Method for computing the three-dimensional capacity dimension from two-dimensional projections of fractal aggregates

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The current theory of projections of fractals is considered in this paper with application to fractal aggregates. In particular, this theory does not accurately enable the computation of the capacity dimension of three-dimensional aggregates from the capacity dimension of their two-dimensional projections. Herein we propose to compute the three-dimensional capacity dimension from the perimeter-based fractal dimension, using a semiempirical equation, an approach not applied earlier.

I. INTRODUCTION

Fractal geometry is widely recognized to be fundamental in scaling a variety of properties of aggregates of various nature. The fractal approach has been fruitfully employed, for instance, in sedimentology (mud flocs, bed structure [1–3]), chemistry (polymers, colloidal aggregates [4,5]), medicine (cancer growth, cell structure [6]), cosmology (galaxy distributions and patterns at large scales [7,8]), and many other disciplines dealing with fractal aggregates.

In general, analysis of fractal aggregates in is based on optical projections in . However, the transformation of projection ) where is the number of measuring points in the th box and is the total number of points of the fractal, then determines the probability of a measuring point lying in the th box. Consequently, the generalized dimensionality of the th order is written as follows:

where is a moment that gives strength to the probability . The capacity dimension [10], the information dimensions [11–13] and the correlation dimension [14] are special cases of where and stands for fully self-similar and homogeneous fractals (monofractal sets), while an infinite number of dimensions (all represented by ) is required to describe statistical self-similar, nonhomogeneous fractals (multifractal sets).

The dimensionality has been further elaborated for application purposes into the corresponding multifractal spectrum ) and singularity strength . These quantities describe any arbitrary mass-density distribution of a nonhomogeneous fractal and its growth rate . The relevance of evaluating the fractal properties of aggregates stems from those fundamental works. In substance, the possibility of determining the fractal dimensions of an aggregate from its projection would make the characterization of the aggregate more complete. In particular, the capacity dimension should be accessed as it plays a role in the nonlinear relationship between the mass of an aggregate and its length scale , and nevertheless in a number of other quantities such as the effective density, porosity, etc. [3,5,23].

However, direct computation of from projections of real aggregates is limited by geometric constraints. This was shown by early investigations which were addressed to understand mathematically how projections affect [8,24]. In particular, Hunt and Kaloshin [24] have elaborated that preserves the 3D information only for a limited range of moments (1), thus leaving unsolved questions related to , which have direct implication to applied sciences and measurement techniques.

The main contribution of this paper is to show that it is possible to extract the 3D capacity dimension of fractal aggregates from their projections by following an alternative path. The paper is organized as follows. Section II summarizes a literature-based survey on the theoretical limits to which is subject in the case of projections of nonhomogeneous, extensive fractals. Furthermore, we give numerical evidence that the application of the theory to nonhomogeneous finite fractals yields distorted results, especially for the capacity dimension . We infer that the finiteness of the sets causes these distortions (real fractal aggregates in contrast to extensive fractals). Section III is dedicated to the analysis of those results. In addition, we propose an analytical formulation capable of circumventing the limits exposed in Sec. II. This formulation is founded upon geometry arguments and fitting numerical results.

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II. PROJECTIONS AND GENERALIZED DIMENSIONALITY

A. Problem definition

Hunt and Kaloshin [24] have observed that the projection $\mathcal{P}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ of a fractal $S_m \subset \mathbb{R}^m$ of generalized dimensionality $d_q(S_m)$ onto $\mathbb{R}^n$ (with $n < m$), yields to

$$d_q(S_n) = \mathcal{P}(S_m) = \min\{n, d_q(S_m)\}. \quad (2)$$

If $d_q(S_m) > n$ then the projection $S_n$ has dimensionality $d_q(S_n) = n$, otherwise $d_q(S_n) = d_q(S_m)$. When the projection has the same dimensionality as the original ($d_q(S_m) \leq n$), then $\mathcal{P}$ is called a dimension-preserving transformation. This relation has been proven analytically in [24] only for values of the moment $q: 1 < q < 2$. $\mathcal{P}$ is not dimension-preserving for $q \equiv 1$ and $q > 2$. In other words, only the correlation dimension $d_2$ and the infinite number of dimensions between $d_2$ and the information dimension $d_1$ are preserved. All other dimensions are not preserved, the capacity dimension $d_0$ included. This implies that for projections of real objects ($m = 3$ and $n = 2$), the capacity dimension $d_0(S_3)$ ($q = 0$) cannot be found from Eq. (2):

$$d_0(S_2) \neq \min\{2, d_0(S_3)\}. \quad (3)$$

It follows that we cannot use the 2D capacity dimension $d_0(S_2)$ of a projection to characterize the 3D capacity dimension $d_0(S_3)$ of the original object in a direct way, at least theoretically, even when $d_0(S_3) < 2$.

