A connected $F$-space

Non impeditus ab ulla scientia

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Oxford, 10 August, 2006: 14:30-14:55
Outline

1. The main result

2. Why?
   - $d$-independent sets and $d$-bases
   - What does our space do then?

3. The construction
   - Intuition
   - Starting point
   - Thin out $S_u$
   - Create $X$

4. Sources

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A connected $F$-space
There is a compact Hausdorff space, $X$, that is connected and an $F$-space.
A space and a function

- There is a compact Hausdorff space, $X$, that is \textit{connected} and an \textit{F-space}.
- It supports a continuous real-valued function, $f$, that is not \textit{essentially constant}.
Contrasting behaviour of functions

For every continuous function \( g : X \to \mathbb{R} \) and every \( t \) in the interior of the interval \( g[X] \) the interior of \( g^{←}(t) \) is nonempty. (Follows from connected plus \( F \).)
The main result

Why?
The construction
Sources

Contrasting behaviour of functions

- For every continuous function $g : \mathbb{X} \to \mathbb{R}$ and every $t$ in the interior of the interval $g[X]$ the interior of $g^{-1}(t)$ is nonempty. (Follows from connected plus $F$.)
- Yet, for $f$ we have: $\Omega_f = \bigcup_t \text{int} f^{-1}(t)$ is not dense. (This is not essentially constant.)
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$D$, a subset of $C(X)$, is \textit{d-independent} if for every nonempty open set $O$ the nonzero elements in $\{d \upharpoonright O : d \in D\}$ are linearly independent.
A $d$-independent set $D$ is a $d$-basis if for every $g \in C(X)$ there is a disjoint family $\mathcal{O}$ of open sets, with dense union, such that for every $O$ the restriction $g \upharpoonright O$ is a linear combination of (finitely many members of) \( \{d \upharpoonright O : d \in D \} \).
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A connected $F$-space
Maximally independent does not mean base

- The family \{1\} is maximally $d$-independent.
  
  (For every continuous function $g : X \to \mathbb{R}$ and every $t$ in the interior of the interval $g[X]$ the interior of $g^{-1}(t)$ is nonempty.)
Maximally independent does not mean base

- The family \{1\} is maximally \(d\)-independent. (For every continuous function \(g : X \rightarrow \mathbb{R}\) and every \(t\) in the interior of the interval \(g[X]\) the interior of \(g^{-1}(t)\) is nonempty.)

- Yet, the family \{1\} is not a \(d\)-basis. (For \(f\) we have: \(\Omega_f = \bigcup_t \text{int} f^{-1}(t)\) is not dense.)
Using a $d$-basis that contains 1 one can project $C(X)$ onto the subspace of essentially constant functions, in case $X$ is extremally disconnected.
Using a $d$-basis that contains 1 one can project $C(X)$ onto the subspace of essentially constant functions, in case $X$ is extremally disconnected. Unknown (but wanted) for basically disconnected spaces.
No (easy) projection

Using a $d$-basis that contains $1$ one can project $C(X)$ onto the subspace of essentially constant functions, in case $X$ is extremally disconnected. Unknown (but wanted) for basically disconnected spaces. Apparently even more difficult for $F$-spaces.
1. **The main result**

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A connected $F$-space
Think of $X$ as the following subspace of $S$:

\[( [0, 1] \times \{0\} ) \cup ( C \times [0, 1] ) \]

($C$ is the Cantor set)
Think of $X$ as the following subspace of $S$:

$$([0, 1] \times \{0\}) \cup (C \times [0, 1])$$

($C$ is the Cantor set)

Think of $f$ as resulting from the map from $C$ onto $[0, 1]$ and constant on complementary intervals in bottom line.
Think of \(X\) as the following subspace of \(S\):

\[
([0, 1] \times \{0\}) \cup (C \times [0, 1])
\]

\((C\) is the Cantor set\)

Think of \(f\) as resulting from \(the\) map from \(C\) onto \([0, 1]\) and constant on complementary intervals in bottom line.

This ‘\(X\)’ is not an \(F\)-space . . .
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A connected $F$-space
Let $S$ be the unit square $[0, 1]^2$
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Let $S = \omega \times S$
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Let $\mathbb{S} = \omega \times S$

Define $p : \mathbb{S} \to [0, 1]$ by $p(n, x, y) = x$
A particular $\beta$

- Let $S$ be the unit square $[0, 1]^2$
- Let $\mathcal{S} = \omega \times S$
- Define $p : \mathcal{S} \to [0, 1]$ by $p(n, x, y) = x$
- and extend to $\beta p : \beta \mathcal{S} \to [0, 1]$. 
A component of $\beta S$ and a function

- $\beta \pi : \beta S \rightarrow \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$. 

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A connected $F$-space
A component of $\beta S$ and a function

- $\beta \pi : \beta S \to \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$.
- Pick one $u \in \beta \omega \setminus \omega$. 

