ON THE DESTRATIFICATION OF LAKES AND RESERVOIRS
USING BUBBLE COLUMNS

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Symbols

- $a$: total depth
- $a_b$: depth of flow towards bubble column (near field)
- $a_{e}, a_h$: initial depths of epilimnion and hypolimnion
- $a_1, a_2, a_3$: depths of epilimnion, interlayer and hypolimnion
- $a_{21} = a_e - a_l$
- $a_{22} = a_h - a_3$
- $b$: coefficient
- $c$: wave celerity
- $c_f$: front celerity
- $f$: Coriolis parameter
- $f_i, f_i^*$: forces
- $g$: gravitational constant
- $h$: height of interface above a reference level
- $h_b$: height of bottom
- $k$: entrainment coefficient (near field), friction coefficient (far field)
- $m$: coefficient (near field)
- $n$: scale factor
- $p$: pressure
- $q$: flow rate per unit of width
- $r$: radial co-ordinate
- $r_o$: initial radius of fluid element
- $s_r, s_\phi$: shear stress components
- $t$: time
- $u, u_r, u_\phi$: local horizontal velocity components
- $u_{\infty}$: horizontal velocity component outside Ekman Layer
- $v, v_r, v_\phi$: depth averaged velocity components
- $w$: vertical velocity component, dimensionless velocity (app. II)
- $x, z$: horizontal and vertical co-ordinates
\( E_{i,j} \)  
entrainment velocity from \( j^{th} \) layer to \( i^{th} \) layer

\( F_i \)  
(square of) densimetric Froude number

\( K_z \)  
vertical eddy diffusivity

\( P \)  
pressure at interface

\( Q_a \)  
air flow rate under standard conditions

\( Q_i \)  
radial flow rate in \( i^{th} \) layer

\( Q_b \)  
rate of flow directed towards bubble column (near field)

\( Q_{i,j} \)  
rate of entrainment flow from \( j^{th} \) to \( i^{th} \) layer

\( R \)  
radius of near field

\( R_e \)  
effective radius of far field

\( R_m \)  
radius of near field in homogeneous situation

\( R_o \)  
radius of far field

\( T \)  
time scale, absolute temperature

\( V \)  
volume

\( \alpha \)  
coefficient (near field)

\( \beta \)  
entrainment coefficient (transition from near field to far field)

\( \beta_1, \beta_2 \)  
coefficients

\( \delta \)  
displacement thickness of Ekman Layer

\( \varepsilon = \rho_3 - \rho_1 \)/\( \rho_3 \) = \( \varepsilon_1 + \varepsilon_3 \), relative density difference between hypolimnion and epilimnion

\( \varepsilon_1 = \rho_2 - \rho_1 \)/\( \rho_3 \), relative density difference between interlayer and epilimnion

\( \varepsilon_3 = \rho_3 - \rho_2 \)/\( \rho_3 \), relative density difference between hypolimnion and interlayer

\( \varepsilon_z \)  
vertical eddy viscosity

\( \zeta = a_o/a \)

\( \zeta_o = a_o/a \)

\( \eta = r_o/R^2 \)

\( \theta, \theta_o \)  
shape factors for radial velocity profile

\( \nu \)  
molecular viscosity

\( \xi = r^2/R^2 \)

\( \xi_o = R_o^2/R^2 \)
\( \rho \) local density
\( \rho_i \) density of \( i \)th layer
\( \phi \) polar angle
\( \phi_i \) (square of) Froude number
\( \psi \) parameter (app. II)
\( \omega \) rotation

**subscripts**

\( \cdot \cdot i \) related to \( i \)th layer (epilimnion: \( i=1 \), interlayer: \( i=2 \), hypolimnion: \( i=3 \))
\( \cdot \cdot r \) radial component
\( \cdot \cdot \phi \) tangential component

**superscripts**

\( \cdot \cdot ^0 \) outer far-field
\( \cdot \cdot ^R \) at \( r=R \)
1. Introduction

Thermal stratification in spring and summer may cause severe deterioration in water quality in the hypolimnion of eutrophic lakes and reservoirs. The stratification, once it has formed under the influence of insolation and wind, reduces the vertical exchange of mass and momentum. As regards the water quality of the hypolimnion, in particular the decrease in the vertical transport of dissolved oxygen is detrimental.

The water quality can be improved by aerating the hypolimnion. This can be achieved, among other methods, by mixing the hypolimnion water with the epilimnion water usually having a higher dissolved-oxygen content, thereby destratifying the lake (e.g. 1, 2, 3, 4). This type of mixing can be brought about by mechanical pumping (3), or by injecting compressed air at the bottom of the lake. In the latter case a bubble plume results. When air is injected through a perforated tube, a bubble curtain is formed; local injection of air produces a bubble column (4). Neilson (2) shows that direct aeration as a result of air injection is negligible when compared with the uptake of oxygen from the atmosphere in the epilimnion and subsequent mixing with hypolimnion water. Air injection therefore also functions as a pumping device.

In this report the attention is directed to some hydraulic aspects of local air injection as a means to the destratification of lakes and reservoirs. Because of the complexity of the various phenomena involved a relatively simple situation is considered. The severest restrictions are:

- The initial stratification (i.e. before the air injection is started) is that of a two layer system: epilimnion and hypolimnion each have a constant temperature, and the thermocline is sharp.
- During destratification no heat exchange through the free surface takes place, and wind is absent (artificial
destratification is required particularly in periods of calms or light wind).

- The flow caused by a single bubble column only is considered. Inlet currents are left out of account.

The flow field induced by air injection at the bottom depends to a large extent on the fact whether the lake is stratified or not. In the case of a homogeneous ambient the flow field may be divided into two, more or less distinct, regions (fig. 1.1):

- A region with vertical two-phase flow of rising air bubbles and entrained ambient water (the bubble column).
- A region with radial outflow (radial jet) at the free surface, and the returning flow underneath of water entrained by the bubble column and the radial jet (the near field).

The horizontal reach of the near field is finite, its radius amounting to approximately five to seven times the total depth. At larger distances from the injection point (the far field) the water is practically stagnant.

fig. 1.1 - Air injection in a homogeneous ambient
In the case of a stratified ambient the flow pattern is different (fig. 1.2). The bubble column entrains hypolimnion water in the lower part of the vertical, and water from the returning flow above. The momentum of the outflowing water at the free surface forces the epilimnion to remove over such distance as to equilibrate the gravity force owing to density differences. At the plunge point (the point at the free surface where radial jet and epilimnion meet), part of the outflowing water mixes with epilimnion water; the mixed water forms an interlayer of intermediate density between epilimnion and hypolimnion. As a consequence of the mixing the flow in the epilimnion is directed towards the bubble column, as it is in the hypolimnion. Another portion of the outflowing water recirculates in the near field. The radius, $R$, of the near field is less than in the case of a homogeneous ambient. The far field ($r > R$) is characterized by a nearly horizontal flow in epilimnion, hypolimnion and interlayer. The interlayer initially spreads as an internal front after the air injection has been started. This front reflects at the banks (provided these are not too distant) and returns as an internal wave. Damping causes the wave phenomenon to develop gradually into a quasi-steady flow. The interlayer thickens during destratification.

This picture of the stratified-flow field will be refined in some respects at the appropriate places in the report.

Research as regards the bubble column and the near field was carried out at the Laboratory of Physical Technology of the Delft University of Technology (5, 6, 7). The investigation of the flow in the far field and the coupling with the near field was a research project of the Laboratory of Fluid Mechanics of this University (sections 2 and 3). The mathematical model is verified with experiments in a physical model (section 4) and in nature (section 5).

The research reported was carried out for the purpose of designing and operating air injection systems in man-made reservoirs situated in the south-west of the Netherlands. These reservoirs,
the contents of which varies from $13 \times 10^6 \text{ m}^3$ to $38 \times 10^6 \text{ m}^3$, serve the water supply of the town of Rotterdam and neighbouring municipalities (4).

fig. 1.2 - Air injection in a stratified ambient

2. Mathematical model of the far field

2.1. Equations

The simplest model giving a realistic description of the actual phenomena in the far field is a layer model (14) comprising, in this case, three layers: epilimnion, interlayer and hypolimnion. The temperature differences between the layers cause corresponding density differences. A layer model applies provided the densimetric Froude numbers are not too large - roughly speaking the flow should be subcritical (15) - since otherwise interfacial mixing would lead to such thick interfaces that a vertically continuous model would be more appropriate. Also background turbulence owing to wind may cause continuous velocity and temperature profiles (8). As explained in the Introduction, however, the occurrence of wind is not considered here. A severe problem when adopting a continuous model,
would be to define appropriate relationships between the turbulent-transport coefficients and the mean-flow variables. It seems to be sensible to examine the utility of the simpler model before turning to a turbulence model.

In a layer model the density is taken constant over the depth of a layer. Usually, the same is assumed as regards the water velocity, but a slight modification is introduced here in view of the influence of the Coriolis force. This force causes a marked rotation of the layers (8), the vector of rotation being vertical, and, in consequence, so-called Ekman (boundary) layers (e.g. 16, 17, 24). Ekman layers occur at the bottom owing to the no-slip condition, but possibly also at the interfaces between the layers since the flow in the layers is in opposite directions. The interfaces would then act more or less rigid boundaries. The velocity profiles outside the Ekman layers are essentially uniform (fig. 2.1). The thickness of these layers are assumed to be less than the layer depths (an estimate is given in appendix III), so that they do not overlap.

\[ \begin{align*}
\text{fig. 2.1 - Radial velocity profile in a layer}
\end{align*} \]
Two forms of the equations of motion may be adopted, viz. an integral form as regards the depth of a layer, and a form applying to points outside the Ekman layers, where vertical transport of momentum (shear stress) is absent (which is related to the locally uniform velocity profile). Disregarding transport of mass through the interfaces (see appendix IV, however) assuming axial symmetry and disregarding shape factors in the quadratic velocity terms, these equations may be written (fig. 2.2)

**depth-averaged form**

\[
\frac{\partial v_{ri}}{\partial t} + v_{ri} \frac{\partial v_{ri}}{\partial r} + \frac{1}{\rho_i} \frac{\partial p_{i-1}}{\partial r} + g \frac{\partial h_{i-1}}{\partial r} = \frac{v_{ri}^2}{r} + f v_{ri} - \frac{s_{r,i} - s_{r,i-1}}{\rho_i a_i} 
\]

\[
\frac{\partial \phi_i}{\partial t} + v_{ri} \frac{\partial \phi_i}{\partial r} = -\frac{v_{ri} v_{ri}}{r} - f v_{ri} - \frac{S_{\phi,i} - S_{\phi,i-1}}{\rho_i a_i} \tag{2.1.2}
\]

(no summation over i)

**local form outside Ekman layers**

\[
\frac{\partial u_{ri}}{\partial t} + u_{ri} \frac{\partial u_{ri}}{\partial r} + \frac{1}{\rho_i} \frac{\partial p_{i-1}}{\partial r} + g \frac{\partial h_{i-1}}{\partial r} = \frac{u_{ri}^2}{r} + f u_{ri} \tag{2.1.3}
\]

\[
\frac{\partial u_i}{\partial t} + u_{ri} \frac{\partial u_i}{\partial r} = -\frac{u_i u_{ri}}{r} - f u_{ri} \tag{2.1.4}
\]

in which \( r, \phi = \) polar co-ordinates, \( i = \) number of the layer (numbering from top to bottom, \( i=1 \) for the epilimnion), \( p_{i-1} = \) pressure at
interface between \((i-1)^{\text{st}}\) and \(i^{\text{th}}\) layer, \(h_{i-1}\) = height of this interface above a reference level, \(s_{r,i-1}\), \(s_{\varphi,i-1}\) = shear stress components at this interface, \(u\) = local velocity, \(v\) = depth-averaged velocity.

fig. 2.2 - Definition sketch of three-layer model

Assuming a hydrostatic pressure distribution yields

\[
P_i = p_{i-1} + \rho_i g a_i \tag{2.1.5}
\]

Assuming the density, \(\rho_i\), to be constant, the conservation of mass for the \(i^{\text{th}}\) layer is expressed by

\[
\frac{3}{r} r a_i + \frac{3}{\varphi} r a_i v_{ri} = 0 \tag{2.1.6}
\]

Equations (2.1.1) through (2.1.6) represent external as well as internal wave phenomena. It is shown in appendix I that approximate equations describing internal phenomena only, can be derived in the case where the relative density differences are small. The equations of interest here are

\[
h_b + a_1 + a_2 + a_3 = \text{constant} \tag{2.1.7}
\]

\[
a_1 v_{r1} + a_2 v_{r2} + a_3 v_{r3} = 0 \tag{2.1.8}
\]
and the depth-averaged equations of motion in radial direction:

\[
\frac{\partial}{\partial t} (v_r^1 - v_r^2) + \frac{1}{2} \frac{\partial}{\partial r} (v_r^2 - v_r^2) - \epsilon_1 g \frac{\partial}{\partial r} (a_2 + a_3) = f_1 - f_2 + \frac{\partial}{\partial r} \frac{d h_b}{d r} \tag{2.1.9}
\]

\[
\frac{\partial}{\partial t} (v_r^2 - v_r^3) + \frac{1}{2} \frac{\partial}{\partial r} (v_r^2 - v_r^2) - \epsilon_3 g \frac{\partial}{\partial r} a_3 = f_2 - f_3 + \frac{\partial}{\partial r} \frac{d h_b}{d r} \tag{2.1.10}
\]

\[
\frac{\partial}{\partial t} (v_r^2 - \frac{\epsilon_3 v_r^1 + \epsilon_1 v_r^3}{\epsilon}) + \frac{1}{2} \frac{\partial}{\partial r} (v_r^2 - \frac{\epsilon_3 v_r^1 + \epsilon_1 v_r^3}{\epsilon}) + \frac{\epsilon_1 \epsilon_3}{\epsilon} g \frac{\partial}{\partial r} a_2 = f_2 - \frac{\epsilon_3 f_1 + \epsilon_1 f_3}{\epsilon} \tag{2.1.11}
\]

The terms \( f_i \) (\( i = 1, 2, 3 \)) represent the right-hand sides of equations (2.1.1). The relative density differences, \( \epsilon_1 \) and \( \epsilon_3 \), are defined by

\[
\epsilon_1 = \frac{\rho_2 - \rho_1}{\rho_2}, \quad \epsilon_3 = \frac{\rho_3 - \rho_2}{\rho_2}, \quad \text{and} \quad \epsilon = \epsilon_1 + \epsilon_3 \tag{2.1.12}
\]

The continuity equations (2.1.6) and the equations of motion in tangential direction, (2.1.2), apply to this approximation also. The equations of motion (2.1.3) in radial direction outside the Ekman layers take forms similar to (2.1.9) through (2.1.11).

2.2. Double two-layer model

The restriction to internal phenomena only already introduces a simplification of the problem, but considerable difficulties remain.

Therefore further simplifications are desirable. The model experiments (section 4, also fig. 1.2) show that the interlayer forms more or less at the level of the original interface, and that afterwards net vertical motions of this layer are hardly perceptible. The interlayer appears to thicken only. It therefore seems worth examining the conditions under which the three-layer model allows
a solution in which the interlayer does not change in height. More precisely, the case is considered where no vertical transport of mass through the plane coinciding with the undisturbed thermocline takes place. The resulting model is referred to as the double two-layer model, since the flows at both sides of this plane may now be regarded as two-layer systems. These two-layer systems are coupled by the requirement that the velocities in the interlayer be equal in both systems (implying an identical pressure distribution at both sides of the plane of the undisturbed thermocline).

![Diagram of double two-layer model]

**Fig. 2.3 - Definition sketch of double two-layer model**

The conservation of mass then yields, analogous to (*I*.10) and (2.1.8) (fig. 2.3),

\[
\begin{align*}
a_1 v_1 + a_{21} v_2 &= 0 \\
 a_3 v_3 + a_{22} v_2 &= 0
\end{align*}
\]

(2.2.1) (2.2.2)

with

\[X\]

It is shown in section 3.4 that this model yields a good approximation as regards the growth of the interlayer and that a simple modification can be introduced to account for the correct level of this layer.
\( a_{21} + a_{22} = a_2 \) \hspace{1cm} (2.2.3)

\( a_1 + a_{21} = a_e \) \hspace{1cm} (2.2.4)

\( a_{22} + a_3 = a_h \) \hspace{1cm} (2.2.5)

\( a_e \) and \( a_h \) represent the undisturbed depths of epilimnion and hypolimnion. Furthermore (2.1.6) applies.

