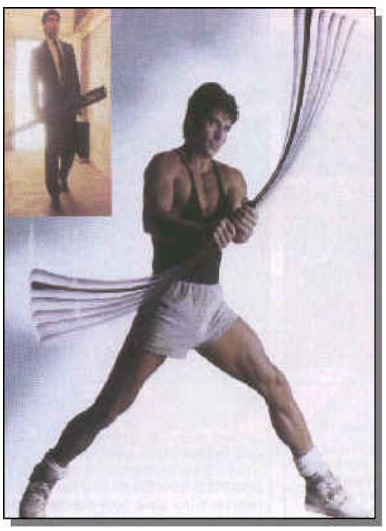
Delft University of Technology
Faculty of Mechanical Engineering
and Marine Technology
Mekelweg 2, 2628 CD Delft

Notes on Linear Vibration Theory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* Body Blade picture taken from: http://www.starsystems.com.au

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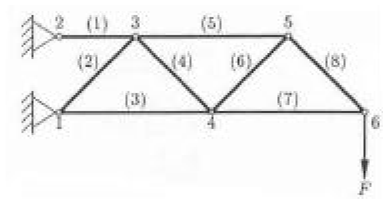
PART THREE: FINITE ELEMENT MODELLING



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1. INTRODUCTION

1. INTRODUCTION

Within the realm of Mechanical Engineering the subject of "Dynamics" is of considerable importance. Dynamics describes the interplay between loads and motion.

Given external loads produce motion, to be determined.

Conversely, given a motion, one may wish to determine the loads responsible for that motion.

External loads produce motion. Motion, in turn, produces inertial loads. As a result, the loads within the system vary during motion. And they modify the motion. If the resulting system motion is to develop according to certain performance criteria, the evolution of the system motion as determined by the system dynamics must be well analyzed.

Given the external loads and the resulting system motion, one may wish to determine the internal loads acting in the system, both in structural components and in devices that join those components. Also, one may wish to determine structural deformations experienced in structurally flexible system components.

Students preparing for engagement in Dynamics projects are in the very first place expected to deeping their understanding of basic concepts in engineering mechanics.
These concepts concern modelling and analysis in Statics (a special case of Dynamics), in Stress and Strain, and in principles of Dynamics.
In addition they must now apply their earlier knowledge of and insight into mathematics concepts, especially those related to ordinary differential equations, partial differential equations, approximation theory, and numerical integration. Indeed, Dynamics relies heavily on mathematics. Therefore the engagement in research in Dynamics requires not only good mathematical abilities but at the same time sufficient self-discipline and a burning desire towards understanding nature and technology.

Mechanical vibrations cover a large part of the entire field of Dynamics. Mechanical vibrations are present everywhere in daily life. Mechanical vibrations are the fluctuations of a mechanical system about its equilibrium configuration.
- Vibrations may be sought for. Examples are vibrations of musical instruments, vibrations of equipment to help improve the condition of the human body, vibrations of transportation equipment, vibrations required in the operation of extremely accurate (atomic) microscopes, vibrations of sensors and motors on micro- and nano-scale.
- Vibrations may be uncalled for and even be a nuisance. Examples are vibrations of chimneys and bridges and skyscrapers and high-voltage lines in the wind, mechanical vibrations as well as the resulting noise experienced by a human in an automobile or train or airplane or ship, vibrations experienced by manufacturing equipment (lathe, grinder, steel mill, wafer stepper) and by production equipment (oil-drillstrings, pipes carrying liquids), vibrations in mechatronic equipment during heavy-duty operations, vibrations in structures leading to failure to satisfy performance criteria if not leading to catastrophic break-up.

Clearly: vibrations galore.

Vibrations must be analyzed, and were and when necessary effective measures must be taken to modify those vibrations. To take measures towards modification, one must first attempt to understand that which one is trying to modify. In the present course the assiduous student will be given a helping hand to assist him in the acquisition of that understanding.

The present Course Notes on Dynamics address the following topics:
- vibrations of systems with a single degree-of-freedom
  (planar translation or rotation)
- vibrations of systems with multiple degrees-of-freedom
  (discrete systems in planar translation and/or rotation)
- vibrations of continuum systems (modelled exactly)
- vibrations of continuum systems (modelled as a collection of finite elements).

In all cases treated the equations of motion will be *linear*; an approximation usually valid for small amplitudes of displacements.

The course on Dynamics is best studied using a textbook as reference material.
An attractive list of possible Course Textbooks is contained in Chapter Two. Of the books listed there, especially the books by D.J. Inman and by S.G. Kelly should be mentioned. These two books make extensive use of numerical computations with MATLAB (Student Edition, version 5). The appropriate script files that come with those books are either to be downloaded through Internet (for Inman's book) or are contained in a CD-ROM provided with the textbook (Kelly's book).
For the present course the teacher will use Kelly's book as Course Textbook.

The Course Notes presented here serve to provide the student with a compact summary of applicable mechanical and mathematical concepts. Its objectives are:
- to clarify or amplify mathematical concepts already presented in earlier courses on Linear Algebra and on Differential Equations;
- to clarify or amplify engineering mechanics concepts already presented in earlier courses on Statics, Elasticity Theory, and Dynamics;
- to clarify, amplify, and extend the material in the Course Textbook.

However, the course notes presented therefore DO NOT replace the Course Textbook.

In addition, the student is to develop insight and experience by energetically trying his hand at dynamics exercises. These exercises can be found e.g., in the Course Textbook, in the book by Meriam and Kraige (Vol. 2, chapter 8), and on Blackboard (worked-out exam assignments).

The following activities are ESSENTIAL if the student is to master the subject:
- clear-minded attendance of the classes, actively writing comments in his own course notebook;
- thorough study of the relevant material in the Course Textbook;
- thorough study of the relevant material in the present Course Notes;
- energetic development of experience in solving recommended dynamics exercises.

We now sketch the "landscape" surveyed in the present Course Notes:

- chapter 2 contains a list of recommended literature for background study or for further study. These may also be found to constitute a source of much inspiration to the assidious reader. Two outstanding books in the context of the present course are those by Inman and by Kelly. Of these, the book by Kelly has been selected as Course Textbook for the present course.

- chapters 3 and 4 develop the equations of motion (dynamics equations) for rigid bodies displaying pure linear motion and pure angular motion, and for those rigid bodies moving in a single, inertially fixed plane (thereby displaying up to three degrees-of-freedom).

- an important case of system excitation is that in which a periodic load acts on the system. In chapter 5 it is shown how a periodic signal (such as a periodically varying external load) can be decomposed into an infinite series of harmonic excitation terms (Fourier decomposition). For linear systems, the response to a series of excitation terms is equal to the sum of the responses of the system to each of those excitation terms separately. As the case of excitation by a single harmonic term will be worked out in detail in later chapters, the extension to excitation by more general, periodic signals then becomes immediate.

- chapter 6 investigates the solution of a system of linear algebraic equations. The results obtained are fundamental for the study of the dynamics of systems with multiple degrees-of-freedom (cf. chapters 8, 12, 13, 14, 15, and 16).

- in chapter 7 the solution of a single, linear, second-order, ordinary differential equation is developed in detail. The results obtained are fundamental to the study of the dynamics of all systems studied in these course notes.

- in chapter 8 the solution of a system of linear, second-order, ordinary differential equations is developed in detail. The results obtained are fundamental to the study of the dynamics of all multi-degree-of-freedom systems studied in these course notes.

- chapter 9 considers the bending of a slender, prismatic beam. The results obtained are used in chapters 10, 13, and 15.

- in chapter 10 the equations of motion of several continuum bodies are derived. The derivation is always carried out in a direct manner; in some cases however also by considering the continuum body as the limit case of a collection of many very small rigid bodies connected by very stiff linear springs. Bodies considered are the string (or cable, displaying small transverse motion), the straight rod (or bar, displaying small longitudinal extension only), and the straight shaft (a straight rod displaying small torsion only).

- in chapter 11 the equation of motion of another important continuum body is developed: the initially straight beam (displaying small transverse displacement only).

- in chapter 12 the solution to a second-order partial differential equation is developed. The specific differential equation is the one arising in the motion of strings, rods, and shafts.

- in chapter 13 the solution to a fourth-order partial differential equation is developed. The specific differential equation is the one arising in the motion of beams.

- in chapter 14 the principle of virtual work is revisited. Here it is introduced with an eye to application to the analysis of the dynamics of structures obtained through finite element modelling. The principle of virtual work underlies various dynamics formalisms, such as the Euler-Lagrange formalism and the Hamilton formalism. Parenthetically we show that the development of the latter two formalisms is not necessary and in fact rather circuitous. Furthermore, it will be seen that the application of the principle of virtual work naturally leads to a formulation that would also result from the more contrived Galerkin approach involving certain "weighting functions" (see Section 15.6).

- in chapter 15 the subject of finite element modelling is introduced. For tutorial reasons the discussion is restricted to a single, nominally straight rod modelled as a single finite element only. The reason for this choice is that in this way the amount of space dedicated in these course notes to the finite element subject remains moderate while physical insight is maximized.

- in chapter 16 the subject of finite element modelling is again introduced. For tutorial reasons the discussion is restricted to a single, nominally straight beam modelled as a single finite element only. The reason for this choice is again that in this way the amount of space dedicated in these course notes to the finite element subject remains moderate while physical insight is maximized.

- in chapter 17 the concept of finite element modelling is applied to structures consisting of multiple rod elements and/or multiple beam elements.

- in chapters 16 and 17 only simple structural components were considered. Rods deform only in the direction of their longitudinal axis; beams deform only in the direction perpendicular to their longitudinal axis. These are idealizations. The general case, displaying deformation in three spatial directions simultaneously, is introduced in chapter 18. In addition, the possibility of including material damping is introduced.

- in all cases the resulting equations of motion are of the second-order, linear, ordinary type. Generally these differential equations are solved with the aid of numerical integration techniques. Treatment of the subject of numerical integration is beyond the scope of the present course notes. However: in general it can be said (somewhat roughly) that numerical stability of the numerical solution increases for given numerical values for integration parameters if the maximum value of the eigenfrequencies in the system is "sufficiently low". Therefore, one would like to throw away system components displaying high eigenfrequencies. Moreover, may of those high-frequency components will display relatively large numerical errors. Little or nothing will be lost by throwing them out. Also, by throwing away those high-frequency components, computational effort per integration step will be reduced. Chapter 19 outlines two commonly used techniques for reducing the complexity of the system equations (mainly by throwing away the high frequency system components).

- chapter 20 contains some concluding remarks.

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2. A SELECTION OF USEFUL LITERATURE

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Nijgh & Van Ditmar, Rijswijk, 1996.

Inman, D.J.
Engineering Vibration, 2nd edition.
Prentice-Hall, Upper Saddle River, N.J., 2000.
(Remark: software for the execution of the exercises in the book may be downloaded from the website http://www.cs.wright.edu/people/faculty/jslater/vtoolbox/vtoolbox.html)

Kelly, S.G.
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McGraw-Hill International Editions, Boston, Mass., 2000.
(Remark: software for the execution of the exercises in the book is contained in a CD-ROM that is provided with the book.)

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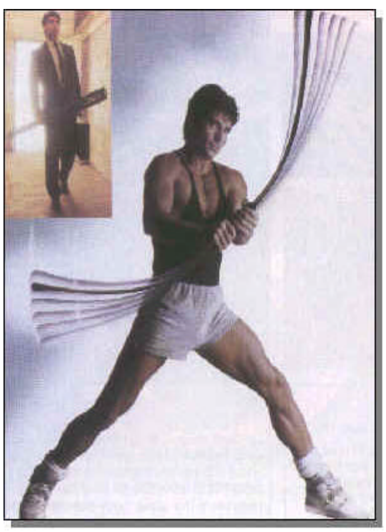

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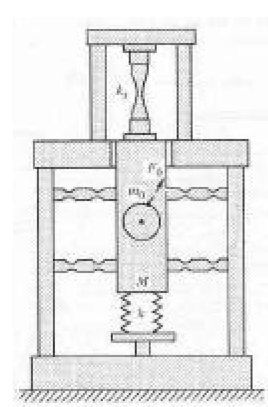
PART ONE:

MATHEMATICS FOR FINITE-DIMENSIONAL SYSTEMS

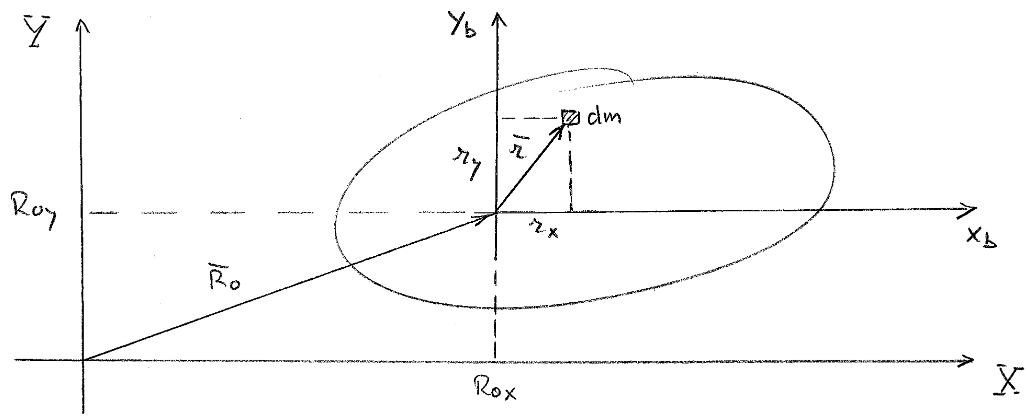


3. DYNAMICS OF A RIGID BODY IN ONE-DIMENSIONAL SPACE

3.1 Linear motion

3.2 Angular motion

Linear motion



Consider motion in $\bar{X}$ - direction only
(i.e. no rotation)

Newton: $\ddot{\bar{R}}\, dm = d\bar{f}$ for elementary mass dm.

$\bar{R}$ w.r.t. inertial space

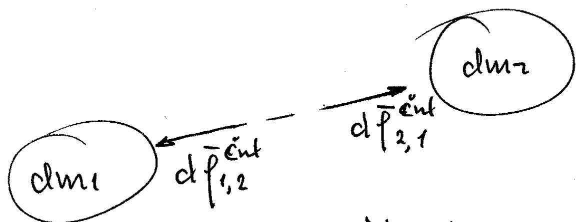
$\{\bar{X}, \bar{Y}\}$ = inertially fixed reference frame

$\{x_b, y_b\}$ = body-fixed reference frame

Position of dm: $\bar{R} = \bar{R}_o + \bar{r}$

$$(\bar{R}_o + \bar{r})^{\cdot\cdot}\, dm = d\bar{f} = d\bar{f}^{\,applied} + d\bar{f}^{\,internal}$$

Constraint forces $d\bar{f}^{int.}$: keep the body rigid



Newton: $d\bar{f}^{int}_{1,2} + d\bar{f}^{int}_{2,1} = \bar{0}$

Integrate over all mass elements:

$$\int (\ddot{\bar{R}}_0 + \ddot{\bar{r}})\, dm = \int d\bar{f}^{a} + \int d\bar{f}^{int}$$

$$M\ddot{\bar{R}}_0 + \int \ddot{\bar{r}}\, dm = \bar{F}^{a} + \bar{0}$$

In $\underline{X}$- direction:

$$M\ddot{R}_{0x} + \int \ddot{r}_x\, dm = F^a_x$$

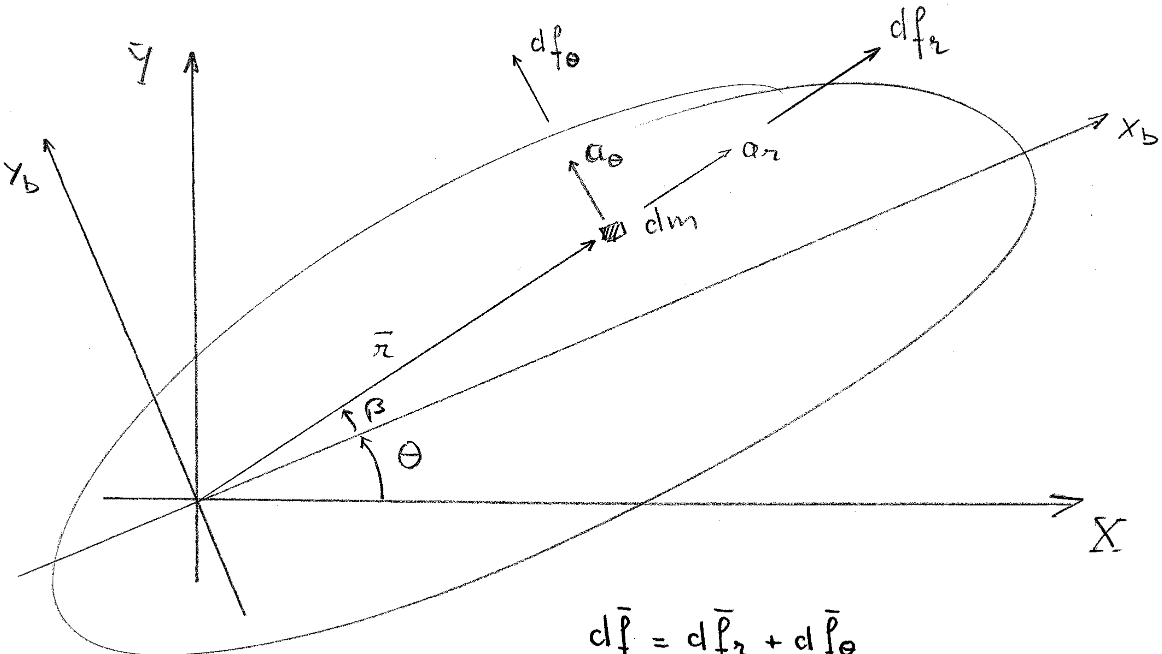
but r_x = constant (rigid body)

$$\Rightarrow \quad \boxed{M\ddot{R}_{0x} = F^a_x} \qquad \text{Newton}$$

Hence, for a non-rotating body: the body behaves like a point mass.

(similar equation for motion in $\underline{Y}$- direction).

Angular motion.



$$d\bar{f} = d\bar{f}_r + d\bar{f}_\theta$$

$\{X, Y\}$ = inertially fixed reference frame

$\{X_b, Y_b\}$ = body fixed reference frame

origin of both frames on rotation axis.

Define: $\varphi = \theta + \beta$ (polar angle)

$$\Rightarrow \begin{cases} X = r \cdot \cos\varphi \\ Y = r \cdot \sin\varphi \end{cases}$$ where r = constant.

$$\begin{cases} \dot{X} = -r \sin\varphi \cdot \dot{\varphi} \\ \dot{Y} = r \cdot \cos\varphi \cdot \dot{\varphi} \end{cases}$$

$$\begin{cases} \ddot{X} = -r.\cos\varphi.\dot{\varphi}^2 - r.\sin\varphi.\ddot{\varphi} \\ \ddot{Y} = -r.\sin\varphi.\dot{\varphi}^2 + r.\cos\varphi.\ddot{\varphi} \end{cases}$$

Component of inertial acceleration along radius r:

$$a_r = \ddot{X}\cos\varphi + \ddot{Y}\sin\varphi \overset{!}{=} -r\dot{\varphi}^2$$

Component of inertial acceleration in direction of motion:

$$a_\theta = -\ddot{X}\sin\varphi + \ddot{Y}\cos\varphi \overset{!}{=} r\ddot{\varphi}$$

Newton, in direction of motion:

$$a_\theta\, dm = df_\theta = df_\theta^{a} + df_\theta^{int.}$$

$$r\ddot{\varphi}\, dm = df_\theta^{a} + df_\theta^{int.} \qquad (\ddot{\varphi} = \ddot{\theta})$$

Multiply by r:

$$r^2\ddot{\theta}\, dm = r.df_\theta^{a} + r\,df_\theta^{int.}$$

Integrate over all mass elements:

$$\int r^2\ddot{\theta}\, dm = \int r\, df_\theta^{a} + \underbrace{\int r\, df_\theta^{int.}}_{=0}$$

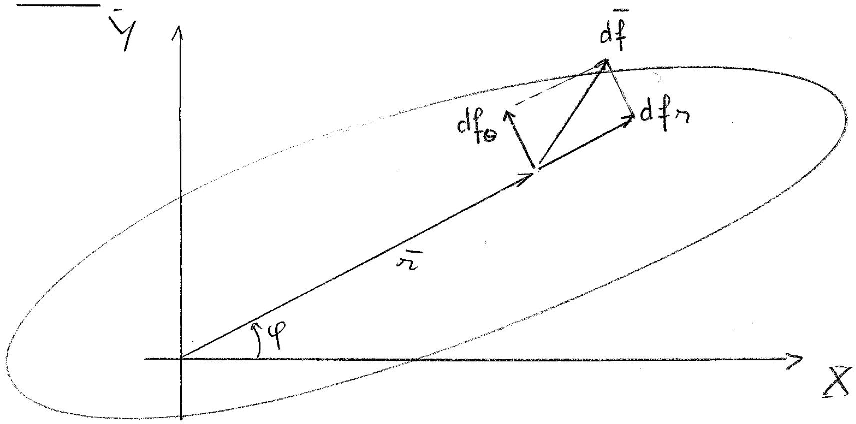
$$\left(\int r^2 dm\right)\ddot{\theta} = T_\theta^{a}$$

$$\boxed{I\,\ddot{\theta} = T_\theta^{\,a}} \qquad \text{Euler}$$

I = mass moment-of-inertia w.r.t. origin

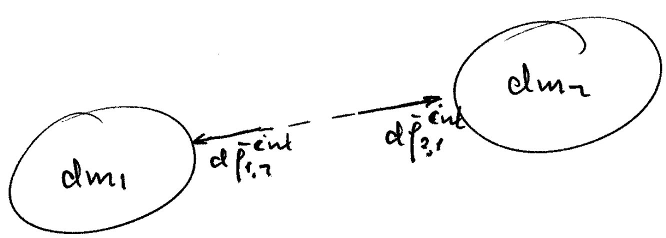
$T_\theta^{\,a}$ = net applied torque, taken w.r.t. origin.

Note:



$r \cdot df_\theta^{\,a}$ = torque due to elementary force $df_\theta^{\,a}$ exerted on mass element dm.

