(W_D, Q_D, D) + G_k\Big(a, \min_{j=0,\dots,L-1} x^{\mathrm{d}}(L, j), L\Big) \\ & + G_k\Big(\max_{j=0,\dots,L-1} x^{\mathrm{a}}(L, j), b, L\Big). \end{aligned} \tag{3.33}$$

The computation of $C_k(x_1, x_2, t_m)$ in (3.33) is as in (3.21). The G_k differ from the expressions (3.16)–(3.19), which will be described in detail in Section 3.4.1.

In the Newton procedure to find the points $x^{\mathrm{d}}(D_i, D_j)$ and $x^{\mathrm{a}}(D_i, D_j)$ we need to find the values of $c(x, t_m)$, $g(x, t_m, D)$, $\partial c/\partial x$ and $\partial g/\partial x$ as in Section 3.2. The values of $\phi_D^{t_m}(x, t_m)$ and $\partial \phi_D^{t_m}/\partial x$ are found by

$$\begin{aligned} \phi_D^{t_m}(x, t_m) = {} & \mathrm{e}^{-r\tau_{\mathrm{R}}(D)} \\ & \times \sum_{k=0}^{N-1}{}' \operatorname{Re}\Big\{\varphi\Big(\frac{k\pi}{b-a}; x, \tau_{\mathrm{R}}(D)\Big) \exp\Big\{-\mathrm{i}k\pi \frac{a}{b-a}\Big\}\Big\} \\ & \times V_k(t_m + \tau_{\mathrm{R}}(D)), \end{aligned}$$

$$\frac{\partial \phi_D^{t_m}}{\partial x} = \mathrm{e}^{-r\tau_{\mathrm{R}}(D)} \times \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{\varphi\left(\frac{k\pi}{b-a}; x, \tau_{\mathrm{R}}(D)\right)\mathrm{i}\beta\frac{k\pi}{b-a}\exp\left\{-\mathrm{i}k\pi\frac{a}{b-a}\right\}\right\} \times V_k(t_m + \tau_{\mathrm{R}}(D)).$$

REMARK 3.5 (Computation of $V_k(t_m + \tau_{\mathrm{R}}(D))$) To calculate $V_k(t_m + \tau_{\mathrm{R}}(D))$, we determine a time step, Δt, so that $T - t$ and $\tau_{\mathrm{R}}(D)$ are both time points. So, we set $\mathcal{M} = T - t/\Delta t$, $N_D = \tau_{\mathrm{R}}(D)/\Delta t$, $D = 1, \dots, L$. For $t_m + \tau_{\mathrm{R}}(D) = t_m + N_D \Delta t \leqslant T$, the value $V_k(t_m + \tau_{\mathrm{R}}(D)) = V_k(t_{m+N_D})$. The values $V_k(t_m + \tau_{\mathrm{R}}(D)) = 0$ for all k if $t_m + N_D \Delta t > T$. In that case, $\phi_D^{t_m}$ and $\partial \phi_D^{t_m}/\partial x$ are zero, as they are linear combinations of $V_k(t_m + \tau_{\mathrm{R}}(D))$. In this setting, $V_k(t_m)$ and $V_k(t_m + \tau_{\mathrm{R}}(D))$, $D = 1, \dots, L$ can be determined in one recursion, in which the intermediate values of V_k need to be stored for later use.

3.4.1 Calculation of $G_k(x_1, x_2, D)$

The terms G_k in (3.31) are split into two parts, ie,

$$G_k(x_1, x_2, D) = G_{k,g}(x_1, x_2, D) + G_{k,c}(x_1, x_2, D),$$

with $G_{k,g}$ from an instantaneous profit $g(x, t_m, D)$, and $G_{k,c}$ the part generated by $\phi_D^{t_m}(x, t_m)$, ie, the continuation value from time point $t_m + \tau_{\mathrm{R}}(D)$, as defined in (2.4).