Moreover, Eq. (2) is deduced from the literature to be valid for indefinitely extensive fractals. However, no precise distinction has been made in literature for finite fractals, such as aggregates. For this reason, we must first consider whether Eq. (2) is applicable to 2D projections of fractal aggregates, because these are nonhomogeneous, finite, and closed (compact) sets. Indeed, images of fractal aggregates consist of the complete closed sets, that is the sets themselves and their boundaries. Furthermore, such sets are self-similar only over limited ranges of length scales. This is in contrast to self-similar (mono) fractal sets. Monofractals are, at least theoretically, open and homogeneous sets because any observation window replicates any other window, at any scale, according to the concept of self-similarity. The consequence of dealing with non-homogeneous finite and closed objects is that the application of Eq. (2) becomes less clear and liable to divergencies and misunderstandings [22].

We therefore compare in the next section theoretical values of $d_0(S_2)$ [via Eq. (2)] and numerical values (via computer simulations) of nonhomogeneous random fractals. From this test we will learn that Eq. (2) is not able to return accurate results, thus preventing the computation of $d_0(S_2)$ from $d_0(S_2)$. Next, we propose an alternative semi-empirical equation to compute $d_0(S_2)$ accurately.

B. Application of Eq. (2) to artificial fractal aggregates

Fractal aggregates are nonhomogeneous, finite, and closed random sets of connected seeds distributed within the domain. Furthermore, they are statistically self-similar with multifractal properties. Herein, we test the applicability of Eq. (2) on artificial aggregates.

We first generate $n_j$ fractal aggregates $S_j \subset \mathbb{R}^3$ by means of a simple algorithm which produces self-correlated random structures with known capacity dimension $d_0$. This technique is a static aggregation algorithm. Seed-by-seed diffusion or cluster-cluster reactions are not accounted for. Starting with a single (cubical) seed $i = 1$, a second seed $i = 2$ is placed randomly in one of the 3-by-3-by-3 free, neighbor locations. Then, one of the existing seeds is chosen from an exponential distribution, and a new seed is attached to it. In this algorithm, recent seeds (large indexes $i$) have higher probabilities to receive a new seed. This procedure is repeated for 1000 seeds. The capacity dimension $d_0(S_j)$ of the aggregate under construction is tuned by means of the exponent of the exponential distribution. The resulting sets $S_j$ are aggregates with few open branches, more similar to CCA aggregates than DLA aggregates [10,21]. The $n_j$ sets $S_j$ are afterwards projected along the three Cartesian directions, thus obtaining the projections

$$\{S_j^{(1)}\}, \{S_j^{(2)}\}, \{S_j^{(3)}\}, \quad (4)$$

with $j = \{1, \ldots, n_j\}$ the repetition index. Three examples of $S_j^{(1)}$ and their projections are given in Fig. 1. For each of the $3n_j$ projections, we compute the capacity dimension $d_0$ according to Vicsk [10]:

$$d_0(S_j^{(l)} \times X) = \frac{\log [N]}{\log [X]}, \quad (5)$$

where $N$ is the number of seeds in the projection and $X = \{L_2, L_3, D_2, D_3\}$ are the length scales taken into account. $L$ is the size of the minimum hypercube enveloping $S$ and $D$ is the hydraulic diameter, in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The length scales $\{L_2, D_2\} \subset \mathbb{R}^2$ are known from the construction of the aggregates while the length scale $\{L_2, D_2\} \subset \mathbb{R}^3$ result from the transformation $\mathcal{P}$. Next, we compute the average capacity dimensions $d_0(S_j^{(l)})$ of the projections for each $j^{th}$ aggregate as follows:

$$d_0(S_j^{(l)} \times X) = \frac{1}{3} [d_0(S_j^{(2)} \times X) + d_0(S_j^{(3)} \times X) + d_0(S_j^{(1)} \times X)]. \quad (6)$$

depending on the used length scales. The reason to consider different length scales comes from a misuse of them in the application to real cases.

