# A strain gradient approach to the analysis of nanoarches: Formulation and numerical solution with NURBS interpolation

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SUMMARY. Size effects observed in the mechanical behavior of micro- and nanostructures cannot be modeled by means of classical continuum mechanics. Enhanced continuum models based on nonlocal integro-differential or higher-order differential formulations have shown, on the contrary, that size effect can be modeled when appropriate modifications to the constitutive model are considered. The use of these models has been rapidily growing but their application to nanoarches appears not to have been fully investigated so far. In this contribution, we focus on a strain gradient formulation for thin arches. NURBS interpolations are considered in order to satisfy the high-order continuity requirement imposed by the strain gradient formulation and to interpolate the exact geometry of circular nanobeams.

## 1 INTRODUCTION

The growing interest in nanotechnology has fuelled the study of nanostructures such as nanotrusses, nanobeams and nanoshells. Classical continuum mechanics cannot fully describe the mechanical behavior of these structures due to the absence of an internal material length scale in the constitutive law. Since the works of Tupin [1] and Mindlin [2], higher-order gradient terms and internal length scale parameters have allowed the study of size effects at micro- and nanoscale. More recently, Aifantis and coworkers developed a strain gradient theory [3] in which microscale deformation is introduced by means of a higher-order strain tensor in the governing equations. An alternative to gradient approaches is Eringen's nonlocal continuum model where the stress at a point depends upon the strain field at all other points in the domain [4]. While higher-order differential theories are widely used for beams, their application to problems formulated in curvilinear coordinates is scarce. In this contribution we focus on the development of a strain gradient formulation in cylindrical coordinates with the aim of studying curved circular beams like nanoarches and nanorings. Strain gradient formulations require high-order continuity thus motivating the use of Hermitian finite elements, element-free Galerkin methods and staggered formulations [5]. Here we propose the use of NURBS basis functions which allow the construction of higher-order basis functions while exactly interpolating the arch geometry. A variational framework suitable for the exploitation of NURBS properties is proposed. The effective capability of the proposed strain gradient approach to capture size effects observed in nanostructures is assessed by comparing the analytical solution of a strain gradient quarter arch subjected to constant moment with the corresponding local solution obtained by means of classical continuum mechanics.

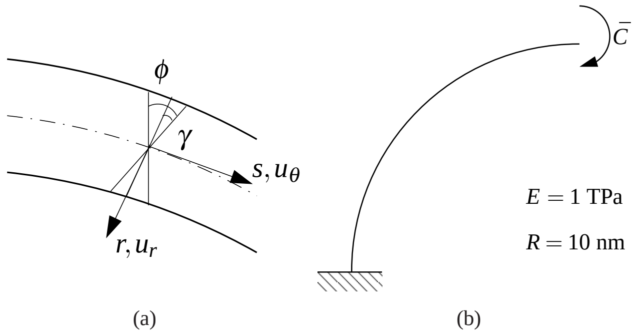


Figure 1: Timoshenko beam model (a) and cantilever arch with tip couple (b).

## 2 MODEL PROBLEM

A planar curved thin Timoshenko beam ($h << R$) is considered. The kinematics of the beam is expressed through [6, 7]

$$v(s,r) = u_r\,\underline{e}_r + (u_\theta + r\,\phi)\,\underline{e}_\theta, \tag{1}$$

with $s$ the curvilinear coordinate, $u_\theta$ the axial displacement, $u_r$ the radial displacement, and $\phi$ the rotation as shown in Figure 1(a). The non-zero components of the strain tensor $\underline{\underline{\varepsilon}}$ in curvilinear coordinates are [6]

$$\varepsilon_{\theta\theta} = u_{\theta,s} - \frac{u_r}{R} + \phi_{,s} r \qquad \text{and} \qquad \varepsilon_{\theta r} = \varepsilon_{r\theta} = u_{r,s} + \frac{u_\theta}{R} + \phi. \tag{2}$$