Equations (3.16) and (3.17) can be used to compute $G_{k,g}(a, \min_{j<L} x^{\mathrm{d}}(L, j), L)$ and $G_{k,g}(P_D, U_D, D)$, $D = 1, \dots, L$, unless $P_D > \ln(S_{\min})$, where we use

$$G_{k,g}(P_D, U_D, D) = D\frac{2}{b-a}(K_{\mathrm{d}}\psi_k(P_D, U_D) - \chi_k(P_D, U_D)).$$

Similarly, the quantities $G_k(\max_{j<L} x^{\mathrm{a}}(L, j), b, L)$ and $G_k(W_D, Q_D, D)$, $D = 1, \dots, L$ can be computed by (3.18) and (3.19), unless if $Q_D < \ln(S_{\max})$ for which we have

$$G_{k,g}(W_D, Q_D, D) = D\frac{2}{b-a}(\chi_k(W_D, Q_D) - K_{\mathrm{a}}\psi_k(W_D, Q_D)).$$

Finally, the quantity $G_{k,c}(x_1, x_2, D)$ can be obtained by (3.21), replacing Δt and $V_l(t_{m+1})$ by $\tau_{\mathrm{R}}(D)$ and $V_l(t_m + \tau_{\mathrm{R}}(D))$, respectively.

REMARK 3.6 (Early-exercise points and convergence) The accurate determination of the early-exercise points, and the consistent pricing of Bermudan-style swing options forms the basis for the valuation of the American-style swing options by means of the Richardson extrapolation scheme. Only with an accurate location of the early-exercise points we can benefit from extrapolation techniques which rely heavily on (consistent) asymptotic expansions.

The main components of the swing option pricing algorithm here are those that have also been used for pricing Bermudan and barrier options with Fourier cosine expansions in Fang and Oosterlee (2009). The convergence of the swing option algorithm is therefore expected to be the same as that for Bermudan options, which has been studied in detail in Fang and Oosterlee (2009).

REMARK 3.7 (Constant recovery time) If the recovery time does not depend on D, we call the recovery time constant. This can be viewed as a special case of the pricing method discussed above. As additional profit is not related to an extra penalty, if it is profitable to exercise the swing option, we have $D_{\mathrm{opt}} \equiv L$ from a profit maximizing point of view. Hence, at any point in time, we have either $D = 0$ or $D = L$.

Newton's method is now applied to determine two early-exercise points x_m^{d} and x_m^{a}, so that

$$c(x_m^{\mathrm{d}}, t_m) = g(x_m^{\mathrm{d}}, t_m, L) + \phi_L^{t_m}(x_m^{\mathrm{d}}, t_m),$$

and

$$c(x_m^{\mathrm{a}}, t_m) = g(x_m^{\mathrm{a}}, t_m, L) + \phi_L^{t_m}(x_m^{\mathrm{a}}, t_m),$$

with $D = L$ and $\tau_{\mathrm{R}}(D)$ constant. Then $V_k(t_m)$ is split into three parts,

$$V_k(t_m) = G_k(a, x_m^{\mathrm{d}}, L) + C_k(x_m^{\mathrm{d}}, x_m^{\mathrm{a}}, t_m) + G_k(x_m^{\mathrm{a}}, b, L),$$

which can be calculated as in the case of state-dependent recovery time.

4 NUMERICAL RESULTS

In this section we demonstrate the performance of our pricing algorithm for swing options with constant and dynamic recovery times. The CPU used is an Intel Core 2 Duo CPU E6550 at 2.33GHz, Cache size 4MB, and the algorithm is programmed in MATLAB 7.5. The two subsections to follow present results with two different types of recovery time.

- Constant recovery time is presented in Section 4.1. If $D \neq 0$, we set $\tau_{\mathrm{R}}(D) = \frac{1}{4}$, as in Dahlgren (2005). In other words, the option holder needs to wait three months between two consecutive swing actions, independent of the time point of exercise or the size D.

- State-dependent recovery time is presented in Section 4.2. We assume $\tau_{\mathrm{R}}(D) = \frac{1}{12}D$, which implies that if the option holder exercises the swing option with D units, they have to wait D months before the option can be exercised again.

Parameter sets used for numerical examples are (unless stated otherwise):

$$\text{CGMY}: \quad C = 1, \quad G = 5, \quad M = 5, \quad Y = 1.5, \quad r = 0.05, \tag{4.1}$$

$$\text{OU}: \quad \kappa = 0.301, \quad \bar{x} = 3.150, \quad \sigma = 0.334, \quad r = 0.05, \tag{4.2}$$

where for the OU process the value of $\bar{x}$ is defined under the Q-measure. The value set for the OU process is as in Dahlgren (2005). The values for CGMY, in particular $Y > 1$ (infinite activity jump process), are known to be particularly difficult for PIDE solvers. We will see here that these CGMY parameters do not pose any problem for the swing option COS method.

