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INTEGRAL METHOD OF SOLUTION
FOR COMPRESSIBLE LAMINAR BOUNDARY LAYERS AND APPLICATIONS

by

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Abstract

The basic equations based on the integral method for solving compressible, laminar, boundary-layer flows are considered in some detail. The moment of momentum equation is added to the usual boundary-layer equations in order to form a complete set of equations. The governing equations for shock-wave or expansion-wave interactions with the boundary-layer are reformulated.

The cold-wall similarity model of the interaction of a compressible, laminar, boundary-layer flow with a corner-expansion wave is investigated using certain approximations and initial conditions. The results compare well with other analytical models and with existing experimental data.

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NOTATION

a	velocity profile parameter
b	total enthalpy profile parameter
C_h	heat transfer coefficient
C_f	skin friction coefficient
f	velocity profile
g	total enthalpy profile
H	total enthalpy
L	characteristic length
M	Mach number
p	pressure
Pr	Prandtl number
T	temperature
u	velocity along the surface
v	velocity in the direction normal to the surface
x	distance along the surface
y	distance normal to the surface
γ	ratio of specific heats
δ	boundary layer thickness in (x,y)-plane
$\bar{\delta}$	boundary layer thickness defined by Eq. 13
δ	boundary layer displacement thickness in (x,y)-plane
Re_∞	Reynolds number at the corner
ξ	coordinate defined by Eq. 11
η	coordinate defined by Eq. 12
ϕ_w	corner turning angle

Subscripts

e	edge of boundary layer

α	freestream value
w	wall value
c	value calculated at the corner

1. INTRODUCTION

Since the original developments by Karman and Pohlhausen, the momentum-integral method of solving compressible, laminar, boundary-layer flows has been studied by many authors and extended to other fields. During the period 1950~1960, Morduchow,[1] Libby[2] and others have made a critical study of the integral method for a compressible, laminar boundary layer. In this case, the energy equation is also introduced in the integral form for heat transfer problems. A review of this work is given in Ref. 2. The principle of the integral method is as follows. First, the set of partial differential equations for the boundary layer is reduced to a set of ordinary differential equations with appropriate form parameters and assumed polynomials of appropriate degree for the velocity and enthalpy (or temperature) profiles. Second, the set of ordinary differential equations is integrated along the surface of body.

For a problem without interactions, the compressible, laminar boundary-layer equations reduce to two ordinary differential equations of momentum and energy. However, there are three unknowns: one velocity-profile parameter, one enthalpy (or temperature)-profile parameter and the boundary layer thickness. The solutions can be obtained by making some approximations. Morduchow, Libby,[1,2] and others in their group have presented an approximation using an average-value-parameter concept. The reason is that the variation of the two parameters (one for the velocity-profile and the other for the enthalpy profile) along the surface is smooth and not very large. Another approximation method was developed by Thwaites,[3] Cohen and Reshotko[4] and Chan[5] from the concept of a combination of the integral method and similar solutions. Chan[5] has extended this approximation to the interaction of a laminar boundary layer and a shock wave.

Later, Tani[6] and other authors[7] have added another equation, the moment of momentum equation, to the usual boundary-layer equations in order to provide another ordinary differential equation. It is now possible to solve the three unknowns based on the three equations. Finlayson[8,9] has given a detailed account of this method, which is the so-called method of weighted residuals. This method can be generalized to the Navier-Stokes equations.[10] In the frame of classical boundary layer flow, some applications of this method to the interaction problems have been made. However, so far a detailed study of this method is not available.

The purpose of this report is to provide some details of the basic equations for the compressible, laminar, boundary-layer flow with or without interactions. Using some approximations, the cold-wall similarity model of Sullivan,[11] on the interaction of a compressible laminar boundary layer and a corner-expansion wave is reviewed and extended to the integral method.

2. BASIC EQUATIONS OF BOUNDARY-LAYER FLOW

The steady, laminar, compressible boundary-layer flow of a perfect gas over a smooth surface is considered. By taking the x-axis along the surface of the body and the y-axis perpendicular to it, the equations of overall continuity, momentum, energy and moment of momentum are written as follows:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) \tag{2}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr} \frac{\partial H}{\partial y} + \mu \left(1 - \frac{1}{Pr} \right) \frac{\partial}{\partial y}\left(\frac{u^2}{2} \right) \right] \tag{3}$$

$$\rho u^2 \frac{\partial u}{\partial x} + \rho uv \frac{\partial u}{\partial y} = - u \frac{dp_e}{dx} + u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) \tag{4}$$

where, the moment of momentum equation, Eq. 4, is obtained from momentum equation, Eq. 2, multiplying by u. The Prandtl number, Pr, is assumed constant.

The boundary conditions for the velocity and total enthalpy are:

$$\text{at } y = 0: \quad u = v = 0, \quad H = H_w(x) \tag{5}$$

$$y = \delta: \quad u = u_e(x), \quad H = H_e = \text{constant} \tag{6}$$

By using the boundary condition of Eq. 5, the equation of continuity, Eq. 1, is integrated with respect to y from y = 0 to y = y, giving:

$$\rho v = - \int_0^y \frac{\partial}{\partial x}(\rho u) dy \tag{7}$$

Similarly, Eqs. 2 to 4 are integrated with respect to y, from y = 0 to y = δ, and using Eqs. 5 to 7, yield the following results:

$$\frac{d}{dx}\int_0^\delta \rho u(u - u_e) dy + \frac{du_e}{dx}\int_0^\delta \rho u \, dy = - \delta \frac{dp_e}{dx} - \left[\mu \frac{\partial u}{\partial y} \right]_w \tag{8}$$

$$\frac{d}{dx}\int_0^\delta \rho u(H - H_e) dy = - \left[\frac{\mu}{Pr} \frac{\partial H}{\partial y} \right]_w \tag{9}$$

$$\frac{d}{dx}\int_0^\delta \rho u(u^2 - u_e^2)dy + \frac{du_e^2}{dx}\int_0^\delta \rho u\, dy = -\,2\,\frac{dp_e}{dx}\int_0^\delta u\, dy - 2\int_0^\delta \mu \left(\frac{\partial u}{\partial y}\right)^2 dy \quad (10)$$

The usual transformations of compressible, boundary-layer theory are applied:

$$\xi = \frac{x}{L} \quad (11)$$

$$\eta = \frac{1}{\bar{\delta}}\int_0^y \frac{\rho}{\rho_e}\, dy \quad (12)$$

where, L is a characteristic length and

$$\bar{\delta} = \int_0^\delta \frac{\rho}{\rho_e}\, dy \quad (13)$$

We note that under present transformation, $y = 0$ and $y = \delta$, correspond to $\eta = 0$ and $\eta = 1$, respectively.

From the equation of state, $p = \rho RT$, and the relation

$$\rho_e\, u_e \frac{du_e}{d\xi} = -\,\frac{dp_e}{d\xi}$$

the following equation can be derived:

$$u_e \frac{d\rho_e}{d\xi} = -\,\rho_e\, M_e^2 \frac{du_e}{d\xi} \quad (14)$$

where, M_e is the local Mach number at the edge of boundary layer.

The following notations are defined:

$$f = \frac{u}{u_e}$$

$$g = \frac{H}{H_e}$$

$$F_1 = \int_0^1 f(1 - f)\, d\eta$$

$$F_2 = \int_0^1 f(1 - g)\, d\eta$$

$$F_3 = \int_0^1 f(1 - f^2)\, d\eta$$

$$E_0 = \int_0^1 f\, d\eta$$

$$E_1 = \int_0^1 f^2\, d\eta$$

$$E_2 = \int_0^1 \frac{\rho_e}{\rho}\, d\eta$$

$$E_3 = \int_0^1 f^3\, d\eta$$

$$E_4 = \int_0^1 f\,\frac{\rho_e}{\rho}\, d\eta$$

$$E_5 = \int_0^1 \left(\frac{\partial f}{\partial \eta}\right)^2 d\eta \qquad (15)$$