Note:



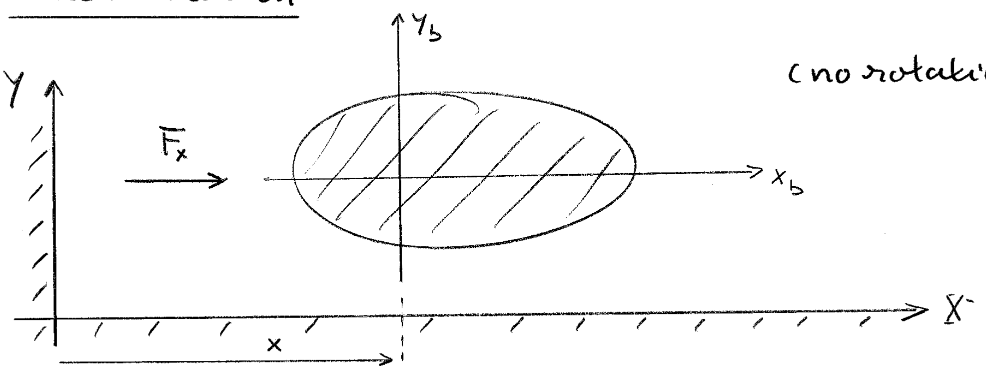
$$r_1\, df_{1,2_\theta}^{\,int} + r_2\, df_{2,1_\theta}^{\,int} \cong r_1\left(df_{1,2_\theta}^{\,int} + df_{2,1_\theta}^{\,int}\right) = r_1 \cdot 0 = 0$$

$$\Rightarrow \int r\, df_\theta^{\,int} = 0.$$

Executive Summary

Linear motion

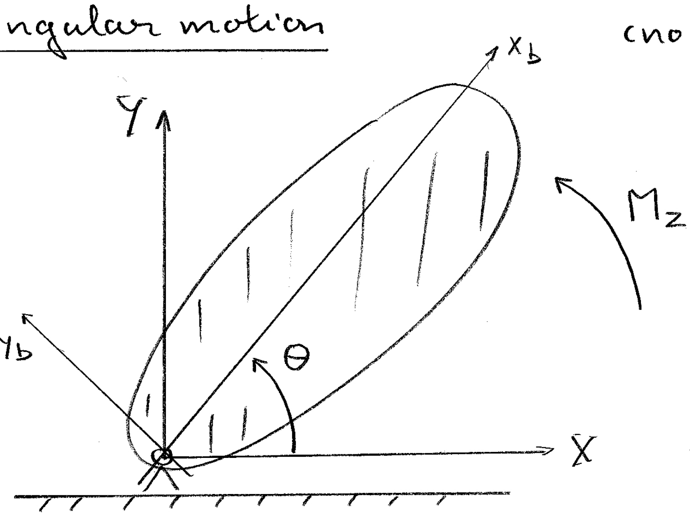
(no rotation)



Newton : $m \ddot{x} = F_x$

Angular motion

(no translation of hinge)



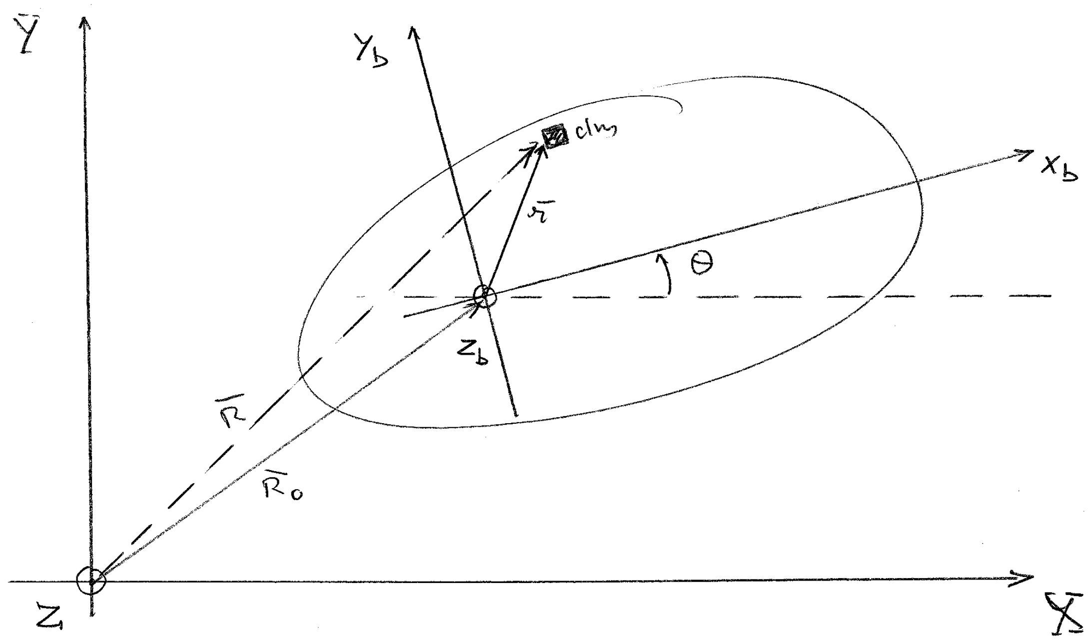
Euler : $I_z \ddot{\Theta} = M_z$

4. DYNAMICS OF A RIGID BODY IN TWO-DIMENSIONAL SPACE

4.1 General motion

4.2 Special cases

General motion



$\{\bar{X}, \bar{Y}, \bar{Z}\}$ = inertially fixed reference frame

$\{x_b, y_b, z_b\}$ = body-fixed reference frame.

(i) Newton for mass element dm:

$$\ddot{\bar{R}}\, dm = d\bar{f}$$

$$(\bar{R}_o + \bar{r})^{\cdot\cdot}\, dm = \underset{\text{(applied)}}{d\bar{f}^{a}} + \underset{\text{(internal)}}{d\bar{f}^{int.}}$$

Integrate over all mass elements:

$$\int (\ddot{\bar{R}}_o + \ddot{\bar{r}})\, dm = \int d\bar{f}^{a} + \int d\bar{f}^{int.}$$

$$M\ddot{\bar{R}}_o + \int \ddot{\bar{r}}\, dm = \bar{F}^a + \underset{\text{as before}}{\bar{0}} \qquad \triangleleft$$

$\bar{r} = \bar{r}(t)$ due to rotation $\Theta(t)$ (in $\{X, Y, Z\}$)

$$d\bar{r} = d\bar{\Theta} \times \bar{r}$$

$$\Rightarrow \dot{\bar{r}} = \dot{\bar{\Theta}} \times \bar{r} = \bar{\omega} \times \bar{r}$$

$$\frac{d\dot{\bar{r}}}{dt} = \frac{d}{dt}(\bar{\omega} \times \bar{r}) = \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \frac{d\bar{r}}{dt}$$

$$\ddot{\bar{r}} = \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

Substitute:

$$M\ddot{\bar{R}}_o + \dot{\bar{\omega}} \times \int \bar{r}\, dm + \bar{\omega} \times \left\{ \bar{\omega} \times \int \bar{r}\, dm \right\} = \bar{F}^a$$

Define position of center of mass:

$$\bar{c} \triangleq \frac{1}{m} \int \bar{r}\, dm$$

Note: $\bar{c}$ is constant in body-fixed frame

$\Rightarrow$ write out the vector eq. in body-fixed frame:

$$\boxed{\begin{aligned} M\,\ddot{R}_{o,xb} - \ddot{\theta}\,(m\,c_{yb}) - \dot{\theta}^2\,(m\,c_{xb}) &= F^a_{xb} \\ M\,\ddot{R}_{o,yb} + \ddot{\theta}\,(m.c_{xb}) - \dot{\theta}^2\,(m.c_{yb}) &= F^a_{yb} \end{aligned}}$$

components in body-fixed frame.

Next, consider again Newton for mass element dm:

$$(\ddot{\bar{R}}_o + \ddot{\bar{r}})\,dm = d\bar{f}^a + d\bar{f}^{int.}$$

Take outer product:

$$\bar{r} \times (\ddot{\bar{R}}_o + \ddot{\bar{r}})\,dm = \bar{r} \times d\bar{f}^a + \bar{r} \times d\bar{f}^{int.}$$

Integrate over all mass elements:

$$\int \bar{r} \times (\ddot{\bar{R}}_o + \ddot{\bar{r}})\,dm = \int \bar{r} \times d\bar{f}^a + \int \bar{r} \times d\bar{f}^{int.}$$

$$\left(\int \bar{r}\,dm\right) \times \ddot{\bar{R}}_o + \int \bar{r} \times \ddot{\bar{r}}\,dm = \bar{T}^a + \underbrace{\bar{0}}_{\text{as before}} \quad \triangleleft$$

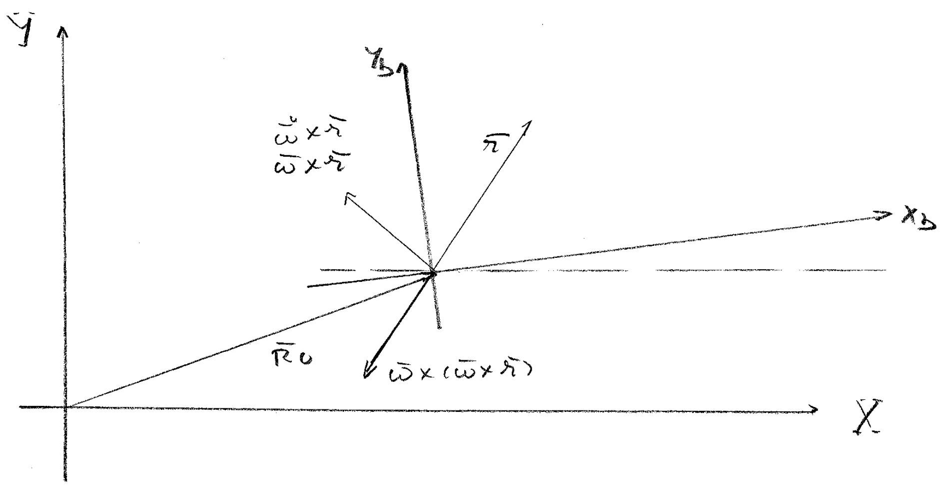
Substitute: $\ddot{\bar{r}} = \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$

and $\int \bar{r}\,dm = m\bar{c}$

$$(m\bar{c}) \times \ddot{\bar{R}}_0 + \int \bar{r} \times (\dot{\bar{\omega}} \times \bar{r})\, dm +$$

$$+ \int \bar{r} \times \{\bar{\omega} \times (\bar{\omega} \times \bar{r})\}\, dm = \bar{T}^a$$

Note: $\bar{r}$ is constant in body fixed frame!



$$\boxed{(m C_{xb}) \ddot{R}_{0,yb} - (m C_{yb}) \ddot{R}_{0,xb} + \ddot{\Theta} \underbrace{\int r^2\, dm}_{I_z} = T_{zb}^a}$$

components in body-fixed frame.

note: $\bar{r} \times \{\bar{\omega} \times (\bar{\omega} \times \bar{r})\} \equiv \bar{0}$

Note: in body-fixed frame: I_z = constant!

Special cases

Special case: pure translation

Θ = constant.

$$\Rightarrow \boxed{\begin{aligned} M.\ \ddot{R}_{o,xb} &= F^{a}_{xb} \\ M\ \ddot{R}_{o,yb} &= F^{a}_{yb} \end{aligned}}$$

The third equation gives the torque to be applied in order to keep Θ = const.

Special case: pure rotation

$\bar{R}_o$ = constant (more relaxed condition: $\dot{\bar{R}}_o$ = constant)

Third equation: $\boxed{I_{zb}\ \ddot{\Theta} = T^{a}_{zb}}$

The first and second equations gives the force to be applied in order to keep the origin of the body-fixed frame stationary in space.

Special case: origin of body fixed reference frame center in center-of-mass of the body.

$$\bar{C} = \bar{O}$$

Hence:

$$\boxed{\begin{aligned} M \cdot \ddot{R}_{o,xb} &= F_{xb}^{a} \\ M \cdot \ddot{R}_{o,yb} &= F_{yb}^{a} \\ \hline I_z \, \ddot{\theta} &= T_z^{a} \end{aligned}}$$

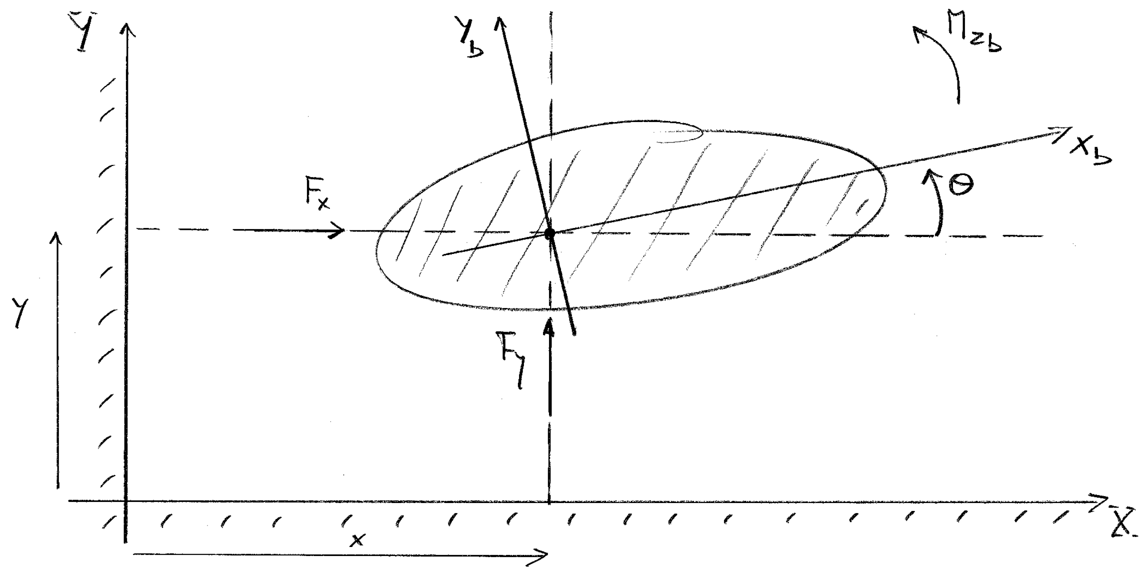
Note: the equations are now uncoupled. ("pure Newton" and "pure Euler").

Cautionary note:
to obtain velocity and position in the inertially fixed frame, one must first decompose the acceleration:

$$\begin{cases} \ddot{R}_{o,\bar{x}} = \ddot{R}_{o,xb} \cos\theta - \ddot{R}_{o,yb} \sin\theta \\ \ddot{R}_{o,\bar{y}} = \ddot{R}_{o,xb} \sin\theta + \ddot{R}_{o,yb} \cos\theta \end{cases}$$

Then: $\dot{R}_{o,\bar{x}} = \int \ddot{R}_{o,\bar{x}} \, dt$ and so on.

Executive Summary



Newton: $\int (\bar{R}_o + \bar{r})^{\cdot\cdot}\, dm = \bar{F}$

Euler : $\int \bar{r} \times (\bar{R}_o + \bar{r})^{\cdot\cdot}\, dm = \bar{M}$

Special case: origin of body-fixed reference frame coincides with center-of-mass.

Then:

Newton: $m\ddot{x} = F_x$

$m\ddot{y} = F_y$

Euler : $I_{zb}\ddot{\theta} = M_{zb}$

"de-coupled" equations

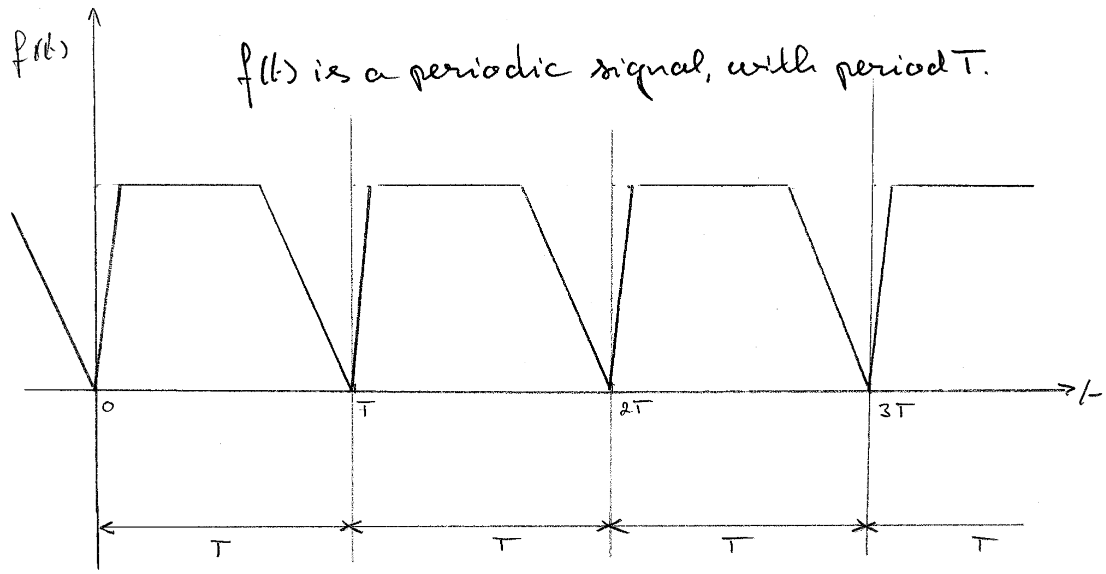
5. DECOMPOSITION OF A PERIODIC SIGNAL

5.1 Fourier series

5.2 Least square approximation

Fourier series.

$f(t)$ is a periodic signal, with period T.



$T =$ fundamental period

$$\boxed{f(t+T) = f(t)}$$

Special case: harmonic signal:

$$f(t) = a \cdot \sin \omega t + b \cdot \cos \omega t$$

$$= \underbrace{\sqrt{a^2+b^2}}_{\text{amplitude}} \cdot \sin(\omega t + \phi)$$

ϕ – phase shift

fundamental period: $T = 2\pi/\omega$

Fourier decomposition:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cdot \cos(n\omega t) + b_n \cdot \sin(n\omega t) \right\}}$$

"Fourier series".

Find the values of the Fourier coefficients!

Consider the following properties ("orthogonality"):

$$\int_0^T \sin(n\omega t) \cdot \sin(m\omega t)\, dt = \frac{T}{2} \cdot \underbrace{\delta_{mn}}$$

Kronecker delta

$$\int_0^T \cos(n\omega t) \cdot \cos(m\omega t)\, dt = \frac{T}{2} \cdot \delta_{mn}$$

$$\int_0^T \sin(n\omega t) \cdot \cos(m\omega t) \cdot dt = 0$$

where $n, m = 1, 2, 3, \ldots$

Note: $\int_0^T dt = T.$

(i) Integrate : $\int_0^T f(t) \cdot dt$

$$\int_0^T f \cdot dt = \frac{a_0}{2} \cdot T + \sum \left\{ a_n \cdot \int_0^T \cos n\omega t \cdot dt + b_n \cdot \int_0^T \sin n\omega t \cdot dt \right\}$$

$$\Rightarrow \boxed{a_0 = \frac{2}{T} \int_0^T f \cdot dt}$$

(ii) Integrate: $\int_0^T f(t) \cdot \sin n\omega t \cdot dt$

$$\Rightarrow \boxed{b_m = \frac{2}{T} \int_0^T f \cdot \sin n\omega t \cdot dt}$$

(iii) Integrate: $\int_0^T f(t) \cdot \cos n\omega t \cdot dt$

$$\Rightarrow \boxed{a_m = \frac{2}{T} \int_0^T f \cdot \cos n\omega t \cdot dt}$$

Conclusion: periodic signal with period T
behaves like the sum of harmonic
signals with period T/n $(n = 1, 2, 3, \cdots)$

Least square approximation

Consider a periodic signal $f(t)$ with fundamental period T.

Approximate this signal by the finite series:

$$\boxed{f^{appr.}_{(t)} = \frac{a_0}{2} + \sum_{n=1}^{N} \left\{ a_n \cdot \cos(n\omega t) + b_n \cdot \sin(n\omega t) \right\}}$$

The coefficients are to be chosen such that the following quantity is minimized.

$$\boxed{R \triangleq \int_0^T \left\{ f(t) - f^{appr.}_{(t)} \right\}^2 \cdot dt}$$

"residual".

Note:

$$dR = 2 \cdot \int_0^T \left\{ f - f^{appr.} \right\} \cdot \left\{ -df^{appr} \right\} \cdot dt$$

$$= -2 \cdot \int_0^T \left\{ f - f^{appr.} \right\} \left\{ \frac{1}{2} da_0 + \sum_{n=1}^{N} da_n \cdot \cos n\omega t + \right.$$

$$\left. + db_n \cdot \sin n\omega t \right\} dt.$$

$$\frac{\partial R}{\partial a_0} \doteq 0 \quad \Rightarrow \quad \int_0^T \left\{ f - f^{appr.} \right\} \cdot dt = 0$$

$$\Rightarrow \int_0^T f \cdot dt = \int_0^T d f^{appr.} dt \stackrel{!}{=} a_0 \cdot \frac{T}{2} \quad . \quad \Rightarrow a_0$$

$$\frac{\partial R}{\partial a_n} \doteq 0 \quad \Rightarrow \quad \int_0^T \left\{ f - f^{appr.} \right\} \cos n\omega t \cdot dt = 0$$

$$\Rightarrow \int_0^T f \cdot \cos n\omega t \cdot dt = \int_0^T f^{appr.} \cos n\omega t \cdot dt$$

$$\stackrel{!}{=} a_n \cdot \frac{T}{2} \quad \Rightarrow a_n$$

$$\frac{\partial R}{\partial b_n} \doteq 0 \quad \Rightarrow \quad \int_0^T \left\{ f - f^{appr.} \right\} \cdot \sin n\omega t \cdot dt = 0$$

$$\Rightarrow \int_0^T f \cdot \sin n\omega t \cdot dt = \int_0^T d f^{appr.} \sin n\omega t \cdot dt$$

$$\stackrel{!}{=} b_n \cdot \frac{T}{2} \quad \Rightarrow b_n$$

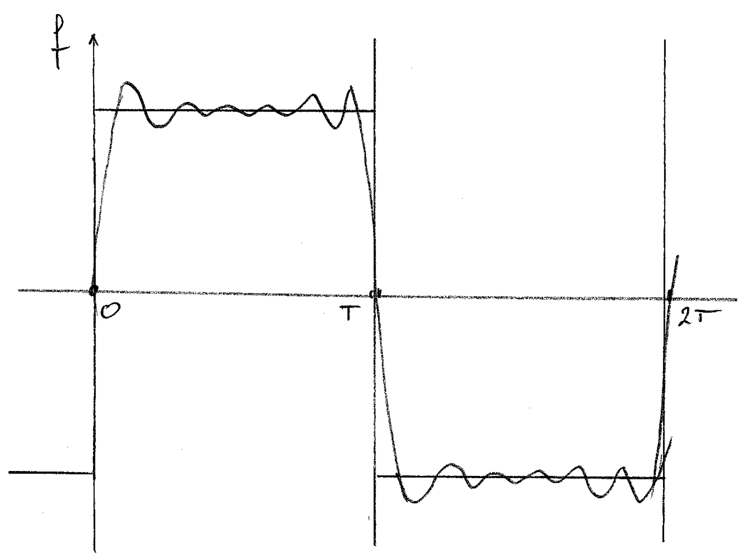
Conclusion: optimal values for a_0, a_n, b_n are precisely the Fourier coefficients computed earlier.

Finite M $\Rightarrow$ least square fit.

The fit improves as N increases.
(except at isolated points where f "jumps").
"convergence theorem".

f must be piecewise continuous.

"Gibbs phenomenon" is a result from discontinuities.