According to the strain gradient formulation [2], the virtual work done by body and surface forces on a body $\Omega$ with boundary $\Gamma$ is

$$W_i = \int_\Omega \underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}}\,\mathrm{d}\Omega + \int_\Gamma \underline{\underline{\varepsilon}} \cdot \underline{\underline{\underline{\mu}}}\,\underline{n}\,\mathrm{d}\Gamma \tag{3}$$

where $\underline{\underline{\varepsilon}}$ is the Cauchy stress tensor, $\underline{\underline{\underline{\mu}}}$ the higher-order stress tensor, $\underline{\underline{\varepsilon}}$ the strain tensor and $\underline{n}$ the outward unit vector normal to the boundary of the body. According to Aifantis [3], the constitutive equations are

$$\underline{\underline{\sigma}} = \lambda\,\mathrm{tr}\big(\underline{\underline{\varepsilon}}\big)\underline{\underline{I}} + 2G\big(\underline{\underline{\varepsilon}} - g^2\nabla^2\underline{\underline{\varepsilon}}\big) \qquad \text{and} \qquad \underline{\underline{\underline{\mu}}} = 2g^2 G\nabla\underline{\underline{\varepsilon}} \tag{4}$$

in which $\lambda$ and $G$ are the two Lamé's constants and $g$ is the material length scale related to the volumetric strain energy. The derivation of the gradient and the Laplacian of the strain tensor in curvilinear coordinates is non trivial (more details can be found in [4]).

The equilibrium equations and the boundary conditions can be calculated by setting the first variation of the total potential energy, $\delta\Pi = \delta W_i - \delta L_e$, equal to zero —in $\delta\Pi$ the external virtual work $L_e = \int \big(q_\theta u_\theta + q_r u_r + c\phi\big)\mathrm{d}s$, $q_\theta$ and $q_r$ are the axial and radial distributed load, respectively, and $c$ is the distributed bending moment along the centroid axis of the curved beam. Replacing the strain terms (2) in equation (3), the first variation of the elastic strain energy $W_i$ (3) yields

$$\begin{aligned}\delta W_i = \int_\Omega \Bigg\{ & E\left[\varepsilon_{\theta\theta} - g^2\left(u_{\theta,sss} + \phi_{,sss} r - \frac{\phi_{,s}}{R} - 2\frac{\phi_{,s}}{R^2} r - 3\frac{u_{\theta,s}}{R^2} + \frac{u_r}{R^3} - 3\frac{u_{r,ss}}{R}\right)\right]\delta\varepsilon_{\theta\theta} \\ & + G\left[\varepsilon_{r\theta} - g^2\left(u_{r,sss} + \phi, ss - 7\frac{u_{r,s}}{R^2} - 3\frac{u_\theta}{R^3} + 5\frac{u_{\theta,ss}}{R} + 4\frac{\phi_{,ss}}{R}\right)\right]\delta\varepsilon_{\theta r}\Bigg\}\mathrm{d}\Omega + \int_\Gamma \delta\underline{\underline{\varepsilon}} \cdot \underline{\underline{\underline{\mu}}}\,\underline{n}\,\mathrm{d}\Gamma.\end{aligned} \tag{5}$$

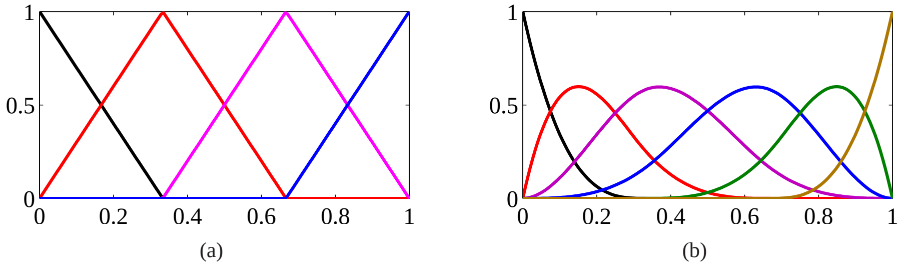


Figure 2: $k$-refinement from linear ($C^0$) to cubic ($C^2$) NURBS basis functions in a one-dimensional domain.

The continuity requirement of the first variation of the elastic strain energy, related to the order of the derivative of the problem fields, can be satisfied by means of high-order NURBS basis functions which can also interpolate, exactly, the geometry of circular nanoarches. Robust and efficient algorithms to compute high-order NURBS basis functions and their derivatives are well known in the literature [8].

The weak form of the problem can be derived employing high-order NURBS basis functions to interpolate the discrete displacement fields in terms of

$$u_\theta^h(s) = \sum_{i=1}^{n} N_{i,p}(s) u_{\theta\, i}, \qquad u_r^h(s) = \sum_{i=1}^{n} N_{i,p}(s) u_{r\, i} \qquad \text{and} \qquad \phi^h(s) = \sum_{i=1}^{n} N_{i,p}(s) \phi_i \tag{6}$$

where $p$ is the polynomial degree of the basis functions which are $p-1$ continuous. Since the solution will be investigated by increasing the number of the degrees of freedom and the polynomial order of the interpolation functions, the continuity will be increased as well by means of the $k$-refinement method. This method builds a homogeous structure of highly continuous basis functions along the whole one-dimensional domain as shown in Figure 2.