By using the transformation, Eqs. 11 to 13, and the notations of Eq. 15, the integral forms, Eqs. 8 to 10, can be reduced to the following simple forms:

$$\lambda \frac{dF_1}{d\xi} + \frac{F_1}{2}\frac{d\lambda}{d\xi} + \frac{\lambda}{u_e}\frac{du_e}{d\xi}\left[(1 - M_e^2)F_1 - E_1 + E_2\right] = \bar{R}\left[\frac{\partial f}{\partial \eta}\right]_w \qquad (16)$$

$$\lambda \frac{dF_2}{d\xi} + \frac{F_2}{2}\frac{d\lambda}{d\xi} + \frac{\lambda}{u_e}\frac{du_e}{d\xi}(1 - M_e^2)\, F_2 = \frac{\bar{R}}{Pr}\left[\frac{\partial g}{\partial \eta}\right]_w \qquad (17)$$

$$\lambda \frac{dF_3}{d\xi} + \frac{F_3}{2}\frac{d\lambda}{d\xi} + \frac{\lambda}{u_e}\frac{du_e}{d\xi}\left[(1 - M_e^2)F_3 - 2E_3 + 2E_4\right] = 2\bar{R}\, E_5 \qquad (18)$$

where,

$$\lambda = Re_\infty \left(\frac{\bar{\delta}}{L} \right)^2$$

$$Re_\infty = \frac{\rho_\infty u_\infty L}{\mu_\infty} \tag{19}$$

$$\bar{R} = Re_\infty \frac{\rho_w \mu_w}{\rho_e^2 u_e L}$$

By making the assumption of a linear viscosity-temperature relation,

$$\frac{\mu}{\mu_\infty} = C^* \left(\frac{T}{T_\infty} \right) \tag{20}$$

$$C^* = \left(\frac{T_w}{T_\infty} \right)^{1/2} \frac{T_\infty + S}{T_w + S}$$

$$S = 216^\circ R \text{ for air}$$

the $\bar{R}$ value, defined in Eq. 19, becomes

$$\bar{R} = C \frac{u_\infty}{u_e} \frac{p_e}{p_\infty} \left(\frac{\rho_\infty}{\rho_e} \right)^2 \tag{21}$$

3. PARAMETER-FORM OF BASIC EQUATIONS AND ITS FUNCTIONS

Morduchow and others have shown that the sixth and seventh polynomials can represent accurately the profiles of velocity and temperature, respectively, in compressible, laminar boundary-layer flows:

$$f(\xi,\eta) = \sum_{n=0}^{6} a_n(\xi)\eta^n \tag{22}$$

$$g(\xi,\eta) = \sum_{n=0}^{7} b_n(\xi)\eta^n \tag{23}$$

The coefficients of this polynomial of n-th degree can be determined from the boundary conditions, resulting either from physical considerations, experimental work or from mathematical considerations, and the basic equations. Increasing the degree of the polynomial and hence the number of boundary conditions is expected to yield more correct results. However, increasing the degree does not mean unequivocally a resulting improvement. Kovacs and Palancz[12,13] have investigated the relation between accuracy and the degree of the polynomial in the laminar, non-Newtonian, boundary-layer flow and have shown that a sixth polynomial is accurate for the velocity profile.

The boundary conditions, Eqs. 5 and 6, can be written as follows:

$$\begin{aligned} \text{at } \eta = 0: \quad & f = 0 \\ & g = g_w \\ \text{at } \eta = 1: \quad & f = 1 \\ & \frac{\partial f}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2} = \frac{\partial^3 f}{\partial \eta^3} = 0 \\ & g = 1 \\ & \frac{\partial g}{\partial \eta} = \frac{\partial^2 g}{\partial \eta^2} = \frac{\partial^3 y}{\partial \eta^3} = 0 \end{aligned} \qquad (24)$$

Substituting Eq. 24 into Eqs. 22 and 23, yields the following relations:

$$\begin{aligned} a_o &= 0 \\ a_3 &= 20 - 10a_1 - 4a_2 \\ a_4 &= -45 + 20a_1 + 6a_2 \\ a_5 &= 36 - 15a_1 - 4a_2 \\ a_6 &= -10 + 4a_1 + a_2 \\ b_o &= g_w \\ b_4 &= 35(1 - g_w) - 20b_1 - 10b_2 - 4b_3 \\ b_5 &= -84(1 - g_w) + 45b_1 + 20b_2 + 6b_3 \\ b_6 &= 70(1 - g_w) - 36b_1 - 15b_2 - 4b_3 \\ b_7 &= -20(1 - g_w) + 10b_1 + 4b_2 + b_3 \end{aligned} \qquad (25)$$

The additional boundary conditions for a_n and b_n can be found from Eqs. 2 and 3 and their differentiation with respect to y at the wall surface, y = 0. From Eq. 2 and its differentiation with respect to y,

$$\frac{\partial}{\partial\eta}\left[C \frac{\partial f}{\partial\eta} \right] = \frac{1}{\bar{R}}\left[- \frac{\rho_e}{\rho_w} \frac{\lambda}{u_e} \frac{du_e}{d\xi} \right] \tag{26}$$

at $\eta = 0$, or

$$a_2 = - \frac{C}{2\bar{R}} \frac{\rho_e}{\rho_w} \frac{\lambda}{u_e} \frac{du_e}{d\xi} \tag{27}$$

and

$$\frac{\partial^2}{\partial\eta^2}\left[C \frac{\partial f}{\partial\eta} \right] = \frac{1}{\bar{R}} \left[\frac{\rho_e}{\rho_w} \right]^2 \frac{\partial}{\partial\eta}\left(\frac{\rho}{\rho_e} \right) \frac{\lambda}{u_e} \frac{du_e}{d\xi}$$

at $\eta = 0$, or

$$a_3 = \frac{C}{6\bar{R}} \left[\frac{\rho_e}{\rho_w} \frac{\lambda}{u_e} \frac{du_e}{d\xi} \right] \left[\frac{\rho_e}{\rho_w} \frac{\partial}{\partial\eta}\left(\frac{\rho}{\rho_e} \right) \right]_w = \frac{a_2}{3} \frac{b_1}{g_w} \tag{28}$$

From Eq. 3 and its differentiation with respect to y,

$$\frac{\partial}{\partial\eta}\left[\frac{C}{Pr} \frac{\partial}{\partial\eta} \right] = - \frac{u_e^2}{H_e} \left[\left(1 - \frac{1}{Pr} \right)\right] \left(\frac{\partial f}{\partial\eta} \right)_w^2$$

at $\eta = 0$, or

$$b_2 = \frac{u_e^2}{2H_e} (1 - Pr)\, a_1^2 \tag{29}$$

and

$$\frac{C}{Pr} \frac{\partial^3 g}{\partial\eta^3} = \frac{1}{\bar{R}} \left[\lambda \frac{\partial f}{\partial\eta} \frac{\partial g}{\partial\xi} \right]$$

$$- \frac{u_e^2}{H_e} \left[\frac{\partial}{\partial\eta} \left\{ \left(1 - \frac{1}{Pr} \right) \left(\frac{\partial f}{\partial\eta} \right)^2 \right\} + \frac{\partial f}{\partial\eta} \frac{\partial}{\partial\eta} \left\{ \left(1 - \frac{1}{Pr} \right) \frac{\partial f}{\partial\eta} \right\} \right]$$

at $\eta = 0$, or

$$b_3 = \frac{Pr}{6\bar{R}} \lambda\, a_1 \frac{\partial g}{\partial\xi} + \frac{u_e^2}{H_e} (1 - Pr)\, a_1\, a_2 \tag{30}$$

where

$$C = \frac{\rho_w\, \mu_w}{\rho_e\, \mu_e}$$

We note that if the temperature at the wall is constant along the survace, then b_o = constant, or

$$\left[\frac{\partial g}{\partial \xi} \right]_w = \frac{\partial b_o}{\partial \xi} = 0$$

in Eq. 30.

The relationship between a_1 and a_2 can be obtained from Eqs. 25, 27 and 28 as follows:

$$a_1 = 2 - \frac{a_2}{30} \left(12 + \frac{b_1}{g_w} \right) \tag{31}$$

For the compressible, laminar, boundary-layer flow without any interaction, i.e., $p_e = p$ and $T_e = T$, we have three unknowns, a_2 (or a_1), b_1 and λ, which can be determined from three ordinary differential e quations, Eqs. 16, 17 and 18, since p_e (or u_e) is given in general. However, Eqs. 16, 17 and 18 are still too complicated in the actual numerical calculation. We must transfer these equations in terms of three unknowns, a_2 (or a_1), b_1 and λ.