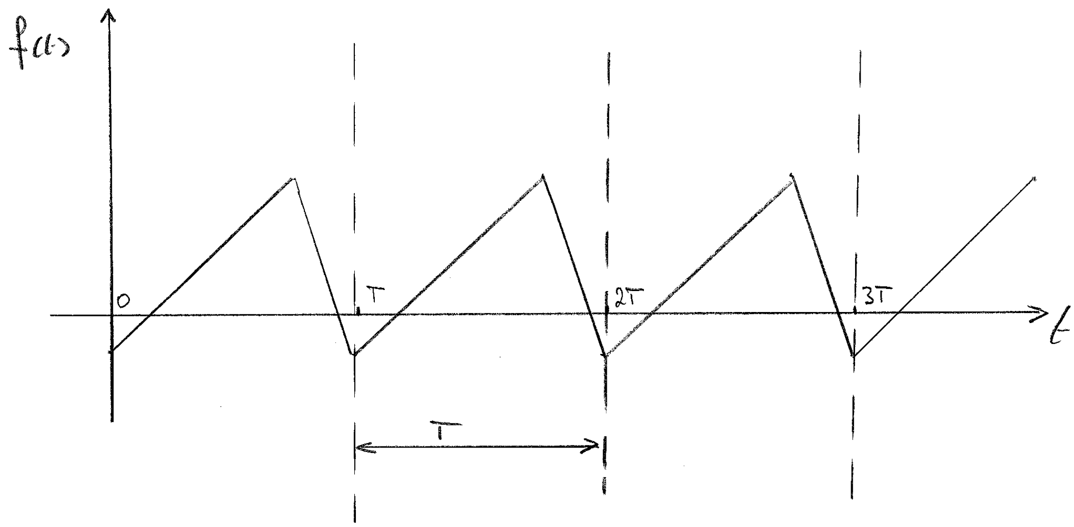
Ref. Boyce - DiPrima, p. 566.

Square wave:

$$f(t) = f_{\text{amplitude}} \cdot \frac{4}{\pi} \left\{ \sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5 \omega t + \cdots \right\}$$

where $\omega = 2\pi / T$

Executive Summary



Periodic signal: $f(t+T) = f(t)$

(T = period)

Fourier expansion:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos n\Omega t + b_n \cdot \sin n\Omega t \right)$$

where $\Omega = 2\pi / T$ (fundamental frequency)

Then:

$$a_0 = \frac{2}{T} \int_0^T f \cdot dt$$

$$a_n = \frac{2}{T} \int_0^T f \cdot \cos(n\Omega t) \cdot dt$$

$$b_n = \frac{2}{T} \int_0^T f \cdot \sin(n\Omega t) \cdot dt.$$

6. SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Problem statement

System of n simultaneous, linear, algebraic equations with n unknowns:

$$
\begin{aligned}
a_{11}x_1 + \cdots &+ a_{1n}x_n = b_1 \\
&\vdots \\
a_{n1}x_1 + \cdots &+ a_{nn}x_n = b_n
\end{aligned}
$$

Concise: $\boxed{A\bar{x} = \bar{b}}$

$\dim A = n \times n$
$\dim \bar{x} = n$
$\dim \bar{b} = n$

Given $A, \bar{b} \Rightarrow$ find $\bar{x}$!

Formal solution:

define matrix inverse A^{-1}:

$$A^{-1}A = AA^{-1} = I_{n\times n} \quad \text{(unit matrix)}$$

$$\Rightarrow A^{-1}(A\bar{x}) = A^{-1}\bar{b} \qquad \Rightarrow \boxed{\bar{x} = A^{-1}\bar{b}}$$

However: how can A^{-1} be computed?

Ref. Hadley.

Gaussian elimination procedure

System:
$$\begin{cases} a_{11} x_1 + \cdots & + a_{1n} \cdot x_n = b_1 \\ \vdots & \\ a_{n1} x_1 + \cdots & + a_{nn} x_n = b_n \end{cases}$$

First equation: solve for x_1

$$x_1 = \frac{1}{a_{11}} \left(-a_{12} x_2 \cdots - a_{1n} x_n + b_1 \right)$$

substitute in 2^{nd}, 3^{rd}, ..., n^{th} eq.

There remains a system of $n-1$ equations, to be solved for $x_2, \ldots, x_n$.

Proceed, untill there remains a single equation in x_n

$$\Rightarrow x_n \Rightarrow x_{n-1} \text{ ----- } \Rightarrow x_2 \Rightarrow x_1$$

Example:
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

From first equation: $x_1 = \dfrac{1}{a_{11}} \left(-a_{12} x_2 + b_1 \right)$

Substitute into 2^{nd} equation. There results,

$$a_{21}\left(\frac{-a_{12}x_2+b_1}{a_{11}}\right)+a_{22}x_2=b_2$$

Solve $\Rightarrow$ $\qquad x_2=\dfrac{a_{11}b_2-a_{21}b_1}{a_{11}a_{22}-a_{12}a_{21}}$

Substitute $\Rightarrow$ $\qquad x_1=\dfrac{a_{22}b_1-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}$.

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\underbrace{\frac{1}{a_{11}\cdot a_{22}-a_{12}\cdot a_{21}}\begin{bmatrix}a_{22}&-a_{12}\\-a_{21}&a_{11}\end{bmatrix}}_{\overset{D}{=}A^{-1}}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

Requirement: A^{-1} exists $\qquad$ ($\Rightarrow$ $\det A\neq 0$).

A^{-1} exists if $\det(A)\neq 0$.

Here: $\det(A)=a_{11}\cdot a_{22}-a_{12}\cdot a_{21}$.

Cramer's rule

Considere the square matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

<u>Co-factor</u> A_{ij} of element a_{ij}:

= $(-1)^{i+j}$ times determinant of the submatrix obtained from A by deleting row i and column j.

<u>Adjoint</u> A^+ of A:

$$A^+ = \text{adjoint}(A) = \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix}$$

Property of adjoint:

$$A A^+ = A^+ A = \{\det(A)\} I_{n \times n}$$

Note: $A^{-1}(A A^+) = A^{-1} \det(A)$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} A^+$$

provided $\det(A) \neq 0$.

Hence: $\bar{x} = A^{-1}\bar{b}$

$$= \frac{1}{\det(A)} A^{+} \bar{b}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad x_i = \frac{1}{\det(A)} \sum_{j=1}^{n} A_{ji} b_j$$

One can subsequently show:

$$\boxed{x_i = \frac{1}{\det(A)} \det \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}}$$

i-th column

i.e. replace i-th column by $\bar{b}$, and take the determinant.

"Cramer's rule".

(compare with earlier example).

Suitable mainly for theoretical studies.

Homogeneous case

Homogeneous case: $\boxed{A\bar{x} = \bar{0}}$

Recall: $A\bar{x} = \bar{b}$

$\Rightarrow A^{+}A\bar{x} = A^{+}\bar{b}$

$\{\det(A)\}\bar{x} = A^{+}\bar{b} = \bar{0}$

Trivial solution: $\bar{x} = \bar{0}$

Non-trivial solution: provided $\det(A) = 0$
(in fact, in that case there are infinitely many solutions)

Properties:

- if $\bar{x}_1$ is a solution,
  then $\alpha\bar{x}_1$ is also a solution.

- if $\bar{x}_1$ and $\bar{x}_2$ are distinct solutions,
  then $\alpha_1\bar{x}_1 + \alpha_2\bar{x}_2$ is also a solution.

$A\bar{x} = \bar{0} \Rightarrow$ set of basisvectors $\bar{x}_i$

$\boxed{\text{Eigen problem}}$

Definition of "eigen problem" ("characteristic $\overbrace{\text{}}^{\text{value}}$ problem")

$$\boxed{A\hat{\bar{x}} = \lambda\hat{\bar{x}} \qquad \Rightarrow \lambda ? \quad \hat{\bar{x}} ?}$$

$$(A - \lambda I)\hat{\bar{x}} = \bar{0}$$

Nontrivial solution provided $\det(A - \lambda I) = 0$

algebraic equation (polynomial) in λ $\Rightarrow \lambda_i$ "eigenvalue"

$(A - \lambda_i I)\hat{\bar{x}} = 0 \Rightarrow \hat{\bar{x}}_i$ "eigenvector"

We know:
- $\hat{\bar{x}}_i$ is a solution $\Rightarrow \alpha_i \hat{\bar{x}}_i$ is also a solution
- $\hat{\bar{x}}_i$ and $\hat{\bar{x}}_j$ are distinct solutions $\Rightarrow$
  $\alpha_i \hat{\bar{x}}_i + \alpha_j \hat{\bar{x}}_j$ is also a solution.

In general: λ_i is complex $\Rightarrow \hat{\bar{x}}_i$ is complex.

$\boxed{\text{Special case: symmetric matrix}}$

Symmetric matrix: $\boxed{A = A^T}$

Properties:

- λ_i is real! $\Rightarrow$ $\hat{\underline{x}}_i$ is also real!
- all $\hat{\underline{x}}_i$ are linearly independent
- if $\lambda_i \neq \lambda_j$ then $\hat{\underline{x}}_i^T A \hat{\underline{x}}_j = 0$ (orthogonality)

Consider $A \hat{\underline{x}}_i = \lambda_i \hat{\underline{x}}_i$

(i) $\hat{\underline{x}}_j^T (A \hat{\underline{x}}_i) = \hat{\underline{x}}_j^T (\lambda_i \hat{\underline{x}}_i)$

$= \lambda_i (\hat{\underline{x}}_j^T \hat{\underline{x}}_i)$

(ii) $\hat{\underline{x}}_i^T (A \hat{\underline{x}}_j) = \hat{\underline{x}}_i^T (\lambda_j \hat{\underline{x}}_j)$

$= \lambda_j (\hat{\underline{x}}_i^T \hat{\underline{x}}_j)$

Take transpose:

$$\hat{\underline{x}}_j^T \underbrace{A^T}_{=A} \hat{\underline{x}}_i = \lambda_j (\hat{\underline{x}}_j^T \hat{\underline{x}}_i)$$

Substract results: $(\lambda_i - \lambda_j)(\hat{\underline{x}}_j^T \hat{\underline{x}}_i) = 0$

if $\lambda_i \neq \lambda_j$ then $\hat{\underline{x}}_j^T \hat{\underline{x}}_i = 0$

Hence $\hat{\underline{x}}_j^T A \hat{\underline{x}}_i = \lambda_i (\hat{\underline{x}}_j^T \hat{\underline{x}}_i) = 0$ q.e.d.

$\hat{\underline{x}}_j^T \hat{\underline{x}}_i = 0$ for $j \neq i$: "orthogonal" eigenvectors

when $\hat{\underline{x}}_i$ has been normalised such that
$\hat{\underline{x}}_i^T \hat{\underline{x}}_i = 1$ $\Rightarrow$ "ortho-normal" eigenvectors.

Notation: $\hat{\underline{x}}_j^T \hat{\underline{x}}_i = \delta_{ji}$ (Kronecker delta)

Consider the transformation

$$\bar{x} = T\bar{y} \quad \text{where} \quad T = [\hat{\bar{x}}_1 \; \cdots \; \hat{\bar{x}}_n]$$

Construct $\bar{x}^T A \bar{x}$

$$\bar{x}^T A \bar{x} = \bar{y}^T (T^T A T) \bar{y}$$

$$= \bar{y}^T \begin{bmatrix} \hat{\bar{x}}_1^T \\ \vdots \\ \hat{\bar{x}}_n^T \end{bmatrix} A [\hat{\bar{x}}_1 \cdots \hat{\bar{x}}_n] \bar{y} = \bar{y}^T \begin{bmatrix} \hat{\bar{x}}_1^T \\ \vdots \\ \hat{\bar{x}}_n^T \end{bmatrix} [A\hat{\bar{x}}_1 \cdots A\hat{\bar{x}}_n] \bar{y}$$

$$= \bar{y}^T \begin{bmatrix} \hat{\bar{x}}_1^T A \hat{\bar{x}}_1 & \cdots & \hat{\bar{x}}_1^T A \hat{\bar{x}}_n \\ \vdots & & \\ \hat{\bar{x}}_n^T A \hat{\bar{x}}_n & \cdots & \hat{\bar{x}}_n^T A \hat{\bar{x}}_n \end{bmatrix} \bar{y}$$

$$= \bar{y}^T \underbrace{\begin{bmatrix} \lambda_1 \hat{\bar{x}}_1^T \hat{\bar{x}}_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \hat{\bar{x}}_n^T \hat{\bar{x}}_n \end{bmatrix}}_{\text{diagonal.}} \bar{y}$$

Conclusion: if all $\lambda_i > 0$ then $\bar{x}^T A \bar{x} > 0$

(positive-definite matrix A).

Executive Summary

System: $A\bar{x} = \bar{b}$ $\Rightarrow$ solve for $\bar{x}$.

(dim $\bar{x}, \bar{b} = n$).

Case $\bar{b} \neq \bar{0}$.

Gaussian elimination of elements of $\bar{x}$

Cramer's rule:

$$x_i = \frac{1}{\det(A)} \det \underbrace{\begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & & b_n & & a_{nn} \end{bmatrix}}$$

i.e. replace i-th column in A by $\bar{b}$.

Case $\bar{b} = \bar{0}$.

Solution exists only <u>if</u> $\det(A) = 0$.

$\Rightarrow$ solve for $\bar{x}$ (there will be a free multiplicative constant).

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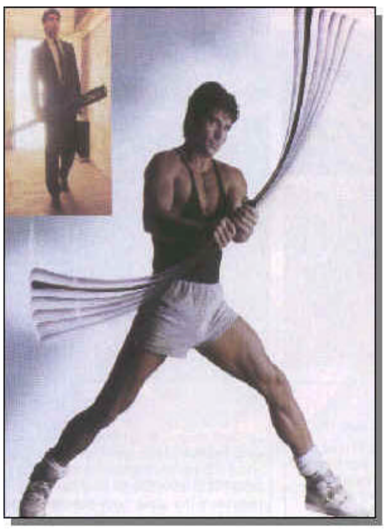
Notes on Linear Vibration Theory



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* Body Blade picture taken from: http://www.starsystems.com.au

7. SINGLE SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

Problem statement.

Canonical form:

$$\boxed{\begin{array}{l} \ddot{x} + 2\zeta\omega_0 \cdot \dot{x} + \omega_0^2 x = f(t) \\ \left.\begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{array}\right\} \text{ initial conditions} \end{array}}$$

Solve for $x(t)$.

Solution procedure:

(i) find general solution to the homogeneous differential equation ("complementary solution")

(ii) find the (a) particular solution (the solution to the inhomogeneous differential equation.

(iii) form the complete solution (the sum of the complementary solution and the particular solution)

(iv) apply the initial conditions.

$\boxed{\text{Complementary solution}}$

The general solution to the homogeneous differential equation is called the "complementary solution".

$$\boxed{\ddot{x} + 2\zeta.\omega_0.\dot{x} + \omega_0^2\, x = 0}$$

Euler method: try $x(t) = c.\,e^{\lambda t}$

Substitute: $(\lambda^2 + 2\zeta\omega_0.\lambda + \omega_0^2).\underbrace{c.\,e^{\lambda t}}_{\neq 0} = 0$

$\Rightarrow \boxed{\lambda^2 + 2\zeta.\omega_0.\lambda + \omega_0 = 0}$ "characteristic equation".

$\Rightarrow \lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}$

Three cases: $0 \leq \zeta^2 < 1$; $\zeta^2 = 1$; $\zeta^2 > 1$

(i) <u>Case $0 < \zeta < 1$</u> "Subcritical damping".

$\lambda_{1,2} = -\zeta\omega_0 \pm i\,\omega_0\sqrt{1-\zeta^2}$ where $i \triangleq \sqrt{-1}$.

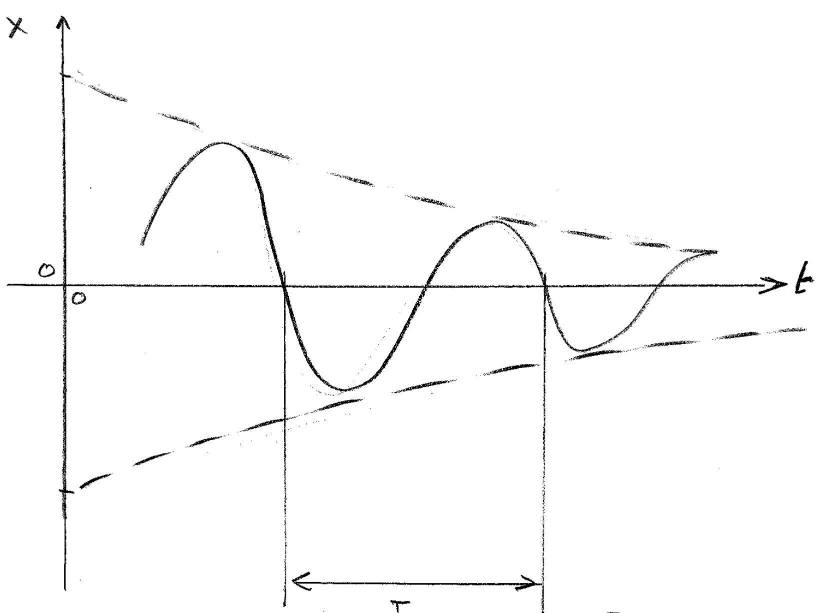
$x = c_1.e^{\lambda_1 t} + c_2\, e^{\lambda_2 t}$

$$x = e^{-\zeta\omega_0 . t}\left\{C_1 . e^{i\omega_0\sqrt{1-\zeta^2}.t} + C_2 . e^{-i\omega_0\sqrt{1-\zeta^2}.t}\right\}$$

Use: $\quad e^{i\beta} = \cos\beta + i\sin\beta \qquad$ (Euler)

There results:

$$\boxed{x(t) = e^{-\zeta\omega_0 . t}\left\{A . \sin(\omega_0 . \sqrt{1-\zeta^2} . t) + B . \cos(\omega_0 . \sqrt{1-\zeta^2} . t)\right\}}$$



$$T = \frac{2\pi}{\omega_0 . \sqrt{1-\zeta^2}}$$

Rewrite the solution:

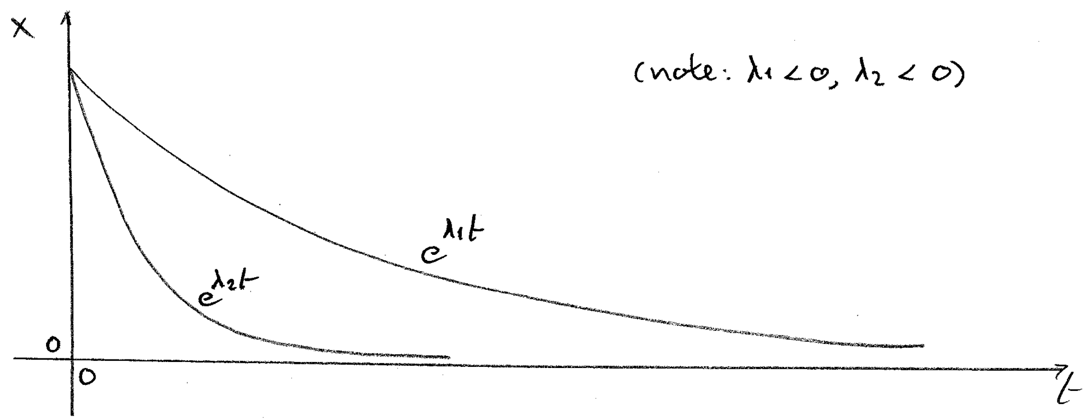
$$x(t) = \underbrace{\left\{e^{-\zeta . \omega_0 . t}\sqrt{A^2+B^2}\right\}}_{\text{amplitude (decaying)}} . \sin\{\omega_0 . \sqrt{1-\zeta^2} . t + \underset{\substack{\uparrow \\ \text{phase angle}}}{\phi}\} \qquad \triangleleft$$

(ii) <u>Case $\zeta > 1$</u> "supercritical damping"

$$\lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1} \quad \text{(both real) and negative}$$

$$x = C_1 \cdot e^{\lambda_1 t} + C_2 \cdot e^{\lambda_2 t}$$

$$x(t) = e^{-\zeta\omega_0 \cdot t}\left\{C_1 \cdot e^{\omega_0 \cdot \sqrt{\zeta^2-1} \cdot t} + C_2 \cdot e^{-\omega_0 \cdot \sqrt{\zeta^2-1} \cdot t}\right\}$$



(iii) <u>Case $\xi = 1$</u> "Critical damping"

$\lambda_1 = \lambda_2 = -\omega_0$ coinciding roots of the characteristic equation.

Two methods:
- undetermined multiplier
- limit case $\xi \downarrow 1$ or $\xi \uparrow 1$.

(iii-a) Method of undetermined multiplier.

One solution is: $x_1 = c_0 \, e^{-\omega_0 t}$

As second solution, try:

$x_2 = \gamma \cdot e^{-\omega_0 t}$ where $\gamma = \gamma(t)$, to be determined

Then: $\dot{x}_2 = \dot{\gamma}\, e^{-\omega_0 t} - \gamma \cdot \omega_0 \cdot e^{-\omega_0 t}$

$\ddot{x}_2 = \ddot{\gamma}\, e^{-\omega_0 t} - 2\dot{\gamma}\,\omega_0\, e^{-\omega_0 t} + \gamma \cdot \omega_0^2 \cdot e^{-\omega_0 t}$

Substitute assumed solution into differential equation:

$$\ddot{x} + 2 \cdot 1 \cdot \omega_0 \cdot \dot{x} + \omega_0^2 \cdot x = 0$$

$$(\ddot{\gamma} - \cancel{2\dot{\gamma}\omega_0} + \cancel{\gamma\omega_0^2})\, e^{-\omega_0 t} + 2\omega_0\,(\cancel{\dot{\gamma}} - \cancel{\gamma\omega_0})\, e^{-\omega_0 t} + \cancel{\gamma \cdot \omega_0^2} \cdot e^{-\omega_0 t} = 0$$

There remains: $\ddot{\gamma} = 0$

$$\Rightarrow \gamma(t) = a_0 + b_0 \cdot t$$

$$\Rightarrow x_2 = (a_0 + b_0 \cdot t) \cdot e^{-\omega_0 t}$$

Complete solution:

$$x = x_1 + x_2 = C_0 \cdot e^{-\omega_0 \cdot t} + (a_0 + b_0 \cdot t)\, e^{-\omega_0 \cdot t}$$

$$\Rightarrow \boxed{x(t) = (A_0 + B_0 \cdot t) \cdot e^{-\omega_0 \cdot t}}$$

(iii-b) Limit case $\xi \downarrow 1$.

