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Assessment of Errors in Approximate Solutions of Differential Equations.

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- SUMMARY -

The term assessment is applied to any process which enables us to set rigid bounds to the error or to estimate its value. It is shown that upper and lower bounds can be assigned whenever the Green's function of the problem is one-signed; this is true in many important problems. Another method is applicable to step by step solutions of ordinary differential equations, linear or non-linear, and depends on use of the "index" of the process of integration. Lastly, the error in a linear problem can be estimated when an approximation to the Green's function is known.

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1. INTRODUCTION: MEANING OF ASSESSMENT.

The word assessment is used to cover any process which enables us to estimate or delimit. There are two principal kinds of assessments of error:

(a) the fixing of rigid upper and lower bounds to the error.

(b) the estimation, more or less closely, of the error.

When it is possible to fix rigid bounds to the error these are usually rather widely separated and the assessment is correspondingly crude, but when the bounds are both rigid and close we have a most useful assessment. However, when we have not discovered how to set rigid bounds, or when these are insufficiently close, we must have recourse to some method of estimation. Such an estimation may yield results of ample accuracy for the purposes of applied mathematics even when lacking ideal precision. It is to be remarked that we exclude strict evaluation of the error from consideration since, if this could be done, we should have a method of exact solution. An essential requirement is that any method of assessment shall be applicable throughout the whole region of integration.

We shall here confine attention to differential equations with boundary conditions which render the solution unique. No attempt at a general treatment will be made, but the following items will be discussed:

(1) Bounds to the errors of ordinary and partial differential problems having one-signed Green's functions.

(2) Estimation of the errors of the step by step solutions of ordinary differential equations, or sets of these, with one point boundary conditions.

(3) Estimation of the errors of linear problems when an approximation to the Green's function is known.

Item (1) is concerned only with linear problems and may appear of very limited applicability; in fact it covers some very important problems such as Poisson's equation with fixed boundary values of the unknown. Item (2) covers non-linear problems.

2. BOUNDS TO THE ERRORS OF PROBLEMS HAVING ONE-SIGNED GREEN'S FUNCTIONS.

2.1. Outline of the Method.

We consider an ordinary or partial linear differential equation with boundary conditions of the linear and homogeneous type which render the solution unique. We suppose further that the Green's function of the problem is one-signed.

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Let the differential equation be

$$\Delta \varphi = f \qquad \ldots\ (2.1,1)$$

where Δ is a linear ordinary or partial differential operator, φ is the unknown and f is a given function of the independent variable or variables. Also let

$$\Delta \sigma = 1 \qquad \ldots\ (2.1,2)$$

with the same boundary conditions as for φ. We shall call σ the basic solution, and it is one-signed on account of our assumption about the Green's function. Suppose that φ_a is some approximation to φ which exactly satisfies the boundary conditions and let

$$\Delta \varphi_a - f = \epsilon , \qquad \ldots\ (2.1,3)$$

so ϵ is the residual in the differential equation corresponding to φ_a and is, in general, a function of the independent variables. Let

ϵ_1 = absolute maximum value of ϵ in the region of integration

and ϵ_2 = absolute minimum value of ϵ in the region of integration.

Then $(\epsilon - \epsilon_1)$ is everywhere negative or zero and $(\epsilon - \epsilon_2)$ everywhere positive or zero in the region.

For definiteness let the Green's function be everywhere positive in the region. Then by equations (2.1,1) - (2.1,3)

$$\Delta (\varphi - \varphi_a + \epsilon_1 \sigma) = \epsilon_1 - \epsilon \geqslant 0$$

and consequently

$$\varphi - \varphi_a + \epsilon_1 \sigma \geqslant 0$$

or $$\varphi \geqslant \varphi_a - \epsilon_1 \sigma .$$

Also $$\Delta (\varphi - \varphi_a + \epsilon_2 \sigma) = \epsilon_2 - \epsilon \leqslant 0$$

and $$\varphi \leqslant \varphi_a - \epsilon_2 \sigma .$$

Finally

$$\varphi_a - \epsilon_1 \sigma \leqslant \varphi \leqslant \varphi_a - \epsilon_2 \sigma . \qquad \ldots(2.1,4)$$

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If the Green's function were everywhere negative we should have

$$\varphi_a - \epsilon_2 \sigma \leqslant \varphi \leqslant \varphi_a - \epsilon_1 \sigma. \qquad \ldots.(2.1,5)$$

These inequalities call for two remarks:-

(a) Provided that ϵ_1 and ϵ_2 are small, a rough approximation to σ will usually give sufficient information about the errors in φ_a.