## 3 QUARTER-CIRCULAR CANTILEVER NANOARCH

The quarter-circular cantilever nanoarch loaded by a tip couple in Figure 1(b) is studied. It can be demonstrated that the equilibrium problem of a strain gradient arch can be conveniently replaced by an equivalent problem formulated in terms of the local/nonlolcal stress-strain law proposed by Eringen [4] (the proof and the circumstances under which such a result holds are discussed in detail in a forthcoming paper [9]). Figure 3 shows that the displacement and curvature fields obtained with the strain gradient approach for $g = 2$ nm are smaller than those obtained with a classical continuuum model for which $g = 0$ nm. These results are in agreement with experimental results on micro- and nanostructures and numerical results in literature [10].

## 4 CONCLUSIONS

A strain gradient model has been formulated in cylindrical orthogonal coordinates in order to study the static response of a circular nanonarch. A suitable NURBS variational framework was defined. The development of the numerical formulation is still in progress.

*References*


[1] R. Tupin. Elastic materials with couple-stresses. *Archive for Rational Mechanics and Analysis*, 11:385–414, 1962.

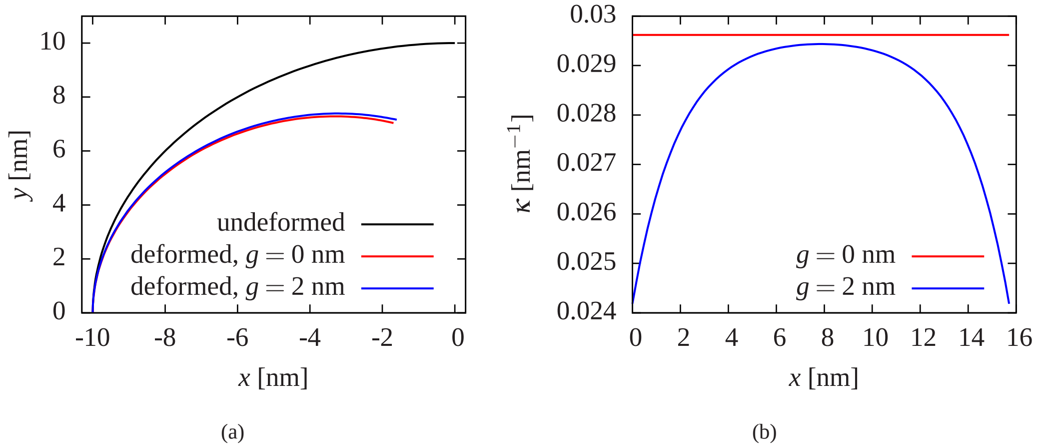


Figure 3: Solution of the flexural nanoarch in terms of displacement (a) and curvature (b).


[2] R. D. Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis*, 16:51–78, 1961.

[3] E. Aifantis. On the role of gradients in the localization of deformation and fracture. *International Journal of Engineering Science*, 30:1279–1299, 1992.

[4] A. C. Eringen. *Mechanics of continua*. John Wiley and Sons, Inc., 1995.

[5] H. Askes and E. C. Aifantis. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *International Journal of Solids and Structures*, 48:1962–1990, 2011.

[6] L. Ascione and F. Fraternali. Sulla statica delle travi ad asse curvo. Technical report, University of Salerno, Faculty of Engineering, 1986.

[7] A. Benedetti and A. Tralli. A new hybrid f.e. model for arbitrarily curved beam-I. Linear analysis. *Computers and Structures*, 33:1437–1449, 1989.

[8] C. de Falco, A. Reali, and R. Vázquez. GeoPDEs: A research tool for Isogeometric Analysis of PDEs. *Advances in Engineering Software*, 42:1020–1034, 2011.

[9] E. Benvenuti and A. Simone. A unified analytical approach to nonlocal and gradient elasticity: The case of a tensile rod. In preparation, 2012.

[10] D. C. C. Lam, F. Yang, A. C. M. Chong, J. Wang, and P. Tong. Experiments and theory in strain gradient elasticity. *Journal of the Mechanics and Physics of Solids*, 51(8):1477–1508, 2003.