Now, we consider the capacity dimension $d_0(S_j, \{L_3, D_3\})$, computed as a function of $L_3$ and $D_3$ solely.

The relationship between $d_0(S_j^{(1)} \times X)$ and $d_0(S_j^{(1)} \times L_3)$ is given in Fig. 2, while the relationship between $d_0(S_j^{(1)} \times X)$ and $d_0(S_j^{(1)} \times D_3)$ is given in Fig. 3. Both these experimental sets deviate largely from Eq. (2) for all the length scales here considered. In particular, Eq. (2) tends to overestimate the real values both for low- and high-dimensional aggregates.

Thus we have shown that, for nonhomogeneous, finite, and closed fractal aggregates, Eq. (2) does not enable a direct
extraction of \(d_0(S_3)\) from \(d_0(S_2)\), even for \(d_0(S_3)<2\). This was already stated in Eq. (2) and derived analytically in [24] for extensive fractals.

### III. DIRECT COMPUTATION OF \(d_0(S_3)\) FROM THE PROJECTION \(S_2\)

#### A. Perimeter of fractal sets

From the previous results, we have found confirmation that information concerning the capacity (the capacity dimension, that is the space-filling ability) is polluted by the projection itself, even for \(d_0(S_3)<2\). Hence, we analyze another set belonging to the projection, which is independent or nearly independent of the transformation: the contour of the projected set \(S_2\). The contour is a subset of the surface of \(S_3\). The measure of the contour, that is the perimeter, does not represent a capacity of \(S_3\). The perimeter, or better the perimeter segmentation reflects the roughness of the object in \(\mathbb{R}^3\).

Herein, we investigate to which extent the information of the structure in \(\mathbb{R}^3\) can be found in the projected, perimeter-based fractal dimension \(d_p\), which is defined for instance in [21]. \(d_p\) does not belong (or give evidence of belonging) to the set of dimensions in \(d_q\). As a consequence, the theory of

![Image](96x520 to 516x733)

**FIG. 1.** (a) Example of a high-fractal-dimension aggregate, \(S_3^{(1)}\), \(d_0(S_3^{(1)})=2.49\). The projections show a massive and round-shaped organization of the primary particles. (b) Example of a mid-fractal-dimension aggregate, \(S_3^{(14)}\), \(d_0(S_3^{(14)})=2.09\). The projections show a less massive and irregular-shaped organisation of the primary particles. (c) Example of a low-fractal-dimension aggregate, \(S_3^{(30)}\), \(d_0(S_3^{(30)})=1.81\). The projections show a weak and irregular-shaped organization of the primary particles.

![Image](54x108 to 294x302)

![Image](318x108 to 558x304)

**FIG. 2.** 2D capacity dimension \(d_0(S_2^j, X)\) of the projections \(S_2^j\) (dots) as a function of the 3D capacity dimension \(d_0(S_3^j, L_3)\). They have been compared to the theoretical relation, computed through Eq. (2) (solid line).

![Image](318x108 to 558x304)

**FIG. 3.** 2D capacity dimension \(d_0(S_2^j, X)\) of the projections \(S_2^j\) (dots) as a function of the 3D capacity dimension \(d_0(S_3^j, D_3)\). They have been compared to the theoretical relation, computed through Eq. (2) (solid line).
projection does not apply to \(d_P\), and therefore it is not subject to the rule of Eq. (2). However, it still gives information on the fractal structures of aggregates. For this reason, and because of a lack of theoretical work dealing with this problem, we perform a simple numerical experiment on the correlation between \(d_P(S_3^{(j)}, L_2)\) and \(d_P(S_3^{(j)})\), thus neglecting the length scales \(L_2\), \(D_2\), and \(D_3\).

B. Perimeter-based fractal dimension of the projections

The perimeter-based fractal dimension \(d_P\) is defined according to [21]:

\[
d_P = 2 \frac{\log [P]}{\log [A]},
\]

(7)

where \(P\) and \(A\) represent the perimeter and the area of a projection. Within our context, \(A\) is given by the number of seeds within the projected area and \(P\) is given by the number of seeds on the contour.