Define $\varepsilon \triangleq \xi - 1$. Hence: $\varepsilon \downarrow 0$

$$x = e^{-\xi \omega_0 t} \left[C_1 \, e^{\omega_0 \sqrt{\xi^2 - 1} \cdot t} + C_2 \cdot e^{-\omega_0 \cdot \sqrt{\xi^2 - 1} \cdot t} \right]$$

$$= e^{-(1+\varepsilon)\omega_0 \cdot t} \left\{ C_1 \cdot e^{\omega_0 \sqrt{2\varepsilon + \varepsilon^2}\, t} + C_2 \cdot e^{-\omega_0 \sqrt{2\varepsilon + \varepsilon^2}\, t} \right\}$$

Expand: $e^{\omega_0 \sqrt{2\varepsilon + \varepsilon^2}\, t} = 1 + \left(\omega_0 \cdot \sqrt{2\varepsilon + \varepsilon^2} \cdot t \right) + \text{h.o.t.}$

$$x = e^{-(1+\varepsilon)\omega_0 t}\left[C_1\left\{1+(\omega_0\sqrt{2\varepsilon+\varepsilon^2}\cdot t)\right\} + \right.$$

$$\left. + C_2\left\{1-(\omega_0\sqrt{2\varepsilon+\varepsilon^2}\, t)\right\}\right]$$

$$= \underbrace{e^{-(1+\varepsilon)\omega_0 t}}_{\lim = 1}\left[\underbrace{(C_1+C_2)}_{\lim \hat{=} A_1} + \underbrace{(C_1-C_2)\omega_0\cdot\sqrt{2\varepsilon+\varepsilon^2}\cdot t}_{\lim \hat{=} B_1}\right]$$

$$\boxed{x(t) \to e^{-\omega_0 \cdot t}(A_1 + B_1 t)}$$

(iii-c) Limit case $\xi \uparrow 0$

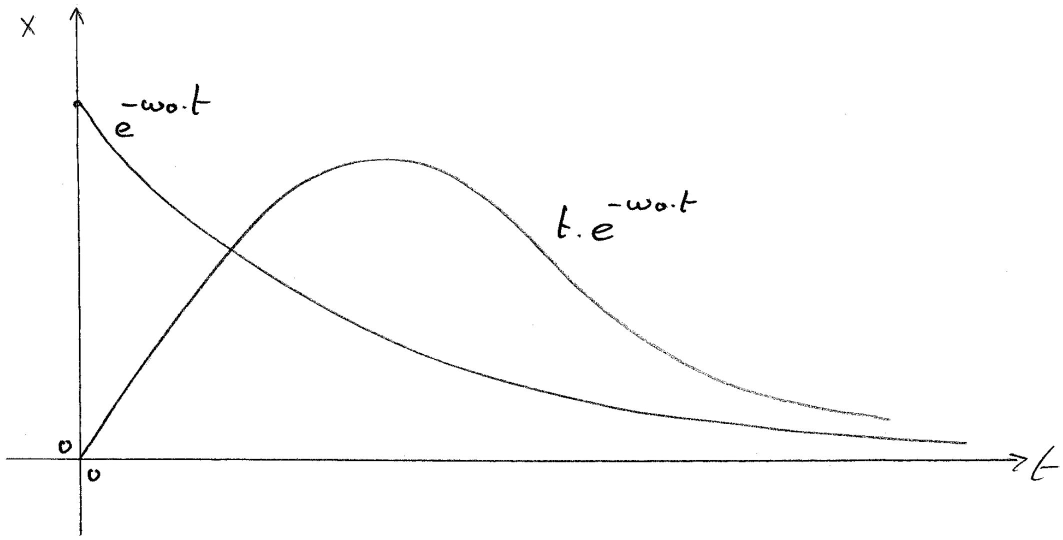
Hence $\varepsilon \uparrow 0$

$$x(t) = e^{-(1+\varepsilon)\omega_0 t}\left[A\cdot\sin\left\{\omega_0\sqrt{+2\varepsilon+\varepsilon^2}\, t\right\} + \right.$$

$$\left. + B\cos\left\{\omega_0\sqrt{+2\varepsilon+\varepsilon^2}\right\} t\right]$$

$$= \underbrace{e^{-(1+\varepsilon)\omega_0 t}}_{\lim = 1}\left[\underbrace{A\cdot\omega_0\cdot\sqrt{2\varepsilon+\varepsilon^2}\cdot t}_{\lim \hat{=} A_2}\cdot \underbrace{\frac{\sin\left\{\omega_0\sqrt{2\varepsilon+\varepsilon^2}\, t\right\}}{\omega_0\sqrt{2\varepsilon+\varepsilon^2}\, t}}_{\lim = 1} + B\right]$$

$$x(t) \to e^{-\omega_0 \cdot t}\left[A_2 t + B\right]$$


x
$e^{-\omega_0 \cdot t}$
$t \cdot e^{-\omega_0 \cdot t}$
0
0
t

Particular solution.

The solution ("a" solution) to the inhomogeneous differential equation is called the "particular solution".

$$\ddot{x} + 2\varphi . \omega_0 . \dot{x} + \omega_0^2 . x = f(t).$$

Various methods for finding the particular solution:

- method of undetermined coefficients (e.g. Ref. Boyce and DiPrima, Section 3.6)

- impulsive analysis ("Duhamel integral", "convolution integral")

- by inspection if the structure of $f(t)$ is simple.

(i). <u>Impulsive analysis.</u>

We shall only consider the case $0 < \varphi < 1$.

$$x(t) = e^{-\varphi \omega_0 . t} \left\{ A . \sin(\omega_0 \sqrt{1-\varphi^2} . t) + B . \cos(\omega_0 \sqrt{1-\varphi^2}\, t) \right\}.$$

$$\Rightarrow \quad x(0) = B = x_0$$

$$\dot{x}(t) = e^{-\xi\omega_0 t}(-\xi\omega_0)\{A\sin\ldots + B\cos\ldots\} +$$

$$+ e^{-\xi\omega_0 t}\,\omega_0\sqrt{1-\xi^2}\{A\cos\ldots - B\sin\ldots\}$$

$$\Rightarrow \dot{x}(0) = -\xi\omega_0 B + \omega_0\sqrt{1-\xi^2}\,A = \dot{x}_0$$

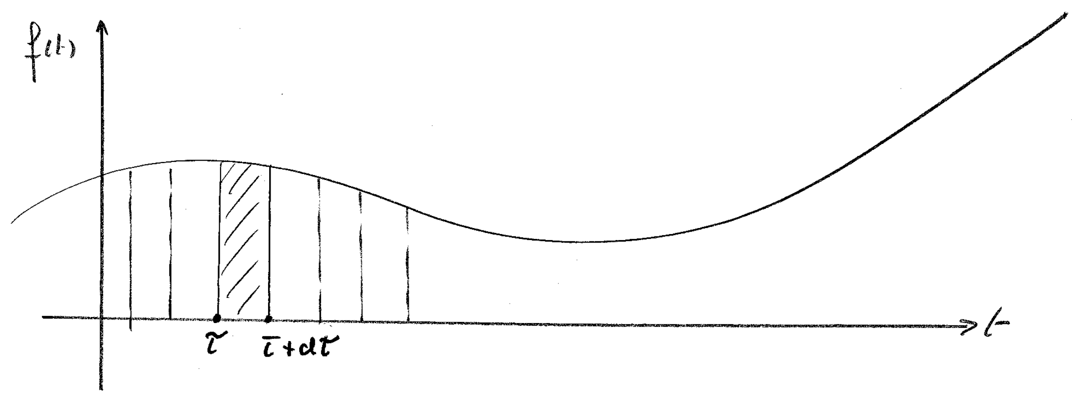
Solve for A, B and substitute:

$$x(t) = e^{-\xi\omega_0 t}\left\{\frac{\xi\omega_0 x_0 + \dot{x}_0}{\omega_0\sqrt{1-\xi^2}}\,\sin(\omega_0\sqrt{1-\xi^2}\,t) + x_0\cos(\omega_0\sqrt{1-\xi^2}\,t)\right\}.$$

Consider the special case: $x(\tau) = 0$

$$\Rightarrow x(t) = \frac{\dot{x}_0}{\omega_0\sqrt{1-\xi^2}}\,e^{-\xi\omega_0(t-t_0)}\sin\{\omega_0\sqrt{1-\xi^2}\,(t-t_0)\}$$

Furthermore, let the forcing term consist of a concatenation of briefly applied loads:



$$\ddot{x} + 2\zeta\omega_0 \cdot \dot{x} + \omega_0^2 \cdot x = f$$

$$\int_{\tau}^{\tau+d\tau} (\ddot{x} + 2\zeta\omega_0 \cdot \dot{x} + \omega_0^2 x)\, dt = \int_{\tau}^{\tau+d\tau} f \cdot dt$$

$$\dot{x}\Big|_{\tau}^{\tau+d\tau} + 2\zeta\omega_0 \, x\Big|_{\tau}^{\tau+d\tau} + \omega_0^2 \int_{\tau}^{\tau+d\tau} x \cdot dt = \int_{\tau}^{\tau+d\tau} f \cdot dt$$

$$\left\{\dot{x}_{(\tau+d\tau)} - \underbrace{\dot{x}_{(\tau)}}_{\text{assume} = 0}\right\} + 2\zeta\omega_0 \{\text{h.o.t.}\} + \omega_0^2 \{\text{h.o.t.}\} = f_{(\tau)} \cdot dt$$

Hence, if the system is at rest at $t = \tau$, then upon application of load f during time interval dt the new initial conditions become:

$$x(\tau) = 0 \qquad \dot{x}(\tau) = f \cdot dt$$

The resulting motion is:

$$x(t > \tau) = f \cdot dt \, \frac{1}{\omega_0\sqrt{1-\zeta^2}} \, e^{-\zeta\omega_0(t-\tau)} \cdot \sin\left\{\omega_0\sqrt{1-\zeta^2} \cdot (t-\tau)\right\}$$

Add contributions of all briefly applied loads:

$$\boxed{x(t) = \int_0^t e^{-\zeta\omega_0(t-\tau)} \cdot \sin\left\{\omega_0 \cdot \sqrt{1-\zeta^2} \cdot (t-\tau)\right\} \cdot \frac{f(\tau)}{\omega_0 \cdot \sqrt{1-\zeta^2}} \cdot d\tau}$$

This is the "Duhamel integral", or "convolution integral". It gives the particular solution for arbitrary load profile $f(t)$.

(ii) <u>Inspection, when $f(t)$ is simple.</u>

(ii-a) Case $f(t) =$ constant $\triangleq f_o$

Try: $x = x_p$ (constant)

$$0 + 2\zeta\omega_o \cdot 0 + \omega_o^2 \cdot x_p = f_o$$

$$\Rightarrow \boxed{x_p = f_o / \omega_o^2}$$

(ii-b) Case $f(t) =$ harmonic $\triangleq f_o \cdot \cos \Omega t$.

Try: $x = a \cdot \sin \Omega t + b \cdot \cos \Omega t$

Substitute: $(-a\Omega^2 \sin - b\Omega^2 \cos) +$

$$+2\zeta\omega_o (a\Omega \cos - b\Omega \sin) + \omega_o^2 (a \sin + b \cos) = f_o \cdot \cos$$

Re-arrange:

$$\{(-\Omega^2+\omega_o^2)\, a - 2\zeta\omega_o.\Omega.b\} \sin \Omega t +$$

$$+ \{(-\Omega^2+\omega_o^2).b + 2\zeta.\omega_o.\Omega.a - f_o\} \cos \Omega t = 0$$

It follows:

$$\begin{bmatrix} (-\Omega^2+\omega_0^2) & -2\zeta\omega_0\Omega \\ 2\zeta\omega_0\Omega & (-\Omega^2+\omega_0^2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ f_0 \end{bmatrix}$$

Solve for a, b (Cramer's rule)

Then:

$$a = \frac{f_0}{\omega_0^2} \cdot \frac{2\zeta(\Omega/\omega_0)}{\Delta}$$

$$b = \frac{f_0}{\omega_0^2} \cdot \frac{1-(\Omega/\omega_0)^2}{\Delta}$$

where $\Delta \triangleq \left\{1-\left(\frac{\Omega}{\omega_0}\right)^2\right\}^2 + \left(2\zeta\cdot\frac{\Omega}{\omega_0}\right)^2$

Finally:

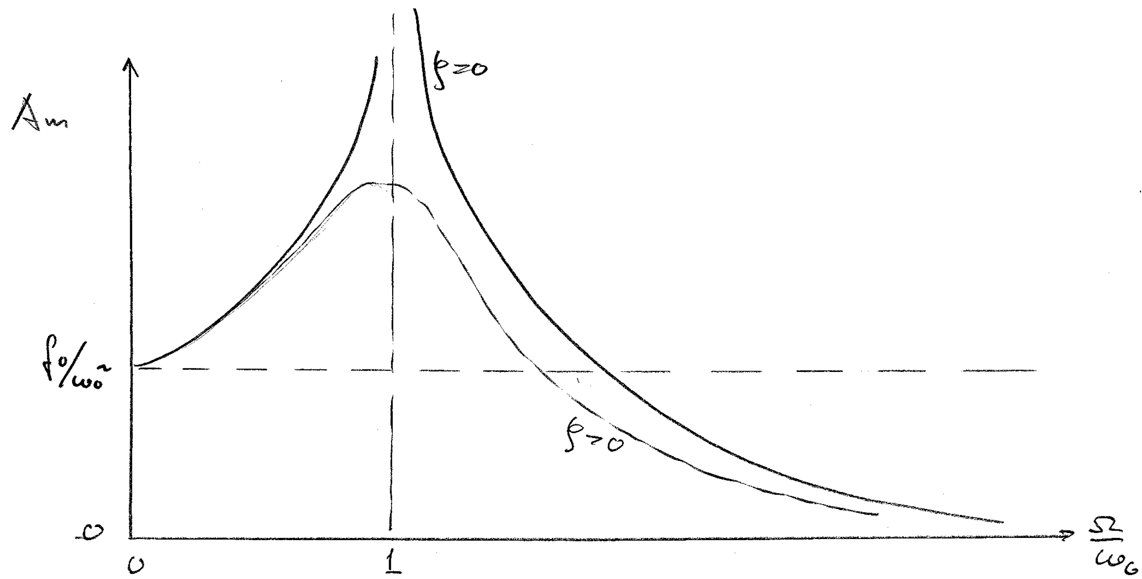
$$\boxed{x(t) = A_m \cdot \cos(\Omega t + \phi)}$$

where A_m = amplitude
ϕ = phase shift.

$$A_m = \frac{f_0}{\omega_0^2} \cdot \frac{1}{\sqrt{\left\{1-\left(\frac{\Omega}{\omega_0}\right)^2\right\}^2 + \left(2\zeta\frac{\Omega}{\omega_0}\right)^2}}$$

Note: $\frac{f_0}{\omega_0^2}$ = static displacement

$\frac{1}{\sqrt{\cdots}}$ = dynamic correction factor.



Peak amplitude: $\dfrac{dA_m}{d(\Omega/\omega_0)} = 0$

This gives: $A_{peak} = \dfrac{f_0}{\omega_0^2} \cdot \dfrac{1}{2\xi \cdot \sqrt{1-\xi^2}}$

for $\dfrac{\Omega}{\omega_0} = \sqrt{1-2\xi^2}$

provided $\xi < \dfrac{1}{\sqrt{2}}$

Limit $\xi \downarrow 0$:

$A_{peak} \cong \dfrac{f_0}{\omega_0^2} \cdot \dfrac{1}{2\xi}$ for $\dfrac{\Omega}{\omega_0} \cong 1.$

(ii-c) Case $f(t)$ = periodic, with period T

Write $f(t)$ in the form of a Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cdot \cos(n\omega t) + b_n \cdot \sin(n\omega t)\}$$

Response of system to this series is the sum of the responses of the system to each of the terms in the series individually.

$$a_0 = \text{constant} \Rightarrow x_{p,0} = \frac{1}{\omega_0^2} \cdot \frac{a_0}{2}$$

$$a_n \cdot \cos(n\omega t) + b_n \cdot \sin(n\omega t) =$$

$$= c_n \cdot \cos(n\omega t + \psi_n)$$

$$\Rightarrow x_{p,n} = A_{m,n} \cos\{(n\omega t + \psi_n) + \phi_n\}$$

$$\text{where } A_{m,n} = \frac{c_n}{\omega_0^2} \cdot \frac{1}{\sqrt{\left\{1 - \left(\frac{n\omega}{\omega_0}\right)^2\right\}^2 + \left(2\zeta \cdot \frac{n\omega}{\omega_0}\right)^2}}$$

Finally: $$x_p = x_{p,0} + \sum_{n=1}^{\infty} x_{p,n}$$

Resonance

Consider the case of negligible damping: $\zeta \to 0$

Then: $\ddot{x} + \omega_0^2 x = f_0 \cdot \cos \Omega t$

has as particular solution:

$$x_p = A_m \cdot \cos(\Omega t + \phi)$$

where $A_m = \dfrac{f_0}{\omega_0^2} \cdot \dfrac{1}{\sqrt{\left\{1-\left(\frac{\Omega}{\omega_0}\right)^2\right\}^2 + \left(2\zeta\frac{\Omega}{\omega_0}\right)^2}}$

$$\to \frac{f_0}{\omega_0^2} \cdot \frac{1}{\left|1-(\Omega/\omega_0)^2\right|}$$

Resonance where $\Omega/\omega_0 \to 1.$

Consider the complete solution:

$$x = A \cdot \sin\omega_0 t + B \cdot \cos\omega_0 t + \frac{f_0}{-\Omega^2+\omega_0^2} \cdot \cos\Omega t$$

Apply initial conditions:

$$\Rightarrow x = \frac{\dot{x}_0}{\omega_0} \sin\omega_0 t + x_0 \cdot \cos\omega_0 t + \frac{f_0}{-\Omega^2+\omega_0^2}\left(\cos\Omega t - \cos\omega_0 t\right)$$

Let $\Omega = \omega_0 + \varepsilon$ where $|\varepsilon| \to 0$

$$\frac{f_0}{-\Omega^2+\omega_0^2}\left(\cos\Omega t - \cos\omega_0 t\right) =$$

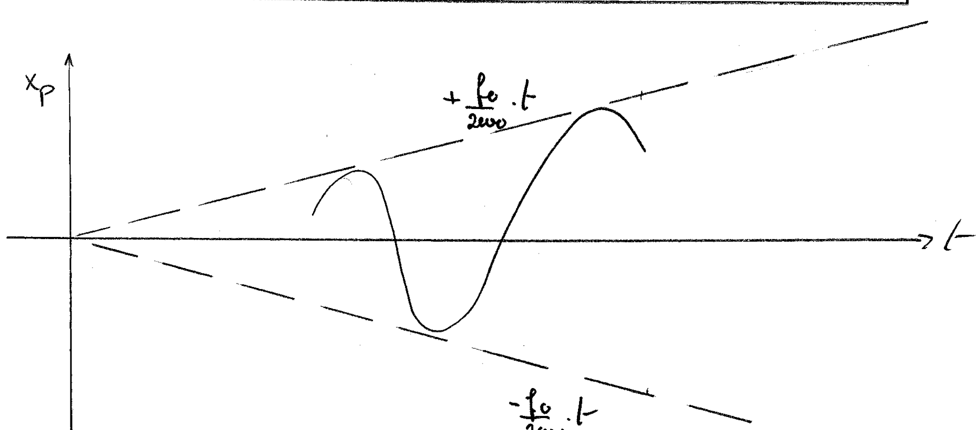
$$= \frac{f_0}{-2\varepsilon\omega_0 + \cdots}\left\{\cos(\omega_0+\varepsilon)t - \cos\omega_0 t\right\}$$

$$= \frac{-f_0}{2\varepsilon\omega_0 + \cdots}\left\{(\cos\omega_0 t\cdot\underbrace{\cos\varepsilon t}_{\lim=1} - \sin\omega_0 t\cdot\sin\varepsilon t) - \cos\omega_0 t\right\}$$

$$= \frac{f_0}{2\omega_0}\cdot t\cdot\left\{\sin\omega_0 t\cdot\underbrace{\frac{\sin\varepsilon t}{\varepsilon t}}_{\lim=1}\right\} = \frac{f_0\cdot t}{2\omega_0}\cdot\sin\omega_0 t$$

$$\Rightarrow x(t) = \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + x_0\cdot\cos\omega_0 t + \overbrace{\frac{f_0\cdot t}{2\omega_0}\cdot\sin\omega_0 t}^{\text{particular solution } x_p(t)}$$

$$\boxed{x(t) = \left(\frac{\dot{x}_0}{\omega_0} + \frac{f_0}{2\omega_0}\cdot t\right)\sin\omega_0 t + x_0\cdot\cos\omega_0 t}$$



Executive Summary

Canonical form: $\ddot{x} + 2\zeta \cdot \omega_0 \cdot \dot{x} + \omega_0^2 \cdot x = f(t)$

General solution = solution of homogeneous equation plus particular solution

$$x(t) = x_c(t) + x_p(t)$$

$\uparrow$ contains integration constants.

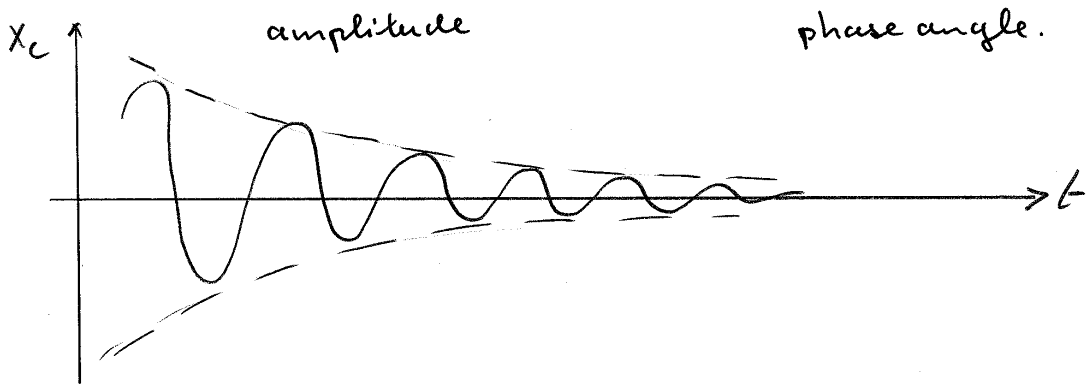
<u>Homogeneous equation</u> (free motion).

$$\ddot{x} + 2\zeta \cdot \omega_0 \cdot \dot{x} + \omega_0^2 \cdot x = 0$$

(i) Sub-critical damping $(0 \leq \zeta < 1)$

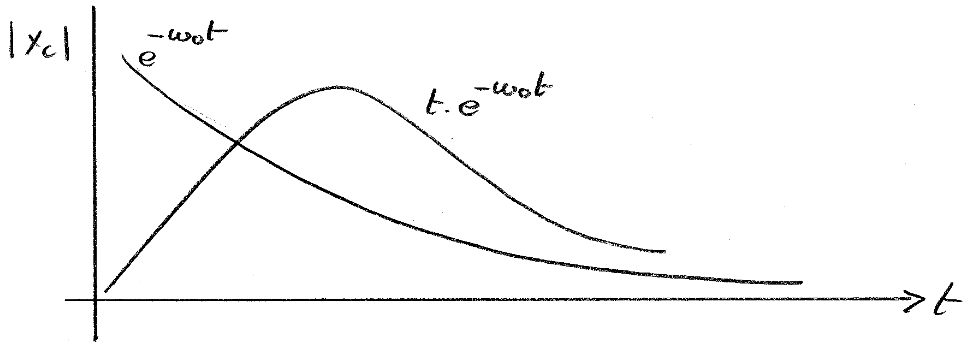
$$x_c(t) = e^{-\zeta \cdot \omega_0 \cdot t} \cdot \left\{ A \cdot \sin(\omega_0 \cdot \sqrt{1-\zeta^2} \cdot t) + B \cos(\omega_0 \cdot \sqrt{1-\zeta^2} \cdot t) \right\}$$

$$= \underbrace{e^{-\zeta \cdot \omega_0 \cdot t} \; C}_{\text{amplitude}} \cdot \sin(\omega_0 \cdot \sqrt{1-\zeta^2} \cdot t + \underset{\substack{\uparrow \\ \text{phase angle.}}}{\phi})$$



(ii) Critical damping ($\zeta = 1$)

$$x_c(t) = e^{-\omega_0 . t}(A + B . t)$$



(iii) Super-critical damping ($\zeta > 1$).

$$x_c(t) = A . e^{\lambda_1 t} + B\, e^{\lambda_2 t}$$

where $\lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$.