Substituting (2.2.1) through (2.2.5) into the equation of motion (2.1.9) and (2.1.10) yields, together with (2.1.7),

\[- \frac{3}{\theta t} \left( \frac{a_e}{a_1} v_{2r} \right) + \frac{1}{2} \frac{3}{\theta r} \left( \frac{a_e}{a_1} \left( \frac{a_e}{a_1} - 2 \right) v_{2r}^2 \right) + \frac{3}{\theta r} (\varepsilon_1 a_1) = f_1 - f_2 \]

\[= -\frac{3}{\theta t} \left( \frac{a_h}{a_3} v_{2r} \right) + \frac{1}{2} \frac{3}{\theta r} \left( \frac{a_h}{a_3} \left( \frac{a_h}{a_3} - 2 \right) v_{2r}^2 \right) + \frac{3}{\theta r} \left[ \varepsilon_3 (h_b + a_3) \right] = -f_2 + f_3 \]

(2.2.6)

(2.2.7)

A change in \((h_b + a_3)\) is now coupled with a change in \(a_1\), and \((h_b + a_3)\) may be conceived of as a function of \(a_1\). (2.2.7) may therefore be seen as an equation in \(v_2\) and \(a_1\), as is (2.2.6). These equations will be compatible, provided

\[\frac{a_e}{a_1} = \frac{a_e}{a_1} \left( \frac{a_e}{a_1} - 2 \right) \]

\[\frac{a_h}{a_3} = \frac{a_h}{a_3} \left( \frac{a_h}{a_3} - 2 \right) \]

\[\frac{\varepsilon_1 a_1}{\varepsilon_3 (h_b + a_3)} = \frac{f_1 - f_2}{f_3 - f_2} \]

(2.2.8)

which gives the conditions

\[\frac{a_1}{a_e} = \frac{a_3}{a_h} \]

(2.2.9)
\[
\frac{a_l}{h_b + \frac{a_h}{a_e} a_l} = \frac{\varepsilon_3}{\varepsilon_1} \quad (2.2.10)
\]

\[f_1 = f_3 \quad (2.2.11)\]

Since \(\varepsilon_1\) and \(\varepsilon_3\) are constants, (2.2.19) shows that the bottom should be horizontal for the double two-layer model to be applicable (the depth of the undisturbed hypolimnion then is constant also). Henceforth it is assumed that

\[h_b = 0 \quad (2.2.12)\]

Consequently, (2.2.10) yields as a condition for the relative density of the interlayer

\[\frac{\varepsilon_3}{\varepsilon_1} = \frac{a_e}{a_h} \quad (2.2.13)\]

Defining the (known) relative density

\[\varepsilon = \frac{\rho_3 - \rho_1}{\rho_3} \quad (2.2.14)\]

(2.2.13) yields

\[\varepsilon_1 = \frac{a_h}{a} \varepsilon \quad \text{and} \quad \varepsilon_3 = \frac{a_e}{a} \varepsilon \quad (2.2.15)\]

The applicability of the double two-layer model therefore requires the interlayer to have the density arising when epilimnion and hypolimnion are completely mixed (\(\rho_2 a = \rho_1 a_e + \rho_3 a_h\)).
A somewhat restrictive condition is represented by (2.2.11) indicating that the force terms related to top layer (i=1) and bottom layer (i=3) should be equal (see (2.1.1)), i.e.

$$\frac{v_{\phi 1}^2}{r} + f \frac{v_{\phi 1}}{\rho_1 a_1} = \frac{v_{\phi 3}^2}{r} + f \frac{v_{\phi 3}}{\rho_3 a_3} - \frac{s_{r,1}}{r} - \frac{s_{r,2}}{r}$$

(2.2.16)

In general, this requirement is not satisfied. In section 2.3 it is shown that $v_{\phi 1}$ and $v_{\phi 3}$ differ little so that centrifugal forces and Coriolis forces in (2.2.16) are approximately equal, but the friction terms may differ considerably. In a prototype situation, however, the friction term in (2.2.16) presumably is relatively small when compared with the other two terms on the RHS (appendix III). In a small scale model the tangential velocities practically vanish, and (2.2.11) is no doubt violated. The condition for critical flow, which controls part of the destratification process (section 3), is not influenced by friction, however. Also the velocities in the far field are small after the interlayer has formed. For these reasons, friction is assumed to be of minor importance in (2.2.16), and (2.2.11) and (2.2.16) are assumed to be (approximately) satisfied.

At the end of this section equations are derived in which the effect of friction (and Ekman layers) is incorporated in a somewhat different way.

Equations (2.2.1) through (2.2.5) give the following results.

$$a_1 = \frac{a}{a} (a - a_2)$$
$$a_3 = \frac{a}{a} (a - a_2)$$

(2.2.17)

$$v_{r1} = v_{r3} = - \frac{a_2}{a - a_2} v_{r2}$$

(2.2.18)

The latter equation indicates that the water velocities in epilimnion and hypolimnion are equal in this model. The equation of motion for
the interlayer, (2.1.11), changes into
\[
\frac{\partial}{\partial t} \left( \frac{a}{a - a_2} \nu_r^2 \right) + \frac{1}{2} \frac{\partial}{\partial r} \left[ \frac{a(a-2a_2)}{(a-a_2)^2} \nu_r^2 \right] + \frac{a}{a_2} \frac{a e h}{2} \epsilon g \frac{\partial a_2}{\partial r} = f_2 - \frac{a}{a} f_1 - \frac{a}{a} f_3
\]
(2.2.19)

and the continuity equation for the interlayer,
\[
r \frac{\partial a_2}{\partial t} + \frac{\partial}{\partial r} \left( a_2 \nu_r^2 \right) = 0
\]
(2.2.20)

form a system of two equations for the variables \(a_2\) and \(\nu_{r2}\). The equations of motion in tangential direction remain unaltered in this model.

The celerities of internal waves in the double two-layer model can be determined from (2.2.19) and (2.2.20) using, for instance, the method of characteristics (10). The result is
\[
c = \frac{a_2}{a - a_2} \nu_r^2 \pm \sqrt{\frac{a_2}{a - a_2} \left[ \frac{a e h}{a_2} \epsilon g a \left( 1 - \frac{a_2}{a} \right)^2 - \frac{2}{\nu_r^2} \right]}
\]
(2.2.21)

Comparing the number of celerities of internal waves with that of the general three-layer model, namely four (appendix I), it is found that two celerities have been eliminated. This is related to the disregarding of waves displacing the interlayer in vertical direction.

The flow is critical when \(c = 0\), or
\[
\nu_{r2}^2 = \frac{a e h}{a^2} \epsilon g a_2 \frac{(a - a_2)^3}{(a - a_2)^3 + a_2^3}
\]
(2.2.22)

The second, not vanishing, celerity then is
\[ c = 2(a - 2a_2) \sqrt{\frac{e a h}{a^2} \epsilon g a_2} - \frac{a_2}{(a - a_2)^3 + a_2^3} \quad (2.2.23) \]

The celerity changes its sign when \( a_2 = \frac{1}{2} a \). The flow is therefore double-critical when

\[ \frac{a_2}{a} = \frac{1}{2}, \quad \frac{v_2}{a} = \frac{1}{4} \frac{e a h}{a^2} \epsilon g a \quad (2.2.24) \]

In this special case it is possible to obtain all four celerities from (I.19) explicitly. The result is

\[ c_1 = -\frac{1 + \sqrt{2}}{2} \sqrt{\frac{e a h}{a^2} \epsilon g a}, \quad c_2 = c_3 = 0, \quad c_4 = -\frac{1 - \sqrt{2}}{2} \frac{e a h}{a^2} \epsilon g a \]

The celerities are real; two of them are zero, consequently this flow type is on the boundary of stability.

So far, continuous variations in layer depths and velocities were considered, which made possible a description using differential equations. The interlayer spreads as an internal front (e.g. 18, 19), which may be conceived of as a discontinuity within the framework of a layer model. Equations for such a discontinuity may be derived following the approach pursued by Benjamin (20), and Wilkinson and Wood (21) for a two-layer system. Applying the results to upper and lower two-layer systems yields for the celerity \( c_{f1} \) of the front (fig. 2.4), respectively,

\[ c_{f1}^2 = \frac{e a_2}{a e} \left( \frac{e - a_2}{e + a_2} \right) \left( \frac{2a - a_2}{e} \right) \quad (2.2.25) \]
\[ c_{f3}^2 = \epsilon_3 \frac{a_{22}}{a_h} \frac{(a_h - a_{22})(2a_h - a_{22})}{a_h + a_{22}} \quad (2.2.26) \]

It is required for the double two-layer model to be consistent, that the celerities \( c_{f1} \) and \( c_{f3} \) be equal. Substituting from (2.2.1), (2.2.2), (2.2.15), (2.2.17) and (2.2.18) it is found that this requirement is satisfied indeed, and

\[ c_f = c_{f1} = c_{f3} = \sqrt[4]{\frac{a_2 e a_h}{a} \left( \frac{a_2}{a} \left( 1 - \frac{a_2}{a} \right) \left( 2 - \frac{a_2}{a} \right) \right)} \frac{a_2}{1 + \frac{a_2}{a}} \quad (2.2.27) \]

![Diagram](image)

**Fig. 2.4 - Definition sketch of internal front**

(Shape of front is schematized)

\( c_f \) is maximum for \( a_2 \approx 0.347a \). The maximum celerity, \((c_f)_{\text{max}}\), amounts to

\[ (c_f)_{\text{max}} = 0.527 \sqrt{\frac{a_2 e a_h}{a} \frac{a_h}{a}} \quad (2.2.28) \]

\((c_f)_{\text{max}}\) is slightly greater than the double-critical flow velocity given by (2.2.24).

If the thickness of the front would be greater than 0.347a, its celerity would be less than \((c_f)_{\text{max}}\). As a result, a part of the
front with a thickness of 0.347a would start to travel with a celerity equal to \((c_f)_{\text{max}}\). The remaining part of the front would flatten and become a continuous wave. Consequently, a front is stable only if

\[ a_2 < 0.347a \]  \hspace{1cm} (2.2.29)

This section is concluded with the adaptation of the equations of motion (2.1.3) outside the Ekman layers to the double two-layer model. Substituting from (2.2.25), a procedure similar to that explained above yields

\[ u_{r1} = u_{r3} \]  \hspace{1cm} (2.2.30)

\[ \frac{\partial}{\partial t} (u_{r2} - u_{r1}) + \frac{1}{2} \frac{\partial}{\partial r} (u_{r2}^2 - u_{r1}^2) + \frac{a}{r} \frac{\partial}{\partial r} \left( \frac{a}{2} \frac{\partial a_2}{\partial r} \right) = f' - \frac{a}{r} \frac{\partial f'}{\partial r} - \frac{a_h}{a} f' \]  \hspace{1cm} (2.2.31)

\[ f'_{1} = f'_{3} \]  \hspace{1cm} (2.2.32)

in which \( f'_{i} \) (\(i = 1, 2, 3\)) represents terms on the RHS of (2.1.3). (2.2.32) implies that

\[ u_{\phi1} = u_{\phi3} \]  \hspace{1cm} (2.2.33)

In section 2.3 this requirement is shown to be satisfied. The effect of friction is absent in (2.2.31). The equations of continuity, (2.2.18) and (2.2.20), however, are in the variables \(v_{ri}\), which differ from the velocities \(u_{ri}\) owing to friction. The effect of friction may be introduced by putting
\[ v_{ri} = \theta u_{ri} \]

in which \( \theta \) is a positive coefficient \(^*)\). An advantage of this formulation of the friction effect is, that the coefficient \( \theta \) can be easily determined from measured velocity profiles, whereas friction terms in (2.1.1) cannot. In fact, different coefficients should be introduced for each layer. Since the intention here is to show only the tendency of the effect of friction, however, this is not done. Equations (2.2.31) and (2.2.20) now become

\[
\frac{\partial}{\partial t} \left( \frac{a}{a-a_2} u_{r2} \right) + \frac{1}{2} \frac{\partial}{\partial r} \left[ \frac{a(a-2a_2)}{2} \right] \frac{\partial}{\partial r} u_{r2} + \frac{a}{2} \frac{a}{a} \frac{\partial}{\partial r} a_2 - \frac{a_2}{a} f_1 + \frac{a_2}{a} f_3 = 0
\]

(2.2.35)

\[
r \frac{\partial a_2}{\partial t} + \frac{\partial}{\partial r} \theta r a_2 u_{r2} = 0
\]

(2.2.36)

These equations are solved in the next section.

2.3. Solution of the far field equations in the case of quasi-steady flow

The travelling time, \( T_t \), of an internal wave is given by

\[
T_t = \frac{R}{c}
\]

(2.3.1)

\(^*)\) The coefficient \( \theta \) would be unity if friction were absent, and less than unity owing to friction in a non-rotating fluid. Ekman layer suction (appendix III), however, may cause significant radial mass transport in the Ekman layers (e.g. 39). In that case \( \theta \) may become much larger than unity.
in which $R_o =$ radius of the far field (assumed to be circular),
and $c =$ wave celerity. The order of magnitude of the time $T_d$, needed
to destratify the lake is

$$T_d \approx \frac{\pi R^2 a}{Q^R_2}$$

(2.3.2)

in which $Q^R_2 =$ mean flow rate entering the interlayer at the
transition ($r=R$) from near field to far field. The order of magnitude
of this flow rate is

$$Q^R_2 \sim 2\pi R \cdot \frac{1}{2} \cdot a \cdot \frac{v^R}{r_2}$$

(2.3.3)

in which $v^R_{r_2} =$ mean radial velocity in the interlayer at the
transition. The order of magnitude of $v^R_{r_2}$ is (see section 3 for
the relevance of the condition of critical flow at the transition)

$$\frac{v^R}{r_2} \sim c$$

(2.3.4)

The above equations yield as an order of magnitude of the ratio
of $T_t$ and $T_d$

$$\frac{T_t}{T_d} \sim \frac{R}{R_o}$$

(2.3.5)

In the case of stratified flow the radius, $R$, of the near field
usually amounts to one to three times the total depth. Considering
only relatively shallow lakes or reservoirs, it follows that this
ratio is small and that many wave reflections take place during
the destratification process. Since the initial front of the
developing interlayer spreads radially, it is already subject to
considerable weakening during propagation. In principle the front reflects at the banks generating an internal wave travelling back towards the bubble column. Owing to weakening of the front and possible additional damping, however, reflected waves were not observed in the physical model (section 4). It may therefore be assumed that a state of quasi-steady flow is reached in an early stage of the destratification process. As regards the equations derived in section 2.2 this is tantamount to neglecting the time derivative in the equation of motion (2.2.19) or (2.2.35). The latter equation then becomes

$$\frac{\partial}{\partial r} \left[ \frac{a(a-2a_z)}{2(a-a_z)^2} u_x^2 r + \frac{a a_x}{a_z} \varepsilon g a_z \right] = f'_{2} - \frac{a}{a} f'_{1} - \frac{a}{a} f'_{3} \quad (2.3.6)$$

Next, the terms on the RHS of (2.3.6) are considered. As the original equation of motion (2.1.3) shows, the forces $f'_i$ ($i = 1, 2, 3$) represent centrifugal forces and Coriolis forces. The tangential velocities $v_{\phi i}$ in (2.1.1) are determined as follows. As a first step (2.1.4) is solved for $u_{\phi i}$. The characteristics of this first order equation are given by

$$\frac{dt}{u_{ri}} = - \frac{du_{\phi i}}{u_{ri} (\frac{u_{\phi i}}{r} + f)} \quad (2.3.7)$$

Prescribing initial conditions on the characteristics, two cases must be distinguished depending on the sign of $u_{ri}$ (fig. 2.5 a and b):

(i) Interlayer ($i=2, u_{r2} > 0$). Assuming the outflow in the near field radially directed, the boundary condition at $r=R$ is

$$u^R_{\phi 2} = 0 \quad (2.3.8)$$
The superscript $R$ denotes a variable at the transition ($r=R$). Integrating (2.3.7) then yields

$$u_{\phi 2} = -\frac{1}{2} f \frac{r^2 - R^2}{r}$$  \hspace{1cm} (2.3.9)

(ii) Epilimnion and hypolimnion ($i=1$ or $3$, $u_{r i} < 0$). The initial condition at $t=0$ is

$$u_{\phi i} = 0$$  \hspace{1cm} (2.3.10)

Integrating (2.3.7) now yields

$$u_{\phi i} = \frac{1}{2} f \frac{(r_o^2 - r^2)}{r}$$  \hspace{1cm} (2.3.11)

The radius $r_o$ denotes the position at which the initial condition is prescribed, and is also the starting position of the fluid particles travelling along the projection in the $r,t$-plane of a particular characteristic considered. Formally, it is given by

$$r_o = r - \int_0^t (u_{r i})_{\text{char}} \, dt$$

in which $(u_{r i})_{\text{char}}$ is the radial velocity component along the characteristic considered. This relationship shows that $r_o$ may be seen as a function of $r$ and $t$. (2.3.11) shows that (2.2.33) is satisfied.

A more useful expression for the radius $r_o$ can be derived by setting up a Lagrangian continuity equation (fig. 2.6).
Fig. 2.5 - Projections of characteristics in r,t-plane, a. interlayer, b. epilimnion and hypolimnion

Fig. 2.6 - Deformation of an annular fluid element in the epilimnion

Consider an annular fluid element in the epilimnion. Its initial volume, \( dV_o \), is given by, since \( a_2 = 0 \) at \( t = 0 \),

\[
dV_o = (a_e - \delta) 2 \pi r_0 \ dr_o
\]

(2.3.12)
in which $\delta_1 = \text{displacement thickness of the Ekman layer in the epilimnion.}$ For the sake of convenience, all displacements thicknesses are assumed to be constant in place and time (see the comment on friction at the end of section 2.2). When time elapses, the interlayer thickens and the fluid element moves in the direction of the bubble column. Its volume, $dV,$ is then given by

$$dV = (a_e - \frac{a}{a} a_2 - \delta_1) \ 2\pi \ r \ dr$$

(2.3.13)

Disregarding Ekman layer suction *) the volume of the fluid element remains constant, i.e. $dV = dV_0,$ one obtains

$$\frac{r_o}{r} \ \frac{\partial r_0}{\partial r} = 1 - \frac{1}{\theta_0} \frac{a_2}{a}$$

(2.3.14)

in which

$$\theta_0 = 1 - \frac{\delta_1}{a_e}$$

(2.3.15)

(2.3.14) applies to the hypolimnion also.

Substituting (2.3.9) and (2.3.11), and making use of the double two-layer model, (2.3.6) becomes

$$\frac{\partial}{\partial r} \left[ \frac{a(a - 2a_2)}{2(a - a_2)^2} \ \frac{u^2}{r^2} + \frac{a_e a_h cga_2}{a^2} \right] = -\frac{1}{4} \ \frac{r^2}{r^3} \ \frac{r_o^4 - R^4}{r^4}$$

(2.3.16)

*) Incorporating Ekman layer suction in the analysis would reduce the effect of the Coriolis force. Consequently, an upper bound of this effect is considered here.
The velocity, \( u_{r2} \), in (2.3.16) can be related to the Lagrangian formulation of the continuity equation. The total differential of \( r_o \) is
\[
dr_o = \frac{\partial r_o}{\partial r} dr + \frac{\partial r_o}{\partial t} dt
\]
\( r_o \) is a constant along a particle trajectory given by \( dr/dt = u_{r1} \).
In that case \( dr_o = 0 \), and
\[
0 = \frac{\partial r_o}{\partial r} u_{r1} + \frac{\partial r_o}{\partial t}
\]
(2.3.17)

On substitution from (2.3.14), (2.2.34) and (2.2.18) this gives
\[
ra_2u_{r2}(1 - \frac{1}{\delta o} a_2) = (a - a_2)r_o \frac{\partial r_o}{\partial t}
\]
(2.3.18)

To check this result, \( r_o \) may be eliminated between (2.3.14) and (2.3.18). Using the relationship
\[
\delta = 1 - \frac{\delta 1}{a_2} = \frac{\delta a - a_2}{a - a_2}
\]
(2.3.19)
the Eulerian continuity equation (2.2.36) is then recovered.