(b) A large but highly localised error in φ_a or its derivatives, will yield numerically large values for one or both of ϵ_1, ϵ_2 and will greatly widen the bounds in the inequalities. It is therefore important to avoid such large local errors when the present method of assessment is used.

2.2. Errors in the Basic Solution.

Let σ_a be an approximation to σ and

$$\Delta\sigma_a - 1 = \eta \qquad \ldots.(2.2,1)$$

Also let η_1, η_2 be the absolute maximum and absolute minimum values, respectively, of η. Then

$$\Delta (\sigma - \sigma_a + \eta_1 \sigma) = \eta_1 - \eta \geqslant 0$$

and $$\Delta (\sigma - \sigma_a + \eta_2 \sigma) = \eta_2 - \eta \leqslant 0.$$

Hence if the Green's function is positive

$$\frac{\sigma_a}{1+\eta_1} \leqslant \sigma \leqslant \frac{\sigma_a}{1+\eta_2} \qquad \ldots.(2.2,2)$$

but if the Green's function is negative

$$\frac{\sigma_a}{1+\eta_2} \leqslant \sigma \leqslant \frac{\sigma_a}{1+\eta_1}. \qquad \ldots.(2.2,3)$$

2.3. Cases where the Boundary Conditions are Not Homogeneous.

When the boundary conditions are linear but not homogeneous we can reduce the problem to one with homogeneous conditions as follows. Let β be a convenient function which satisfies the boundary conditions. Then

$$\psi = \varphi - \beta \qquad \ldots.(2.3,1)$$

/satisfies ...

satisfies linear and homogeneous boundary conditions and

$$\Delta \psi = f - \Delta \beta \qquad \ldots\ldots (2.3,2)$$

where the function on the right hand side of the equation is known.

2.4. An Important Case where the Green's Function is One-Signed.

By way of example we shall show that the Green's function for Poisson's equation is negative, the region of integration being the space interior to a closed surface B upon which the solution is to vanish.

Let the differential equation to be solved be

$$\nabla^2 \varphi = \psi \qquad \ldots\ldots (2.4,1)$$

where ψ is everyshere positive within B and φ is zero on B. Then either φ is everywhere negative within B or it is positive at some point P within B. If it is positive at P it must be also positive everywhere within a closed surface C surrounding P and vanish on C ; moreover C is entirely within B or coincides with it wholly or partly and ψ is therefore everywhere positive within C. Apply Green's theorem to the space within C.

$$\iiint \left\{ \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 + \varphi \nabla^2 \varphi \right\} dxdydz$$

$$= - \iint \varphi \frac{\partial \varphi}{\partial \nu} d\sigma = 0 \qquad \ldots\ldots (2.4,2)$$

where $\frac{\partial \varphi}{\partial \nu}$ is the rate of change of φ along the inward drawn normal to C and $d\sigma$ is the element of the surface of C. But according to our hypothesis the integrand is everywhere positive within C and the integral cannot vanish. Accordingly φ cannot be positive anywhere within B , and, since ψ is an arbitrary positive function, it follows that the Green's function is always negative. The same theorem is true in two dimensions.

This proposition was applied by the writer some years ago to delimit the errors in approximate solutions of special problems in the theory of elasticity [1,2]. In place of the basic function σ the Prandtl torsional stress function Ψ was used; this satisfies the two-dimensional equation:

$$\nabla^2 \Psi + 2 = 0 \qquad \ldots\ldots (2.4,3)$$

and vanishes on the closed boundary, so

$$\Psi = - 2\sigma \qquad \ldots\ldots (2.4,4)$$

It was shown to be possible to place very close bounds to the errors in approximations to Ψ itself [1,2] and to a stress function arising in the St. Venant theory of flexure [2].