After some rearrangement, Eqs. 16, 17 and 18 can be written in the following forms:

$$\lambda F_{11} \frac{da_2}{d\xi} + \lambda F_{12} \frac{db_1}{d\xi} + \frac{F_1}{2} \frac{d\lambda}{d\xi} = x_1$$

$$\lambda F_{21} \frac{da_2}{d\xi} + \lambda F_{22} \frac{db_1}{d\xi} + \frac{F_2}{2} \frac{d\lambda}{d\xi} = x_2 \tag{32}$$

$$\lambda F_{31} \frac{da_2}{d\xi} + \lambda F_{32} \frac{ab_1}{d\xi} + \frac{F_3}{2} \frac{d\lambda}{d\xi} = x_3$$

where,

$$F_{11} = \frac{\partial F_1}{\partial a_2}$$

$$F_{12} = \frac{\partial F_1}{\partial b_1}$$

$$x_1 = \bar{R} a_1 - \frac{\lambda}{u_e} \frac{du_e}{d\xi} [(1 - M_e^2)F_1 - E_1 + E_2]$$

$$F_{21} = \frac{\partial F_2}{\partial a_2}$$

$$F_{22} = \frac{\partial F_2}{\partial b_1}$$

$$x_2 = \frac{\bar{R}}{Pr} b_1 - \frac{\lambda}{u_e} \frac{du_e}{d\xi} (1 - M_e^2) F_2$$

$$F_{31} = \frac{\partial F_3}{\partial a_2}$$

$$F_{32} = \frac{\partial F_3}{\partial b_1}$$

$$x_3 = 2\bar{R}\, E_5 - \frac{\lambda}{u_e} \frac{du_e}{d\xi} [(1 - M_e^2) F_3 - 2E_3 + 2E_4] \qquad (33)$$

The explicit expressions for F_n, E_n, $\partial F_n/\partial a_2$ and $\partial F_n/\partial b_1$ are quite complex. However, their analytical expressions can be evaluated without difficulty. They are,

$$F_1 = A_o - (A_1 + a_1 A_2 + a_2 A_3)$$

$$F_2 = (1 - b_o)(A_o - B_o) - (b_1 B_1 + b_2 B_2 + b_3 B_3)$$

$$F_3 = A_o - E_3$$

$$E_1 = A_1 + a_1 A_2 + a_2 A_3$$

$$E_2 = \frac{T_o}{T_e} (G - E_1) + E_1$$

$$E_3 = L_o + 3L_1 a_1 + 3L_2 a_1^2 + 3L_3 a_2 + 6L_4 a_1 a_2 + 3L_5 a_2^2 + L_6 a_1^3 + 3L_7 a_1 a_2^2 + 3L_8 a_1^2 + L_9 a_2^3$$

$$E_4 = \frac{T_o}{T_e} [F_9 - E_3] + E_3$$

$$F_9 = A_o - F_2$$

$$E_5 = E_{50} + E_{51} a_1^2 + E_{52} a_2^2 + 2E_{53} a_1 + 2E_{54} a_1 a_2 + 2E_{55} a_2$$

$$G = g_w + \frac{1}{2}(1 - g_w) + \frac{3}{28} b_1 + \frac{1}{42} b_2 + \frac{1}{280} b_3$$

$$\frac{\partial F_1}{\partial a_2} = A_{03} - Q_2 - \frac{1}{30}\left(12 + \frac{b_1}{g_w}\right)(A_{02} - Q_1)$$

$$\frac{\partial F_1}{\partial b_1} = -\frac{1}{30}\frac{a_2}{g_w}(A_{02} - Q_1)$$

$$\frac{\partial F_2}{\partial a_2} = P_2 - \frac{P_1}{30}\left(12 + \frac{b_1}{g_w}\right)$$

$$\frac{\partial F_2}{\partial b_1} = -B_1 - \frac{a_2}{30}\frac{P_1}{g_w}$$

$$\frac{\partial F_3}{\partial a_2} = S_{12} - \frac{S_{11}}{30}\left(12 + \frac{b_1}{g_w}\right)$$

$$\frac{\partial F_3}{\partial b_1} = -\frac{a_2}{30}\frac{S_{11}}{g_w}$$

$$Q_1 = A_{12} + a_1 A_{22} + A_2 + a_2 A_{32}$$

$$Q_2 = A_{13} + a_1 A_{23} + A_3 + a_2 A_{33}$$

$$P_1 = (1 - g_w)(A_{02} - B_{02}) - b_1 B_{12} - b_2 B_{22}$$
$$- K B_2 a_1 - b_3 B_{32} - K B_3 a_2$$

$$P_2 = (1 - g_w)(A_{03} - B_{03}) - b_1 B_{13} - b_2 B_{23} - b_3 B_{33} - K B_3 a_1$$

$$S_{12} = A_{03} - 3L_3 - 6L_4 a_1 - 6L_5 a_2 - 6L_7 a_1 a_2 - 3L_8 a_1^2 - 3L_9 a_2^2$$

$$S_{11} = A_{02} - 3L_1 - 6L_2 a_1 - 6L_4 a_2 - 3L_6 a_1^2 - 3L_7 a_2^2 - 6L_8 a_1 a_2$$

$$A_i = A_{i1} + A_{i2} a_1 + A_{i3} a_2, \quad i = 0,1,2,3$$

$$B_i = B_{i1} + B_{i2} a_1 + B_{i3} a_2, \quad i = 0,1,2,3$$

$$K = \frac{u_e^2}{H_e}(1 - Pr)$$

where, the numerical values A_{ij}, B_{ij}, L_i and E_{ij} are given in Appendix A.

As seen from Eq. 31, a_1 is a function of $a2$ and b_1; therefore, all the explicit functions in Eq. 32, can be calculated in terms of a_2 and b_1.

The solution of the physical problem is then reduced to integrating the set of equations, Eq. 32, simultaneously, for the three dependent variables a_2, b_1 and λ with ξ (or x) as the independent variable.

The heat transfer coefficient is defined by

$$C_h = \frac{-\dot{q}_w}{\rho_\infty u_\infty (H_\alpha - H_w)}$$

where, $\dot{q}_w$ is the rate of heat transfer to the surface of the body:

$$\dot{q}_w = -\left[k \frac{\partial T}{\partial y} \right]_w$$

and k is the thermal conductivity.

Using the basic assumption, $\rho\mu = \rho_e \mu_e$, and after a transformation, we can write

$$C_h = \frac{1}{Pr\, Re_\infty} \frac{T_e}{T_\infty} \frac{1}{\bar{\delta}/L} \frac{b_1}{1 - g_w}$$

The skin friction coefficient, C_f is defined as,

$$C_f = \frac{2\tau_w}{\rho_\infty u_\infty^2}$$

where, τ_w is the shear stress on the surface of the body:

$$\tau_w \left[\mu \frac{\partial u}{\partial y} \right]_w$$

Similarly, the skin friction coefficient can be written in the form,

$$C_f = \frac{2}{Re_\infty} \frac{M_e}{M_\infty} \left(\frac{T_e}{T_\infty} \right)^{3/2} \frac{a_1}{\bar{\delta}/L}$$

4. GENERAL EQUATIONS FOR INTERACTION PROBLEMS

Two types of interaction problems will be considered, namely, (1) Interaction of a shock wave and a compressible, laminar, boundary-layer flow, and (2) Interaction of a compressible, laminar, boundary-layer flow and a corner expansion (or compression) wave. No extensive numerical calculations are made, but the general equations are rewritten in the context of the present theory.

For the interaction problem, the pressure and temperature of the freestream are quite different from that at the edge of the boundary layer in the region of interaction, i.e., $p_e \neq p_\infty$ and $T_e \neq T_\infty$. In general, we have four unknowns, a_2, b_1, λ and p_e and, therefore, another equation must be provided.

4.1 Interaction of Shock-Wave and Boundary-Layer Flows

Many papers have been published concerning the interaction shock-wave and compressible, boundary-layer flows. A discussion of the problem with a comprehensive review is given in Ref. 5.

The boundary-layer thickness δ^* can be written in this form:

$$\delta^* = F_4 \, \bar{\delta} \tag{35}$$

where,

$$F_4 = \int_0^1 \left[\frac{\rho_e}{\rho} - \frac{u}{u_e} \right] d\eta$$

Differentiating Eq. 35, we obtain the following relation:

$$\frac{d}{d\xi}\left(\frac{\bar{\delta}}{L}\right) = \frac{1}{F_4}\frac{d}{d\xi}\left(\frac{\delta^*}{L}\right) - \frac{1}{F_4}\left(\frac{\bar{\delta}}{L}\right)\frac{dF_4}{d\xi} \tag{36}$$

If the effective body is slender, the pressure at the edge of the boundary layer is accurately given by the hypersonic, small-disturbance solution for oblique shocks. For example, the tangent-wedge relation is used as the solution for the external, inviscid flow because of its simplicity and the explicit relation between the local pressure and the local flow inclination. The relation between the local pressure and the growth rate of the boundary layer thickness, δ^*, is given by,

$$\frac{d\delta^*}{dx} = \frac{1}{M_\infty}\left[\frac{2}{\gamma(\gamma+1)}\right]^{1/2}\left(\frac{p_e}{p_\infty} - 1\right) x \left(\frac{p_e}{p_\infty} + \frac{\gamma - 1}{\gamma + 1}\right)^{-1/2} - \alpha_w$$

$$= \Theta\left(\frac{p_e}{p_\infty}, M_\infty, \gamma, \alpha_w\right) \tag{37}$$

where, α_w is the local geometric slope.