By means of simple geometry arguments, we compute the values of \(d_P\) for the two extreme cases of thin line and massive box projections. Let us therefore consider an \(\epsilon\) covering of the set \(S_3\) of length scale \(L_2\) by means of boxes of size \(\epsilon\), corresponding to a resolution \(\ell = L_2/\epsilon\). The values of \(d_P\) then depend on the resolution \(\ell\), as elaborated in the two following cases.

Thin line. Let us consider the case of a projection which becomes a thin line for increasing resolution \(\ell \in [1, \infty)\), Fig. 4(a). In that case

\[
P = A = \ell, \quad \ell \in [1, \infty),
\]

and, using Eq. (7), the resulting \(d_P\) becomes

\[
d_P = 2 \frac{\log [P]}{\log [A]} = 2 \frac{\log [\ell]}{\log [\ell]} = 2, \quad \ell \in [1, \infty).
\]

Massive box. Let us now consider a projection consisting of a massive box for resolutions \(\ell \in [1, \infty)\), Fig. 4(b). The generalized forms expressing the perimeter \(P\) and the area \(A\) are

\[
P = A = \ell, \quad \ell = 1
\]

\[
P = 4 \ell - 4, \quad A = \ell^2, \quad \ell \in [2, \infty),
\]

from which we write Eq. (7) as a function of \(\ell\)

\[
d_P(\ell = 1) = 2 \frac{\log [\ell]}{\log [\ell]} = 2, \quad \ell = 1
\]

\[
d_P(\ell = 2) = 2 \frac{\log [4 \ell - 4]}{\log [\ell^2]} = 2, \quad \ell = 2
\]

\[
d_P(\ell \geq 3) = 2 \frac{\log [4 \ell - 4]}{\log [\ell^2]} < 2, \quad \ell \in [3, \infty),
\]

(9)

where the cases \(\ell = 1\) (\(\epsilon = L\)) and \(\ell = 2\) (\(\epsilon = L/2\)) represent two trivial solutions for \(d_P\) that can be referred to as a pathological effect caused by the low resolution. It is possible to see from Fig. 4(b) that \(P \neq A\) for resolutions \(\ell \geq 3\) (\(\epsilon = L/3\)). Hence, \(d_P\) decreases for increasing resolution. For \(\ell \to \infty\) (\(\epsilon \to 0\)) we obtain the lower limit

\[
\lim_{\ell \to \infty} d_P = \lim_{\ell \to \infty} \frac{\log [4 \ell - 4]}{\log [\ell]} = \lim_{\ell \to \infty} \left( \frac{\log [4] + \log [\ell - 1]}{\log [\ell]} \right) = 1,
\]

(10)

which represents an asymptotic case for infinitely high resolutions of fully massive aggregates.

The limiting values of \(d_P\) are then represented by \(d_P = 2\) for linelike projections and \(d_P = 1\) for massive projections and infinitely high resolution.

C. Correlation analysis of \(d_P(S_3)\) and \(d_P(S_2)\)

In order to investigate how \(d_P(S_3)\) relates to \(d_P(S_3)\), we first normalize the projections of Eq. (4) with a reference resolution \(\ell_r\). This is performed by using a magnification factor \(f_m\) defined as

\[
f_m = \frac{\ell_r}{\ell},
\]

(11)

in such a way that

\[
L_{2m} = f_m L_2 = \ell_r \epsilon \forall S_3^{(j)}.
\]

(12)

We compute the average perimeter-based fractal dimension \(d_P(S_2^{(j)})\) for the \(j\)th set \(S_2^{(j)}\) as follows:

\[
\overline{d_P(S_2^{(j)})} = \frac{1}{3} [d_P(S_2^{(j)} - 1) + d_P(S_2^{(j)} + 1) + d_P(S_2^{(j)})],
\]

(13)
where $d_P$ is defined in Eq. (7). In this computation we consider the external perimeter only, therefore neglecting inner empties.

