Propagation and band width of smeared cracks
Propagation and band width of smeared cracks

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## Symbols and abbreviations

### Roman symbols
- $a_n, b_n$: Fourier series coefficients
- $A_e$: element area
- $A^*$: numerically obtained crack area
- $d_{\text{crit,crk}}$: damage threshold value for crack path fixation
- $d_{ij}$: damage values in integration point $j$, $j = 1, \ldots, n_{ip}$
- $D$: constitutive stiffness matrix
- $e$: normalized vector or propagation path direction at exit point
- $E_0$: Young’s modulus
- $f_c$: compressive strength
- $f_t$: tensile strength
- $f_y$: yield stress
- $F_x$: reaction force
- $G_0$: shear retention
- $G^*_f$: apparent, i.e., numerically obtained, fracture energy
- $G_{\text{red}}$: reduced shear retention
- $k_n$: normal interface stiffness
- $k_t$: tangential interface stiffness
- $l$: crack band width
- $l_g$: crack band width according to Govindjee et al.
- $l_h$: crack band width according to Oliver
- $l^*_p$: new proposed crack band width formulation
- $L_i$: numerically obtained localization band width
- $n(x)$: crack normal at point $x$
- $n_{cn}$: number of corner nodes
- $n_{ip}$: number of integration points in an element
- $n_{\text{newt}}$: number of saw-teeth
- $n_{ij}$: normal of potential crack in integration point $j$, $j = 1, \ldots, n_{ip}$
- $N_i$: standard shape function in corner node $i$, $i = 1, \ldots, n_{cn}$
- $N$: coefficient matrix with evaluated shape functions
- $N_{\text{inp}}$: coefficient matrix with evaluated shape functions at input positions
- $p_{\text{u, pl}}$: upper and lower strength percentages
- $P$: tensile load
- $r_{\text{excl}}$: user-defined exclusion radius
- $r_i$: normalized propagation vector in corner node $i$, $i = 1, \ldots, n_{cn}$, added for regularization
SYMBOLS AND ABBREVIATIONS

\( R^2 \)  \hspace{1cm} \text{coefficient of determination} \\
\( R^k \)  \hspace{1cm} \text{standard rotation matrices, } k = 1, 2 \\
\( s \) \hspace{1cm} \text{normalized start vector or propagation path direction at entry point} \\
\( S_{j1}, S_{j2} \) \hspace{1cm} \text{opposite boundary edges/faces, } j = x, y, z, c \\
\( t \) \hspace{1cm} \text{thickness} \\
\( t_j \) \hspace{1cm} \text{normalized propagation vector in integration point } j, j = 1, \ldots, n_{ip} \\
\( t^k \) \hspace{1cm} \text{normalized orthogonal to maximum principal strain direction, } k = 1, 2 \\
\( T \) \hspace{1cm} \text{transformation matrix} \\
\( T(\xi, \eta) \) \hspace{1cm} \text{estimated normalized propagation vector field} \\
\( u_{cr} \) \hspace{1cm} \text{total inelastic (crack) displacement} \\
\( u_{el} \) \hspace{1cm} \text{elastic displacement} \\
\( u_{j1,i}, u_{j2,i} \) \hspace{1cm} \text{nodal displacements in } i \text{ direction for corresponding nodes on edges/faces} \\
\( S_{j1} \text{ and } S_{j2} \text{ respectively, } j = x, y, z, c \) \\
\( u_{\Delta_j} \) \hspace{1cm} \text{degree of freedom that represents } \Delta_{ji} \\
\( w \) \hspace{1cm} \text{crack opening or width} \\
\( w_{ult} \) \hspace{1cm} \text{ultimate crack opening at which no stresses can be transferred anymore} \\
\( w \) \hspace{1cm} \text{weight factors} \\
\( W \) \hspace{1cm} \text{total amount of dissipated energy} \\
\( W \) \hspace{1cm} \text{diagonal matrix that contains weight factors} \\
\( x \) \hspace{1cm} \text{considered point at which } l \text{ is evaluated} \\
\( x_c \) \hspace{1cm} \text{center point of element belonging to } x \\
\( x_i \) \hspace{1cm} \text{corner node } i \text{ of element belonging to } x \\

Greek symbols \\
\( \alpha \) \hspace{1cm} \text{factor for strain localization within an element} \\
\( \beta \) \hspace{1cm} \text{shear retention factor} \\
\( \gamma \) \hspace{1cm} \text{alignment factor} \\
\( \gamma_c \) \hspace{1cm} \text{orientation factor according to Cervenka et al.} \\
\( \delta \) \hspace{1cm} \text{displacement difference} \\
\( \delta_{\text{average}} \) \hspace{1cm} \text{the average value of displacement differences} \\
\( \Delta_{ji} \) \hspace{1cm} \text{constant displacement difference in } i \text{ direction, } i = x, y, z, \text{ between opposite edges/faces } S_{j1} \text{ and } S_{j2}, j = x, y, z, c \\
\( \varepsilon \) \hspace{1cm} \text{strain} \\
\( \varepsilon_{\text{ult}} \) \hspace{1cm} \text{ultimate strain} \\
\( \eta \) \hspace{1cm} \text{multiplication factor of the regularized fracture energy} \\
\( \theta \) \hspace{1cm} \text{crack or element orientation angle} \\
\( \theta_{j} \) \hspace{1cm} \text{potential crack normal angle in integration point } j, j = 1, \ldots, n_{ip} \\
\( \lambda_{\text{crit}} \) \hspace{1cm} \text{critical load multiplier} \\
\( \nu_0 \) \hspace{1cm} \text{Poisson’s ratio} \\
\( \sigma \) \hspace{1cm} \text{stress tensor} \\
\( \sigma \) \hspace{1cm} \text{stress tensor} \\
\( \tau_i \) \hspace{1cm} \text{crack indicator of corner node } i \text{ according to Govindjee et al.} \\
\( \tau_{\text{max}} \) \hspace{1cm} \text{maximum shear stress} \\
\( \phi_1 \) \hspace{1cm} \text{crack indicator of corner node } i \text{ according to Oliver} \\
\( \Phi(\xi, \eta) \) \hspace{1cm} \text{estimated scalar propagation direction field} \\
\( \Phi_{cn} \) \hspace{1cm} \text{estimated propagation directions in corner nodes} \\
\( \varphi \) \hspace{1cm} \text{input propagation directions} \\
\( \partial_x \) \hspace{1cm} \text{partial derivatives to } x \\

Subscripts \\
\( \text{cn} \) \hspace{1cm} \text{corner node}
SYMBOLS AND ABBREVIATIONS

ep  exit point
inp  input
ip  integration point
n  normal direction in crack coordinate system
sp  starting point
swt  saw-tooth
\(t\)  tangential direction in crack coordinate system

Abbreviations
BVP  boundary value problem
CMSD  crack mouth sliding displacement
CPA  crack propagation algorithm
DEN  double-edge-notched
E-FEM  elemental enrichment finite element method
FE  finite element
NLFEA  (conventional) nonlinear finite element analysis
RC  reinforced concrete
SDA  strong discontinuity approach
SEN  single-edge-notched
SLA  sequentially linear analysis
X-FEM  extended finite element method
Chapter 1

Introduction

1.1 Background

A number of uncertainties have risen regarding the shear capacity of existing infrastructural reinforced concrete (RC) structures in different countries [3, 153]. It raises the question whether these bridges and viaducts are still sufficiently safe. The answer to this question is relevant for decision-making by the responsible authorities concerning maintaining, upgrading or even demolishing and possibly rebuilding these structures. Obviously, the last option will include significant investments and serious traffic disruption.

In order to give a proper answer to the question posed, one needs to come up with reliable predictions of the actual bearing capacities of the existing infrastructural RC structures. These predictions can be based on testing, either laboratory testing or proof loading [47], on numerical simulations, or on a combination of testing and simulation. In this thesis the focus is on the numerical approach, by means of nonlinear finite element (FE) analysis. Here, nonlinearity is related to the cracking or softening behavior of the concrete material. This requires proper kinematic and constitutive descriptions and an analysis procedure that leads to equilibrium.

Within the field of concrete mechanics many common strategies for the modeling of concrete fracture are available [1, 113]. The strategy selected in this work is the classical crack band approach [10] in the smeared crack concept [115]. This strategy allows for analysis at structural level, while simultaneously the modeling of single cracks is still possible, provided that a suitable mesh fineness is adopted. Other strategies, such as nonlocal models [8] and Partition of Unity methods [12], are also applicable in this scale of interest.
1. INTRODUCTION

Significant progress within these advanced strategies has been made over the last decades. However, the engineering community generally has not adopted them for analyzing large-scale RC structures with multiple cracking. Most FE packages available to the engineering practice do not (yet) offer them. On the other hand, the crack band approach is widely used and still an important analysis technique within advanced engineering.

Nonlinear FE analyses on aforementioned shear critical structures face difficulties that arise from the nature of shear failure. Typically, such failure modes are rather brittle and are frequently accompanied by the sudden appearance of a major diagonal crack. These inelastic processes indicate that initially smooth strain patterns suddenly change into strain distributions with rapidly increasing strain values in a narrow localized band and decreasing strain values in the adjacent parts of the structure. For that reason such phenomena are called strain localizations [77].

When modeling strain localizations numerically with FE discretizations and standard continuum models, generally the problem of mesh dependency is observed [37]. This means that the spatial discretization affects the numerical results, e.g. load - displacement responses and crack patterns, and mesh objective energy dissipation of the fracture process cannot be guaranteed. Mesh dependency can be subdivided into a sensitivity with respect to the size of the finite element (mesh size sensitivity) and in a sensitivity with respect to the structure and orientation of the FE mesh (directional mesh bias). These sensitivities are reflected in the frequently reported phenomena that strains tend to localize in the smallest possible band of elements across the FE discretization and the strain localization bands prefer to propagate along continuous mesh lines. The crack band approach has solved the issue of mesh size sensitivity when strains localize in a single row of elements and the elements are aligned with their principal directions. However, for the general case of strain localizations in meshes that are not aligned, the results show mesh dependence, since the crack band approach still suffers from directional mesh bias. This important and challenging flaw needs to be tackled. As long as numerical simulations of quasi-brittle structures are impeded by mesh-induced directional bias, reliable predictions of ultimate load capacities, failure mechanisms and post-peak behaviors are difficult to make.
1.2 Objectives

The aim of this thesis is to improve the numerical simulations with the crack band approach in the smeared crack concept to obtain accurate and reliable predictions of brittle (shear) failure problems. Special attention is given to the mesh-induced directional bias, which needs to be reduced as much as possible. Simultaneously with increased accuracy for the modeling of concrete fracture, it is the intention to preserve the relative simplicity of the crack band approach. Such improvements will be valuable for engineering practice in general, and for the re-assessment of shear critical RC structures more specifically.

To this end, the work distinguishes the following four objectives:

- to find a way for a systematic and quantitative assessment of mesh-induced directional bias;
- to develop enhancements of the crack band approach in the smeared crack concept to reduce the mesh-induced directional bias;
- to investigate the influence of the proposed enhancements on the modeling of brittle failure in plain concrete;
- to explore different modeling strategies, material and mesh sensitivities and the influence of the proposed enhancements on the simulation of brittle shear failure in reinforced concrete.

1.3 Scope

The work in this thesis is realized within the context of quasi-static and two-dimensional (plane stress) analyses. Developments and numerical simulations are respectively embedded in and performed with the Sequentially Linear Analysis (SLA) method, and focus on the modeling of brittle shear failure. It is expected that the research findings are also valuable outside this scope. Results will be applicable to conventional nonlinear FE analysis, three-dimensional problems and to other brittle failure mechanisms.

The motivation to select the SLA method and a further specification of the modeling of brittle shear failure are discussed in the two following paragraphs.
Sequentially Linear Analysis method as a solution procedure

Nonlinear FE analyses not only require proper kinematic and constitutive descriptions, but at least as important is the solution procedure in order to obtain equilibrium. Different solution procedures are available.

The conventional way of doing nonlinear FE analysis is by using incremental-iterative procedures, like regular or modified Newton-Raphson. However, in case of modeling brittle failure these procedures may be unstable and convergence problems can be faced. This can be explained by the following: when a sudden large crack in a concrete structure appears, the material tangent stiffness becomes negative (softening) in many integration points simultaneously. The global structural stiffness in a consecutive load increment can therefore be strongly affected and the iterative procedure finds difficulties to obtain an equilibrium state. Furthermore, softening behavior may involve snap-backs on structural level and may also result in non-uniqueness of the solution or bifurcation of the equilibrium path [31]. Although solution procedures and control schemes (like arc-length control) have been considerably improved [57, 146], still problems exist to capture strong nonlinear structural behavior and to obtain accurate equilibrium situations in the post-peak regime.

Another way of performing nonlinear FE analysis is by using the SLA method [123]. This relatively new solution procedure performs a series of scaled linear analyses. For every analysis the global load is scaled in such a way that only one integration point reaches a critical stress state. Subsequently, damage is applied in this critical point according to a discretized stress-strain softening law based on secant stiffnesses, and the process is repeated. Due to the absence of an iteration process and negative tangent stiffnesses, aforementioned difficulties related to the control of the solution process are circumvented.

Avoidance of numerical instability is attractive, both for the modeling of brittle failure behavior and for evaluation of the proposed enhancements (see objectives in Section 1.2). Therefore, the SLA method is chosen in this work as solution procedure.

Shear critical RC beam as a benchmark

To fulfill the last objective, mentioned in Section 1.2, there is a need for a well-conditioned fracture test, representative for the type of problems we are interested in. Given the context of this study, as described in Section 1.1, an experiment that contains a series of six identical shear critical RC beams is selected [130]. The experiment was performed in the Stevin Laboratory at Delft University of Technology. The specimens, with three longitudinal reinforcing
bars and without shear reinforcement, were simply supported and loaded by a concentrated load at midspan. All six beams revealed the same brittle shear failure mode during testing and the global load-deflection responses were in good agreement. Hence, this experiment is a well-defined benchmark for our research.

1.4 Thesis approach and outline

After this introductory chapter, the thesis is organized as described below. An overview is presented in Figure 1.1.

Chapter 2 reviews relevant literature for the topic of this thesis. Firstly, the classical constitutive modeling of concrete fracture is briefly recapitulated. Secondly, the problem of mesh-induced directional bias is addressed and an overview of remedies is provided. Subsequently, the general concept of the crack band approach is explained and details of various band width estimators are given. Finally, the main issues of the SLA method are discussed.

Chapter 3 proposes a dedicated numerical testing procedure to assess the directional mesh bias of constitutive models in a systematic way. It enables to quantify mesh-induced directional bias, rather than only get a qualitative impression. The test makes use of periodic boundary conditions, by which strain localization can be analyzed for different mesh alignments with preservation of mesh uniformity and with exclusion of boundary disturbances. Adopting certain specimen and element sizes, this testing framework is applied to the element projection based crack band width formulation according to Govindjee et al. [51] that has been implemented in the SLA method. The results are presented for different conventional element types, which are commonly used in practice. The mesh-induced directional bias is identified quantitatively and its relation to the finite element characteristics is evaluated. Finally, the effect of the specimen size is investigated in this chapter.

Subsequently, two different enhancements of the crack band approach in the smeared crack concept are developed, using conventional finite elements. One enhancement is related to the transverse direction, the other to the longitudinal direction of a single crack:

1. Chapter 4 proposes an extended element projection based crack band width formulation, in which the effects of mesh alignment and all finite element characteristics, as shape, interpolation function and integration
1. INTRODUCTION

Figure 1.1: Thesis overview.
1.4 Thesis approach and outline

scheme, are taken into account. This enhancement is based on the results in Chapter 3 and aims for maximization of the mesh objectivity via improvement of the fracture energy regularization. The new formulation is validated by bending and mixed-mode plain concrete fracture tests.

2. **Chapter 5** proposes a new local crack propagation algorithm that blends with the damage controlled analysis procedure of the SLA method. This enhancement aims to reduce mesh-induced directional bias by the determination of crack paths across the FE discretization. Only elements crossed by a crack path experience nonlinear material behavior. The crack paths are constructed element by element, based on calculated normalized propagation vector fields. These vector fields are obtained from the principal strain and damage states in the integration points and the propagation direction with which the crack path enters an element. By taking into account the propagation direction at the entry point explicitly, smoothly curved \( C^1 \)− continuous crack paths can be obtained. Furthermore, this \( C^1 \)− continuous crack propagation algorithm enables to delay the moment of crack path fixation, allowing for possible redistribution of stresses and strains in front of the crack path tip prior to this moment. The proposed algorithm is validated by the same experimental fracture tests as in Chapter 4.

**Chapter 6** switches from *plain* concrete to *reinforced* concrete, with the focus on the aforementioned benchmark. Attention is paid to important FE modeling aspects that arise from the SLA method and the material modeling of concrete and reinforcement with their interaction, i.e. bond-slip behavior. Furthermore, an exploratory numerical study to the RC shear critical beam is conducted and presented, with variations on input parameters and meshes, temporarily using simplifications as a crack band width formulation based on the element area and perfect bond. Since SLA has hardly been applied to this type of failure problems, its ability to capture the brittle shear failure behavior is investigated and compared to the conventional nonlinear FE analysis procedure. Finally, results of analyses with the proposed enhancements are presented and discussed. Specifically for the crack propagation algorithm, its combination with the way of modeling bond behavior is examined in detail.

**Chapter 7** summarizes the main conclusions of the thesis and gives a brief outlook for potential future use and further developments of the proposed enhancements.
1. INTRODUCTION

Foregoing thesis approach and outline include a wide range of numerical simulations on different fracture tests, using different crack band width formulations and with variations of mesh and finite element characteristics. To categorize and list all these simulations systematically, an overview table for the reader is added in Appendix A.
Chapter 2

Literature review

This chapter reviews four main issues that were touched on in the first chapter. Section 2.1 starts with a brief introduction of some key notions regarding the classical constitutive modeling of concrete fracture within the context of standard FE discretizations. The numerical solutions of these models are inherently biased by the orientation of the mesh. The background of this mesh bias is discussed in Section 2.2 and subsequently a concise overview of the proposed remedies in literature is provided. In Section 2.3 we continue with the classical smeared way of modeling concrete fracture, where the general concept of the crack band approach is further explained and different common band width estimators are reviewed in detail. Section 2.4 describes the Sequentially Linear Analysis (SLA) method, with attention to the global procedure, the SLA specific treatment of the constitutive behavior and the adopted material model. This chapter ends with concluding remarks in Section 2.5.

2.1 Classical constitutive FE modeling of concrete fracture

Figure 2.1 summarizes the content of Sections 2.1 and 2.2 by means of five topics, which are treated one by one in the sequel. References to five subsections are included in the figure. Completeness is not claimed and other perspectives to structure the relevant literature are certainly possible.

Many different strategies for the FE modeling of strain localizations in quasi-brittle materials have been developed in the last decades. This wide diversity can be considered as a result of the broad range of applications, with varying structural sizes or scales, loadings (e.g. duration, speed, repetition), and failure modes. Another driving force to invent new models arose from the
2. LITERATURE REVIEW

**Cohesive zone model (§ 2.1.1)**

![Diagram of cohesive zone model]

**Classical crack representations (§ 2.1.2)**

- Discrete / Discontinuous crack concept with interface elements
- Smeared / Continuous crack concept with crack band approach

**Constitutive frameworks (§ 2.1.3)**

- Theory of plasticity
- Damage mechanics
- Combined damage-plasticity

**Mesh-induced directional bias (§ 2.2.1)**

![Diagram of mesh-induced directional bias]

**Remedies (§ 2.2.2)**

- [A] Enriched continua:
  - Nonlocal models
  - Gradient models
  - Micropolar models
  - Viscous models
- [B] Cracks tracking algorithms (level sets, global, nonlocal, local)
- [B] Finite elements with inserted discontinuities:
  - X-FEM
  - Hansbo method
  - Cohesive segment method
- [B] Remeshing
- [A] Enhanced crack band width formula
- [B] Mixed (stabilized) FE method

**Figure 2.1:** Important aspects of the modeling of concrete fracture in a nutshell.
observed mesh dependency in simulations of, for instance, crack propagation in concrete when using the classical models. From the latter point of view, several modeling strategies as remedies for mesh-induced directional bias will be discussed in Section 2.2. The current section reviews firstly their common conceptual aspects. The discussion is limited to continuum models (smeared crack concept) and continuum models with discontinuities (discrete crack concept). Discrete models as distinct element methods or particle methods and lattice models are not considered. Also the wide-ranged meshfree or meshless methods are excluded from the discussion.

2.1 Cohesive zone model

The formation of a (tensile) crack is a gradual process, starting from the development of a fracture process zone in which micro-cracking takes place, and after progressive damaging resulting in a stress-free macro-crack. Generally, for ordinary concrete the length of the fracture process zone can be measured in dozens of centimeters \([5, 46]\), so that for nonlinear FE analyses of structures with standard dimensions this quasi-brittle fracture needs to be taken into account. A common way of modeling the nonlinear fracture process zone is by using the cohesive zone model, which was introduced by Dugdale \([42]\) and Barenblatt \([4]\) for elastic - perfect plastic material behavior (e.g. metals). The model was later adapted to quasi-brittle material behavior (e.g. concrete, rock) by Hillerborg et al. \([63]\), who called it fictitious crack model.

The essence of the cohesive zone model is shown in the first picture of Figure 2.1, adapted from \([63]\), for mode I (tensile) cracking. Its basic assumptions are:

1. the crack propagates when the principal stress \(\sigma\) at the tip exceeds the tensile strength \(f_t\) (for crack initiation the same criterion is used);
2. the cohesive force transfer is lumped into a fictitious crack line or plane, where the usual tortuosity of cracks is not taken into account;
3. the cohesive force or stress transfer along the crack line or plane is a function of the crack width \(w\), \(\sigma = f(w)\).

The function \(f(w)\), called tension softening curve, is typically a monotonically decreasing function from \(f(0) = f_t\) to \(f(w_{\text{ult}}) = 0\), where \(w_{\text{ult}}\) is the ultimate crack opening at which no stresses can be transferred anymore. Several types of softening functions for concrete exist, e.g. simplified linear softening or nonlinear softening according to Hordijk \([66]\) or Moelands & Reinhardt \([116]\) that are obtained by curve fitting from experimental test results. The area under these curves represent the fracture energy \(G_f\), being the energy required
2. LITERATURE REVIEW

to create a unit surface area of a fully developed crack. Recently, a new curve with bilinear softening was proposed in Reference [64], based on data from a comprehensive test program [65], which distinguishes an initial and a total fracture energy.

2.1.2 Classical crack representations

Generally, there are two ways to incorporate the cohesive zone model in FE analyses: with discrete and smeared crack representations, e.g. [1, 23, 34, 121].

The discrete or discontinuous crack concept is usually based in the fracture mechanics theory. In its original form [98] the crack is directly modeled via separation of continuum element nodes that initially have the same position in the mesh. The resulting geometric discontinuity represents the final damage stage of a crack formation process most closely. After the onset of cracking, the displacements across the discrete crack show a jump and the associated, but in fact non-existing, strains become unbounded. This is schematically indicated for the one-dimensional situation in Figure 2.1. For specific cases, when the crack path is known in advance due to preliminary (smeared) analyses, experimental results or engineering judgment, the potential crack can be predefined by conventional interface elements in the mesh. A more flexible approach is to insert these interfaces between all continuum elements [152]. Quasi-brittle cohesive zone modeling is included by adopting a traction-separation law as constitutive relation for the interface elements. In order to avoid deformations before the onset of cracking the initial elastic stiffness of the interface requires a high dummy value. This high dummy stiffness may result in significant oscillations in the normal traction profile, depending on the stiffness value and the numerical integration scheme [120, 131].

The smeared or continuous crack concept is based in the continuum mechanics theory. Instead of modeling cracks by changing the mesh topology, cracks are modeled by changing only the constitutive (stiffness and strength) properties of conventional finite elements, keeping the cracked material as continuum. Hence, this concept leads to distributed or smeared cracks that represent the initial diffused damage stage of a crack formation process most closely. After fully softening, the displacements across the smeared crack remain continuous with two kinks at the band sides (weak discontinuities) and the associated strains show a jump, see Figure 2.1. Originally, the smeared crack concept aimed to describe large areas of distributed cracking and is consequently specifically applicable to global analysis of large-scale reinforced concrete structures [115]. However, since the introduction of the crack band approach [10], the smeared crack concept can also be used for detailed analysis of individual
2.1 Classical constitutive FE modeling of concrete fracture

Crack localizations [122]. In the crack band approach the fracture energy \( G_f \) is consumed over a certain crack band width (sometimes called a characteristic length) \( l \). This crack band width needs to be known at the moment of cracking, and is originally based on the element area or volume. Incorporation of \( l \) in the constitutive relation, turns the stress-separation softening curve into a stress-strain softening curve. Hence, the crack band approach can be seen as the cohesive zone model in a continuum setting. Furthermore, it can be considered as the first successful attempt that has brought fracture mechanics and continuum mechanics together [23]. Section 2.3 discusses the crack band approach and different band width estimators in more detail.

2.1.3 Constitutive frameworks

The overview in Figure 2.1 further summarizes the constitutive frameworks that are used in the discrete (left column) and smeared (right column) crack representations of the cohesive zone model. Both crack representations can adopt the plasticity, damage mechanics and combined damage-plasticity frameworks. Specifically for the smeared crack concept, which is in our terminology equivalent to any continuum modeling of cracking in materials, also the smeared crack models* and the microplane theory can be used. In the remaining of this subsection the aforementioned constitutive frameworks are described in the context of the smeared crack concept.

The two main constitutive frameworks for the description of material nonlinearity are the theory of plasticity and damage mechanics. In the plasticity framework [62] the initial elastic stiffnesses remain constant during un- and reloading, which results in the occurrence of permanent strains. Furthermore, the strains are usually decomposed into an elastic and a plastic part. Currently, its most widely used form is the flow theory of plasticity, with the yield condition, flow rule and hardening law as key ingredients. The damage mechanics framework [89], pioneered by Kachanov [85], describes the constitutive behavior by an irreversible, progressive degradation of the stiffnesses that involves secant un- and reloading. Generally, a total strain is used in its formulations. Both the plasticity and damage mechanics frameworks can describe isotropic and anisotropic material behavior. A third framework arises from the combination of plasticity and damage mechanics [56, 90], allowing for permanent

---

*The expression “smeared” here might be redundant, since this type of constitutive framework is already categorized in the smeared crack concept (as counterpart of the discrete crack concept). However, these models are generally denoted with “smeared crack models” in the literature, and for that reason we use this duplication.
strains and reductions of the stiffness moduli in the constitutive behavior. This
leads to unloading branches between elastic and secant unloading.

The smeared crack models [120] typically adopt a crack coordinate sys-

tem at the onset of cracking, in which the corresponding strains components
are directly coupled to the stresses transmitted across and along the crack.
The stiffness moduli degrade progressively and generally this is taken into
account in the unloading branch. The smeared crack models can be subdi-
vided into decomposed strain based and total strain based models. The de-
composed strain based models basically distinguish crack strains and strains
in the solid material [128]. On the one hand, this formulation allows for a
further sub-decomposition of the crack strain into separate contributions of
multi-directional fixed cracks that simultaneously occur at the same point. On
the other hand, cracking can for instance be combined with plastic, creep or
thermal effects by sub-decomposition of the strains in the solid material. The
total strain based models describe the material via stress - total strain rela-
tions and hence do not require internal iterations. This robust approach can
adopt either a fixed or a rotating crack coordinate system with mutually per-
pendicular axes. For the fixed crack model this system is consolidated upon

crack initiation, while for the rotating crack models this system is aligned with
the continuously rotating maximum principal strain direction. Models with
transition from rotating to fixed also exist [114].

The microplane theory [18, 108] obtains the tensorial stress-strain relation
from stress and strain vectors on planes of various orientations (microplanes).
It implies that stiffness moduli can degrade (although the unloading path is
not trivial to indicate) and that the strain tensor or macroscopic strain is not
decomposed. Similarities exist with the aforementioned multi-directional fixed
crack model.

Note that the smeared crack models and the microplane theory are distin-
guished here as separate constitutive frameworks, since they have some partic-
ular features. Nevertheless, both frameworks are closely related to the damage
mechanics framework. They can be conceived as special cases of the anisotropic
damage models and their formulations can be written into the damage mechan-
ics format, as shown in Reference [33].

2.2 Reducing mesh-induced directional bias

The modeling of strain localizations in concrete with the classical crack rep-
resentations of the cohesive zone model suffers from mesh-induced directional
bias, independent of the adopted constitutive framework, e.g. [24, 80, 120].
2.2 Reducing mesh-induced directional bias

This section discusses the flaw, explanations for its underlying reason and the proposed remedies in the literature.

2.2.1 Mesh-induced directional bias

Mesh-induced directional bias means that the numerical results depend on the structure and orientation of the FE discretization. The mesh bias is reflected in the phenomenon that strain localization bands prefer to propagate along continuous mesh lines rather than propagating across them in a zig-zag manner. Figure 2.2 illustrates the mesh dependency for the classical smeared crack concept, by sketches of localization bands in two differently oriented structured meshes and their load - displacement responses, in case a fictitious concrete specimen is loaded by a horizontal tensile load. As a consequence of the difference in crack patterns, the global energy dissipation will be different in mesh A and the slanted mesh B. In this example the post-peak behavior is only affected, but more generally predictions of ultimate load capacities and even failure mechanisms could be biased as well. Hence, mesh-induced directional bias is a major problem. Not only the classical smeared crack concept, but also the classical discrete crack concept can suffer to directional mesh bias [144].

The overview in Figure 2.1 illustrates this for the fictitious example with uniaxial tensile loading. Both the mesh with interfaces between the continuum elements (left column) and the mesh with continuum elements only (right column) may typically show an aligned crack, while vertical cracking would be expected.

In the remaining of this subsection explanations of the underlying reason for mesh-induced directional bias are considered. For the classical discrete crack concept this mesh bias is caused by the initial mesh design [34], as has been demonstrated in Reference [144]. However, for the classical smeared crack concept, which is based in the continuum mechanics theory, the underlying reason
appears to be less trivial. Recent publications show that this issue is still a subject of debate. Two main views can be distinguished from the literature. To explain the differences between them it is important to mention the global procedure of solving a continuum mechanics problem. In short, to solve a continuum boundary value problem (BVP) a numerical approximation technique or discretization procedure is used to cast the continuum differential equations into discrete algebraic equations, see Figure 2.3. In the context of the classical smeared crack concept, the spatial discretization of the continuum model is approximated by conventional finite elements with continuous displacement fields (see Section 2.1).

**View 1: the continuum description as cause of mesh bias**

According to the first view, elaborated by De Borst et al. [35, 37] in the nineties, the observed mesh bias is attributed to mathematical problems in the continuum description of strain localizations. For a long time this was a common and shared viewpoint, used in many publications. The way of reasoning takes its starting point in continuum models without the crack band approach and in a standard incremental-iterative setting. It can be briefly recapitulated as follows. The modeling of a strain localization involves local softening of the material, negative tangent stiffnesses and a discontinuity in the strain field. Due to the loss of positive-definiteness of the material tangent stiffness matrix the rate or incremental equations may lose ellipticity. This local change in character of the continuum differential equations leads to an ill-posed BVP, meaning that uniqueness of the solution is lost and an infinite number of possible solutions remains. Discretization of the continuum problem with the FE method results in reduction to a finite number of possible solutions. This number of solutions is obviously related to the fineness of the FE discretization, and so the numerical solution depends on the discretization, irrespective of the adopted type of spatial discretization procedure. Using conventional finite elements for
2.2 Reducing mesh-induced directional bias

the spatial discretization, the strain localization is typically captured with the smallest possible band/volume of elements and following the continuous mesh lines.

**View 2: the discretization procedure as cause of mesh bias**

About ten years ago the aforementioned view was questioned by Cervera et al. [25] for the first time. They explain the underlying reason for mesh-induced directional bias from deficiencies in the spatial discretization of the differential equations rather than in the differential equations themselves, considering the fact that well-aligned meshes produce good results. The term “well-aligned mesh” means that the orientation of the mesh lines coincides with the (experimentally) expected orientation of the localization band, e.g. mesh A in Figure 2.2. In more recent publications this view has been elaborated and further illustrated by means of examples [23, 26, 27]. It is posed that due to the local nature of a strain localization problem, the solution of the discretized problem is largely affected by the local discretization error. This local discretization error arises from differentiation of the directly approximated displacement field and is displayed in two different ways. Firstly, the differentiation process leads to inaccuracy of the strain and implicitly the stress fields during the elastic stage, especially for areas with strong displacement gradients. Hence, the evaluated strain and stress fields around the tip of a propagating localization band or smeared crack in the discrete problem can differ significantly from those in the continuum problem. According to Cervera et al. this local discretization error is the main reason that from the existing finite number of possible discrete solutions, the one that is obtained depends on the mesh. Secondly, in a later stage of the localization process the local discretization error can be observed by the inability of conventional finite elements to capture the displacement discontinuity properly. Where Cervera et al. consider the limited capacity of the elements as “an approximability shortage of the discrete solution spaces” [27], recently Oliver et al. [103] conclude that specifically this aspect of the local discretization error is the main cause of mesh bias dependency.

Finally, the nature of strain localization would indeed give rise to mathematical problems in the continuum description. Nevertheless, view 1 cannot entirely explain the origin of directional mesh bias. For nonlinear FE analyses with the SLA method and the classical smeared crack concept, a dependence on the mesh structure and orientation is observed [137]. However, issues as loss of ellipticity and ill-posedness of the rate BVP cannot play a role here,
since only (a series of) linear analyses, total strains and secant stiffnesses are involved. In other words, also well-posed problems can suffer from mesh bias. Therefore, it is the author’s opinion that both the continuum description and the discretization procedure contribute to the cause of mesh bias.

2.2.2 Remedies

To overcome the issue of mesh-induced directional bias many different solutions have been proposed. Overviews with different ways of categorization are available in the References [1, 23, 77, 113]. The overview in Figure 2.1 presents a selection of remedies, which are related to the discrete (left column) or the smeared (right column) or the combined (at the bottom) crack concepts. In this subsection the remedies belonging to the discrete and smeared crack concepts are arranged according to the foregoing views on the origin of mesh bias in the categories A and B. The remedies belonging to the combined crack concept are denoted with category C. The main features of each remedy is explained briefly. More detailed descriptions of the remedies with pros and cons can be found in the provided references and references herein.

Category A: Enhancing the continuum description

Associating mesh dependency with the mathematical problem of local loss of ellipticity in the continuum description (view 1) has logically resulted in remedies that modify the continuum problem, known collectively as enriched, higher order or regularized continuum models. Regularization (i.e. the introduction of additional information) is a common way to solve ill-posed problems in mathematics. Enriched continua change the constitutive relations by incorporating a characteristic length of the material, which enforces a certain minimum width of the strain localization band (localization limiters) [78]. Discontinuities are represented by smooth displacement and strain fields, and hence loss of ellipticity can be avoided. Examples of this class of models are the nonlocal [8, 75, 112], gradient [36, 111], micropolar [32] and viscous or rate-dependent models [97, 140].