The three equations (2.3.14, 2.3.16) and (2.3.18) in the three dependent variables \( r_o, a_2 \) and \( v_{r2} \) form a nonlinear system of second order (this can be seen by eliminating \( a_2 \) and \( v_{r2} \), for instance). The initial conditions are
\[
r_o(r,0) = r
\]
(2.3.19)
\[
a_2(r,0) = 0
\]
The boundary conditions are

\[ r_0(R_0, t) = R_0 \]

\[ a_2(R, t) = a_2^R(t) \]  \hspace{1cm} (2.3.20)

\[ v_{r2}(R_0, t) = 0 \]

The solution of the equations yields the variables \( r_0, a_2 \) and \( v_{r2} \) as functions of place and time. The main interest, however, concerns the values of the variables at the transition from near field to far field to render possible the coupling with the near field (section 3). Also an effective radius, \( R_e \), of the far field is defined according to

\[ \pi (R_e^2 - R^2) a_2^R = 2\pi \int_R^{R_0} a_2 r \, dr \]  \hspace{1cm} (2.3.21)

If \( a_2 \) would be equal to \( a_2^R \), the effective radius would equal the radius, \( R_0 \), of the far field. The Coriolis force causes a decrease of \( a_2 \) with increasing \( r \), however, so that the effective radius is less than the actual radius. The effective radius is a convenient parameter when computing the accumulation of mixing water in the far field (section 3). Using (2.3.14) and (2.3.20) it is easily shown that

\[ R_e^2 = \frac{a}{a_2^R} (r_0^R)^2 - (\frac{a}{a_2^R} - 1)R^2 \]  \hspace{1cm} (2.3.22)

An approximate solution to the above equations is obtained in appendix II using the method of matched asymptotic expansions (e.g., 22, 23). The effect of the Coriolis force in a prototype situation turns out to be considerable. This is elucidated in
fig. 2.7 - Typical particle trajectories, a. epilimnion and hypolimnion, b. interlayer

Fig. 2.7 showing typical particle trajectories based on the outer far-field solution given in appendix II (also 8). Fig. 2.8 shows that the effective radius decreases when the influence of the Coriolis force increases (increasing $\psi$). It is notable that for large actual radius, the effective radius becomes independent of the actual radius. Apparently, the effective reach of the far field is finite, however large the lake may be.

3. Coupling conditions at the transition from near field to far field

3.1. Physical considerations

The flow at the transition from near field to far field (fig. 1.2) shows an analogy with the free-surface flow of a
fig. 2.8 - Effective radius of far field \( a_2^R/a = 0.5 \)

homogeneous liquid over a broad-crested weir. The hydraulics of flow over weirs is analyzed in \((28)\), for instance. In the present case, it is worth devoting some attention to the situation where the flow at some distance upstream from the weir is supercritical and a hydraulic jump has formed in front of the weir (fig. 3.1). This flow configuration will occur, provided the height of the weir is greater than a critical value depending on the total head of the upstream flow. Directly downstream from the jump the flow is subcritical. The flow over the weir can be an unsubmerged flow (in the sense that downstream conditions do not influence the flow at the weir), or a submerged flow. In the latter case the downstream water level is so high, that it does influence the flow over the weir. If the flow over the weir is unsubmerged, it will be critical (i.e. the celerity of long-wave disturbances in upstream direction is zero). The downstream flow then is supercritical (fig. 3.1a). Subcritical flow over and downstream of the weir will occur, if the flow is submerged (fig. 3.1b).
fig. 3.1 - Flow over a broadcrested weir,
a. unsubmerged flow, b. submerged flow

Returning to the flow caused by a bubble column in a stratified ambient (fig. 1.2), some analogy with the problem discussed above may be observed. In this case it is the epilimnion that obstructs the outflow of the radial jet at the free surface of the near field. The epilimnion in this sense plays the part of the weir in the one-layer problem. The observed eddying flow in front of the epilimnion \((r < R)\) is the counter part of the hydraulic jump. A difference is, that water from the epilimnion is now entrained by the radial jet. The flow at the transition \((r = R)\) may be expected to be internally critical as long as it is free, i.e. as long as the accumulation of mixing water in the far field does not influence the flow at the transition (this phase will be indicated as phase I). When the interlayer has thickened to a certain depth, the flow becomes 'submerged' and, in consequence, subcritical (phase II). It is shown in section 3.2. that critical flow would occur a second time in a later phase of the destratification process, since the flow rates
in the (thinned out) epilimnion and hypolimnion then become extreme (phase III). The situation then resembles that considered by Rigter (29). Rigter examined the two-layer flow resulting from the (horizontal) discharge of fresh water at the bottom of an outlet channel containing water of larger density (fig. 3.2, \( \rho_0 < \rho_2 \)). The turbulent fresh water tends to move upwards, but meanwhile it entrains water from the channel. The mixing water forms an upper layer of intermediate density (\( \rho_0 < \rho_1 < \rho_2 \)). Rigter obtained reasonable to good agreement with experimental evidence by assuming an extreme flow rate entrained from the lower layer. Fig. 3.2 has to be turned upside down for comparison with fig. 1.2.

![Two-layer flow diagram](image)

**fig. 3.2 - Two-layer flow in an outlet channel, \( \rho_0 < \rho_2 \)**

In the case under consideration, phase III would not occur until the epilimnion and hypolimnion had become very thin. It is likely, however, that a recirculation phenomenon spoiling the picture outlined (section 4.4.1) has then already come into existence.

The velocity in the hypolimnion is determined mainly by the amount of water entrained by the bubble column. Also the eddy in front of the epilimnion intermittently entrains some hypolimnion water (section 4.4.1).

The velocity in the epilimnion is determined by entrainment into the radial jet in the near field.
The near field is assumed to be essentially identical to the corresponding part in the homogeneous case. Some modifications accounting for density differences are introduced later. The homogeneous near field data are taken from Goossens and Van Pagee (6); these data, which do not include the effect of the Coriolis force, are summarized in appendix V.

The radius, R, of the near field is not known beforehand. It is an important parameter when calculating the flow rates in the layers, and is determined by applying an integral momentum balance to a small region including the transition (section 3.3).

The results of appendix I can be used to determine the required number of boundary conditions at the transition. According to equation (I.21) the number of non-positive wave celerities has to be determined. In phase I one of the celerities vanishes (critical flow), one is negative, and the remaining two are positive. Consequently, two boundary conditions are needed. It is obvious from the foregoing to prescribe the velocities in the hypolimnion (coupling with entrainment into the bubble column) and in the epilimnion (coupling with entrainment into the radial jet). These two boundary conditions also apply to phase II, since two celerities are then negative, and two are positive.

In phase III the situation is different: two celerities are negative, one vanishes, and the fourth is positive. Consequently, only one boundary condition is required, and it seems difficult to judge which of the two aforementioned conditions applies here. This problem is rather irrelevant, however, because of the occurrence of recirculation.

The double two-layer model is subject to an implicit condition, viz. no vertical mass transport through the plane of the undisturbed thermocline. In consequence, the above numbers of boundary conditions have to be decreased by one when using this model. Obviously this is contradictory. Nevertheless it turns out to be possible to use the double two-layer model as a first approximation. One of the boundary
conditions is then to be prescribed in phases I and II. Prescribing the boundary condition for the velocity in the hypolimnion was found to yield predictions which are already close to the experimental results. In a second, correcting step the second boundary condition can be satisfied. The reason for applying this artifice lies in the simplicity of the double two-layer model. The correction is also simple. The double two-layer model is used in the next two sections; the correction is dealt with in section 3.4.

3.2. The three phases of the destratification process

For ease of survey the results obtained in section 2.2 are summarized here as far as relevant to the coupling problem. The superscript, $R$, denotes a quantity at the transition.

$$
\varepsilon_1 = \frac{a}{a} \varepsilon \\
\varepsilon_3 = \frac{a}{a} \varepsilon \\
(3.2.1)
$$

$$
a_1^R = \frac{a}{a} (a - a_2^R) \\
a_3^R = \frac{a}{a} (a - a_2^R) \\
(3.2.2)
$$

$$
v_{r1}^R = v_{r3}^R = -\frac{a}{a} \frac{a_2^R}{a - a_2^R} v_{r2}^R \\
(3.2.3)
$$

$$
q_1 = -\frac{a}{a} q_2^R \\
q_3 = -\frac{a}{a} q_2^R \\
(3.2.4)
$$

in which $q_1 = a_1 v_{r1}$. Eqs. 3.2.4 imply conservation of volume of water from the epilimnion and hypolimnion when it is mixed in the near field and flows back into the far field as an interlayer. The assumption of conservation of volume during mixing is permitted, provided the relationship between density differences and temperature differences may be taken linear. It is assumed here that the temperature differences are sufficiently small for this condition to be satisfied.

Equations (3.2.1) and (3.2.4) are found to satisfy the mass balance at the transition (assuming a quasi-stationary near field),
\[ \rho_1 q_1^R + \rho_2 q_2^R + \rho_3 q_3^R = 0 \quad (3.2.5) \]

Regarding for the moment the radius, \( R \), of the near field as known, two unknowns remain: \( a_2^R \) and \( v_{r2}^R \). Consequently, two more equations are required.

One boundary condition is obtained by assuming a relationship between the rate of flow, \( Q_b^R \), entrained by the bubble column and the radial jet at the free surface (appendix V), and the rate of flow \( Q_3^R = 2\pi R a_3^R \) in the hypolimnion. In the case of a homogeneous ambient the velocity profile below the radial jet is nearly uniform. This suggests the relationship

\[ Q_3^R = \frac{a_3^R}{a_b^R} Q_b^R \quad (3.2.6) \]

in which \( a_b^R \) = height of the portion of the near field in which the flow is directed towards the bubble column (appendix V). In the case of a stratified ambient this velocity profile is not uniform: the hypolimnion extends up to the bubble column (fig. 1.2), and the velocity in this part of the hypolimnion is less than that above it (section 4.4.1). On the other hand, hypolimnion water is also entrained by the eddy in front of the epilimnion. Since a better relationship than (3.2.6) is not available, it will be accepted for the stratified case also (in fact only for the phases I and II discussed below). A coefficient in the expression for the flow rate \( Q_b^R \) will be determined from model experiments (section 4.4.3) to correct for the ad hoc character of this equation.

The second equation needed to arrive at a determinate problem, follows either from the condition of critical flow in the case of unsubmerged flow (phase I) or from a volume balance as regards the far field in the case where the flow is submerged (phase II). The condition for critical flow is
\[(v^R_{r2})^2 = \left(\frac{a_e}{a^2} \frac{a_h}{a^3} a_2^R \epsilon g a_2^R \right) \frac{(a - a_2^R)^3}{(a - a_e^R)^3 + (a - a_2^R)^3} \]  

(3.2.7)

A volume balance for the far field gives

\[
\frac{d}{dt} \left[ \pi (R_e^2 - R^2) a_2^R \right] = Q_{r2}^R = 2\pi R a_2^R v^R_{r2} \]  

(3.2.8)

in which the effective radius, \(R_e\), of the far field is defined by (2.3.21). Equation (II.14) shows that this radius depends on \(a_2^R\) and \(v^R_{r2}\); (3.2.8) therefore is another relationship between these two variables. It is shown below that, if recirculation would not occur, equations (3.2.7) and (3.2.8) would apply at the end of the destratification process (phase III), whereas (3.2.6) would have to be dropped.

Next, the three phases are discussed in greater detail.

**Phase I**

This phase follows the starting of the air injection. The flow of mixing water at the transition is unsubmerged, since the depth of the interlayer is still relatively small. In consequence, the flow at the transition is critical, and the flow rate of mixing water is extreme (appendix I). One of the celerities is zero; the second celerity given by (2.2.23) is positive, which is in agreement with the fact that a flow rate in positive direction is extreme.

The flow rates follow from (3.2.6), (3.2.2) and (3.2.4),

\[
Q_{1}^R = 2\pi R q_{1}^R = \frac{a_e}{a} \left(1 - \frac{a_2^R}{a}\right) Q_{b}^R a_b
\]
\[ Q_2^R = 2\pi R q_2^R = -\frac{a^R}{a_b^R} (1 - \frac{a_2^R}{a}) Q_b^R \quad (3.2.9) \]

\[ Q_3^R = 2\pi R q_3^R = \frac{a_h^R}{a_b^R} (1 - \frac{a_2^R}{a}) Q_b^R \]

The depth, \( a_2^R \), of the interlayer follows from (3.2.9) and (3.2.7); eliminating the velocity, \( v_2^R = \frac{q_2^R}{a_2^R} \), yields

\[ \sqrt{a_2^R e \frac{h}{a^2} g(a_2^R)^3 \frac{(a - a_2^R)^3}{(a - a_2^R)^3 + (a - a_2^R)^3}} = \frac{a^R}{a_b^R} (1 - \frac{a_2^R}{a}) \frac{Q_b^R}{2\pi R} \quad (3.2.10) \]

This depth is constant during phase I. Consequently, \( v_2^R \) and \( Q_2^R \) are constant also. Equation (3.2.10), however, does not always yield real solutions. The LHS and RHS of (3.2.10) are shown in fig. 3.3 as functions of \( a_2^R \). Two real solutions of (3.2.10) will exist, if (fig. 3.3a)

\[ -\frac{a}{a_b^R} \frac{Q_b^R}{2\pi R} < 0.556 \sqrt{\frac{a e h}{a^2}} e g a^3 \quad (3.2.11) \]

---

fig. 3.3 - LHS and RHS of equation (3.2.10),
- a. two real solutions, b. no real solutions
In that case a root \(a_2^R < 0.616a\) and a root \(a_2^R > 0.616a\) exist. The smaller root applies to phase I, which can be explained as follows: suppose the entrained flow rate, \(-Q_b^R\), is small so that \(a_2^R < 0.5a\). Increasing this flow rate increases \(a_2^R\) (RHS of (3.2.10)) until the value \(a_2^R = 0.5a\) is reached. According to (2.2.24) the flow then is double-critical, i.e. both celerities vanish. Further increase of \(-Q_b^R\) has no effect at the transition, since disturbances cannot propagate in either direction (the radius of the near field may change, however). Consequently, double-critical flow will occur in phase I, if (fig. 3.3b)

\[
- \frac{a}{a_b^R} \frac{Q_b^R}{2\pi R} > \frac{1}{2} \sqrt{\frac{a}{a_e h}} c g a^3
\]

(3.2.12)

The flow rates in interlayer as well as in epilimnion and hypolimnion would then be extreme. In general (3.2.12) to be satisfied would require excessive air flow rates (appendix VI). Therefore, the possibility of double-critical flow at the transition is not investigated further.

The larger root comes up for discussion below.

During phase I the interlayer is filling up. After a certain time interval, \(t_I\), the volume of the interlayer has increased such that (3.2.8) and (II.14) yield a value of \(a_2^R\) which equals (is no longer less than) that given by (3.2.10). All quantities at the transition being constant in phase I, (3.2.8) gives for \(t_I\)

\[
t_I = \frac{1}{2} \left[ \frac{R_e^2 - R^2}{R_v R} \right]_I
\]

(3.2.13)

The subscript I indicates quantities in phase I; \((v_{r_2}^R)_I\) is given by (3.2.7) and \((a_2^R)_I\) by (3.2.10). Substituting these quantities into (II.14) yields \((R_e)_I\).
Phase II

Phase II follows phase I when

\[ t = t_I \]  \hspace{1cm} (3.2.14)

The flow at the transition is now influenced by the far field and becomes subcritical (fig. 3.3a). The flow rates are still given by (3.2.9). The depth, \( a_2^R \), of the interlayer at the transition follows from the volume balance for the far field, equation (3.2.8).

The accumulation of mixing water in the far field now causes a gradual increase in \( a_2^R \) as time elapses. Fig. 3.3a shows that a situation is reached at a certain instant, \( t = t_{II} \), where \( a_2^R \) equals the larger root of (3.2.10).

Phase III

This phase is discusses for the sake of completeness only, since under practical circumstances recirculation causes a different type of flow. Phase III would set in when

\[ t > t_{II} \]  \hspace{1cm} (3.2.15)

The condition for critical flow (3.2.7) now again controls the flow rates at the transition. (3.2.10) is not satisfied, the LHS being less than the RHS (see also fig. 3.3a). Apparently, the flow rates in epilimnion and hypolimnion according to (3.2.9) cannot be supplied by the three-layer system. The condition for critical flow is more restrictive. Inspecting (2.2.23) shows that the non-zero celerity now is negative. This indicates that in phase III the critical flow concerns the flow towards the bubble column, i.e. the flow rates in epilimnion and hypolimnion are extreme.
The depth, $a_2^R$, of the interlayer at the transition increases also during phase III, it follows from the condition for critical flow and the far field equation (3.2.8).

The rate of destratification goes to zero when $a_2^R$ tends to the total depth, $a$.

3.3. The radius of the near field

The radius of the near field, which is the distance from the bubble column to the transition from near field to far field, is determined by applying the conservation laws of mass, volume and momentum to the fluid in a domain enclosing the cross-section $r = R$ (fig. 3.4).

![Control domain](image)

fig. 3.4 - Control domain to determine the radius of the near field

To simplify the calculations, a quasi-steady state is assumed. The fact that the flow is radial may be disregarded, if the width of the fluid element in radial direction is taken as infinitesimal. Dropping, for the sake of convenience, the superscript $R$ the resulting equations are

**Conservation of mass**

$$\int_0^a \, dz \, u_r(R,z) \rho(R,z) + \rho_1 v_{r1} a_1 + \rho_2 v_{r2} a_2 + \rho_3 v_{r3} a_3 = 0 \quad (3.3.1)$$
or, using (3.2.5),

\[ 0 \int_{a}^{b} dz \ u_r(R,z) \ \rho(R,z) = 0 \quad (3.3.2) \]

**Conservation of volume**

\[ 0 \int_{a}^{b} dz \ u_r(R,z) + v_r a_1 + v_r a_2 + v_r a_3 = 0 \quad (3.3.3) \]

or, using (2.1.8),

\[ 0 \int_{a}^{b} dz \ u_r(R,z) = 0 \quad (3.3.4) \]

Equation (V.1) satisfies (3.3.4).