Figure 5 shows the relationship between $d_P(S_2)$ and $d_0(S_3)$ for various resolutions, $\ell_r=\{16,256,1024\}$ pixels. Therein, we have evaluated the boundary points $Z$ at $d_0(S_3)=3$ (massive box),

1. $Z_{16}=(3,z(\ell_r=16))$,
2. $Z_{256}=(3,z(\ell_r=256))$,
3. $Z_{1024}=(3,z(\ell_r=1024))$.

known by the analytical solution of Eq. (9), where we have applied the notation

$$z(\ell)=d_p(S_2,\ell) = \frac{\log[4\ell - 4]}{\log[\ell]}.$$  

There are three major features that we can observe from the results given in Fig. 5. The first is that low dimensional structures, with a high level of branching at the left-hand side of the plot, possess projections with high values of $d_P$. In contrast, high dimensional structures, with massive and round-shaped masses at the right-hand side of the plot, have low values of $d_P$. The second is that $d_P(S_2)$ does not reach a constant value for $d_0(S_3)>2$, in contrast to the rule of Eq. (2). Rather, a hyperbolic-like correlation does appear in the full range $1 \leq d_0(S_3) \leq 3$. The third is that low resolutions (16 pixels, for instance) move the points towards the upper limit $d_P=2$. An increase in resolution lowers the points asymptotically towards the limit $d_P=1$, as shown in Eq. (10).

These are valuable results that can be used to derive a semiempirical equation to relate $d_0(S_3)$ and $d_P(S_2)$ as a function of the resolution and with a hyperbolic-like structure.

D. Semiempirical relation for $d_p(S_2)$ and $d_0(S_3)$

By considering the fully known points $Z$ of Eq. (14) and assuming a function of the form

$$d_p(S_2) = \frac{a}{[d_0(S_3)]^2} + b,$$  

we correlate the results in Fig. 5 by solving the following system in correspondence of the two points $Z$ and $K$:

$$z(\ell) = \frac{a}{3^2} + b \quad \text{at} \quad Z=(3,z(\ell)),$$

$$2 = \frac{a}{[k(\ell)]^2} + b \quad \text{at} \quad K=(k(\ell),2),$$

with $z(\ell)$ defined in Eq. (15). The coordinates $k(\ell)$ of the boundary points $K$ at $d_P=2$ for a given resolution $\ell$ have been expressed as a function of $z(\ell)$ by fitting the data points in Fig. 5 at the upper limit $d_P=2$:

$$k(\ell)=k(\ell)=z(\ell)[z(\ell) - 1] + 1,$$

which results in

$$K_{16}=(k(\ell_r=16),2),$$

$$K_{256}=(k(\ell_r=256),2),$$

$$K_{1024}=(k(\ell_r=1024),2).$$

Hence, the coefficients $a$ and $b$ are

$$a(\ell) = 9z(\ell) - \frac{2k(\ell)^2 - 9z(\ell)}{k(\ell)^2 - 9},$$

$$b(\ell) = \frac{2k(\ell)^2 - 9z(\ell)}{k(\ell)^2 - 9}.$$  

Finally, Eq. (16) reads as a function of $d_0(S_3)$ and $\ell$.

$$d_p(S_2) = \begin{cases} 
\frac{a(\ell)}{[d_0(S_3)]^2} + b(\ell) \quad \text{for} \quad d_0(S_3) > k(\ell), \\
2 \quad \text{for} \quad d_0(S_3) \leq k(\ell).
\end{cases}$$

Figure 6 shows the numerical results (dots) and the empirical fit (solid curves) obtained from Eq. (21) for resolutions $\ell_r=\{16,256,1024\}$ pixels. The fit for $\ell_r=16$ pixels is acceptable though not perfect ($R^2=0.970$). A better fit is obtained for resolutions $\ell_r=256$ pixels ($R^2=0.975$) and for $\ell_r=1024$ pixels ($R^2=0.973$), see Fig. 7. The correlation coefficients $R^2$ appear to have a maximum for a given resolution ($\ell_r=256$ in this case). Consequently, the reader can argue that the optimal determination of $d_0(S_3)$ occurs for a resolution $\ell < \infty$. However, we note that the fluctuation of the correlation coefficient is in the order of $10^{-3}$; therefore,
statistically it is not relevant to infer any systematic trend or behavior. Besides this, the appreciable alignment of the data point supports the goodness of the technique proposed here.