Some authors classify the in SubSection 2.1.2 mentioned crack band approach also as a localization limiter in the smeared crack concept, e.g. [1, 22]. However, a true localization limiter prevents both a zero volume strain localization and a zero energy dissipation upon mesh refinement. The crack band approach objectifies only with respect to the dissipated energy, but strain localization still occurs in a single row of elements, even if the size of the elements
becomes infinitely small [113, 141]. Moreover, a strain localization is still represented by two weak discontinuities in the displacement field and a jump in the associated strain field. So, the crack band approach uses just a partially regularization. Although the crack band approach suffers from mesh-induced directional bias, as discussed before, it has been shown that an accurate formulation of the crack band width can significantly reduce it [19, 22]. For this reason, the \textit{enhanced crack band width formula} is added separately in the list of remedies in Figure 2.1. The remedy will be further explored in Chapter 4.

\textbf{Category B: Enhancing the FE discretization}

Associating mesh dependency with local discretization errors in the strain and stress fields (view 2) has resulted in several types of remedies that enhance the FE discretization. Firstly, the local discretization error can be alleviated by using \textit{remeshing}. These techniques originate from the discrete crack concept [17, 68], but are also used to reduce the mesh bias in the context of standard continuum models [155]. Starting from a relatively coarse mesh the idea is to automatically refine the mesh in areas with highly localized strains, which continues until the discretization error is reduced satisfactorily. Since adaptive remeshing procedures enable the modeling of strain localizations in fine meshes, they are combined with nonlocal models [110] as well.

Secondly, in the smeared crack concept, the \textit{mixed FE method} [154] has been recently used for the elimination of mesh bias in strain localization problems [26, 27, 28]. The idea is to approximate both a stress (or strain) field and a displacement field as primary variable, instead of obtaining the former from the latter by differentiation. Local discretization errors are avoided due to convergence of the stress or strain values at local level.

Thirdly, attributing the directional mesh bias explicitly to the inability of conventional finite elements to capture the displacement discontinuity has inspired the development of \textit{finite elements with inserted discontinuities}. The idea is to enrich the kinematics of conventional finite elements by describing the displacement jump directly. Two classes can be distinguished, namely the eXtended Finite Element Method (X-FEM) approaches [12, 95] and the embedded crack models (E-FEM) or Strong Discontinuity Approaches (SDA) [2, 11, 76, 102, 105]. The X-FEM approaches [12, 95, 149] describe the displacement jumps via nodal enrichments and are based on the discrete crack concept (left column of Figure 2.1). Originally, the standard X-FEM adopted stress-free cracks only, but later cohesive cracks were also included [94, 149]. Alternative formulations to standard X-FEM are the Hansbo method [58] and the cohesive segment method [117]. E-FEM or SDA describes the displacement jumps
2. LITERATURE REVIEW

via elemental enrichments and are based on the smeared crack concept (right column of Figure 2.1).

Aforementioned nodal and elemental enrichments allow discontinuities in the elements with arbitrary orientations, and hence these approaches circumvent the need for remeshing. However, the use of crack tracking algorithms, to determine which elements need to be enriched, appears to be inevitable to capture localized deformation patterns properly. When using E-FEM without a crack tracking technique, directional mesh bias similar to the bias for the classical smeared crack concept has been observed [96]. And also within the framework of X-FEM the level set concept [142, 145] has become a commonly used technique for an accurate description of the crack propagation process. So, the crack tracking algorithms are helpful to overcome the local discretization error, and, in view of the discussion of the underlying reason for mesh-induced directional bias, they help to ignore “undesired alternative solutions of the nonlinear discrete problem” and to select “the appropriate solution among the many possible ones” [27].

Finally, crack propagation or crack tracking algorithms have also been successfully applied in standard smeared crack approaches, and particularly in the framework of continuum damage models [23, 29, 55]. For that reason we consider this modeling strategy here also as a separate remedy, although it does not enhance the FE discretization as the other remedies in this category. The main principle of the crack tracking technique is to trace and designate potential crack paths within an arbitrary FE discretization [106]. Elements crossed by a crack path are allowed to damage, while the others are restrained from that, keeping their constitutive relation linearly elastic. The determination of the crack propagation direction should be in line with the adopted failure criterion. For instance, once the maximum principal stress exceeds the tensile strength, it is a natural choice to define the crack propagation direction perpendicular to this maximum principal tensile stress direction. Other crack propagation criteria are reported in References [43, 113]. Besides aforementioned level set method in X-FEM, the following types of crack tracking algorithms can be distinguished: global [106], nonlocal [49] and local strategies [148]. The terminology global, nonlocal and local indicates the domain where information is obtained from in order to predict the crack path propagation. Each type has its own advantages and disadvantages [49, 72, 113]. Generally, the trend is that a wider considered domain increases the complexity of implementation and the computational costs, but also the robustness. The remedy will be further explored in Chapter 5, where a new propagation algorithm is proposed.
Category C: Continuous-Discontinuous modeling

We started the literature review with descriptions of the crack formation process (SubSection 2.1.1) and the classical crack representations (SubSection 2.1.2). The initial stage of diffuse micro-cracking is most closely represented by the smeared or continuous crack concept. The final stage of a stress-free macro-crack is most closely represented by the discrete or discontinuous crack concept. So far, the models discussed either use a continuous or a discontinuous crack representation. For a more realistic modeling of the entire fracture process the two concepts should be combined [133]. This can avoid a too stiff mechanical response and/or spurious damage growth for the pure continuous concept and it can avoid initial mispredictions of the crack orientation for the pure discontinuous concept.

Continuous-discontinuous modeling approaches merge the left and right column of the overview in Figure 2.1. Several combinations of models from both concepts have been used [30, 82, 133, 143, 151]. A key aspect in all these approaches is the moment at which the switch from the continuous to the discontinuous stage is made. When this transition occurs in an early stage during strain softening, the orientation of the discrete crack might be incorrect. When this transition occurs in a late stage during strain softening, issues with the correctness of energy dissipation arise. Therefore, some continuous-discontinuous modeling approaches adopt a smooth transition, via a gradual reduction of the involved characteristic length scale parameter [14, 104].

Finally, a rigorous way of continuous-discontinuous modeling is the element erosion technique, in which fully softened elements are removed from the mesh or their stresses set to zero (e.g. [122]).

2.3 Crack band approach*

In SubSection 2.1.2 the crack band approach was already briefly introduced as the smeared representation of the cohesive zone model. The current section further reviews this approach by a discussion of its general concept in SubSection 2.3.1 and a comparison of different band width estimators in SubSection 2.3.2.

*Based on a section in Reference [137].
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2.3.1 General concept

Many authors already discussed the crack band or fracture energy approach to a greater or lesser extent [1, 22, 24, 51, 77, 79, 109]. Here the main issues will be recapitulated. After the introduction of the smeared crack concept [20, 115] it was discovered that the dissipated energy during the cracking process appeared to depend on the size of finite elements. More than ten years later Bazant and Oh solved this problem of mesh size sensitivity by introducing the crack band approach [10]. In this approach, the slope of the stress-strain softening curve is adapted to a crack band width parameter \( l \), regularizing in this way the energy \( G_f \) required to create a unit surface area of a fully developed crack. Where the fracture energy \( G_f \) itself can be considered as a material property, the crack band width \( l \) is seen as a finite element discretization parameter. This parameter \( l \), which has to be specified upon cracking, is an a priori estimation for the actual localization band width \( l^* \) that will emerge a posteriori. Usually, a crack will localize in the smallest possible band of elements across the FE discretization. More specifically it can be stated that all possible finite element characteristics (i.e. shape, size, orientation, interpolation function and numerical integration scheme) affect the actual localization band width \( l^* \), and so the crack band width \( l \) should be a function of them all.

Mesh size objectivity is only reached when the a priori defined \( l \) agrees with the numerically obtained \( l^* \). This can be derived by considering the total amount of energy \( W \) dissipated in a fully developed, perfectly aligned, single smeared crack band:

\[
W = \left( \frac{G_f}{l} \right) \cdot l^* \cdot A^*
\]

where \( A^* \) indicates the numerically obtained surface area of the crack. With \( l \) equal to \( l^* \) the solution for \( W \) will match the energy dissipation of an equivalent discrete crack. Incorrect a priori estimations for \( l^* \) may be dangerous. It may lead to incorrect energy consumptions of numerical models that can significantly affect the prediction of the failure load, the post-peak behavior and even the failure mode. Based on the author’s experience, this holds especially true in the context of localized failure problems, such as analyses on shear critical beams with low reinforcement ratios [135].

Finally, the main advantage of the crack band approach is its relative simplicity, which subsequently makes the implementation in FE codes relatively easy. It explains why the vast majority of nonlinear FE packages used in the engineering community adopt this crack band approach [5, 22, 24, 39]. On the other hand the approach has its limitations. Apart from a restriction of the maximum element size in order to prevent a snap-back at local constitutive
level, the crack band approach suffers from directional mesh bias, see Section 2.2. Although the crack band approach cannot eliminate this mesh bias, it has been shown that an accurate formulation for \( l \) can reduce it (e.g. [19, 22]).

### 2.3.2 Band width estimators

Band width estimators aim at equalizing the ratio \( l^*/l \) from Equation (2.1) to 1.0 in order to obtain a mesh objective energy dissipation. In the past different methods were proposed for the determination of \( l \) as a priori estimation for \( l^* \). They can be grouped into three categories [79]. The methods in the first category are based on element area or volume, or more general on finite element characteristics. For instance, Rots made the crack band width dependent on the size, the shape and the interpolation function of the used finite element [120]. Based on trial-and-error fitting of numerical results obtained from models within the discrete crack concept, recommendations were given for a limited number of element types. Although these recommendations can be useful in case of regular meshes, for irregular meshes and arbitrary crack directions they are less applicable. This was solved by Oliver [101], who presented two more general approaches. The methods of Oliver, considered as a second category of band width estimators, were recently reviewed in detail by Jirasek and Bauer [79]. Here, we focus on his first method. Using isoparametric mapping a crack band width in global terms can be calculated for each cracked integration point at \( \mathbf{x} \), according to:

\[
\ell_o(\mathbf{x}) = \left( \sum_{i=1}^{n_{cn}} [\partial_\mathbf{x} N_i(\mathbf{x}) \phi_i] \cdot \mathbf{n}(\mathbf{x}) \right)^{-1} \tag{2.2}
\]

in which the subscript ‘o’ stands for Oliver, \( n_{cn} \) is the number of corner nodes, \( N_i \) the standard shape function in corner node \( i \) belonging to an element with \( n_{cn} \) nodes and \( \phi_i \) a crack indicator value in corner node \( i \). The \( \phi \)-values are obtained by assuming the crack at the midpoint of an isoparametric element and subsequently set them to 1 if the corner node is in front of the crack and to 0 if the corner node is behind the crack. Here, the terms “in front of” and “behind” are determined by the pointing direction of the crack normal vector \( \mathbf{n}(\mathbf{x}) \), see Figure 2.4. The direction of \( \mathbf{n}(\mathbf{x}) \) is integration point dependent and is determined by the crack angle \( \theta \) of that point with:

\[
\mathbf{n}(\mathbf{x}) = \begin{cases} 
\cos(\theta) \\
\sin(\theta)
\end{cases} \tag{2.3}
\]
Based on two numerical tests with different FE discretizations, it has been shown that mesh objectivity can be obtained for this crack band width formulation \( l_0 \) [101]. It should be noted that only elements with linear interpolation functions were used and that in general cracks in the integration points were perfectly aligned with at least one of the element edges. For the sake of completeness it is mentioned briefly that in Oliver’s second method the crack band width is equal to the element area divided by an equivalent crack length.

A few years later, the formulation by Oliver in Equation (2.2) was adapted by Govindjee et al. [51]. They have further generalized the estimation of \( l \) and made it applicable for three-dimensional problems. The formulation by Govindjee et al. equals:

\[
\tau_g(\mathbf{x}) = \left( \sum_{i=1}^{n_c} [\partial_\mathbf{x} N_i(\mathbf{x}) \tau_i] \cdot \mathbf{n}(\mathbf{x}) \right)^{-1}
\]

with:

\[
\tau_i = \frac{(\mathbf{x}_i - \mathbf{x}_c) \cdot \mathbf{n}(\mathbf{x}) - \tau_{\text{min}}}{\tau_{\text{max}} - \tau_{\text{min}}}
\]

Herein \( \mathbf{x}_c \) and \( \mathbf{x}_i \) are the center point and the corner nodal points of the corresponding finite element respectively. The parameters \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) are the minimum and maximum values of \( [(\mathbf{x}_i - \mathbf{x}_c) \cdot \mathbf{n}(\mathbf{x})] \) with \( i = 1 \ldots n_{\text{eq}} \). So the binary crack indicator function \( \phi_i \) (either 0 or 1) in Equation (2.2) is replaced by a continuous crack indicator function \( \tau_i \). The reason for this change lies in the fact that the discontinuous nature of \( \phi_i \) resulted in a discontinuous function for the crack band width \( l_0 \) with respect to a varying \( \theta \). This difficulty
2.3 Crack band approach

was observed in case of three-dimensional problems [51], but also for situations in two-dimensional problems [79] as will be shown at the end of this section. Equations (2.2) and (2.4) show that the determination of the crack band width is based on the spatial position of the crack with respect to the element edges. This includes the element dimensions, the element shape (quadrilateral or triangular) and the crack or element orientation. Finite element characteristics as the numerical integration scheme and the interpolation function are not taken into account.

The formulation by Govindjee et al. can be considered as a projection method, since $l_g$ corresponds to the element dimension in the crack normal direction. These projection methods have been indicated as a third category of band width estimators. Similar element projection based crack band width estimations have been used by Cervenka et al. [21, 22]. Additionally to Govindjee et al., Cervenka et al. multiply the crack band width with an orientation factor $\gamma_c$ [19, 22]. This factor takes into account the effect that, when a crack is not aligned with one of the element edges, the localization band width $l^*$ can extend over more than one single element. A linear variation for $\gamma_c$ is assumed between a value of 1.0 for a perfectly aligned crack, and 1.5 for a crack with an inclination angle of $\pi/4$ with respect to the element edge. The need for such an orientation factor is not supported by Jirasek and Bauer [79].

To illustrate the differences between the band width estimators of Rots, Oliver and Govindjee et al. a non-square rectangular element with one integration point in the center is considered (see Figure 2.5(a)), for which the crack band widths are calculated by a varying crack angle in a range of $0 \leq \theta \leq \pi/2$. The results are presented in the graph of Figure 2.5(b). The horizontal dashed and dotted black lines belong to the formulation by Rots, indicating an independence of the crack angle and a dependency of the used interpolation function. For Oliver and Govindjee et al. it is the other way around. In case of $l_o$ according to Oliver, for this non-square rectangle, a clear discontinuity is visible at the $\theta$-value where the crack is aligned with the top left - bottom right diagonal of the element. This is the angle at which two of the four binary crack indicators will change. In case of $l_g$ according to Govindjee et al. a continuous curve is observed, being the result of the continuous crack indicator function from Equation (2.5). Furthermore, the crack band width $l_o$ of Oliver can be interpreted as the distance between boundary edges of the element in crack normal direction. Govindjee et al. measures the crack band width $l_g$ between the two most distant element corner nodes in crack normal direction. This is visualized in Figure 2.5(a) for a crack angle $\theta$ of $\pi/8$. Consequently, $l_o$ depends on the location of the considered integration point in the element (keeping $\theta$ fixed).
Figure 2.5: Determination of crack band widths with different band width estimators for a rectangular quadrilateral element.

constant), where \( l_g \) results in identical band widths irrespective of the location of the integration point in the element.

2.4 Sequentially Linear Analysis method*

Quasi-brittle fractures in concrete structures not only involve discontinuities in geometry, given the separation of crack surfaces and as discussed in the previous sections, but also discontinuities in (pseudo) time, given the jumpy structural response during the loading process. With increasing loading, cracks may occur in an explosive discontinuous fashion, with alternating local peaks and snap-throughs or snap-backs in the load-displacement response. SLA aims at simulating such locally, brittle snap-type response of structures by capturing the brittle events directly, rather than trying to iterate around them in a Newton-Raphson scheme.

The principles of SLA are extensively discussed in several papers, e.g. [38, 52, 60, 123, 124]. This section recapitulates the general concept of the solution procedure in SubSection 2.4.1, the method’s specific saw-tooth constitutive relation in SubSection 2.4.2 and the currently adopted material model in Sub-Section 2.4.3.

*Partially based on a section in Reference [135].
2.4 Sequentially Linear Analysis method

2.4.1 General procedure

SLA is a non-iterative solution procedure consisting of a series of scaled linear analyses. Between successive analyses a damage increment is applied in terms of a discrete stiffness and strength reduction in a single integration point of a single element. This critical integration point is selected by considering the ratio of the local principal stress versus the local current strength in all integration points of the entire FE model. To apply the stepwise stiffness and strength reductions in such event-by-event procedure the nonlinear stress-strain softening curve is discretized to a saw-tooth curve. In general, the SLA method can be thought of as discretizing the space via finite elements, discretizing the local softening via a saw-tooth diagram and re-computing the load via a scaling technique. Standard incremental-iterative solution procedures rather discretize the space via finite elements, discretize the load via increments and re-compute the local softening on a continuous smooth diagram [126].

The global solution procedure of SLA in case of proportional loading can be summarized as follows [123].

- apply a reference load;
- calculate the (principal) strains and stresses in all integration points through a linear-elastic analysis;
- determine the critical integration point in the structure. This is the point with the largest principal stress - current strength ratio;
- determine the critical load multiplier $\lambda_{\text{crit}}$, belonging to the critical integration point, i.e. the current strength divided by the stress level;
- scale the reference load proportionally with $\lambda_{\text{crit}}$ and calculate the strains and stresses in all integration points, belonging to the scaled load (for post-processing of the results);
- increase the damage in the critical integration point by reducing the stiffness and strength according to the saw-tooth constitutive relation;
- repeat this cycle of steps continuously until the damage has spread sufficiently into the structure.

The nonlinear response is extracted by consecutively linking the results of each cycle.

Currently, SLA only allows a secant unloading/reloading scheme. Other schemes as elastic or plastic unloading/reloading are not straightforward to implement. Due to this the range of physical nonlinear problems that can be captured with SLA is limited. Another drawback of the solution procedure is the relatively long computation time as a result of applying a (small) damage
increment in only one integration point of the FE model at a time. Furthermore, for proportional loading, the identification of the critical integration point is straightforward, but for non-proportional loading, like combinations of self-weight and prestress, the selection procedure becomes more complex. The treatment of non-proportional loading is still a subject of ongoing research [38, 44, 45, 52, 53, 60].

Primary advantages of this purely damage controlled analysis procedure are its robustness on the one hand, because of the absence of equilibrium iterations, and its effectiveness and relative simplicity on the other hand, as a relatively small number of control parameters is involved. The SLA method is particularly attractive for analyzing brittle failure behavior, as convergence problems due to sudden changes are avoided. Alternative equilibrium states or bifurcations of the equilibrium path are circumvented [126], due to the scaled loading procedure that allows only one integration point to change its status from elastic to softening at a time. Unique solutions are even achieved in targeted homogeneous strain fields, where the selection of the critical point is now determined by numerical round-off. For standard incremental-iterative solution procedures, it can be different. The use of load increments implies that multiple integration points may crack simultaneously, through which the local stiffnesses at these points switches from positive to negative, following the softening constitutive laws for quasi-brittle materials. As a consequence of multiple softening points, the system of equations can have more than one solution. Although the incremental-iterative procedure can converge to one of the possible equilibrium states, it will not automatically pick up the most critical or the lowest equilibrium path. Finally, note that for SLA the use of reduced element integration schemes does not pose problems, since the SLA method always uses a positive secant stiffness. For standard incremental-iterative solution procedures, often the tangent stiffness is adopted. In case of softening the use of under-integrated elements will then lead to spurious kinematic modes [120].

Recently, an incremental - non-iterative method has been proposed [53, 54], which combines two solution procedures. In each load step an (non-iterative) incremental procedure is adopted, but in times of convergence problems the procedure temporarily switches to the SLA method for the corresponding step.

2.4.2 Saw-tooth constitutive relation

The crux of the SLA method lies in the way the strength and stiffness properties for the critical integration point are reduced. In order to apply such damage increments the constitutive stress-strain relation is discretized and approximated by a series of positive secant stiffness saw-teeth. Several techniques
2.4 Sequentially Linear Analysis method

were developed in the past to construct these so-called saw-tooth curves, of which the main ones are briefly discussed in the following.

Figure 2.6 provides an overview of five different approximation techniques for a linear softening curve in the tension regime. The reference or base curve is shown by a solid line in the first picture and by dashed lines in the other pictures, and is defined by Young’s modulus $E_0$, tensile strength $f_t$, and the area under the curve that represents the fracture energy density $G_f$ divided by the crack band width $l$. Table 2.1 summarizes the SLA specific input parameters and the output details of the five approximation techniques. Subscript 'swt' denotes the features of the saw-tooth softening curve.

Early approximation techniques simply defined the strength $f_{t,i}$, the maximum strain $\varepsilon_i$ and the stiffness $E_i$ of each consecutive tooth via reductions of $f_t$ or $E_0$ [124]. In approach (1) the tensile strength is equally reduced and $f_{t,i}$ is computed from

$$f_{t,i} = f_{t,1} - \Delta f_t, \text{ for } i = 1 \ldots n_{swt} \tag{2.6}$$

with $\Delta f_t$ equal to $f_t/n_{swt}$. The user-specified parameter $n_{swt}$ represents the number of saw-teeth. Corresponding reduced Young’s moduli $E_i$ are subsequently calculated from intersections with the base curve, via $f_{t,i} / \varepsilon_i$. Approach (2) starts with stepwise reductions of Young’s modulus by

$$E_i = E_{i-1} - a, \text{ for } i = 1 \ldots n_{swt} \tag{2.7}$$

with the user-specified parameter $a$ as a constant, and computes $f_{t,i}$ from the base curve. Both saw-tooth approximation techniques have the drawback that the area under the saw-tooth curve $(G_f/l)_{swt}$ underestimates the regularized fracture energy $G_f/l$ belonging to the base curve. It has been shown that due to this feature analyses suffered from mesh dependency in the global responses.

Approximation technique (3) solves this issue by upscaling the $f_{t,i}$ and $\varepsilon_i$ values, obtained from for instance approach (2), so that $(G_f/l)_{swt}$ becomes equal to $G_f/l$ [123, 124]. Alternatives with only the scaling of $f_{t,i}$ or $\varepsilon_i$ were also proposed, but they resulted in too high values of the saw-tooth curve tensile strength $f_{t,swt}$ or saw-tooth curve ultimate strain $\varepsilon_{u,swt}$.

A more elegant formulation to obtain regularized saw-tooth curve approximations was shown in Reference [123], which is called the ripple approach. In this approach a band is introduced by defining two curves parallel and equidistant to the base curve. The width of the band is equal to $2p f_t$, where $p$ is
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Table 2.1: Input and output for different saw-tooth curve approximation techniques, see Figure 2.6.

<table>
<thead>
<tr>
<th>Techniques</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Equidistant ( f ) reductions [124]</td>
<td>( n_{\text{swt}} ) ( f_{\text{swt}} = f ) ( (G_f/l)_{\text{swt}} \neq G_f/l )</td>
<td></td>
</tr>
<tr>
<td>(2) Stepwise reduction of ( E_0 ) [124]</td>
<td>( a, n_{\text{swt}} ) ( f_{\text{swt}} = f ) ( (G_f/l)_{\text{swt}} \neq G_f/l )</td>
<td></td>
</tr>
<tr>
<td>(3) Regularized saw-tooth curve [123, 124]</td>
<td>( a, n_{\text{swt}} ) ( f_{\text{swt}} \neq f ) ( (G_f/l)_{\text{swt}} = G_f/l )</td>
<td></td>
</tr>
<tr>
<td>(4) Ripple approach [123]</td>
<td>( p ) ( f_{\text{swt}} \neq f ) ( (G_f/l)_{\text{swt}} \approx G_f/l )</td>
<td></td>
</tr>
<tr>
<td>(5) Improved ripple approach [52]</td>
<td>( n_{\text{swt}} ) ( f_{\text{swt}} \neq f ) ( (G_f/l)_{\text{swt}} = G_f/l )</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.6: Different saw-tooth curve approximation techniques.
2.4 Sequentially Linear Analysis method

user-specified percentage of the tensile strength. Without presenting the equations, the procedure is as follows: find the point of intersection between the secant elastic branch and the upper curve of the band, determine the tensile strength belonging to the lower curve for the same strain $\varepsilon_i$-value, calculate the reduced stiffness for the next tooth and repeat these steps until negative $f_{\varepsilon_i}$-values are obtained. The procedure results generally in a certain overshoot of $f_t$ and $(G_f/l)_{swt}$ is virtually the same as $G_f/l$, considering the pairs of equally shaped triangles above and below the base curve. Note that in this approximation technique (4) the number of saw-teeth $n_{swt}$ is not an explicit input parameter anymore. The fineness of the saw-tooth curve approximation is now determined by the adopted $p$-value. Obviously, there is a close relation between $p$ and the resulting $n_{swt}$. For a lower strength percentage the number of saw-teeth increases and vice versa. Hence, the ripple approach “can be interpreted as setting the tolerance at local constitutive level for SLA, rather than setting the convergence tolerance at global level for standard nonlinear analysis” [123].

Recently, an improved version of the ripple approach, approximation technique (5), has been developed in Reference [52], moving back to the single input parameter $n_{swt}$ as in approximation technique (1). The width of the ripple band is now defined with two unknown strength percentages, $p_u$ for the upper curve and $p_l$ for the lower curve, which need not necessarily be the same. Subsequently, the idea is to search for $p_u$ and $p_l$ values in such a way that $(G_f/l)_{swt}$ and $\varepsilon_{u,swt}$ becomes respectively equal to $G_f/l$ and $\varepsilon_u$ for a given $n_{swt}$. Within this new saw-tooth curve formulation the direct solution procedure in the previous approximation technique is replaced by an iterative solution procedure. The improved ripple approach works well for linear softening curves, but may sometimes fail for nonlinear softening curves. Therefore a modification of the iterative solution procedure is proposed in SubSection 3.2.2, where also more details of this approximation technique are provided.

Finally, since the SLA method is mainly focused on the modeling of quasi-brittle materials the consecutive strength and stiffness reduction is often based upon the concept of tensile strain softening. Besides the previously shown linear tension softening curve, saw-tooth curve approximations of exponential tension softening and tension softening with a local snap-back [71] were made. Generally, the behavior of concrete in compression is then assumed to be linearly elastic, following the (reduced) secant stiffness. However, the sequentially linear saw-tooth curve approximations are also employed for other types of constitutive relations, e.g. linear elastic - ideally plastic stress-strain curves for concrete in compression and the behavior of steel and reinforcement [123].
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Saw-tooth modeling combined with creep effects has also been explored [59]. Furthermore, in the context of interface elements saw-tooth curve approximations are possible for normal stress - relative normal displacement (discrete cracking) [125], tangential stress - relative tangential displacement (bond-slip), and Coulomb-friction models [52]. In SubSection 3.2.2 the aforementioned list is further extended with some more commonly used nonlinear softening curves in tension and compression.

2.4.3 Fixed smeared crack model

The SLA method can be combined with different constitutive formulations. In the past, analyses were performed by using an isotropic damage formulation [124]. Later, orthotropic crack models were adopted to enable compressive strut action parallel to the cracks, which is relevant when modeling reinforced concrete [38, 123]. Throughout the present thesis all analyses are based on a total strain based orthogonal fixed smeared crack model, meaning that a crack is represented by a damaged element area, the stresses are evaluated in the crack directions given by the onset of cracking, the stresses are functions of total strains and finally that orthogonal secondary cracking is possible. The fixed crack model is preferred over a rotating crack model, despite its additional sensitivity to stress locking by the presence of shear stiffness [81, 120]. The reason is that the fixed crack model in its SLA format is conceptually simpler and computationally more efficient. A rotating crack model in its SLA format needs, besides damage events, also events for the rotations of the cracks [134], which will generally lead to a larger series of linear analyses. Note that recently an alternative smeared crack model in the SLA format has been proposed in Reference [61]. This so-called elastic-brittle fraction model splits the material cross section into a preset number of parallel fractions that will break sequentially. Since each fraction can have its own fixed crack direction, the model combines the fixed and rotating crack concepts in an elegant way.

Orthotropic damage

Referring to a plane stress situation, the concrete prior to cracking is represented as a linear elastic isotropic material for which the constitutive relation equals

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \frac{E_0}{1 - \nu_0^2} \begin{bmatrix}
1 & \nu_0 & 0 \\
\nu_0 & 1 & 0 \\
0 & 0 & 1 - \nu_0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

(2.8)
2.4 Sequentially Linear Analysis method

Figure 2.7: Plane stress element including its fixed crack coordinate system.

where \( E_0 \) and \( \nu_0 \) are the initial Young’s modulus and initial Poisson’s ratio respectively. As soon as the principal stress violates the tensile strength in an integration point, the isotropic formulation is switched into an orthotropic formulation and a local \( nt\)-coordinate system along the crack plane is adopted, see Figure 2.7. This orthotropic stress - total strain relation is defined as

\[
\begin{bmatrix}
\sigma_{nn} \\
\sigma_{tt} \\
\sigma_{nt}
\end{bmatrix} =
\begin{bmatrix}
\frac{E_n}{1-\nu_n\nu_m} & \frac{\nu_n E_m}{1-\nu_n\nu_m} & 0 \\
\frac{\nu_m E_n}{1-\nu_n\nu_m} & \frac{E_m}{1-\nu_n\nu_m} & 0 \\
0 & 0 & G_{red}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{nn} \\
\varepsilon_{tt} \\
\gamma_{nt}
\end{bmatrix}
\tag{2.9}
\]

or in more compact form

\[
\sigma_{nt} = D_{nt} \varepsilon_{nt} \tag{2.10}
\]

The subscripts \( n \) and \( t \) in the orthotropic scheme above refer to the normal and tangential crack directions. Via transformations of the strain and stress tensors, Equation (2.10) can be transposed to stresses and strains in a global \( xy\)-coordinate system, using

\[
\sigma_{xy} = T^{-1}_\sigma(\theta) D_{nt} T_\varepsilon(\theta) \varepsilon_{xy} \tag{2.11}
\]

\( E_n \) and \( E_t \) in Equation (2.9) represent the (reduced) Young’s moduli normal and tangential to the crack face respectively. After initiation of the first crack in an integration point, the stiffness \( E_n \) and tensile strength \( f_n \) will be reduced conform the saw-tooth tension softening curve, while \( E_t \) and \( f_t \) still equal the initial values \( E_0 \) and \( f_t \). Due to principal strain rotations a secondary
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crack, perpendicular to the fixed primary crack, may arise in a later stage of the analysis. This occurs when the stress in tangential direction exceeds the tensile strength, and results in a reduction of $E_t$ and $f_t$. Orthogonal secondary cracking is especially important when modeling splitting cracks in compression struts. From a computational point of view, the consequence of orthotropic damage is that two saw-tooth diagrams per integration point have to be adopted and hence the number of linear analyses may increase before complete failure is obtained.

Finally, the effect of lateral contraction is assumed to reduce at a similar rate as the Young's moduli, according to

$$
\nu_{nt} = \nu_0 \cdot \frac{E_n}{E_0}, \quad \nu_{tn} = \nu_0 \cdot \frac{E_t}{E_0}
$$

(2.12)

Shear behavior

Fixed crack models require the specification of the behavior between shear strains and stresses. Due to potential redistributions of strains and stresses after cracking, rotations of the principal strain and stress directions with respect to the previous principal directions can occur, through which shear stresses on the crack face will be generated. Extensive research has been carried out regarding experimental determination and FE modeling of stress transfer mechanisms across cracks, see e.g. References [7, 48, 73, 127, 147] and references herein. Generally, the experimental results show interactions between the normal and tangential components of the stresses and strains (crack dilatancy), or in other words, between the phenomena strain softening and aggregate interlock. However, these interactions are difficult to model numerically. In the adopted fixed crack model the strain softening and aggregate interlock phenomena are fully decoupled, as can be observed from the zero entries in the stiffness matrix of Equation (2.9). This simplification is in fact only valid for relatively small parallel crack displacements with respect to the normal crack displacements [127].

The shear behavior at the onset of cracking is often described with a single reduction of the shear stiffness $G_{\text{red}}$ in the form

$$
G_{\text{red}} = \beta G_0
$$

(2.13)

where $G_0$ equals the stiffness between $\sigma_{xy}$ and $\gamma_{xy}$ in Equation (2.8) and $\beta$ is a constant shear retention factor, which could be interpreted as the modeling of aggregate interlock. However, this linear relation between the shear stresses
2.4 Sequentially Linear Analysis method

and strains has both computational and physical problems. From a computational point of view, the choice for a proper $\beta$-value is rather arbitrary, while it can significantly affect the predictions of the failure load and global structural response [120]. Moreover, for relatively large $\beta$-values stress locking can occur, whereas relatively low $\beta$-values may lead to convergence problems in the context of an incremental-iterative solution procedure [38]. From a physical point of view, a constant shear stiffness during an ongoing cracking process is an oversimplification of the reality, generally leading to an underestimation of the shear stiffness for a small crack width $w$ and to an overestimation of the shear stiffness for a larger $w$.

A more realistic description between shear strains and stresses is obtained by using a variable shear stiffness. Such a variable shear stiffness can be expressed in different ways, e.g. via a nonlinear relation between the shear stresses $\sigma_{nt}$ and the shear strains $\gamma_{nt}$, via a (non)linear relation between $\beta$ and $\gamma_{nt}$, or by making $\beta$ a function of the strain $\varepsilon_{nn}$ in normal direction of the crack. For the latter type of expression, different relations were formulated in the past, see References [86, 120, 127]. They have a clear physical meaning, since they include the principle that by an increasing crack width the shear stiffness decreases. Moreover, they circumvent dilemmas on selecting proper constant $\beta$-values.

In the present thesis, an equivalent formulation to the latter type is adopted, which was proposed in Reference [38]. According to this formulation the shear stiffness for a cracked material is reduced by

$$G_{\text{red}} = \frac{E_{\text{min}}}{2 \cdot \left(1 + \nu_0 \cdot \frac{E_{\text{min}}}{E_0} \right)}$$

(2.14)

with $E_{\text{min}} = \min(E_n, E_t)$. From Equation (2.14) a similar expression as the isotropic definition of the shear stiffness in Equation (2.8) can be recognized, but with the difference that $E_0$ and $\nu_0$ are replaced by their reduced counterparts. Using $E_{\text{min}}$, the shear stiffness is now related to the maximum of the orthotropic damage. The effectiveness of this simple variable shear stiffness relation has been demonstrated in [39].

Figure 2.8 illustrates the way the shear stiffness varies during the opening of a crack for linear softening. The gray lines indicate the reference tension softening stress-strain curve (dashed) and its saw-tooth curve approximation (solid). The black lines indicate their corresponding shear stiffnesses according to Equation (2.14), which is expressed in terms of $\beta$ versus strains. The shape of the $\beta$-$\varepsilon$ curve reveals a sharp initial drop followed by a gradual shear retention factor reduction to zero for a completely open crack, which corresponds to
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Figure 2.8: A linear softening curve and its corresponding variable shear retention factor $\beta$ versus $\varepsilon$ according to Equation (2.14).

recommendations of Kolmar and Mehlhorn in Reference [87]. Note that the stepwise reduction of $\beta$ arises from the discretized stiffness degradation.

2.5 Concluding remarks

This chapter presented a concise overview of the relevant literature in light of the subject of the thesis. The classical FE strategies to model concrete fracture were explained and the important issue of mesh-induced directional bias was addressed. Given the different views on the underlying reason for this bias and the wide diversity of proposed remedies, it can be concluded that a comprehensive solution is not available yet.