From Eqs. 36 and 37, and the definition of λ, the following equation is obtained:

$$\lambda F_{41}\frac{da_2}{d\xi} + \lambda F_{42}\frac{db_1}{d\xi} + \frac{F_4}{2}\frac{d\lambda}{d\xi} = \sqrt{Re_\infty \lambda}\;\Theta \tag{38}$$

where, $dF_4/d\xi$ given in Θ is written in the form,

$$\frac{dF_4}{d\xi} = F_{41}\frac{da_2}{d\xi} + F_{42}\frac{db_1}{d\xi}$$

The explicit expressions of F_{41} and F_{42} can be evaluated as follows:

$$F_{41} = \frac{T_o}{T_e}(G_{11} - E_{11}) - F_{11}$$
$$F_{42} = \frac{T_o}{T_e}(G_{12} - E_{12}) - F_{12} \qquad (39)$$

where,

$$G_{11} = \frac{K}{280}\ a_1 - \frac{K}{30}\left(12 + \frac{b_1}{g_w}\right)\left(\frac{a_1}{42} + \frac{a_2}{280}\right)$$

$$G_{12} = \frac{3}{28} - \frac{a_2}{30}\frac{K}{g_w}\left(\frac{a_1}{42} + \frac{a_2}{280}\right)$$

$$E_{11} = Q_2 - \frac{Q_1}{30}\left(12 + \frac{b_1}{g_w}\right)$$

$$E_{12} = -\frac{Q_1}{30}\frac{a_2}{g_w} \qquad (40)$$

Equations 32 and 38 can be written in the following simple form:

$$W_{i1}\frac{da_2}{d\xi} + W_{i2}\frac{db_1}{d\xi} + W_{i3}\frac{d\lambda}{d\xi} + W_{i4}\frac{d}{d\xi}\left(\frac{p_e}{p_\infty}\right) = W_i \qquad (41)$$

where, i = 1, 2, 3 and 4, and

$$W_{i1} = \lambda F_{i1}$$

$$W_{i2} = \lambda F_{i2}$$

$$W_{i3} = \frac{F_i}{2}$$

$$W_{i4} = -\lambda\frac{p_\infty}{p_e}\frac{Z_i}{\gamma M_e^2} \quad (i = 1,2,3)$$

$$W_{44} = 0$$

$$W_1 = \bar{R}\, a_1$$

$$W_2 = \frac{\bar{R}}{Pr}\, b_1$$

$$W_3 = 2\bar{R}\, E_5$$

$$W_4 = \sqrt{Re_\infty \lambda}\,\Theta$$

$$Z_1 = (1 - M_e^2)\, F_1 - E_1 + E_2$$

$$Z_2 = (1 - M_e^2)\, F_2$$

$$Z_3 = (1 - M_e^2)\, F_3 - 2E_3 + 2E_4 \qquad (42)$$

Equation 41 is the basic equation for the interaction of shock-wave and the boundary-layer flows.

4.2 Interaction of Corner-Expansion Wave and Boundary-Layer Flows

For the interaction of a boundary-layer and a corner-expansion (or compression) wave, Eq. 41 is still applicable except that $d\delta^*/dx$ is replaced by,

$$\frac{d\delta^*}{dx} = \tan\left[\pm\,\phi_w + \tan^{-1}\left(\frac{d\delta^*}{dx}\right)_c - \theta\right] = \Theta \qquad (43)$$

where, ϕ_w is the corner turning angle of the surface body, (+)-sign denotes an expansion corner, and (-)-sign for compression corner, $\tan^{-1}(d\delta^*/dx)_c$ denotes the inclination angle of the boundary-layer thickness at the corner, and θ is the Prandtl-Meyer deflection angle,

$$\theta = \sqrt{\frac{\gamma + 1}{\gamma - 1}}\left[\tan^{-1}\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}}\sqrt{M_e^2 - 1}\right) - \tan^{-1}\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}}\sqrt{M_\infty^2 - 1}\right)\right]$$

$$+ \tan^{-1}\sqrt{M_\infty^2 - 1} - \tan^{-1}\sqrt{M_e^2 - 1} \qquad (44)$$

The main task now is to find the initial values of a_2, b_1, λ and p_e at the corner. Due to upstream-influence effects, the flat-plate solutions at the corner can not be correctly taken as the initial conditions for the downstream-interaction region. This task will be attempted in a future research paper. A simple example for finding the initial conditions is given in Chapter 5.

5. COLD-WALL SIMILARITY MODEL OF INTERACTION OF CORNER-EXPANSION WAVE AND BOUNDARY-LAYER FLOWS

5.1 General Considerations

The numerical complexities involved in the solution of the interaction of a hypersonic, laminar boundary layer and a corner-expansion or corner-compression wave (described in Chapter 4) necessitate reliable approximation schemes. Previous authors have approached this problem in different ways.[10,11,14] A simple method has been presented by Sullivan[11] based on the cold-wall similarity model. It was assumed that flat-plate solutions can be applied in the interaction region and that the integration of the total enthalpy profiles yields a constant value. The latter is due to the cold-wall assumption. The cold-wall similarity model is self consistent since the calculations confirm the basic assumption and it can be regarded as a first approximation to a complete solution.

In this chapter, we point out that the simple results of the cold-wall similarity model can be obtained and improved directly from the integral method.

5.2 Basic Assumptions and Equations

Self-similarity exists only if $p_e \, \alpha \, x^n$. If self-similarity exists, the velocity profile, expressed by Eq. 22 is independent of ξ, or

$$f(\eta) = \sum_{n=0}^{6} a_n \eta^n$$

From Eq. 27, we note that a_2 is very small when the velocity u_e is very large, thus,

$$a_2 \approx 0 \tag{45}$$

and, from Eq. 31,

$$a_1 \approx 2 \tag{46}$$

The velocity profile then becomes,

$$f = 2\eta - 5\eta^4 + 6\eta^5 - 2\eta^6 \tag{47}$$

Similarly, the parameter b_1 of g, expressed by Eq. 23, can be obtained as follows:

$$b_1 = \frac{\frac{31}{252}(1 - g_w) - \frac{151}{9009} K}{\frac{1}{Pr\Lambda_o} + \frac{821}{24024}}$$

$$b_2 = 2K$$

$$b_3 = 0 \tag{48}$$

where

$$\lambda_o = 36036/985$$

The cold-wall assumption has been discussed by Sullivan. The cold-wall-similarity concept should be applicable over a wider range of g_w. If $da_2/d\xi$ and $db_1/d\xi$ are very small (from the similarity solution of Eq. 47, we have $da_2/d\xi = 0$ and from Eq. 48, we know $ab_1/d\xi \approx 0$), then $dF_4/d\xi \approx 0$. From Eq. 36, we obtain the following approximate result:

$$\frac{d}{d\xi}\left(\frac{\bar{\delta}}{L}\right) \approx \frac{1}{F_4}\frac{d}{d\xi}\left(\frac{\delta *}{L}\right) \tag{49}$$

The solution of the physical problem is then reduced to find two unknowns, p_e/p_∞ and $\bar{\delta}/L$ (or λ). From Eq. 41 with $i = 1$, $da_2/d\xi = db_1/d\xi = 0$, we obtain the following momentum equation:

$$\frac{d}{d\xi}\left(\frac{p_e}{p_\infty}\right) = \frac{\left(\frac{p_e}{p_\infty}\right)\gamma M_e^2}{\lambda[(1 - M_e^2)F_1 - E_1 + E_2]}\left[\sqrt{\lambda\, Re_\infty}\, F_1 \frac{d}{d\xi}\left(\frac{\bar{\delta}}{L}\right) - 2\bar{R}\right] \tag{50}$$

The explicit functions can be evaluated by substituting Eqs. 47 and 48 and we have,

$$F_1 = 985/9009$$

$$E_1 = 5450/9009$$

$$E_2 = T_o/T_e\,(G - E_1) + E_1$$

$$F_4 = T_o/T_e\,(G - E_1) - F_1$$

$$G = 1/2\,(1 + g_w) + 3/28\, b_1 + b_2/42$$

$$T_o/T_e = 1 + \gamma\text{-}1/2\, M_e^2 \tag{51}$$

The initial conditions for p_e/p_∞ and $\bar{\delta}/L$ (or λ) at the corner can be calculated by the following three methods:

Method 1: By assuming that the upstream-influence effects for locally hypersonic flow and for small turning angles can be neglected, the initial conditions are obtained from the solutions of the flat-plate, boundary-layer equations just upstream of the corner and are given as follows:

At the corner,

$$\frac{p_e}{p_\infty} = 1$$

$$\frac{\bar{\delta}}{L} = \sqrt{\frac{\lambda_o}{Re_\infty}} \qquad (52)$$

This assumption has been applied by Sullivan in his calculations.