In the numerical experiments we further choose $S_{\text{min}} = 10, K_{\text{d}} = 20, K_{\text{a}} = 25, S_{\text{max}} = 50, T_0 = 0$. The choice $T_0 = 0$ does not pose any restrictions on the algorithm, as we can simply change it to any $T_0 > 0$.

4.1 Constant recovery time

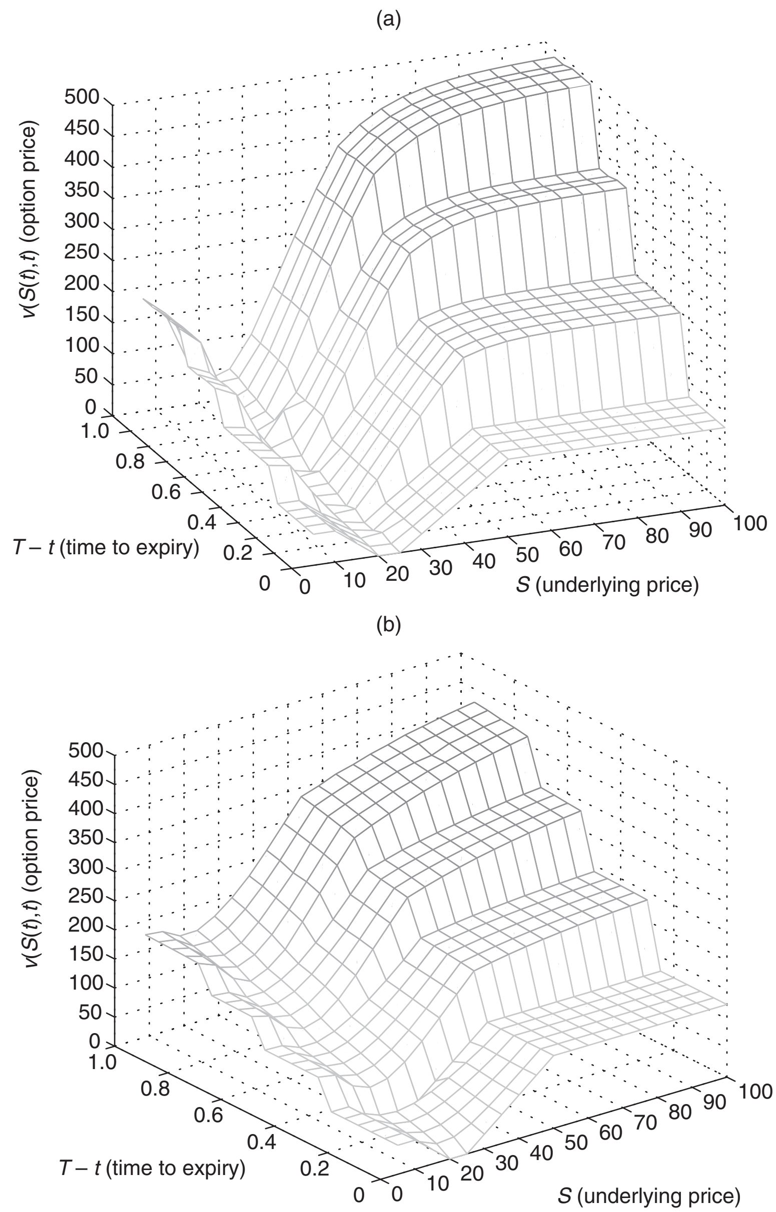
First of all, American-style swing option values under the CGMY and OU processes, with $L = 5$, are presented in Figure 5 on the facing page, with S and t as independent variables; $v(S, t)$ is the swing option value. Jumps in the swing option values are observed at $t = 0.25$, $t = 0.5$ and $t = 0.75$. This can be explained by the fact that at these time points the maximum number of times the holder can exercise, n_{s}, is reduced by one. For instance, time point $t = 0.5$ is the last time point at which an option holder can exercise up to three times. For any $t > T - 0.5$, the holder cannot exercise more than twice.

Due to the constant recovery time, we should exercise $L = 5$ units whenever it is profitable to exercise. Hence for $S > 50$, with $K_{\text{a}} = 25$, the profit would be $L(50 - 25) = 125$. When $T - t \in [0.75, 1)$, we have at maximum four possibilities to exercise, which is the reason for option values as high as 500 in Figure 5 on the facing page.