Nowadays, the X-FEM and E-FEM strategies are very popular for the simulation of quasi-brittle materials. For good reasons, since the results are promising and generally mesh objectivity is obtained. However, it should be noticed that validation of these advanced methods is frequently limited to localized fracture problems in plain concrete only. Applications to reinforced concrete are rarely seen in literature. This is remarkable from the viewpoint of engineering practice, since most of the concrete structures or structural members are at least lightly reinforced. Moreover, the analysis of RC structures poses challenging issues as the formation of multiple cracks, like flexural cracks, debonding cracks, shear cracks, which are propagating and possibly intersecting with other cracks and reinforcing bars. These issues require serious efforts concerning the kinematic and constitutive aspects of the modeling strategy and also regarding the numerical stability and robustness of the solution procedure.
2.5 Concluding remarks

X-FEM and E-FEM strategies that can properly capture all these issues have not been devised yet.

In the author’s view, the combination of the crack band approach in the smeared crack concept with the SLA method can result in an efficient, effective, robust and accurate solution strategy for this type of problems, provided that the inherent mesh-induced directional bias can be sufficiently reduced. To that end, two enhancements are proposed in Chapters 4 and 5.
Chapter 3

Systematic assessment of mesh-induced directional bias

In the previous chapter the problem of mesh-induced directional bias was described in a qualitative sense. For a clear understanding it is required to study the phenomenon in a more quantitative way. This chapter presents a dedicated numerical testing procedure that enables a systematic assessment of mesh-induced directional bias for constitutive models. Section 3.1 explains the background and the concept of the test. Subsequently, necessary SLA extensions regarding an element projection based crack band width estimation and saw-tooth curve approximations for nonlinear softening are described in Section 3.2. Section 3.3 shows a study in which the proposed numerical test is systematically applied to the crack band approach with aforementioned implemented band width estimator. The sensitivity of the numerical “specimen” size on the results is examined in Section 3.4. Finally, Section 3.5 summarizes the main findings of this chapter.

3.1 Test with periodic boundary conditions

In order to assess the influence of directional mesh bias, authors usually perform analyses on one or more fracture tests. Very popular are the single-edge-notched (SEN) and double-edge-notched (DEN) specimens, uni-axial tensile tests, strips with a hole and a three-point bending test. Commonly one compares the numerical results obtained from a structured versus an unstructured finite element mesh, or from a regular versus a slanted finite element mesh, after which conclusions are drawn whether the directional bias of the mesh is

*Based on a section in Reference [137].
3. SYSTEMATIC ASSESSMENT OF MESH-INDUCED DIRECTIONAL BIAS

eliminated or not, e.g. References [6, 9, 23, 80]. Without objecting to the correctness of these conclusions we propose to study the sensitivity with respect to the orientation of the crack versus the mesh lines in a more elaborated and systematic way. Therefore a dedicated numerical test is presented, making use of the concept of periodicity in the field of strain localization analysis.

The general concept of periodicity in FE simulations is discussed in SubSection 3.1.1. Also some specific possibilities are mentioned when using this concept in strain localization analysis. SubSection 3.1.2 describes briefly an implementation of periodic boundary conditions in FE models.

3.1.1 General concept

Starting from the concept of periodicity, known in the field of numerical homogenization of the elasticity tensor, e.g. References [13, 16, 156], periodic boundary conditions are used for studying the influence of finite element characteristics on directional mesh bias and the localization behavior. The idea is that a finite piece is cut from an assumed infinite, initially homogeneous, discretized periodic medium. Within a two-dimensional $\mathbb{R}^2$ space the discretized periodic medium is represented by an infinite (flat) plane, see Figure 3.1, while in a three-dimensional $\mathbb{R}^3$ space an infinite volume is used. The isolated finite plane or volume is now considered as a separate FE discretization with periodic length scales $L_i$, $i = x, y, z$, by which constraints will be added at the opposite boundaries $S_{j1}$ and $S_{j2}$, $j = x, y, z, c$. When these constraints or periodic boundary conditions are properly assigned, the behavior of a finite plane/volume within an infinite periodic medium under certain loading conditions can be simulated exactly with the FE model of just the isolated finite plane/volume. Actually, knowing the behavior of the finite plane/volume also the behavior of the infinite medium is known, due to the assumption of periodicity. Note that due to the imposed periodicity in principle it is not important how the boundary edges are shaped, since they do not have a physical meaning anymore, as long as they are periodic. This means that the distance between each pair of opposite boundary edges/faces is equal. The periodic boundary conditions in terms of displacements at the opposite edges/faces can be expressed in a general format:

$$d_i(S_{j1}) = d_i(S_{j2}) - \Delta_{ji}$$

(3.1)

In words, the displacement in $i$ direction of edge/face $S_{j2}$ is equal to the displacement in $i$ direction of edge/face $S_{j1}$ with a constant displacement difference $\Delta_{ji}$ (for each pair of opposite points at the edges/faces). Depending on
3.1 Test with periodic boundary conditions

Figure 3.1: An example (2D) of a finite plane, taken from an infinite discretized periodic medium, with periodic length scales $L_x$ and $L_y$.

what kind of loading situation is desired one could prescribe one or more of the constant displacement differences $\Delta_{ji}$. For a two-dimensional situation as shown in Figure 3.1, Equation (3.1) can be elaborated to the following set of equations:

$$
\begin{align*}
\frac{d_x}{dx}(S_{x1}) &= \frac{d_x}{dx}(S_{x2}) - \Delta_{xx} \\
\frac{d_x}{dx}(S_{y1}) &= \frac{d_x}{dx}(S_{y2}) - \Delta_{yx} \\
\frac{d_y}{dy}(S_{x1}) &= \frac{d_y}{dy}(S_{x2}) - \Delta_{xy} \\
\frac{d_y}{dy}(S_{y1}) &= \frac{d_y}{dy}(S_{y2}) - \Delta_{yy} \\
\frac{d_x}{dx}(S_{c1}) &= \frac{d_x}{dx}(S_{c2}) - \Delta_{xx} + \Delta_{yx} \\
\frac{d_y}{dy}(S_{c1}) &= \frac{d_y}{dy}(S_{c2}) + \Delta_{yy} - \Delta_{xy}
\end{align*}
$$

(3.2)

The last two equations of Equation (3.2) express the periodic boundary conditions of the diagonally opposite edges/faces $S_{c1}$ and $S_{c2}$, indicated by the orange lines in Figure 3.1.

Due to inclusion of periodic boundary conditions, different mesh alignments (element orientations) with respect to the loading direction can be adopted. In contrary to standard tests this can be done without disturbance of the localization process by the model boundaries and without loss of mesh uniformity. Mesh uniformity in this sense means that all characteristics of each finite element (i.e. shape, size, orientation, interpolation function and numerical integration scheme) in a specific mesh are identical.

The attractiveness of the above described test is further explained on the basis of an example. Suppose, a specimen that is subjected to an uni-axial
3. SYSTEMATIC ASSESSMENT OF MESH-INDUCED DIRECTIONAL BIAS

Figure 3.2: Uni-axial loaded specimen (a) and two possible FE discretizations (b) and (c).

tensile force, as depicted in Figure 3.2(a), is modeled with a certain mesh alignment and without the use of periodic boundary conditions. Considering only FE discretizations with two-dimensional elements, the meshes might be generated in a way as shown in Figures 3.2(b) and 3.2(c). Figure 3.2(b) shows a FE model with straight model edges, but where mesh uniformity is lost since both quadrilateral and triangular elements are required. Consequently, a clean-cut study to the influence of one specific element characteristic on directional mesh bias and the localization behavior is not possible. The opposite is true for the FE model shown in Figure 3.2(c). Mesh uniformity is now maintained, but the model edges are tooth-shaped. These irregular boundaries will disturb the targeted uniform strain pattern before cracking. The consequence is that both initiation and propagation of strain localization are mainly triggered by this structural effect, making a study to directional mesh bias difficult.

The proposed test with periodic boundary conditions can be seen as an alternative to the mesh in Figure 3.2(c). The price to pay is that additional (non-physical) constraints are included. For complex loading situations this may lead to results that are hard to interpret. On the other hand, due to the imposed periodicity the aforementioned structural effect will not occur. Consequently, the numerical test with periodic boundary conditions enables:
3.1 Test with periodic boundary conditions

- to study strain localization in uniform meshes;
- to assess directional mesh bias of constitutive models in a systematic way, i.e. arbitrary mesh alignments (element orientations) with respect to the loading direction can be adopted without loss of mesh uniformity;
- to study the influence of a separate finite element characteristic on directional mesh bias and the localization behavior in uniform meshes;
- to identify the directional mesh bias quantitatively and subsequently calibrate specific length scale parameters of the used constitutive models, such as a crack band width (crack band approach) or a width of a non-local averaging zone (e.g. nonlocal models), that aim to guarantee mesh objectivity. This may lead to improvement of their ability to deal with directional mesh bias.

The absence of physical boundaries in this numerical test might be appealing for the nonlocal models and the gradient-enhanced models.

3.1.2 Implementation in FEM environment

In order to include periodic boundary conditions in FE models, two additional requirements should be introduced. Firstly, Equation (3.1) requires the specification of linear dependencies between degrees of freedom (d.o.f.’s) of the nodal pairs at opposite edges/faces and their corresponding constant displacement difference $\Delta_{ji}$. These linear dependencies are often expressed in a master-slave format. Using this format, Equation (3.1) can be reformulated to (for a two-dimensional situation):

$$
\begin{align*}
    u_{j2,i} &= 1.0 \cdot u_{j1,i} + 1.0 \cdot u_{\Delta_{ji}} , \text{ for } i = x, y \text{ and } j = x, y \\
    u_{c2,x} &= 1.0 \cdot u_{c1,x} + 1.0 \cdot u_{\Delta_{xx}} - 1.0 \cdot u_{\Delta_{yx}} \\
    u_{c2,y} &= 1.0 \cdot u_{c1,y} - 1.0 \cdot u_{\Delta_{yy}} + 1.0 \cdot u_{\Delta_{xy}}
\end{align*}
$$

in which $u_{j1,i}$, $u_{c1,x}$ and $u_{c1,y}$ are the master nodal displacement components and $u_{j2,i}$, $u_{c2,x}$ and $u_{c2,y}$ their coupled slave nodal displacement components. The parameter $u_{\Delta_{ji}}$ is the master d.o.f. that represents the constant displacement difference $\Delta_{ji}$.

Secondly, in order to incorporate $u_{\Delta_{ji}}$ in the system equations one needs additional d.o.f.’s: four in $\mathbb{R}^2$ and nine in $\mathbb{R}^3$. These d.o.f.’s must be physically isolated from the FE model that represents the periodic infinite medium, which can be practically done by adding for instance external one-node and zero stiffness spring elements or by adding fictitious nodes. Alternatively, $u_{\Delta_{ji}}$ could be incorporated in the system equations by using linear multi-point constraints. In this case no additional d.o.f.’s are required.
To illustrate this, two FE discretizations with different mesh alignments are shown in Figure 3.3. The two-dimensional meshes consist of quadrilateral eight-node elements. As can be seen, the shapes of the FE-models are periodic of nature: the distances between the red vertical boundary lines are everywhere equal to the periodic length scale $L_x$ and the distances between the blue horizontal boundary lines are everywhere equal to the periodic length scale $L_y$. Note that the element sizes relative to these periodic length scales determine the fineness of the mesh. To obtain periodic behavior of the models, the linear dependencies should be specified as shown in the pictures. The symbol ‘•’ stands for a master node and the symbol ‘◦’ for a slave node. Each slave node at the red or blue lines is coupled to its corresponding master node at the opposite boundary edge on a distance $L_x$ or $L_y$. The midnodes at the orange lines in the top-left and bottom-right corners do not have a corresponding node at the opposite boundary edge. Instead, they are coupled to each other, as shown in the last two equations of Equation (3.2). Finally, supports need to be added in the FE-model in order to prevent rigid body motions.

3.2 SLA prerequisites

Prior to an application of the proposed numerical test with periodic boundary conditions, first the implementation and development of some required features.
within the SLA framework are described. These SLA extensions are also required for the work in the following chapters. SubSection 3.2.1 briefly discusses the implementation of the element projection based crack band width estimation, according to Govindjee et al. SubSection 3.2.2 focuses on saw-tooth curve approximations for the nonlinear Hordijk and Moelands & Reinhardt softening.

3.2.1 Element projection based crack band width estimation

In previous SLA versions crack band width estimations were based on the element area or volume. As shown in SubSection 2.3.2, this estimator is less applicable in case of non-aligned cracks. For that reason the more accurate element projection based crack band width estimation, according to Govindjee et al. [51], has been implemented. The formulation of Govindjee et al. can be considered as an improved and generalized version of the formulation by Oliver [101].

Figures 3.4 and 3.5 present the schemes of the SLA procedures for an element area / volume based and an element projection based crack band width estimation respectively. The main implication of the latter estimator, from an implementation point of view, is the difference in the moment of computing the crack band width $l$. Consequently, the moment of saw-tooth curve generation differs, since $l$ is one of its input parameters, see SubSection 2.4.2. For the previously used element area / volume based formulation, $l$ depends on finite element characteristics only, and so for each element the crack band width and the saw-tooth curve can already be determined in the initialization stage of the analysis. In this case, all integration points in the element have the same $l$ and the same saw-tooth curve. For the newly implemented formulation of Govindjee et al., according to Equation (2.4), $l$ is both element shape and crack orientation dependent. Hence, crack band widths and saw-tooth curves are determined during the execution stage of the analysis, once a critical integration point starts to crack.

Finally, an equivalent element projection based method has been implemented to estimate crush band widths for compression failure [84].

3.2.2 Saw-tooth curve approximations for nonlinear softening

An important aspect for the modeling of concrete fracture is the use of a realistic description of the material behavior. From experimental test data the shape of the stress $\sigma$ versus crack opening $w$ curve can be characterized by an
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Initialization stage:
Read input data (mesh details, material properties, etc.)

Compute \( l \) for each element based on element size, shape and interpolation function

Generate saw-tooth curves for all elements

Store \( l \) and saw-tooth curve data on element level

Execution stage:
Perform sequence of linear-elastic analyses according to §2.4.1

Figure 3.4: Existing scheme of SLA procedure with an element area / volume based crack band width estimation.

Initialization stage:

Execution stage:
Perform linear-elastic analysis and determine the critical integration point

Yes
Critical integration point already damaged?

No

Compute \( l \) for this point based on \( \theta \) element size and shape, using Equation (2.4)

Generate the saw-tooth curve

Store \( l \) and saw-tooth curve data on integration point level

Increase damage in critical integration point and repeat these steps

Figure 3.5: Proposed and implemented scheme of SLA procedure with an element projection based crack band width estimation.
initial steep decrease followed by a relatively long softening tail. This subsection discusses the implementation of two such nonlinear softening curves in the SLA method, one according to Hordijk [66] and another one according to Möe-lands & Reinhardt [116]. The smooth \( \sigma(w) \) reference curves are transformed into discretized saw-tooth \( \sigma(\varepsilon) \) curves. Since the improved ripple approach as saw-tooth curve approximation technique [52] sometimes diverges when generating nonlinear softening curves, the procedure is locally modified. Note that robustness of the approximation technique becomes especially important for an element projection based crack band width estimation, by which saw-tooth curves are generated during the execution stage of the analysis.

Solution procedure

Recalling from SubSection 2.4.2, the improved ripple approach uses \( \sigma(w) \), \( E_0 \), \( f_t \), \( G_t \), \( l \) and the number of saw-teeth \( n_{swt} \) as input and searches for the strength percentages \( p_u \) and \( p_l \) that lead to a \( (G_t/l)_{swt} \) and a \( \varepsilon_{utswt} \) equal to their corresponding quantities \( G_t/l \) and \( \varepsilon_u \) of the reference curve. Mathematically, the construction of a saw-tooth curve in this way can be considered as an optimization or minimization problem with two boundary conditions.

Figure 3.6 shows the scheme of the iterative solution procedure for saw-tooth curve construction with the improved ripple approach. Basically, it consists of an outer loop that generates saw-tooth curves for certain values of \( p_u \) and \( p_l \) and an inner loop that goes over the individual saw-teeth \( i = 1 \ldots n_{swt} \) to determine their total strains \( \varepsilon_i \) and tensile strengths \( f_{ti}^{+} \). In the proposed modified version this global scheme remains the same. What changes compared to the original procedure in Reference [52] are the goal function \( f(p_u, p_l) \) in step \{10\} and the update algorithms in steps \{5\} and \{11\}. These changes are further explained in a more detailed discussion of Figure 3.6 below.

The construction of a saw-tooth curve starts with initial estimations for \( p_u \) and \( p_l \), step \{1\}. Subsequently, for each saw-tooth \( i \) the \( \varepsilon_i \) is determined at which the secant branch with stiffness \( E_i \) intersects with the uplifted softening curve (loop 2). Figure 3.7 illustrates that this point of intersection can be obtained from the equations \( f_{ti}^{+} = E_i \varepsilon_i \) and \( f_{ti}^{+} = \sigma_i + p_u f_t \), resulting in

\[
\sigma_i + p_u f_t - E_i \varepsilon_i = 0 \quad (3.4)
\]

Since the considered softening curves are nonlinear and expressed in a \( \sigma(w) \) relation, \( \sigma_i \) is in general a non-explicit function of the total strain \( \varepsilon_i \). The total strain \( \varepsilon_i \) then needs to be determined iteratively. This can be done by
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Loop 1: saw-tooth curve

1. Estimate $p_u$ & $p_l$

2. $i = 1$

Loop 2: saw-tooth $i = 1 \ldots n_{swt}$

3. Estimate $f_{cr}$

4. $f(f_{cr}) \leq \text{tolerance}$?

5. Update $f_{cr}$

6. $f_{cr} > f_u$?

7. Save $f_{cr}$ & $f_u$?

8. $i = n_{swt}$?

9. $i = i + 1$

10. $f(p_u, p_l) \leq \text{tolerance}$?

11. Update $p_u$ & $p_l$

12. Store saw-tooth curve data

Figure 3.6: Solution procedure to construct saw-tooth curve approximations with the improved ripple approach.
rewriting Equation (3.4) with $\sigma_t = E_0 \varepsilon_{d;i}$ and decomposing the strain in an elastic and crack part as $\varepsilon_i = \varepsilon_{d;i} + \varepsilon_{cr;i}$. The goal function becomes

$$f(\varepsilon_{cr;i}) = p_u f_t + \left(1 - \frac{E_i}{E_0}\right) \sigma_i - E_i \varepsilon_{cr;i} \leq \text{tolerance}$$

(3.5)

So, loop 2 starts with an initial estimation for the crack strain $\varepsilon_{cr;i}$, step {3}. Changing the known $\sigma(w)$ relation into a $\sigma(\varepsilon_{cr})$ relation, via $w = \varepsilon_{cr}$, the stress $\sigma_i$ can be computed. Hence, the goal function $f(\varepsilon_{cr;i})$ from Equation (3.5) can be evaluated in step {4}. When $f(\varepsilon_{cr;i})$ is not fulfilled, within a certain small tolerance, a new prediction for $\varepsilon_{cr;i}$ is made in step {5}. Previously, this was done by using a Newton-Raphson algorithm. However, it appeared that specifically in case of the Moelands & Reinhardt curve this algorithm was not able to find the points of intersection for the first saw-teeth. Therefore, the generally more robust Bisection method is implemented to update the estimation for $\varepsilon_{cr;i}$ in such cases. Once Equation (3.5) is satisfied, the total strain $\varepsilon_i = \sigma_i / E_0 + \varepsilon_{cr;i}$ and strength $f_{cr;i}^{-1} = E_i \varepsilon_i$ of the current saw-tooth can be computed and temporarily saved, step {7}. When $i$ is unequal to $n_{swt}$, step {8}, the $\varepsilon_i$ for the next saw-tooth will be determined. This requires the secant stiffness $E_{i+1}$, which can be calculated via $E_{i+1} = f_{i+1}^- / \varepsilon_i$ with the lower curve strength value $f_{i+1}^- = f_{i+1}^- - (p_u + p_l) f_t$. 

Figure 3.7: Construction of a saw-tooth curve with nonlinear softening by initial mispredictions for $p_u$ and $p_l$. 

[Diagram of saw-tooth curve with nonlinear softening]
Note that the goal function $f(\varepsilon_{cr};i)$ is only evaluated for $\varepsilon_{cr;i}$ smaller than the ultimate strain $\varepsilon_u$. For $\varepsilon_{cr;i}$ larger than $\varepsilon_u$ the update algorithm is ignored and $\varepsilon_{cr;i}$ is set to $\varepsilon_u$, see step \{6\}. In the next step $\varepsilon_i$ and $f_{t,i+1}$ are computed, as explained before, and the procedure continues. Figure 3.7 shows the cut-off at $\varepsilon_u$ for the saw-teeth $i+2$ and $i+3$. Although this cut-off is reasonable from a physical point of view, since the nonlinear softening curves are only valid for the range $0 \leq \varepsilon_{cr} \leq \varepsilon_u$, the main argument to include it is provided from an algorithmic point of view. Due to initial mispredictions for $p_u$ and $p_l$ some of the last saw-teeth may temporarily have negative secant stiﬀnesses. When such negative stiffness becomes too large no intersection between the secant branch and the uplifted softening curve exists anymore, as shown for $E_{i+3}$, and the procedure would abort. The introduction of the cut-off always leads to an intersection and the procedure can continue until the complete saw-tooth curve is constructed.

Once the saw-tooth curve based on the estimations for $p_u$ and $p_l$ is known, the goal function $f(p_u, p_l)$ of loop 1 in step \{10\} can be evaluated:

$$f(p_u, p_l) = \left( \frac{(G_l/l)_{swt}}{G_t/l} - 1 \right)^2 + \left( \frac{\varepsilon_{u,swt}}{\varepsilon_u} - 1 \right)^2 + \left( \frac{f_{t,n_{swt}}}{f_{t}} \right)^2 \leq \text{tolerance (3.6)}$$

Herein, $(G_l/l)_{swt}$, $\varepsilon_{u,swt}$ and the lower curve strength of the last saw-tooth $f_{t,n_{swt}}$ are the outcome of loop 2, being dependent on $p_u$ and $p_l$. The area under the saw-tooth curve can be computed with

$$\left( \frac{G_l/l}{G_t/l} \right)_{swt} = \sum_{i=1}^{n_{swt}} \frac{1}{2} (f_{t+1} - f_{t}) \varepsilon_i$$

where the $f_{t+1}$-values belong to the lower curve, except the final value $f_{t,n_{swt}}$ that is equal to $f_{t,n_{swt}} = f_{t,n_{swt}} - p_u f_t$.

In the first two terms of Equation (3.6) one can recognize the two boundary conditions that are used in the original improved ripple approach, as mentioned at the beginning of this subsection. A third dimensionless term is added here to the goal function in order to prevent saw-teeth with negative secant stiﬀnesses in the final saw-tooth curve, see Figure 3.7. This term represents the boundary condition that $f_{t,n_{swt}}$ at $\varepsilon_u$, should approximately be equal to zero.

When the result of $f(p_u, p_l)$ exceeds a certain tolerance the three boundary conditions are not properly satisfied and the steering parameters $p_u$ and $p_l$ needs to be updated, step \{11\}. It appeared that the Newton-Raphson scheme used in Reference [52] fails to find appropriate $p$-values for nonlinear softening curves, most likely due to the involved evaluations of the derivatives of the
goal function. Therefore, the Newton-Raphson scheme is replaced and the slower but more robust Downhill Simplex method is implemented as update algorithm for \( p_u \) and \( p_l \). This method can be used for minimization problems with multiple independent variables and is comparable to the “single - variable” Bisection method, used in loop 2, since it also requires functional evaluations only.

With new predictions for \( p_u \) and \( p_l \) a new saw-tooth curve is constructed and the goal function of loop 1 is again evaluated. This process continues until Equation (3.6) is fulfilled and the final saw-tooth curve data can be stored in step \{12\}.

**Hordijk and Moelands & Reinhardt softening curves**

The solution procedure discussed above is applicable to construct saw-tooth curves with linear and nonlinear softening behavior. Frequently used nonlinear tension softening relations for the constitutive modeling of concrete fracture are the ones proposed by Hordijk [66] and Moelands & Reinhardt [116]. Derived from deformation controlled uni-axial tensile tests, Hordijk’s relation in terms of stress and crack opening equals

\[
\frac{\sigma (w)}{f_t} = \begin{cases} 
(1 + (c_1 \frac{w}{w_{ult}})^3) \exp (-c_2 \frac{w}{w_{ult}}) & \text{for } 0 \leq w \leq w_{ult} \\
-\frac{w}{w_{ult}} (1 + c_3^3) \exp (-c_2) & \text{for } w > w_{ult} 
\end{cases}
\]

(3.8)

with \( c_1 = 3.0, c_2 = 6.93 \) and

\[ w_{ult} = 5.136 \frac{G_{fl}}{f_t} \]  

(3.9)

The relation of Moelands & Reinhardt is defined as

\[
\frac{\sigma (w)}{f_t} = \begin{cases} 
1 - \left( \frac{w}{w_{ult}} \right)^k & \text{for } 0 \leq w \leq w_{ult} \\
0 & \text{for } w > w_{ult} 
\end{cases}
\]

(3.10)

with \( k = 0.31 \) and

\[ w_{ult} = 4.226 \frac{G_{fl}}{f_t} \]  

(3.11)

Figure 3.8 shows the resulting saw-tooth curve approximations of both softening relations, using the same material input and number of saw-teeth as
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Figure 3.8: Saw-tooth curve approximations for Hordijk (a) and Moelands & Reinhardt softening (b).

the linear softening curve in Figure 2.8. The final p-values are in the range of 7% to 8.5%. For the Hordijk curve this results in an overshoot of the initial tensile strength. The Moelands & Reinhardt curve does not show such overshoot. Instead, its first saw-tooth has a strength $f_t^+; 1$ lower than $f_t$. This strength reduction after the point $(f_t/E_0, f_t)$ is a consequence of the transformation from a $\sigma(w)$ to a $\sigma(\varepsilon)$ curve. Since the Moelands & Reinhardt $\sigma(w)$ relation is characterized by an initial decreasing slope of $-\infty$, the $\sigma(\varepsilon)$ curve with total strains will reveal a snap-back. Hence, the intersection of the initial secant stiffness branch with the uplifted snap-back curve occurs at a lower value of $f_t$.

Finally, in addition to the above described SLA extensions also a saw-tooth curve approximation for parabolic softening in compression and a simplified combined tension-compression saw-tooth model have been developed in collaboration with others [84, 93].

3.3 Testing an existing element projection based crack band width formulation*

In this section the numerical testing procedure introduced in Section 3.1 is used to assess the mesh-induced directional bias of the element projection based

*Based on a section in Reference [137].
crack band width formulation of Govindjee et al. A series of analyses on uniform discretized “specimens” is conducted, with variations of mesh alignment (element orientation), element shape, interpolation function and numerical integration scheme. To study the influence of each of the last three mentioned finite element characteristics on directional mesh bias and on the localization behavior in a systematic way, five different element types are defined. Compared to a study by Oliver [101] with similar variations, this study focuses also on the localization behavior in meshes with higher order elements and with a localization band nonparallel to the element edges. Although the test with periodic boundary conditions can be used in three-dimensional models and with different loading cases as well, the study is limited to two-dimensional plane stress situations with uni-axial tensile loading. Further information of the geometry, boundary conditions and the used material properties are provided in SubSection 3.3.1. SubSection 3.3.2 describes the mesh variation study or test series in more detail, and SubSection 3.3.3 presents and discusses the results.

3.3.1 Finite element modeling aspects

Figure 3.9 shows the geometry, load and the expected crack orientation of the numerical test with periodic boundary conditions. It models an infinite two-dimensional periodic medium by FE discretizations with periodic length scales $L_x$ and $L_y$ of approximately 1000 mm and 450 mm. Element sizes of 50 mm by 50 mm and a thickness $t$ of 150 mm are chosen. Periodic boundary conditions are applied according to SubSection 3.1.2. The numerical models are loaded by a uni-axial tensile load in the horizontal direction. For that only the term $\Delta_{xx}$ in Equation (3.2) is prescribed. The other three constant displacement differences are kept free.

The material behavior is modeled with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, see SubSection 2.4.3. Fictitious material properties are adopted according to parameters of the numerical tests from Oliver [101]: Young’s modulus $E_0$ of 10,000 N/mm$^2$, tensile strength $f_t$ of 1.0 N/mm$^2$, fracture energy $G_f$ of 0.125 N/mm and Poisson’s ratio $\nu_0$ of 0.0. Different from Oliver, here a Moelands & Reinhardt nonlinear softening behavior [116] is used instead of a linear softening behavior. This is done for stimulation of strain localization, since a Moelands & Reinhardt $\sigma(\varepsilon)$ relation involves an initial steeper slope of the softening branch. As explained in SubSection 3.2.2, the use of Moelands & Reinhardt softening in a stress - total strain relation will generally lead to a snap-back on constitutive level. However, thanks to the damage controlled nature of the
adopted SLA solution procedure these local snap-backs do not lead to numerical problems [71]. The crack opening $u_{ab}$ according to Equation (3.11) is equal to 0.528 mm.

Using the SLA method the nonlinear stress-strain relation needs to be discretized, which is done according to SubSection 3.2.2. The used saw-tooth curve approximations in the test series consist of 20 teeth. Note that due to the event-by-event strategy in the adopted SLA method it is not necessary to include material imperfections in the FE models in order to trigger a localization. When using an incremental-iterative solution procedure the addition of an imperfection is inevitable.

Finally, from the uni-axial loading situation and the material input the theoretical solution of the equivalent one-dimensional problem can be derived for the global behavior in terms of load $F_x$ and inelastic (crack) displacement $u_{cr}$ with

$$F_x (u_{cr}) = \sigma (u_{cr}) L_y t$$

(3.12)

where $\sigma (u_{cr})$ can be evaluated from Equation (3.10). This theoretical solution is based on the assumption that a single vertical crack will appear in the “specimen”. For a meaningful comparison of this solution with the global responses obtained from the numerical tests, the numerically obtained crack patterns should also reveal a single global crack. Anticipating on the results that are presented in SubSection 3.3.3, we indeed observe subjectively only one global localized crack per plot. However, that need not always be the case. Since the strain localization behavior may depend on the adopted periodic length.

Figure 3.9: Geometry, load and expected crack orientation of the test with periodic boundary conditions.
3.3 Testing an existing element projection based crack band width formulation

scales $L_x$ and $L_y$, element sizes, material properties and stress-strain law, multiple global cracks may appear in the numerical solution. In such cases one should objectify $u_{cr}$, which can be done by dividing it with the number of observed global cracks. Note that the numerical inelastic displacement is obtained by subtracting the elastic displacement from the prescribed displacement, so $u_{cr} = \Delta_{ex} - u_{el}$. Actually, it is more precise then to define $u_{cr}$ as the total inelastic displacement.

3.3.2 Mesh variations

The test series consists of variations of element types, each type with its specific shape, interpolation function and numerical integration scheme, and variations of mesh alignments (element orientations). For a straightforward generation of all various FE discretizations including proper periodic boundary conditions, the test series is based on the considerations below.

Regarding the element types, five standard isoparametric two-dimensional plane stress elements are considered that are commonly used in practice. Furthermore, only square and right-angled isosceles triangle shaped elements are adopted. Hence, the meshes in this test series are all structured. Figure 3.10 shows the different finite elements, which have the following characteristics:

- a linear quadrilateral with $2 \times 2$ Gauss integration and a selective reduced integration for the shear terms;
- a quadratic quadrilateral with $2 \times 2$ (reduced) Gauss integration;
- a quadratic quadrilateral with $3 \times 3$ (full) Gauss integration;
- a linear triangular element (constant strain triangle) with 1-point Gauss integration;
- a quadratic triangular element with 3-point Gauss integration.

Regarding the variation of mesh alignments, different $\theta$-values are adopted. According to the convention in Figure 2.4 $\theta$ is defined by the angle between the isoparametric $\xi$-axis and the crack normal $\mathbf{n}(\mathbf{x})$. The $\xi$-axis, indicated as the alignment axis, is aligned with one of the four element edges for quadrilaterals and with the longest element edge for triangular elements. The crack normal $\mathbf{n}(\mathbf{x})$ points in the direction perpendicular to the crack. The variation of the mesh alignment by different crack or element orientations $\theta$ is possible, since the applied load on the FE models triggers only vertical cracking, see Figure 3.9. Hence, $\mathbf{n}(\mathbf{x})$ will always point in the global $x$ direction and can be considered as “fixed” for all various tests.
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For a complete variation of $\theta$ it is not necessary to cover the whole range $(0, 2\pi)$. Based on (double) symmetry considerations of the spatial crack orientation within the used element shapes and making use of $\theta = \min(\theta_1, \theta_2)$, as shown in Figure 2.4, a range of $(0, \pi/4)$ will suffice for the squared quadrilaterals, whereas the right-angled isosceles triangular elements require the double range of $(0, \pi/2)$. Within these ranges we consider five different $\theta$-values in case of the quadrilateral element types and ten different $\theta$-values in case of the triangular element types. The $\theta$-values are the result of five adopted $a$-values, namely $\infty$, 9, 4, 2 and 1. As can be observed from Figure 3.9, these $a$-values arise from the geometry of the tooth-shaped boundary edges ($1 : a$) and are directly coupled to $\theta$. Using “upward diagonal triangles” and “downward diagonal triangles”, two times the same five $a$-values can be adopted in order to cover the required range of $(0, \pi/2)$ for the triangular element types.

Based on the mentioned considerations above, a mesh generator is developed that produces the FE models of the numerical “specimens” (including the periodic boundary conditions) automatically. The resulting fifteen uniform and structured meshes are depicted in Figure 3.11.

In summary, this variation study involves two different element shapes. Three different element types are considered in case of the squared quadrilaterals and two different element types in case of the right-angled isosceles triangular elements. For the quadrilateral element types FE discretizations with five different alignments are adopted, for the triangular element types ten. So, in total this test series consists of $3 \times 5 + 2 \times 10 = 35$ analyses. Note that for the analyses with a triangular element type and $\theta$ equal to $\pi/4$ there
3.3 Testing an existing element projection based crack band width formulation

Figure 3.11: Uniform meshes (including supports) with five different element orientations for quadrilateral (a) and ten for triangular element types (b).
are two different FE discretizations. These analyses should give the same results due to the symmetry. Note further that because of the different \( a \)-values it is impossible to keep the periodic length scales \( L_x \) and \( L_y \) exactly the same for all the FE discretizations. Since actually an infinite plane is represented by these models this is seen as a minor issue. However, with respect to the fineness of the mesh it is aimed to keep the periodic length scales more or less equal.

### 3.3.3 Numerical results

This paragraph shows and discusses the results of the test series that is performed with the element projection based crack band width formulation according to Govindjee et al., following the Equations (2.4) and (2.5). To present the 35 analyses in a compact way, an outcome \( G_r^* \) is introduced. The apparent fracture energy \( G_r^* \) is defined by

\[
G_r^* = \int_0^{u_{\text{ult}}} \frac{F_x}{L_y} du_{cr}
\]  

(3.13)

and indicates the area under the numerically obtained load - total inelastic displacement curve for the range \( 0 \leq u_{cr} \leq u_{\text{ult}} \), divided by the theoretically expected crack area \( L_y t \). Subsequently, the ratio \( G_r^* / G_t \) is calculated for all the analyses and plotted against the element orientation angle \( \theta \). The ratio \( G_r^* / G_t \) can be seen as a measure for the deviation of the numerically obtained fracture energy to the specified fracture energy.