Method 2: For a large turning-angle case, the initial value of $\bar{\delta}/L$ can be obtained simply from the geometry at the edge of the boundary layer (see Fig. 1):

$$\frac{\bar{\delta}}{L} \approx \sqrt{\frac{\lambda_o}{Re_\infty}} \; [1 + \tan \phi_w \tan \Delta\theta] \qquad (53)$$

where,

$$\Delta\theta \approx \frac{\phi_w}{2} + \tan^{-1}\left(\frac{d\delta^*}{dx}\right)_c$$

There is no evidence that the upstream effect on p_e/p_∞ is significant in hypersonic flow. From the experimental results (Ref. 16) it appears that p_e/p_∞ does approach unity, as expected. Consequently, it is assumed that at the corner,

$$\frac{p_e}{p_\infty} = 1$$

Method 3: Some initial arbitrary values of p_e/p_∞ and $\bar{\delta}/L$ are assumed and Eqs. 49 and 50 are integrated when the results are close to the experimental results, then the matched values are used as the initial conditions. However, although this method can be made to fit the experimental data very well, the calculated initial values are still questionable owing to the assumption of cold-wall similarity in the theory. Since the initial conditions for p_e/p_∞ and $\bar{\delta}/L$ at the corner are dependent on the freestream conditions, especially Mach number, no unique parameter can be obtained, as for example, in the case of finding the atom-atom cross-section constant from a match of experimental data with shock-structure analysis.[16]

We also note that the flow inclination before and after the corner is assumed to have the same value. The effect of this assumption is negligible in the actual calculations, as the value of $\tan^{-1}(d\delta^*/dx)_c$ is very small compared with ϕ_w. For example, the flow inclination angle is about $1°$ for $Re_\infty = 2 \times 10^5$.

The effect of using different values for the initial condition of $\bar{\delta}/L$ on the $d/d\xi \; (p_e/p_\infty)$ can be seen from Eq. 50. Since $\lambda \propto (\bar{\delta}/L)^2$, therefore, the calculated absolute values of $d/d\xi \; (p_e/p_\infty)$ using Method 1 are larger than from Method 2. This means that the p_e/p_∞ value calculated by Method 1 is smaller than from Method 2. However, the difference is actually small for small turning angle. For example, if $\phi_w = 5°$, then,

$$\frac{\left.\frac{d}{d\xi}\left(\frac{p_e}{p_\infty}\right)\right|_{\text{Method 1}}}{\left.\frac{d}{d\xi}\left(\frac{p_e}{p_\infty}\right)\right|_{\text{Method 2}}} \approx 1.004$$

5.3 Comparison With Other Theories and Experimental Results

A special example is computed in order to show the difference between the present method and Sullivan's method: $g_w = 0.2$, $\phi_w = 5°$, $Re_\infty = 1.644 \times 10^5$, $M_\infty = 10$ and $Pr = 1$. The initial conditions are calculated by Method 1, as in Sullivan's paper.[11] The displacement-thickness ratio downstream of the corner and the pressure ratio are shown in Fig. 2 where the results based on Sullivan's model[11] without the simple-wave assumption are also given. It is seen from Fig. 2 that a small change of the displacement-thickness ratio results in a significant change in the pressure ratio. The main difference is due to the assumption made in Sullivan's method, that is,

$$F_4 \approx \frac{T_o}{T_\infty} (G - E_1)_c \tag{55}$$

where, $(G - E_1)_c$ is calculated at the corner, based on the assumption of hypersonic flow. A small change in F_4, from Eqs. 55 and 51, can result in a significant change in p_e/p_∞, as seen from Fig. 2. The corresponding heat-transfer and skin-friction coefficients are shown in Fig. 3, for comparison.

Koziak and Sullivan[16] have obtained some experimental data for the following conditions: $M_\infty = 6.5$, $g_w = 0.22$, $Re_\infty = 8 \times 10^5$ and $\phi_w = 5°$. In Fig. 4, the displacement thickness ratio and the density ratio are compared with their experimental measurements and with other theoretical results[10] with $Pr = 0.7$. It is shown that the present calculations with initial conditions calculated by Methods 1 and 2 agree very well with the experimental data. The corresponding pressure ratio and Mach number at the edge of boundary layer are shown in Fig. 5. One can note that the present results are close to the results of Lo[10] obtained from a more complex model. The experimental data[16] shows that there is no evidence that an upstream effect is significant for the present conditions and the initial pressure ratio at the corner appears to be unity. In the model of Lo,[10] with the upstream effect taken into account, the initial pressure ratio is obtained from the perturbation of the upstream pressure value in order to have stability in the calculations.

The effects of the initial values of the boundary-layer displacement-thickness at the corner are shown to be small in Figs. 4 and 5 for a turning angle $\phi_w = 5°$. However, Method 2, which provides for better agreement with experiment, should be used for larger turning angles.

5.4 Discussions

The present analysis of the cold-wall similarity model gives some physical insight into the problem. The advantages of this method are: (i) the calculation is simpler than the method of Sullivan[11] and is always stable, and (ii) the agreement with experimental data for the present analysis is even better than the complex model of Lo.[10] One can quickly obtain numerical results to compare with experimental data. Finally, the present method complements that of Sullivan, in some ways and is in surprisingly better agreement with experiment than the other methods.

6. CONCLUSIONS

The present report considered in some detail the basic equations based on the integral method for solving compressible, laminar, boundary-layer flows. Even though no extensive numerical calculations were made, the present method may be applied to other problems, such as shock-wave or expansion-wave interactions with the boundary layer, boundary-layer separation problems, and heat-transfer problems.

Using some approximations, the cold-wall similarity model of Sullivan, on the interaction of a compressible, laminar, boundary-layer flow with a corner-expansion wave, can be obtained from the present analysis. Three advantages over Sullivan's method are: (i) the calculation is simpler, (ii) the analysis gives some physical insight into the problem, and (iii) with the initial conditions provided by Method 2, the present analysis agrees very well with the experimental data. Additional calculations for the interaction of a corner-expansion wave with the boundary-layer flow and separation in heat-transfer problems are under way. The results will be compared with existing exact numerical calculations and experimental data where possible.

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APPENDIX A: NUMERICAL FACTORS OF FUNCTIONS A_{ij}, B_{ij}, L_i and E_{ij}

Substituting Eqs. 22 and 23 into the relations for F_1 and F_2, yields,

$$F_1 = A_0 - A_1 - a_1 A_2 - a_2 A_3$$

$$F_2 = (1 - b_0)(A_0 - B_0) - (b_1 B_1 + b_2 B_2 + b_3 B_3) \qquad (A.1)$$

where,

$$A_j = \sum_{n=1}^{6} A_n^{(j)} a_n$$

$$B_j = \sum_{n=1}^{6} B_n^{(j)} a_n \qquad (A.2)$$

and j = 0, 1, 2 and 3. The $A_n^{(j)}$ and $B_n^{(j)}$ are given by

$$A_n^{(0)} = \frac{1}{n+1}$$

$$A_n^{(1)} = \frac{20}{n+4} - \frac{45}{n+5} + \frac{36}{n+6} - \frac{10}{n+7}$$

$$A_n^{(2)} = \frac{1}{n+2} - \frac{10}{n+4} + \frac{20}{n+5} - \frac{15}{n+6} + \frac{4}{n+7}$$

$$A_n^{(3)} = \frac{1}{n+3} - \frac{4}{n+4} + \frac{6}{n+5} - \frac{4}{n+6} + \frac{1}{n+7}$$

$$B_n^{(0)} = \frac{35}{n+5} - \frac{84}{n+6} + \frac{70}{n+7} - \frac{20}{n+8}$$

$$B_n^{(1)} = \frac{1}{n+2} - \frac{20}{n+5} + \frac{45}{n+6} - \frac{36}{n+7} + \frac{10}{n+8}$$

$$B_n^{(2)} = \frac{1}{n+3} - \frac{10}{n+5} + \frac{20}{n+6} - \frac{15}{n+7} + \frac{4}{n+8}$$

$$B_n^{(3)} = \frac{1}{n+4} - \frac{4}{n+5} + \frac{6}{n+6} - \frac{4}{n+7} + \frac{1}{n+8} \qquad (A.3)$$

From Eqs. A.2 and 34, A_{ij} and B_{ij} are given by the following expressions:

$$A_{j1} = 20A_3^{(j)} - 45A_4^{(j)} + 36A_5^{(j)} - 10A_6^{(j)}$$

$$A_{j2} = A_1^{(j)} - 10A_3^{(j)} + 20A_4^{(j)} - 15A_5^{(j)} + 4A_6^{(j)}$$

$$A_{j3} = A_2^{(j)} - 4A_3^{(j)} + 6A_4^{(j)} - 4A_5^{(j)} + A_6^{(j)}$$

$$B_{j1} = 20B_3^{(j)} - 45B_4^{(j)} + 36B_5^{(j)} - 10B_6^{(j)}$$

$$B_{j2} = B_1^{(j)} - 10B_3^{(j)} + 20B_4^{(j)} - 15B_5^{(j)} + 4B_6^{(j)}$$

$$B_{j3} = B_2^{(j)} - 4B_3^{(j)} + 6B_4^{(j)} - 4B_5^{(j)} + B_6^{(j)} \qquad (A.4)$$

where, j = 0, 1, 2 and 3.