By inversion of Eq. (21), we can write the following equation:

\[
d_0(S_3) = \sqrt{\frac{a(\ell)}{d_p(S_2) - b(\ell)}} \quad \text{for} \quad d_p(S_2) < 2,
\]

which gives the 3D capacity dimension of the aggregates from the perimeter-based fractal dimension of their projections and the adopted resolution.

**E. The case of infinite resolution**

For \( \ell \to \infty \), the coordinates \( z(\ell) \) and \( k(\ell) \) of the boundary points \( Z \) and \( K \) of Fig. 6 become

\[
z_\infty = \lim_{\ell \to \infty} z(\ell) = \lim_{\ell \to \infty} \frac{\log(4\ell - 4)}{\log(\ell)} = 1,
\]

\[
k_\infty = \lim_{\ell \to \infty} k(z(\ell)) = \lim_{\ell \to \infty} z(\ell)[z(\ell) - 1] + 1 = 1.
\]

and the coefficients of Eq. (20) are consequently

\[
a = \frac{9}{8}, \quad b = \frac{7}{8}.
\]

Equation (22) in the asymptotic limit \( \ell \to \infty \) then becomes

\[
d_0(S_3) = \sqrt{\frac{9/8}{d_p(S_2) - 7/8}} \quad \text{for} \quad d_p(S_2) < 2.
\]

It matches the theoretical points \( Z = (3, z_\infty) \) and \( K = (k_\infty, 2) \) as shown in Fig. 6.

**F. Critical resolution**

Equation (22) allows us to detect a critical resolution \( \ell_c \) below which the computation of \( d_0(S_3) \) is corrupted by low resolution. Let us consider a fractal aggregate with capacity dimension \( d_0(S_3) = d^* \). If we want to be able to detect \( d^* \) by means of Eq. (22), then the condition

\[
k(\ell) \ll d^*
\]

must be satisfied. By expanding we obtain

\[
[z(\ell)]^2 - z(\ell) + 1 - d^* \leq 0.
\]

Its corresponding solution is

\[
z_1(d^*) \leq z(\ell) \leq z_2(d^*),
\]

with

\[
\begin{align*}
z_\infty &= \lim_{\ell \to \infty} z(\ell) = \lim_{\ell \to \infty} \frac{\log(4\ell - 4)}{\log(\ell)} = 1, \\
k_\infty &= \lim_{\ell \to \infty} k(z(\ell)) = \lim_{\ell \to \infty} z(\ell)[z(\ell) - 1] + 1 = 1.
\end{align*}
\]
If we consider that real aggregates possess capacity dimensions in the range \(1 \leq d^* \leq 3\), then the discriminant \(\Delta = -3 + 4d^*\) is limited to the range \(1 \leq \Delta \leq 9\). In fact, if \(S_3 \subseteq \mathbb{R}^3\) then \(d^* \approx 3\) for obvious physical limits. At the same time, if \(d^* < 1\) then the aggregate would consist of, at least, two disjointed, thin masses. This is not a unique aggregate anymore but two or more individual aggregates, with independent fates. Therefore, for the considered range of \(\Delta\), we obtain the ranges of validity of \(z_1(d^*)\) and \(z_2(d^*)\):

\[
\begin{align*}
-1 = z_1^{\text{inf}}(d^* = 3) & \leq z_1(d^* = 3) = z_1^{\text{sup}}(d^* = 1) = 0, \\
1 = z_2^{\text{inf}}(d^* = 1) & \leq z_2(d^* = 1) = z_2^{\text{sup}}(d^* = 3) = 2.
\end{align*}
\]

The quantity \(z(\ell)\) is \(z(\ell) = d_P(S_2)\) as defined in Eq. (15); it is proven to lie in the range \([1, 2]\) in Sec. III B. The resulting solutions of \(z(\ell)\) are then limited to the positive range:

\[
1 \leq z(\ell) = z_2(d^*), \quad (27)
\]

as represented in Fig. 8. In particular, since

\[
z(\ell) \geq 1 \quad \forall \ell \in [1, \infty),
\]

we must satisfy only the condition

\[
z(\ell) \leq z_2(d^*). \quad (28)
\]