In Figures 3.12(a) and 3.12(b) the computed ratios are plotted for the quadrilateral element types, showing three times five points, and for the triangular element types, showing two times ten points, respectively. The points at \( \theta = \pi/4 \) for both the “upward diagonal triangles” and the “downward diagonal triangles” overlap, which means that the results of these two analyses give indeed identical results. Ideally, the five curves should be horizontal lines at \( G_r^* / G_t = 1.0 \). However, from the trends of the dashed lines it can be concluded that all considered element types from Figure 3.10, using the band width estimator \( l_b \) of Govindjee et al., show a significant spread around the target line. The actual ratio of \( G_r^* / G_t \) ranges from approximately 0.5 to 1.9. Hence, there is significant directional mesh bias, depending on element shape, interpolation function and numerical integration scheme.

Considering the adopted element shape it can be observed that for quadrilateral elements the curves are generally increasing with an increasing \( \theta \)-value, while the curves for triangular elements show more fluctuations in the bias with respect to \( \theta \). The background of this difference can be explained by the fact
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![Graphs showing deviation of obtained apparent fracture energies to theoretical solution for quadrilateral and triangular element types.](image)

Figure 3.12: Deviation of the obtained apparent fracture energies $G^*_f$ to the fracture energy $G_f$ for quadrilateral (a) and triangular element types (b) using $l_h$.

that the crack directions are parallel with at least one of the element edges for $\theta = 0$ in case of quadrilateral elements and for $\theta = \pi/4$ and $\pi/2$ in case of triangular elements. Generally, it appears that with an increasing misalignment of the cracks with respect to the element edges, the value $G^*_f$ also increases.

Considering the adopted interpolation function it can be observed that the curves belonging to the linear element types seem like shifted upwards compared to the curves belonging to the quadratic element types. In anticipation on the results that are discussed at the end of this section, the reason for these shifts can be attributed to strain localization within only a part of the element width. The shift is almost perfect over the entire range of $\theta$ for the triangular elements in Figure 3.12(b). To a lesser extent it is also seen for the quadrilateral elements in Figure 3.12(a). Here, the curve ‘quads, linear, 2 × 2 Gauss’ is monotonically increasing, whereas the curves of the quadratic elements show an initial increase until approximately $\theta = \pi/8$ and a subsequent plateau until $\theta = \pi/4$. Apparently, the quadratic quadrilaterals are able to break through the aforementioned tendency of an increasing bias with an increasing misalignment.

Finally, considering the adopted numerical integration scheme it can be observed that for the quadratic quadrilaterals the differences in mesh bias remain relatively small. The analyses with $3 \times 3$ Gauss integration gives slightly better results compared to the analyses with $2 \times 2$ Gauss integration.

To investigate the results of the test series in more detail, for some analyses the global responses and crack patterns are shown. The global responses of
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The FE models are presented in terms of normalized loads and total inelastic displacements. The normalization of $F_x$ allows for a fair comparison between the results, since the periodic length scales are not exactly the same for all the adopted meshes, as mentioned at the end of the previous subsection.

Figure 3.13 shows a graph with the normalized load - total inelastic displacement curves of the analyses with ‘quads, linear, $2 \times 2$ Gauss’ by five different mesh alignments (element orientations). Furthermore, the theoretical solution from Equation (3.12) is normalized and shown by the black line. It can be observed that the numerically obtained curves are of a spiky shape, which is typical for the SLA method due to its saw-tooth softening input. The curves
3.3 Testing an existing element projection based crack band width formulation

reveal the trend that with an increasing $\theta$, the deviation from the theoretical solution becomes larger and a more ductile post-peak behavior is obtained. Generally, the deviation from the theoretical curve happens already in the initial stage of the softening part, but becomes more pronounced towards the end of the softening branch, as reflected in the height of the residual plateau.

Figure 3.13 shows further the corresponding crack patterns/widths at $u_{cr} \approx w_{ult}$. The crack widths are obtained from the multiplication of the crack strains with the crack band width. In plotting crack patterns, three different line widths are used for crack widths in the ranges $[0, 1/2 w_{ult})$, $[1/2 w_{ult}, w_{ult})$, $[w_{ult}, \rightarrow)$. The crack patterns are plotted in deformed meshes with displacement amplification factors around 140. Noteworthy is the observed periodicity at the opposite boundary lines even after localization, which indicates that the periodic boundary conditions are properly applied. The crack plot belonging to $\theta = 0$ shows a vertical global crack that is perfectly aligned with the mesh lines. In the other plots the crack patterns are increasingly diffuse for higher $\theta$-values. There, the local crack directions in the elements within the final localized global cracks are generally vertical, whereas the global cracks themselves clearly propagate along the inclined mesh lines (except for $\theta = \pi/4$). Two different locations of the global cracks in the plots are visible. For $\theta = 0$, the strains localize approximately in the middle of the numerical “specimen”, while in case of the nonzero $\theta$-values the localization is observed near the vertical model boundaries. This difference is a coincidence, since the location depends on where the first critical event occurs. In an initial homogeneous strain field this is determined by numerical round-off, as explained in Subsection 2.4.1. Note that the global crack in case of $\theta = \pi/4$ is not yet fully developed.

Above mentioned observations from Figure 3.13 indicate mesh-induced directional bias. The increasing deviation of the numerically obtained curves from the theoretical solution for an increasing $\theta$ can be explained by considering their corresponding crack plots. The global cracks generally tend to propagate along mesh lines, rather than vertically. This results in a larger global crack area $A^*$, see Equation (2.1), and more cracked integration points. Consequently, the amount of dissipated energy is overestimated. The existence of a residual normalized load for curves belonging to nonzero $\theta$-values is mainly caused by “spurious stress transfer” in the elements with strain localization [81, 120, 122]. Additionally, locking might occur by spurious cracking at the spots where the global crack zig-zags from one to another column of elements.
Figure 3.14 shows the normalized load - total inelastic displacement curves of the analyses with a constant element orientation of \( \theta = 0 \) by the five different element types of Figure 3.10. The theoretical solution is indicated by the black line. Furthermore, the corresponding crack patterns/widths at \( u_{cr} \approx w_{ult} \) are plotted in undeformed meshes. Again, three line widths are used to differentiate the local crack widths.

Also the graph in Figure 3.14 shows reasonable differences between the numerically obtained curves and the theoretical solution. While in the previous presented set of analyses the differences were caused by directional mesh bias, in this set of analyses they are mainly driven by mispredictions of the numerical localization band widths \( \ell^* \). This is clearly visible from the corresponding crack plots, where the crack band width estimation \( \ell_b \) of Govindjee et al. is depicted for reference. As discussed in SubSection 2.3.1, disagreement between \( \ell_b \) and \( \ell^* \) leads to incorrect energy dissipation, which can subsequently result in a wrong post-peak behavior.

From the comparison of the five crack plots in Figure 3.14 it can be observed that \( \ell^* \) is influenced by the interpolation function and the numerical integration scheme. Finite elements with a quadratic interpolation function show strain localization within only a part of the element width. On the other hand for finite elements using a linear interpolation function, strain localization occurs at least over the entire element width. This phenomenon, which has also been observed by other authors [79], can be explained by considering the derivatives of the interpolation functions (i.e. strain field functions). Figure 3.15 depicts these strain field functions for the quadrilateral element types. Due to the linear strain field function, one column of integration points in the quadratic element can have strain values that remain in the linear elastic regime during the whole localization process. With a constant strain field function in the linear element this is not possible and all the integration points will crack.

Note, the effect of strain localization within an element should be taken into account when interpreting the differences of the numerically obtained curves from the target line in Figure 3.12.

3.4 Effect of “specimen” size

In each of the crack plots shown before, we observed subjectively a single localized crack to which all the apparent fracture energy \( G^*_f \) can be attributed. However, the imposed periodicity on the “specimens” implies that not a single crack, but actually patterns of infinite numbers of periodically repeating cracks are obtained. For the considered uni-axial tensile load, see Figure 3.9, this leads
3.4 Effect of "specimen" size

Figure 3.14: Normalized load - total inelastic displacement curves of analyses with a constant element orientation $\theta = 0$ and varying element types + their crack plots in undeformed meshes at $u_{cr} \approx w_{ult}$. The thickest lines correspond to $w \geq w_{ult}$.

Figure 3.15: Strain field development within the quadrilateral element types by progressive localization.
3. SYSTEMATIC ASSESSMENT OF MESH-INDUCED DIRECTIONAL BIAS

to an infinite series of vertical or aligned parallel cracks with crack distances equal to \( L_x \) in the horizontal direction. Hence, the question raises whether the localization behavior in one patch could be influenced by the periodic length scales \( L_x \) and \( L_y \).

This subsection discusses the results of a concise study to the “specimen” size objectivity of the numerical test with periodic boundary conditions. The same material properties and loading conditions are adopted as in SubSection 3.3.1. As reference the analysis with ‘quads, linear, 2 × 2 Gauss’ and \( a = 2 \), corresponding with a \( \theta \) of about 0.46 rad, from Figure 3.13 is selected. Subsequently, several analyses are performed with variations of \( L_x \) and \( L_y \). Since the fineness of the mesh is determined by the element sizes relative to these periodic length scales, additionally the element size is also varied. Table 3.1 provides an overview of all the analyses. The analyses are compared by considering the normalized loads \( F_x \) versus total inelastic displacements \( u_{cr} \) responses and the crack patterns/widths at \( u_{cr} \approx w_{ult} \). The crack patterns are plotted in deformed meshes with a displacement amplification factor of 100 and distinguish crack widths in the ranges \( \langle 0, 1/2 w_{ult} \rangle, [1/2 w_{ult}, w_{ult}] \), \( [w_{ult}, \rightarrow] \) by using three different line widths. All these results are reported in Appendix B. A selection of the results are presented in Figure 3.16.

From the results it can be observed that the considered range of variations with \( L_x \) and \( L_y \) do not affect the localization behaviors in the “specimens” and so the represented parallel cracking behaviors of infinite discretized periodic media remain consistent. The normalized load - total inelastic displacement curves show similar responses, indicating an objectivity and a “specimen” size independence of the numerical outcome \( G^*_f \). A particular result is obtained from analysis 4a, where the post-peak response is rather ductile and the crack plot reveals nine parallel cracks. This different failure behavior in a relatively fine mesh might occur since the use of smaller element sizes leads to smaller crack band widths and subsequently to more ductile local stress-strain curves. Relating \( G^*_f \) also in case of multiple cracks to a single crack, one should objectify the total inelastic displacements by dividing them with the number of observed global cracks.

3.5 Summary

This chapter presented a dedicated numerical test with periodic boundary conditions that enables a systematic assessment of mesh-induced directional bias of constitutive models, like the crack band approach, nonlocal and gradient-enhanced models. The test can be performed for two- or three-dimensional FE
3.5 Summary

Table 3.1: Analyses in “specimen” size objectivity study.

<table>
<thead>
<tr>
<th>reference</th>
<th>1a</th>
<th>1b</th>
<th>2a</th>
<th>2b</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_x$ [mm]</td>
<td>1006</td>
<td>559</td>
<td>2012</td>
<td>1006</td>
<td>1006</td>
<td>670</td>
<td>1006</td>
</tr>
<tr>
<td>$L_y$ [mm]</td>
<td>447</td>
<td>447</td>
<td>447</td>
<td>223</td>
<td>223</td>
<td>447</td>
<td>447</td>
</tr>
<tr>
<td>Element size [mm]</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>25</td>
<td>100</td>
</tr>
</tbody>
</table>

![analysis 1b](image1.png)

![analysis 4a](image2.png)

![reference](image3.png)

Figure 3.16: A selection of crack plots in deformed meshes ($\times$ 100) at $u_{cr} \approx w_{ult}$ belonging to the analyses from Table 3.1. The thickest lines correspond to $w \geq w_{ult}$.

discretizations under different loading conditions, with or without reinforcement. Since the test includes the concept of periodicity, strain localization can be studied in meshes with different alignments, with preservation of mesh uniformity and with exclusion of boundary disturbances.

The testing procedure is applied to the crack band approach, using an existing element projection based crack band width formulation, for a two-dimensional plane stress situation with uni-axial tensile loading. Simultaneously, different finite element characteristics are varied in a clean-cut way. The test clearly identifies and quantifies a significant directional mesh bias, depending on element shape, interpolation function and numerical integration scheme. Additionally, a variation study with different periodic length scales $L_x$ and $L_y$ is conducted, which shows an objectivity of the results to the size of the numerical “specimen”.

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3. SYSTEMATIC ASSESSMENT OF MESH-INDUCED DIRECTIONAL BIAS
Chapter 4

Enhanced crack band width formulation

Section 3.3 presented a systematic assessment of the mesh-induced directional bias of the crack band width formulation according to Govindjee et al. The results of the test series showed deviations from the expected energy dissipation and the a priori estimated band widths. Aiming for maximization of mesh objectivity via improvement of the fracture energy regularization, this chapter proposes an extension of Govindjee et al. by including the effect of mesh alignment and the observed influences of different finite element characteristics. Section 4.1 describes the derivation of this enhanced element projection based crack band width formulation from the results in Section 3.3. Subsequently, the test series is carried out for a second time, now with the new formulation. The results, showing the remaining residuals, are briefly discussed in Section 4.2. Section 4.3 validates the new formulation by simulations of bending and mixed-mode plain concrete fracture tests. Section 4.4 ends with the main conclusions of this chapter.

4.1 Derivation of element specific factors*

Section 3.3 showed that the deviation of the five curves from the horizontal line with $G_i^* / G_i$ equal to 1.0 in Figure 3.12 is mainly caused by two effects. Firstly, both the numerical crack surface area and the number of cracked integration points can become larger when the smeared crack path is not aligned with the mesh lines. This may generally result in an overestimation of the dissipated energy. Secondly, strain localization can occur within a part of the

*Based on a section in Reference [137].
4. ENHANCED CRACK BAND WIDTH FORMULATION

Table 4.1: \(\alpha\)-values for the considered element types of Figure 3.10.

<table>
<thead>
<tr>
<th>Element types</th>
<th>quadrilateral (2 \times 2) Gauss</th>
<th>quadrilateral (3 \times 3) Gauss</th>
<th>triangle 1-point intgr.</th>
<th>triangle 3-point intgr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>quadratic</td>
<td>1/2</td>
<td>13/18</td>
<td>-</td>
<td>2/3</td>
</tr>
</tbody>
</table>

The element width rather than the entire element width. Since the crack band width formulation of Govindjee et al. does not include this effect, it can lead to an incorrect prediction of the numerically obtained localization band width \(l^*\) with subsequent underestimation of the energy dissipation. In this section the observed deviation of \(G^*_f\) to \(G_f\) in Figure 3.12 due to the aforementioned two effects are incorporated in the estimation of the crack band width in order to obtain more mesh objective results with the crack band approach.

As starting point the formulation of Govindjee et al. from Equation (2.4) is extended by two element type related factors, making the new proposed band width estimator \(l_p\) equal to

\[
l_p = \alpha \cdot \gamma \cdot l_g (4.1)
\]

Herein, \(l_g\) takes into account the element shape and element aspect ratio, the factor \(\alpha\) includes the effect of possible strain localization within an element and \(\gamma\) is a so-called alignment factor. The idea of the \(\alpha\)-factor is also mentioned by others, e.g. [79]. The idea of the \(\gamma\)-factor is comparable to the \(\gamma_c\)-factor of Cervenka et al., as described in Subsection 2.3.2, but the derivation differs. Cervenka’s orientation factor is based on the observation that the band width of a smeared crack becomes larger in case the crack direction is inclined with respect to the element edges. On the other hand, the introduced alignment factor is based on a quantitative compensation for the numerically obtained incorrect energy dissipation induced by directional mesh bias.

The \(\alpha\)-factor is determined by considering the numerical localization band width \(l^*\) from Figure 3.14 and the location of the integration points in the Gauss numerical integration scheme of each element type. Its values are presented in Table 4.1. The \(\alpha\)-factor directly depends on the interpolation function and numerical integration scheme, but is considered as a constant over a varying mesh alignment or element orientation angle \(\theta\).

Knowing the ratio \(G^*_f / G_f\) and the \(\alpha\)-factor from Figure 3.12 and Table 4.1 respectively, the \(\gamma\)-values for each of the 35 analyses can be computed with

\[
\frac{G^*_f}{G_f} = \alpha \cdot \gamma (4.2)
\]
4.1 Derivation of element specific factors

Figure 4.1: Computed \( \gamma \)-values and their fitted curves for quadrilateral (a) and triangular element types (b).

Figure 4.1 plots the \( \gamma \)-values for the quadrilateral element types, showing three times five points, and for the triangular element types, showing two times ten points (with overlapping points at \( \theta = \pi/4 \)). The \( \gamma \)-factor is made dependent on both the element type and \( \theta \). To obtain a continuous function of \( \gamma \) over a varying \( \theta \), an unweighted least squares curve fitting on the separate data points belonging to each element type is performed. A Fourier series is adopted as a base function, using

\[
\gamma(\theta) = a_0 + \sum_{n=1}^{N} \left[ a_n \cos(nw\theta) + b_n \sin(nw\theta) \right] \tag{4.3}
\]

where \( N = 1 \) for quadrilateral element types and \( N = 2 \) for the triangular element types. The resulting Fourier series coefficients of each obtained fitted curve are shown in Table 4.2. The “goodness of the fit” for each curve is quantified by the calculation of the coefficient of determination \( R^2 \) in case of nonlinear regression. Figure 4.1 presents these \( R^2 \)-values, together with the continuous lines of the derived functions for \( \gamma(\theta) \). Note that the functions of \( \gamma(\theta) \) are only valid for \( (0, \pi/4) \) and \( (0, \pi/2) \), and so \( \theta \) should always be mapped to these ranges (see Figure 2.4).

It is important to realize that the above derived and calibrated element type related factors \( \alpha \) and \( \gamma \) from Equation (4.1) are based on regular and uniform meshes with square and right-angled isosceles triangle shaped elements, as shown in Figure 3.11, under mode I loading conditions. These factors may be different for three-dimensional element types and other loading conditions.
The values for $\alpha$ and especially $\gamma$ may also be different when the considered two-dimensional element types have irregular shapes or aspect ratios not close to unity. Note that in case of irregular element shapes with nonparallel edges, the *alignment* $\xi$-axis can be based on an average element edge direction [22].

The enhancement of an element projection based crack band width formulation by two element type related factors actually contradicts the findings and recommendations that are recently published by Jirasek and Bauer [79]. Firstly, the addition of an $\alpha$-factor, primarily motivated to obtain a proper band width estimation with higher order elements, has been questioned for the general case with non-aligned crack bands. Since stress fields become highly disturbed due to the phenomenon of strain localization within an element, Jirasek and Bauer conclude that higher order elements are not suitable for crack band simulations at all. Although we agree on the issue of disturbed stress fields, it is the author’s opinion that higher order elements are still useful and advantageous in FE simulations with the crack band approach. Quadratic elements, for instance, perform generally better in describing more complex deformation modes and failure modes compared to the linear elements, and dilemmas on selecting assumed strain approaches, relevant for linear quadrilaterals, can be avoided. For these reasons quadratic elements have a recommended position in guidelines for engineers [118]. Secondly, also the need for an orientation factor has been criticized by Jirasek and Bauer. Such a factor appeared to be unnecessary and would only lead to unrealistic reductions of the dissipated energy, especially in case of inclined cracks with respect to the element edges. This finding is based on numerical simulations of a notched beam subjected to three-point bending, using an isotropic damage model. It is in marked contrast with observations from the test with periodic boundary conditions using orthotropic damage, as discussed in Section 3.3. Further research is needed to understand the background of these opposite findings.
4.2 Verification for test with periodic boundary conditions

Before the new proposed crack band width formulation will be validated in the next section, this concise section first presents the remaining residuals of Equation (4.1). Using the calibrated formula $l_p$ for the estimations of the crack band widths, the test series in Section 3.3 is conducted again. The results are given in Figure 4.2. It can be observed that the computed ratios $G^*_f / G_f$ are closer to the target line at 1.0 compared to the ones obtained with the formulation $l_g$ of Govindjee et al., shown in Figure 3.12. Typical spread ranges now from approximately 0.9 to 1.35 rather than 0.5 to 1.9. From the normalized load - total inelastic displacement curves of the analyses with $l_p$ it is seen that they are all close to the theoretical solution, especially in the initial part of the softening stage. Hence, the remaining residuals can mainly be attributed to stress locking, which is considered as an inherent deficiency of the adopted kinematics and constitutive model. The crack patterns belonging to the analyses in the second test series do not show substantial differences compared to the ones of the test series in Section 3.3, indicating that cracks still prefer to propagate along continuous mesh lines.

Based on a section in Reference [137].

*Figure 4.2: Deviation of the obtained apparent fracture energies $G^*_f$ to the fracture energy $G_f$ for quadrilateral (a) and triangular element types (b) using $l_p$.\*
4.3 Validation on plain concrete fracture tests

This section validates the new proposed crack band width formulation by numerical simulations of the following three experimental fracture tests:

1. the three-point bending test of a plain concrete notched beam with mode I fracture;
2. the four-point shear test of a plain concrete single-edge-notched (SEN) beam with combined mode I and mode II fracture;
3. the plain concrete double-edge-notched (DEN) specimen with again mixed-mode fracture.

Since these tests showed different (curved) shaped failure cracks, the crack orientation dependent formulation $l_p$ can be effectively evaluated by using structured meshes with square plane stress elements. The size of the elements is set to 5 mm. For each test three different element types are adopted: the linear quadrilateral with $2 \times 2$ Gauss integration, the quadratic quadrilateral with $2 \times 2$ Gauss integration and the quadratic quadrilateral with $3 \times 3$ Gauss integration. The linear quadrilaterals do not use selective reduced integration for the shear terms (or constant shear), except in case of the three-point bending test. Furthermore, the simulations are performed both with the formulation $l_g$ of Govindjee et al. according to Equation (2.4) and with the enhanced formulation $l_p$ according to Equation (4.1), resulting in six analyses for each of the fracture tests.

The material behavior in all analyses is modeled with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, see SubSection 2.4.3. The results are assessed by considering the global behavior in terms of a load versus a displacement (difference) and by considering the crack patterns in undeformed meshes. In plotting crack patterns, three different line widths are used for crack widths in the ranges $(0, \frac{1}{2} w_{ult})$, $[\frac{1}{2} w_{ult}, w_{ult}]$, $[w_{ult}, \to]$, where $w_{ult}$ depends on the stress-strain softening curve and the material properties $G_1$ and $f_k$. The crack widths are obtained from the multiplication of the crack strains with the crack band width. The fracture tests and numerical results are presented and briefly discussed in the following subsections.

4.3.1 Three-point bending test

The first experiment is a plain concrete notched beam subjected to three-point bending, tested by Körneling and Reinhardt [88]. Figure 4.3 shows the geometry and experimental setup. The beam with a thickness of 100 mm
has been analyzed by many authors, e.g. [29, 79, 120]. The adopted material properties are: Young’s modulus $E_0$ of 20,000 N/mm$^2$, tensile strength $f_t$ of 2.4 N/mm$^2$, fracture energy $G_f$ of 0.113 N/mm and Poisson’s ratio $\nu_0$ of 0.2. The shape of the softening curve is taken according to Hordijk and is discretized with 20 damage increments, see SubSection 3.2.2. Based on these material properties, the crack opening $w_{\text{ult}}$ is equal to 0.242 mm.

Figures 4.4(a) and 4.4(b) present the load - deflection curves at midspan of aforementioned six analyses using $l_g$ and $l_p$ respectively. The range of experimentally obtained load - deflection curves is indicated by the gray shaded area. For the three analyses with $l_g$ the values of the peak loads reveal a variation from approximately 1250 to 1450 N, whereas for the analyses with $l_p$ all the peak loads are about 1400 N.

Figure 4.5 presents the corresponding crack width plots at a midspan deflection of 0.5 mm. No clear differences between the results of the two band width estimators can be observed. Generally, the plots show that the local cracks in the integration points just above the notch start more or less vertically, but with an increasing distance from the notch the local crack directions become more diffuse. This holds especially true for the quadratic elements. The meshes with quadratic elements show further that the strains tend to localize within a part of the element width.

Foregoing results indicate that $l_p$ improves the objectivity of the fracture energy dissipation when different element types are used. It leads to consistent predictions of the maximum load capacity for this notched beam. The observed scatter of the maximum load capacities in Figure 4.4(a) can be attributed to incorrect estimations of the numerical localization band widths $l^*$ by $l_g$ for the quadratic element types, since $l_g$ does not take into account the effect of strain localization within an element.
4. ENHANCED CRACK BAND WIDTH FORMULATION

Figure 4.4: Load - deflection responses of analyses with three different quadrilateral element types on the three-point bending test.

Figure 4.5: Crack width plots at a deflection of 0.50 mm belonging to the analyses on the three-point bending test from Figure 4.4. The thickest lines correspond to $w \geq w_{\text{ult}}$. 

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4.3 Validation on plain concrete fracture tests

Figure 4.6: Geometry, experimental setup and failure cracks of the four-point shear test.

4.3.2 Four-point shear test

The second experiment is a four-point shear test on a plain concrete SEN beam with a thickness of 100 mm, tested by Schlangen [132]. Figure 4.6 shows the dimensions of the specimen, the test setup and the experimentally obtained failure crack patterns. Its behavior has been simulated by several authors, e.g. [19, 36, 150], using different values for the material properties. Here, the following material properties are adopted, based on the con@ material from the original report of the test [132]: Young’s modulus $E_0$ of 30,000 N/mm$^2$, tensile strength $f_t$ of 3.44 N/mm$^2$, fracture energy $G_f$ of 0.115 N/mm and Poisson’s ratio $\nu_0$ of 0.2. Again, a Hordijk softening stress-strain diagram is used, which is approximated by a saw-tooth curve of 20 teeth. The crack opening $\omega_{ult}$ belonging to these properties is equal to 0.172 mm.

Figure 4.7 presents the global responses of the six analyses using $l_g$ and $l_p$, together with the range of experimentally obtained responses (gray shaded areas), in terms of load and crack mouth sliding displacement (CMSD). The CMSD is measured as the displacement difference in vertical direction between the points A and A’, indicated in Figure 4.6. Generally, all curves reveal higher failure loads compared to the experimental results, which may be the reason that some other researchers use a lower tensile strength as was reported [19, 36, 150]. Reasonable agreement with the experiment is only observed for the orange curve ‘quadratic, $3 \times 3$ Gauss, □’ in case of $l_g$, whereas the other curves reveal too brittle post-peak behaviors that accompany with sudden load drops. An extreme brittle failure is visible for the analysis ‘linear, $2 \times 2$ Gauss, □’ with $l_p$, showing even a global snap-back response. From the comparison between the three analyses with $l_g$ and with $l_p$ it can be seen that the values of the peak loads are closer to each other in case of the latter band width estimator.
4. ENHANCED CRACK BAND WIDTH FORMULATION

Figure 4.7: Load - CMSD responses of analyses with three different quadrilateral
element types on the four-point shear test.

Figure 4.8: Crack width plots at a CMSD of 0.1 mm belonging to the analyses
on the four-point shear test from Figure 4.7. The thickest lines correspond to
\( w \geq w_{\text{ult}} \).
4.3 Validation on plain concrete fracture tests

Figure 4.8 presents the corresponding crack patterns/widths at a CMSD of 0.1 mm. All plots reveal local cracks in the critical region near the notch and local cracks in the two bending zones outside this critical region. In five plots the failure crack starts from the notch according to the experimental result, although the shapes of these failure cracks are not or less curved as seen in Figure 4.6. Only in case of analysis ‘quadratic, 2×2 Gauss, □’ with $l_g$ the predicted failure mode of the beam is caused by a developed flexural crack, outside the critical region near the notch. From the comparison between both sets of three plots it can be seen that, specifically for quadratic element types, the enhanced formulation $l_p$ results in more realistic crack patterns. The analysis ‘quadratic, 2×2 Gauss, □’ with $l_p$ predicts the failure crack in the critical region of the beam and the analysis ‘quadratic, 3×3 Gauss, □’ with $l_p$ shows somewhat more curvature in the failure crack. The two plots belonging to the linear quadrilaterals are more or less similar, showing straight failure cracks that follow the mesh lines, even though the local cracks in the integration points are at an angle.

From the aforementioned observations it can be concluded that the new proposed crack band width formulation has a positive effect on the consistency for maximum load capacity predictions and on the accuracy of the crack patterns after failure by FE discretizations with varying element types. Furthermore, the results show clearly a relation between the brittleness of the post-peak response on the one hand and the location and shape of the failure crack on the other. Remarkable is the observed influence of the crack band width formulation on the predicted failure mode in case of the two analyses ‘quadratic, 2×2 Gauss, □’. The incorrect failure mode that is obtained with $l_g$ might be caused by its overestimation of the numerical localization band widths $l^*$ for the initial flexural cracks, left from the critical region near the notch. This overestimation is a consequence of ignoring the effect of strain localization within an element. Hence, the dissipation of the regularized fracture energy $G_f/l_g$ in the elements of the flexural cracks is underestimated and the material in this region behaves relatively more brittle than in the region near the notch. So, the flexural cracks are not arrested, as in the experiment, but propagation of these flexural cracks is stimulated, which leads to premature failure before the development of a shear crack in the critical region. Better a priori estimations of $l^*$ result in failure crack locations at the notch. This is shown for the analysis ‘quadratic, 2×2 Gauss, □’ with $l_p$, but in fact also for the analysis ‘quadratic, 3×3 Gauss, □’ with $l_g$. 

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4.3.3 Double-edge-notched specimen

The last experiment in this validation study is a plain concrete DEN specimen, tested by Nooru-Mohamed [99]. Different load paths were applied on specimens with different sizes, aiming for broadening the knowledge of mixed-mode cracking. Here, the specimens with label ‘46-05’ and ‘47-01’ are studied, which were subjected to the non-proportional loading path ‘4b’. The test setup and the geometry of the specimens are shown in Figure 4.9. Also the experimentally obtained failure crack pattern of specimen ‘46-05’ is depicted, being representative for the test ‘47-01’ [100]. On the square shaped specimens with a thickness of 50 mm, first a lateral shear was applied in displacement control until \( P_s = 10 \text{ kN} \). During this time the specimen was allowed to deform in vertical direction (so \( P = 0 \text{ kN} \)). Subsequently, \( P_s \) was switched to load control and kept constant, and an axial tensile load \( P \) was applied under displacement control. This well-known test has been numerically simulated by several authors, e.g. [24, 38, 40, 100].

For the modeling of this experiment a simplification has been made. The vertical constraints, needed to apply the displacement controlled \( P \) in the second loading stage, are also present during the initial shear loading stage. Contrary to the experiment, this does not allow for free vertical deformation of
the specimen when $P_s$ is applied. However, since the experimental results show negligible vertical deformations during this initial shear loading stage, the used simplification is acceptable.

The assumed concrete material properties are: Young’s modulus $E_0$ of 30,000 N/mm$^2$, tensile strength $f_t$ of 3.0 N/mm$^2$, fracture energy $G_f$ of 0.1 N/mm and Poisson’s ratio $\nu_0$ of 0.2. A Hordijk stress-strain softening curve is adopted, discretized by 20 damage increments. The crack opening $w_{ult}$ is equal to 0.171 mm.

Figure 4.10 presents the graphs with the $P - \delta_{average}$ responses of the six analyses on the DEN specimen using $l_k$ and $l_p$. The $\delta_{average}$ is the mean value of the two measured $\delta$’s shown in Figure 4.9. Experimentally obtained responses are indicated by the gray shaded areas. The numerically obtained curves of both band width estimators reveal the same responses and only the analysis ‘linear, $2 \times 2$ Gauss, □’ with $l_k$ gives a somewhat higher peak load. Compared to the experimental results, all numerical results significantly overestimate the peak load. Similar overestimations were reported by some of the aforementioned referred authors, whereas others changed the material properties to decrease them. The overestimation could be a result of unintended deficiencies in the experimental test setup that are not incorporated in the FE model [24]. Furthermore, all the curves show a relatively brittle behavior in the post-peak regime. The red curves belonging to ‘linear, $2 \times 2$ Gauss, □’ even intersect with the horizontal axis at $\delta_{average} = 0.054$ mm and $\delta_{average} = 0.045$ mm.

Figure 4.11 presents the corresponding crack width plots at $\delta_{average} = 0.1$ mm. All plots reveal strong sensitivity to mesh alignment, i.e. cracks prefer to propagate along continuous mesh lines, and none of them show good agreement with the experimentally obtained failure crack patterns. The mesh alignment is observed especially for the analysis ‘linear, $2 \times 2$ Gauss, □’ with $l_k$. The local cracks in the integration points are all under an angle, but the two global failure cracks propagate horizontally from both notches towards the center of the specimen. Although severely biased, the crack pattern of the analysis ‘linear, $2 \times 2$ Gauss, □’ with $l_p$ shows two distinct global cracks, indicating a small improvement.

Foregoing results hardly show differences between the band width estimators $l_k$ and $l_p$. Only for the linear elements the results are slightly improved with the enhanced formulation. Generally, the crack developments in all these analyses are strongly affected by the horizontal mesh lines. Less curved failure cracks, or the absence of any curvature in the failure cracks, cause a decreased resistance after the peak, which explains the more brittle softening responses compared to the experimental results.
4. ENHANCED CRACK BAND WIDTH FORMULATION

Figure 4.10: $P - \delta_{\text{average}}$ responses of analyses with three different quadrilateral element types on the DEN specimen.

Figure 4.11: Crack width plots at a $\delta_{\text{average}}$ of 0.1 mm belonging to the analyses on the DEN specimen from Figure 4.10. The thickest lines correspond to $w \geq w_{\text{ult}}$. 

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4.4 Summary

This chapter presented an enhanced element projection based crack band width formulation, which aims to maximize a mesh objective energy consumption for the different elements types. By including an element type specific strain localization factor and an element type specific mesh alignment factor, the crack band width becomes a more complete finite element discretization parameter. The two additional factors are empirically derived from the series of analyses of the test with periodic boundary conditions, as introduced in the previous chapter. Since only square and right-angled isosceles triangle shaped elements were considered in this test series, and especially the mesh alignment factor might be element shape dependent, the applicability of the new proposed formulation is limited. Moreover, the values of the mesh alignment factor might become different when another constitutive framework is used. Nevertheless, the verification and validation of the new proposed formulation show that it is profitable to invest in improvements of crack band width estimators. Using a calibrated fracture energy regularization, the predictions of maximum load capacities become more consistent for varying element types. Crack patterns remain still biased, although in some analyses more realistic failure cracks were obtained.

Finally, this chapter illustrates a way how the results of a systematic assessment of mesh-induced directional bias can be used to optimize the model parameter that has to ensure objectivity of the crack band approach. In an equivalent way, the testing procedure as presented in Chapter 3 might be attractive to assess also more advanced fracture models and calibrate their involved length scale parameters.
4. ENHANCED CRACK BAND WIDTH FORMULATION
Chapter 5

A delayed $C^1$ – continuous crack propagation algorithm∗

The enhancement proposed in the previous chapter focuses on a mesh objective energy consumption in the crack band approach via more accurate a priori estimations of crack band widths. Its validation showed consistency in the failure load predictions, but the obtained failure crack patterns were still influenced by continuous mesh lines. This chapter proposes a second enhancement for the adopted smeared crack approach, which primarily focuses on an elimination of the observed mesh bias in the crack patterns via the determination of smoothly curved $C^1$ – continuous propagation paths within and across the conventional continuum elements. Section 5.1 starts with a motivation for the new proposed crack propagation algorithm (CPA). Subsequently, a detailed description of the proposed algorithm is provided in Section 5.2. Section 5.3 validates the CPA by simulations of the same fracture tests as in Chapter 4. The mesh objectivity is studied with structured and unstructured meshes of linear or quadratic elements. Section 5.4 explores the concept of delayed crack path fixation, being a specific feature of the proposed CPA. Finally, a summary and the main conclusions of this chapter are provided in Section 5.5.