The numerical results of Eq. A.4 are given in the following table, Table A1,

TABLE A1

	A_{ij}		
i \ j	1	2	3
0	0.571428	0.714285×10^{-1}	0.952375×10^{-2}
1	0.471504	0.248179×10^{-1}	0.393236×10^{-2}
2	0.248911×10^{-1}	0.850946×10^{-2}	0.111353×10^{-2}
3	0.393224×10^{-2}	0.112551×10^{-2}	0.155032×10^{-3}

TABLE A1 - CONTINUED

B_{ij}			
i \ j	1	2	3
0	0.433787	0.172367×10^{-1}	0.281143×10^{-2}
1	0.436628×10^{-1}	0.123383×10^{-1}	0.168580×10^{-2}
2	0.114415×10^{-1}	0.265729×10^{-2}	0.385940×10^{-3}
3	0.188750×10^{-2}	0.378430×10^{-3}	0.558496×10^{-4}

From Eq. 22, f can be rewritten as,

$$f = J_0 + J_1 a_1 + J_2 a_2 \tag{A.5}$$

where,

$$J_0 = 20\eta^3 - 45\eta^4 + 36\eta^5 - 10\eta^6$$

$$J_1 = \eta - 10\eta^3 + 20\eta^4 - 15\eta^5 + 4\eta^6$$

$$J_2 = \eta^2 - 4\eta^3 + 6\eta^4 - 4\eta^5 + \eta^6$$

E_3, defined in Eq. 15, is calculated as follows:

$$\begin{aligned} E_3 = L_0 &+ 3L_1 a_1 + 3L_2 a_1^2 + 3L_3 a_2 \\ &+ 6L_4 a_1 a_2 + 3L_5 a_2^2 + L_6 a_1^3 \\ &+ 3L_7 a_1 a_2^2 + 3L_8 a_1^2 + L_9 a_2^3 \end{aligned} \tag{A.6}$$

where,

$$L_0 = \int_0^1 J_0^3 \, d\eta = 0.419722$$

$$L_1 = \int_0^1 J_0^2 J_1 \, d\eta = 0.0139592$$

$$L_2 = \int_0^1 J_0 J_1^2 \, d\eta = 0.254502 \times 10^{-2}$$

$$L_3 = \int_0^1 J_0^2 \, J_2 \, d\eta = 0.233197 \times 10^{-2}$$

$$L_4 = \int_0^1 J_0 \, J_1 \, J_2 \, d\eta = 0.383891 \times 10^{-3}$$

$$L_5 = \int_0^1 J_0 \, J_2^2 \, d\eta = 0.588166 \times 10^{-4}$$

$$L_6 = \int_0^1 J_1^3 \, d\eta = 0.11376 \times 10^{-2}$$

$$L_7 = \int_0^1 J_1 \, J_2^2 \, d\eta = 0.203184 \times 10^{-4}$$

$$L_8 = \int_0^1 J_1^2 \, J_2 \, d\eta = 0.149506 \times 10^{-3}$$

$$L_9 = \int_0^1 J_2^3 \, d\eta = 0.283514 \times 10^{-5} \qquad (A.7)$$

The value of $\partial f/\partial\eta$ can be found to be,

$$\frac{\partial f}{\partial \eta} = N_0 + a_1 \, N_1 + a_2 \, N_2 \qquad (A.8)$$

where,

$$N_0 = 60\eta^2 - 180\eta^3 + 180\eta^4 - 60\eta^5$$

$$N_1 = 1 - 30\eta^2 + 80\eta^3 - 75\eta^4 + 24\eta^5$$

$$N_2 = 2\eta - 12\eta^2 + 24\eta^3 - 20\eta^4 + 6\eta^5$$

Therefore, E_5 is given by,

$$E_5 = E_{50} + E_{51} \, a_1^2 + E_{52} \, a_2^2 + 2E_{53} \, a_1 + 2E_{54} \, a_1^2 + 2E_{55} \, a_2 \qquad (A.9)$$

where,

$$E_{50} = \int_0^1 N_0^2 \, d\eta = 1.55843$$

$$E_{51} = \int_0^1 N_1^2 \, d\eta = 0.173159$$

$$E_{52} = \int_0^1 N_2^2 \, d\eta = 0.288601 \times 10^{-2}$$

$$E_{53} = \int_0^1 N_0 \, N_1 \, d\eta = -0.170995$$

$$E_{54} = \int_0^1 N_1 \, N_2 \, d\eta = 0.194805 \times 10^{-1}$$

$$E_{55} = \int_0^1 N_0 \, N_2 \, d\eta = -0.129870 \times 10^{-1} \qquad \text{(A.10)}$$

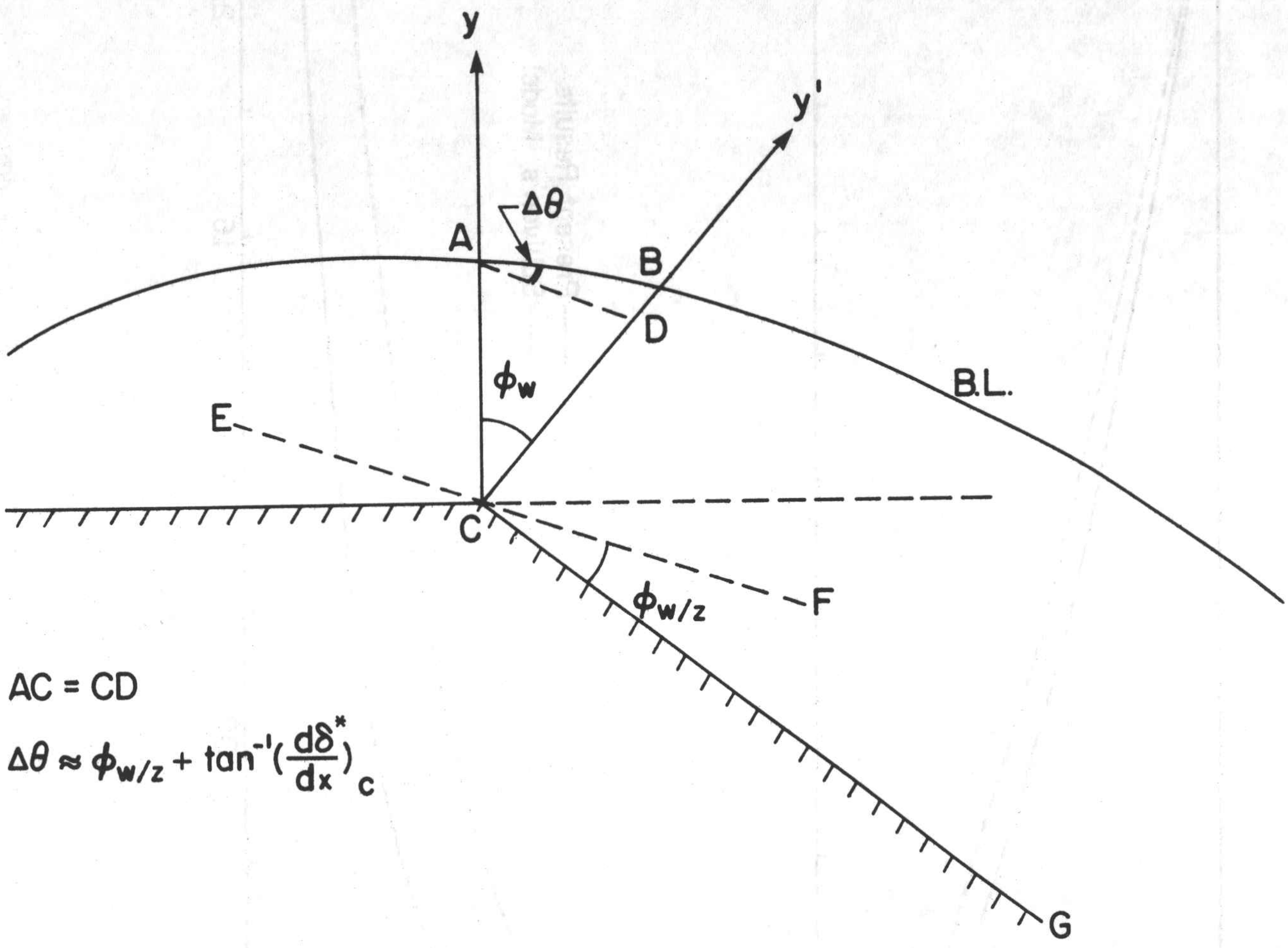

FIG. 1 GEOMETRY OF THE EDGE OF THE BOUNDARY LAYER AT THE CORNER. AC = CD IS THE INITIAL VALUE OF $\bar{\delta}/L$ BEFORE TURNING THE STREAMLINE. AD IS PARALLEL TO EF WHICH MAKES $\phi_w/2$ ANGLE WITH CG.