Next, we discuss the convergence behavior of the option values over N, the number of terms in the Fourier cosine series. The CGMY and OU processes are used with the parameters in (4.1) and (4.2). The remaining parameters are $\tau_{\text{R}} = 0.25$, $T = 1$, $\mathcal{M} = 12$ and $S_0 = 8$.

In Table 1 on page 28 it is shown that the swing option pricing algorithms for the CGMY and OU processes, with the parameters chosen, take 0.024 and 0.19 seconds, respectively, to converge to one basis point accuracy. The CPU time is higher for the OU process as its computational complexity is of higher order than for the Lévy processes. The convergence behavior for both processes is very similar, as shown in Table 1 on page 28.

FIGURE 5 American-style swing option values under the OU and CGMY processes with constant recovery time, $\tau_{\mathrm{R}}(D) = 0.25$.



(a) OU process. (b) CGMY process.

TABLE 1 Swing option prices and CPU time under the CGMY and OU processes, with parameter sets (4.1) and (4.2).

	N				
	64	**96**	**128**	**160**	**192**
CGMY option value	190.2045	229.0515	229.0515	229.0515	229.0515
CPU time (s)	0.0191	0.0234	0.0266	0.0304	0.0402
OU option value	225.9100	225.9100	225.9100	225.9100	225.9100
CPU time (s)	0.1891	0.3994	0.7147	1.1638	1.7590

TABLE 2 Convergence over $\mathcal{M}$ and comparison between two approximation methods for American-style swing options.

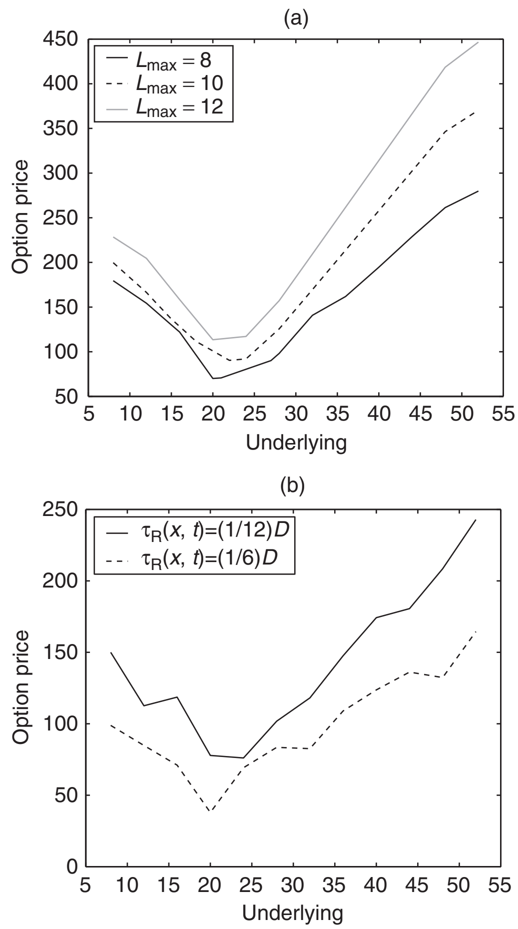
	$P(N/2)$		Richardson	
$n = \log_2 N$	**Option value**	**CPU time (s)**	**Option value**	**CPU time (s)**
7	137.423	0.27	137.395	0.59
8	137.408	0.53	137.390	0.99
9	137.399	2.00	137.390	1.79
10	137.394	8.39	137.390	3.40
11	137.392	39.55	137.390	6.68
12	137.391	203.27	137.390	13.21

An American option can be viewed as a Bermudan option with $\mathcal{M} \to \infty$. In Table 2 the performance of two methods to approximate an American-style swing option is compared. One method is the direct approximation by means of Bermudan-style options by increasing $\mathcal{M}$, whereas the second method is based on the repeated Richardson four-point extrapolation technique (3.7) on Bermudan-style swing options with four different numbers of exercise opportunities. In Table 2, the columns labeled "$P(N/2)$" give the computed values of the Bermudan-style options with $\mathcal{M} = N/2$. For the values obtained with the Richardson extrapolation we use $\mathcal{M} = 16$ in (3.7) (so, $2\mathcal{M} = 32$, $4\mathcal{M} = 64$, $8\mathcal{M} = 128$).

The CGMY model is used here with the parameters r, C, G, M, Y, from (4.1), and $T = 0.5$, $S_0 = 8$, $S_{\min} = 10$, $S_{\max} = 50$, $K_{\mathrm{d}} = 20$, $K_{\mathrm{a}} = 25$.