5.1 Motivation

Crack propagation or crack tracking algorithms trace and designate potential crack paths within an arbitrary FE discretization in order to determine which elements are allowed to damage and which are not. These algorithms are introduced in SubSection 2.2.2 as remedy to eliminate mesh-induced directional

∗This chapter is based on Reference [139].
bias. As discussed, the tracking strategies appeared to be indispensable for X-FEM and E-FEM approaches to obtain mesh objective and realistic results from simulations of crack propagation processes. Hence, they might be useful for approaches in the standard smeared crack concept as well.

Recently, a local crack tracking technique in the context of a continuum damage model was proposed by Cervera et al. [29]. The algorithm generates continuous crack paths that are constructed from straight line segments and is specifically suitable for meshes with constant strain triangles. The relative simplicity and low computational costs make this technique attractive. However, the algorithmic robustness can be violated by sudden and unrealistic sharp changes of crack paths, as shown in Figure 5.1(a). To remedy such U-turns of the propagation direction, Cervera et al. limit the maximum allowable curvature of the crack path. Figure 5.1(b) illustrates this for the potential crack path element at the tip. Firstly, the vectors $V_e$ and $V_c$ are determined. $V_e$ is based on the direction perpendicular to the maximum principal tensile stress in the element and $V_c$ is the average crack direction vector of the elements within a user-defined radius $r_{neigh}$. Subsequently, the angle $\alpha$ between $V_e$ and $V_c$ is computed and compared with the user-defined maximum allowable curvature angle $\alpha_{max}$. In case $\alpha \leq \alpha_{max}$, the crack path will propagate with the natural direction $V_e$, whereas in case $\alpha > \alpha_{max}$, the crack path is corrected and will propagate with $V_c$. From the results it was observed that foregoing crack path correction strategy can effectively circumvent the occurrence of U-turns, although more sensitivity analyses are required due to the introduction of additional model parameters.

This chapter presents a new local, i.e. within the element, crack tracking technique that treats the path propagation and the important issue of U-turns in a different way. Firstly, the algorithm aims for smoothly curved crack paths within and across conventional continuum elements. Secondly, the algorithm enables to postpone the moment of crack path fixation until a certain level of damage is reached, using a distinction between damage paths and crack paths. Both the $C^1$–continuity of the crack path across elements and the concept of delayed crack path fixation can be seen as ingredients to avoid unrealistic U-turns. $C^1$–continuity may avoid U-turns, since the path propagation within the considered element starts in the same direction as the path entered this element. Delayed crack path fixation contributes, since it corrects initial poor predictions of the crack path by possible stress and strain redistributions during the evolution of the fracture process [129].

The proposed algorithm for curved $C^1$–continuous crack paths is naturally best suited to quadratic displacement fields. As discussed in Section 4.1,
5.1 Motivation

Figure 5.1: U-turn of a crack path (a) and the maximum curvature criterion with crack path correction (b) (pictures adapted from [29]).

quadratic elements with full numerical integration perform generally better in describing more complex deformation modes and failure modes, and dilemmas on selecting assumed strain approaches, e.g. relevant for linear quadrilaterals, can be avoided. Nevertheless, the CPA can be used for linear elements as well. In addition, besides the possibility for an immediate or postponed crack path fixation, the algorithm has flexibility regarding the shape of the crack path. Without changes in the framework of the formulation, it is optional to replace the curved $C^1$ - continuous crack paths by $C^0$ - continuous crack paths constructed from either curved or straight line segments.

In line with the aims and scope of the present thesis (Sections 1.2 and 1.3), the proposed CPA is implemented in the Sequentially Linear Analysis (SLA) method and applied for the fixed smeared crack model (Section 2.4). Note that the CPA can be implemented in standard incremental-iterative solution procedures as well. Nevertheless, using the purely damage controlled SLA method enables to find snap-backs in load - displacement responses accurately and crack tracking algorithm related issues concerning load increment size dependency and prediction-correction strategies are circumvented. Note further that the combination of a smeared crack model and a crack tracking technique could seem to be a paradox. Originally, the smeared crack model aims to describe large areas of distributed cracking by an averaging procedure and is consequently specifically applicable to global analysis of large-scale reinforced concrete structures [115]. This would not fit with crack tracking, which concerns
5. A DELAYED $C^1$– CONTINUOUS CRACK PROPAGATION ALGORITHM

the determination of propagation directions for individual cracks. However, since the introduction of fracture mechanics in the smeared crack concept [10], smeared crack models are also used for detailed analysis of individual crack localizations [122]. At this scale a crack tracking technique can be applied.

5.2 Proposed algorithm

In order to reduce the influence of a mesh topology on the propagation direction of a crack, the proposed algorithm aims to steer the cracking process by pointing elements with propagation paths. Administration of these propagation paths and corresponding elements during the simulation is handled via an element-label system. Figure 5.2 shows the composition and the element labels of such a propagation path. They are smoothly curved and $C^1$– continuous, although constructed in an element by element way (see segmented red and blue lines). Generally, each of them is composed of a crack path (red line) that starts in a root element, and a damage path (dark blue line) that ends at a potential damage path tip element. Crack and damage paths are distinguished in order to include the concept of delayed crack path fixation, which means that the moment of crack path fixation can be postponed to a later stage in the fracture process. Crack paths are fixed. Damage paths are not fixed, but they can “wag” over load increments or time steps during an analysis (light blue lines). Consequently, they enable progressive damaging and reorientation of the propagation path in front of a consolidated crack path tip, following the principles of the CPA. Depending on the evolution of the fracture process, it is possible that propagation paths consist exclusively of a damage path starting in a root element, or exclusively of a crack path ending at a potential crack path tip element.

In which elements the crack paths end and the damage paths begin is defined by a damage threshold for the elements. This damage threshold is denoted as $d_{\text{crit,crk}}$, being a user-defined parameter between 0 and 1. The damage of an element $d_e$ is defined as the maximum damage level in terms of relative stiffness reduction of all its integration points $j$. So, $d_e = \max(d_j)$ for $j = 1, \ldots, n_{ip}$, with $n_{ip}$ being the number of integration points in the element. When $d_e$ exceeds the threshold value $d_{\text{crit,crk}}$, transition from damage path to crack path in the element takes place.

At any time step during the analysis only root elements, elements on the composed propagation paths and the potential crack or damage path tip elements can (further) damage. Unlabeled elements are excluded to experience nonlinear material behavior, keeping their constitutive relation linearly elastic.
5.2 Proposed algorithm

Note that this may lead to physically unrealistic high principal tensile stresses in these elements.

The proposed local CPA has been implemented for two-dimensional plane stress situations. Some similarities exist with the algorithm proposed by Cervera et al. [29], mainly regarding the element-label system and the way of starting a propagation path. The primary differences between this work and Cervera et al. are twofold. Firstly, the proposed algorithm determines smoothly curved and $C^1$-continuous propagation paths, either with immediate ($d_{\text{crit,crk}} = 0$) or postponed ($d_{\text{crit,crk}} > 0$) fixation, which bypasses the need of an explicit crack path correction strategy. Secondly, in this work SLA is used as solution procedure. The algorithm is not restricted to SLA. Note however that in the more common incremental-iterative solution procedure, both the development (i.e. propagation and reorientation) of the damage paths and associated damage increments should be part of a prediction-correction algorithm during the equilibrium iterations. Cervera et al. simplified the procedure by determining damage paths at the beginning of a step and not update them in subsequent iterations, which could result in load increment size dependency. From this point of view the adopted SLA method blends well with the local CPA, by the avoidance of equilibrium iterations. However, for SLA damage increment size dependency might occur.

Figure 5.2: Composition and corresponding element labels of a propagation path.
5. A Delayed $C^1$− Continuous Crack Propagation Algorithm

Section 5.2 is further subdivided into the following subsections. SubSection 5.2.1 explains the way initiation, element-wise propagation and completion of a crack or damage path is treated. SubSection 5.2.2 discusses the procedure for $C^1$− continuous crack or damage path propagation on element level. SubSection 5.2.3 provides more details of the concept of delayed crack path fixation and the used fixation criterion. SubSection 5.2.4 briefly addresses how the proposed algorithm works in a SLA framework.

5.2.1 Element by element construction of the propagation path

The composed propagation paths are constructed element by element. Each segment has a starting point and exit point. In the starting point a normalized start vector $s$ is located. The definitions of the starting point and the start vector $s$ depend on whether a new path initiates or an existing path propagates, see Figure 5.3.

In case of path initiation, crack or damage paths start from root elements. Currently, it is assumed that these root elements are located at the mesh boundary. Their corresponding starting point is subsequently assumed as the midpoint of the root element edge at the mesh boundary. Furthermore, the start vector $s$ is defined as the inwards normal of the root element edge at the mesh boundary. For corner elements with two element edges at the mesh boundary, the starting point and $s$ are respectively assumed as the centroid of

Figure 5.3: Path initiation in root elements at the mesh boundary and subsequent element-wise $C^1$− continuous path propagation.
5.2 Proposed algorithm

the element and the average of the inwards normals of both mesh boundary edges.

In case of path propagation, the starting point and $s$ for the new element are passed on by the preceding element on the path. They correspond respectively to the point at which the crack or damage path enters the element and the propagation direction of the crack or damage path at this point. Crack or damage path propagation is completed when their paths arrive at the mesh boundary. Completion also occurs when two paths intersect and, in the particular case, when a path crosses itself. In case two paths intersect, it is assumed that they merge.

Shielding of a propagation path [106], in the sense that other paths are precluded to propagate in its neighborhood, is not used in the proposed algorithm. Nevertheless, it is optional to specify an exclusion radius $r_{\text{excl}}$ in order to enforce a minimum crack spacing, see Figure 5.2. When surrounding mesh boundary elements have their starting points within $r_{\text{excl}}$ from a new crack path origin, they will be excluded from the list of potential root elements. For multiple crack problems this way of shielding could be helpful as shown by Cervera et al. [29]. In the analyses throughout this chapter $r_{\text{excl}}$ is not used (i.e. $r_{\text{excl}} = 0$).

Finally, the implementation of the proposed CPA requires the initialization of additional topological information at the start of the analysis. This information includes the connectivity between elements and edges, and the connectivity between edges and nodes. The “edge-level” data enables a straightforward administration and update of new and existing propagation paths.

5.2.2 Crack and damage path propagation within an element

The previous subsection explained the procedure of path propagation across elements. This subsection discusses the determination of smooth curved crack and damage paths (that are treated in the same way) within an element. Its procedure can be subdivided into three subsequent steps. Table 5.1 summarizes the input, algorithmic aspects and output for each step and illustrates the procedure for an isoparametric element with quadratic interpolation and full numerical integration. The general idea is that path propagation within an element is based on a calculated local propagation field, which is obtained from the principal strain and damage states in the integration points and the propagation direction at the entry point. By taking the propagation direction at the entry point explicitly into account, $C^1$ – continuity across elements can be obtained. More details of Table 5.1 are provided per step in the sequel. Step 1 determines the path propagation directions in the Gauss points of the
Table 5.1: Procedure for smooth curved path propagation within an element.

<table>
<thead>
<tr>
<th>Step 1: Path propagation directions in the integration points</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td>- starting propagation direction ( \overline{\rho}_{sp} ) or ( s )</td>
</tr>
<tr>
<td>- principal strains ( \varepsilon_{1,i} ), ( i = 1, \ldots, n_{ip} )</td>
</tr>
<tr>
<td><strong>Algorithmic aspects:</strong></td>
</tr>
<tr>
<td>- procedure in Figure 5.4</td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
<tr>
<td>- propagation directions in the integration points ( \overline{\rho}_{ip,j} ) or ( t_j )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 2: Path propagation field</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td>- ( \overline{\rho}_{sp} ) or ( s ), and damage ( d_j )</td>
</tr>
<tr>
<td>- ( \overline{\rho}<em>{ip,j} ) or ( t_j ), and start location ( (\xi</em>{sp}, \eta_{sp}) )</td>
</tr>
<tr>
<td><strong>Algorithmic aspects:</strong></td>
</tr>
<tr>
<td>- linear ( \phi )-field using standard ((b)linear) shape functions ( N_i(\xi,\eta) ) (curved path)</td>
</tr>
<tr>
<td>- weighted linear least squares method:</td>
</tr>
<tr>
<td>- ( w_{ip,j} = d_j )</td>
</tr>
<tr>
<td>- ( w_{ip} = \sum w_{ip,j} ) (( C^1 )-continuity)</td>
</tr>
<tr>
<td>- ( \sum w_{ip,j} = 1 % ) of ( \sum w_{ip} ) (regularization)</td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
<tr>
<td>- estimated linearized propagation field ( \hat{\overline{\rho}}(\xi,\eta) ) or ( \hat{T}(\xi,\eta) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 3: Crack or damage path</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td>- ( (\xi_{sp}, \eta_{sp}) )</td>
</tr>
<tr>
<td>- ( \hat{T}(\xi,\eta) )</td>
</tr>
<tr>
<td><strong>Algorithmic aspects:</strong></td>
</tr>
<tr>
<td>- particle tracking</td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
<tr>
<td>- smooth curved path</td>
</tr>
<tr>
<td>- exit location ( (\xi_{ep}, \eta_{ep}) ) and propagation direction ( \hat{\rho}_{ep} ) or ( e )</td>
</tr>
</tbody>
</table>
5.2 Proposed algorithm

Figure 5.4: Procedure to obtain propagation directions in the integration points.

Step 1: Path propagation directions in the integration points

The first step in the procedure of Table 5.1 is to determine the propagation direction $\varphi_{ip;j}$ in each integration point $j$, $j = 1, \ldots, n_{ip}$, of the element. This direction and its corresponding unit vector $t_j$ are obtained from the start vector $s$ and the maximum principal strains $\varepsilon_{1;j}$, according to Figure 5.4.

Suppose a certain strain state in point $j$. Its maximum principal strain direction coincides with the direction of the potential crack normal $n_j$. The corresponding potential crack normal angle $\theta_j$ in the global $xy$-coordinate system is depicted in Figure 5.4(a). Using the four-quadrant inverse tangent function, $\theta_j$ is equal to $\text{atan2}(n_{j;y}, n_{j;x})$ with a possible range of $[-\pi, \pi]$. Herein, $n_{j;x}$ and $n_{j;y}$ are the coordinates of the normal $n_j$.

Subsequently, assuming the propagation direction orthogonal to $n_j$, two possible propagation vectors exist in the considered integration point. These normalized vectors are called $t^1$ and $t^2$, see Figure 5.4(b), and can be computed with

$$ t^k = R^k n_j, \text{ for } k = 1, 2 \quad (5.1) $$

This results in $t^1 = -t^2$, since $R^1$ and $R^2$ are the standard rotation matrices. The corresponding propagation directions are $\varphi_{ip;j}^1$ and $\varphi_{ip;j}^2$, which are wrapped to the range $[0, 2\pi]$. 

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5. A DELAYED $C^1-$ CONTINUOUS CRACK PROPAGATION ALGORITHM

The final propagation vector $t_j$ in the considered integration point is now selected from $t^1$ and $t^2$, where $t_j$ is assumed to be the one that makes the smallest angle with $s$. The normalized start vector $s$ is defined by its corresponding propagation direction $\varphi_{sp}$ in the global $xy$-coordinate system, within the range of $[0, 2\pi]$. Figure 5.4(c) shows an example where $s$ is located at $(-0.5, -1.0)$ in isoparametric coordinates $(\xi_{sp}, \eta_{sp})$ and $\varphi_{sp}$ is equal to $\pi/1.5$. This results in $t_j = t^f$ and $\varphi_{ip,j} = \varphi^f_{ip,j}$. The propagation direction angle $\varphi_{ip,j}$ is in the global $xy$-coordinate system.

Note that due to above mentioned selection method the maximum absolute difference between $t_j$ and $s$ is implicitly limited to $\pi/2$. Note further that for the next step in the procedure it is important that $\varphi_{ip,j}$ mod $2\pi$ should be wrapped within the range $[\varphi_{sp} - \pi/2, \varphi_{sp} + \pi/2]$.

Step 2: Path propagation field

The second step in the procedure of Table 5.1 is the determination of the path propagation field $\varphi(\xi, \eta)$ within an element. Before proceeding to a detailed description, two important algorithmic aspects are briefly introduced.

The first algorithmic aspect concerns the choice for the type of propagation field, which subsequently determines the final shape of the path. When adopting constant propagation fields within elements, e.g. with $\varphi(\xi, \eta) = \varphi_{qp,average}$ and $\varphi_{qp,average}$ as a local measure [29] or even as a nonlocal measure [149], the path will propagate by straight line segments. In the present thesis, however, linear propagation fields are considered in order to obtain curved crack and damage paths. The linear propagation field within an element is defined by standard (bi)linear shape functions and their corresponding propagation directions in the corner nodes. Hence, the main task is to determine these unknown propagation directions in the corner nodes.

The second algorithmic aspect concerns the calculation of the unknown corner nodal propagation directions. In the general case the input for the linear propagation field is provided by the propagation directions at the integration points and the starting point. The unknown propagation directions at the corner nodes are estimated by extrapolation from this input. It requires a regression technique, since the number of corner nodes $n_{cn}$ (representing the unknown values) will generally be lower than the number of integration points plus the starting point (representing the known values). Therefore, the corner nodal propagation directions are approximated by using a weighted linear least squares method. The weights are used to include the algorithmic feature of $C^1-$continuity and to serve the concept of delayed crack path fixation.
The estimated linearized propagation direction field \( \hat{\varphi}(\xi, \eta) \) is represented in isoparametric coordinates, with scalar values for the propagation directions in the global xy-coordinate system. It is calculated by

\[
\hat{\varphi}(\xi, \eta) = N(\xi, \eta) \hat{\varphi}_{cn}
\]  

(5.2)

where \( \hat{\varphi}_{cn} \) contains the unknown propagation directions at the corner nodes and \( N(\xi, \eta) \) contains their corresponding evaluated standard (bi)linear shape functions at each considered isoparametric point in the element. Note that any midnodes in case of quadratic elements do not have to be considered. Subsequently, \( \hat{\varphi}_{cn} \) is computed with a weighted linear least squares approximation via

\[
\hat{\varphi}_{cn} = (N_{imp}^T W N_{imp})^{-1} N_{imp}^T W \varphi
\]

(5.3)

Herein, \( \varphi \) contains the input propagation directions and is expressed by the \((1 + n_{ip} + n_{cn}) \times 1\) vector

\[
\varphi = \begin{bmatrix} \varphi_{sp}^T \varphi_{ip}^T \varphi_{cn}^T \end{bmatrix}^T = \begin{bmatrix} \varphi_{sp} \{ \varphi_{ip,1} \cdots \varphi_{ip,n_{ip}} \} ; \{ \varphi_{cn,1} \cdots \varphi_{cn,n_{cn}} \} \end{bmatrix}^T
\]

(5.4)

where \( \varphi_{sp} \) is the propagation direction in the starting point and \( \varphi_{ip} \) represents the collection of propagation directions of all integration points in the considered element, according to Step 1. Although not mentioned before, \( \varphi \) also contains input propagation directions \( \varphi_{cn,i} \) from the corner nodes \( i, i = 1, \ldots, n_{cn} \). Each \( \varphi_{cn,i} \) and its corresponding unit vector \( r_i \) is set to \( \varphi_{sp} \) and \( s \) respectively. The corner nodal contribution \( \varphi_{cn} \) is added for regularization of the propagation field, and may influence the unknown \( \hat{\varphi}_{cn} \). This will be further discussed in the following, when the selection of the weight matrix \( W \) is addressed. The added regularization thus means that the number of knowns in Equation (5.3) increases to \( 1 + n_{ip} + n_{cn} \). For triangular elements the number of corner nodes \( n_{cn} = 3 \), for quadrilateral elements \( n_{cn} = 4 \).

Note that the linear propagation fields are actually based on the strain fields of the elements, mainly through the input propagation directions at the integration points \( \varphi_{ip} \) that are directly coupled to the maximum principal strain directions. Hence, a linear propagation field is most in line with those types of elements that also have a linear strain field. For that reason, the linear propagation fields and their corresponding curved crack paths are naturally best suited to quadratic elements.
Equation (5.3) further uses the $(1 + n_{ip} + n_{cn}) \times n_{cn}$ matrix $N_{inp}$, which is equal to

$$N_{inp} = \begin{bmatrix} N_{sp} \\ N_{ip} \\ N_{cn} \end{bmatrix}$$

(5.5)

Herein, the coefficient matrices $N_{sp}$ and $N_{ip}$ contain the evaluated standard (bi)linear shape functions $N_i$ at the starting point and integration points in isoparametric coordinates, being

$$N_{sp} = \{ N_1(\xi_{sp}, \eta_{sp}) \ldots N_{n_{cn}}(\xi_{sp}, \eta_{sp}) \}$$

(5.6)

and

$$N_{ip} = \begin{bmatrix} N_1(\xi_{1}, \eta_{1}) \ldots N_{n_{cn}}(\xi_{1}, \eta_{1}) \\ \vdots & \ddots & \vdots \\ N_1(\xi_{n_{ip}}, \eta_{n_{ip}}) \ldots N_{n_{cn}}(\xi_{n_{ip}}, \eta_{n_{ip}}) \end{bmatrix}$$

(5.7)

The coefficient matrix $N_{cn}$, with evaluated standard (bi)linear shape functions at the corner nodes, is equal to the $n_{cn} \times n_{cn}$ identity matrix $I$.

Finally, Equation (5.3) uses the diagonal $(1 + n_{ip} + n_{cn}) \times (1 + n_{ip} + n_{cn})$ weight matrix $W = diag(w)$, which contains the following diagonal elements:

$$w = \{ w_{sp}, w_{ip}; w_{cn}^T \}^T$$

(5.8)

$$= \{ w_{sp}, \{ w_{ip;1} \ldots w_{ip;n_{ip}} \}, \{ w_{cn;1} \ldots w_{cn;n_{cn}} \} \}^T$$

The different weights are shown in Table 5.1 and will be addressed one by one in the sequel.

Regarding the weight factors at each integration point, they are simply equated with their damage values in terms of relative stiffness reduction by $w_{ip,j} = d_j$. In the example of Table 5.1 the damage is visualized by the colors in the propagation vectors $t_1 \ldots t_9$, according to the color map in the second picture. Considering that more damage in a Gauss point tends towards a more pronounced crack propagation direction in that point, the damage-dependent weights $w_{ip,j}$ include the principle that the influence of a Gauss point on the final propagation field within the element is proportionally related to its damage. This principle makes especially sense for higher order elements, where the strain and damage state per integration point in an element can vary significantly. A minimal influence of the integration points is maintained by adopting an arbitrary $w_{ip,j}$ of $10^{-3}$. The maximum value for $w_{ip,j}$ is generally equal to the threshold damage value $d_{crit,crk}$ (see the introduction of Section 5.2).

Regarding the weight factor belonging to the propagation direction at the starting point, its value is important for the smoothness of the path at the
5.2 Proposed algorithm

Aiming for $C^1$-continuity at the entry point, being one of the ingredients to avoid U-turns (see Section 5.1), $w_{sp}$ is set to the sum of the weights of all integration points:

$$w_{sp} = \sum_{j=1}^{n_{ip}} w_{ip;j} \quad (5.9)$$

Although other expressions for $w_{sp}$ are equally possible, Equation (5.9) always ensures a relatively high weight factor at the starting point for the different element types and integration schemes that can be adopted. Since the values of the weights only have a relative meaning in a weighted least squares approximation, the used definition will generally be sufficient to impose $C^1$-continuity. Note that when $w_{sp}$ is set to 0, $C^0$-continuous curved propagation paths are obtained.

Contributions of only the integration points and the starting point to establish the propagation field in an element are generally sufficient, but can lead to problems in some cases with delayed crack path fixation ($d_{\text{crit,crk}} > 0$). The problems are typically observed for higher order elements with only a few damaged Gauss points ($d_j > 0$), which are usually located near the starting point. Although the $\bar{\varphi}_{ip;j}$ of these damaged Gauss points are more weighted compared to the $\bar{\varphi}_{ip;j}$ of the remaining undamaged Gauss points ($d_j = 0$, so $w_{ip;j} = 10^{-3}$), the undamaged Gauss points can exert a significant influence on the final propagation field if the number of damaged Gauss points is low. This is seen as unwanted. Moreover, it can lead to spurious tortuosity of the propagation path within an element and across several elements, with subsequent locking effects.

To attenuate the sensitivity to spurious tortuosity of path propagation the corner nodal contribution $\bar{\varphi}_{cn}$ and its weight $w_{cn}$ are added in the equations. As mentioned at the begin of Step 2, $\bar{\varphi}_{cn;i}$ is set to $\bar{\varphi}_{sp}$. The sum of the weights $w_{cn;i}$ is set to an arbitrary small fraction of 1% of the total sum of weights at the integration points $w_{ip;j}$ and starting point $w_{sp}$. Using Equation (5.9), this results in

$$w_{cn;i} = 0.01 \frac{2w_{sp}}{n_{cn}} \quad (5.10)$$

The regularization via Equation (5.10) can be illustrated by means of two examples. Firstly, suppose an element as in Table 5.1 with $d_{1-9} = 0.5$. Using the previously defined weight factors, the ratio $w_{sp} : \sum w_{ip,1-9} : \sum w_{cn,1-4}$ is equal to 4.5 : 4.5 : 0.09. Note that the same ratios are obtained for any other damage value $d_{1-9}$. Secondly, suppose an element as in Table 5.1 with $d_{1-3} = 0.5$ and $d_{4-9} = 0$. Now, the ratio $w_{sp} : \sum w_{ip,1-3}$ (damaged points)
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: $\sum w_{ip;4-9}$ (undamaged points): $\sum w_{cn;1-4}$ is equal to 1.506 : 1.5 : 0.006 : 0.03012. In the first example the weight in the corner nodes is relatively small compared to the weight in the Gauss points and hence the effect of regularization is negligible. In the second example the weight in the corner nodes is relatively large compared to the weight of the undamaged Gauss points. Consequently, the influence of the undamaged Gauss points on the propagation field is corrected by the propagation direction with which the path enters the element via $\varphi_{cn;1-4} = \varphi_{sp}$.

After determination of the estimated propagation directions $\varphi_{cn}$ in the corner nodes from Equation (5.3), the linearized propagation direction field $\varphi (\xi, \eta)$ from Equation (5.2) can be calculated. The corresponding normalized propagation vector field $\hat{T}(\xi, \eta)$ within the element equals

$$\hat{T}(\xi, \eta) = \{\cos (\varphi (\xi, \eta)), \sin (\varphi (\xi, \eta))\}$$  (5.11)

and is visualized by the field of small black arrows in the example of Table 5.1.

For crack path determination, in all integration points of the considered element the (potential) crack normal angles $\theta_j$ from Step 1 are aligned with the propagation vector field and fixed for the rest of the analysis. In this way, consistency between the local (potential) crack normal angles and the consolidated crack path propagation field within the element is ensured.

So far, the procedure for the general case of path propagation is discussed. In case of path initiation in a root element (see SubSection 5.2.1) the calculation of the propagation field slightly differs. Due to the absence of an entering crack or damage path, the contributions of the starting point and the corner nodes in Equations (5.4), (5.5) and (5.8) are disabled by setting $w_{sp}$ to 0. So, the propagation field $\hat{T}(\xi, \eta)$ in case of path initiation is based on the Gaussian input only. Note that the root elements still need a start vector $s$, as shown in Figure 5.3, in order to determine $\varphi_{sp}$ according to Step 1.

Step 3: Crack or damage path

The last step in the procedure of Table 5.1 concerns the derivation of the crack or damage path through the propagation vector field $\hat{T}(\xi, \eta)$. This task is analogous to finding the streamline of a particle in a two-dimensional steady flow field. The motion of a single particle is obtained by a stepwise movement, based on the explicit forward Euler method. Steps are taken with relatively small sizes compared to the element dimensions. The 'particle tracking' starts
in the point with coordinates $(\xi_{sp}, \eta_{sp})$, indicated by the symbol ‘•’ in the third picture of Table 5.1. Connection of all evaluated points in the unit vector field $\mathbf{T}(\xi, \eta)$ results in a smooth curved crack or damage path line. When the curved crack or damage path line intersects with one of the element edges ‘particle tracking’ stops. The point of intersection is called exit point, indicated by the symbol ‘◦’, and has the coordinates $(\xi_{ep}, \eta_{ep})$. Evaluation of $\hat{\varphi}(\xi_{ep}, \eta_{ep})$ gives the propagation direction in the exit point and its corresponding unit vector $e$. The value of this propagation direction is wrapped, if necessary, to the range $[0, 2\pi]$. In case of a crack path, $\hat{\varphi}(\xi_{ep}, \eta_{ep})$ is fixed and stored for the rest of the analysis. It will be used as $\varphi_{sp}$ in the element on the other side of the intersected edge, once a crack or damage path needs to be determined there.

5.2.3 Delayed crack path fixation

Delayed crack path fixation allows possible stress and strain redistributions in front of the consolidated crack path tip, prior to fixing the path. Naturally, the stress and strain redistributions in this fracture process zone are accompanied with progressive damaging. In the proposed algorithm the idea is to include delayed crack path fixation, entirely in line with the principles of the CPA, by preliminary and variable crack paths. These so-called damage paths start from the consolidated crack path tips and are updated in every step. Hence, they can experience progressive damaging (i.e. growing or further localizing) and reorientation until a certain fixation criterion is exceeded. After that, the damage path (or part of it) is consolidated for the rest of the analysis.

As explained in the introduction of Section 5.2, the proposed algorithm consolidates the damage path in an element when the damage level, in terms of relative stiffness reduction, in one integration point of the element exceeds the damage threshold $d_{crit,crk}$. Here, $d_{crit,crk}$ is a user-defined parameter between 0 and 1. Consequently, crack path fixation is only delayed when $d_{crit,crk}$ is set unequal to zero. When $d_{crit,crk}$ is set to zero, the path in the considered element is consolidated immediately once the element becomes critical for the first time in the SLA procedure (see SubSection 5.2.4). In this undelayed crack path fixation the predictive and variable aspects of the damage paths do not exist.

The adopted fixation criterion is simple and somewhat arbitrarily chosen. Other fixation criteria are also possible, for instance a criterion that is related to a minimum number of damaged integration points or a minimum release of fracture energy in an element. Another kind of criterion would be to consolidate
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after a certain time (i.e. load increment or number of steps) if the path in an

element hardly changes anymore.

Most of the existing crack tracking or propagation algorithms do not in-
clude delayed crack path fixation. Often, the crack path within an element
is consolidated immediately (i.e. undelayed) once the local failure criterion
has been fulfilled, e.g. [29, 102]. It assumes that the crack path prediction
in an initial stage of the fracture process is accurate enough and can be used
throughout the rest of the analysis. However, in analogous settings without
tracking strategies postponement of crack fixation has been applied previously.
E.g. models have been proposed that provide a transition from continuous
to discontinuous fracture, see SubSection 2.2.2. These models use different
transition criteria, for instance: exceeding a certain crack width in the de-
layed embedded crack model [82, 143], reaching a remaining yield stress equal
to a percentage of the initial yield stress in a model that combines Perzyna
viscoplasticity with X-FEM [151], and exceeding a critical damage value in
models that combine nonlocal or gradient-enhanced isotropic damage with X-
FEM [30, 133]. The issue is also touched in combined rotating-fixed smeared

crack models [114] and embedded crack models with crack adaptation [129],
where the smeared or embedded crack is allowed to rotate with the principal
tensile stress direction until a certain crack width is reached, after which the
orientation is fixed.

5.2.4 Path propagation with the SLA framework

The global solution procedure of the SLA method is discussed in SubSec-
tion 2.4.1. The proposed CPA blends well with SLA. After strain and stress
evaluation, but prior to selecting the critical integration point, the damage
paths will be recalculated as continuations of the fixed crack paths. Subse-
quently, the critical integration point is only sought in potential root elements,
elements along crack and damage paths, and in crack and damage tip elements
(see Figure 5.2). Hence, the notion of events by damage increments from stan-
dard SLA are coupled in an elegant way to events as initiation, localization or
propagation of a crack or damage path.

5.3 Validation on (plain concrete) fracture tests

This section validates the proposed CPA and investigates whether the mesh-
induced directional bias in a classical smeared crack model can be reduced or
eliminated. The validation study consists of three fracture tests with different
failure behaviors:
5.3 Validation on (plain concrete) fracture tests

1. the test with periodic boundary conditions with pure mode I fracture from Section 3.3;
2. the four-point shear test of a plain concrete single-edge-notched (SEN) beam with combined mode I and mode II fracture from SubSection 4.3.2;
3. the plain concrete double-edge-notched (DEN) specimen with again mixed-mode fracture from SubSection 4.3.3.

The study is carried out for linear and quadratic plane stress elements. All tests are analyzed with the following meshes:

- structured mesh of linear quadrilaterals with $2 \times 2$ Gauss integration and without selective reduced integration for the shear terms;
- structured mesh of quadratic quadrilaterals with $3 \times 3$ Gauss integration;
- un- or structured mesh of linear triangular elements (constant strain triangles) with 3-point Gauss integration;
- un- or structured mesh of quadratic triangular elements with 3-point Gauss integration.

The numerical simulations are performed with and without the proposed CPA, resulting in eight analyses for each of the fracture tests. Note that 3-point Gauss integration for the constant strain triangles is used in order to calculate the propagation direction fields in these elements.

In anticipation on the results, it appears that for the three aforementioned fracture tests undelayed crack path fixation can be used satisfactorily. This is further demonstrated for only the four-point shear test in SubSection 5.4.2. Hence, for all analyses with the CPA in this mesh objectivity study $d_{\text{crit,crk}}$ is set to 0.

As in the previous chapters, the material behavior is modeled with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, see SubSection 2.4.3. Unless otherwise specified, the crack band widths are estimated according to the element projection based formulation of Govindjee et al. (SubSection 2.3.2). Furthermore, the results of all analyses are assessed by considering global load - displacement (difference) responses and crack patterns. The crack patterns are plotted in undeformed meshes, using three different line widths for crack widths in the ranges $(0,\frac{1}{2}w_{\text{ult}})$, $[\frac{1}{2}w_{\text{ult}}, w_{\text{ult}})$, $[w_{\text{ult}}, \rightarrow)$.

5.3.1 Test with periodic boundary conditions

The first “experiment” in the mesh objectivity study is the specially designed numerical test that was proposed in Section 3.1. Details of this test and the
adopted material properties for the analyses are provided in Figure 3.9 and Section 3.3. The “specimen” is loaded by a uni-axial tensile load in x-direction, triggering vertical cracking (mode I fracture). According to the idea of the test, each mesh of the “specimen” is designed in such a way that the characteristics of all finite elements, i.e. shape, size, orientation, interpolation function and numerical integration scheme, are identical. Due to this imposed mesh uniformity the model boundaries become tooth-shaped for mesh orientations \( \theta \) unequal to 0 and \( \pi/2 \). Aiming for an initial homogeneous strain field in such “specimens”, periodic boundary conditions between the opposite boundaries are applied. In this subsection the analyses are limited to meshes with an \( a \)-value of 2 (different from Figure 3.9, where \( a = 4 \)), corresponding to \( \theta = 0.46365 \) rad for the quadrilaterals and \( \theta = 0.32175 \) rad for the triangular elements in Figure 3.11. It results in periodic length scales \( L_x \) and \( L_y \) of 1006 mm and 447 mm respectively. Furthermore, the used plane stress elements have sides of 50 mm. To stimulate strain localizations at the same location an imperfection is applied in the mid bottom element of the model by reducing the tensile strength \( f_t \) with 1%.