Non-homogeneous equation (forced motion).

Consider three special cases.

(i) $f(t) = \text{constant} = f_0$

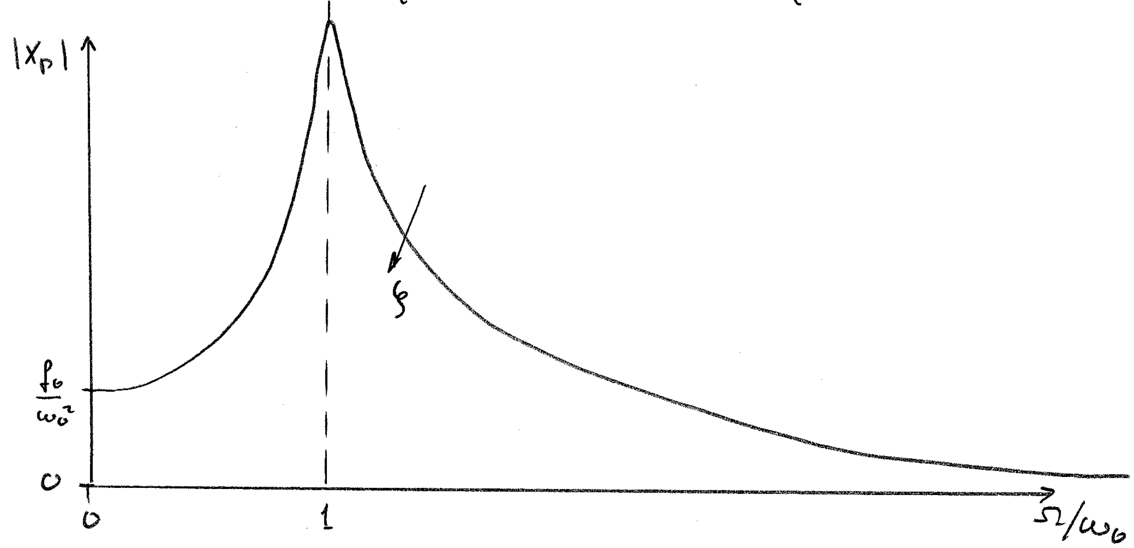
$\Rightarrow$ particular solution: $x_p = \text{constant} = \dfrac{f_0}{\omega_0^2}$.

(ii) $f(t)$ = harmonic = $f_0 \cdot \cos \Omega t$

$\Rightarrow$ particular solution:

$$x_p = a \cdot \sin \Omega t + b \cdot \cos \Omega t$$

$$= \underbrace{\frac{f_0}{\omega_0^2}}_{\text{static displacement}} \cdot \underbrace{\frac{1}{\sqrt{\left\{1-(\Omega/\omega_0)^2\right\}^2 + (2\zeta \cdot \Omega/\omega_0)^2}}}_{\text{dynamic correction factor}} \cos(\Omega t + \underbrace{\phi}_{\text{phase angle.}})$$



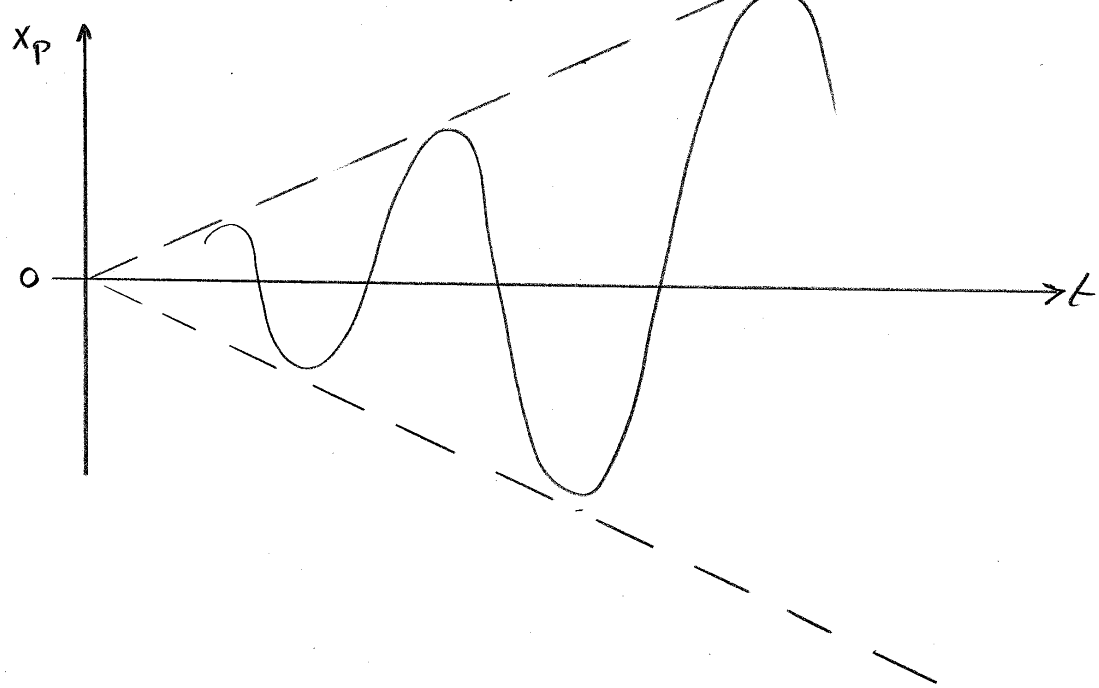
(iii) Resonance

$$f(t) = \text{harmonic} = f_0 \cos \Omega t$$

where $\Omega = \omega_0$, and $\varphi = 0$.

$$\Rightarrow x(t) = \left\{ \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t) + x_0 \cdot \cos(\omega_0 t) \right\} +$$

$$+ \underbrace{\frac{f_0}{2\omega_0} \cdot t \cdot \sin \omega_0 t}_{x_p}$$



8. SYSTEM OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Problem statement

Dynamics equation:

$$\boxed{M\ddot{\bar{u}} + K\bar{u} = \bar{f}(t)}$$

$\dim \bar{u} = n$
$\dim M, K = n \times n$

initial conditions: $\bar{u}(0) = \bar{u}_0$
$\dot{\bar{u}}(0) = \dot{\bar{u}}_0$

$\bar{f}(t)$ prescribed.

Solve for $\bar{u}(t)$

Solution procedure:

(i) find general solution to the homogeneous differential equation ("complementary solution")

(ii) find the/a particular solution (the solution to the inhomogeneous differential equation)

(iii) form the complete solution (the sum of the complementary solution and the particular solution)

(iv) apply the initial conditions.

We shall start with the <u>assumptions:</u>

$$\begin{cases} M \text{ is symmetric and positive definite.} \\ K \text{ is symmetric and positive semi-definite.} \end{cases}$$

or:
$$M = M^T > 0$$
$$K = K^T \geq 0$$

The more general case of nonsymmetric matrices will be treated subsequently.

Complementary solution

The general solution to the homogeneous equation is called the "complementary solution".

$$\boxed{M\ddot{\bar{u}} + K\bar{u} = \bar{0}}$$

Try a solution in the form:

$$\bar{u}(t) = \hat{\bar{u}}\, q(t) \qquad \text{where } \hat{\bar{u}} = \text{constant.}$$

$$\Rightarrow \text{find } \hat{\bar{u}} \text{ and } q(t)$$

Substitute: $M(\hat{\bar{u}}\ddot{q}) + K(\hat{\bar{u}}q) = \bar{0}$

$$\left\{ M\left(\frac{\ddot{q}}{q}\right) + K \right\}\hat{\bar{u}} = \bar{0}$$

Conclusion: $\dfrac{\ddot{q}}{q} \doteq \text{constant} \triangleq \lambda$ (possibly complex)

$$\Rightarrow \boxed{(\lambda M + K)\hat{\bar{u}} = \bar{0}}$$

Note: $\bar{u}$ = vector of generalised coordinates u_i
$\hat{\bar{u}}_i$ = mode shape
q_i = principal coordinate

$\hat{\bar{u}}q$ = "normal mode".

This is an "eigenproblem":

$$K \hat{\bar{u}} = -\lambda M \hat{\bar{u}}$$

Trivial solution: $\hat{\bar{u}} = \bar{0}$

Nontrivial solution: if $\det(\lambda M + K) = 0$

$\Rightarrow$ polynomial in λ

$\Rightarrow$ eigenvalues λ_j

$(\lambda_j M + K)\hat{\bar{u}} = \bar{0} \Rightarrow$ eigenvectors $\hat{\bar{u}}_j$

where $j = 1, 2, 3, \ldots, n$.

<u>Property</u>: λ_j and $\hat{\bar{u}}_j$ are real.

Proof.

Let λ_j and $\hat{\bar{u}}_j$ be complex:

$\lambda_j = a_j + i\, b_j$ and $\hat{\bar{u}}_j = \bar{\alpha}_j + i\, \bar{\beta}_j$, where $i \triangleq \sqrt{-1}$

$(\lambda_j M + K)\hat{\bar{u}}_j = \bar{0} \Rightarrow$

$$\{(a_j + i\, b_j) M + K\}(\bar{\alpha}_j + i\, \bar{\beta}_j) = \bar{0}$$

$$\{(a_j M + K)\bar{\alpha}_j - b_j M \bar{\beta}_j\} +$$

$$+ i \{(a_j M + K)\bar{\beta}_j + b_j M \bar{\alpha}_j\} = \bar{0}$$

Hence: $Re + i\, Im = \bar{0} \Rightarrow Re = 0$ and $Im = 0$

$$\Rightarrow Re - i\, Im = \bar{0}$$

Replace $i \Rightarrow -i$

$$(\lambda_j^* M + K)\,\hat{\bar{u}}_j^* = \bar{0}$$

where $\lambda_j^* = a_j - i b_j$ and $\hat{\bar{u}}_j^* = \bar{\alpha}_j - i\bar{\beta}_j$

Hence: if $\lambda_j, \hat{\bar{u}}_j$ are a solution,
then $\lambda_j^*, \hat{\bar{u}}_j^*$ are also a solution.

Consider: $(\hat{\bar{u}}_j^*)^T \{(\lambda_j M + K)\hat{\bar{u}}_j\} = 0$

and $\hat{\bar{u}}_j^T \{(\lambda_j^* M + K)\hat{\bar{u}}_j^*\} = 0$

Subtract, and use $M^T = M$, $K^T = K$:

$$(\lambda_j - \lambda_j^*)\,(\hat{\bar{u}}_j^*)^T M \hat{\bar{u}}_j = 0$$

$$(2i\,b_j)\ (\bar{\alpha}_j - i\bar{\beta}_j)^T M (\bar{\alpha}_j + i\bar{\beta}_j) = 0$$

$$\Rightarrow \quad b_j \{ \underbrace{\bar{\alpha}_j^T M \bar{\alpha}_j}_{+} + \underbrace{\bar{\beta}_j^T M \bar{\beta}_j}_{+} \} = 0$$

$$\Rightarrow b_j = 0$$

Hence $\lambda_j = a_j = \text{real}$ $\triangleleft$

From $(\lambda_j M + K)\hat{\bar{u}}_j = \bar{0} \Rightarrow \hat{\bar{u}}_j = \text{real}.$ $\triangleleft$

(without loss of generality)

<u>Property</u>: $\hat{\bar{u}}_j^T M \hat{\bar{u}}_k = 0$ and $\hat{\bar{u}}_j^T K \hat{\bar{u}}_k = 0$ for $j \neq k$.

Proof.

Consider: $\hat{\bar{u}}_k^T \{ (\lambda_j M + K) \hat{\bar{u}}_j \} = 0$

and $\hat{\bar{u}}_j^T \{ (\lambda_k M + K) \hat{\bar{u}}_k \} = 0$

Subtract, and use $M^T = M$, $K^T = K$:

$$(\lambda_j - \lambda_k)\ \hat{\bar{u}}_j^T M \hat{\bar{u}}_k = 0.$$

(i) Case $j \neq k$

(i-a) If $\lambda_j \neq \lambda_k \Rightarrow \hat{u}_j^T M \hat{u}_k = 0$

$\hookrightarrow \hat{u}_j^T K \hat{u}_k = 0$

The eigenvectors are orthogonal (with respect to M and K)

(i-b) $\lambda_j = \lambda_k$ degenerate case (see literature)

<u>Property:</u> $\lambda_j \leq 0$

Consider $\hat{u}_j^T \{ (\lambda_j M + K) \hat{u}_j \} = 0$

$$\Rightarrow \lambda_j \underbrace{(\hat{u}_j^T M \hat{u}_j)}_{+} + \underbrace{(\hat{u}_j^T K \hat{u}_j)}_{\geq 0} = 0$$

$$\Rightarrow \lambda_j = - (\hat{u}_j^T K \hat{u}_j) / (\hat{u}_j^T M \hat{u}_j) \leq 0$$

Write: $\lambda_j = -\omega_j^2$ where ω_j = real, positive.

Write in ascending order:

$$0 \leq \omega_1 < \omega_2 < \omega_3 < \cdots \cdots < \omega_n$$

Note: $\hat{u}_j^T K \hat{u}_j = \omega_j^2 \, \hat{u}_j^T M \hat{u}_j$ Δ

Return to original problem:

$$\left\{ M \underbrace{\left(\frac{\ddot{q}}{q}\right)}_{\lambda \, = -\omega^2} + K \right\} \hat{u} = \bar{0}$$

$$\boxed{(-\omega^2 M + K)\,\hat{u} = \bar{0}}$$

Trivial solution: $\hat{u} = \bar{0}$

Nontrivial solution: if $\det(-\omega^2 M + K) = 0$

$\Rightarrow$ polynomial in ω:

$$a_n \cdot (\omega^2)^n + a_{n-1} (\omega^2)^{n-1} + \cdots + a_1 (\omega^2) + a_0 = 0$$

Solve for ω^2 $\Rightarrow$ ω_j "eigen frequency"
"modal frequency"
"natural frequency".

Then: $(-\omega_j^2 M + K)\,\hat{u}_j = \bar{0}$

$\Rightarrow$ $\hat{u}_j$ "eigenvector"
"mode shape"

$$\frac{\ddot{q}}{q} = -\omega^2 \quad \Rightarrow \quad \ddot{q}_j + \omega_j^2 q_j = 0 \qquad \triangleleft$$

(i) Case $\omega_j = 0 \quad \Rightarrow \quad q_j = a_j \cdot + b_j t$

$\Rightarrow$ "rigid body mode"

(ii) Case $\omega_j > 0 \quad \Rightarrow \quad q_j = a_j \cdot \sin(\omega_j t) + b_j \cdot \cos(\omega_j t)$

"oscillatory mode"

where $j = 1, 2, \ldots, n$

Hence: $\quad \hat{\bar{u}}_j \, q_j(t)$ is a solution

General solution:

$$\hat{\bar{u}}(t) = \sum_{j=1}^{n} \hat{\bar{u}}_j \, q_j(t)$$

$q_j(t)$ – modal coordinate

$\hat{\bar{u}}_j$ – mode shape

$$\boxed{\begin{aligned}\bar{u}(t) = {} & \sum_j \hat{\bar{u}}_j (a_j + b_j \cdot t) + \\ & + \sum_k \hat{\bar{u}}_k \{a_k \cdot \sin(\omega_k t) + b_k \cdot \cos(\omega_k t)\}\end{aligned}}$$

give interpretation!

Particular solution

$$\boxed{M\ddot{\bar{u}} + K\bar{u} = \bar{f}(t) \qquad \text{(prescribed)}}$$

(i) <u>Special case</u>: $\bar{f}(t) = \text{constant} \hat{=} \bar{f}_0$

if $K = K^T > 0$ then $\boxed{\bar{u} = K^{-1}\bar{f}_0}$ (constant).

if $K = K^T \geq 0$ then there is also a rigid body mode. The body accelerates

(ii) <u>Special case</u>: $\bar{f}(t) = \bar{f}_0 \cos \Omega t$ (harmonic)

Try $\bar{u}(t) = \hat{\bar{u}}_p \cdot \cos \Omega t$

Substitute: $\{(-\Omega^2 M + K)\hat{\bar{u}}_p - \bar{f}_0\} \cos \Omega t = 0$

$$\Rightarrow \hat{\bar{u}}_p = (-\Omega^2 M + K)^{-1} \bar{f}_0$$

$$\boxed{\bar{u}(t) = (-\Omega^2 M + K)^{-1} \bar{f}_0 \cos \Omega t}$$

(iii) <u>Special case:</u>

$$\bar{f}(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \{ a_j \cdot \cos(j\Omega t) + b_j \cdot \sin(j\Omega t) \}$$

periodic excitation, with period $T = 2\pi/\Omega$

compare Section 7.3.

(iv) <u>General case</u> : arbitrary $\bar{f}(t)$

Seek a solution in the form:

$$\boxed{\bar{u}(t) = \sum_{j=1}^{n} \hat{\bar{u}}_j \; Q_j(t)}$$

where $\hat{\bar{u}}_j$ is the eigenvector obtained from analysis of the homogeneous equation, and where $Q_j(t)$ is an unknown function, to be determined.

"modal expansion".

This constitutes a transformation from geometric coordinates to new "modal coordinates".

"modal superposition method".

Substitute:

$$M\left(\sum_j \hat{u}_j \ddot{Q}_j\right) + K\left(\sum_j \hat{u}_j Q_j\right) = \bar{f}$$

Pre-multiply by $\bar{u}_k^T$: Only the terms with $j = k$ survive (due to orthogonality):

$$(\hat{u}_k^T M \hat{u}_k)\, \ddot{Q}_k + (\hat{u}_k^T K \hat{u}_k)\, Q_k = \bar{u}_k^T \bar{f}$$

Divide by $\hat{u}_k^T M \hat{u}_k$:

$$\boxed{\ddot{Q}_k + \omega_k^2\, Q_k = \frac{1}{\hat{u}_k^T M \hat{u}_k}\, \hat{u}_k^T \bar{f}} \quad \triangleleft$$

This is a scalar, inhomogeneous differential equation for $Q_k(t)$ only!

Solution: $Q_k(t) = \underbrace{q_k(t)\Big|_{\text{complem.}}}_{\text{unforced}} + \underbrace{q_k(t)\Big|_{\text{particular}}}_{\text{forcing}}$

See Section 7.

$$\boxed{\bar{u}(t) = \sum_{j=1}^{n} \hat{u}_j \left\{ q_j(t)\Big|_{\text{compl.}} + q_j(t)\Big|_{\text{partic.}} \right\}}$$

$Q_j(t)$ = "amplification coefficient".

Initial conditions

Initial conditions: $\bar{u}(0) = \bar{u}_0$
$\dot{\bar{u}}(0) = \dot{\bar{u}}_0$

General solution:

$$\bar{u}(t) = \sum_{j=1}^{n} \hat{\bar{u}}_j \left\{ q_j^{comp.}(t) + q_j^{part.}(t) \right\}$$

$$\begin{cases} \bar{u}(0) = \sum_{j=1}^{n} \hat{\bar{u}}_j \left\{ q_j^{comp}(0) + q_j^{part}(0) \right\} = \bar{u}_0 \\ \dot{\bar{u}}(0) = \sum_{j=1}^{n} \hat{\bar{U}}_j \left\{ \dot{q}_j^{comp}(0) + \dot{q}_j^{part}(0) \right\} = \dot{\bar{u}}_0 \end{cases}$$

Pre-multiply by $(M\bar{U}_k)^T$:

$$\begin{cases} \hat{\bar{u}}_k^T M \bar{U}_0 \overset{!}{=} \hat{\bar{U}}_k^T M \hat{\bar{U}}_k \left\{ q_k^{comp.}(0) + q_k^{part}(0) \right\} \\ \bar{U}_k^T M \dot{\bar{U}}_0 \overset{!}{=} \hat{\bar{U}}_k^T M \hat{\bar{U}}_k \left\{ \dot{q}_k^{comp}(0) + \dot{q}_k^{part}(0) \right\} \end{cases}$$

Solve for $q_k^{comp}(0)$ and $\dot{q}_k^{comp}(0)$

Damping

We considered: $M\ddot{\bar{u}} + K\bar{u} = \bar{f}(t)$

Now, add a linear damping term:

$$\boxed{M\ddot{\bar{u}} + C\dot{\bar{u}} + K\bar{u} = \bar{f}(t)}$$

We consider the following <u>three cases</u>:

(i) $\bar{f}(t) = \bar{f}_0$ constant

(ii) $\bar{f}(t) = \bar{f}_0 \cdot \cos \Omega t$ harmonic load

(iii) $\bar{f}(t)$ arbitrary

(i). <u>Case $\bar{f}(t) = \bar{f}_0$.</u> Find <u>stationary</u> solution.

Solution: $\bar{u} = K^{-1}\bar{f}_0$

(provided K^{-1} exists).

This is the familiar case of statics.

(iii) <u>Case $\bar{f}(t) = \bar{f}_o . \cos \Omega t$</u> Find <u>stationary</u> solution.

Try: $\bar{u}(t) = \bar{\alpha} \cos \Omega t + \bar{\beta} \sin \Omega t$

where $\bar{\alpha}$ and $\bar{\beta}$ are to be determined.

Substitute into the equation of motion:

$$M \Omega^2 (-\bar{\alpha} \cos - \bar{\beta} \sin) + C \Omega (-\bar{\alpha} \sin + \bar{\beta} \cos) + $$
$$+ K (\bar{\alpha} \cos + \bar{\beta} \sin) = \bar{f}_o \cos.$$

$$\{ (-\Omega^2 M + K) \bar{\alpha} + \Omega C \bar{\beta} - \bar{f}_o \} \cos +$$
$$+ \{ - \Omega C \bar{\alpha} + (-\Omega^2 M + K) \beta \} \sin = 0.$$

This expression is to hold for all t

$$\Rightarrow \quad (-\Omega^2 M + K) \bar{\alpha} + \Omega C \bar{\beta} - \bar{f}_o = 0$$
$$- \Omega C \bar{\alpha} + (-\Omega^2 M + K) \bar{\beta} = 0.$$

$$\begin{bmatrix} -\Omega^2 M + K & \Omega C \\ - \Omega C & -\Omega^2 M + K \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} = \begin{bmatrix} \bar{f}_o \\ 0 \end{bmatrix}$$

Solve: $$\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} = \begin{bmatrix} -\Omega^2 M + K & \Omega C \\ - \Omega C & -\Omega^2 M + K \end{bmatrix}^{-1} \begin{bmatrix} \bar{f}_o \\ 0 \end{bmatrix}$$

If $C = 0$, we retrieve:

$$\begin{cases} \bar{\alpha} = (-\Omega^2 M + K)^{-1} \bar{f}_0 \\ \bar{\beta} = \bar{0} \end{cases}$$

(provided Ω does not coincide with any of the modal frequencies!).