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3. A METHOD FOR ESTIMATING THE ERRORS IN STEP BY STEP SOLUTIONS OF SETS OF ORDINARY DIFFERENTIAL EQUATIONS.

3.1. Outline of the Method.

Briefly, the method is based on the idea of extrapolation towards the limit of the step by step solution corresponding to a vanishingly small interval. Suppose that the independent variable is t, the range of integration a to b, and n the number of equal intervals used in the step by step process. Then, provided that the process is completely regular and the boundary values of the unknowns are all given for a (say), we may assume that the error in the value of the dependent variable x_r can be expanded in the series

$$\epsilon_r(t) = n^{-k}\left(e_0 + e_1 n^{-1} + e_2 n^{-2} + \text{etc.}\right) \qquad \ldots\ (3.1,1)$$

provided also that n is not too small. The number k is characteristic of the particular process used and is called its index. Now when the interval is sufficiently small we may as a first approximation retain only the first or dominant term in the expansion and write

$$\epsilon_r(t) = e_0 n^{-k}. \qquad \ldots\ (3.1,2)$$

Since k will be known it becomes possible to calculate e_0, when the approximate solution has been obtained for two values of n. Thus we shall have

$$x_r(t) = x_{r1}(t) + e_0 n_1^{-k} = x_{r2}(t) + e_0 n_2^{-k}$$

where $x_{r1}(t)$, $x_{r2}(t)$ are the approximations to $x_r(t)$ with n_1 and n_2 intervals respectively. The last equations yield

$$e_0 = \frac{x_{r2}(t) - x_{r1}(t)}{n_1^{-k} - n_2^{-k}} \qquad \ldots\ (3.1,3)$$

and a first approximation to the error is obtained. With 3 values of n we can similarly calculate e_0 and e_1 and obtain a next approximation, and so on.

The method just described occurred to the writer after noticing that the error in the step by step solution of a differential equation given by the process of Euler (of index unity) was almost exactly halved when the number of intervals was doubled. After a general account [3] of the method had been prepared and circulated to the Aeronautical Research Council it was pointed out by R.A. Fairthorne that the same method had been proposed earlier by L. F. Richardson [4,5] who gave it as an example of what he called " the deferred approach to the limit". The practical value of the method, which is applicable to non-linear equations, has been demonstrated by a number of examples. It may be noted that the index varies from 1 in the original and relatively crude process given by Euler in the eighteenth century to 4 in the process of Runge and Kutta [6]. The various methods are briefly reviewed in the writer's paper [3].

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When the method just described is applied an estimate should first be made of the number of steps needed to provide the desired accuracy. Then a first calculation should be made with say half this number of steps. A comparison of the results obtained with the two numbers of steps allows the error to be assessed and partially corrected. If need be, a still larger number of intervals must finally be used.

3.2 Extension of the Method to Partial Differential Equations.

The method just described can sometimes be applied to partial differential problems. It is essential that a perfectly regular process be used and that an index should exist. When this is so, the error can be assessed as before when the results for two similar lattices are known. It may even be possible to dispense with this condition of similarity.

4. ESTIMATION OF THE ERRORS IN APPROXIMATE SOLUTIONS OF LINEAR PROBLEMS WHEN AN APPROXIMATION TO THE GREEN'S FUNCTION IS KNOWN.

We shall suppose that we have an approximation ψ_a to the solution of (2.3,2) which satisfies the linear and homogeneous boundary conditions exactly and gives a residual

$$\epsilon = f - \Delta(\psi_a + \beta). \qquad \ldots\ (4,1)$$

Then, if there is an approximation G_a to the Green's function appropriate to the homogeneous boundary conditions, we may derive an approximation to the error of deficiency in ψ_a at any point by forming the corresponding integral of the product ϵG_a throughout the region.

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