Figures 5.5(a) and 5.5(b) present the load - total inelastic displacement curves of aforementioned eight analyses without and with the proposed CPA respectively. Generally, the curves are of a spiky shape, which is typical for the SLA method due to its saw-tooth softening input. The theoretically expected solution is represented by the black line. It can be observed that the four curves of analyses with quadratic elements are quite consistent and show good agreement with the theoretical solution. Addition of the proposed algorithm slightly changes the global responses. Differences between quadrilateral elements and triangular elements remain negligible. For the analyses using elements with linear interpolation less consistency is observed. The red and orange curves belonging to the analyses without the CPA show more ductile post-peak behaviors compared to the theoretical solution and the analyses that use quadratic elements, with residual plateaus at 11.5 kN and 14.5 kN respectively. In case the CPA is adopted, the red curve ‘linear, 2 × 2 Gauss, □’ comes close to the theoretical solution, while the orange curve ‘linear, 3-point intgr., △’ still shows a residual plateau at about 13.5 kN for total inelastic displacements \( u_{cr} \) beyond \( w_{ult} \).

Figure 5.6 shows the corresponding crack width plots at \( u_{cr} = w_{ult} \) for the analyses from Figure 5.5. The shaded elements in the plots are the elements with 1% lower \( f_t \). Analyses without the proposed CPA show significant spurious cracking and the global cracks are aligned to the inclined mesh lines rather than being vertical. In contrast, analyses including the proposed CPA reveal vertical
5.3 Validation on (plain concrete) fracture tests

Figure 5.5: Load - total inelastic displacement responses of analyses on the test with periodic boundary conditions.

Figure 5.6: Crack width plots at $u_{cr} = w_{ult}$ belonging to the analyses on the test with periodic boundary conditions from Figure 5.5. The thickest lines correspond to $w \geq w_{ult}$. 

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cracking, both locally in the integration points and globally across the FE mesh. This observation is irrespective of the element shape and interpolation order.

Foregoing observations clearly indicate that the proposed CPA in this problem eliminates the mesh-induced directional bias. The simulations with linear and quadratic elements show improvement of both the load - total inelastic displacement responses and crack patterns. Since the remaining $F_t$ at $u_{cr} = u_{ah}$ for three of the four analyses with the CPA in Figure 5.5(b) are close to zero, it can be concluded that the observed higher residual plateaus of the corresponding analyses without the CPA in Figure 5.5(a) are mainly caused by the mesh alignment of the global crack. The area of an aligned global crack is larger compared to a vertical global crack, and hence there are more integration points that dissipate energy. The exception to this is the analysis with the CPA belonging to the constant strain triangles. Its residual plateau at about 13.5 kN can be attributed to the occurrence of stress locking [81, 120, 122], being “spurious stress transfer across a widely open crack” according to Jiřísek and Zimmermann [81]. Such stress locking is an intrinsic deficiency of smeared crack models. It generally occurs by incorrect utilization of shear stiffness along the crack and tangential stiffness parallel to the crack (compressive strut action), due to the kinematic description of this crack with a continuous displacement field. Note that in the current case the adopted variable shear retention relation, which gradually reduces the shear stiffness to zero with an opening crack, prevents a contribution of shear stiffness to stress locking for fully localized cracks. So, the tangential stiffness parallel to the crack must be responsible for it.

5.3.2 Four-point shear test

The second experiment in the mesh objectivity study is a four-point shear test on a plain concrete SEN beam, tested by Schlangen [132]. Details of this test and the adopted material properties for the analyses are provided in Figure 4.6 and SubSection 4.3.2. The FE models use plane stress elements with sides of 5 mm. Analyses on meshes consisting of triangular elements use the crack band width formulation according to Govindjee et al., whereas in case of quadrilaterals the enhanced formulation from Section 4.1 is adopted. Note that the results of the two analyses with quadrilaterals and without CPA were also presented in Figures 4.7(b) and 4.8(b).

Figure 5.7 shows the global responses of the analyses without and with the proposed CPA, together with the range of experimentally obtained responses (gray shaded areas), in terms of load and crack mouth sliding displacements (CMSD). The CMSD is measured as the displacement difference in vertical
5.3 Validation on (plain concrete) fracture tests

Figure 5.7: Load - CMSD responses of analyses on the four-point shear test.

Figure 5.8: Crack width plots at a CMSD of 0.1 mm belonging to the analyses on the four-point shear test from Figure 5.7. The thickest lines correspond to $w \geq w_{ult}$. 
direction between the points A and A', indicated in Figure 4.6. Generally, all curves reveal higher failure loads compared to the experimental results, which may be the reason that some other researchers use a lower tensile strength as was reported [19, 36, 150]. The four curves belonging to analyses with quadratic elements show all reasonable agreement with the experiment. For the analyses with linear elements the results are less consistent. The obtained curve 'linear, $2 \times 2$ Gauss, □' from the analysis without CPA exhibits rather brittle post-peak behavior, while the two orange curves belonging to the constant strain triangles show relatively high peak loads and more ductile post-peak behaviors. Furthermore, the curves 'linear, $2 \times 2$ Gauss, □' and 'linear, 3-point intgr., △' of the analyses with CPA reveal an increasing resistance after the load drop.

Figure 5.8 depicts the crack width plots at a CMSD of 0.1 mm for the analyses above. From the comparison between both sets of four plots the following two issues can be observed. Firstly, all plots from analyses with the proposed CPA show good agreement with the experimentally obtained curved failure crack. For analyses without the proposed CPA only the plot from quadratic triangular elements is comparable to the experimental result. The other plots without CPA show erroneous crack patterns. For instance, the result for linear quadrilaterals reveals a straight failure crack that follows the mesh lines, even though the local cracks in the integration points are under an angle, and this explains the too brittle load - CMSD response. Secondly, the plots without CPA reveal significant spurious cracking around the failure crack, whereas plots of analyses with the CPA show sharp localization.

From aforementioned observations it is concluded that the proposed CPA improves the mesh objectivity of the cracking behavior. However, for the load - CMSD response no significant improvement of mesh independence is observed. The two curves belonging to the linear elements with CPA reveal the occurrence of a locking mechanism. A closer look to the results of these analyses shows that in both cases the last part of the failure crack is tortuous. Subsequently, further crack propagation to the bottom of the beam becomes difficult and a significant force transmission across the undamaged zone remains.

5.3.3 Double-edge-notched specimen

The last experiment in the mesh objectivity study is a plain concrete DEN specimen, tested by Nooru-Mohamed [99]. Details of this test and the adopted material properties for the analyses are provided in Figure 4.9 and SubSection 4.3.3. Preliminary analyses with crack tracking on the tests with load paths '4a', '4b' and '4c' were conducted by the author in Reference [136]. Here, the tests with load path '4b' are only considered. The numerical models
are meshed by plane stress elements with sides of 5 mm. Note that the results of the two analyses with quadrilaterals and without CPA were also presented in Figures 4.10(a) and 4.11(a).

Figure 5.9 presents the graphs with the $P - \delta_{\text{average}}$ responses of the analyses without and with the proposed CPA. The $\delta_{\text{average}}$ is the mean value of the two measured $\delta$'s shown in Figure 4.9. Experimentally obtained responses are indicated by the gray shaded areas. It can be seen that all numerical results significantly overestimate the peak values from experimental results. The reason for this overestimation is explained in SubSection 4.3.3. Furthermore, the curves belonging to analyses with the proposed CPA reveal generally more ductile behavior in the post-peak regime and consequently higher residual plateaus compared to the analyses without the proposed CPA. Nevertheless, the global responses of the four analyses with quadratic elements are reasonably consistent, while the curves belonging to linear elements show more diversity. Specifically the red curve of the analysis ‘linear, 2 $\times$ 2 Gauss, □’ without the CPA exhibits a remarkable response. The curve intersects with the horizontal axis at $\delta_{\text{average}} = 0.054$ mm and subsequently continues with negative $P$-values by a further increasing $\delta_{\text{average}}$ (not shown in the graph).

The corresponding crack width plots at $\delta_{\text{average}} = 0.1$ mm of these eight analyses are given in Figure 5.10. Generally, all plots are point symmetric with respect to the specimen’s midpoint. Now, the positive effect of the CPA is remarkable. Crack width plots without CPA reveal strong sensitivity to mesh alignment, i.e. cracks prefer to propagate along continuous mesh lines. This mesh alignment is observed especially for the analysis ‘linear, 2 $\times$ 2 Gauss, □’, as for the four-point shear test in SubSection 5.3.2. The local cracks in the integration points are all under an angle, but the two global failure cracks propagate horizontally from both notches towards the center of the specimen. Furthermore, without CPA a lot of spurious cracks between the two curved failure cracks can be observed. It should be noted that these cracks have hardly dissipated energy. With CPA, crack width plots are mesh independent and clearly show two separate curved cracks. All these four plots are in good agreement with each other and with the experimentally obtained failure cracks.

Foregoing results illustrate that addition of the proposed CPA in the numerical simulations on this DEN specimen leads to mesh objective cracking behavior, irrespective of the element order. The differences in heights of residual plateaus in the global responses between analyses without and with the proposed algorithm are related to the differences in curvatures of the failure cracks. Analyses with CPA show more curved failure cracks than analyses
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Figure 5.9: $P - \delta_{\text{average}}$ responses of analyses on the DEN specimen.

Figure 5.10: Crack width plots at a $\delta_{\text{average}}$ of 0.1 mm belonging to the analyses on the DEN specimen from Figure 5.9. The thickest lines correspond to $w \geq w_{\text{ult}}$. 
5.4 The moment of crack path fixation

without CPA, as was also observed for the four-point shear test in SubSection 5.3.2. On the one hand, these more curved failure cracks increase the structural resistance. On the other hand, these more curved failure cracks result in deformation modes at element level that are less suitable to capture properly with the used element formulations, which increases the effect of stress locking. Especially the global responses of the linear elements with CPA show relatively large $P$-values at $\delta_{\text{average}} = 0.1$ mm. Since the increase of the structural resistance is the same for all four analyses, this indicates that linear elements are less capable to handle the force transmission along curved and non-aligned cracks compared to quadratic elements.

5.4 The moment of crack path fixation

At the beginning of Section 5.2 and in SubSection 5.2.3 it was explained that the proposed CPA includes the possibility of postponing the moment of crack path fixation to a later stage in the fracture process. This section demonstrates the concept of delayed crack path fixation by means of two fracture tests:

1. the three-point bending test of a plain concrete notched beam with mode I fracture from SubSection 4.3.1;
2. the four-point shear test of a plain concrete single-edge-notched (SEN) beam with mixed-mode fracture from SubSections 4.3.2 and 5.3.2.

Undelayed and delayed crack path fixation are demonstrated for each of the fracture test by setting the $d_{\text{crit,crk}}$ values equal to 0.00 and 0.60 respectively.

The material behavior in these four analyses is modeled with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, as in Section 5.3. The results of the analyses are assessed by considering the global load - displacement (difference) responses and plots with crack and damage paths.

5.4.1 Three-point bending test

The first experiment is a plain concrete notched beam subjected to three-point bending, tested by Körneling and Reinhardt [88]. Details of this test and the adopted material properties for the analyses are provided in Figure 4.3 and SubSection 4.3.1. The beam is analyzed by a symmetric and structured mesh, using 5 mm by 5 mm square linear quadrilateral plane stress elements in the critical region of the beam. The elements adopt $2 \times 2$ Gauss integration, with selective reduced integration for the shear terms (constant shear). Crack
band widths in the integration points are estimated according to the element projection based formulation of Govindjee et al., see SubSection 2.3.2.

Figure 5.11 shows the adopted (saw-tooth) stress-strain curve with Hordijk softening and 20 damage increments (gray lines), assuming a crack band width of 5 mm. The corresponding evolution of the damage in terms of relative stiffness reduction is plotted in the same graph by the black line. Interesting to note is the interpretation of the damage threshold value \( d_{\text{crit,crk}} \) in terms of energy dissipation and stiffness change. Figure 5.11 helps to comprehend that a threshold of 0.60 corresponds to a dissipated energy of just about 0.9% from the total amount of energy, while only 40% of the original secant stiffness \( E_0 \) is left. Taking for instance a threshold of 0.95, the dissipated energy is about 9% and only 5% of the original secant stiffness \( E_0 \) is left. So, postponement of crack or damage path determination by using relatively low threshold damage values implies negligible small energy dissipation, but significant local stiffness changes. The local stiffness changes may lead to local changes in strains and stresses as well. Note further that the amount of energy dissipation prior to fixation belonging to \( d_{\text{crit,crk}} = 0.60 \) is in line with the suggestion of Sancho et al. [129] for the used threshold in their embedded crack model with crack adaptation.

Figure 5.12 shows the resulting load - deflection curves at midspan of the two analyses with linear quadrilaterals. The range of experimentally obtained load - deflection curves is indicated by the gray shaded area. Good agreement with these experimental results is observed for the analysis using \( d_{\text{crit,crk}} = \)
0.60. The analysis with $d_{\text{crit,crk}} = 0.00$ shows a strange plateau that completely neglects the structural softening behavior.

Figure 5.13 presents the evolution of the crack and damage paths for these two simulations by means of three plots at different deflections. The crack paths are indicated by red lines. For the damage paths, only present in the analysis with delayed crack path fixation, a distinction is made between damage paths of previous steps (light blue lines) and the damage paths of the current step (dark blue lines). Considering the evolution belonging to the analysis with undelayed crack path fixation, the crack path starts with a deviation to the left in the third element above the notch. Subsequently, the deviation becomes more pronounced in the fourth element above the notch and the path propagates to the column of elements on the left side. The last two plots reveal a “U-turn” of the crack path and a subsequent intersection with itself. Considering the evolution belonging to the analysis with delayed crack path fixation, from the first plot (just before the peak load) one can observe that in the fourth element above the notch initial predictions of the crack path showed a tendency to deviate from the vertical line. However, during a continuous damage process the strains redistribute and once the fixation criterion is reached the crack path within this element is vertical. The damage path belonging to the considered step in the first plot shows again a “U-turn”. Note that this damage path has already propagated over a few elements. All the elements that are crossed by the blue line are on the list of potential critical elements for this step and have already experienced nonlinear material behavior. From the second plot, at the peak load, it can be noticed that after some damage occurred in the fifth element above the notch the crack path propagates in the end again vertically. The third plot shows the continuous tendency to deviate from the vertical line, resulting in a pattern that reminds of the “tail wagging of a dog”. Nevertheless, the crack path remains more or less straight. At the considered step belonging to this plot no new damage path is determined.

Foregoing results demonstrate the concept of delayed crack path fixation. For this three-point bending test on a notched beam, the simulations clearly show a better performance when the moment of fixation is postponed. Although not presented, similar conclusions can be drawn from analyses with a different mesh, element type and $d_{\text{crit,crk}}$ value. The analysis with undelayed crack path fixation suffer from a “U-turn” of the crack path, which causes crack path completion by self-intersection and a subsequent locking of the system. This behavior explains the observed absence of structural softening behavior for the red curve in Figure 5.12. On the other hand, although initial path
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![Graph showing load-deflection responses with different $d_{\text{crit,crk}}$ values.]

**Figure 5.12:** Load - deflection responses of analyses with two different $d_{\text{crit,crk}}$ values on the three-point bending test.

(a) $d_{\text{crit,crk}} = 0.00$ (undelayed crack path fixation)

(b) $d_{\text{crit,crk}} = 0.60$ (delayed crack path fixation)

**Figure 5.13:** Crack (red lines) and damage (light and dark blue lines) path evolution belonging to the analyses on the three-point bending test from Figure 5.12. The plots are taken at deflections of 0.085 mm, 0.11 mm and 0.30 mm.
predictions in the analysis with delayed crack path fixation also show a continuous tendency to “U-turns”, it appears that postponement of fixation can be an adequate strategy to find the true crack path.

The plots with crack and damage paths depicted in Figure 5.13 provide a remarkable insight in the numerically obtained fracture process. Moreover, these plots reveal smoothly curved lines, indicating that $C^1$ continuity is obtained with the proposed CPA. The observed “wagging” behavior of the damage paths for a nonzero $d_{\text{crit,crk}}$ value can be interpreted as an indication that the stress and strain states at the consolidated crack path tip are considerably changing during the simulation of the fracture process. There might be two possible reasons that causes this “wagging” behavior and the corresponding tendency to “U-turns”.

Firstly, the complexity of the problem. Although the geometry, loading condition and experimentally obtained failure of the specimen look simple, from a numerical point of view the three-point bending test on a notched beam poses difficult issues. The path propagation is for instance complicated, since the stress state in the critical region above the notch involves biaxial tension. Moreover, this stress state is affected by the load on top of the beam and by (undamaged) elements around the main vertical crack that induce additional stresses due to confinement. These issues were also mentioned by Jirasek and Bauer [79]. Especially biaxial tension may cause “U-turns” in the crack and damage path predictions.

Secondly, the “wagging” behavior and the “U-turns” in the symmetric meshes of the three-point bending test may also be triggered by the adopted SLA procedure and its combination with the CPA. Finding non-symmetric modes is typically for the damage-driven SLA method, even for symmetric FE models [126, 135]. Nevertheless, the occurrence of “U-turns” has been noticed in analyses on non-symmetric and unstructured meshes as well. Cervera et al. [29] reported similar observations from standard incremental-iterative analyses of the three-point bending test, using unstructured meshes.

5.4.2 Four-point shear test

The second experiment is again the four-point shear test on a plain concrete SEN beam from SubSections 4.3.2 and 5.3.2. In this subsection the two analyses for $d_{\text{crit,crk}}$ values of 0.00 and 0.60 are performed on a structured mesh of quadratic quadrilaterals with $3 \times 3$ Gauss integration. Note that the results of the analysis with $d_{\text{crit,crk}} = 0.00$ were also presented in Figures 5.7(b) and 5.8(b).
Figure 5.14 shows the load - CMSD responses of the two analyses and the range of experimentally obtained responses (gray shaded areas). The response belonging to $d_{crit;crk} = 0.00$ is comparable to the experiment, while the curve belonging to $d_{crit;crk} = 0.60$ reveals a higher residual load and remarkable spiky behavior at the end.

Figure 5.15 presents the corresponding plots with crack and damage paths at a CMSD of 0.1 mm. As before, the crack paths are indicated by red lines, the damage paths of previous steps by light blue lines and the damage paths of the current step by dark blue lines. Generally, both plots reveal two clouds of cracks in the bending zones outside the critical region and one curved crack in the critical region starting from the notch. Last mentioned crack is for both cases similarly shaped as the experimentally obtained failure crack. Moreover, the lines of this crack are smoothly curved, indicating $C^1 −$ continuity. A closer look to the failure crack paths of both analyses reveals some interesting issues. Firstly, the moment of fixation influences the starting position of the failure crack around the notch detail. Secondly, the damage paths for $d_{crit;crk} = 0.60$ hardly show any “wagging” behavior (except in the last part, see the detail) and so the final crack path follows virtually the same track as the damage path. Thirdly, for $d_{crit;crk} = 0.60$ the crack path starts to meander in the last part and does not fully propagate to the bottom of the beam, as observed for $d_{crit;crk} = 0.00$.

From the foregoing analyses it is concluded that in case of the four-point shear test the undelayed or immediate crack path fixation provides the best results. This is the opposite of what was concluded from the three-point bending test in SubSection 5.4.1, where delayed crack path fixation was in favor. Here, the analysis with $d_{crit;crk} = 0.60$ shows that the crack path follows virtually the same track as the damage path, supporting the conclusion that for this problem the initial predictions are indeed accurate. Furthermore, delaying the moment of fixation seems not helpful when the crack path reaches the area above the bottom support. The stress state in that zone becomes rather complex, due to an increasing opening and propagation of the failure crack and the still existing reaction force from the support. Hence, the “wagging” behavior of damage paths occur and the crack path becomes tortuous. From the detail in Figure 5.15(b) it can be seen that the crack path in its last element propagates to the column of elements on the left side, making further propagation to the bottom of the beam difficult. The remaining undamaged elements in front of the crack path allow significant force transmission, which explains the substantial gap between the levels of the residual plateaus of the two analyses in Figure 5.14. Note that during the spiky behavior at the end of
5.4 The moment of crack path fixation

![Graph showing load vs CMSD responses with different critical crack depths](image)

**Figure 5.14:** Load - CMSD responses of analyses with two different $d_{\text{crit,crk}}$ values on the four-point shear test.

![Crack paths with different critical crack depths](image)

**Figure 5.15:** Crack (red lines) and damage (light and dark blue lines) paths belonging to the analyses on the four-point shear test from Figure 5.14. The plots are taken at a CMSD of 0.1 mm.

The blue curve in Figure 5.14 the second path from the notch in Figure 5.15(b) temporarily further localizes for a few steps.

Finally, the results show that multiple cracking is properly handled by the proposed CPA. Closely spaced crack paths initiate at three different spots before the peak load is reached. After the peak, the groups of crack paths at the two spots in the bending zones outside the critical region do not further localize and become inactive, whereas the curved crack path between the mid supports remains active and develops to a failure crack.
5.5 Summary and conclusions

This chapter presented a new local crack tracking technique for numerical simulations of localized fracture processes in quasi-brittle structures. The proposed technique, indicated by delayed $C^1$-continuous crack propagation algorithm (CPA), is combined with a fixed smeared crack model and is developed in the framework of the Sequentially Linear Analysis (SLA) method. The enhanced smeared crack model can be positioned between approaches in the standard smeared crack concept and E-FEM or X-FEM. Cracks are still modeled as smeared rather than discrete discontinuities, but they are forced to propagate along a single row of conventional continuum elements rather than potentially originating anywhere in the FE mesh. This strategy helps to overcome the influence of a mesh alignment on the propagation direction of a crack, as observed for the general case of crack growth simulation with the standard smeared crack concept in meshes that are not aligned. More fundamentally, associating the underlying reason for mesh-induced directional bias in deficiencies of the spatial discretization [27, 103], the crack propagation algorithm helps to ignore “undesired alternative solutions of the nonlinear discrete problem” and to select “the appropriate solution among the many possible ones” [27].

The proposed CPA constructs smoothly curved $C^1$-continuous crack paths element by element. Curved paths are obtained from the determination of linear propagation fields within the elements. $C^1$-continuity is obtained by taking explicitly into account the propagation direction with which the path enters an element and using a relatively high weight factor at this entry point. Validation of this CPA on three plain concrete tests with mode I and mixed-mode fracture leads to the following conclusions:

- Enhancement of the fixed smeared crack model by the proposed CPA leads generally to the best results for quadratic quadrilaterals and triangular elements. The load - displacement (difference) responses and failure crack patterns show mesh independence and good agreement with expected or experimentally obtained results. On the other hand, results of corresponding analyses without the proposed algorithm show that cracks prefer to propagate along continuous mesh lines. Hence, it is concluded that the proposed CPA has reduced significantly the mesh-induced directional bias.

- For linear quadrilaterals and triangular elements the addition of the proposed CPA has a similar positive effect on the mesh objectivity of the cracking behavior, as observed for quadratic elements. However, since these element formulations are less suitable to capture the deformation
modes at element level belonging to non-aligned and curved failure cracks, stress locking becomes pronounced, resulting generally in stiffer and more ductile load - displacement (difference) responses. Especially the constant strain triangles suffer from this phenomenon.

- The proposed CPA avoids spurious cracking and enables to obtain $C^1$ - continuous crack paths within FE discretizations. Furthermore, multiple cracking is properly handled.

Foregoing conclusions are based on analyses with undelayed or immediate crack path fixation. The proposed CPA also enables to delay crack path fixation, by making a distinction between damage paths and crack paths and by introducing a user-defined damage threshold $d_{\text{crit,crk}}$. From analyses on two plain concrete fracture tests it is shown that postponement of the moment of fixation can be an adequate strategy to find a more realistic response and crack path, but not necessary. Furthermore, tracking the crack and damage path evolution provides a new way to obtain insight in the numerically obtained fracture process, and more specifically in the stress and strain state development in front of a propagating crack path tip during the simulation.

Relevant aspects for the presented CPA that are not treated in this chapter, but that may affect aforementioned conclusions, are: isotropic versus orthotropic material modeling, rotating versus fixed smeared crack models, other criteria to define the moment of crack path fixation and the applicability for three-dimensional problems.

Finally, the delayed $C^1$ - continuous CPA does not affect the robustness of the SLA method and no significant additional computation costs are required.
5. A DELAYED $C^1$– CONTINUOUS CRACK PROPAGATION ALGORITHM
Chapter 6

Extension towards shear failure in reinforced concrete

The new proposed crack band width estimator and crack propagation algorithm were validated by simulations of plain concrete fracture tests in the Chapters 4 and 5. However, the engineering practice generally deals with reinforced concrete (RC) structures or structural members. Numerical analysis of RC poses challenging issues as the formation of multiple cracks, like flexural cracks, debonding cracks, shear cracks, which are propagating and possibly intersecting with other cracks and reinforcing bars. This chapter presents a study in which the aforementioned enhancements are used to analyze a shear critical RC beam without shear reinforcement. Section 6.1 describes the experiment and its test results. Section 6.2 discusses different strategies for the modeling of a RC shear beam, especially regarding its bond-slip behavior. Subsequently, exploratory analyses for the identification of material and mesh sensitivities are shown in Section 6.3, temporarily using simplifications as a crack band width formulation based on the element area and perfect bond. Comparisons with conventional nonlinear FE analysis are also made. The analyses with the proposed enhancements are presented in the Sections 6.4 and 6.5. Section 6.6 summarizes the main findings of this chapter.

6.1 Benchmark description and experimental results*

The selected shear critical RC beams, introduced in Section 1.3, took part of an experimental study to the shear capacity of concrete beams under sustained

*Based on a section in Reference [135].
loads. The experiment was conducted in the Stevin Laboratory at Delft University of Technology [130]. Beams were tested in long-term loading, ranging from 87% to 97.5% of the ultimate shear capacity. To obtain a high-confidence value of the ultimate capacity of the specimens, six static short term tests on identical shear critical RC beams were performed in advance (test series 1). The results of these six tests, denoted with S1B1 - S1B6, are used in this chapter.

Figure 6.1 shows the test setup, the relevant dimensions and a failure crack pattern of beam S1B5. The specimens were simply supported and loaded by a concentrated load at midspan, which led to a constant shear force and a bi-linear variation of the bending moments over the length of the beam. The beams had a thickness of 200 mm and contained one layer of three longitudinal ribbed reinforcing bars Ø 20. These bars were anchored on the beam ends with anchorage plates. Shear reinforcement was not present.

Figure 6.1: Geometry, experimental setup and failure crack pattern (S1B5) of the RC shear beam.

Figure 6.2(a) presents the experimentally obtained load - deflection responses of test series 1. The curves reveal good agreement, with just a small scatter in the ultimate load capacities. The beams failed by an average load of 184.7 kN and an average deflection at midspan of 3.57 mm. Five of the six beams revealed a similar failure process. Until the peak load only flexural cracks were visible, but then failure suddenly occurred in a split of a second by the appearance of a large diagonal crack, immediately followed by a collapse of the compression zone and a horizontal debonding crack that propagated just above the reinforcement toward one end of the beam. Figure 6.2(b) shows this sequence of fracture types at the moment of failure by three consecutive frames from a video footage of beam S1B6, with time steps of 0.03 s. After the occurrence of the ‘no warning’ failure, the beams did not show any post cracking capacities. These brittle failure modes can be recognized as flexural shear failures, sometimes called diagonal tension failures, e.g. Reference [153] and references herein.
6.2 Modeling strategies

This section treats several strategies for the modeling of the benchmark, making use of the smeared crack concept and the proposed enhancements from the previous chapters. Two relevant mechanisms are the shear transfer in the beam and the bond-slip behavior between concrete and reinforcement. SubSection 6.2.1 starts with a brief explanation of these mechanisms. Subsequently, the FE modeling of both mechanisms are discussed in SubSections 6.2.2 and 6.2.3. Finally, SubSection 6.2.4 presents three modeling strategies that will be used in the following sections of this chapter.

6.2.1 Mechanisms of shear transfer and bond-slip

This subsection briefly introduces shear transfer in cracked concrete and bond related cracks and failures, being two relevant mechanisms in a RC beam without shear reinforcement. Understanding of these mechanisms supports the interpretation of modeling assumptions and the interpretation of the results of numerical simulations on the benchmark in the Sections 6.3 to 6.5.
Shear transfer in cracked concrete

Shear transfer in a RC beam with flexural cracks can be attributed to several mechanisms. Generally, four types of mechanisms have been widely accepted [83, 153]: shear in the uncracked zone $V_c$, interface shear transfer along the crack $V_i$, residual tensile stresses across the crack $V_r$ and dowel action of the longitudinal reinforcing bars $V_d$. These four shear transfer mechanisms are illustrated in Figure 6.3.

Shear in the uncracked zone is transferred by the inclination of the principal stresses. The magnitude of this shear force depends on the depth of the compression zone. In a slender beam, this force has a minor contribution to the shear capacity, since the depth of the compression zone is relatively small.

Interface shear transfer or aggregate interlock arises from frictional forces, which are generated by relative tangential displacements between the surfaces with protruding aggregate particles in a crack, e.g. [147]. Relevant parameters for its contribution to the shear capacity are the tangential and normal crack displacements (or slip and crack opening respectively), and the roughness of the crack surfaces. The latter depends on the aggregates, the aggregate size, the compressive strength and the fracture mode. In case of normal strength concrete, the strength of the cement matrix is commonly lower than the strength of the aggregates, which leads to cracking through the cement matrix and subsequently to relatively rough crack surfaces. In case of high strength concrete and lightweight aggregate concrete, the strength of the cement matrix is commonly higher than the strength of the aggregates, which leads to cracking through the aggregate particles and subsequently to relatively smooth crack surfaces. The interface shear transfer decreases for decreasing slip, for increasing crack opening and for smoother crack surfaces.

Residual tensile stresses are transmitted across cracks, since cracks in concrete originate via a gradual process from micro-cracking to a stress-free macro-
crack rather than an immediate clean break, see SubSection 2.1.1. As long as the widths of flexural cracks remain small, this mechanism can contribute to the shear transfer in a beam.

*Dowel action* stands for the contribution of the reinforcing bars to the shear transfer. In case of transverse displacement differences along a crack that is crossed by reinforcing bars, the bars are subjected to bending and shear via their interaction with the surrounding concrete and will act as dowels. For RC beams without shear reinforcement, dowel action can occur at the longitudinal reinforcing bars. The maximum shear in the reinforcing bars at a crack in such beams is limited by the tensile strength of the concrete cover. Dowel action may be significant when large amounts of longitudinal reinforcement are applied, particularly when the longitudinal reinforcement is distributed in more than one layer [83].

**Bond related cracks and failures**

Bond behavior between concrete and ribbed reinforcing bars is a complex topic and discussed in many publications. Here, different types of bond related cracks and failure mechanisms are briefly explained. For detailed descriptions of bond behavior is referred to e.g. References [15, 50, 91, 120].

Three types of bond related cracks can be distinguished:

*Internal cracks*, or sometimes called *Goto cracks*, are small cone-shaped cracks that initiate from the ribs of the reinforcing bar and generally tend to propagate towards the nearest primary crack. Their angles with respect to the bar axis can vary from 45 to 80 degrees. Figure 6.4(a) shows two-dimensional representations of these cracks.

*Secondary cracks* are internal cracks that have propagated to the concrete surface and that behave as primary cracks, see below.

*Splitting cracks* are longitudinal cracks caused by too high circumferential tensile stresses, see Figure 6.4(b). Splitting cracks start at the concrete - reinforcement interface and propagate in radial direction.

*Primary cracks* can be mentioned as the fourth crack type in the context of reinforced concrete. They arise from the mechanical behavior of the specimen and are not directly induced by interaction between the concrete and ribbed bar, as the aforementioned crack types. The primary cracks are formed in the plane orthogonal to the bar axis and intersect with the reinforcing bar. Their crack distances are typically determined by bond behavior. Figure 6.4(a) shows a primary crack in case of a tension-pull test. For a three-point bending

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*Based on a section in Reference [138].*
6. EXTENSION TOWARDS SHEAR FAILURE IN REINFORCED CONCRETE

Figure 6.4: Visualizations of internal and primary cracks (a) splitting cracks (b) and pull-out bond failure (c). Pictures obtained from References [15, 50].

test, flexural or shear cracks that cross the reinforcement could be denoted as primary cracks.

Two types of bond related failure mechanisms can be distinguished:

Splitting bond failure occurs when splitting cracks propagate through the entire concrete cover to the outer surface.

Pull-out bond failure occurs when a sliding plane develops along the reinforcing bar, shearing off the concrete, see Figure 6.4(c). This failure mode will occur in case of sufficient confinement by the concrete cover and the shear reinforcement, preventing splitting cracks to reach the outer surface.

Finally, it should be noted that, specifically for pull-out tests, the bond related cracks and failure mechanisms govern the global behavior of the test. In general for RC structures or structural members, these bond phenomena occur only at local spots, e.g. on both sides of an open crack that is bridged by a reinforcing bar.

6.2.2 Modeling shear transfer

This subsection discusses the FE modeling of the four shear transfer mechanisms in a cracked RC beam from SubSection 6.2.1.

The force transfer $V_c$ in the uncracked compression zone does not require any particular modeling issues, due to the linear elastic behavior of the con-
crete. Modeling the concrete by continuum elements, $V_c$ arises from the shear stresses that are calculated at the integration points in this zone.

Modeling the concrete fracture with a fixed smeared crack model, the interface shear transfer along the crack $V_i$ is mainly determined by the shear retention relation. As mentioned in SubSection 2.4.3, a constant and a variable shear retention can be adopted at the onset of a local crack. In case of the constant shear retention relation, the shear stiffness experiences a single reduction to a certain percentage of the original shear stiffness. In case of the variable shear retention relation, the shear stiffness depends on the damage and reduces gradually to zero for an increasing damage. The used fixed smeared crack model does not include a coupling of the normal and tangential components of the local stresses and strains.

The force transfer by residual tensile stresses across the crack $V_r$ is mainly determined by the adopted stress-strain tension softening curves at the integration points. From SubSection 2.1.1 it can be noticed that the type of softening function and the fracture energy $G_f$ are the important model parameters. Specifically for SLA, also the number of saw-teeth $n_{swt}$ can be relevant, since it affects the accuracy of the softening curve approximation.

The simulation of the shear transfer by dowel action $V_d$ requires at least the addition of a bending and shear stiffness of the reinforcing bars in the FE model. This can be done by modeling the bars with Mindlin beam elements. Note that with the more commonly used embedded reinforcement or truss elements dowel action cannot be simulated, since they only include the axial stiffness of a bar.

Finally, three-dimensional effects play a minor role in the considered benchmark, and hence the shear transfer mechanisms can be adequately simulated with a two-dimensional FE model.