(iii) <u>Case $\bar{f}(t)$ arbitrary</u>.

Consider the modal transformation:

$$\bar{u} = \sum_j \hat{u}_j \; Q_j(t)$$

where $\hat{u}_j$ are the eigenvectors for the original case of <u>zero</u> damping.

Substitute:

$$M(\sum_j \hat{u}_j \ddot{Q}_j) + C(\sum_j \hat{u}_j \dot{Q}_j) + K(\hat{u}_j Q_j) = \bar{f}$$

Pre-multiply by $\hat{u}_k^T$:

$$(\hat{u}_k^T M \hat{u}_k)\ddot{Q}_k + \sum_j (\hat{u}_k^T C \hat{u}_j)\dot{Q}_j +$$

$$+ (\hat{u}_k^T K \hat{u}_k) Q_k = \hat{u}_k^T \bar{f}$$

$$\ddot{Q}_k + \left\{ \frac{1}{\hat{\bar{u}}_k^T M \hat{\bar{u}}_k} \sum_j (\hat{\bar{u}}_k^T C \hat{\bar{u}}_j) \dot{Q}_j \right\} + \omega_k^2 Q_k = \frac{\hat{\bar{u}}_k^T \bar{f}}{\hat{\bar{u}}_k^T M \hat{\bar{u}}_k}$$

(i) Assume : $|\hat{\bar{u}}_k^T C \hat{\bar{u}}_j| << |\hat{\bar{u}}_k^T C \hat{\bar{u}}_k|$ for $k \neq j$

Then, approximately:

$$\sum_j (\hat{\bar{u}}_k^T C \hat{\bar{u}}_j) \dot{Q}_j \cong (\hat{\bar{u}}_k^T C \hat{\bar{u}}_k) \dot{Q}_k$$

i.e. we assume that the modal transformation approximately diagonalizes C.

Define the modal damping ratio:

$$\xi_k \triangleq \frac{1}{2\omega_k} \frac{\hat{\bar{u}}_k^T C \hat{\bar{u}}_k}{\hat{\bar{u}}_k^T M \hat{\bar{u}}_k}$$

Then:

$$\boxed{\ddot{Q}_k + 2\xi_k \omega_k \dot{Q}_k + \omega_k^2 Q_k = \frac{1}{\hat{\bar{u}}_k^T M \hat{\bar{u}}_k} \hat{\bar{u}}_k^T \bar{f}(t)}$$

$\Rightarrow$ solve for $Q_k(t)$

Finally: $\bar{u}(t) = \sum_k \hat{\bar{u}}_k Q_k(t)$

(ii) Another approximation: Rayleigh damping

Assume: $C = \alpha M + \beta K$

Then: $\sum_j \hat{\bar{u}}_k^T (C \hat{\bar{u}}_j) \dot{Q}_j = \sum_j \hat{\bar{u}}_k^T (\alpha M + \beta K) \hat{\bar{u}}_j \dot{Q}_j$

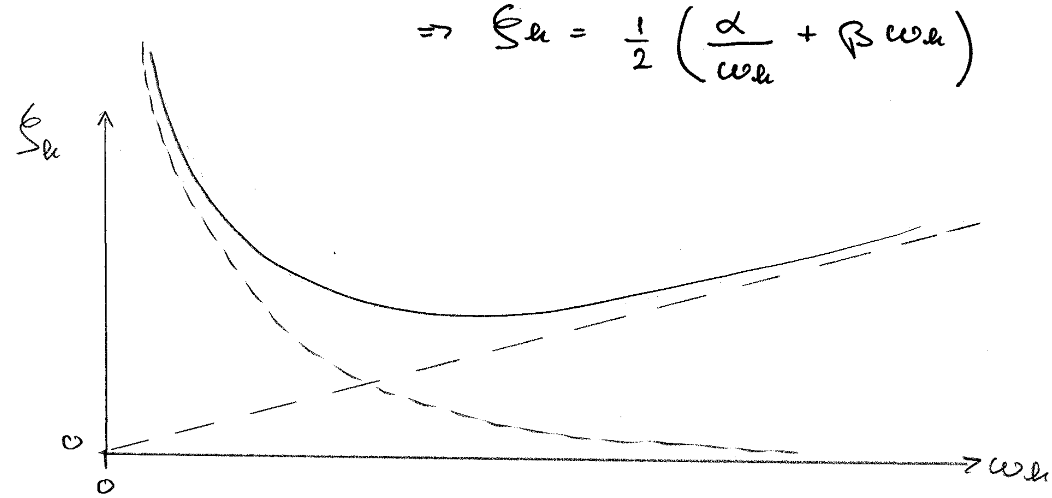
$$= (\alpha \hat{\bar{u}}_k^T M \hat{\bar{u}}_k + \beta \hat{\bar{u}}_k^T K \hat{\bar{u}}_k) \dot{Q}_k$$

$$= \hat{\bar{u}}_k^T M \hat{\bar{u}}_k \left(\alpha + \beta \omega_k^2\right) \dot{Q}_k$$

$$\Rightarrow \quad \ddot{Q}_k + (\alpha + \beta \omega_k^2) \dot{Q}_k + \omega_k^2 Q_k = \frac{\hat{\bar{u}}_k^T \bar{f}}{\hat{\bar{u}}_k^T M \hat{\bar{u}}_k}$$

Note: $2 \xi_k \omega_k = \alpha + \beta \omega_k^2$

$$\Rightarrow \quad \xi_k = \frac{1}{2} \left(\frac{\alpha}{\omega_k} + \beta \omega_k \right)$$



General case

$$M\ddot{\bar{u}} + C\dot{\bar{u}} + K\bar{u} = \bar{f}(t)$$

Write in state-space form.

$$\begin{bmatrix} \bar{u} \\ \dot{\bar{u}} \end{bmatrix}^{\bullet} = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \bar{u} \\ \dot{\bar{u}} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ M^{-1} \end{bmatrix} \bar{f}$$

This equation is of the type:

$$\dot{\bar{x}} = A\bar{x} + \bar{g}(t)$$

Homogeneous equation: $\dot{\bar{x}} = A\bar{x}$

$$A \Rightarrow \left.\begin{array}{l} \text{eigenvalues } \lambda_j \text{ (complex)} \\ \text{eigenvectors } \hat{u}_j \text{ (complex)} \end{array}\right\} j = 1, \dots, 2n$$

and so on.

Alternatively, retaining second-order form.

Try $\quad \bar{u}(t) = \hat{u}\, q(t)$

Substitute: $\quad (M\ddot{q} + C\dot{q} + Kq)\hat{u} = \bar{0}$

Consider $\quad q = e^{\lambda t}$

$$(\lambda^2 M + \lambda C + K)e^{\lambda t}\hat{u} = \bar{0}$$

$$(\lambda^2 M + \lambda C + K)\,\hat{u} = 0$$

Nontrivial solution: if $\det(\lambda^2 M + \lambda C + K) = 0$.

"quadratic eigenproblem".

λ_j is complex.

Ref. Gatti and Ferraro
Ref. Inman.

Special case: non-symmetric matrices

Consider: $M\ddot{\bar{u}} + K\bar{u} = \bar{f}$

where $M > 0$ but not necessarily symmetric
$K \geq 0$ " " " "

(i) First step: diagonalise M.

$\det(\lambda I - M) = 0 \Rightarrow \lambda_j, \hat{\bar{m}}_j$

Construct $T_1 \triangleq [\hat{\bar{m}}_1 \dots \hat{\bar{m}}_n]$

Transform: $\bar{u} = T_1 \bar{u}_1$

$$\underbrace{T_1^{-1} M T_1}_{\substack{\text{diagonal!} \\ \triangleq M_1}} \ddot{\bar{u}}_1 + \underbrace{T_1^{-1} K T_1}_{\triangleq K_1} \bar{u}_1 = \underbrace{T_1^{-1} \bar{f}}_{\triangleq \bar{f}_1}$$

(ii) Second step: transform M_1 to unit matrix

$$\bar{u}_2 = M_1 \bar{u}_1$$

$$\ddot{\bar{u}}_2 + K_2 \bar{u}_2 = \bar{f}_1$$

where $K_2 = K_1 \cdot M_1^{-1}$.

(iii) Third step: diagonalise K_2

$\det(\mu I - K_2) = 0 \Rightarrow \mu_j,\ \hat{\underline{k}}_j \qquad (\hat{\underline{k}}_j^T \hat{\underline{k}}_j \doteq 1)$

Construct: $T_2 \triangleq [\hat{\underline{k}}_1 \cdots \hat{\underline{k}}_n]$

Transform: $\bar{u}_2 = T_2 \bar{u}_3$

$$\underbrace{T_2^{-1} \cdot T_2}_{I_{n\times n}} \ddot{\bar{u}}_3 + \underbrace{T_2^{-1} \cdot K_2 T_2}_{\triangleq K_3} \bar{u}_3 = \underbrace{T_2^{-1}}_{\triangleq \bar{f}_3} \bar{f}_1$$

$$\ddot{\bar{U}}_3 + K_3 \bar{U}_3 = \bar{f}_3 \quad \text{with } K_3 \text{ diagonal.}$$

The differential equations are decoupled:

$$\ddot{u}_{3,j} + \omega_j^2 u_{3,j} = f_{3,j}$$

Solve: $u_{3,j}(t) \Rightarrow \bar{u}_3(t)$

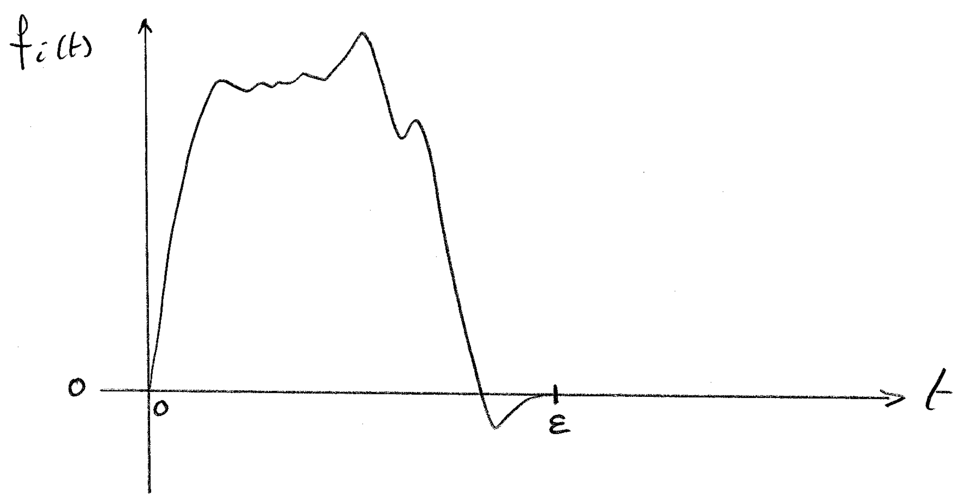
Finally: $\bar{u}(t) = T_1 \cdot M_1^{-1} \cdot T_2 \cdot \bar{u}_3(t)$.

Effect of application of an impulse.

System equation: $M\ddot{\bar{u}} + C\dot{\bar{u}} + K\bar{u} = \bar{f}(t)$

Initial conditions: $\bar{u}(0) = \bar{u}_0$; $\dot{\bar{u}}(0) = \dot{\bar{u}}_0$.

Apply an impulsive load $\bar{f}(t)$ at $t = 0^+$:



where $\varepsilon \downarrow 0$

Determine the new initial conditions immediately after application of the impulse.

Integrate:

$$\int_0^{\varepsilon} (M\ddot{\bar{u}} + C\dot{\bar{u}} + K\bar{u} - \bar{f})\, dt = \bar{0}.$$

Define:

$$\bar{f}_I \triangleq \lim_{\varepsilon \downarrow 0} \int_0^{\varepsilon} \bar{f}(t) \cdot dt \qquad \text{"generalized impulse".}$$

(i) $$\int_0^{\varepsilon} M\ddot{\bar{u}}\, dt = M\dot{\bar{u}}\Big|_0^{\varepsilon} = M\{\dot{\bar{u}}(\varepsilon) - \dot{\bar{u}}(0)\}$$

(ii) $$\left\|\int_0^{\varepsilon} C\dot{\bar{u}}\, dt\right\| \leq \|C\|\, \|\dot{\bar{u}}\|_{max.}\, \varepsilon$$

(iii) $$\left\|\int_0^{\varepsilon} K\bar{u}\, dt\right\| \leq \|K\| \int_0^{\varepsilon} (\|\dot{\bar{u}}\|_{max}\, t)\, dt$$

$$= \|K\|\, \|\dot{\bar{u}}_{max}\| \cdot \frac{\varepsilon^2}{2}.$$

Hence:

finite impulse $\bar{f}_I \Rightarrow$ change in velocity is finite $\Rightarrow \|\dot{\bar{u}}\|_{max}$ is finite

Take $\lim_{\varepsilon \downarrow 0}$

$$\Rightarrow \quad M\{\dot{\bar{U}}(0^+) - \dot{\bar{U}}(0)\} + \bar{0} + \bar{0} - \bar{f}_I = \bar{0}$$

$$\boxed{\begin{array}{ll} \dot{\bar{U}}(0^+) = \dot{\bar{U}}_0 + M^{-1}\,\bar{f}_I & \triangleleft \\ \bar{U}(0^+) = \bar{U}_0 & \triangleleft \end{array}}$$

Executive Summary

$$M\ddot{\bar{u}} + K\bar{u} = \bar{f}(t)$$

M = generalized mass matrix
K = " stiffness "
f = " force vector.

$\dim \bar{u}, \bar{f} = n$

For suitable choice of coordinates:

$M = M^T \; (>0)$ $\quad\quad$ $K = K^T \; (\geq 0)$

<u>Homogeneous equation</u> (free motion)

$$\bar{u}(t) = \hat{u}\, q(t) \quad \Rightarrow \quad \hat{u}, q \; ?$$

$$\det(-\omega^2 M + K) = 0 \quad \Rightarrow \quad \omega_i \quad (i = 1, 2, \dots, n)$$

$$(-\omega_i^2 M + K)\,\hat{u}_i = \bar{0} \quad \Rightarrow \quad \hat{u}_i$$

(one free multiplicative factor)

$$\ddot{q}_i + \omega_i^2 q_i = 0 \quad \Rightarrow \quad \bar{q}_i$$

If $\omega_i = 0 \Rightarrow$ rigid body motion.

$$\Rightarrow \quad \bar{u}(t) = \sum_{i=1}^{n} \hat{U}_i \, q_i(t)$$

Orthogonality: $\left.\begin{array}{l} \hat{U}_i^T M \hat{U}_j = 0 \\ \hat{U}_i^T K \hat{U}_j = 0 \end{array}\right\}$ for $i \neq j$.

Non-homogeneous equation.

Seek: $\bar{u}(t) = \sum_{i=1}^{n} \hat{U}_i \, Q_i(t) \quad \Rightarrow \quad Q_i$?

"modal expansion".

$$\Rightarrow \ddot{Q}_i + \omega_i^2 Q_i = \frac{1}{\hat{U}_i^T M \hat{U}_i} \hat{U}_i^T \bar{f}(t)$$

$$\hat{=} \; g_i(t).$$

$$\Rightarrow Q_i(t) = q_i(t) + Q_i^{\text{particular}}(t)$$

$$\Rightarrow \bar{U}(t) = \sum_{i=1}^{n} \hat{U}_i \left\{ q_i(t) + Q_i^{\text{part.}}(t) \right\}$$

Initial conditions: $\bar{U}(0) = \bar{U}_0$; $\dot{\bar{U}}(0) = \dot{\bar{U}}_0$

$$\Rightarrow \quad \hat{U}_i^T M \bar{U}_0 = (\hat{U}_i^T M \hat{U}_i) \left\{ q_i(0) + Q_i^{part.}(0) \right\}$$

$$\hat{U}_i^T M \dot{\bar{U}}_0 = (\hat{U}_i^T M \hat{U}_i) \left\{ \dot{q}_i(0) + \dot{Q}_i^{part}(0) \right\}$$

$\Rightarrow$ solve for integration constants in $q_i(t)$

Nomenclature:

symbol	mathematics	dynamics
ω_i	eigen frequency	modal frequency
q_i	principal coordinate	modal coordinate
$\hat{U}_i$	eigenvector	mode shape

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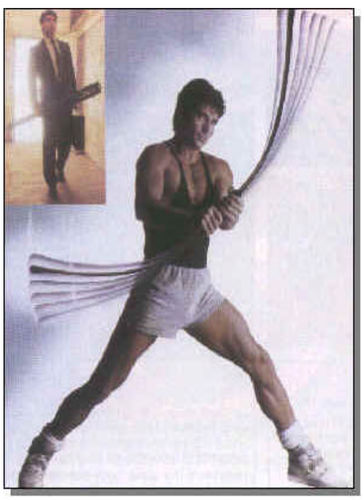
Notes on Linear Vibration Theory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* Body Blade picture taken from: http://www.starsystems.com.au

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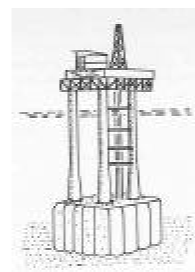

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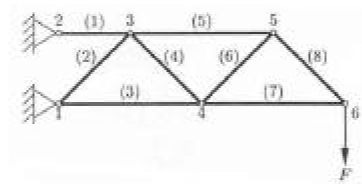

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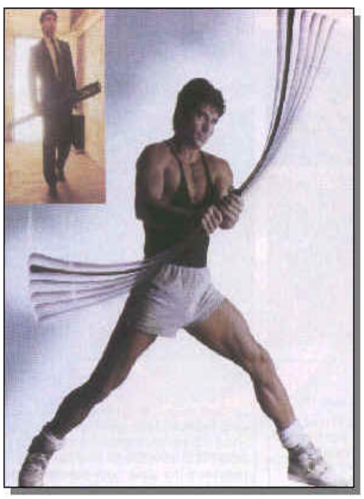

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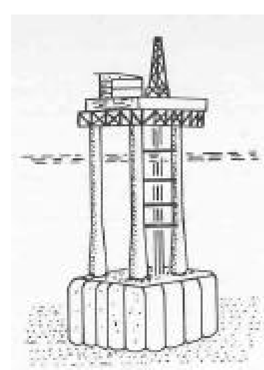
PART TWO:

INFINITE-DIMENSIONAL SYSTEMS



9. NOTES ON THE EULER-BERNOULLI BEAM

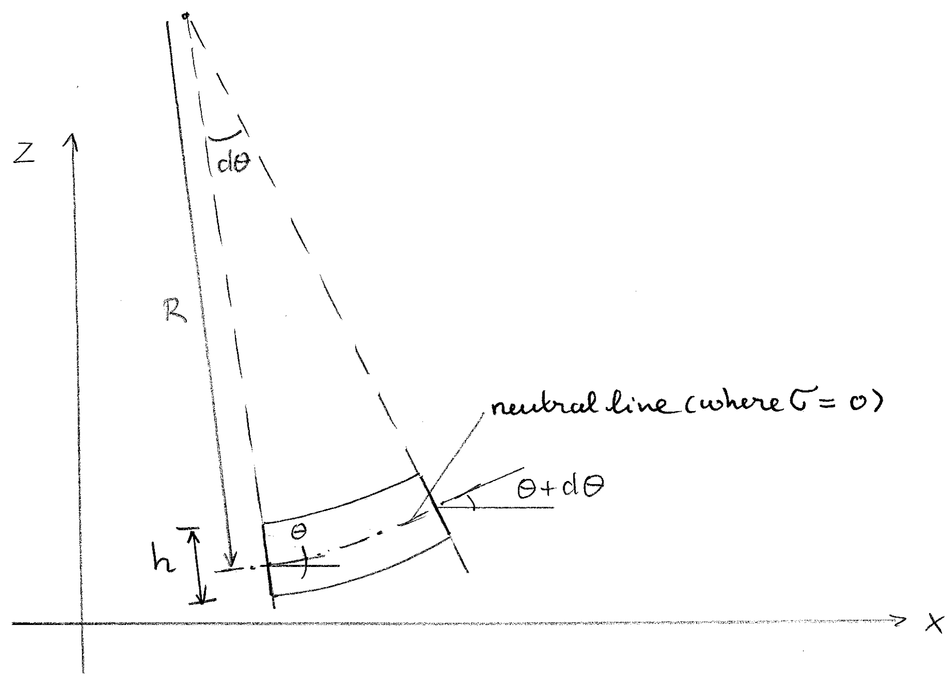
9.1 Pure bending

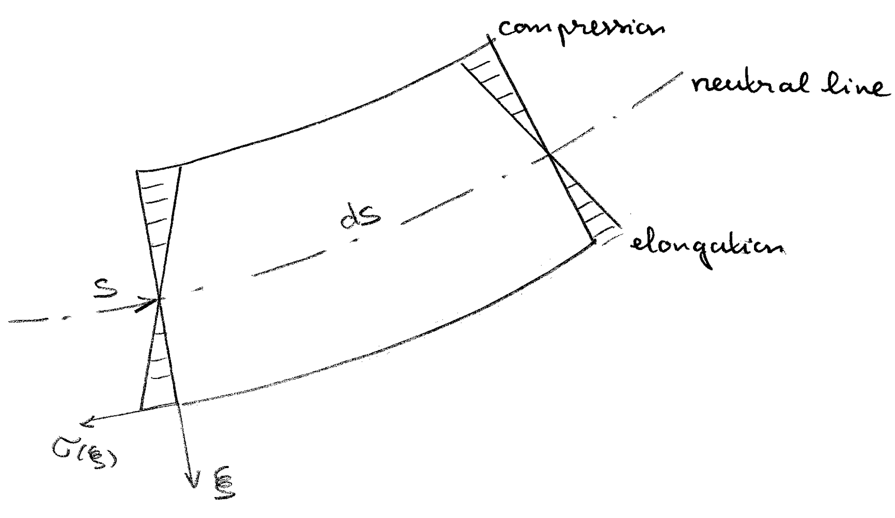
9.2 Static equilibrium under various loads

9.3 Static displacement field

Pure bending

Initially straight beam
Slender ($h/L < 1/10$)
plane cross-sections remain plane
parallel planes remain parallel





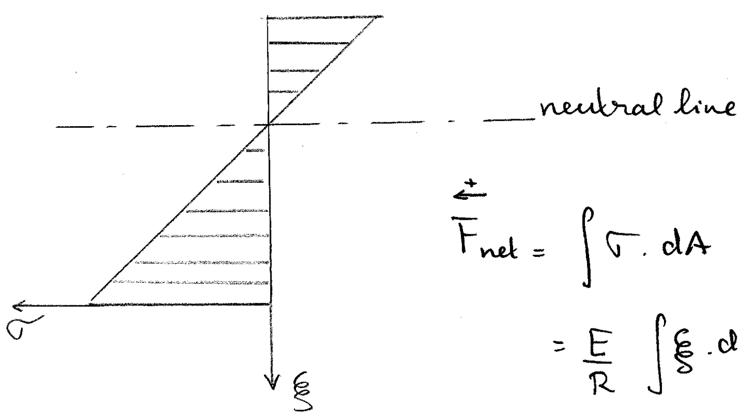
Undeformed fibre: length $= ds_0 = R \cdot d\theta$