As illustrated in Table 2, to converge to an error of $O(10^{-4})$, we would require 203 seconds with the direct approximation method, and approximately one second with the extrapolation technique. The convergence observed here is in accordance with the behavior observed in Fang and Oosterlee (2009).

FIGURE 6 CGMY process, $T - t = 1$.



(a) Varying amount L, and fixed $\tau_R(D, t) = \frac{1}{12}D$. (b) Varying recovery time, and fixed $L = 5$.

4.2 State-dependent recovery time

We now consider the case where the recovery time, τ_R, depends on the amount D. We first use the CGMY model with the parameters from (4.1). Part (a) of Figure 6 compares the swing option prices with three upper bounds for D: $L = 8, 10, 12$. A higher upper bound results in higher option values, because a higher upper bound implies more possibilities for an option holder at each exercise date.

Part (b) of Figure 6 illustrates the influence of the recovery time on the swing option value. Here we compare $\tau_R(D) = \frac{1}{12}D$ with $\tau_R(D) = \frac{1}{6}D$, which corresponds to one

TABLE 3 D_{opt} over time $L = 8$, $S_0 = 8$, $\tau_{\mathrm{R}}(D) = \frac{1}{12}D$.

$T-t$	Option value	D_{opt}	$T-t$	Option value	D_{opt}
0	80	8	8/24	110.587	4
1/24	80	8	9/24	111.556	4
2/24	85.489	1	10/24	120.572	5
3/24	85.794	1	11/24	121.806	5
4/24	92.441	2	12/24	130.769	6
5/24	93.116	2	13/24	132.224	6
6/24	101.058	3	14/24	141.051	7
7/24	102.371	3	15/24	142.690	7

month (solid line) or two months (dashed line) penalty time for each unit exercised. Part (b) of Figure 6 shows that longer recovery times lead to lower option prices. In other words, if we can wait after exercising then we can pay less for the swing option.[4]

Table 3 shows how the option value and optimal value of D (ie, D_{opt}) change over time. Here we take $L = 8$ and $S_0 = 8$, a case where the option is deep in-the-money. As expected, jumps in the optimal D-values are observed at $t_n^* = T - n\tau_{\mathrm{R}}(1)$.

Recovery time $\tau_{\mathrm{R}}(D) = \frac{1}{12}D$ implies that if we exercise n or fewer units at t_n^*, we can exercise once more before expiry T, whereas if we exercise more than n units, we cannot exercise again before T. In other words, at t_n^*, $\phi_D^t > 0$ for $D \leqslant n$ and $\phi_D^t = 0$ otherwise.

Note that at the time points $t = T$ and $t = T - \frac{1}{24}$, the optimal value equals $D_{\mathrm{opt}} = L = 8$. For $t = T$ this is due to the arbitrage-free condition and the profit maximization principle, whereas for $T = t - \frac{1}{24}$ the time left is so small that, in our present setting, there is only one opportunity left for a swing action ($\phi_D^t = 0$ for all D, n). We should then choose the largest D-value allowed for an optimal profit.

Figure 7 on the facing page shows how D_{opt} changes with respect to the underlying price, with $L = 8, t = 0, \tau_{\mathrm{R}}(D) = \frac{1}{12}D$. As S goes beyond K_{d} and K_{a}, D_{opt} tends to increase, because in this region the payoff $g(x, t, D)$ dominates in the term $g(x, t, D) + \phi_D^t(x, t)$. Between $S = 20$ and $S = 25$, $D_{\mathrm{opt}} = 0$, since $g(x, t, D) = 0$ for all $D > 0$ in this interval.

In another experiment, the convergence of the swing option value with respect to parameter N, with the corresponding CPU time, for the CGMY and the OU processes with $S_0 = 8$, $T = 1$, $M = 12$ and different upper bounds L, are presented in Table 4 on the facing page and Table 5 on page 32, respectively. With $N = 256$ the swing option prices are accurate up to a basis point for both stochastic processes.

[4] Similarly, smaller recovery times result in higher option prices with constant recovery time.

FIGURE 7 D_{opt} over underlying price, $L = 8$, $t = 0$, $\tau_{\mathrm{R}}(D) = \frac{1}{12}D$.