**Conservation of momentum**

\[ 0 \int_{a}^{b} dz \left[ g \int_{a}^{b} dz_1 \ \rho(R,z_1) + u_r^2(R,z) \ \rho(R,z) \right] = \Delta p \ a \ + \\
+ \frac{1}{2} \rho_1 g a_2^2 + \frac{1}{2} (\rho_2 - \rho_1) g (a_2 + a_3)^2 + \frac{1}{2} (\rho_3 - \rho_2) g a_3^2 + \\
+ \rho_1 v_r a_1 + \rho_2 v_r a_2 + \rho_3 v_r a_3 \quad (3.3.5) \]

in which \( \Delta p \) is a nonnegligible pressure difference between near field and far field, which arises as a consequence of the neglect of disturbances in the free surface. The nonuniformity of the velocity profiles in the layers (see (2.2.34)) is disregarded here, since the relevant momentum transport terms in (3.3.5) are of secondary importance.

The pressure difference, \( \Delta p \), is determined by applying Bernoulli's equation along the free surface (20, 21). In the
near field this gives, since the velocity at the plunge point (in fact plunge circle) may be assumed to vanish (fig. 3.5),

\[ p_A + \frac{1}{2} \rho_A u_r^2 (R,a) = p_B \]  
\[ (3.3.6) \]

On the far field side one obtains, since at the plunge point only the radial velocity component vanishes,

\[ p_C + \frac{1}{2} \rho_1 (u_{r1}^2 + u_{\phi1}^2) = p_B + \frac{1}{2} \rho_1 u_{\phi1}^2 \]  
\[ (3.3.7) \]

Again disregarding the nonuniformity of the velocity profiles, (3.3.6) and (3.3.7) give

\[ \Delta p = p_C - p_A = \frac{1}{2} \rho (R,a) u_r^2(R,a) - \frac{1}{2} \rho_1 v_{r1}^2 \]  
\[ (3.3.8) \]

fig. 3.5 - Flow pattern at free surface near plunge circle

The density distribution in the near field is not uniform, and a departure from the homogeneous case will be introduced. Since the hypolimnion extends up to the bubble column and its depth varies little with distance, it is assumed that

\[ \rho(R,z) = \rho_3 \quad \text{if} \quad 0 \leq z < a_3 \]  
\[ (3.3.9) \]
The density of the fluid between the hypolimnion and the radial jet at the free surface is close to that in the interlayer (section 4.4.2),

\[ \rho(R, z) = \rho_2 \quad \text{if} \quad a_3 \leq z < a - a_0 \quad (3.3.10) \]

The depth \( a_0 \) is introduced to satisfy (3.3.2), which required the density at the free surface to be greater than \( \rho_2 \). This phenomenon was observed experimentally also, and can be explained by considering the entrainment mechanism of the bubble column: hypolimnion water is entrained and transported upwards in the centre of the bubble column, and subsequently flows out horizontally at the free surface. Assuming that the density of this water is still equal to the density of the hypolimnion (which is not completely true because of mixing), and using approximations for small \( r \) (e.g. \( r < 3a \)) in the equations in appendix V, (3.3.2) yields

\[ a_0 = \frac{kR}{a} a_3 \quad (3.3.11) \]

Consequently

\[ \rho(R, z) = \rho_3 \quad \text{if} \quad a - a_0 \leq z \leq a \quad (3.3.12) \]

The use of the homogeneous results as regards the radial jet is justified by the small (absolute) values of the local Richardson numbers.

Substituting (3.3.8) through (3.3.12) into the momentum equation, (3.3.5), neglecting the density differences in the velocity terms, and using the double two-layer model, yields
\[ \frac{1}{2} \frac{a e a_h}{\alpha^2} e \sigma a \left( \frac{a}{a} \right)^2 \left[ 2 - \frac{a}{a} \left( 1 - \frac{a}{a} \right) + 2 \frac{R}{a} \right] = \]

\[ = \frac{1}{2} \frac{u_r^2(R,a)}{a} - \frac{1}{a} \int_0^a dz u_r^2(R,z) + \frac{1}{2} \frac{a_z(2a - 3a)}{(a - a_z)^2} v_r^2 \]

(3.3.13)

The velocity \( v_{r2} \) in (3.3.13) is given by (3.2.7) in phases I and III defined in section 3.2, and by (3.2.9) in phase II. Together with the homogeneous near field data given in appendix V, an implicit relationship between the radius, \( R \), and the depth, \( a_z \), of the interlayer at the transition is then obtained in either case. In consequence, the radius of the near field can be determined simultaneously with layer depths, flow rates and velocities in a (numerical) simulation of the destratification process as outlined in section 3.2.

3.4. Correction to double two-layer model

The theory developed so far does not explicitly account for the entrainment of epilimnion water into the radial jet at the free surface in the near field. Anticipating the experimental results (section 4.4.3) it may be noted that nevertheless a good agreement between theory and experiment is obtained, particularly as regards the depth of the interlayer as a function of time. Except at large air flow rates, there are some secondary discrepancies in other respects.

The radial velocities in the epilimnion are larger than those in the hypolimnion in the early phases of the destratification process, which contradicts (2.2.18). Related to this, the density of the interlayer and the depth of the epilimnion are less than predicted by the double two-layer model.

Starting from the relationships derived it is possible to show that the growth of the interlayer is rather insensitive to departures
from the double two-layer model. In phase I the flow at the transition is critical. The depth of the interlayer is then relatively small, since the air flow rates applied are much less than that required for double-critical flow (eq. 2.2.24 and appendix VI). Consequently, the radial velocities in epilimnion and hypolimnion are then small compared to that in the interlayer. Neglecting these velocities, equations (2.1.6) and (2.1.11), which are not affected by the assumptions leading to the double two-layer model, yield as a condition for critical flow

\[ v_{2r} = \sqrt{\frac{\varepsilon_1}{\varepsilon} - \frac{\varepsilon_3}{\varepsilon} - \varepsilon g a_2} \]

(3.4.1)

The double two-layer model, equation (3.2.7), also gives this result for small \( a_2 \), since in that case \( \varepsilon_1/\varepsilon = a_{eh}/a \) and \( \varepsilon_3/\varepsilon = a_e/a \). The critical velocity according to (3.4.1) is independent of the level of the interlayer (i.e. of the depths of epilimnion and hypolimnion) and insensitive to the actual density of the interlayer within a fairly large range, e.g. \( 0.2 < \varepsilon_1/\varepsilon < 0.8 \) (fig. 3.6).

![Graph showing \( \varepsilon_3/\varepsilon \) as a function of \( \varepsilon_1/\varepsilon \) and \( \varepsilon_3/\varepsilon \)]

**fig. 3.6** - \( \varepsilon_1 \varepsilon_3/\varepsilon^2 \) as a function of \( \varepsilon_1/\varepsilon \) and \( \varepsilon_3/\varepsilon \)

The depth of the interlayer in phase I follows from (3.2.10), which also contains the factor \( a_{eh}/a^2 = \varepsilon_1 \varepsilon_3/\varepsilon^2 \).
Equation (3.2.6) is influenced by deviations from the double two-layer model, since \( a_3^R \) then changes. The physical basis of this equation is rather weak, however, and adaptation to experiment is required, as explained before.

The rate of growth of the interlayer is influenced in all phases by the radius, \( R \), of the near field, equation (3.3.13). This equation also contains the factor \( a_n a_h / a^2 \). The form of the last term in (3.3.13) also results from the double two-layer model, but this term is of minor importance. The expression in brackets contains \( a_e / a = \varepsilon_3 / \varepsilon \). It is not sensitive to this parameter, however. The factor \( a_n a_h / a^2 \) is also found in the far field solution, appendix II.

Summarizing it can be concluded that the growth of the interlayer is only slightly influenced by (not too large) departures from the double two-layer model.

To improve upon the theory as regards the depth of the epilimnion, etc., the double two-layer model can be considered as a first approximation to which a correction is to be applied. This correction can be obtained by involving in the analysis the entrainment mechanism of epilimnion water into the near field. To this end the usual assumption is made that the entrainment velocity is proportional to that of the flow causing the entrainment,

\[
\frac{R}{v_{r1}} = - \beta u_r(R,a) \tag{3.4.2}
\]

in which \( u_r(R,a) \) is given by (V.1). A similar relationship was proposed by Van Pagee (32). The entrainment coefficient, \( \beta \), is estimated from the model experiments (sections 4.4.1 and 4.4.2).

Assuming a quasi-steady near field, the integral expression for the conservation of mass in the far field is

\[
2\pi \int_{R_0}^{R} (\rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3) rdr = (\rho_e a_e + \rho_h a_h) \pi (R_0^2 - R^2) \tag{3.4.3}
\]
Since most of the water is in the outer far-field (appendix II), this equation may be approximated by

\[ \rho_1 a_1^0 + \rho_2 a_2^0 + \rho_3 a_3^0 = \rho_1 a_e + \rho_3 a_h \]  

(3.4.4)

The superscript 0 indicates layer depths in the outer far-field, which do not depend on the radius, \( r \). The approximating character of (3.4.4) is admissible the more so as only a correction to a first approximation is concerned. Eliminating the density \( \rho_2 \) between (3.4.4) and (3.2.5) yields \( q_i = a_i v_{r1} \)

\[ \rho_1 \left( \frac{R}{a_2} \frac{R}{v_{r2}} + \frac{a_e}{a_2} - \frac{a_1}{a_2} \right) \rho_3 \left( \frac{R}{a_2} \frac{R}{v_{r2}} \frac{R}{v_{r3}} + \frac{a_h}{a_2} - \frac{a_3}{a_2} \right) = 0 \]  

(3.4.5)

Since (3.4.5) is obtained from the conservation of mass only, it must be true for any \( \rho_1 \) and \( \rho_3 \). This condition gives

\[ \frac{a_1}{a_2} \frac{R}{v_{r1}} + \frac{a_e}{a_2} \frac{R}{v_{r2}} - \frac{a_1}{a_2} = 0 \]  

(3.4.6)

\[ \frac{a_3}{a_2} \frac{R}{v_{r3}} + \frac{a_h}{a_2} \frac{R}{v_{r2}} - \frac{a_3}{a_2} = 0 \]  

(3.4.7)

The double two-layer model gives for the ratio of the depths of the epilimnion at the transition and in the outer far-field

\[ \frac{a_1}{a_2} = \frac{a - a_2}{a - a_2} \]  

(3.4.8)

These depths are now given by (3.4.6), (3.4.8), (3.4.2) and the
results obtained for the interlayer using the double two-layer model. The depths, $a_3^R$ and $a_3^O$, of the hypolimnion are then also known. Equation (3.4.7) gives a corrected value of the radial velocity, $v_{r3}^R$, in the hypolimnion. The density of the interlayer, which now becomes a slowly varying function of time, follows from (3.4.4).

Fig. 3.7 shows the uncorrected and corrected layer depths in the outer far-field in the case of a negligible Coriolis force and a relatively low air flow rate (corresponding with run No. 2b of the model experiments).

![Diagram showing the correction to double two-layer model](image)

fig. 3.7 - Correction to double two-layer model

4. Model experiments

4.1. Model laws

The shape of the model reservoir is that of a circular constant-depth basin. The only geometrical quantity then is its radius. The initial stratification is a two-layer system at rest,
which is characterized by the total depth, the initial depth of
the epilimnion and the reduced gravitational acceleration, \( e_g \).
Equations (V.1) and (V.2) suggest that the air injection is
sufficiently characterized by the total depth and the quantity
\( (gQ/a)^{1/3} \). The far field solution (section 2.3 and appendix II)
indicates a marked Coriolis effect. Furthermore viscosity may play
a part. The variables, such as layer depths and flow rates, depend
on place and/or time. Thus the quantities determining the
destratification process become

\[
R_o, a, a_e, e_g, (-\frac{a}{a})^{1/3}, f, v, r, t
\]

in which \( v \) = kinematic viscosity. A dependent variable is a function
of these quantities. The relationship for the depth of the inter-
layer, for instance, takes the form

\[
a_2 = F_1 \left[ R_o, a, a_e, e_g, (-\frac{a}{a})^{1/3}, f, v, r, t \right]
\]  \hfill (4.1.1)

From dimension considerations it then follows that (4.1.1) can
be written in a form containing dimensionless parameters only, i.e.

\[
\frac{a_2}{a} = F_2 \left[ \frac{R_o}{a}, \frac{a_e}{a}, \frac{(-\frac{a}{a})^{1/3}}{\sqrt{e_g a}}, \frac{fR_o}{\sqrt{e_g a}}, \frac{a(-\frac{a}{a})^{1/3}}{\sqrt{e_g a}}, \frac{r}{R_o}, \frac{t}{R_o^2} \frac{a\sqrt{e_g a}}{R_o^2} \right]
\]  \hfill (4.1.2)

The third parameter on the RHS of (4.1.2) may be seen as a
densimetric Froude number related to the near field, the fourth
as a Rossby number (far field), and the fifth as a Reynolds number
(near field). The radial coordinate and time have been normalized
so as to make the corresponding dimensionless variables of order
one.

(4.1.2) yields, in the usual way, the model laws
\[ n_{R_0} = n_{a_e} = n_{a_z} = n_r = n_a = n_{a_1} = n_{a_3} = n_{a_h} \]  
\[ n_{Q_a} = n_{\varepsilon_1} \frac{3/2}{n_a} \]  
\[ n_{\varepsilon_1} = n_{\varepsilon_2} \frac{1/2}{n_a} \]  
\[ n_{\varepsilon_2} = n_{\varepsilon_3} \frac{1/2}{n_a} \]  
\[ n_{\varepsilon_3} = n_{\varepsilon_4} \frac{1/2}{n_a} \]  
\[ n_{\varepsilon_4} = n_{\varepsilon_5} \frac{3/2}{n_a} \]  

in which \( n \) is a scale factor defined by, in the case of an arbitrary quantity \( \phi \),

\[ n_{\phi} = \frac{\text{value of } \phi \text{ in prototype}}{\text{value of } \phi \text{ in model}} \]  

Generally speaking, the model law for the viscosity cannot be satisfied. A departure from this law will be permitted, however, if the model Reynolds numbers related to regions with turbulent mixing are sufficiently large. This in turn requires large velocities and therefore a large relative density in the model so that \( n_{\varepsilon_1} < 1 \). The model law for the Coriolis parameter then yields (since \( n_{a_1} > 1 \)) \( n_{\varepsilon_2} < 1 \), which can be realized only in a rotating model. In a non-rotating model the Coriolis effect caused by the rotation of the earth practically vanishes, and a non-negligible scale effect results.

4.2. Physical model

The possibility of a rotating model was rejected because of excessive cost and construction problems. Consequently, the Coriolis effect is not reproduced. The utility of a non-rotating model is concerned mainly with the flow in the near field and that at the
transition to the far field.

The physical model was a circular basin, inner diameter 5.00 m and (maximum) depth 0.35 m, constructed of a concrete bottom and a brick side-wall (figs. 4.1 and 4.2). The inner side of the wall was plastered. Seams in the model were caulked with silicone sealant to prevent leakage. Bottom and side-wall were coated with an epoxy paint. A glass window was fixed in the side-wall for visual observations and filming.

![Diagram](image)

fig. 4.1 - Physical model

To avoid the difficult task to control temperatures as required in a model with thermal stratification, sodium chloride dissolved in tap water was used to obtain density differences. In all experiments the salinity (grammes of NaCl per kilogramme of saline water) of the hypolimnion was about 7.5 per mil and that of the epilimnion about 1 per mil resulting in a relative density difference of about 0.005. The slight salinity of the epilimnion was necessary, since the relationship between output voltage of the
fig. 4.2 - Photograph of the physical model

measuring system pertaining to the conductivity probes and salinity was markedly non-linear at lower salinities. The epilimnion water was prepared in the model itself. After a stilling period the hypolimnion water, which had been prepared earlier separately, was supplied through four diffusers mounted flush with the bottom. Thus the hypolimnion water lifted the epilimnion. The filling process took about six hours. The interface had then thickened to some centimetres as a result of molecular diffusion and possibly some mixing in the early stages of the filling process. Delaying the experiment to the next day would lead to a significant further thickening of the interface.

Compressed air was injected through a porous glass filter, diameter 50 mm, which was mounted in the bottom. A filter is necessary to produce small air bubbles. All experiments but one were run with the air injection in the centre of the basin.
The air was supplied by a standard compressor. The air flow rate was measured with a free-float flowmeter, and the pressure difference with a static head pressure gauge (U-tube).

The model Reynolds numbers are less than those in prototype by a factor \( n^{1/2} \cdot n^{3/2} \), which is about 40 in the case of the prototype experiments discussed in section 5. Nevertheless the air injection was found to produce sufficiently intensive turbulence in the near field to adequately model the mixing of epilimnion and hypolimnion water (section 4.3). The flow in almost all of the far field, however, was laminar. Since the prototype far field is turbulent, if only in consequence of the large tangential velocities, viscosity also causes a scale effect so far as the far field is concerned.

Density profiles were measured with conductivity probes of the type shown in fig. 4.3, see also (31). During an experiment the plexiglass tube was attached in vertical position to a point gauge which could be mechanically moved up and down with a constant speed (fig. 4.4). The electrodes (platinum wires) protruded horizontally to minimize the disturbing effect of the wake of the plexiglass tube. The density profiles were taken with a vertical speed of about 2 mm/s. In stagnant water and in laminar flow a retardation effect was observed, which was caused presumably by the dragging along of ambient fluid. It was found necessary to average the conductivity recordings obtained during downward and subsequent upward motions of the probes to reduce this effects. The readings were made on a U.V. recorder.

fig. 4.3 - Photograph of conductivity probe
The flow in the far field was visualized by injecting a dye, methylene blue, through a row of small holes in plexiglass tubes, diameter 5 mm, placed vertically (fig. 4.5). To reduce buoyancy effects of the dye two tubes were used, one with holes in the lower half of the vertical through which dye of hypolimnetic density was injected, and a tube with holes in the upper half of the vertical for the injection of dye of epilimnetic density.