Deformed fibre: length $= ds = (R+\xi)d\theta$

$\Rightarrow$ Strain in fibre: $\varepsilon(\xi) \triangleq \dfrac{ds - ds_0}{ds_0} = \dfrac{\xi}{R}$

$\Rightarrow$ Stress in fibre (Hooke): $\sigma_{(\xi)} = E \cdot \dfrac{\xi}{R}$

Neutral line: defined by $\sigma = 0$.



$$\overset{\leftarrow +}{F_{net}} = \int \sigma \cdot dA$$

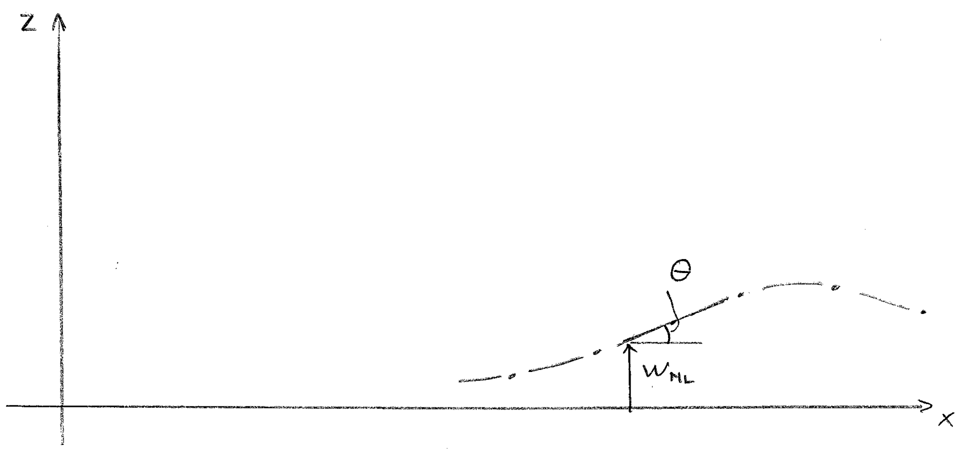
$$= \frac{E}{R} \int \xi \cdot dA$$

Hence: neutral line from $\int \xi \cdot dA = 0$.

$$\overset{\curvearrowleft +}{M} = \int (\sigma \cdot dA) \cdot \xi = \frac{E}{R} \underbrace{\int \xi^2 \cdot dA}_{I}$$

I = area moment-of-inertia
with respect to neutral line.

$$\overset{\curvearrowright +}{M} = \frac{EI}{R} \qquad \triangle$$



Small deformations: $|w_{NL}(x)/L| \ll 1$

$|\theta| \ll 1$

Then: $ds = \sqrt{(dx)^2 + (dz)^2}$

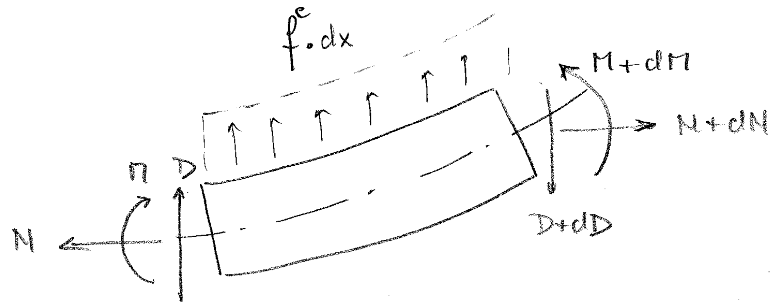
$= dx\sqrt{1 + (dz/dx)^2} \cong dx$

$ds = R \cdot d\theta \quad \Rightarrow \quad \frac{1}{R} = \frac{d\theta}{ds} \cong \frac{d\theta}{dx} = \frac{d^2 w_{NL}}{dx^2}$

Hence:

$$\boxed{\begin{gathered} M = EI\frac{d^2 w_{NL}}{dx^2} \\ \varepsilon = \xi \cdot \frac{d^2 w_{NL}}{dx^2} \; ; \; \sigma = E \cdot \xi \cdot \frac{d^2 w_{NL}}{dx^2} \end{gathered}}$$

Static equilibrium under various loads



transversal load D
normal load N
distributed external load $f^e \cdot dx$

Vertical equilibrium: $\cancel{D} + f^e \cdot dx - (\cancel{D} + dD) = 0$

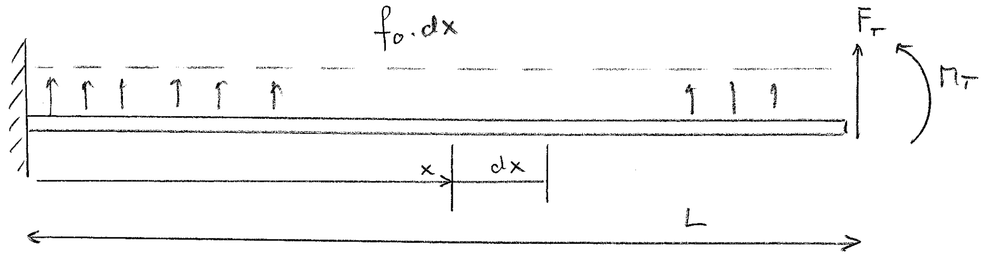
Horizontal equilibrium: $-N + dN = 0$

Angular equilibrium :

$$-\cancel{M} + (\bar{f}^e \cdot dx) \cdot \frac{dx}{2} - (D + dD)\,dx + (\cancel{M} + dM) + $$
$$- (N + dN)\, dw_{NL} = 0$$

$$\boxed{\frac{dD}{dx} = f^e \qquad N = \text{constant.} \qquad \frac{dM}{dx} = D + N \cdot \frac{dw_{NL}}{dx}}$$

Static displacement field

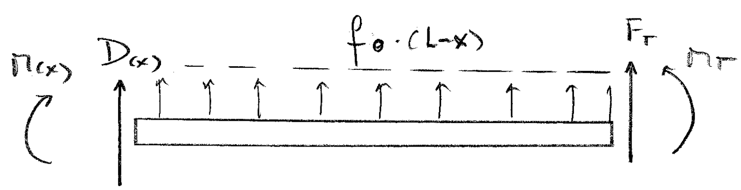


Clamped at $x=0$ $\Rightarrow$ $W_{ML}(0)=0$ and $\dfrac{dW_{ML}}{dx}(0)=0$.

Concentrated tip loads F_T and M_T

Distributed load $f_0 \cdot dx$ (f_0 = constant).

Consider a finite beam element:



Vertical equilibrium: $D + f_0 \cdot (L-x) + F_T = 0$

$$D(x) = -f_0 \cdot (L-x) - F_T$$

Angular equilibrium:

$$-M + \{f_0 \cdot (L-x)\} \cdot \frac{L-x}{2} + F_T \cdot (L-x) + M_T = 0$$

$$M(x) = f_0 \cdot \frac{(L-x)^2}{2} + F_T \cdot (L-x) + M_T.$$

$$M(x) = E\,I.\ \frac{d^2 W_{ML}}{dx^2}(x)$$

$$E\,I.\ \frac{d^2 W_{ML}}{dx^2} = f_0 \cdot \frac{(L-x)^2}{2} + F_T \cdot (L-x) + M_T$$

Integrate: $E\,I.\ W_{ML}(x) = f_0 \cdot \frac{(L-x)^4}{24} + F_T \cdot \frac{(L-x)^3}{6} + M_T \cdot \frac{x^2}{2} +$

$$+ C_1 \cdot x + C_2$$

Boundary conditions: $W_{ML}(0) = 0$ and $\frac{dW_{ML}}{dx}(0) = 0$

$$\Rightarrow C_1, C_2$$

Finally:

$$E\,I.\ W_{ML}(x) = f_0 \cdot \left\{ \frac{(L-x)^4}{24} + \frac{L^3}{6} \cdot x - \frac{L^4}{24} \right\} +$$

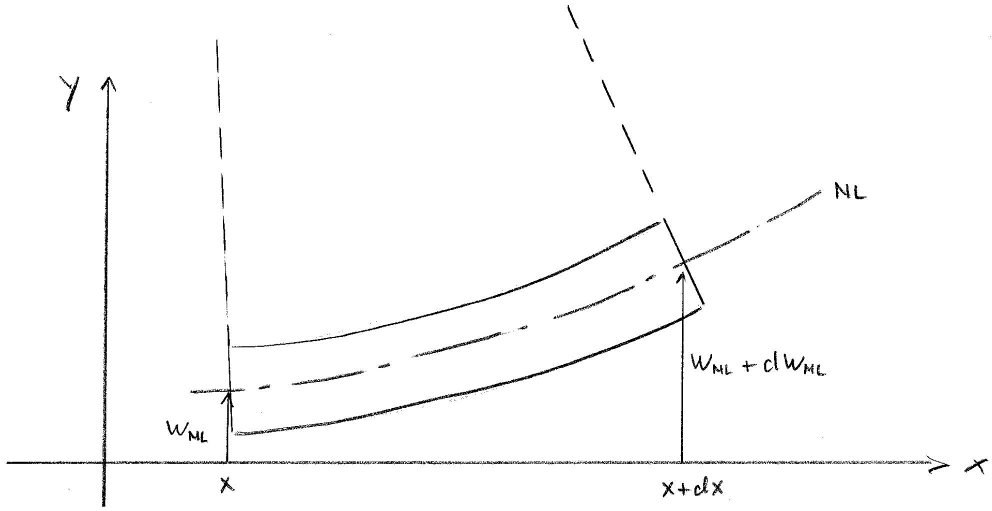
$$+ F_T \cdot \left\{ \frac{(L-x)^3}{6} + \frac{L^2}{2} \cdot x - \frac{L^3}{6} \right\} + M_T \cdot \frac{x^2}{2}.$$

At the <u>tip</u>:

$$\boxed{\begin{aligned} W_{NL}(L) &= \frac{1}{EI}\left\{ f_0 \cdot \frac{L^4}{8} + F_T \cdot \frac{L^3}{3} + M_T \cdot \frac{L^2}{2} \right\} \\ \Theta(L) &= \frac{1}{EI}\left\{ f_0 \cdot \frac{L^3}{6} + F_T \cdot \frac{L^2}{2} + M_T \cdot L \right\} \end{aligned}}$$

"Vergeet-mij-nietjes" !

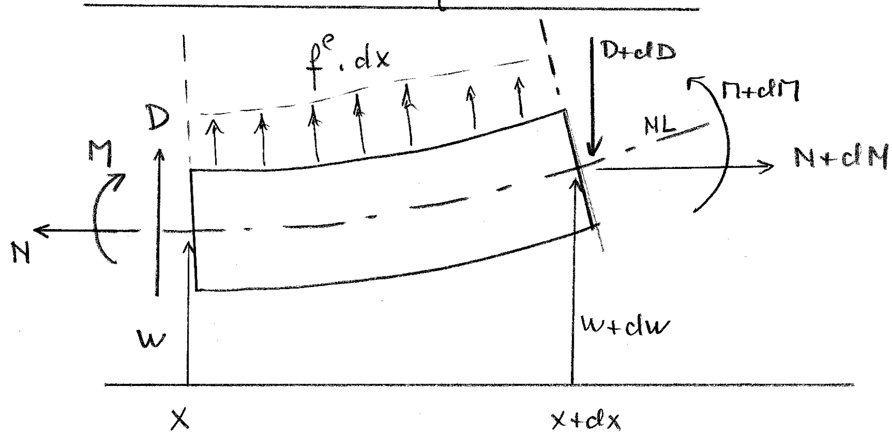
Executive Summary



Euler: $M = EI \dfrac{d^2 w_{ML}}{dx^2}$

$$w_{ML}(x) \Rightarrow \varepsilon(x,\xi) = \xi \cdot \frac{d^2 w_{ML}}{dx^2} \Rightarrow \sigma(x,\xi)$$

<u>Beam in static equilibrium.</u>



Newton, horizontal: $N = \text{constant} = N_0$

Newton, vertical : $\dfrac{dD}{dx} = f^e$

Euler : $\dfrac{dM}{dx} = D + N_0 \dfrac{dw}{dx}$

Special case:
$$\begin{cases} \text{tip force } F_T \\ \text{tip moment } M_T \end{cases}$$

Then ("vergeet-mij-nietjes"):

$$\begin{cases} w_{Tip} = \dfrac{F_T \cdot L^3}{3EI} + \dfrac{M_T \cdot L^2}{2EI} \\[2ex] \theta_{Tip} = \dfrac{F_T \cdot L^2}{2EI} + \dfrac{M_T \cdot L}{EI}. \end{cases}$$

10. DYNAMICS OF CONTINUUM BODIES - A

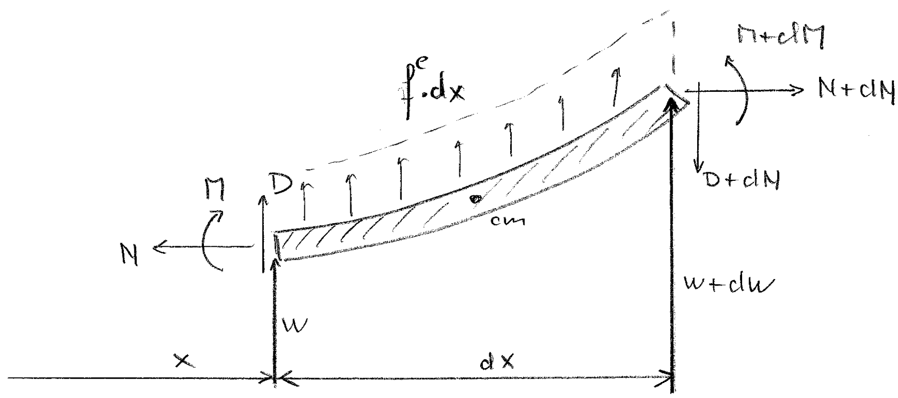
10.1 Dynamics of a string
- direct derivation
- limit derivation

10.2 Dynamics of a rod
- direct derivation
- limit derivation

10.3 Dynamics of a shaft
- direct derivation
- influence of geometry

Dynamics of a string

Direct derivation.



Euler : $M = EI \, \dfrac{d^2w}{dx^2}{}_{xx}$

Approximation: bending stiffness can be neglected.

$EI \to 0$ Hence $M \to 0$.

(i) Horizontal dynamics (Newton):

ignore horizontal displacement (higher order effect)

$\Rightarrow \quad -\cancel{N} + (\cancel{N} + dN) = 0$

$\Rightarrow \quad N = \text{constant} \triangleq N_0 \triangleleft$

(ii) Vertical dynamics (Newton).

$$(\rho \cdot A \cdot dx) \cdot \frac{\partial^2 w_{cm}}{\partial t^2} = \cancel{D} + f^e \cdot dx - (\cancel{D} + dD)$$

$$\text{where } w_{cm} = w + \frac{\partial w}{\partial x} \cdot \frac{dx}{2}$$

$$\Rightarrow (\rho \cdot A \cdot dx) \cdot \frac{\partial^2 w}{\partial t^2} = f^e \cdot dx - dD \qquad \triangleleft$$

(iii) Rotational dynamics (Euler):

$$I_{cm} \cdot \frac{\partial^2 \theta_{cm}}{\partial t^2} = -D \cdot \frac{dx}{2} - M \cdot \left\{ \frac{\partial w}{\partial x} \cdot \frac{dx}{2} \right\} +$$

$$-(D + dD) \frac{dx}{2} - (M + dM) \left\{ \frac{\partial w}{\partial x} \cdot \frac{dx}{2} \right\} +$$

$$+ f^e \cdot dx \cdot \{\text{Order } dx\}$$

where: $I_{cm} = \frac{1}{12} \cdot (\rho \cdot A \cdot dx) \cdot h^2$

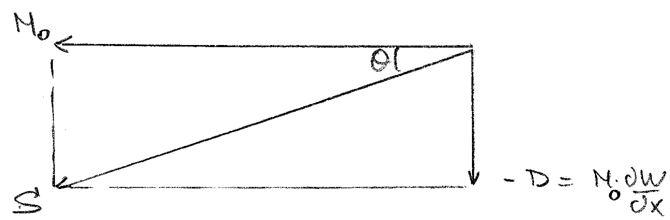
(square wire with height h)

$$\theta_{cm} = \frac{\partial w_{cm}}{\partial x} = \frac{\partial}{\partial x} \left\{ w + \frac{\partial w}{\partial x} \cdot \frac{dx}{2} \right\}$$

Approximation: neglect rotational dynamics (valid for low frequency motions).

Hence: $-D \cdot dx - N_o \cdot \frac{\partial w}{\partial x} dx = 0 \Rightarrow D = N_o \cdot \frac{\partial w}{\partial x}$. $\triangleleft$

Consider the resultant load S:



For small displacements: $|\theta| \ll 1$

$$S = \sqrt{N_o^2 + \left(-N_o \frac{\partial w}{\partial x}\right)^2} = N_o \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2}$$

$$S \cong N_o = \text{constant.} \qquad S = S_o$$

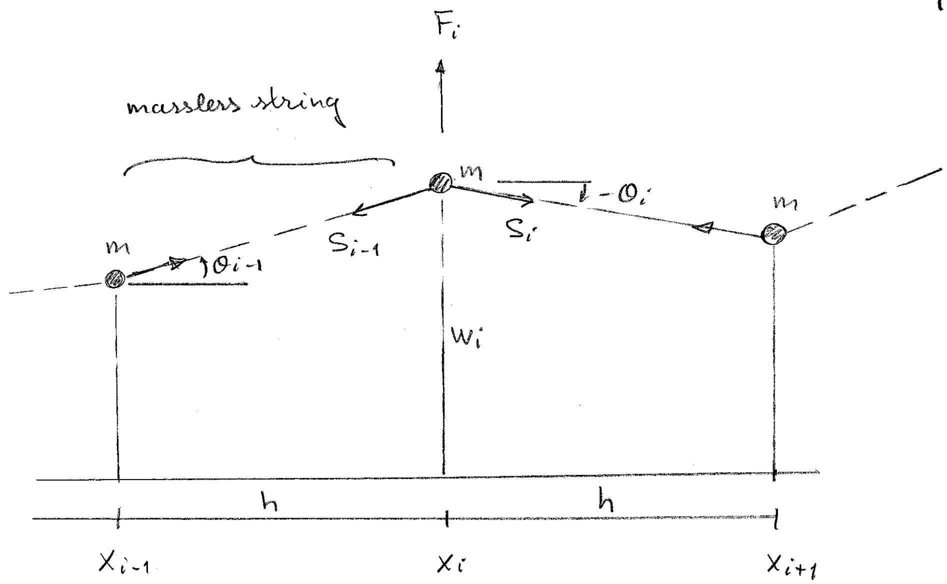
The resultant load is directed along the centerline of the cable.

$$\boxed{\rho \cdot A \cdot dx \cdot \frac{\partial^2 w}{\partial t^2} = S_o \cdot \frac{\partial^2 w}{\partial x^2} \cdot dx + f^e dx}$$

Note: we proved that the internal force S_o is an axial force; see figure above.

Limit derivation

vertical motion only



Horizontal: $-S_{i-1}\cos\theta_{i-1} + S_i\cos\theta_i = 0$

Vertical: $m\cdot\dfrac{d^2w_i}{dt^2} = -S_{i-1}\sin\theta_{i-1} + S_i\sin\theta_i + F_i$

Small displacements $\Rightarrow$ $|\theta_i| \ll 1$

$$-S_{i-1} + S_i = 0 \quad\Rightarrow\quad S_i = \text{constant} \triangleq S_0$$

$$m\cdot\frac{d^2w_i}{dt^2} = -S_0\cdot\theta_{i-1} + S_0\cdot\theta_i + F_i$$

$$= S_0\left\{\frac{w_{i+1}-w_i}{h} - \frac{w_i - w_{i-1}}{h}\right\} + F_i$$

$m = \rho \cdot A \cdot h \qquad F_i = f^e h$

$$\rho \cdot A \cdot h \cdot \frac{d^2 w_i}{dt^2} = (S_0 \cdot h) \cdot \frac{1}{h} \left\{ \frac{w_{i+1} - w_i}{h} - \frac{w_i - w_{i-1}}{h} \right\} + f^e h$$

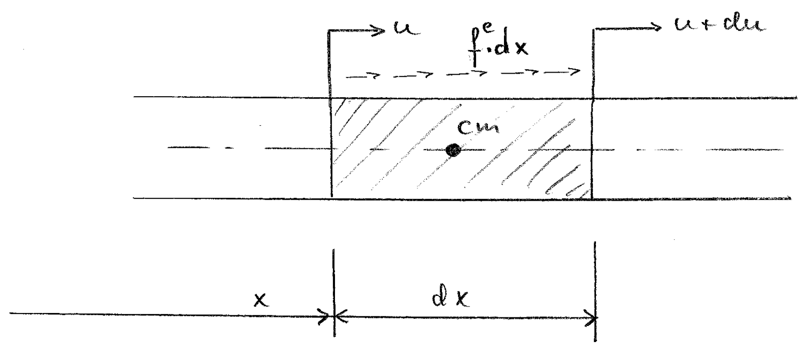
Limit $h = dx \rightarrow 0$

$$\rho A dx \cdot \frac{\partial^2 w}{\partial t^2} = S_0 \cdot \frac{\partial^2 w}{\partial x^2} \cdot dx + f^e dx \quad \triangleleft$$

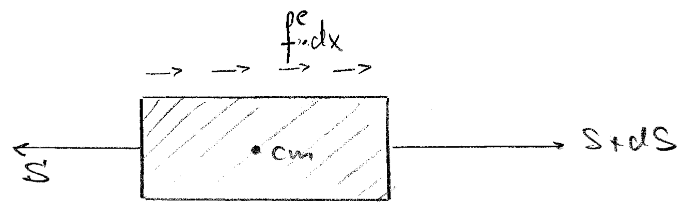
as before

Dynamics of a rod

Direct derivation.



undeformed rod: u = constant ($= 0$)



Newton, horizontal:

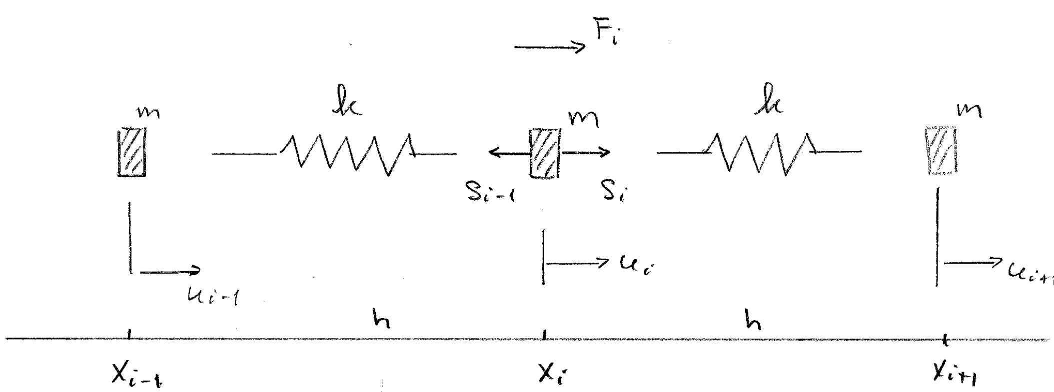
$$(\rho\cdot A\cdot dx)\cdot\frac{\partial^2}{\partial t^2}\left\{\underbrace{u+\frac{\partial u}{\partial x}\cdot\frac{dx}{2}}_{u_{cm}}\right\} = -\cancel{S} + (\cancel{S}+dS) + f^e dx$$

$$S = \sigma\cdot A \quad \text{where } \sigma = E\cdot\varepsilon \text{ and } \varepsilon = \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad S = AE\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \boxed{\rho A\,dx \cdot \frac{\partial^2 u}{\partial t^2} = AE \cdot \frac{\partial^2 u}{\partial x^2}\,dx + f^e \cdot dx}$$

Limit derivation



Newton, horizontal:

$$m \cdot \frac{d^2 u_i}{dt^2} = -S_{i-1} + (S_i + F_i)$$

Spring: $\quad S_i = k\,(u_{i+1} - u_i)$

$$\Rightarrow \quad m \cdot \frac{d^2 u_i}{dt^2} = k \cdot \left\{ (u_{i+1} - u_i) - (u_i - u_{i-1}) \right\} + F_i$$

As before: $\quad m = \rho \cdot A \cdot h \qquad\qquad F_i = p \cdot h$

$$\rho \cdot A \cdot h \cdot \frac{d^2 u_i}{dt^2} = (k \cdot h^2) \cdot \frac{1}{h} \left\{ \frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h} \right\} + f^e \cdot h$$

Let $k \cdot h = E \cdot A$

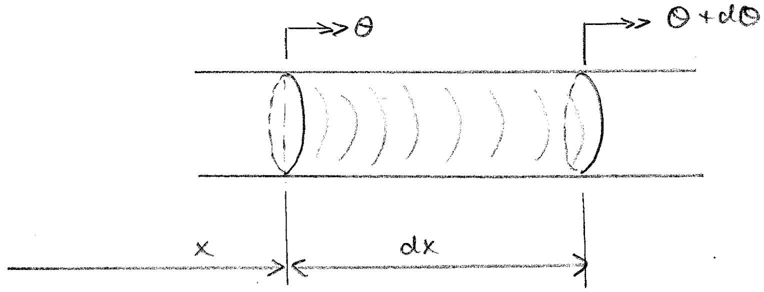
Limit $h = dx \rightarrow 0$:

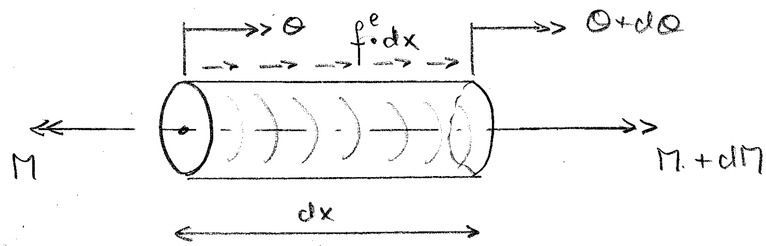
$$\rho A dx \cdot \frac{\partial^2 u}{\partial t^2} = EA \cdot \frac{\partial^2 u}{\partial x^2} \cdot dx + f^e \cdot dx$$

as before.