### 6.2.3 Modeling bond behavior*

This subsection further explores the FE modeling of bond behavior, as explained in SubSection 6.2.1. The discussion focuses on simulations with the proposed CPA from Chapter 5. Crack tracking strategies face important challenges when developing flexural or shear cracks cross the bond induced damage area along reinforcing bars.

The FE modeling of bond behavior is difficult and controversial as well [1]. It is difficult, since bond behavior is accompanied with adhesion and friction between concrete and reinforcement, and with crushing of concrete in front

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*Based on a section in Reference [138].
of the ribs. Moreover, it involves cone-shaped internal cracks and possible in radial direction propagating longitudinal splitting cracks, which may interact with each other. The complexity is further increased in case of multiple reinforcing bars. FE modeling of bond behavior is also controversial, since simulations that do not allow bond-slip between concrete and reinforcement show frequently good agreement with experimentally obtained results.

The bond behavior between concrete and ribbed reinforcing bars is obviously a three-dimensional phenomenon. In the “ideal” FE model one should adopt a very detailed mesh with element sizes on the scale of the bar’s ribs. Both concrete and ribbed bars should be modeled with solid elements, and frictional contact should be added to simulate the interface behavior. A CPA in a three-dimensional setting may be used to track the bond induced cracks and the primary cracks.

Although this modeling strategy is appealing, it faces some practical concerns. Besides the fact that such a detailed level is not feasible for structural analysis, given the enormous computational costs, the implementation of the proposed enhancements in the previous chapters is currently not devised for three-dimensional simulations. Hence, we search for alternative strategies in a two-dimensional context, using conventional continuum elements with sizes in order of magnitude equal to the bar’s diameter. Two-dimensional FE models on this scale make both structural analysis and tracking of individual (primary) cracks possible. The price to pay is that, strictly speaking, an accurate modeling of bond cracks becomes impossible. Internal and secondary cracks cannot initiate, since the ribs of bars are not included, and splitting cracks cannot propagate in the radial direction, since this requires the third dimension. In fact, two main effects of bond behavior can only be incorporated in simplified ways. These effects are

- the mutual interaction between concrete and reinforcement (slip);
- the fracture energy dissipation in bond cracks.

The remaining of this subsection discusses three different approaches, which includes one or both aforementioned effects.

**Perfect bond**

The approach with perfect bond directly connects the concrete, modeled with continuum elements, to the reinforcing bars, modeled with truss or beam elements. Alternatively, perfect bond is obtained by modeling the reinforced concrete with continuum elements and embedded reinforcement. This approach
implies that slip between concrete and the reinforcing bars cannot occur. Furthermore, fracture energy dissipation of bond cracks is not deliberately modeled, although it may partially be captured via cracking in the continuum elements along the reinforcement. FE models with perfect bond are frequently seen in literature and engineering practice. In general, their global responses behave too stiff and their crack patterns reveal a lot of small cracks along the reinforcing bars. The latter issue makes the perfect bond approach less attractive for analyses that include a CPA.

Non-explicit bond-slip modeling

The bond-slip mechanism is modeled in a non-explicit way if all bond behavior is treated on the constitutive level. Such an approach was used in Reference [107], where the continuum strong discontinuity approach with a global crack tracking algorithm was applied to reinforced concrete specimens. In that publication, the reinforced concrete was considered as a composite material of a matrix and two orthogonal fibers. Subsequently, the bond-slip was included by a modification of the uni-axial elasto-plastic stress-strain curve of the fiber, assuming the fiber deformation and the bond-slip between matrix and fiber as a two-component serial system. The material characteristics of the modified stress-strain curve were calibrated from a pull-out test.

Non-explicit bond-slip modeling is attractive due to its computationally efficiency. Moreover, the presented results in Reference [107] show that this approach can be properly combined with a crack tracking strategy. The non-explicit bond-slip modeling approach may be less applicable for problems with a pronounced bond failure.

Explicit bond-slip modeling

The bond-slip mechanism is modeled in an explicit way if bond-specific kinematic and constitutive descriptions are used. A particular way of explicit bond-slip modeling is presented in Reference [67], using specifically X-FEM for the modeling of bond-slip. A more classical way is to use conventional interface elements for the connection between the continuum elements (concrete) and truss or beam elements (reinforcing bars). Figure 6.5 shows a representation of this classical way. The addition of degrees of freedom in the system via interface elements enables to simulate the slip directly by displacement differences between the continuum and the truss or beam elements. Note that the mesh generation in the the explicit bond-slip modeling approach can become
cumbersome for most existing automatic mesh generators. This issue is circumvented when using an embedded bar element that includes the bond-slip simultaneously [74].

An important aspect in this modeling approach is the adopted constitutive relation for the interface elements in terms of bond stress ($\tau$) and slip ($\Delta u_s$). In general, two different types of curves can be distinguished: a splitting curve (schematized by the light gray line in Figure 6.5) and a pull-out curve (schematized by the dark gray line in Figure 6.5). These curves are related to the two different failure mechanisms, as mentioned in SubSection 6.2.1. Corresponding analytical expressions can be found in e.g. References [15, 41, 70]. Such expressions, based on experimental results, commonly depend on many influencing factors as concrete strength, concrete cover, bar diameter and boundary conditions. Note that the explicit bond-slip modeling approach needs a $\tau$ - $\Delta u_s$ curve as input, whereas in case of the “ideal” FE model with a very detailed three-dimensional mesh the $\tau$ - $\Delta u_s$ curve and the failure mechanism could be an outcome of the simulation [119].

The primary advantage of explicit bond-slip modeling is its potential to simulate the mutual interaction between concrete and reinforcement more accurately. Nevertheless, a dilemma arises regarding the a priori selection of the type of $\tau$ - $\Delta u_s$ curve for the modeling of arbitrary RC structures or structural members. The true (local) bond-slip behavior is difficult to determine, which complicates a rational choice in advance. Moreover, both the continuum elements along the reinforcement and the interface elements can capture the energy dissipation of bond related cracks [119]. It will be difficult to make an a priori distinction which element type will capture the energy dissipation of these cracks, and, most likely, this will also depends on the adopted $\tau$ - $\Delta u_s$ curve.
6.2 Modeling strategies

Table 6.1: Overview of selected modeling strategies.

<table>
<thead>
<tr>
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<th>Strategy A</th>
<th>Strategy B</th>
<th>Strategy C</th>
</tr>
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<tbody>
<tr>
<td>Bond-slip</td>
<td>perfect bond</td>
<td>explicit modeling and a</td>
<td>explicit modeling and a</td>
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<td></td>
<td></td>
<td>$\tau - \Delta u$ curve with</td>
<td>$\tau - \Delta u$ curve</td>
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<td></td>
<td></td>
<td>linear softening</td>
<td>plastic $\tau - \Delta u$ curve</td>
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<tr>
<td>Root elements</td>
<td>all</td>
<td>along mesh boundary</td>
<td>along mesh boundary</td>
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<td></td>
<td></td>
<td>and reinforcement</td>
<td>and reinforcement</td>
</tr>
<tr>
<td>Reinforcement</td>
<td>embedded truss element</td>
<td>beam element</td>
<td></td>
</tr>
<tr>
<td>Fixed smeared cracks</td>
<td>$l = \sqrt{\frac{A}{\varepsilon}}$ (§ 6.3)</td>
<td>$l_q$ or $l_p$ (§ 6.4)</td>
<td>$l_q$ and CPA (§ 6.5)</td>
</tr>
</tbody>
</table>

An additional dilemma arises when the explicit bond-slip modeling is combined with the proposed CPA. The CPA initializes a set of root elements, i.e. continuum elements where cracks may initiate, at the start of each analysis. For the analyses of the plain concrete specimens in the previous chapter, the set of root elements consisted of only the elements that were located at the mesh boundary, see SubSection 5.2.1. Assuming that all the energy dissipation of the bond related cracks from SubSection 6.2.1 is captured by the interface elements, this set of root elements may also be valid for RC specimens. In fact, each row of interface elements is now considered as a predefined bond crack path. However, from a physical point of view it is more realistic to allow also crack (path) initiation along the reinforcement, considering for instance the development of a secondary crack. This implies that the continuum elements along the reinforcing bars should also be marked as root elements.

6.2.4 Overview of selected strategies

SubSection 6.2.3 reveals a range of possibilities for the modeling of the RC shear beam in our scope of interest. Table 6.1 provides an overview of three selected strategies that will be used in the following sections of this chapter. These strategies A, B and C are distinguished by means of four modeling aspects, which are related to the bond-slip, the root elements, the reinforcement and the smeared cracks. Regarding the smeared cracks, each strategy uses the crack band approach in the fixed crack model, but the adopted crack band width estimator differs and the proposed CPA is not always included.

Strategy A is a common way to model RC structures in practice. It assumes perfect bond between the concrete and the reinforcing bars. The reinforcing
bars are modeled by an embedded truss reinforcement. The crack band widths are estimated with the element area \( A_e \) based formulation, the element projection based formulation \( l_p \) of Govindjee, et al. or the new proposed formulation \( l_p' \). The proposed CPA is not used in this strategy and so cracks may arise in all continuum elements of the FE model.

The strategies B and C combine the explicit bond-slip modeling with the proposed CPA and model the reinforcement by Mindlin beam elements. The difference between the strategies B and C lies in the way they aim to simulate the bond related cracks and their energy dissipation. Note that the combination of an explicit bond-slip modeling with a crack tracking strategy is new, according to the author’s knowledge.

Strategy B adopts a \( \tau - \Delta u_s \) (splitting) curve with linear softening in the interface, see Figure 6.5, and has only root elements along the mesh boundary. The idea is that all the fracture energy dissipation from internal, secondary and splitting cracks is lumped into the interface constitutive behavior. Only the primary cracks, e.g. flexural or shear cracks, should be tracked by the CPA and their fracture energy should dissipate via the continuum elements. Basically, this strategy neglects the interaction between the bond induced cracks and the primary cracks that originate from the mechanical behavior of the structure.

Strategy C adopts a linear elastic - ideally plastic \( \tau - \Delta u_s \) (simplified pull-out) curve, see Figure 6.5, and has root elements along the mesh boundary and the reinforcement. The idea is that the bond-slip interface should only dissipate the fracture energy from internal cracks and the formation of a (local) sliding plane. Secondary cracks may initiate in the continuum elements along the reinforcing bars and should be tracked by the CPA, as the primary cracks. So, this strategy allows interaction between the bond induced cracks and the primary cracks.

Finally, the strategies A, B and C will be used in the Sections 6.3 to 6.5. An overview of the different numerical simulations is provided in Appendix A.

6.3 Exploratory study to material and mesh sensitivities∗

This section presents an exploratory study on the shear critical RC beam without shear reinforcement. The analyses are performed with modeling strategy A from Table 6.1, using a crack band width formulation based on the element area and perfect bond. Firstly, SubSection 6.3.1 treats the material input,

∗Based on sections in Reference [135].
the mesh details and the results of a reference analysis. Subsequently, the input parameters and the mesh are varied in SubSections 6.3.2 and 6.3.3. Since the Sequentially Linear Analysis (SLA) method has hardly been applied to RC shear beams, its ability to capture the brittle shear failure behavior is investigated and for some aforementioned variations compared to conventional nonlinear finite element analyses (NLFEA) with an incremental-iterative procedure. Finally, the main lessons learned from all observations and comparisons between the results are briefly described in SubSection 6.3.4.

6.3.1 Reference analysis

Exploiting symmetry, the benchmark is simulated by a FE model with the left half of the beam geometry and with horizontal constraints at the midspan cross section. The concrete is represented by a structured mesh, consisting of quadrilateral elements based on quadratic interpolation and using a $3 \times 3$ Gauss integration scheme. Their element sizes are set to 25 mm by 25 mm, being in the same order of magnitude as the diameter of the reinforcing bars. The steel plates, that introduce the load and support forces, are modeled by the same elements. The reinforcing bars are modeled by embedded reinforcement elements with 3-point Gauss integration. The anchorage plate on the beam end is not included in the model.

The cracking behavior in the concrete is captured with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, see SubSection 2.4.3. From tests on casted concrete cubes, the compressive strength $f_c$ and tensile strength $f_t$ were determined on 38 N/mm$^2$ and 3.5 N/mm$^2$ respectively \cite{130}. Hence, the Young’s modulus $E_0$ and the fracture energy $G_f$ are derived from formulas of the CEB-FIP Model Code 1990 \cite{69} and are respectively equal to 33,551 N/mm$^2$ and 0.0765 N/mm. In the reference analysis, the regularized fracture energy $G_f/l$ is multiplied by a factor $\eta = 1.2$. Poisson’s ratio $\nu_0$ is set to 0.15 for SLA and to 0.0 for the conventional NLFEA. For SLA the effect of lateral contraction reduces with an increasing damage, following Equation (2.12), whereas in the used conventional NLFEA the Poisson’s ratio remains constant. The adopted stress-strain tension softening curve is according to Hordijk and for concrete under compression a linear elastic behavior is assumed. Furthermore, the material behavior of the steel plates is modeled with a linear-elastic stress-strain relation, using the elastic properties $E_0$ of 210,000 N/mm$^2$ and $\nu_0$ of 0.3. The reinforcing bars adopt linear elastic - ideally plastic behavior with $E_0$ of 210,000 N/mm$^2$ and a yield stress $f_y$ of 500 N/mm$^2$.  

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Regarding the SLA specific input, the Hordijk softening curve for the concrete and the elastic-plastic curve for the reinforcement are discretized with 16 and 35 damage increments respectively, see SubSections 2.4.2 and 3.2.2. Regarding the conventional NLFEA specific input, the analyses are performed in displacement control with step sizes of 0.05 mm and 0.01 mm. A Quasi-Newton or Secant iteration method is used, based on a BFGS algorithm and with maximal 25 iterations per load step. The analyses further adopt an explicit line search technique in order to decrease the number of iterations per increment. Finally, a force tolerance of 1% is used as convergence criterion.

Figure 6.6(a) presents the load - deflection responses of the reference analysis with SLA and NLFEA. For the NLFEA response, the non-converged steps are indicated by the black circles. Furthermore, the curve of beam S1B5 is shown, being closest to the average of the experimentally obtained responses and hence representative for test series 1. The SLA curve reveals the typical spiky shape, which arises from the saw-tooth softening input. After the peak
at 183.1 kN, the load initially drops to 100 kN, and some residual capacity is visible before the beam completely fails. The SLA curve shows further good agreement with the NLFEA curve, both in the pre-peak regime and regarding the prediction of the maximum load capacity. However, their responses after the peak are different. In this post-peak regime none of the load steps in the incremental-iterative solution procedure converged anymore. Note that before the peak approximately 20% of the load steps were not converged. Finally, both numerically obtained curves show good agreement with the experimental curves, although the initial stiffness of the RC beam seems to be overestimated.

Figure 6.6(b) presents a maximum principal strain ($\varepsilon_1$) contour plot for both analyses and their color legend. The plots belong to the load steps that are indicated by the colored dots in Figure 6.6(a). Since the color legend is linked to the Hordijk softening curve for $\eta = 1.0$ with ranges of $(0, f_t/E_0)$, $(f_t/E_0, \varepsilon_u)$, $(\varepsilon_u, \to)$, the $\varepsilon_1$ contour plots provide understanding of the crack patterns after the load drop. The horizontal dashed lines in the plots indicate the position of the reinforcing bars. Both plots reveal some flexural cracks and one clearly localized diagonal crack across them. Considering the location and the shape of the diagonal crack, especially the NLFEA plot shows good agreement with the experimentally obtained failure cracks in Figures 6.1 and 6.2(b). For the NLFEA plot, it can be observed that the horizontal crack just above the reinforcing bars starts to develop toward the end of the beam. Although not shown, this horizontal crack also appears at a later stage in the analysis with SLA.

### 6.3.2 Variations on the material input

This subsection investigates the sensitivity of the numerical results with respect to the regularized fracture energy, the shear retention relation and the number of saw-teeth in the approximation of the stress-strain curve. As mentioned in SubSection 6.2.2, these material input items are closely related to the way of modeling shear transfer by residual tensile stresses and interface shear transfer.

**Fracture energy**

The used element area based formulation estimates crack band widths $l$ that are independent of the crack orientation, see SubSection 2.3.2. This leads to the same $l$, and so the same regularized fracture energy $G_t/l$, for all elements in the structured mesh of the reference analysis. However, the $\varepsilon_1$ contour plots in Figure 6.6(b) show different localization band widths in the elements and
Figure 6.7: Load - deflection responses of analyses on the RC shear beam with varying amounts of regularized fracture energy.

Figure 6.8: $\varepsilon_1$ contour plots for the indicated load steps of the analyses with varying amounts of fracture energy in Figure 6.7. For the color legend, see Figure 6.6(b).
hence the local dissipations of the fracture energy should actually differ as well. Section 6.4 will use the crack orientation dependent band width formulations. Here, the influence of the amount of regularized fracture energy is investigated, by using two different multiplication factors $\eta$ of 1.0 and 1.4 in $\eta \times G_t/l$. Note that for the reference analysis in SubSection 6.3.1 $\eta$ was set to 1.2. The analyses are performed with SLA and NLFEA.

Figure 6.7 shows the load - deflection responses of the analyses with varying amounts of regularized fracture energy. Both graphs reveal the trend that the predicted load capacity increases for an increasing regularized fracture energy. In case of SLA, the peak loads ranges from 167.2 kN to 197.9 kN, whereas for NLFEA a range of 168.6 kN to 213.8 kN is obtained. Regarding the convergence behavior of the analyses with NLFEA, approximately 20% ($\eta = 1.0$) and 5% ($\eta = 1.4$) of the load steps before the peak were not converged. After the peaks no convergence was obtained anymore.

Figure 6.8 provides the $\varepsilon_1$ contour plots of the foregoing analyses. The three SLA plots reveal more or less the same crack patterns beyond the peak. The observed shear cracks, which appear after significant snap-backs in the global responses, indicate that the experimentally obtained failure mode is properly captured. Nevertheless, the locations and shapes of these failure cracks differ from the experiment, as can be noticed by the comparison with Figures 6.1 and 6.2(b). The NLFEA plots reveal less consistency in their obtained failure cracks. The plot belonging to $\eta = 1.0$ deviates significantly from the experimental result, whereas the plot belonging to $\eta = 1.4$ shows reasonable agreement.

Shear retention relation

In the reference analysis a variable shear retention relation according to Equation (2.14) was adopted. Here, a sequentially linear analysis and a conventional nonlinear analysis are performed with a constant shear retention. The shear retention factor $\beta$ in Equation (2.13) is set to a relatively low constant value of 0.005, which means that after the onset of local cracks just 0.5% of the initial shear stiffness $G_0$ remains.

Figure 6.9(a) presents the load - deflection responses of these two analyses. Note that the graph uses different values along the axes, compared to the previous graphs. Both curves are in reasonable agreement, but reveal a completely different behavior as the experimental response. Even without showing a clear failure, the two simulations predict already load capacities of more than 300 kN. Moreover, the SLA response results in a cloud of points that are connected with a rather spiky line and a post-peak behavior cannot be observed.
The analysis with NLFEA has a similar convergence behavior as the reference analysis.

Figure 6.9(b) shows the corresponding $\varepsilon_1$ contour plots at the observed peaks. Again, reasonable agreement is visible between the two analyses, but the plots differ from the experimentally obtained crack pattern. From the development of $\varepsilon_1$ during the simulations appears that the diagonal cracks continuously expand their bandwidths, instead of keeping a single localized shear crack.

Comparisons with the results of the reference analysis reveal an enormous difference between the modeling of interface shear transfer with a constant or a variable shear retention relation. The reference analysis shows a clear failure mechanism near the experimentally obtained load capacity, whereas in the analyses with constant shear retention the failure is postponed to significantly higher load capacities. Even with the relatively low adopted $\beta$-value, the constant shear stiffness apparently enables to generate substantial and unrealistic resistance on the widely open diagonal shear crack. This resistance may be initiated by shear strains that arise from principal strain rotations in the local fixed crack coordinate systems. In case of a variable shear retention relation such a resistance cannot be generated, since the shear stiffness completely vanishes for fully softened cracks.
6.3 Exploratory study to material and mesh sensitivities

The number of damage increments or saw-teeth \( n_{\text{swt}} \), to discretize the adopted stress-strain curves, is actually more a model parameter than a material parameter, as the title of SubSection 6.3.2 may suggest. However, this only SLA specific input parameter for the reference analysis affects the constitutive behavior, due to the relation between \( n_{\text{swt}} \) and the accuracy of the stress-strain curve approximation, and therefore the subject is discussed here. In the following, the number of saw-teeth in the adopted Hordijk softening curve for concrete is varied and the influence on the maximum load capacity is investigated. The reference analysis in SubSection 6.3.1 used a \( n_{\text{swt}} \) equal to 16.

Figure 6.10 presents the predicted load capacities (black circles) of six analyses with \( n_{\text{swt}} \) ranging from 8 to 40. The experimentally obtained peak loads and the peak load from the conventional nonlinear reference analysis are indicated by the gray shaded area and the black line respectively. The SLA results reveal a scatter of 20 kN, with a maximum value of 183.1 kN for \( n_{\text{swt}} = 16 \). The prediction of the peak load from \( n_{\text{swt}} = 16 \) shows the best agreement with the NLFEA result. The increase of load capacities for the analyses with 8, 12 and 16 saw-teeth is remarkable. Considering that the overshoot of the tensile strength by the first saw-tooth increases for a decreasing \( n_{\text{swt}} \), see SubSection 2.4.2, the opposite was expected. The explanation for this may lie in the crack patterns, specifically in the locations and shapes of the failure cracks, but a systematic trend was not observed.

6.3.3 Variations on the mesh

This subsection investigates the sensitivity of the numerical results with respect to the element size and explores the possibility of non-symmetric failure modes.
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Element size

The reference analysis in SubSection 6.3.1 adopted elements with sizes of 25 mm by 25 mm. Here, two sequentially linear analyses are performed with elements of 50 mm by 50 mm and 12.5 mm by 12.5 mm. The three meshes with the different element sizes of 50 mm, 25 mm and 12.5 mm are denoted by coarse, medium and fine meshes respectively.

Figure 6.11(a) shows the load - deflection responses of the analyses with varying element size. The analyses predict load capacities of 140 kN, 183.1 kN and 188.5 kN with increasing mesh fineness, showing a significant deviation for the coarse mesh.

Figure 6.11(b) shows the $\varepsilon_1$ contour plots after the peak of the foregoing analyses. The color legends of the plots are adapted to the different $\varepsilon_u$ values in order to make the correct comparison. In all the plots shear cracks with more or less the same locations and shapes are visible. Compared to the experimental failure cracks of Figures 6.1 and 6.2(b), the numerically observed shear crack initiation points at the reinforcing bars are closer to the midspan of the beams.

Non-symmetric failure modes

The use of symmetry enables to perform more efficient analyses from a computational viewpoint. However, imposing symmetry forces symmetric failure modes, while the actual physical modes are generally non-symmetric as nature is imperfect. As final variation on the reference analysis in SubSection 6.3.1, here the possibility of non-symmetric failure modes is explored by using the entire geometry of the RC shear beam in the FE model. Both a sequentially linear analysis and a conventional nonlinear analysis is performed.

Figure 6.12 presents the load - deflection responses of these two analyses. The responses show good agreement in the pre-peak regime and also the predicted load capacities, 171.4 kN for SLA and 178.4 kN for NLFEA, are relatively close to each other. The NLFEA result showed that approximately 20% of the load steps until the peak did not converge. These non-converged steps are mainly concentrated around the peak. Beyond the peak none of the load steps were converged anymore. In this stage the SLA curve reveals a clear snap-back. Many similarities are observed between the results of these two analyses and the results of the reference analyses in Figure 6.6(a). The most striking difference can be seen in the post-peak regimes of the SLA responses. For the reference analysis in Figure 6.6(a) some residual load capacity is visible, whereas for the analysis with a FE model of the entire geometry in Figure 6.12.
6.3 Exploratory study to material and mesh sensitivities

Figure 6.11: Load - deflection responses of analyses on the RC shear beam with varying element size (a) and the corresponding $\varepsilon_1$ contour plots for the indicated load steps (b). For the color legend, see Figure 6.6(b).

Figure 6.12: Load - deflection responses of analyses on the RC shear beam with a FE model of the entire geometry.
the load “immediately” drops to 15% of the maximum load capacity. The latter brittle behavior is more in line with the experimentally observed failures, since they also showed no remaining capacities after failure.

Figure 6.13 presents the developments of $\varepsilon_1$ from the peak load by means of three contour plots for the sequentially linear analysis and two contour plots for the conventional nonlinear analysis. The corresponding load steps of these plots are indicated by the colored dots in Figure 6.12. In case of SLA, first several flexural cracks can be observed in the first plot at the peak, then one curved localized shear crack is visible in the second plot at 100 kN, which finally has further propagated in the compression zone toward the loading plate, as can be seen in the third plot at 25 kN. In case of NLFEA, first several flexural cracks can be observed in the first plot at the peak, and subsequently two curved localized shear cracks are visible in the second plot at the last load step. The comparison between these SLA and NLFEA results reveals good agreement in the bending stage and in the development, location and shape of the shear crack. However, NLFEA shows a typical symmetric failure crack pattern, whereas SLA shows a more realistic non-symmetric failure crack pattern.

The non-symmetric and brittle failure mode in the SLA result reveals a remarkable consistency with the experimentally obtained result, as described in Section 6.1 and shown in the three images of Figure 6.2(b). Besides the good agreement in the location and the shape of the failure crack, also the sequence ‘flexural cracks - shear crack - collapse of the compression zone’ is properly captured. Only the horizontal crack along the reinforcing bars is not observed in the simulation.

6.3.4 Lessons learned

The exploratory analyses, with a crack band width formulation based on the element area and perfect bond between concrete and reinforcement, shows that SLA is able to model the brittle failure behavior of the shear critical RC beams without shear reinforcement. Their load - deflection responses and $\varepsilon_1$ contour plots reveal generally reasonable agreement with the results of the experimental test and conventional nonlinear analyses.

The main lessons learned from the study to material and mesh sensitivities for this fracture problem are as follows:

1. the amount of regularized fracture energy $\eta \times G_f/l$ has a considerable influence on the maximum load capacity; an $\eta$-value of 1.0 significantly
6.3 Exploratory study to material and mesh sensitivities

(a) SLA - entire geometry

(b) NLFEA - entire geometry

Figure 6.13: $\epsilon_1$ contour plots for the indicated load steps of the analyses with a FE model of the entire geometry in Figure 6.12. For the color legend, see Figure 6.6(b).
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underestimates the load capacity of the beam; the observed scatter emphasizes the importance of an accurate crack band width estimation;

2. a variable shear retention relation, that gradually results in a zero shear stiffness, is essential for realistic simulations of localized brittle shear failure;

3. the adopted number of saw-teeth $n_{swt}$ reveals a limited influence on the predictions of the load capacity;

4. a mesh size sensitivity is observed, but the results clearly converge with mesh refinement; realistic load capacities are obtained for the medium and fine mesh;

5. using the entire geometry of the beam in the FE model improves the results with respect to post-peak load - deflection response, and the location and shape of the failure crack.

Furthermore, the study shows that the strength of SLA in comparison with conventional NLFEA for the RC shear beam is mainly reflected in its robustness and its ability to find non-symmetric failure modes. Regarding the robustness, conventional NLFEA with an incremental-iterative procedure faces difficulties to obtain a converged equilibrium state around and beyond the peak. This hampers to make accurate predictions of the maximum load capacity and to investigate possible post-peak behavior. SLA does not suffer from numerical instability and enables to explore the post-peak regime, even in case of this brittle failure problem. Regarding the ability to find non-symmetric failure modes, conventional NLFEA typically finds a symmetric failure crack pattern in the symmetric FE model. Non-symmetric modes could for instance be triggered by adding material inhomogeneity or by using unstructured meshes. SLA automatically triggers a non-symmetric failure crack pattern in the symmetric FE model, as a result of the procedure to damage only the critical integration point for each step. Note that the symmetric and non-symmetric failure crack patterns can be interpreted as two possible post-bifurcation solutions after a bifurcation point [126]. Considering that the actual physical response is always non-symmetric, since nature is imperfect, it can be concluded that SLA effectively traces the correct equilibrium path.
6.4 Analyses with the proposed crack band width formulation

Using the lessons learned from SubSection 6.3.4, this section presents sequentially linear analyses on the benchmark with the proposed crack band width formulation \( l_p \) from Chapter 4. The analyses are based on modeling strategy A from Table 6.1, with perfect bond between concrete and reinforcement. The adopted FE models include the entire geometry of the RC shear beam and are discretized with structured (medium) meshes that consist of 25 mm by 25 mm plane stress elements. As in Section 4.3, three different element types are adopted: the linear quadrilateral with \( 2 \times 2 \) Gauss integration and constant shear, the quadratic quadrilateral with \( 2 \times 2 \) Gauss integration and the quadratic quadrilateral with \( 3 \times 3 \) Gauss integration. In order to assess the results of the three analyses with \( l_p \), the same three analyses are also performed with the formulation \( l_g \) of Govindjee et al..

The cracking behavior in the concrete is captured with a total strain based orthogonal fixed smeared crack model and a variable shear retention relation, see SubSection 2.4.3. Some of the adopted material properties for concrete are modified with respect to SubSection 6.3.1. Here, the Young’s modulus \( E_0 \) is set to 28,500 N/mm\(^2\), which is derived from the initial branch of the experimentally obtained load - deflection response. The tensile strength \( f_t \) is equal to 3.5 N/mm\(^2\). The fracture energy \( G_f \) and Poisson’s ratio \( \nu_0 \) are taken from the CEB-FIP Model Code 2010 [70] and are respectively equal to 0.141 N/mm and 0.2. The adopted stress-strain tension softening curve is according to Hordijk and is approximated by 16 saw-teeth. Based on these material properties, the crack opening \( w_{ult} \) is equal to 0.207 mm. For concrete under compression a linear elastic behavior is assumed. Finally, the steel plates and the reinforcing bars are modeled in the same way as in the reference analysis.

Figure 6.14 presents the load - deflection responses of the analyses with \( l_g \) and \( l_p \). The results of the analyses with \( l_g \) show significant differences in the maximum load capacities and the post-peak responses. The obtained peak loads for the curves ‘linear, \( 2 \times 2 \) Gauss, constant shear, □’, ‘quadratic, \( 2 \times 2 \) Gauss, □’ and ‘quadratic, \( 3 \times 3 \) Gauss, □’ are equal to 233.8 kN, 178.4 kN and 190.7 kN respectively. As can be observed, the analysis with linear elements overestimates considerably the load capacity of the RC shear beam. On the other hand, the results of the analyses with \( l_p \) are consistent and in agreement

*Based on a section in Reference [137].
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Figure 6.14: Load - deflection responses of analyses on the RC shear beam with three different quadrilateral element types.

with the experimental response. Their peak loads, in aforementioned order, are equal to 188.2 kN, 187.3 kN and 182.1 kN respectively.

Figure 6.15 presents the crack pattern after failure for each analysis. The considered load steps are indicated by the colored dots in Figure 6.14. The crack patterns are plotted in undeformed meshes, using three different line widths for crack widths in the ranges $[0, \frac{1}{2} w_{ult}]$, $[\frac{1}{2} w_{ult}, w_{ult}]$, $[w_{ult}, \rightarrow]$. The horizontal dashed lines in the plots indicate the position of the reinforcing bars. In general, each plot shows several flexural cracks and one clearly localized shear failure crack across them, as in the experimentally obtained failure crack pattern. However, regarding the number of flexural cracks and the locations and shapes of the shear failure cracks some differences are observed. The most remarkable results are the crack plots of the two analyses with linear elements. In case of $l_p$, the shear failure crack shows a zig-zag pattern, indicating a strong tendency to propagate along continuous mesh lines. On the other hand, the shear failure crack in case of $l_p$ is more curved. It is noteworthy that the band width of the latter crack approximately equals two times the adopted element size, which is generally larger compared to the band widths of the observed shear failure cracks in the other analyses.

The observations on the load - deflection responses indicate that the new proposed crack band width formulation $l_p$ leads to an objective energy consumption for the different quadrilateral element types. Predictions of the load
6.4 Analyses with the proposed crack band width formulation

Figure 6.15: Crack width plots for the indicated load steps of the analyses with three different quadrilateral element types in Figure 6.14. The thickest lines correspond to \( w \geq w_{ult} \).
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capacity and the brittle post-peak response of the beam are consistent and realistic, whereas the comparable results of the analyses with \( l_p \) reveal significant differences and specifically for the linear elements they are unrealistic. Note that the positive effect of \( l_p \) on the numerically obtained load-deflection responses here is much more pronounced as for the fracture tests in Section 4.3. Obviously, this stems from the differences in the cracked area that dissipate energy. The relatively small plain concrete notched specimens just have one or two active cracks, while for the larger RC shear beam the number of active cracks is significantly higher. Hence, it can be expected that for the modeling of large-scale RC structures the observed differences between \( l_k \) and \( l_p \) will further increase.

Note further that the overestimation of the failure load for the analysis with linear elements and \( l_k \) could be explained from a structural “locking” effect, caused by the zig-zag pattern in the localized failure crack. More fundamentally, however, the overestimation can be the result of an incorrect energy consumption \( G_f / \overline{G_f} \) in the diagonal cracks at the integration points of the global localized failure crack. As can be observed, these local cracks have orientations \( \theta \) of about \( \pi/4 \). Figure 3.12(a) shows that for \( \theta = \pi/4 \) the ratio \( G_f / \overline{G_f} \) is approximately equal to 2.0, which means that the actual energy dissipation is twice as large than it should be. By comparison, in case of the quadratic elements and \( \theta = \pi/4 \) this ratio \( G_f / \overline{G_f} \) is close to unity.

The observed differences in the number of flexural cracks and the locations and shapes of the shear failure cracks show that \( l_p \) is not fully mesh objective for the different quadrilateral element types regarding their cracking behavior. Nevertheless, the analyses with linear elements illustrate that the new proposed crack band width formulation can improve the cracking behavior by reducing the sensitivity to mesh alignment. The plot ‘linear, \( 2 \times 2 \) Gauss, □ (constant shear)’ for \( l_p \) in Figure 6.15(b) shows a curved failure crack. Its band width of approximately two times the adopted element size indicates that the \( \gamma \)-factor of about 2.0 in Figure 4.1(a) for crack orientations of \( \pi/4 \) is indeed found a posteriori.

6.5 Analyses with the proposed crack propagation algorithm

This section presents four sequentially linear analyses on the shear RC beam with the proposed CPA from Chapter 5. The analyses are based on modeling strategies B and C from Table 6.1, using an explicit bond-slip modeling via an interface between concrete and reinforcement. The adopted FE models for both strategies include the entire geometry and are discretized with a structured and
an unstructured (medium) mesh, consisting of plane stress elements with sides of 25 mm. The out-of-plane thickness of the interface elements is set to the sum of the bar’s perimeters.

The concrete is represented by quadratic quadrilaterals with $3 \times 3$ Gauss integration for the structured meshes, and by quadratic triangular elements with 3-point Gauss integration for the unstructured meshes. The same material assumptions are used as in Section 6.4. Here, the crack band widths are a priori estimated with the formulation $l_0$ of Govindjee et al.. The CPA parameters $\tau_{\text{crit}}$ and $d_{\text{crit,crk}}$ are set to 0. The ribbed reinforcing bars are represented by three-node Mindlin beam elements with 2-point Gauss integration along the bar axis. For the sake of convenience, the three longitudinal reinforcing bars $\emptyset 20$ are replaced by one rectangular shaped cross-section of $17.321 \text{ mm} \times 54.414 \text{ mm}$ (height $\times$ width), keeping the axial stiffness $E_A$ and the bending stiffness $EI$ remain similar. The material behavior of the reinforcement is now modeled with a linear-elastic behavior, using the elastic properties $E_0$ of 210,000 N/mm$^2$ and $\nu_0$ of 0.3. Since yielding was not observed in the experiment and in previous analyses, this simplification is acceptable. Further specific input for each explicit bond-slip modeling strategy is provided in the SubSections 6.5.1 and 6.5.2 respectively.