Repeating an experiment under identical conditions demonstrated that the reproducibility was good.

4.3. Experimental procedure

In preliminary experiments the behaviour of the air injection under homogeneous conditions was investigated. The rate of flow entrained by the bubble column was found to agree with the results obtained by Goossens and Smith (5). The radial jet at the free surface could be satisfactorily described with the expressions given by Goossens and Van Pagee (6) and summarized in appendix V. These
fig. 4.5 - Photographs showing injection of dye through perforated tubes

results indicate that the near field was sufficiently turbulent despite the smaller Reynolds numbers.

A series of experiments under stratified conditions was filmed to obtain an understanding of the mechanism of the mixing process, and the flow in the far field. The pictures made included the flow at the transition, the initial front of the interlayer and the subsequent three-layer flow. In a single case the near field was filmed.

The motion-pictures showed that, once the interlayer had formed, a state arose in which the layer depths did no longer depend on the radial co-ordinate. The same result was obtained from density profiles taken in the far field at various distances from the bubble column (fig. 4.6, cf. 38). Therefore only one probe was placed in the far field (at a distance of 1.25 m from the bubble column) during the final test runs. In some experiments two probes were placed in the near field (at 0.10 and 0.20 m from the bubble column).

Before an experiment was started a surface-active agent was
fig. 4.6 - Density profiles showing horizontal homogeneity in the far field, $Q_a = 29 \times 10^{-6}$ m$^3$/s

sprinkled on the free surface to reduce surface tension effects caused by contamination during the filling period. The conductivity probes were calibrated by immersing in solutions of known salinity. These solutions and the water in the model had the temperature of the surroundings, so that temperature effects on the conductivity and the calibrating were negligible. Next, the initial density profile was recorded.

During destratification density profiles were taken by making the probes move downwards and, after the bottom had been reached, upwards. In early stages of the destratification process the return from the bottom started immediately after arrival; in later stages when the depth of the hypolimnion had become small, the probes were kept at rest at the bottom for one or two minutes to reduce the retardation effect mentioned.

The radius of the near field was measured at various instants using dye to visualize the transition to the far field. The accuracy
of these measurements is low, since the transition was not well defined and appeared to be rather unstable in the later stages of the tests.

A test was ended when the depth of the interlayer, as indicated by the density profiles, had increased to about 90 per cent of the total depth.

The air flow rate, \( Q_a \), and the initial depth, \( a_0 \), of the epilimnion were varied in the test programme. The total depth was 0.20 m, except for one case where it was 0.30 m. The air flow rate was varied around values obtained from the model laws, equations (4.1.3), which were applied to the prototype situations described in section 5. The experimental programme is given in table 4.1.

4.4. Discussion of experimental and theoretical results

4.4.1. Dye experiments

The dye experiments served to obtain a qualitative insight into the flow phenomenon under consideration. The description of the destratification process given in the Introduction is based for the larger part on these experiments. The motion-pictures made did not allow a quantitative analysis. Velocities derived from the displacements of individual puffs of dye showed considerable scatter. The layer depths could not be determined with some accuracy owing to a perspective effect when filming the dye cloud and refraction of light in the inhomogenous liquid.

The instantaneous celerity of the front of the developing interlayer varied between 0.005 m/s and 0.023 m/s in runs No. 1 through 3 (table 4.1). The maximum value equals the prediction according to (2.2.28). Fig. 4.7 shows a typical example of the shape of interlayer and front at three times. It is seen that the tip or nose of the front propagates somewhat above the middle of the halocline.
Table 4.1

<table>
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<th>run No.</th>
<th>$Q_{a_3}^{10^{-6} m^3/s}$</th>
<th>$a_e$ (m)</th>
<th>$a$ (m)</th>
<th>$\varepsilon$</th>
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<tbody>
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<td>20</td>
<td>0.05</td>
<td>0.20</td>
<td>0.0050</td>
</tr>
<tr>
<td>1b</td>
<td>20</td>
<td>0.10</td>
<td>0.20</td>
<td>0.0047</td>
</tr>
<tr>
<td>1c</td>
<td>20</td>
<td>0.15</td>
<td>0.20</td>
<td>0.0053</td>
</tr>
<tr>
<td>2a</td>
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<td>0.20</td>
<td>0.0046</td>
</tr>
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<td>2b</td>
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<td>0.10</td>
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</tr>
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<td>2c</td>
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<td>0.15</td>
<td>0.20</td>
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</tr>
<tr>
<td>2d *)</td>
<td>29</td>
<td>0.10</td>
<td>0.20</td>
<td>0.0050</td>
</tr>
<tr>
<td>3a</td>
<td>56</td>
<td>0.05</td>
<td>0.20</td>
<td>0.0050</td>
</tr>
<tr>
<td>3b</td>
<td>56</td>
<td>0.10</td>
<td>0.20</td>
<td>0.0046</td>
</tr>
<tr>
<td>3c</td>
<td>56</td>
<td>0.15</td>
<td>0.20</td>
<td>0.0043</td>
</tr>
<tr>
<td>3d *)</td>
<td>56</td>
<td>0.12</td>
<td>0.30</td>
<td>0.0050</td>
</tr>
<tr>
<td>4 **)</td>
<td>94</td>
<td>0.10</td>
<td>0.20</td>
<td>0.0049</td>
</tr>
<tr>
<td>5 **)</td>
<td>253</td>
<td>0.10</td>
<td>0.20</td>
<td>0.0049</td>
</tr>
</tbody>
</table>

This peculiarity occurred in most of the experiments. Apparently, the density of the tip is close to that of the epilimnion. It was observed by injecting dye ahead of the front, that at the halocline a thin layer of water was already flowing towards the side-wall before the front had arrived. This phenomenon is related to the fact that the stratification at the halocline was continuous (e.g. 19).

* ) Eccentric air injection.
** ) To suppress instability of the bubble column, which caused standing waves in the model, a horizontal disc, diameter 0.10 m, was fitted over the injection point at the level of the free surface.
fig. 4.7 - Shape of developing interlayer (from motion-picture, \[ Q_a = 29 \times 10^{-6} \, \text{m}^3/\text{s} \])

The depths of the layers at the transition remained approximately constant during formation of the interlayer, which agrees with the viewpoints developed in sections 3.1 and 3.2 (phase I). Fig. 4.8 shows photos of the dyed front. Wave reflection occurring on arrival of the internal front at the side-wall could be observed only in the case of large air flow rates (run No. 5). The reflected wave damped quickly, however, after which the interlayer thickened gradually. In other cases the interlayer thickened without observable wave reflection (fig. 4.5). Entrainment in the far field did not occur.

The water velocities in epilimnion and hypolimnion deduced from the motion-pictures showed a marked difference during a considerable period after the air injection had started (but not in run No. 5). This point was already discussed in section 3.4 in view of the incorporation in the mathematical model of the entrainment of epilimnion water. The entrainment coefficient, \( \beta \), in (3.4.2)
was found to vary between 0.11 and 0.18, which are larger values than that for a plane jet (33), for instance. The larger values may be explained by the fact that the radial jet in the near field is deflected by the epilimnion, thus increasing the mixing.

The hypolimnion extended virtually up to the bubble column, and in the near field a more or less stable interface between hypolimnion and returning flow above it could be observed. Only the eddying motion in front of the epilimnion (fig. 1.2) intermittently entrained significant amounts of water from the hypolimnion. Part of this entrained water flowed into the interlayer, the remaining portion mixed with the returning flow. The efficiency of the air injection may be somewhat less in the stratified situation than in the homogeneous situation, since a (small) fraction of the relatively denser hypolimnion water entrained tended to flow back to the near field after having been lifted above the interface mentioned (fig. 4.9, cf. 40).

The character of the mixing process changed towards the end of an experiment, when the depth of the epilimnion had decreased to a small fraction of the total depth. The radial jet in the near
fig. 4.9 - Flow pattern close to the bubble column in the stratified near field

field then started to act not only on the epilimnion, but also on (an upper) part of the interlayer. The direction of flow in this part of the interlayer then reversed, thus forming a new 'epilimnion' together with the existing epilimnion. The mixing proceeded, but it is doubtful whether the far field would ever become completely homogeneous.

Eccentric air injection (at 0.90 m from the side-wall) was found to have no visible influence on the flow in the near field and at the transition to the far field. It could therefore be expected that the rate of destratification was also not influenced. The density profiles indeed showed little difference with those of centric air injection, particularly in the later phases of destratification. The horizontal flow pattern in the layers in the far field was different, however, and was much like one would compute from potential theory: stagnation points at the side-wall closest to the bubble column and diametrically opposite to it, and curved streamlines. Marked tangential velocities occurred along the larger part of the side-wall in all layers throughout the destratification process.
4.4.2. Density profiles

A typical example of the density profiles taken in the outer far-field is given in fig. 4.10. The profiles in this figure are taken from run No. 2b. The initial profile (t=0) shows a continuous halocline. It is likely that after formation of the interlayer a portion of this continuously stratified water becomes part of the epilimnion; a second (lower) portion becomes part of the hypolimnion. These continuously stratified regions thin out during destratification, which explains the sharpening of the interfaces as time elapses.

![Density profiles at various times (run No. 2b)](image-url)

fig. 4.10 - Density profiles at various times (run No. 2b)
A second point of interest is that the density of the interlayer is not uniform, which indicates that the mixing is incomplete. This particularly concerns the early phases of destratification. Apparently, mixing within the interlayer proceeds during an experiment. The density of the interlayer gradually approaches that arising after complete mixing of epilimnion and hypolimnion.

The initial halocline in fig. 4.10 is at half-depth. The depth of the epilimnion is observed to decrease more rapid than that of the hypolimnion. This phenomenon was already discussed in section 3.4 dealing with a correction to the double two-layer model. This model would predict equal depths in this case. The rate of destratification decreases as the interlayer thickens and tends to zero when the homogeneous situation is approached.

The density profiles obtained from the remaining experiments showed more or less the same features as those discussed above. All far field density profiles observed (except those of run No. 2d, for reasons explained in section 4.4.1) are collected in appendix VII.

Fig. 4.11 shows one of the far field profiles of fig. 4.10, and a density profile in the near field (at 0.20 m from the bubble column) taken simultaneously. Apparently, the depth of the hypolimnion in the near field nearly equals that in the far field. The density fluctuations above it indicate that the returning flow towards the bubble column was turbulent. Unstable stratification \((\text{d} \rho / \text{d}z > 0)\) is seen to occur locally. This is also true for the radial jet at the free surface: the density of the jet was larger than that of the returning flow (but less than that of the hypolimnion). The explanation of this larger density is given in section 3.3. The (mean) density of the returning flow depends on the distance from the bubble column. At the transition it is close to the density of the interlayer, and increases with decreasing radius. This density gradient indicates that entrainment of water from the hypolimnion takes place.
fig. 4.11 - Density profiles in near field and far field at $t = 1540$ s (run No. 2b)

To compare the experimental results with the theory developed, layer depths have to be determined from the observed density profiles. Since the profiles are continuous, there is some ambiguity in doing this. The method pursued was to estimate the mean density of the interlayer and to subsequently determine the levels of the interfaces by equalizing the areas indicated as $A_1$ and $A_2$, and $A_3$ and $A_4$, respectively, in fig. 4.12a. Thus the conservation of mass is warranted. To take into account, to some extent, the finite depth of the initial halocline, this procedure was applied also to the initial profiles (fig. 4.12b). The density of the 'initial
interlayer' was assumed to equal that of the density profile taken next.

![Diagram](image)

**fig. 4.12** - Schematization of density profiles,

a. after formation of interlayer b. initial profile

The experimental layer depths obtained this way are shown in figs. 4.13.1a through 4.13.5 together with theoretical predictions. The theoretical curves were obtained using a lower value of the coefficient $\alpha$ (see equations (V.1) and (V.2) relating air flow rate and water velocities) than given by Goossens and Van Pagee (6) for the homogeneous situation. This was found necessary to account for the nonuniformity of the velocity profile of the returning flow in the near field (discussed in connection with equation (3.2.6) and in section 4.4.1). The possible negative entrainment discussed in section 4.4.1 may also play a part here. The value of this coefficient was obtained from a comparison of experimental and theoretical depths of the interlayer. The coefficient $k$ (equation (V.1)) was not changed, since the desimetric Froude numbers in the radial jet were large. The entrainment coefficient $\beta$ was obtained from numerical
simulations of the experimental depths of epilimnion and hypolimnion. Its value so obtained was in agreement with those from the dye experiments, section 4.4.1. The assumed values of the coefficients are:

\[ \alpha = 0.17 \quad \text{(homogeneous experiments: } \alpha \approx 0.21) \]
\[ \beta = 0.15 \]
\[ k = 0.072 \]

Friction was neglected in the final computations.

The schematized initial density profiles were used as initial conditions for the computations. It has not been attempted to involve in the computations the (relatively small) time the flow needs to become quasi-steady after the air injection has started. The same applies to the recirculation phenomenon discussed in section 4.4.1, because its influence on the rate of mixing is small. Since the air flow rates applied are much less than that required to reach the double-critical flow situation at the transition, phase III (section 3.2) would theoretically set in at a very late stage of the destratification process. Recirculation then already takes place, however. Therefore, only phases I and II were computed, although recirculation occurred also at the end of phase II. The vertical dashed lines in figs. 4.13 mark the transition from phase I to phase II (cf. fig. 3.7).

Generally speaking, the agreement between experimental and theoretical results is satisfactory, despite variation of the air flow rate by a factor greater than twelve. There seems to be a tendency, however, to underestimate the depth of the hypolimnion in the later stages of the computations. This discrepancy may possibly be caused by bottom friction, since in this case bottom friction would tend to cause an increasing depth of the hypolimnion with increasing distance from the bubble column (%a_{3}/%r > 0).
fig. 4.13. la - Observed and theoretical layer depths versus time

lb - Observed and theoretical layer depths versus time

lc - Observed and theoretical layer depths versus time
2a - Observed and theoretical layer depths versus time
2b - Observed and theoretical layer depths versus time
2c - Observed and theoretical layer depths versus time
3a - Observed and theoretical layer depths versus time

3b - Observed and theoretical layer depths versus time

3c - Observed and theoretical layer depths versus time
3d - Observed and theoretical layer depths versus time

4 - Observed and theoretical layer depths versus time

5 - Observed and theoretical layer depths versus time
Comparing in figs. 4.13 the cases with equal air flow rate but varying initial depth, $a_e$, of the epilimnion, it can be seen that this parameter hardly influences the rate of destratification.

Fig. 4.14 shows an example (run No. 2b) of the development of the (mean) density of the interlayer. The density as predicted by the uncorrected double two-layer model ($c_i/c = 0.5$ in this case) is seen to be approached only gradually.

![Graph](image)

fig. 4.14 - Relative density difference between epilimnion and interlayer (run No. 2b)

### 4.4.3. Radius of near field

Figs. 4.15 a, b and c show experimental values and theoretical predictions of the radius, $R$, of the near field for some experiments. The radius increases as time elapses, and eventually tends to the value pertaining to the homogeneous situation. The agreement is seen to decrease when the initial depth, $a_e$, of the epilimnion increases. It must be kept in mind, however, that the experimental values are inaccurate (section 4.3).
fig. 4.15a - Radius of near field versus time, $a_e/a = 0.25$

fig. 4.15b - Radius of near field versus time, $a_e/a = 0.50$
fig. 4.15c - Radius of near field versus time, $a_e/a = 0.75$

5. Prototype experiments

Prototype destratification experiments were carried out in two reservoirs in the south-western part of the Netherlands. The measurements took place in the month of June of the years 1976 and 1977. The reservoirs serve the water supply of the town of Rotterdam and surroundings, and are managed by the "N.V. Watervoordeelbedrijf Brabantse Biesbosch". This institution, in which several municipalities participate, operates as a public utility service.

In this section a short description of the experiments and results is given. A full discussion and the complete data can be found in (35) (1976 experiment) and (36) (1977 experiment).
5.1. 1976 Experiment

The experiment was carried out in the reservoir "Honderd en dertig", see fig. 5.1. The storage capacity of this reservoir is $32 \times 10^6$ m$^3$, the area is $2.2 \times 10^6$ m$^2$, and the depth varies from 13 to 27 m. Under normal circumstances the throughput is 4 m$^3$/s. Destratification can be brought about by three bubble columns, but during the experiment only the deep injection point, marked A in fig. 5.1, with an airflow rate of 0.2 m$^3$/s (under standard conditions), was used.

![Diagram of the reservoir](image)

fig. 5.1 - The reservoir "Honderd en dertig"

Some weeks before the experiment was started the air injection was stopped to allow the reservoir to stratify. Conditions for the experiment to be started were: (i) distinct thermal stratification which should preferably be homogeneous throughout the reservoir to
obtain unambiguous initial conditions, and, to avoid disturbance of the flow by wind, (ii) a weather forecast indicating a period of calms or light wind. Both conditions were satisfied to some extent on June 9. The initial temperature profiles are shown in fig. 5.2.a.

The air injection was started on June 10, after which temperature and velocity profiles were taken at five buoys and two measuring stations (fig. 5.1). The velocities (actually two horizontal components) were measured with an acoustic meter. The measurements were repeated every one or two days. Relatively few velocity profiles are available, however, since only one or two profiles could be taken in one day. The experiment was ended on June 22. The wind was not too high (0 - 8 m/s) until June 19, when the wind increased to a storm (7 - 20 m/s). The total heat content of the reservoir increased during the experiment owing to insolation and the relatively warm inlet flow (fig. 5.3 and 5.4).