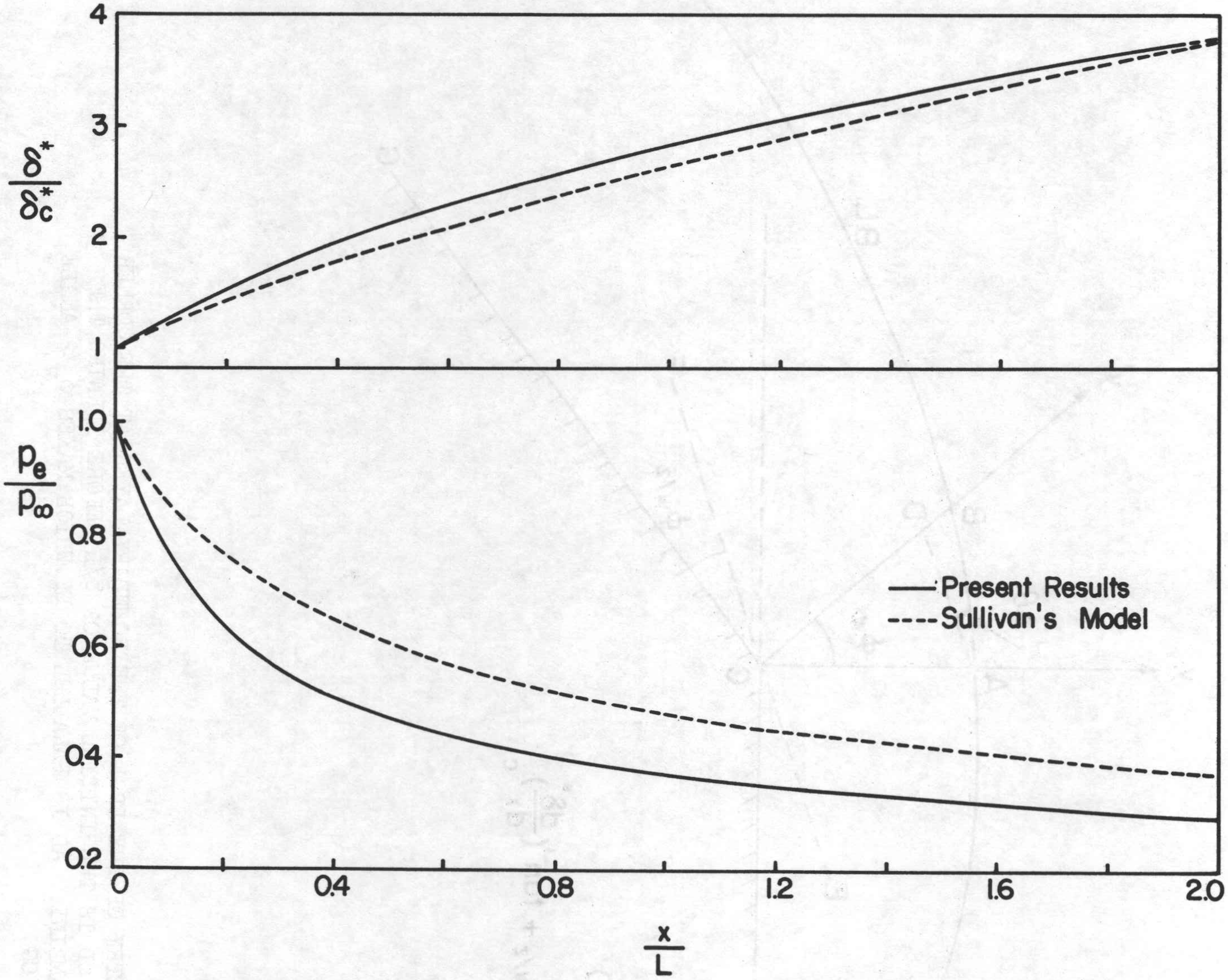

FIG. 2 VARIATIONS OF THE DISPLACEMENT-THICKNESS RATIO, δ^*/δ_c^*, AND THE PRESSURE RATIO, p_e/p_∞, WITH DISTANCE RATIO, x/L, FOR THE CASE OF $g_w = 0.2$, $\phi_w = 5^o$, $R_{e_\infty} = 1.644 \times 10^5$, $M_\infty = 10$ and $Pr = 1$.

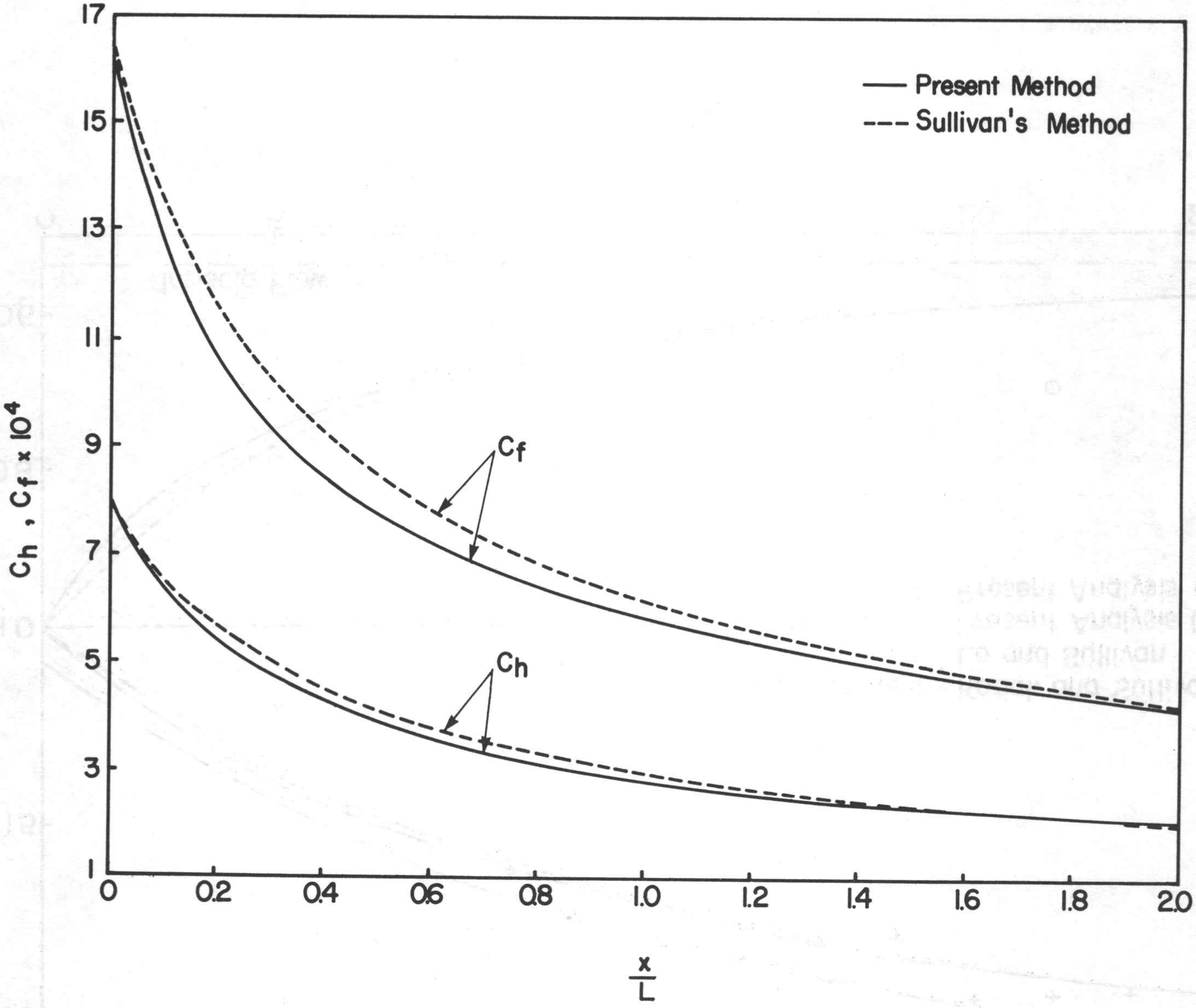

FIG. 3 VARIATIONS OF THE HEAT-TRANSFER AND SKIN-FRICTION COEFFICIENTS WITH THE DISTANCE RATIO FROM THE CORNER. THE INITIAL CONDITIONS ARE GIVEN IN FIG. 2.