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A connected $F$-space
A component of $\beta S$ and a function

- $\beta \pi : \beta S \to \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$.
- Pick one $u \in \beta \omega \setminus \omega$.
- Let $S_u = \beta \pi^{-1}(u)$
A component of $\beta S$ and a function

- $\beta \pi : \beta S \rightarrow \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$.
- Pick one $u \in \beta \omega \setminus \omega$.
- Let $S_u = \beta \pi^{-1}(u)$.
- $S_u$ is a compact connected $F$-space
A component of $\beta S$ and a function

- $\beta \pi : \beta S \to \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$.
- Pick one $u \in \beta \omega \setminus \omega$.
- Let $S_u = \beta \pi^{-1}(u)$.
- $S_u$ is a compact connected $F$-space.
- $\beta p \upharpoonright S_u$ is continuous.
A component of $\beta S$ and a function

- $\beta \pi : \beta S \to \beta \omega$ is the extension of $\pi : \langle n, x, y \rangle \mapsto n$.
- Pick one $u \in \beta \omega \setminus \omega$.
- Let $S_u = \beta \pi^{-1}(u)$.
- $S_u$ is a compact connected $F$-space.
- $\beta p \upharpoonright S_u$ is continuous.

but $S_u$ and $\beta p$ are not good enough . . .
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A connected $F$-space
Set $Y_0 = S_u$ and $q_0 = \beta p \upharpoonright Y_0$ and recursively
Get rid of interiors

Set $Y_0 = S_u$ and $q_0 = \beta p \upharpoonright Y_0$ and recursively

- $Y_{\alpha+1} = Y_\alpha \setminus \bigcup_t \text{int}_\alpha q_\alpha^{-1}(t)$ and $q_{\alpha+1} = q_\alpha \upharpoonright Y_{\alpha+1}$
  (\text{int}_\alpha: \text{interior in } Y_\alpha)
Set $Y_0 = S_u$ and $q_0 = \beta p \upharpoonright Y_0$ and recursively

- $Y_{\alpha+1} = Y_\alpha \setminus \bigcup_t \text{int}_\alpha q_\alpha^{-1}(t)$ and $q_{\alpha+1} = q_\alpha \upharpoonright Y_{\alpha+1}$ (int$\alpha$: interior in $Y_\alpha$)
- $Y_\alpha = \bigcap_{\beta < \alpha} Y_\beta$ and $q_\alpha = q_0 \upharpoonright Y_\alpha$ if $\alpha$ is a limit
Get rid of interiors

Set $Y_0 = S_u$ and $q_0 = \beta p \upharpoonright Y_0$ and recursively

- $Y_{\alpha+1} = Y_{\alpha} \setminus \bigcup_t \text{int}_\alpha q_\alpha^\leftarrow(t)$ and $q_{\alpha+1} = q_\alpha \upharpoonright Y_{\alpha+1}$ (int$_\alpha$: interior in $Y_\alpha$)
- $Y_\alpha = \bigcap_{\beta < \alpha} Y_\beta$ and $q_\alpha = q_0 \upharpoonright Y_\alpha$ if $\alpha$ is a limit

There is a first (limit) $\delta < c^+$ where $Y_\delta = Y_{\delta+1}$, meaning that $\text{int}_\delta q_\delta^\leftarrow(t) = \emptyset$ for all $t$
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Tie everything together

Sadly, $Y_\delta$ is not connected
Tie everything together

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However, take the bottom line of $S_u$:
Tie everything together

 Sadly, $Y_\delta$ is not connected
However, take the bottom line of $S_u$:

$$B_u = S_u \cap \text{cl}(\omega \times [0, 1] \times \{0\}).$$
Here are $X$ and $f$

Finally then

\[ X = B_u \cup Y \delta \]

$f = \beta p \upharpoonright X$

$X$ is connected and $F$-space

All components of $Y$ meet the top line, so $\Omega f \subseteq B_u$ is not dense
Here are $X$ and $f$

Finally then

$$X = B_u \cup Y_\delta$$
Finally then

- $X = B_u \cup Y_\delta$
- $f = \beta p \upharpoonright X$
Here are $X$ and $f$

Finally then

- $X = B_u \cup Y_\delta$
- $f = \beta p \upharpoonright X$

$X$ is connected and $F$
Here are $X$ and $f$

Finally then

- $X = B_u \cup Y_\delta$
- $f = \beta p \upharpoonright X$

$X$ is connected and $F$
int $f^{-1}(t) \subseteq B_u$ for all $t$
Finally then

- $X = B_u \cup Y_\delta$
- $f = \beta p \upharpoonright X$

$X$ is connected and $F$

$\text{int} f^{-1}(t) \subseteq B_u$ for all $t$

All components of $Y_\delta$ meet the top line, so $\Omega_f \subseteq B_u$ is not dense
Light reading

Website: fa.its.tudelft.nl/~hart

- Y. A. Abramovich and A. K. Kitover. 

- K. P. Hart. 
  \textit{A connected F-space}, Positivity, 10 (2006), 607–611.