![Temperature profiles showing temporary horizontal inhomogeneity](image)
Results

The initial temperature profiles in fig. 5.2a indicate that the stratification is rather superficial. The depth of the epilimnion amounts to about 4 m. Below this layer there is a sharp drop in the temperature and a layer with a gradual decrease in temperature (metalimnion). The metalimnion extends to a depth of about 8 to 10 m, and passes gradually into the hypolimnion. The horizontal homogeneity is good, except sometimes at larger distances from the injection point A (fig. 5.2b). This anomaly can be attributed to the temporary presence of the inlet flow at the measuring point concerned.

Fig. 5.3 and 5.4 show that the temperatures of metalimnion and hypolimnion increase in course of time, whereas the temperature (and depth) of the epilimnion do not change significantly, despite the insolation. This tendency was observed throughout the reservoir in the same measure. Presumably, the related heat exchange was therefore caused not only by the bubble column, but also by vertical diffusion due to background turbulence. Assuming gradient-type transport with constant vertical diffusivity $K_z$, this process is described by the equation

$$\frac{\partial T}{\partial t} = K_z \frac{\partial^2 T}{\partial z^2}$$  \hspace{1cm} (5.1)

in which $T$ is the temperature and $z$ a vertical co-ordinate.

Approximating the temperature profiles in metalimnion and hypolimnion by (the transport at the bottom, $z = 0$, is put equal to zero)

$$T(z,t) = T_0 + b_1 t + b_2 z^2$$  \hspace{1cm} (5.2)

in which $T_0$, $b_1$ and $b_2$ are constants, (5.1) yields on substitution

$$K_z = \frac{b_1}{2b_2}$$  \hspace{1cm} (5.3)
Estimates of $b_1$ and $b_2$ obtained from the measured temperature profiles give $K_z = 2 \times 10^{-4} \text{ m}^2/\text{s}$. This result agrees with values in cases where the wind is light found in the literature, e.g. (37). The turbulence in the hypolimnion is probably related to the superficiality of the stratification, since the marked heat exchange was not observed in an earlier experiment in which the depth of the epilimnion was about 10 m.

![Graph showing temperature and velocity profiles at buoy 2](image)

fig. 5.3 - Temperature and velocity profiles at buoy 2

The temperature profiles give little insight into the effect of the bubble column. The velocity profiles are more informative in this respect. Horizontal velocity vectors of two profiles are shown in figs. 5.3 and 5.4 as a function of depth. A vector pointing downward indicates flow directed towards the bubble column, a vector pointing to the left indicates flow in positive tangential direction, etc. The accuracy of these measurements is not high, because of low-frequency oscillations in magnitude and orientation.
fig. 5.4 - Temperature and velocity profiles at buoy 4

Nevertheless, an outflowing interlayer can be discerned in both profiles (fig. 5.3: $a_1 = 4$ m, $a_2 = 6$ m, fig. 5.4: $a_1 = 6$ m, $a_2 = 4$ m). The interlayer coincides more or less with the initial metalimnion. The temperature of the interlayer decreases with depth. In both profiles the radial velocity components in the epilimnion are directed towards the bubble column. This is less clear as regards the hypolimnion, possibly as a result of the interaction of the flow with the bottom topography. There is some deviation from axial symmetry, since the radial velocity components yield a net volume flux when integrated over the total depth.

The lack of axial symmetry makes the estimation of radial flow rates in the layers impossible without introducing crude assumptions. The velocity measurements indicate that the depth of the interlayer is about 5 m after five days of air injection. Even if the interlayer would extend over the whole reservoir, this would correspond with a flow rate of only $25 \text{ m}^3/\text{s}$ fed into the interlayer, whereas the velocity profiles point to considerably larger flow rates ($100 \text{ m}^3/\text{s}$, to an order of magnitude). This means that water from the interlayer recirculates in epilimnion and/or hypolimnion. The recirculation may be a consequence of insolation and the background turbulence already mentioned.

The velocity vectors in figs. 5.3 and 5.4 show large tangential components, much in the sense predicted (fig. 2.7). The tangential component in the epilimnion is positive, that in the interlayer tends to become
negative for larger distances from the bubble column. Again the behaviour of the hypolimnion is more complicated. Nevertheless, modelling the flow as a three-layer system seems appropriate. The velocity profiles at buoys 5 and 6, and measuring station II deviated substantially from those shown in figs. 5.3 and 5.4. A three-layer did not occur, and the directions of the velocities were not in agreement with those predicted. It is therefore likely that the bubble column has induced mixing in the south-western part of the reservoir only.

The stratification was broken down for the larger part by the storm of June 19 and 20. The three-layer flow could no longer be observed. Instead, the velocity profiles appeared to be more erratic.

**Comparison with some theoretical results**

A quantitative comparison between theory and prototype experiment is not possible, since the insolation and recirculation observed are not accounted for in the theory. However, some theoretical results apply in a qualitative sense.

The predicted tangential flow pattern caused by the Coriolis force agrees to some extent with that observed. Table 5.1 gives experimental and theoretical values of the tangential velocity components. The radius, $R_o$, of the far field must be known to make the calculation of the tangential velocities in epilimnion and hypolimnion possible. It can be made plausible theoretically that the presence of walls forces the rotating flow to form a more or less circular far field, the radius of which is of the order of magnitude of the shortest distance to a wall. In the case under consideration this yields a value of $R_o$ of about 500 m. This result is more or less in agreement with the observations as regards the deviating behaviour of the velocity profiles at buoys 5 and 6, and measuring station II. The values assumed for the coefficient $\alpha$, $\beta$ and $k$ are those of section 4.4.2. The Coriolis parameter was put equal to $1.1 \times 10^{-4} \text{ s}^{-1}$; the coefficient $\theta_o$, rather arbitrarily, to 0.8. The theoretical values in table 5.1 are of the same order of magnitude as those found experimentally, and show the same tendency: relatively large tangential velocities in epilimnion and (to a less extent) hypolimnion, and small tangential velocities in the interlayer.

The observed velocity profiles show that boundary layers at the interfaces between epilimnion and interlayer, and between
interlayer and hypolimnion appear to occur. The thicknesses, \( \delta \), of these boundary layers is about 2 m, which compares well with most of the predictions in appendix III.

The radius of the near field could be estimated, since the circle where the radial free surface jet in the near field plunged below the epilimnion was visible. The observed radius was about 25 m; the prediction for phase I (section 3.3.2) is about 30 m. The observed radius did not increase significantly during the experiment. It is therefore likely that, as a result of the recirculation already mentioned, the destratification process remained in phase I during the measurements. This presumption agrees with the fact that the depth of the interlayer remained almost constant during the experiment. The theoretical depth in phase I is about 4.70 m, which is close to the experimental value (about 5 m). The theoretical flux fed into the interlayer in phase I is about 70 m\(^3\)/s; the experiment possibly points to a somewhat larger value (80 to 100 m\(^3\)/s), but the order of magnitude is correct.

The relative density differences decreased during the experiment, which effect is not included in the theory. Changing the relative density difference between epilimnion and hypolimnion, however,
influences the theoretical results only slightly.

5.2. 1977 experiment

This experiment was carried out in the reservoir "Petrus plaat", see fig. 5.5. The storage capacity of this reservoir is \(13 \times 10^6\) m\(^3\), the area amounts to about \(10^6\) m\(^2\), and the depth is about 15 m. The usual throughflow is 4 m\(^3\)/s. During the experiment the injection point marked D in fig. 5.5 with an air flow rate of 0.17 m\(^3\)/s was used.

The procedure prior to the experiment was as in the case of the 1976 experiment. The air injection was started on June 16. The temperature profiles were taken for the larger part in the near field. Far field temperature profiles at distances of 67 m, 200 m and 300 m from the injection point are shown in fig. 5.6. The velocity measurements failed in this case. The experiment was ended on June 20, since near field and far field then had become almost homogeneous.

The local wind velocities varied from 0 to 4.5 m/s during the measuring period. The sky was overcast so that the insolation was small. The total heat content of the reservoir therefore increased only slightly, see fig. 5.6.

Experimental and theoretical results

Since the insolation was small, the 1977 experiment in principle lends itself better to a comparison with the theory developed than the 1976 experiment.

Fig. 5.6 shows that the initial temperatures decrease gradually with depth. To make calculations possible the method to schematize a continuous density profile to a three-layer system, described in section 4.4.2, was applied. The radius, \(R_0\), of the far field was estimated in the way explained in the preceding section (\(R_0 = 400\) m).
fig. 5.5 - The reservoir "Petrus plaat"

fig. 5.6 - Far field temperature profiles
Fig. 5.7 - Radius of near field versus time

Fig. 5.7 shows a comparison between experimental and theoretical values for the radius $R$ of the near field. Fig. 5.8 shows the layer depths as functions of time. The estimates of the experimental layer depths were obtained from the temperature profiles, fig. 5.6. The theoretical depth, $a_2^0$, of the interlayer in the outer far-field is found to be less than the experimental depth. It must be kept in mind, however, that the influence of wind is not included in the theory. This influence could be significant particularly in later stages of the destratification process, since the density difference between free surface and bottom considerably decreased during the experiment (fig. 5.6), apparently as a result of vertical exchange of mass between the layers. Wind, although light, could then amplify the mixing at all depths.

The experimental result that appreciable mass exchange between the layers occur, is in qualitative agreement with the conclusions of appendices III and IV.

The difference between $a_2^0$ and $a_2^R$ (theory) is caused mainly
by the Coriolis force. For the sake of comparison, the layer depths in the outer far-field are shown also for the case where the Coriolis force is disregarded (but $R_o$ remains unchanged). The theory possibly overestimates the effect of the Coriolis force in the hypolimnion. This tendency seems to be present also in table 5.1.

![Diagram](image.png)

fig. 5.8 - Layer depths versus time

6. Summary and conclusions

The aim of this research project was (i) to obtain an understanding of the destratification process caused by the local injection of air at the bottom of a thermally stratified lake or reservoir, and (ii) to make possible quantitative predictions of the rate of destratification in a given situation. To this end a mathematical model was set up, and experiments in a laboratory model and in nature were made. The situation considered in the theory was relatively simple: the initial stratification was schematized as a two-layer system at rest in a constant-depth reservoir; wind and insolation during air injection were disregarded.

The mathematical model distinguishes between a near field and
a far field. The near field theory is taken from research carried out elsewhere (6, 7). The far field is modelled as a three-layer system, since the water mixed in the near field forms an interlayer between the original epilimnion and hypolimnion. Turbulent exchange of mass and heat between the layers is disregarded, although this is not correct in some cases (appendices III and IV). The Coriolis force and the inertial terms are included in the layer model. Coupling conditions between near field and far field can be derived by considering the transition between the two zones as an "internal weir".

The theory was verified with experiments in a physical model. Some coefficients related to entrainment in the near field and at the transition were determined experimentally. Salt was used to obtain the density differences in the model. A recirculation phenomenon observed in later stages of the destratification process, is not accounted for by the theory.

The comparison between theory and (two) prototype experiments was rather cumbersome because of a number of reasons. As regards the first experiment, insolation increased the water temperatures during destratification. The stratification before starting the second experiment was weak. Problems encountered during both experiments concern the vertical exchange of mass and heat, and the determination of the horizontal reach of the far field. Presumably, the vertical exchange was influenced by background turbulence. Nevertheless, the experiments indicate that the theory to a certain extent describes appropriate prototype situations. The marked effect of the Coriolis force predicted was observed also, and the theoretical rate of destratification agrees reasonably with the experimental one in the case where little insolation occurred.

Test calculations (not shown in this report) based on the theory indicate that the rate of destratification depends only slightly on the density difference between epilimnion and hypolimnion,
and on the depth ratio of epilimnion and hypolimnion. The dependence on the air flow rate is also rather weak: increasing, in a particular case, the air flow rate from 0.05 m$^3$/s to 0.17 m$^3$/s increased the rate of destratification by a factor of only 1.5 to 1.7.

The following conclusions may be drawn:

- The flow in the far field exhibits a three-layer structure in experiments on laboratory scale. It is doubtful, however, whether this always holds in prototype situations.
- The agreement between theory and model experiments is satisfactory. A recirculation phenomenon observed in the model experiments when the depth of the epilimnion had decreased to a certain fraction of the total depth is not accounted for by the theory. The total amount of mixing water, however, is hardly influenced by the recirculation. Moreover, it is questionable whether recirculation of this type would occur in a prototype situation, since the Coriolis force then opposes the thinning out of the epilimnion.
- Theory and a prototype experiment show the Coriolis force to cause relatively large tangential velocities in prototype situations. The theory predicts a considerable effect of the Coriolis force on the rate of destratification.
- Further measurements in nature are necessary to verify the applicability of the theory to prototype situations. This in particular concerns the relationship between air flow rate and water flow rates (the coefficient $a$, (17)), the rate of destratification, the vertical exchange between the layers, and the horizontal reach of the far field.
- The theory indicates that the rate of destratification is not sensitive to changes in the density differences between epilimnion and hypolimnion, and to the depth ratio of these layers.
- Increasing the air flow rate increases the rate of destratification less than proportionally. Increasing the number of injection points in a particular reservoir while the total air flow rate is fixed therefore increases the rate of destratification.
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Appendix I - Three-layer model

In the case of nearly horizontal flow the vertical accelerations may be neglected when compared with gravity. As a result the vertical pressure distribution becomes hydrostatic. The waves to be examined later therefore have a long-wave character (wave length large when compared with layer depths). In each layer a uniform velocity profile is assumed. Transport of mass through the interfaces is not considered. The density of each layer is assumed constant.

The flow caused by a bubble column is three-dimensional, but it is sufficient to consider a two-dimensional situation in order to obtain expressions of interest, such as those for (internal) wave celerities and the condition for internally critical flow.

The equations for the three-layer model then are (fig. I.1)

### Continuity Equations

\[
\frac{\partial a_1}{\partial t} + \frac{\partial}{\partial x} a_1 v_1 = 0 \tag{I.1}
\]

\[
\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x} a_2 v_2 = 0 \tag{I.2}
\]

\[
\frac{\partial a_3}{\partial t} + \frac{\partial}{\partial x} a_3 v_3 = 0 \tag{I.3}
\]

### Equations of Motion

\[
\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + g \frac{\partial a_1}{\partial x} + g \frac{\partial a_2}{\partial x} + g \frac{\partial a_3}{\partial x} = f_1 - g \frac{dh_b}{dx} \tag{I.4}
\]

\[
\frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x} + \rho_1 \frac{\partial a_1}{\partial x} + \rho_2 \frac{\partial a_2}{\partial x} + \rho_3 \frac{\partial a_3}{\partial x} = f_2 - g \frac{dh_b}{dx} \tag{I.5}
\]

\[
\frac{\partial v_3}{\partial t} + v_3 \frac{\partial v_3}{\partial x} + \rho_1 \frac{\partial a_1}{\partial x} + \rho_2 \frac{\partial a_2}{\partial x} + \rho_3 \frac{\partial a_3}{\partial x} = f_3 - g \frac{dh_b}{dx} \tag{I.6}
\]
The terms $f_i$ ($i = 1, 2, 3$) on the RHS of (I.4) through (I.6) represent forces, such as interfacial friction, bottom friction and Coriolis force, not depending on derivatives of the variables on the LHS.

The relative density differences

$$
\varepsilon = \frac{\rho_3 - \rho_1}{\rho_2}, \quad \varepsilon_1 = \frac{\rho_2 - \rho_1}{\rho_2}, \quad \text{and} \quad \varepsilon_3 = \frac{\rho_3 - \rho_2}{\rho_2}
$$

are assumed to be small. According to the definitions

$$\varepsilon_1 + \varepsilon_3 = \varepsilon \tag{I.8}$$

In the case of a stable density stratification $\varepsilon_1$ and $\varepsilon_3$ are positive.

fig. I.1 - Definition sketch

Equations (I.1) through (I.6) represent external as well as internal wave phenomena. External waves (wave celerity $\sim \sqrt{ga}$) are not of interest here, and are eliminated. This simplification is made possible by the assumption of small density differences (e.g. 9). Continuity equations for purely internal phenomena can be derived
as follows. Neglecting wave disturbances at the free surface yields

\( a = a_1 + a_2 + a_3 \)

\( \frac{\partial}{\partial x} (h_b + a) = 0 \)

In the case of flow caused by a bubble column the volume of water in the lake is constant, so that

\( h_b + a = \text{constant} \) \hspace{1cm} (I.9)

Adding (I.1) through (I.3) gives, with (I.9) and \( \partial h_b/\partial t = 0 \),

\( \frac{\partial}{\partial x} (q_1 + q_2 + q_3) = 0 \)

in which \( q_i = a_i v_i \). Thus the net flow rate is a function of time only. In view of (I.9) the net flow rate must be zero, however. Consequently

\( q_1 + q_2 + q_3 = 0 \) \hspace{1cm} (I.10)

Equations of motion for internal phenomena can be derived by eliminating the relatively large pressure terms between (I.4) through (I.6). Neglecting density differences in the inertia terms and in the force terms on the RHS (the Boussinesq approximation) the two new equations of motion, viz. \( [\rho_1 \times (I.4) - \rho_2 \times (I.5)]/\rho_2 \) and \( [\rho_2 \times (I.5) - \rho_3 \times (I.6)]/\rho_2 \), become

\( \frac{\partial}{\partial t} (v_1 - v_2) + \frac{1}{2} \frac{\partial}{\partial x} (v_1^2 - v_2^2) - \epsilon_1 g \frac{\partial}{\partial x} (a_2 + a_3) = f_1 - f_2 + \epsilon_1 g \frac{dh_b}{dx} \)

(I.11)
\[ \frac{\partial}{\partial t} (v_2 - v_3) + \frac{1}{2} \frac{\partial}{\partial x} (v_2^2 - v_3^2) - \varepsilon_3 g \frac{\partial a_3}{\partial x} = f_2 - f_3 + \varepsilon_3 g \frac{dh_b}{dx} \quad (I.12) \]

There are again six equations for the six unknowns \( a_1, a_2, a_3, v_1, v_2 \) and \( v_3 \): two of the three continuity equations, (I.9), (I.10), (I.11) and (I.12).