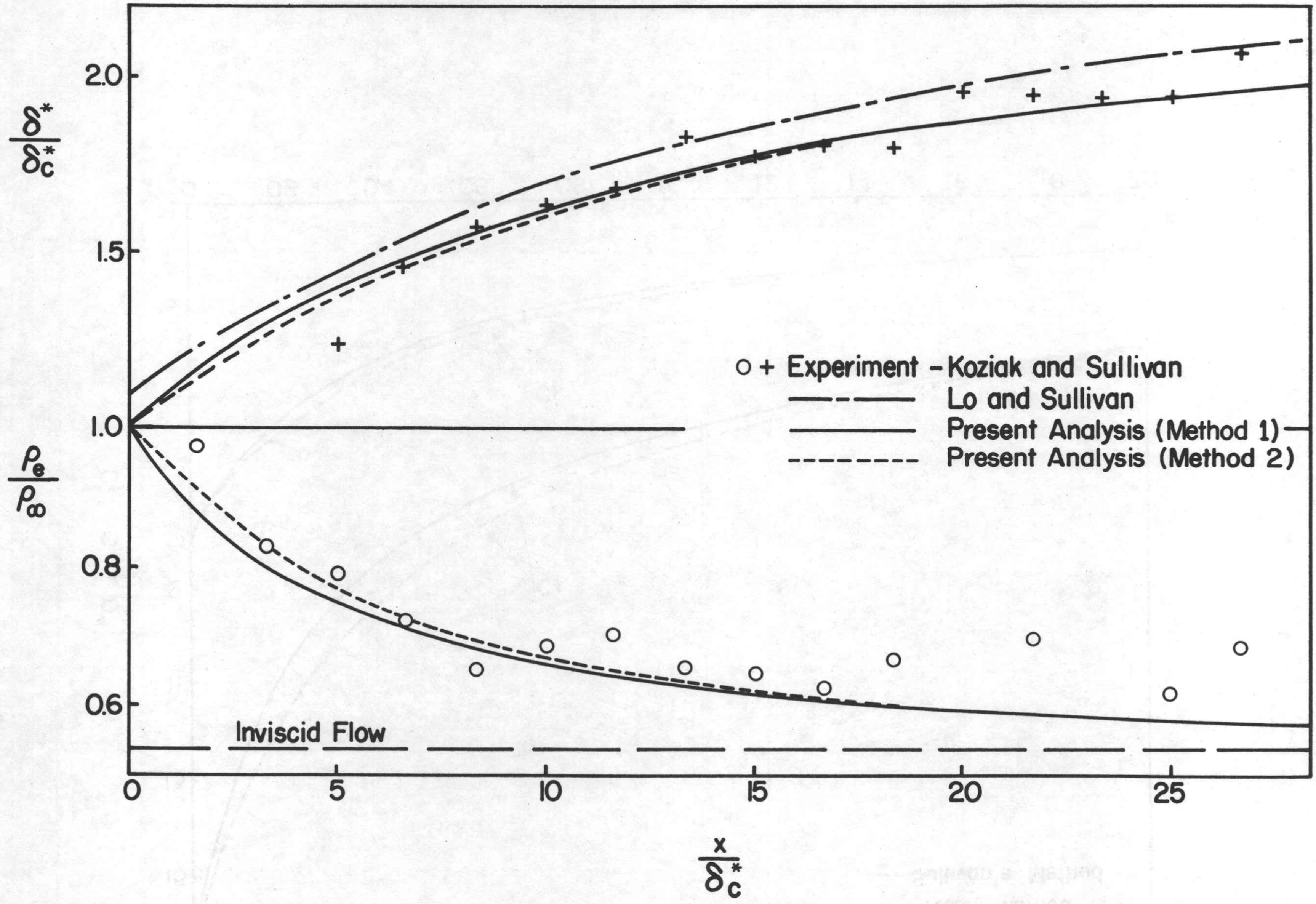

FIG. 4 VARIATIONS OF THE DISPLACEMENT-THICKNESS RATIO, δ^*/δ_c^*, AND THE DENSITY RATIO, ρ_e/ρ_∞, WITH THE DISTANCE RATIO, x/δ_c^*, FROM THE CORNER FOR THE CASE OF $g_w = 0.22$, $\phi_w = 5^o$, $Re_\infty = 8 \times 10^5$, $M_\infty = 6.5$ AND $Pr = 0.7$.

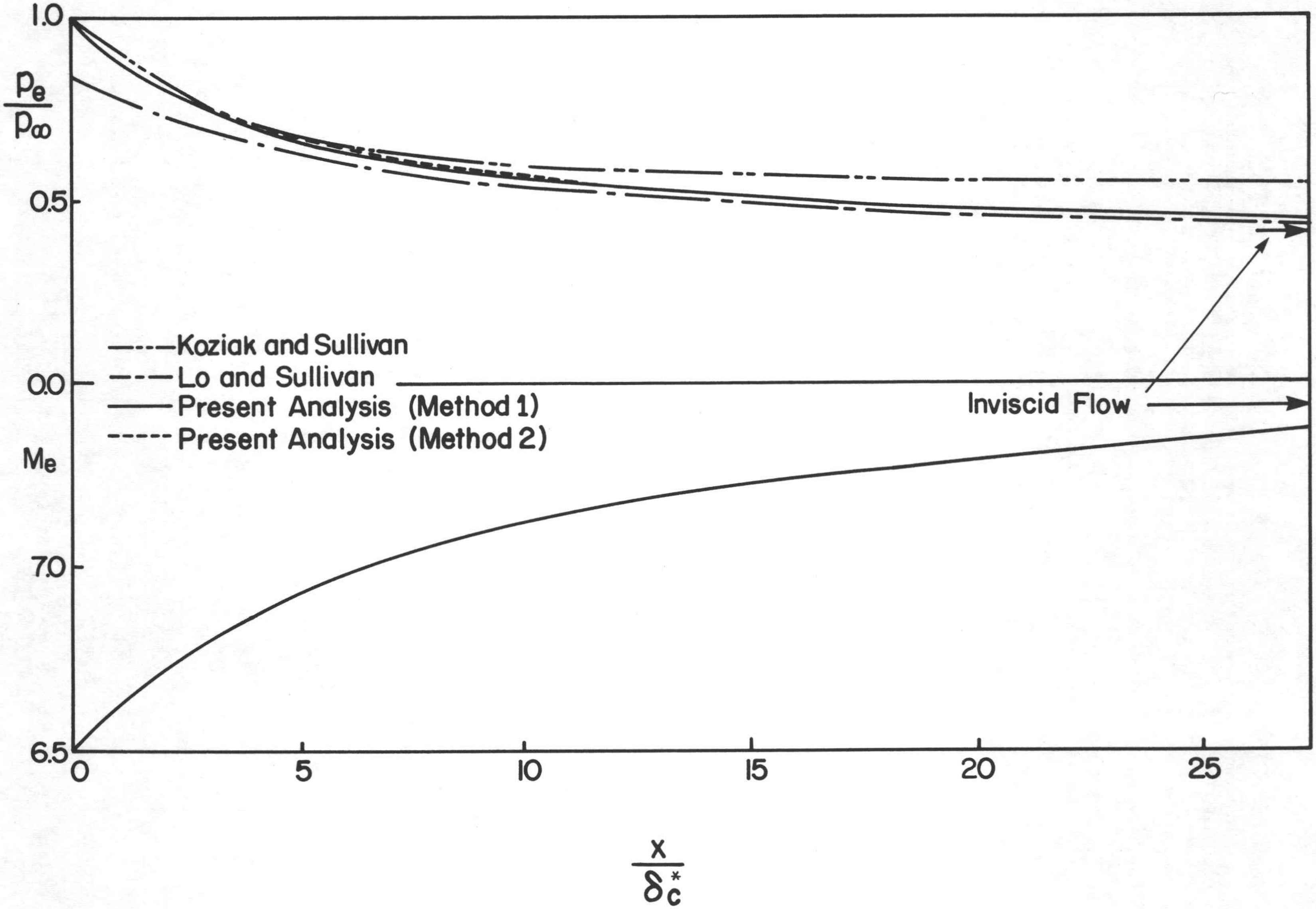

FIG. 5 VARIATIONS OF THE PRESSURE RATIO, p_e/p_∞, AND THE MACH NUMBER AT THE EDGE OF BOUNDARY LAYER, M_e, WITH THE DISTANCE RATIO, x/δ_c^*, FROM THE CORNER. THE INITIAL CONDITIONS ARE GIVEN IN FIG. 4.

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INTEGRAL METHOD OF SOLUTION FOR COMPRESSIBLE LAMINAR BOUNDARY LAYERS AND APPLICATIONS

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4. Cold-wall similarity model

1. Liu, W. S. 2. UTIAS Technical Note No. 195

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The cold-wall similarity model of the interaction of a compressible, laminar, boundary-layer flow with a corner-expansion wave is investigated using certain approximations and initial conditions. The results compare well with other analytical models and with existing experimental data.

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Liu, W. S. 26 pages 5 figures 1 table

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