Therefore, in order to compute \(\ell_c\) we substitute Eqs. (15) and (26) into Eq. (28),

\[
\begin{align*}
z(\ell) = \frac{\log[4 \ell - 4]}{\log[\ell]} & \leq \frac{1 + \sqrt{-3 + 4d^*}}{2} = z_2(d^*), \\
\log[4 \ell - 4] & \leq \frac{1 + \sqrt{-3 + 4d^*}}{2} \log[\ell], \\
(4 \ell - 4) & \leq \ell(1 + \sqrt{-3 + 4d^*})/2. \quad (29)
\end{align*}
\]

Eventually, the transcendental function in \(\ell\) of Eq. (29) can be rewritten for simplicity in the following form:

\[
f(\ell) = g(\ell, d^*), \quad (30)
\]

with \(f(\ell) = (4 \ell - 4)\) and \(g(\ell, d^*) = \ell(1 + \sqrt{-3 + 4d^*})/2\). In Fig. 9 we have represented the critical resolutions \(\ell_c\) computed from the intersection of \(f(\ell)\) and \(g(\ell, d^*)\) through Eq. (30), for aggregates of various capacity dimensions \(d^*\) in \(\mathbb{R}^3\). Figure 9 provides a practical tool for computing the minimum resolution required to be able to extract \(d_P(S_3)\) from \(d_P(S_2)\), once the observer can estimate the minimum expected 

**IV. CONCLUSION**

Current theory of the projection of fractals does not always enable direct computation of the capacity dimension of fractal sets embedded in \(\mathbb{R}^3\) from the capacity dimension of projections in \(\mathbb{R}^2\). This occurs in particular when the fractals under investigation are aggregates, that is finite and closed objects, with nonhomogeneous mass density distributions. In general, theoretical research tends to refer mostly to extensive fractals. However, in practice finite fractals are more likely to occur. Fractal aggregates differ considerably from indefinitely extended fractals. We have given evidence of the impact of the finite extent of fractals by means of comparisons of theoretical and numerical results in Figs. 2 and 3.

For these reasons, we have developed a method, circumventing the rule of Eq. (2) to obtain the 3D capacity dimension of aggregates from their projections. We neglect the information of “capacity” \(d_0(S_3)\) present in the projections \(S_2\), in favor of information concealed in the perimeter of the projection solely. To this end, a correlation analysis has been carried out to relate \(d_0(S_3)\) to \(d_P(S_2)\), using the perimeter-based fractal dimension \(d_P\) in \(\mathbb{R}^2\). The results show that

**FIG. 8.** Representation of the solution interval of the quantity \(z(\ell)\). In particular, the range of validity of \(z(\ell)\) is shown to be bounded in the range \([1, z_2(d^*)]\).

\[
z_1(d^*) = \frac{1 - \sqrt{-3 + 4d^*}}{2}, \\
z_2(d^*) = \frac{1 + \sqrt{-3 + 4d^*}}{2}. \quad (26)
\]

**FIG. 9.** Representation of the functions \(f(\ell)\) and \(g(\ell, d^*)\). The intersection points define the critical resolution \(\ell_c\) below which the estimation of \(d_0(S_3)\) through Eq. (21) is distorted by the low resolution.
$d_0(S_3)$ and $d_F(S_2)$ are related to each other by means of a hyperbolic-like resolution-dependent function, defined in Eq. (21).

The expression here proposed to compute $d_0(S_3)$ from $d_F(S_2)$ allows us to derive analytically a critical resolution below which $d_0(S_3)$ cannot be calculated accurately. This has resulted in the nomogram of Fig. 9, which can be directly employed to estimate $\ell_c$.

The accurate extraction of the capacity dimension of fractal aggregates obtained with Eq. (22) does not mean, however, that it can be successfully applied to any type of aggregated structure. The concept of universality is here involved for two reasons. The first is that Eq. (22) considers the information of perimeter segmentation, so that the internal area of the projections can be of any type: Euclidian or non-Euclidian. For Euclidian aggregates, only the external surface can be considered fractal, and not the complete object, thus there is no sense in computing a fractal dimension of a regular (Euclidian) structure. The second reason is that DLA and CCA processes produce different aggregate geometries [10,21]. At the moment we cannot state whether the perimeter segmentation is effectively capable of incorporating information on the geometrical structure in addition to the capacity of the structure. Therefore, future investigation must be oriented to understand whether Eq. (22) is valid for different aggregation kinematics (that is different structures), and not only for various capacity dimensions.

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