In section 2 a third equation of motion related to internal waves only is used. This equation arises by eliminating \( \partial a_3/\partial x \) between (I.11) and (I.12), i.e.

\[ \frac{\partial}{\partial t} (v_2 - \frac{\varepsilon_3 v_1 + \varepsilon_1 v_3}{\varepsilon}) + \frac{1}{2} \frac{\partial}{\partial x} (v_2^2 - \frac{\varepsilon_3 v_1^2 + \varepsilon_1 v_3^2}{\varepsilon}) + \]

\[ + \frac{\varepsilon_1 \varepsilon_3}{\varepsilon} g \frac{\partial a_2}{\partial x} = f_2 - \frac{\varepsilon_3 f_1 + \varepsilon_1 f_3}{\varepsilon} \quad (I.13) \]

**Celerities of internal waves**

The relationship giving the celerities, \( c \), of internal waves may be derived by using the method of characteristics (e.g. 10).

This method yields, when applied to the original set of equations (I.1) through (I.6),

\[
\begin{array}{ccccccc}
 v_1 - c & 0 & 0 & a_1 & 0 & 0 & 0 \\
 0 & v_2 - c & 0 & 0 & a_2 & 0 & 0 \\
 0 & 0 & v_3 - c & 0 & 0 & a_3 & 0 \\
 \frac{\rho_1}{\rho_2} g & g & g & v_1 - c & 0 & 0 & 0 \\
 \frac{\rho_1}{\rho_3} g & \frac{\rho_2}{\rho_3} g & g & 0 & v_2 - c & 0 & 0 \\
 & & & & & & \end{array} = 0 \quad (I.14)
\]

or

\[
(1 - \phi_1)(1 - \phi_2)(1 - \phi_3) - \frac{\rho_2}{\rho_3} (1 - \phi_1) + \frac{\rho_1}{\rho_3} (1 + \phi_2) - \frac{\rho_1}{\rho_2} (1 - \phi_3) = 0
\]

\[(I.15)\]
in which the (squares of the) Froude numbers \( \phi_i \) are given by

\[
\phi_i = \frac{(v_i - c)^2}{ga_i}, \quad i = 1, 2, 3
\]

(I.15) is an equation of sixth degree in \( c \); Consequently there are six (real or complex) celerities, two of which are related to external waves. A relationship giving the four internal-wave celerities can be found by putting

\[
c - v_i \sim \sqrt{\varepsilon a_i} \quad \text{with} \quad \varepsilon \ll 1
\]

and neglecting higher order (small) terms in (I.15). An equation of fourth degree is then obtained,

\[
\phi_1 \phi_2 + \phi_2 \phi_3 + \phi_3 \phi_1 - \varepsilon \phi_1 - \varepsilon \phi_2 - \varepsilon \phi_3 + \varepsilon_1 \varepsilon_2 = 0 \quad (I.16)
\]

This result can also be obtained directly from the approximation for internal waves derived above.

Complex celerities indicate instability of the system (e.g. 11, art. 205). Using this criterion, some necessary stability conditions may be derived from (I.16). Since \( \phi_1, \phi_2 \) and \( \phi_3 \) will be nonnegative if \( c \) is real, the following conditions must be satisfied for stability (fig. I.2)

\[
\phi_1 \leq \varepsilon_1
\]

\[
\phi_2 \leq \frac{\varepsilon_1 \varepsilon_3}{\varepsilon} \quad (I.17)
\]

\[
\phi_3 \leq \varepsilon_3
\]

(I.17) yields as necessary (but not sufficient) conditions for the velocity differences in a stable situation.
fig. I.2 - Relationship between $\phi_1$, $\phi_2$ and $\phi_3$

$$|v_1 - v_2| \leq \sqrt{\varepsilon_1 g a_1} + \sqrt{\frac{\varepsilon_1 \varepsilon_3}{\varepsilon_2}} g a_2$$

$$|v_2 - v_3| \leq \sqrt{\frac{\varepsilon_1 \varepsilon_3}{\varepsilon_2}} g a_2 + \sqrt{\varepsilon_3 g a_3}$$  \hspace{1cm} (I.18)

$$|v_1 - v_3| \leq \sqrt{\varepsilon_1 g a_1} + \sqrt{\varepsilon_3 g a_3}$$

This concerns long internal waves. Shorter waves may become unstable at smaller velocity differences (11, art. 232); breaking of these waves increases the thicknesses of the interfaces until a stable situation is reached (12, 13).

(I.16) may be written

$$P(c) = b_0 + b_1 c + b_2 c^2 + b_3 c^3 + b_4 c^4 = 0$$  \hspace{1cm} (I.19)

in which
\[ b_0 = F_1 F_2 + F_2 F_3 + F_3 F_1 - \frac{e_3}{e} F_1 - F_2 - \frac{e_1}{e} F_3 + \frac{e_1 e_3}{e^2} \]

\[ b_1 = -\frac{2}{eg} \left[ (F_2 + F_3 - \frac{e_3}{e}) \frac{v_1}{a_1} + (F_1 + F_3 - 1) \frac{v_2}{a_2} + (F_2 + F_1 - \frac{e_1}{e}) \frac{v_3}{a_3} \right] \]

\[ b_2 = \frac{1}{(eg)^2} \left( \frac{v_1^2 + 4v_1 v_2 + v_2^2}{a_1 a_2} + \frac{v_2^2 + 4v_2 v_3 + v_3^2}{a_2 a_3} + \frac{v_3^2 + 4v_3 v_1 + v_1^2}{a_3 a_1} \right) + \]

\[ -\frac{1}{eg} \left( \frac{e_3}{a_1} + \frac{e_1}{a_2} + \frac{e}{a_3} \right) \]

\[ b_3 = -\frac{2}{(eg)^2} \left( \frac{v_1 + v_2}{a_1 a_2} + \frac{v_2 + v_3}{a_2 a_3} + \frac{v_3 + v_1}{a_3 a_1} \right) \]

\[ b_4 = \frac{a}{(eg)^2 a_1 a_2 a_3} \]

and \( F_i \) the (squares of the) densimetric Froude numbers given by

\[ F_i = \frac{v_i^2}{e g a_i} \quad , \quad i = 1, 2, 3 \]

As regards the zero's of the polynomial \( P(c) \) various cases can be distinguished. Three examples with stable flow are considered here.

1. Subcritical flow

There are two positive and two negative wave celerities (fig. I.3a with a graphical representation of the polynomial on the left and a \( x,t \)-diagram with characteristic (wave) directions on the right).

2. Critical flow

One of the celerities vanishes (fig. I.3b). The condition for
critical flow is $b_0 = 0$ or

$$F_1 F_2 + F_2 F_3 + F_3 F_1 - \frac{\epsilon_3}{\epsilon} F_1 - F_2 - \frac{\epsilon_1}{\epsilon} F_3 + \frac{\epsilon_3^2}{\epsilon^2} = 0$$  \hspace{1cm} (I.20)

3. **Double-critical flow**

There is a double zero (fig. I.3c). The conditions for double-critical flow are (I.20) and $b_1 = 0$. This type of flow is on the boundary of stability.

![Diagram](image)

**fig. I.3** - Examples of flow types, 
a. subcritical, b. critical, c. double-critical flow

The examination of the directions of the characteristics renders possible the determination of the number of boundary conditions required in a certain case. Suppose a three-layer system exists in the region $x \geq x_0$ and is to be connected with boundary conditions at $x = x_0$. At this boundary there are six unknowns: the three layer depths, and the three velocities in the layers.
Available equations are: (I.9) and (I.10), and compatibility relationships along characteristics directed towards the boundary or coinciding with it (critical or double-critical flow). The required number of boundary conditions then is

\[ 4 - \text{(number of non-positive wave celerities)} \]  

(I.21)

**Extreme discharges and critical flow**

Integrating equations (I.11) and (I.12) in the case of negligible forces \( f_1 \) yields

\[
\frac{1}{2} \left( \frac{q_1^2}{a_1^2} - \frac{q_2^2}{a_2^2} \right) - \varepsilon_1 g ( a_2 + a_3 ) = C_1 + \varepsilon_1 h_b \]  

(I.22)

\[
\frac{1}{2} \left( \frac{q_2^2}{a_2^2} - \frac{q_3^2}{a_3^2} \right) - \varepsilon_3 g a_3 = C_3 + \varepsilon_3 h_b \]  

(I.23)

Furthermore

\[ a_1 + a_2 + a_3 = a \]  

(I.24)

\[ q_1 + q_2 + q_3 = 0 \]  

(I.10)

\( C_1 \) and \( C_2 \) are integration constants. The above four equations contain six unknowns: \( q_1, q_2, q_3, a_1, a_2 \) and \( a_3 \). Consequently, two of the unknowns must be considered as independent variables when differentiating the equations to find an extreme discharge. As an example, the case where \( q_1 \) is extreme is treated in detail. \( a_2 \) and \( q_2 \) are taken as independent variables. Differentiating with respect to \( a_2 \) yields
\[ \frac{\partial q_1}{\partial a_2} = 0 \]
\[ - \frac{q_1^2}{a_1} \frac{\partial a_1}{\partial a_2} + \frac{q_2^2}{a_2} - \epsilon_1 g - \epsilon_1 g \frac{\partial a_3}{\partial a_2} = 0 \]
\[ - \frac{q_2^2}{a_2} \frac{\partial q_3}{\partial a_2} + \frac{q_2^2}{a_2} \frac{\partial a_3}{\partial a_2} - \epsilon_3 g \frac{\partial a_3}{\partial a_2} = 0 \]  
(I.25)

\[ \frac{\partial a_1}{\partial a_2} + 1 + \frac{\partial a_3}{\partial a_2} = 0 \]

\[ \frac{\partial q_3}{\partial a_2} = 0 \]

Eliminating the partial derivatives yields the compatibility condition

\[ F_1 F_2 + F_2 F_3 + F_3 F_1 - \frac{\epsilon_3}{\epsilon} F_1 - F_2 - \frac{\epsilon_1}{\epsilon} F_3 + \frac{\epsilon_1 \epsilon_3}{\epsilon^2} = 0 \]

which is the condition for critical flow, equation (I.29).

Differentiating with respect to \( q_2 \) yields

\[ \frac{\partial q_1}{\partial q_2} = 0 \]

\[ - \frac{q_1^2}{a_1} \frac{\partial a_1}{\partial q_2} - \frac{q_2^2}{a_2} - \epsilon_1 g \frac{\partial a_3}{\partial q_2} = 0 \]

\[ \frac{q_2^2}{a_2} \frac{\partial q_3}{\partial q_2} - \frac{q_3^2}{a_3} \frac{\partial q_3}{\partial q_2} - \epsilon_3 g \frac{\partial a_3}{\partial q_2} = 0 \]

\[ \frac{\partial a_1}{\partial q_2} + \frac{\partial a_3}{\partial q_2} = 0 \]
\[ 1 + \frac{3q_3}{3q_2} = 0 \]

These expressions are compatible, provided

\[
(F_1 + F_3 - 1) \frac{v_2}{a_2} + (F_1 - \frac{e_1}{c}) \frac{v_3}{a_3} = 0
\]  \hspace{1cm} (I.26)

In the case where \( q_2 \) is extreme one obtains again condition (I.20) for critical flow and, instead of (I.26),

\[
(F_1 - \frac{e_1}{c}) \frac{v_3}{a_3} - (F_3 - \frac{e_3}{c}) \frac{v_1}{a_1} = 0
\]  \hspace{1cm} (I.27)

In the case where \( q_3 \) is extreme, (I.20) is obtained and

\[
(F_1 + F_3 - 1) \frac{v_2}{a_2} + (F_3 - \frac{e_3}{c}) \frac{v_1}{a_1} = 0
\]  \hspace{1cm} (I.28)

These results show that if one of the discharges is extreme, the flow will be critical. The reverse is not true in general, since one of the equations (I.26) through (I.28) also has to be satisfied for one of the discharges to be extreme.

A somewhat different case does yield a one to one relationship between critical flow and an extreme value of one of the discharges. This case occurs when one of the discharges is prescribed and one of the remaining two is extreme. As an example the case is considered where \( q_3 \) is given and \( q_1 \) is extreme. Since the number of unknowns is reduced to five, there is only one independent variable (here \( a_2 \) is chosen arbitrarily). Differentiating now yields only one set of equations, viz. (I.25) with partial derivatives replaced by ordinary derivatives. Thus only the condition for critical flow is obtained. The same result is obtained if \( q_2 \) or \( q_3 \) is extreme and one of the other discharges is kept constant.
Appendix II - Asymptotic solution of the far field equations

The far field equations are normalized by introducing the following dimensionless variables

\[ \eta = \frac{r^2}{r_o^2}, \]
\[ \zeta = \frac{a_r^2}{a}, \quad \omega = \frac{u_r^2}{a e_h \epsilon a}, \]
\[ \xi = \frac{x^2}{R^2}. \]  

(Equation II.1)

(2.3.14) and (2.3.16) then change into

\[ \frac{\partial \eta}{\partial \xi} = 1 - \frac{\zeta}{\zeta_o} \]  

(Equation II.2)

\[ \frac{\partial}{\partial \xi} \left( \frac{1 - 2 \zeta}{2(1 - \zeta)^2} \omega^2 + \zeta \right) = -\psi \frac{\eta^{n-1}}{\xi^2} \]  

(Equation II.3)

(2.3.18) is not needed in first approximation. The parameter \( \psi \) in (II.3) is given by

\[ \psi = \frac{f^2 R^2}{8 \frac{a e_h}{a^2} \epsilon a} \]  

(Equation II.4)

The boundary conditions become \( (\xi_o = R_o^2/R^2) \)

\[ \eta(\xi_o, t) = \xi_o \]
\[ \zeta(1, t) = \frac{a_r(t)}{a}, \]  

(Equation II.5)

\[ \omega(\xi_o, t) = 0 \]
\[ \omega(1, t) = \left( \frac{a e_h}{a^2} \epsilon a \right) - \frac{1}{2} u_r^R(t) \]
$a_2^R(t)$ and $u_{r2}^R(t)$ follow from the coupling of the far field with the near field (section 3).

It will be assumed here that $\xi_0 \gg 1$. The parameter $\psi$ typically attains values much less than unity: e.g. $f = 10^{-4} \text{ s}^{-1}$, $R = 30 \text{ m}$, $a_e/a = a_h/a = 0.5$, $\varepsilon = 8 \times 10^{-4}$ and $a = 15 \text{ m}$ yields $\psi = 4 \times 10^{-5}$. Therefore a solution to the above equations will be derived for the asymptotic case where $\psi \to 0$. It may be noted that the point $\xi = 0$ is a second order turning-point (22, 23) of the system (II.2)-(II.3). Although this point lies just outside the domain of interest, it turns out necessary to construct local ($\xi \sim 1$, the inner far-field) and regular ($\xi \sim \xi_0$, the outer far-field) approximations. A matching condition is imposed to connect the two solutions.

The outer far-field

In this case $\xi \sim \xi_0 \gg 1$ and, according to (II.5), $\eta \sim \xi_0$. Consequently (II.3) gives, in first order approximation and disregarding the velocity term ($w$ decreases with increasing $\xi$, and $w = 0$ at $\xi = \xi_0$),

$$\frac{\partial \xi}{\partial \xi} = 0$$

or

$$\xi = \xi_0(t)$$  \hspace{1cm} (II.6)

(II.2) then yields, together with (II.5),

$$\eta = \xi_0 - \left(1 - \frac{\xi_0}{\xi_0}\right)(\xi_0 - \xi)$$  \hspace{1cm} (II.7)

The outer far-field solution can be verified to be identical with the solution given in (8).
The inner far-field

It is shown below that for large $\xi_o$ one must assume that $\psi \eta^2(1) = O(1)^x$. Since in the inner far-field $\xi \sim 1$, and $\partial \eta/\partial \xi < 1$ (see (II.2)), equation (II.3) may now be approximated by

$$\frac{\partial}{\partial \xi} \left[ \frac{1 - 2\xi}{2(1 - \xi)^2} \omega^2 + \zeta \right] = - \psi \frac{\eta^2(1) - 1}{\xi^2}$$

(II.8)

Upon integration and introduction of the boundary condition at $\xi = 1$ this equation yields

$$\frac{1 - 2\xi}{2(1 - \xi)^2} \omega^2 + \zeta = \frac{1 - 2\xi(1)}{2 \left[ 1 - \xi(1) \right]^2} \omega^2(1) + \xi(1) - \psi \left[ \eta^2(1) - 1 \right] (1 - \frac{1}{\xi})$$

(II.9)

Matching

The matching condition for the first-order solution is (22)

$$\lim_{\xi \to 1} \left[ \text{outer far-field solution} \right] = \lim_{\xi \to \xi_o} \left[ \text{inner far-field solution} \right]$$

(II.10)

This gives

$$\xi_o = \frac{1 - 2 \xi(1)}{2 \left[ 1 - \xi(1) \right]^2} \omega^2(1) + \xi(1) - \psi \left[ \eta^2(1) - 1 \right] (1 - \frac{1}{\xi_o})$$

(II.11)

and

$$\xi_o - (1 - \frac{\xi_o}{B_o})(\xi_o - 1) = \eta(1)$$

(II.12)

$x$: The parameter $t$ (time) is suppressed henceforth.
\( \zeta_0 \) and \( \eta(1) \) are unknowns in (II.11) and (II.12). Solving for \( \eta(1) \) yields

\[
\eta(1) = \frac{2c}{\zeta_0} + \left[ \left( \frac{\theta}{\zeta_0} \right)^2 + 4\psi c \right]^{1/2}
\]

in which

\[
C = \zeta(1) + \frac{1 - 2\zeta(1)}{2\left[1 - \zeta(1)\right]^2} \psi(1) + \frac{\theta}{\zeta_0} + \psi
\]

The parameter \( C \) is \( O(1) \) so that \( \eta(1) = O\left(\frac{1}{\sqrt{\psi}}\right) \) if \( \zeta_0 \to \infty \), or, more accurately, \( \psi^2(1) = O(1) \) if \( \frac{1}{\zeta_0} = O(\sqrt{\psi}) \) \hspace{1cm} (II.13)

Fig. II.1 diagrammatically shows the variables \( \zeta \) and \( \eta \) as functions of \( \xi \).