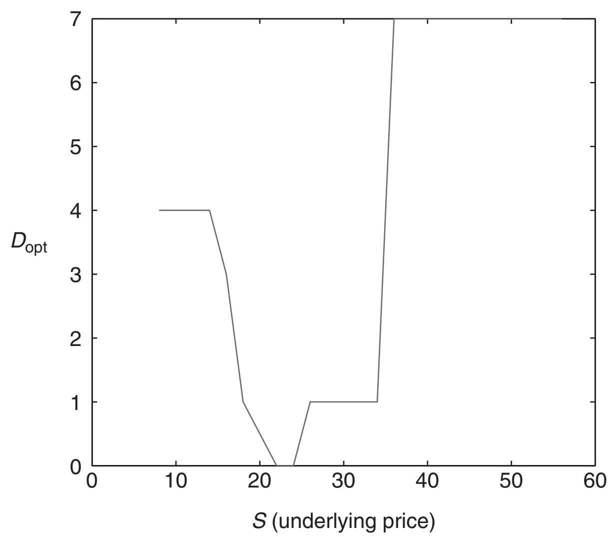


TABLE 4 Swing option values for CGMY process, dynamic recovery time, $S_0 = 8$, $T = 1$, $t = 0$.

		N: 128	N: 256	N: 512
$L = 5$	Option price	153.6884	150.0041	150.0041
	CPU time (s)	0.3293	0.4569	0.7731
$L = 8$	Option price	177.2750	179.5152	179.5152
	CPU time (s)	0.6914	1.1369	1.9020
$L = 10$	Option price	199.4206	199.6870	199.6870
	CPU time (s)	1.0625	1.6609	2.9439

Table 4 and Table 5 on the next page also indicate that the algorithm is flexible with respect to the variation in parameter L. Large L-values result in higher CPU times, because a larger number of early-exercise points needs to be determined, and many C_k and G_k terms have to be computed.

In the final experiment we compare, for American-style swing options with state-dependent recovery time, the approximation obtained by the four-point Richardson extrapolation with direct approximation, obtained with Bermudan option values with

TABLE 5 Swing option values for OU process, dynamic recovery time, $S_0 = 8$, $T = 1$, $t = 0$.

		N: 96	128	160
$L = 5$	Option price	145.5943	153.1150	153.1150
	CPU time (s)	0.5256	0.7854	1.0180
$L = 8$	Option price	172.0567	172.0567	172.0567
	CPU time (s)	0.9182	1.2263	1.5297
$L = 10$	Option price	196.9790	196.9790	196.9790
	CPU time (s)	1.3039	1.6746	2.0252

TABLE 6 Comparison between two approximation methods for American-style swing options, CGMY model, $S_0 = 10$, $L = 5$, $Y = 0.5$.

(a) Bermudan approximation, $\mathcal{M} = N/2$

N	**Option value**	**CPU time (s)**
128	93.9501	5.7391
256	93.9710	20.1821
512	93.9707	77.0859

(b) Richardson approximation, $\mathcal{M} = 6$ in Equation (3.7)

N	**Option value**	**CPU time (s)**
64	93.9710	1.6077
128	93.9707	2.3621
256	93.9707	3.9196

increasing $\mathcal{M}$-values. We use the CGMY model with $Y = 0.5$ (other parameters as in (4.1)). Table 6 shows that the four-point Richardson extrapolation is much more efficient than the direct approximation method, and that both methods converge to the same swing option values. Larger values of $\mathcal{M}$ give the same extrapolation result.

5 CONCLUSIONS

We have presented an efficient, flexible and robust pricing algorithm for swing options with early-exercise features based on Fourier cosine series expansions and backward recursion. The algorithm performs nicely for different types of swing contract with flexibility in the upper bounds for the amount that can be exercised and recovery times. The pricing technique is valid under different stochastic commodity processes, such as the CGMY process, other Lévy processes, and the OU process.

For the Lévy processes, the FFT can be applied in the backward recursion procedure. This gives Bermudan-style swing option prices that are accurate up to a basis point in milliseconds for constant recovery times, and in a fraction of a second ($L = 5$) to 1.7 seconds ($L = 10$) for the dynamic recovery time.

For OU processes, despite the higher computational complexity (compared with Lévy processes), swing option prices can be obtained with basis point precision in less than 0.2 seconds for constant recovery times and within 2 seconds for dynamic recovery times. This is due to the exponential convergence rate of Fourier cosine series expansions.

The Richardson four-point extrapolation technique has been used for pricing the American-style swing options in an efficient way.

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