The four analyses, presented in this section, are selected from a study to the mesh objectivity of the proposed CPA. This mesh objectivity study consists of twelve analyses. For both modeling strategies B and C, six analyses are performed, using coarse, medium and fine structured and unstructured meshes. The obtained crack path plots, after complete failure, for all these twelve analyses are provided in Appendix C.

### 6.5.1 Modeling strategy B

Strategy B only allows crack (path) initiation in the continuum elements at the mesh boundary. The interaction between concrete and reinforcement is modeled by quadratic interface elements with 4-point Newton-Cotes integration and a $\tau - \Delta u_s$ curve with linear softening. The shape of the softening curve corresponds with the shape of the analytical bond-slip relationship for splitting bond failure in Reference [70], see Figure 6.5, and is approximated by 25 saw-teeth. Here, we adopt a tangential interface stiffness $k_t$ of $5 f_t / 0.06 = 292 \text{ N/mm}^3$, according to Reference [41]. The normal interface stiffness $k_n$ is taken as infinitely stiff. Furthermore, we set the characteristic points $(\tau, \Delta u_s)$ of the base curve to $(0, 0)$, $(4.125, 0.0141)$ and $(0, 0.225)$, expressed in the units (N/mm$^2$, mm). This set of values is obtained from a small variation study, giving the most satisfactory results. Note that the results were sensitive to the
variations in curves with linear softening. Consequently, a more in-depth study to the constitutive interface behavior of bond-slip in two-dimensional simulations is necessary, but outside the scope of this chapter.

Figure 6.16 presents the load - deflection responses of the analyses on a structured mesh with quadrilaterals and an unstructured mesh with triangular elements. Both curves reveal initially a small load drop beyond the peak, followed by a small plateau. Subsequently, the curves snap back to load levels of about 10 kN. With peak loads of 207.9 kN and 224.8 kN, the analyses overestimate the maximum load capacity of the beam.

Figure 6.17 presents the undeformed meshes with crack paths (red lines) of the two analyses, after the failure has occurred. The horizontal dashed lines in the plots indicate the position of the reinforcing bars. Each plot shows multiple smooth curved crack paths, which all have their starting point at the mesh boundary. Reasonable agreement can be observed with respect to the number of flexural cracks and their crack distances in the plain concrete area of the beam. Note that these flexural crack distances are obtained with an exclusion radius $r_{\text{excl}}$ of 0. Considering the corresponding principal strain contour plots, both analyses reveal a strong strain localization in the crack path at the right side. These crack paths can therefore be indicated as failure cracks (FC). The location and shape of the failure cracks are consistent with the experimental obtained failure cracks, see Figures 6.1 and 6.2(b). The inset A shows a detail of the failure crack near the reinforcement in the structured mesh. Besides the crack paths, it also presents the cracks in the integration points with different line widths for the different crack width ranges. The local crack widths make clear that the failure crack is formed by multiple crack paths.

Figure 6.18 provides detailed information of the numerically obtained bond behavior for the analysis with the structured mesh. It shows four graphs, which plot the normal and shear strains in the concrete, the axial strain in the reinforcement and the slip in the interface over the whole length of the beam. Each graph shows further two lines that represent the strains or slip for a load step at the peak (gray line) and after failure (black line). These load steps are indicated by the red colored dots in Figure 6.16. The corresponding four graphs of the analysis with the unstructured mesh are presented in Figure C.4 of Appendix C. These results are qualitatively and quantitatively similar to the results of Figure 6.18.

The (global) strain components in the concrete are taken from the row of integration points just above the reinforcing bars. The $\varepsilon_{xx}$ curve at the peak
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Figure 6.16: Load - deflection responses of the analyses on the RC shear beam with modeling strategy B and a structured and an unstructured (medium) mesh.

Figure 6.17: Crack path plots, after complete failure, of the analyses on the RC shear beam with modeling strategy B in Figure 6.16.
load shows clearly the locations of the flexural cracks, as observed in the corresponding crack path plot of Figure 6.17. After failure, these sharp peaks in the \( \varepsilon_{xx,c} \) curve are disappeared and only around \( x = 2220 \text{ mm} \), the location where the failure crack intersects with the reinforcement, high strain values are visible. Note that the ultimate strain for a fully softened crack in an element of 25 mm is equal to \( 0.83 \times 10^{-2} \). The two \( \gamma_{xy,c} \) curves are almost similar and they show only significant values at the location of the failure crack. The presence of these high values in the load step at the peak indicates that the development of the shear failure crack is already started during this stage of the loading process. The axial strains \( \varepsilon_{xx,s} \) in the reinforcement are taken from the integration points at the mid surface of the beam elements. Its corresponding curve at the
peak load shows significant values between the supports at $x = 300$ mm and $x = 2700$ mm, indicating that the reinforcement is properly activated via the bond-slip interface and is able to contribute to the resistance of the RC beam. The local peaks in this $\varepsilon_{\text{xx},s}$ curve correspond with the locations of the flexural cracks. The yield strain, equal to $2.38 \times 10^{-3}$, is not exceeded. The slip values are taken from the integration points of the interface elements. During the load step at the peak, the slip curve reveals steep gradients at the locations of the flexural cracks. After failure, large slip values are only present between $x = 2220$ mm and $x = 3000$ mm, which indicates that the experimentally observed horizontal crack along the reinforcing bars (see Section 6.1) is captured by the interface elements.

From aforementioned observations on the Figures 6.16 to 6.18 it can be concluded that the modeling strategy B leads to consistent and realistic results for the two analyses with different meshes. Both analyses show a brittle failure in the load - deflection response and a cracking process that is accompanied by the development of several flexural cracks with clear crack spacing, one localized failure crack and a horizontal crack along the reinforcement. These results are generally in good agreement with the experimentally obtained loading and failure processes. Nevertheless, also some differences between the simulations and the experiment are observed, e.g. regarding the predictions of the maximum load capacities and the initial small load drop with the subsequent small plateau in the numerical load - deflection responses. The latter issue might typically be a consequence of the quasi-static and sequentially linear analysis procedure, since during this small plateau the interface crack along the reinforcement propagates towards the right side of the beam.

Considering the numerically obtained bond behavior more specifically, it can be concluded that this strategy simulates accurately the interaction between the concrete and reinforcement. The relation between the steel strains, slip and concrete strains along the reinforcement is realistic and the locations of the flexural and failure cracks can be clearly identified from these results. Moreover, the interface not only simulates bond phenomena at local spots, but also captures the horizontal debonding crack along the reinforcement by significant slipping over an length of 780 mm. Note that this full debonding all the way to the right end of the beam would not occur if, according to the experiment, the anchorage plates would have been included in the model.

Despite foregoing conclusions, the modeling strategy B has a conceptual deficiency. As observed from the crack path plots, cracks can only initiate in continuum elements at the mesh boundary. Consequently, internal cracks, that
6. EXTENSION TOWARDS SHEAR FAILURE IN REINFORCED CONCRETE

originate from the ribs of a reinforcing bar and are captured by the interface elements, can never develop to a flexural or shear crack. Although this limitation of the interaction between bond cracks and primary cracks seems not relevant for the current benchmark, the issue is not satisfactory from a physical point of view. Modeling strategy C aims to overcome this deficiency.

6.5.2 Modeling strategy C∗

Strategy C allows crack (path) initiation in the continuum elements along the mesh boundary and the reinforcing bars. The interface constitutive behavior adopts a pull-out bond failure curve according to Reference [41]. The cubic function is simplified by a linear elastic - ideally plastic $\tau - \Delta u_s$ curve, see Figure 6.5, using a maximum shear stress $\tau_{\text{max}}$ of 1.9 $f_t = 6.65$ N/mm². As in SubSection 6.5.1, $k_t$ is set to 292 N/mm³ and $k_n$ is taken as infinitely stiff.

Figure 6.19 presents the load - deflection responses of the analyses on a structured mesh with quadrilaterals and an unstructured mesh with triangular elements. The analyses predict maximum load capacities of 221.2 kN and 198.3 kN respectively. Compared to Figure 6.16, the curves have a rather spiky shape in the post-peak regime. Several sharp peaks and snap-backs are visible before the load remains below 50 kN.

Figure 6.20 shows the two corresponding crack path plots, after a complete failure has occurred. Compared to Figure 6.17, several differences can be observed. Firstly, these two plots reveal less agreement with respect to the number of flexural cracks, and the location and shape of the failure cracks. Secondly, based on the developments of the principal strains, two failure cracks can be marked in each crack path plot. These failure cracks are labeled with “FC_initial” and “FC_final”, indicating their sequence of appearance in the loading process. Note that once the crack “FC_final” is fully developed, the crack “FC_initial” is almost closed. Thirdly, these two plots reveal much more small crack paths in the continuum elements around the reinforcement. Some of them have their starting point at the location of the reinforcing bars. The small Goto look-alike cracks are present in all the continuum elements above the reinforcing bars, between the intersection point of crack “FC_final” with the reinforcing bars and the left end of the beam.

The inset C shows a detail of crack “FC_final” in the unstructured mesh, which illustrates how the CPA in the smeared crack concept handles the crack

∗Based on a section in Reference [138].
6.5 Analyses with the proposed crack propagation algorithm

Figure 6.19: Load - deflection responses of the analyses on the RC shear beam with modeling strategy C and a structured and an unstructured (medium) mesh.

Figure 6.20: Crack path plots, after complete failure, of the analyses on the RC shear beam with modeling strategy C in Figure 6.19.
merging. The left crack path does not intersect directly with the crack path in the middle, but their corresponding row of elements are connected. Hence, both crack paths form together the global localized failure crack, as can be observed from the local crack widths. Note that for discrete crack techniques as X-FEM, the crack paths themselves have necessarily to intersect in order to form a global localized crack from two different crack paths.

Figure 6.21 presents the four graphs with the normal and shear strains in the concrete, the axial strain in the reinforcement and the slip in the interface over the whole length of the beam, for the analysis with the structured mesh. The results are plotted for a load step at the peak (gray line) and after failure (black line), see the red colored dots in Figure 6.19. The results in these four graphs are representative for the analysis with the unstructured mesh, which results are presented in Figure C.6 of Appendix C.

The results of Figure 6.21 reveal a different bond behavior, as observed from Figure 6.18. The $\varepsilon_{xx,c}$ curve at the peak load shows an erratic pattern and the locations of the flexural cracks are more difficult to determine. Furthermore, the shear strains in concrete for the row of integration points just above the reinforcing bars are much more pronounced and they change significantly between the two considered load steps. The gray line shows a remarkable peak around $x = 2130$ mm, the location of crack “FC$_{initial}$”, with an almost linearly descending branch to $x = 2700$ mm, the location of the support at the right side. On the other hand, the black line shows only high $\gamma_{xy,c}$ values between the left end of the beam, and $x = 900$ mm, the location where crack “FC$_{final}$” intersects with the reinforcement. The $\varepsilon_{xx,s}$ curves reveal a similar behavior of the reinforcing bars as in Figure 6.18, with relatively high strain values at the peak load and relatively low strain values after failure. Finally, the slip in the interface elements is rather limited and only small jumps are visible in the curve that belongs to the load step at the peak.

The observations on Figures 6.19 to 6.21 indicate a remarkable sequence in the failure processes of both analyses. Around the peak load, the crack with label “FC$_{initial}$” developed and showed a strong strain localization. From the shear strains in the concrete, it can be observed that a horizontal debonding crack developed simultaneously in the continuum elements above the reinforcement, which tended to propagate towards the right end of the beam. However, during the subsequent spiky post-peak regime this process was interrupted and strains started to localize at the left side of the beam. The new strain localization, labeled with “FC$_{final}$”, was also accompanied by the development of a shear plane in the continuum elements above the reinforcement. Complete
6.5 Analyses with the proposed crack propagation algorithm

Figure 6.21: Strains in concrete, strain in reinforcement and slip in interface of the analysis on the RC shear beam with modeling strategy C and a structured mesh, see Figures 6.19 and 6.20.

failure of the beam occurred, after further development of “FC_{final}” and when the horizontal debonding crack reached the left side of the beam.

According to the idea behind modeling strategy C, the crack path plots reveal indeed crack (path) initiation along the reinforcement. Nevertheless, from the foregoing observations and description of the numerically obtained failure process, it can be concluded that modeling strategy C is less successful in the simulation of the test with the RC shear beam compared to modeling strategy B. The switch from a failure crack at the right side to a failure crack at the left side of the beam is obviously in contradiction with the experimental results. Furthermore, the observed differences in the crack path plots indicate a mesh dependency.
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The reason for the unrealistic failure process and the mesh dependent cracking behavior with modeling strategy C is not obvious. Two issues might play a role. Firstly, the adopted pull-out bond failure curve may not be a proper constitutive relation for the interface in this fracture problem. The interface elements were meant to capture the small Goto cracks and the formation of a (local) sliding plane, but fail to do so and reveal hardly any bond-slip. Instead, the continuum elements above the reinforcing bars show small cracks and significant shear strain values. This is a typical behavior for FE models with perfect bond rather than for the explicit bond-slip modeling approach. Consequently, the fracture mechanics of the bond related cracks is not properly handled. Secondly, the many cracks in the continuum elements above the reinforcing bars, as unintended way of simulating the energy dissipation of bond related internal cracks, could hamper the development of the final shear failure crack, since these cracks are tracked by the crack propagation algorithm. This is illustrated in the inset B of Figure 6.20. A lot of crack paths in the two element rows just above the reinforcing bars intersect with their neighboring element, leading to an early crack path completion (see SubSection 5.2.1). Hence, the rows of elements in the plain concrete area on top of them cannot damage anymore.

6.6 Summary

This chapter presented an extensive study to the FE modeling of brittle shear failure in a RC beam without shear reinforcement. Firstly, the considered benchmark and the mechanisms of shear transfer and bond-slip were introduced. Subsequently, several ways to model these mechanisms were discussed and three strategies were defined. The modeling strategy with perfect bond was adopted in exploratory analyses and in analyses with the proposed crack band width formulation from Chapter 4. Two different explicit bond-slip modeling strategies were adopted in analyses with the proposed CPA from Chapter 5.

The exploratory analyses reveal the importance of an accurate crack band width estimation and a variable shear retention relation in the fixed smeared crack model. Furthermore, comparisons with conventional nonlinear FE analyses show the potential of SLA to explore the post-peak regime and to find realistic non-symmetric failure crack patterns, in absence of convergence and bifurcation problems. The analyses with the proposed crack band width formulation show an objective energy consumption and a reduced tendency of cracks to propagate along continuous mesh lines for the different quadrilateral element types. Nevertheless, the cracking behavior remains not fully mesh
objective. The analyses with the proposed CPA show promising results, but raise new questions as well. In general, issues as multiple cracking and crack merging are properly captured. However, the principal idea of a CPA to track each individual crack causes difficulties in zones along reinforcing bars that are affected by bond mechanisms. The two-dimensional simulations on the RC shear beam provide the most realistic and consistent results for the strategy that lumps all the fracture energy dissipation of the bond related cracks in interface elements and where the CPA focuses only on the tracking of primary cracks.
6. EXTENSION TOWARDS SHEAR FAILURE IN REINFORCED CONCRETE
Chapter 7

Conclusions and Outlook

7.1 Conclusions

Nonlinear FE analyses can provide good understanding of the structural behavior and the local or global failure mechanisms of bridges and viaducts. This knowledge can be employed for an optimal utilization of the capacity in these RC structures, either via clever designs in case of new structures or via reliability checks in case of existing structures with uncertainties regarding the capacity. However, nonlinear FE analysis constitutes a broadly framed concept and the results can strongly depend on the practitioners modeling choices regarding the kinematic description, the constitutive model and the analysis procedure to obtain equilibrium. Since the number of available modeling tools is enormous, even for a specific field as the modeling of quasi-brittle concrete fracture (see Chapter 2), it is not straightforward to make a proper selection from these tools and to set the right model and control parameters. Hence, the strength of nonlinear FE analysis, by showing such a variety in best options for certain classes of problems, is in danger to become simultaneously a weakness from a viewpoint of functionality. This work took the functionality as a starting point. In the context of structural analyses of brittle (shear) failure in RC, it is the author’s opinion that the crack band approach in the smeared crack concept, combined with the SLA method, can serve as an efficient, effective, robust and accurate modeling strategy, provided that the inherent mesh-induced directional bias can be sufficiently reduced. Hence, the aim of this thesis was to improve numerical simulations with the aforementioned modeling strategy, by developing enhancements of the crack band approach that increases its accuracy and preserves its relative simplicity. This aim was decomposed into four objectives, which are evaluated in the sequel.
The first objective, to find a way for a systematic and quantitative assessment of mesh-induced directional bias, was addressed in Chapter 3. A numerical testing procedure is developed that enables to study strain localizations in FE models with uniform meshes. The systematic assessment of the bias is obtained by the variation of the mesh alignment or element orientation in these models, whereas the loading (direction) is kept the same. Boundary disturbances for skewed meshes are avoided by the addition of periodic boundary conditions. The quantitative assessment of the bias is subsequently obtained by the calculation of the ratio between the numerically obtained fracture energy and the specified fracture energy for each considered mesh alignment. Foregoing testing procedure is applied to the crack band approach with an existing element projection based formulation \( l_p \), using two-dimensional rectangular shaped FE models with uni-axial tensile loading. Several FE characteristics and the size of the rectangular shaped FE model are varied. The test could also be applied to other constitutive models. Moreover, it can be conducted with three-dimensional FE discretizations and different loading conditions.

The second objective, to develop enhancements of the crack band approach in the smeared crack concept in order to reduce the mesh-induced directional bias, was addressed in the Chapters 4 and 5. Two different types of enhancements are proposed. Firstly, the element projection based crack band width formulation is extended by an element type specific strain localization factor and an element type specific mesh alignment factor. These multiplication factors are empirically derived from the results of the aforementioned testing procedure that was conducted on \( l_p \). Secondly, a (delayed) \( C^1 \) – continuous CPA is developed for the standard smeared crack concept. The cracks keep their smeared representation, but they are forced to propagate along a single row of conventional continuum elements rather than potentially originating anywhere in the FE mesh. From a more general perspective, the two enhancements have the following features. On the one hand, the enhanced band width estimator \( l_p \) concerns the transverse direction of a smeared crack. It primarily focuses on the maximization of a mesh objective energy consumption for the different elements types. The current formulation of \( l_p \) is relatively easy to implement in existing FE codes. However, its applicability is limited to square and right-angled isosceles triangle shaped elements. On the other hand, the proposed CPA concerns the longitudinal direction of a smeared crack. It primarily focuses on the minimization of the influence of a mesh alignment on the propagation direction of a crack. Although the implementation in FE codes...
requires more effort, the algorithm is more widely applicable and not restricted to some element shapes.

The third objective, to investigate the influence of the proposed enhancements on the modeling of brittle failure in plain concrete, was also addressed in the Chapters 4 and 5. Both enhancements are validated by three well-known plain concrete fracture tests with mode I and mixed mode fracture. Additionally, the CPA is validated by the test with periodic boundary conditions. Compared to $l_p$, the influence of $l_p$ (without CPA) on the results is limited and mainly reflected in more consistent predictions of the maximum load capacities. Crack patterns remain still biased, although in some analyses more realistic failure cracks are visible. The simulations with the CPA show major improvements in the obtained failure crack patterns. These results are mesh objective and in good agreement with the expected or experimental results.

The fourth objective, to explore different modeling strategies, material and mesh sensitivities and the influence of the proposed enhancements on the simulation of brittle shear failure in reinforced concrete, was addressed in Chapter 6. Three strategies are selected for the modeling of a shear critical RC beam without shear reinforcement. The first modeling strategy, with perfect bond, is used to study the sensitivity of the results with respect to the regularized fracture energy, the shear retention relation, the number of saw-teeth, the element size and the included possibility to non-symmetric failure modes. Furthermore, SLA results are compared to the results of conventional nonlinear FE analysis. The modeling strategy with perfect bond is also used to investigate the differences between the band width estimators $l_g$ and $l_p$. Compared to the plain concrete notched specimens, the positive effect of $l_p$ regarding the predictions of the maximum load capacities is much more clear now, since the RC shear beam involves a larger cracked area that dissipate energy. Hence, it can be expected that for the modeling of large-scale RC structures this effect becomes even more pronounced. The second and third modeling strategy explore different ways of bond-slip simulation with the proposed CPA. Generally, crack tracking techniques cannot simply be used in two-dimensional structural analysis of RC. They face challenging issues as the formation of multiple flexural and shear cracks that propagate and possibly intersect other cracks and reinforcing bars. Moreover, it requires further consideration regarding the modeling of bond related cracks and their interaction with the flexural and shear cracks. From the analyses with the two modeling strategies, promising results are especially obtained when the slip and the fracture energy dissipation of all bond related cracks are lumped into bond-slip interface elements and when the CPA is only focused on the tracking of primary cracks.
7. CONCLUSIONS AND OUTLOOK

The main conclusions of this work can be summarized as follows:

1. existing element projection based crack band width formulations can suffer from significant directional mesh bias, depending on element shape, interpolation function and numerical integration scheme (see SubSection 3.3.3);
2. the test with periodic boundary conditions is suitable to optimize the model parameter that has to ensure objectivity of the crack band approach (see Section 4.1);
3. the proposed crack band width estimator $l_p$ results in an objective energy consumption and in a consistent prediction of the maximum load capacity for different quadrilateral element types (see Sections 4.3 and 6.4);
4. although $l_p$ can reduce the tendency of cracks to propagate along continuous mesh lines, the crack patterns remain biased (see Sections 4.3 and 6.4) and it should therefore be used with caution for simulations of localized fracture problems;
5. the proposed crack propagation algorithm reduces significantly the mesh-induced directional bias in the simulations of plain concrete notched specimens (see Section 5.3);
6. quadratic elements are more suitable than linear elements for the simulation of localized fracture problems with an orthotropic fixed crack model (see Sections 3.3, 4.3 and 5.3);
7. an accurate regularized fracture energy $G_f/l$ and a variable shear retention relation are essential for realistic simulations of localized brittle shear failure with the fixed smeared crack model (see Section 6.3);
8. the sequentially linear analysis method is suited to capture brittle shear failure responses, showing the ability to explore the post-peak regime and to find realistic non-symmetric failure crack patterns, in absence of convergence and bifurcation problems (see Section 6.3);
9. the proposed crack propagation algorithm is able to simulate the behavior of the reinforced concrete shear beam, showing realistic and objective results and properly handling aspects as multiple cracking and crack merging (see Section 6.5 and Appendix C);
10. the application of the proposed crack propagation algorithm in two-dimensional simulations of the reinforced concrete shear beam is an intricate modeling task (see Sections 6.2 and 6.5).
7.2 Outlook

This research was realized in the context of two-dimensional simulations of brittle shear failure with the SLA method. In order to make the proposed modeling strategy and enhancements wider applicable to structural analysis in engineering practice, future investigations should focus on their extensions to the three-dimensional context.

In case of the new crack band width estimator $l_p$, such an extension will be rather straightforward. Since the test with periodic boundary conditions can also be applied to three-dimensional FE discretizations, a similar procedure as in the Sections 3.3 and 4.1 can be conducted to derive the strain localization factors and the mesh alignment factors for solid element types.

The extension to 3D in case of the proposed CPA will be more challenging. Specifically, the determination of a curved crack plane through a calculated propagation field within the solid element will address serious issues. The adopted particle tracking technique should probably be replaced by a more sophisticated method. Furthermore, it is not clear if and how the proposed CPA can deal with crack plane propagation in a solid element with 2 or 3 element side planes already intersected. For existing local crack tracking strategies with flat crack planes such a case implies an over-constrained situation, resulting in a crack plane propagation that depends exclusively on the surrounding crack geometry and that is completely independent of principal strain or stress state in the crack tip element [49, 72]. However, the proposed CPA with curved crack planes allows for more flexibility and might therefore be less sensitive to these over-constrained situations.

Besides the material modeling part, also the SLA method should be further devised for three-dimensional structural analysis. Current two-dimensional and three-dimensional simulations can be hampered by too long running times, due to a large number of degrees of freedom and/or number of load steps. A possible interesting solution direction is the development of a tailor-made SLA solver, which utilizes the fact that for each new load step the global stiffness matrix changes just rather locally. Such an optimized solver enables simulations with sufficient mesh densities and still acceptable running times.

Extensions of the proposed enhancements to the conventional nonlinear FE analysis procedure are also useful to broaden the range of applicability. For $l_p$, this is rather straightforward. For the CPA, a proper prediction-correction method is required to blend it with an incremental-iterative approach.
Further study to the modeling of RC when using the proposed CPA is recommended. The role of the parameters $d_{crt,cks}$ and $r_{exc}$ is not yet investigated in this context. Moreover, there is room for improvements regarding the simulation of bond mechanisms. Experimental research, together with analytical and detailed three-dimensional modeling of ribbed reinforcing bars embedded in concrete, could provide more insight in the bond-slip behavior and properties as $k_t$ and $\tau_{max}$. The output of such work should be translated to interface constitutive relations, which can subsequently be used as input for structural analysis [119]. In addition, the use of sophisticated interface bond models, e.g. [92], should be considered.
Appendix A

Overview analyses
### Table A.1: Overview of the numerical simulations in the thesis and references to the sections where they are presented.

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<th>periodic b.c. test</th>
<th>3-point bending test</th>
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Crack Propagation Algorithm

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Superscripts

a Variations on the regularized fracture energy, shear retention, number of saw-teeth and the possibility to non-symmetric failure modes + conventional nonlinear FE analysis.
b Variations on the element size (coarse, medium, fine).
c Analysis with perfect bond.
d 5 or 10 mesh orientations.
e Elements use a selective reduced integration for the shear terms (constant shear).
f Variations on the specimen size.
g Variations on the damage threshold value for crack path fixation.
h Analysis with the enhanced element projection based crack band width estimation.
i Analyses with 2 bond-slip modeling strategies.
A. OVERVIEW ANALYSES
Appendix B

“Specimen” size objectivity study
Figure B.1: Normalized load - total inelastic displacement curves of analyses from Table 3.1 with varying $L_x$ + their crack plots in deformed meshes ($\times$ 100) at $u_{cr} \approx w_{ult}$. The thickest lines correspond to $w \geq w_{ult}$. 
Figure B.2: Normalized load - total inelastic displacement curves of analyses from Table 3.1 with varying $L_y$ + their crack plots in deformed meshes ($\times 100$) at $u_{cr} \approx w_{ult}$. The thickest lines correspond to $w \geq w_{ult}$. 

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Figure B.3: Normalized load - total inelastic displacement curves of analyses from Table 3.1 with varying $L_x$ and $L_y$ and their crack plots in deformed meshes ($\times 100$) at $u_{cr} \approx w_{ult}$. The thickest lines correspond to $w \geq w_{ult}$.
Figure B.4: Normalized load - total inelastic displacement curves of analyses from Table 3.1 with varying element size + their crack plots in deformed meshes (× 100) at $u_{cr} \approx w_{ult}$. The thickest lines correspond to $w \geq w_{ult}$. 

$F_x / (L_y t)^{-1}$ [N/mm²] 

$u_{cr}$ [mm] 

Element size = 25 mm (analysis 4a) 

Element size = 50 mm (reference) 

Element size = 100 mm (analysis 4b) 

Theoretical solution 

Analysis 4a 

Reference 

Analysis 4b 

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B. "SPECIMEN" SIZE OBJECTIVITY STUDY
Appendix C

Mesh objectivity study of the proposed CPA
C. MESH OBJECTIVITY STUDY OF THE PROPOSED CPA

Figure C.1: Crack path plots, after complete failure, of the analyses on the RC shear beam with modeling strategy B and a varying element size in structured (a) and unstructured meshes (b).

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Figure C.2: Crack path plots, after complete failure, of the analyses on the RC shear beam with modeling strategy C and a varying element size in structured (a) and unstructured meshes (b).
Figure C.3: Crack paths, strains in concrete, strain in reinforcement and slip in interface of the analysis on the RC shear beam with modeling strategy B and a structured medium mesh (see Figure C.1(a)).
Figure C.4: Crack paths, strains in concrete, strain in reinforcement and slip in interface of the analysis on the RC shear beam with modeling strategy B and an unstructured medium mesh (see Figure C.1(b)).
C. MESH OBJECTIVITY STUDY OF THE PROPOSED CPA

Figure C.5: Crack paths, strains in concrete, strain in reinforcement and slip in interface of the analysis on the RC shear beam with **modeling strategy C** and a structured medium mesh (see Figure C.2(a)).
Figure C.6: Crack paths, strains in concrete, strain in reinforcement and slip in interface of the analysis on the RC shear beam with modeling strategy C and an unstructured medium mesh (see Figure C.2(b)).
C. MESH OBJECTIVITY STUDY OF THE PROPOSED CPA
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Summary

The crack band approach (in the smeared crack concept) is widely used for the modeling of concrete fracture and is an important analysis technique within advanced engineering. However, the simulations can be impeded by mesh-induced directional bias. Cracks prefer to propagate along continuous mesh lines, which causes a dependency of the numerical results on the structure and orientation of the finite element discretization. In this thesis, two different enhancements of the crack band approach are proposed to reduce the mesh bias as much as possible. The work in this thesis is realized within the Sequentially Linear Analysis method and focuses on accurate and reliable modeling of brittle shear failure.

The first enhancement concerns the transverse direction of a smeared crack. An existing element projection based crack band width formulation is implemented and its mesh-induced directional bias is systematically assessed by a newly developed testing procedure with periodic boundary conditions. Based on the observed and quantified bias, which depends on the element shape, interpolation function and numerical integration scheme, the crack band width formulation is extended by an element type specific strain localization factor and an element type specific mesh alignment factor. Including these two factors, the enhanced band width estimator focuses on the maximization of a mesh objective energy consumption via an improved fracture energy regularization.

The second enhancement concerns the longitudinal direction of a smeared crack. A new crack propagation algorithm is developed that constructs, element by element, smoothly curved $C^1$-continuous crack paths across the finite element discretization. The cracks keep their smeared representation and they are forced to propagate along a single row of conventional continuum elements by considering the principal strains. The algorithm focuses on the minimization of the influence of a mesh alignment on the propagation direction of a crack.
SUMMARY

The performance of both enhancements is investigated by simulations of plain concrete fracture tests and a shear critical reinforced concrete beam without shear reinforcement. Comparisons to standard smeared crack analyses show that the proposed enhancements indeed lead to reduced mesh-induced directional bias. The predictions of maximum load capacities become more consistent and, especially for the crack propagation algorithm, the obtained failure crack patterns are realistic and mesh independent. Detailed analyses of reinforced concrete, with explicit modeling of the bond behavior between reinforcing bars and concrete via cracking, require further developments of the enhancements, including a three-dimensional generalization.

Both enhancements increase the accuracy of the crack band approach in the smeared crack concept and simultaneously preserve its relative simplicity. Hence, they can be valuable for engineering practice, supporting reliable predictions of ultimate load capacities, failure mechanisms and post-peak behaviors of quasi-brittle reinforced concrete structures.
Samenvatting

PROPAGATIE EN BANDBREEDTE VAN UITGESMEERDE SCHEUREN

Het scheurbandmodel (in het uitgesmeerd scheurconcept) wordt op grote schaal toegepast voor het simuleren van breuken in beton en is een belangrijke analysetechniek binnen geavanceerde engineering. De simulaties kunnen echter gehinderd worden door een richtingsvooringenomenheid, geïnduceerd door de eindige-elementendiscretisatie. Scheuren vertonen een voorkeur om zich langs de doorgaande lijnen van het elementennet voort te planten, waardoor de numerieke resultaten afhankelijk zijn van de structuur en de orientatie van het elementennet. In dit proefschrift wordt een tweetal uitbreidingen van het scheurbandmodel voorgesteld om de richtingsvooringenomenheid zoveel mogelijk te reduceren. Het onderzoek is uitgevoerd met de Sequentieel Lineaire Analyse methode en concentreert zich op een nauwkeurige en betrouwbare modellering van bros bezwijken op dwarskracht.

De eerste uitbreiding heeft betrekking op de dwarsrichting van een uitgesmeerde scheur. Een bestaande scheurbandbreedte-formulering, gebaseerd op element projectie, is geïmplementeerd. Door middel van een nieuw ontwikkelde testprocedure, die gebruik maakt van periodieke randvoorwaarden, is deze formulering systematisch beoordeeld op de richtingsvooringenomenheid. Op basis van de waargenomen en gekwantificeerde afwijking, die afhangt van de elementvorm, -interpolatie en -integratie, is de scheurbandbreedte-formulering uitgebreid met een element-specifieke factor voor de lokalisatie van de rek en een element-specifieke factor voor de orientatie van de mesh lijnen. Deze nieuwe schatting van de scheurbandbreedte regulariseert de energieconsumptie en maximaliseert de objectiviteit van de resultaten ten opzichte van het gekozen elementennet.

De tweede uitbreiding heeft betrekking op de langsrichting van een uitgesmeerde scheur. Een nieuw scheurpropagatie algoritme is ontwikkeld dat
SAMENVATTING

Elementsgewijs gladde gekromde, \( C^1 \) – continue scheurpaden door de eindige- elementendiscretisatie construeert. De scheuren behouden hun uitgesmeerde representatie en worden gedwongen zich op basis van hoofdrekken voort te planten langs een enkele rij elementen. Het algoritme richt zich op het minimaliseren van de invloed van de lijnen van het elementennet op de voortplantingsrichting van een scheur.

De prestaties van beide uitbreidingen zijn onderzocht via simulaties van breuktesten op ongewapend beton en op dwarskracht in gewapend beton zonder beugels. Uit vergelijkingen met de analyses die gebruik maken van het standaard uitgesmeerd scheurconcept blijkt dat de voorgestelde uitbreidingen inderdaad leiden tot een reductie van de door de eindige-elementendiscretisatie geïnduceerde richtingsvooringenomenheid. De voorspellingen van het draagvermogen zijn consistent en in het bijzonder voor het scheurpropagatie algoritme zijn de verkregen bezwijkingscheurpatronen realistisch en onafhankelijk van het elementennet. Gedetailleerde analyses van gewapende betonnen constructies, waarin het aanhechtingsgedrag tussen de wapeningsstaven en het beton expliciet wordt gemodelleerd via scheuren, vereisen verdere ontwikkeling van de twee uitbreidingen, waaronder een generalisatie naar 3D.

Beide uitbreidingen verhogen de nauwkeurigheid van het scheurbandmodel in het uitgesmeerd scheurconcept, terwijl de eenvoud behouden blijft. Ze bevorderen het betrouwbaar voorspellen van uiterste draagvermogens, bezwijkmechanismen en na-piek gedrag van quasi-brosse gewapend betonnen constructies, en kunnen hiermee de ingenieurspraktijk een goede dienst bewijzen.
Curriculum Vitae

June 19, 1986  Born in Rotterdam, The Netherlands, as Aart Theodoor Slobbe

2003 – 2007  BSc student in Civil Engineering (graduated)
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2007 – 2010  MSc student in Civil Engineering (graduated cum laude)
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