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**Fig. II.1** - \( \eta \) and \( \zeta \) versus \( \zeta \)
Neglecting some minor contributions, (2.3.22), (II.11) and (II.12) yield for the effective radius of the far field 
\[ (\eta(1) = (r_0^R)^2 / R^2) \]
\[ \frac{\psi (a_2 R^2)}{a} \left( \frac{R}{R} \right) + \theta \left( \frac{R}{R_h} \right) \frac{a_2}{a} \left( \frac{R}{R} \right)^2 - \frac{a_2}{a} \left( \frac{a - 2a_2}{a_2} \right) \frac{R^2}{2(a - a_2)^2} \left( \frac{R}{R} \right)^2 \left( \frac{R}{R_2} \right) \frac{R^2}{a} = 0 \]  
(II.14)

The above method of solving the equations in the case where \( \psi \ll 1 \) was compared with a numerical solution for a simplified but similar problem (all velocity terms were dropped). The solutions agreed favourably (\( \psi < 10^{-4} \)).

Appendix III - Estimates of Ekman layer thickness, Ekman layer suction, and interfacial shear stress

The flows in the layers are more or less in opposite directions. As a consequence the velocities at the interfaces are relatively small (the velocity at the bottom is exactly zero). In order to examine the formation of Ekman layers (16, 17, 24) at interfaces and bottom, it is therefore assumed here that the velocities are zero also at the interfaces*. Introducing a Cartesian co-ordinate system \( x, y, z \) (the \( z \)-axis is directed vertically upwards, \( z = 0 \) at the interface or bottom) and velocity components \( u, v, w \) (the subscript \( i \) denoting the layer number is suppressed), the equations of motion in horizontal directions may be conveniently written

---

*Essentially, the solution given in section 2.3 and appendix II is based on negligible interfacial friction (although a parameter, \( \theta \), was introduced to account for the shape of the radial velocity profile). The present appendix considers the other extreme, viz. no-slip conditions at the interfaces.
\[ \frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p + \mathbf{f} \times \mathbf{u} = \frac{\partial}{\partial z} \varepsilon_z \frac{\partial \mathbf{u}}{\partial z} \]  

(III.1)

in which \( \varepsilon_z \) is an eddy viscosity related to vertical transport of momentum, and

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \frac{\partial}{\partial x} + \mathbf{v} \frac{\partial}{\partial y} + \mathbf{w} \frac{\partial}{\partial z} \]

\( \mathbf{u} = (u, v, 0) \)

\( \mathbf{f} = (0, 0, f) \)

\( \mathbf{v} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right) \)  

(III.2)

Forming the cross-product (curl) of (III.1) and \( \nabla \) yields as an equation in \( z \)-direction (the only non-trivial equation)

\[ \frac{D\omega}{Dt} + \frac{1}{2} \frac{\partial v}{\partial z} \frac{\partial \omega}{\partial x} - \frac{1}{2} \frac{\partial u}{\partial z} \frac{\partial \omega}{\partial y} - (f + \omega) \frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \varepsilon_z \frac{\partial \omega}{\partial z} \]  

(III.3)

in which \( \omega \) is the vertical component of the rotation vector,

\[ \omega = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

An estimate of the Ekman layer thickness can be obtained by considering (III.1) at the interface. Since \( \mathbf{u} \) then vanishes and \( \nabla p \) does not depend on \( z \), this yields

\[ \frac{1}{\rho} \nabla p = \frac{\partial}{\partial z} \varepsilon_z \frac{\partial u}{\partial z} \]

Denoting the Ekman layer thickness by \( \delta \), (III.4) gives

\[ \frac{1}{\rho} |\nabla p| \approx \varepsilon_z \frac{|\mathbf{u}_\infty|}{\delta^2} \]

or

\[ \delta \approx \beta \sqrt{\varepsilon_z \rho \frac{|\mathbf{u}_\infty|}{|\nabla p|}} \]

in which \( \mathbf{u}_\infty \) is the vector \( \mathbf{u} \) outside the Ekman layer (\( z \gg \delta \)), and \( \beta \) a coefficient of order one. This coefficient is estimated by comparing (III.5) with the simple case of a parallel flow outside the Ekman layer and constant \( \varepsilon_z \) (e.g. 16, 24). Conceiving of \( \delta \) as a displacement thickness then gives \( \beta = 1/\sqrt{2} \approx 0.7 \).

It is well known that when the flow outside the Ekman layer is rotational, i.e. \( \nabla \times \mathbf{u}_\infty + \mathbf{Q} \), the Ekman layer induces a vertical flow (velocity \( \mathbf{w}_e \)) outside the Ekman layer. This phenomenon is
called Ekman layer suction (or injection), see (34), for instance, for a simple explanation. It is relevant to determine the order of magnitude of the suction velocity, $w_\infty$, since an interaction with the turbulent exchange of mass between the layers (appendix IV) may occur. (III.3) becomes at the interface

$$-\frac{1}{2} f \frac{3w}{3z} \bigg|_{z=0} = \frac{3}{3z} \varepsilon z \frac{3w}{3z} \bigg|_{z=0} (III.6)$$

which yields

$$\frac{f}{\delta} \left| \frac{w_\infty}{\varepsilon} \right| = z \frac{\left| \omega_\infty \right|}{\delta^2}$$

or

$$w_\infty = \beta_2 \frac{\varepsilon \omega_\infty}{f \delta} (III.7)$$

in which $\omega_\infty$ is the z-component of the rotation vector

$$\frac{1}{2} \nabla \times u_\infty$$

outside the Ekman layer, and $\beta_2$ a coefficient of order one. It can be shown that in the case of gradually varying flow outside the Ekman layer, i.e. the case where the horizontal length scale of the flow field is large, $\delta$ and $w_\infty$ are given by

$$\delta = \sqrt{\frac{\varepsilon z}{2f}}, \quad w_\infty = \delta \omega_\infty = \frac{\varepsilon z}{f} \frac{\omega_\infty}{\delta}$$

Therefore $\beta_2 = 1.0$ is assumed.

Returning to the problem under consideration, the following approximations are introduced

$$u_\infty = u_{\phi i}$$

$$\frac{1}{\rho} \nabla p = \frac{u_{\phi i}^2}{r} + f u_{\phi i}$$
The latter expression follows from (2.1.3), in which the inertial terms are neglected (cf. the outer far-field solution in appendix II). Substituting these expressions into (III.5) and (III.7) yields

\[ \delta_1 = 0.7 \sqrt{\frac{\epsilon}{z} \left( \frac{1}{r} \frac{u_i}{u_\Phi} + \frac{1}{f} \right)} \]  

(III.8)

\[ \omega_{\infty_1} = \frac{\epsilon}{2f \delta_1} \left( \frac{\partial u_i}{\partial r} + \frac{u_i}{r} \right) \]  

(III.9)

Substituting from (IV.2) and (IV.3), (III.8) and (III.9) yields for epilimnion and hypolimnion

\[ \delta_1 = \delta_3 = \sqrt{\frac{\epsilon}{f} \left( \frac{1}{2} + \frac{R}{R_o} \left( \frac{\frac{R}{R_o}}{a^2} \right) \right)} \]  

(III.10)

\[ \omega_{\infty_1} = \omega_{\infty_3} = -\frac{\epsilon}{2f \delta_1} \frac{R}{R_o} \frac{a^2}{a} \]  

(III.11)

For the interlayer one obtains, see (2.3.9) \((r >> R)\)

\[ \delta_2 \approx \sqrt{\frac{\epsilon}{f}} \]  

(III.12)

\[ \omega_{\infty_2} \approx -\frac{1}{2} \sqrt{\frac{\epsilon}{z} \frac{f}{f}} \]  

(III.13)

The signs of \(\omega_{\infty_1}, \omega_{\infty_2}\) and \(\omega_{\infty_3}\) are such that the vertical velocity component is directed towards the Ekman layer, so that Ekman layer suction (and not injection) occurs in any case. The mechanism of Ekman layer suction in principle decelerates the tangential motion of epilimnion, interlayer and hypolimnion, which may be significant at the end of the destratification process. The radial velocities, and consequently the tangential component of the Coriolis force
driving the tangential motion, then tend to zero.

Assuming the numerical values

\[ \epsilon_z = 10^{-4} \text{ m}^2/\text{s} \quad (\text{section 5, } \epsilon_z = K_z) \]
\[ f = 10^{-4} \text{ s}^{-1} \]
\[ R = R_0 = 500 \text{ m} \]
\[ R = a_2 = 10 \text{ m} \]
\[ a = 15 \text{ m} \]

one obtains (the subscripts 1, 2 and 3 indicate the three layers as usual)

for \( r = 100 \text{ m} \): \( \delta_1 = \delta_3 \approx 0.2 \text{ m} \) and \( \omega_{\infty 1} = \omega_{\infty 3} \approx -1.4 \times 10^{-4} \text{ m/s} \approx -12 \text{ m/day} \)

for \( r = 500 \text{ m} \): \( \delta_1 = \delta_3 \approx 0.7 \text{ m} \) and \( \omega_{\infty 1} = \omega_{\infty 3} \approx -0.5 \times 10^{-4} \text{ m/s} \approx -4 \text{ m/day} \)

\( \delta_2 \approx 1 \text{ m} \) and \( \omega_{\infty 2} \approx -0.5 \times 10^{-4} \text{ m/s} \approx -4 \text{ m/day} \)

Since the depths of epilimnion, hypolimnion and interlayer amount to some metres at least, the Ekman layers presumably do not overlap in most cases. This indicates that the concept of (Ekman) boundary layers applies to the problem under consideration. The suction velocities are of the same order of magnitude as the entrainment velocities given by (IV.1). The artificial destratification of a lake or reservoir usually requires some days to one week at least. The suction and entrainment velocities together amount to some metres per day in the examples given, which indicates that considerable vertical mass transport may take place during destratification.

The shear stress at an interface depends on the velocity difference between the layers. Usually a quadratic relationship is assumed for turbulent flow. Since the horizontal velocity vector
rotates over angles of order one in the Ekman layers, an estimate for the radial component of the shear stress is

\[ |s_{r,i}| = \rho i k (|u_{\omega,i+1}| - |u_{\omega,i}|)^2 \]

or, because of the large tangential velocities in epilimnion and hypolimnion,

\[ |s_{r,i}| = \rho i k u_{\phi i}^2 \quad \text{(III.14)} \]

in which \( k \) is a coefficient, \( k \sim 4 \times 10^{-4} \) (27). The ratio of the friction terms in (2.1.1) to the centrifugal term, for instance, becomes for the epilimnion

\[ \frac{|s_{r,1} - 0|/\rho i a_1}{v_{\phi 1}/r} = k \frac{r}{a_1} \quad \text{(III.15)} \]

This ratio is small in most cases: for \( r = 200 \text{ m} \) and \( a_1 = 4 \text{ m} \) one obtains \( kr/a_1 = 0.02 \). A similar result is obtained for the hypolimnion. The result for the interlayer is different:

\[ \frac{|s_{r,2} - s_{r,1}|/\rho_2 a_2}{v_{\phi 2}/r} = 2 \frac{u_{\phi 1}^2}{u_{\phi 2}} k \frac{r}{a_2} \quad \text{(III.16)} \]

In the case of the above numerical example \( u_{\phi 1} = 0.03 \text{ m/s} \) and \( u_{\phi 2} = -0.01 \text{ m/s} \) at \( r = 200 \text{ m} \). This then gives

\[ 2 \frac{u_{\phi 1}^2}{u_{\phi 2}^2} k \frac{r}{a_2} = 0.14 \]
A plausible conclusion is that friction is of minor importance as regards the equations of motion in radial direction for epilimnion and hypolimnion, but it is not entirely negligible as far as the interlayer is concerned.

Appendix IV - Estimate of vertical transport through interfaces

The volumes of the layers change during destratification as a result of inflow or outflow at the transition to the near field, but in addition transport of mass, volume and energy through the interfaces may take place. To obtain an idea about the magnitude of these transports, an expression due to Ottesen Hansen (26) is adopted (similar forms are proposed by others). This expression may be written

\[ E_{i,j} = 2 \times 10^{-3} \frac{u_{i,j}^3}{\varepsilon_{i,j} g a_i} \]  
(IV.1)

in which \( E_{i,j} \) = volume of water per unit area of interface transported from the \( j \)th layer to the \( i \)th layer, \( \varepsilon_{i,j} \) = relative density difference between these layers, and \( u_{i,j} \) = velocity difference between these layers.

Velocity differences arise for the larger part as a consequence of the relatively large tangential velocities in epilimnion and hypolimnion. The solution given in appendix II yields for the outer far-field (\( \Theta_o = 1 \))

\[ u_\phi = u_\phi^3 = \frac{f}{2r} (R_o^2 - r^2) \frac{a_o^2}{a} \]  
(IV.2)

and

\[ a_o^2 = \frac{R}{(R_o^2 - r^2)} a_r^2 \]  
(IV.3)
The total rate of entrainment $Q_{2,1}$ of fluid from epilimnion into the interlayer, for instance, becomes

$$Q_{2,1} = R_0^2 \int_0^R 2\pi r E_{2,1} \, dr$$

Putting $|u| \approx u_\phi$ and evaluating the integral gives ($R_0 >> R$)

$$Q_{2,1} = 1.6 \times 10^{-3} \frac{a_2 R}{a} \left( f \frac{a_2^3}{a} R_0^4 R^2 \right)$$

Assuming, as an example, the numerical values (cf. the 1977 prototype experiment, section 5.2)

$$f = 10^{-4} \text{ s}^{-1}$$
$$a = 15 \text{ m}$$
$$a_2^R = 10 \text{ m}$$
$$a_2^h = 10 \text{ m}$$
$$R_0 = R_e = 400 \text{ m}$$
$$\varepsilon = 4.5 \times 10^{-4} \text{ (temperature difference } 17^\circ \text{C} - 14^\circ \text{C})$$
$$R = 30 \text{ m}$$

the volume flux from epilimnion into interlayer would amount to about $2 \text{ m}^3/\text{s}$. Similar values are obtained for the other rates of entrainment.

Comparing this result with the flow rate withdrawn from the epilimnion at the transition to the near field (about $8 \text{ m}^3/\text{s}$, for instance), entrainment is found to be of secondary importance, but not completely negligible. In fact, the vertical exchange of mass during destratification may cause a noticeable decrease in the density differences. The Coriolis force plays an essential part.
here, since it generates the large tangential velocities in epilimnion and hypolimnion.

In cases where wind is present, the level of turbulence may be higher than is implicitly assumed in (IV.1). The entrainment rates would then be even larger.

Appendix V - Homogeneous near field data

The following results are taken from Goossens and Van Pagee (6). The horizontal, radial velocity component, \( u_r(r, z) \), is given by

\[
\frac{u_r(r, z)}{kr} = \frac{m}{kr} \left( \text{sech} \left( \frac{a - z}{kr} \right) - \frac{kr}{a} \tanh \frac{a}{kr} \right) \tag{V.1}
\]

in which the coefficient, \( k \), is equal to about 0.072, and

\[
m = \alpha \sqrt{k} a \left( \frac{Q_a}{a} \right)^{1/3} \tag{V.2}
\]

\( Q_a \) = air flow rate under standard conditions, and the coefficient, \( \alpha \), is equal to about 0.21\(^{ wired} \). (V.1) yields an expression for the height, \( a_b \), of the flow towards the bubble column (fig. V.1),

\[
\frac{kr}{a} \tanh \frac{a}{kr} \cosh^2 \left( \frac{a_a - a_b}{kr} \right) = 1 \tag{V.3}
\]

The rate of flow, \( Q_b \), directed towards the bubble column follows from

\[
Q_b = 2 \pi r \int_a^{a_b} u_r(r, z) \, dz = -2 \pi r m \left( \tanh \frac{a-a_b}{kr} - \frac{a-a_b}{a} \tanh \frac{a}{kr} \right) \tag{V.4}
\]

\(^{ wired} \) Numerical values were obtained from small scale models and prototype experiments (\( a = 15 \) m). However, the validity as regards prototype experiments is subject to some doubt (7).
The flux of horizontal momentum is
\[ \rho \int_0^a u_r^2(r, z) \, dz = \rho \frac{m^2}{kr} \left( 1 - \frac{kr}{a} \tanh \frac{a}{kr} - \frac{1}{3} \tanh^2 \frac{a}{kr} \right) \tanh \frac{a}{kr} \]
(V.5)

The radius, \( R_m \), of the near field, is
\[ R_m = \frac{0.43}{k} a \]
(V.5)

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**fig. V.1** - Definition sketch

**Appendix VI** - Double-critical flow at the transition from near field to far field

Condition (3.2.12) for double-critical flow at the transition gives together with (V.4) and (V.2)

\[
\sqrt{\frac{a \varepsilon h}{e h}} \geq \frac{1}{2\alpha \sqrt{k}} \frac{a R}{a - a^R} \tanh \frac{a}{kR} - (a - a^R) \tanh \frac{a}{kR}
\]
(VI.1)
Substituting the numerical values of the coefficients, $\alpha$ and $k$, and using (V.3) it can be shown that the RHS of (VI.1) varies from about 9 for small radius, $R$, of the near field to about 16 for the maximum radius, $R_m$, of the near field (which is the radius in the homogeneous situation) given by (V.5). Since (VI.1) will turn out to lead to large air flow rates, the radius of the near field in the stratified case will be close to $R_m$, if the flow is double-critical. The RHS of (VI.1) then is close to 16.

Assuming the numerical values

\[ a = 15 \text{ m} \]
\[ \frac{a_e}{a} = \frac{a_h}{a} = 0.5 \]
\[ e = 8 \times 10^{-4} \]

one obtains $Q_a \geq 32 \text{ m}^3$/s if $R = R_m$, and $Q_a \geq 5.7 \text{ m}^3$/s for small values of $R$. These air flow rates are so large that it may be wondered whether a bubble column will still develop. Anyhow, these air flow rates are one or two orders of magnitude larger than those encountered in practice. In the prototype experiments discussed in section 5 the maximum air flow rate was 0.2 m$^3$/s. Rogers et al. (30) mention a flow rate of about 0.12 m$^3$/s in an even deeper reservoir (depth up to 46 m).