Dynamics of a shaft.

Direct derivation

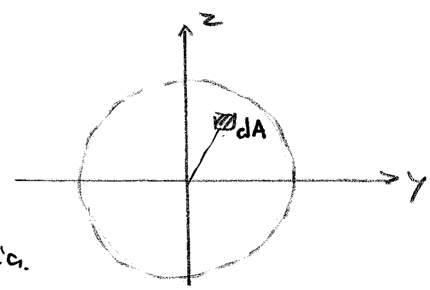




Circular cross-section:

$$I_p = \int (y^2 + z^2)\, dA$$

polar area moment of inertia.



⇒ polar mass moment of inertia:

$$\int (y^2 + z^2)\, dm = \rho \cdot dx \cdot I_p.$$

Angular motion (Euler):

$$(\rho \cdot dx \cdot I_p) \cdot \frac{\partial^2}{\partial t^2} \left\{ \theta + \frac{\partial \theta}{\partial x} \cdot \frac{dx}{2} \right\} = -\cancel{M} + (\cancel{M} + dM) + f^e dx$$

$$\rho \cdot dx \cdot I_p \cdot \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial x} dx + f^e dx$$

Torsion: $d\theta = \frac{M}{I_p \cdot G} \cdot dx$

(Ref. Gere and Timoshenko)

G = "shear modulus"

$I_p \cdot G$ = "torsional rigidity"

"torsional stiffness".

$$\Rightarrow \quad M = I_p \cdot G \cdot \frac{\partial \theta}{\partial x}$$

Substitute:

$$\boxed{\rho \cdot dx \cdot I_p \cdot \frac{\partial^2 \theta}{\partial t^2} = I_p \cdot G \cdot \frac{\partial^2 \theta}{\partial x^2} \cdot dx + f^e dx}$$

Influence of geometry

Recall: $\rho \cdot dx \cdot I_p \cdot \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial x} \cdot dx + f^e dx$

(i) For a circular cross-section:

$$M = I_p \cdot G \cdot \frac{\partial \theta}{\partial x}$$

$$\text{where } I_p = \int r^2 \cdot \underbrace{(r^2 \, d\varphi)}_{dA} = \frac{\pi}{2} \cdot R^4.$$

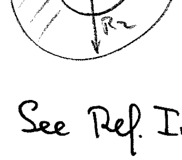
(ii) For a noncircular cross-section:

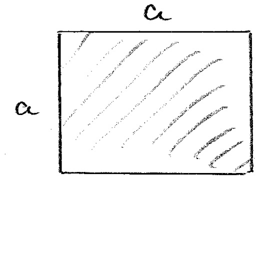
$$M = \gamma \cdot G \cdot \frac{\partial \theta}{\partial x}$$

where the "torsional constant" γ depends on the geometry of the cross-section.

annulus : $\gamma = \frac{\pi}{2} (R_2^4 - R_1^4)$ $(= I_p)$

square : $\gamma = 0.1406 \; a^4$ $(\neq I_p)$

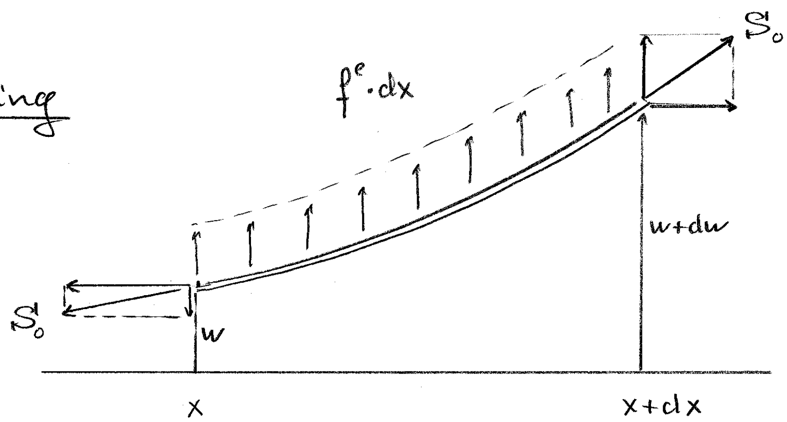




See Ref. Inman.

Executive Summary

String

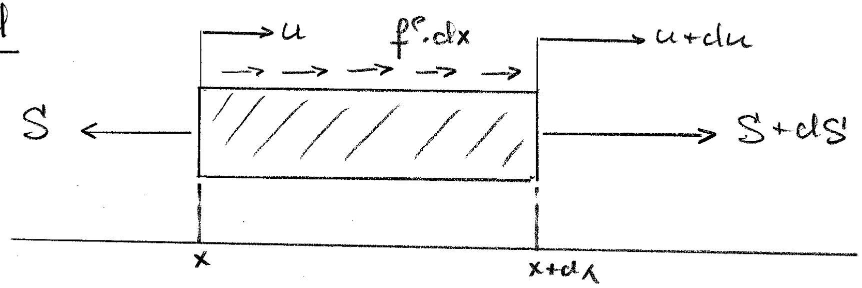


Newton, horizontal: $S = S_0$

Newton, vertical
Euler, static

$$(\rho A\,dx)\cdot\frac{\partial^2 w}{\partial t^2} = S_0\cdot\frac{\partial^2 w}{\partial x^2}\cdot dx + f^e\cdot dx$$

Rod

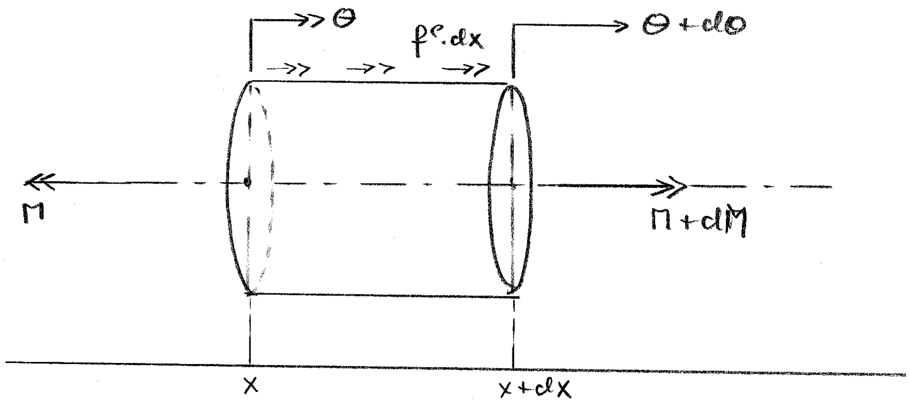


Newton, horizontal
Hooke

$$(\rho\cdot A\,dx)\cdot\frac{\partial^2 u}{\partial t^2} = EA\cdot\frac{\partial^2 u}{\partial x^2}\cdot dx + f^e\cdot dx$$

$$u \Rightarrow \varepsilon \Rightarrow \sigma$$

Shaft (torsion)



Torsion: $M = G.I_p \dfrac{\partial \theta}{\partial x}$

With Euler : $(\rho.I_p.dx).\dfrac{\partial^2 \theta}{\partial t^2} = G.I_p \dfrac{\partial^2 \theta}{\partial x^2} \cdot dx + f^e.dx$

$\theta \Rightarrow \tau,\ M.$

11. DYNAMICS OF CONTINUUM BODIES - B

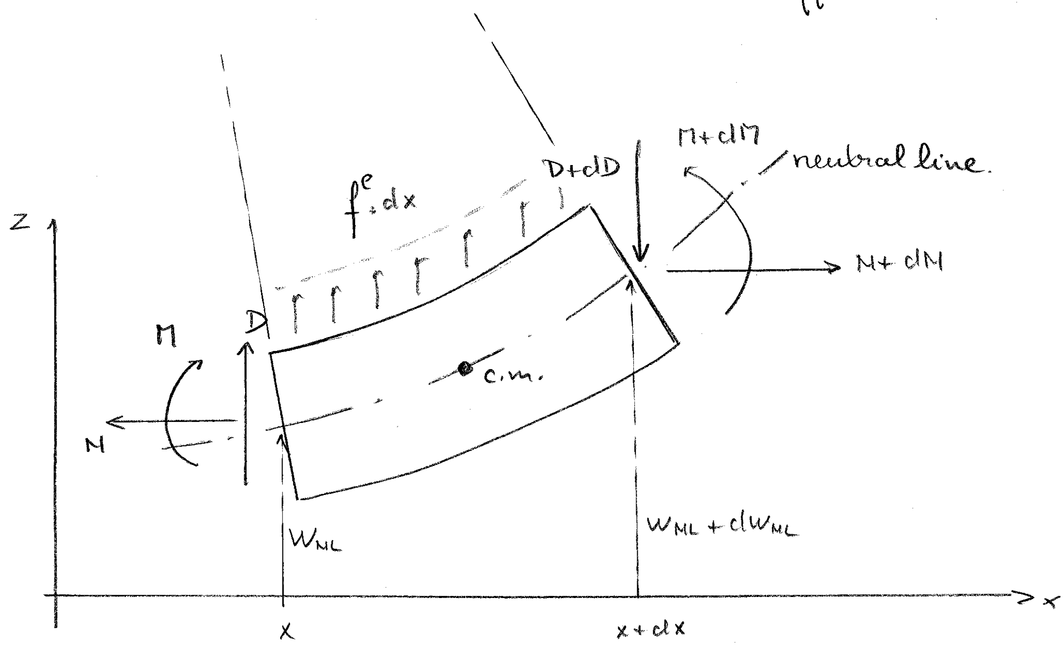
Dynamics of a beam.

Direct derivation.

Simplification: "engineering beam"
or "Euler-Bernoulli beam".

includes these assumptions:
- no shear deformation
- no rotary inertia

("Timoshenko beam" includes these effects)



Neutral line along x-axis when beam is undeformed. ("straight beam").
Displacement of neutral line: $w_{NL}(x,t)$

Consider again small displacements.

(i) Newton, horizontal: $\quad -\cancel{N} + (\cancel{N} + dN) = 0$

as beam theory considers vertical motion only.

Hence: $\quad dN = 0 \quad \Rightarrow \quad N = \text{constant} \mathrel{\hat{=}} N_0 \quad \triangleleft$

(ii) Newton, vertical:

$$\rho A\,dx \cdot \frac{\partial^2}{\partial t^2}\left\{ w_{NL} + \frac{\partial w_{NL}}{\partial x} \cdot \frac{dx}{2} \right\} = \cancel{D} + f^e\,dx - (\cancel{D} + dD)$$

$$\Rightarrow \rho A\,dx \cdot \frac{\partial^2 w_{NL}}{\partial t^2} = -\frac{\partial D}{\partial x} \cdot dx + f^e\,dx \qquad \triangleleft$$

(iii) Euler, with respect to center of mass:

$$I_{cm} \cdot \frac{\partial^2 \Theta}{\partial t^2} = \left[-\cancel{M} - D \cdot \frac{dx}{2} - N \cdot \frac{\partial w}{\partial x} \cdot \frac{dx}{2} \right] +$$

$$+ \left[(\cancel{M} + dM) - (D + dD) \cdot \frac{dx}{2} - (N + \underbrace{dN}_{=0}) \frac{\partial w}{\partial x} \cdot \frac{dx}{2} \right] +$$

$$+ (f^e \cdot dx) \cdot \left\{ \text{Order } dx \right\}$$

Note: $I_{cm} = \frac{1}{12} \cdot (\rho \cdot A \cdot dx) \, h^3$

where h is the height of the beam (assuming a beam with rectangular cross-section).

Approximation: neglect effect of rotational dynamics.

$$\Rightarrow 0 = -D \cdot dx - N_0 \frac{\partial w}{\partial x} \cdot dx + dM = 0$$

$$\Rightarrow D = \frac{\partial M}{\partial x} - N_0 \frac{\partial w}{\partial x} \qquad \triangleleft$$

(iv) Introduce the beam curvature equation (also due to Euler):

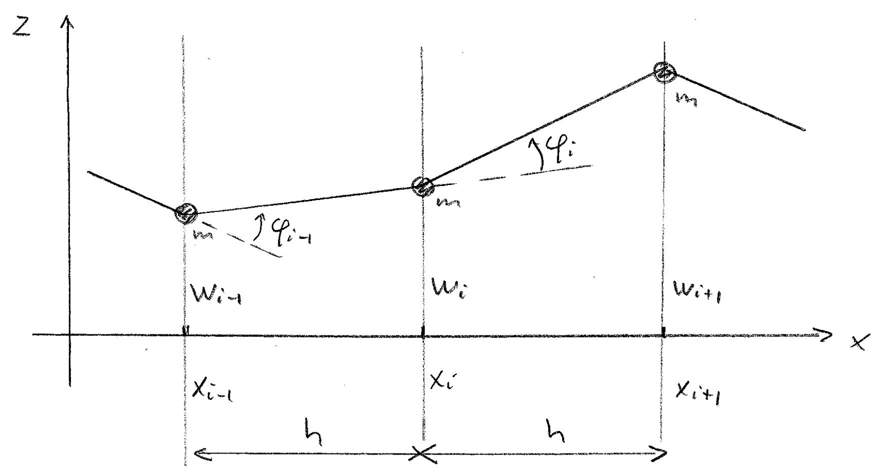
$$M = EI \cdot \frac{\partial^2 w}{\partial x^2} \qquad \triangleleft$$

Substitution of (iii) and (iv) into (ii) finally gives:

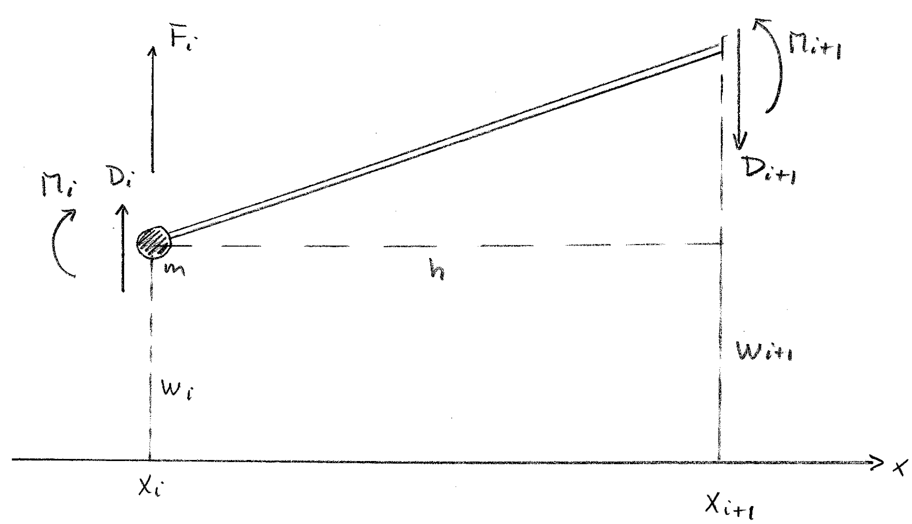
$$\boxed{(\rho \cdot A \cdot dx) \cdot \frac{\partial^2 w}{\partial t^2} = -EI \cdot \frac{\partial^4 w}{\partial x^4} \cdot dx + N_0 \frac{\partial^2 w}{\partial x^2} \cdot dx + f^e \cdot dx}$$

Limit derivation

Consider the following discrete model:



massless beams; point masses; torsion springs.



Torsional spring: $M_i = k \cdot \varphi_i$ $\triangleleft$

where:

$$\varphi_i = \frac{w_{i+1} - w_i}{h} - \frac{w_i - w_{i-1}}{h}$$

Newton, vertical:

$$m \frac{d^2 w_i}{dt^2} = D_i + F_i - D_{i+1} \qquad \triangleleft$$

Euler:

$$-M_i + \{M_{i+1} - D_{i+1} \cdot h\} = 0 \qquad \triangleleft$$

Hence: $D_{i+1} = \dfrac{M_{i+1} - M_i}{h}$

where $M_i = k \cdot \left\{ \dfrac{w_{i+1} - w_i}{h} - \dfrac{w_i - w_{i-1}}{h} \right\}$

Substitute:

$$m \frac{d^2 w_i}{dt^2} = \left\{ \frac{M_i - M_{i-1}}{h} \right\} + F_i - \left\{ \frac{M_{i+1} - M_i}{h} \right\}$$

Define: $\Delta w_i \triangleq w_i - w_{i-1}$

$$\Rightarrow M_i = k \left\{ \frac{\Delta w_{i+1}}{h} - \frac{\Delta w_i}{h} \right\}$$

$$m \cdot \frac{d^2 w_i}{dt^2} + \frac{k}{h}\left[\left\{\frac{\Delta w_{i+2}}{h} - \frac{\Delta w_{i+1}}{h}\right\} - \left\{\frac{\Delta w_{i+1}}{h} - \frac{\Delta w_i}{h}\right\}\right] +$$

$$- \frac{k}{h}\left[\left\{\frac{\Delta w_{i+1}}{h} - \frac{\Delta w_i}{h}\right\} - \left\{\frac{\Delta w_i}{h} - \frac{\Delta w_{i-1}}{h}\right\}\right] = F_i$$

Let: $\begin{cases} m = \rho \cdot A \cdot h \\ k \cdot h = EI \end{cases}$ $\quad F_i = f^e \cdot h$

$$\Rightarrow (\rho \cdot A \cdot h)\frac{d^2 w_i}{dt^2} + EI \cdot h \frac{1}{h}\left[\frac{1}{h}\left(\frac{\frac{\Delta w_{i+2}}{h} - \frac{\Delta w_{i+1}}{h}}{h} - \frac{\frac{\Delta w_{i+1}}{h} - \frac{\Delta w_i}{h}}{h}\right) + \right.$$

$$\left. - \frac{1}{h}\left(\frac{\frac{\Delta w_{i+1}}{h} - \frac{\Delta w_i}{h}}{h} - \frac{\frac{\Delta w_i}{h} - \frac{\Delta w_{i-1}}{h}}{h}\right)\right] =$$

$$= f^e \cdot h$$

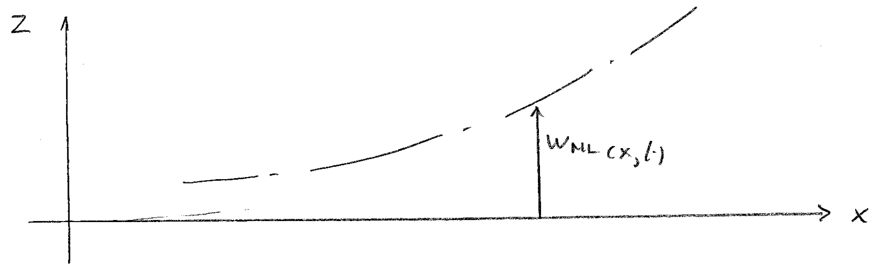
Limit $h = dx \rightarrow 0$:

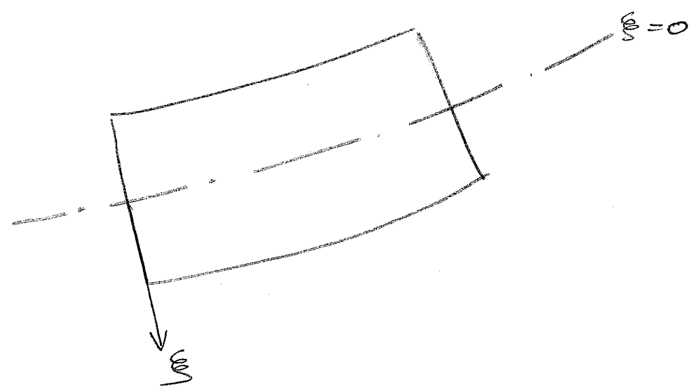
$$(\rho A \cdot dx) \cdot \frac{\partial^2 w}{\partial t^2} + EI \cdot \frac{\partial^4 w}{\partial x^4} \cdot dx = f^e \cdot dx$$

as before.

Exercise: repeat analysis for the case of an additional axial load N_o.

Beam design parameters





Neutral line: $\sigma = 0$

measure distance ξ relative to neutral line.

Curvature: $$\frac{1}{R} = \frac{\partial^2 w_{NL}}{\partial x^2} \qquad \triangleleft$$

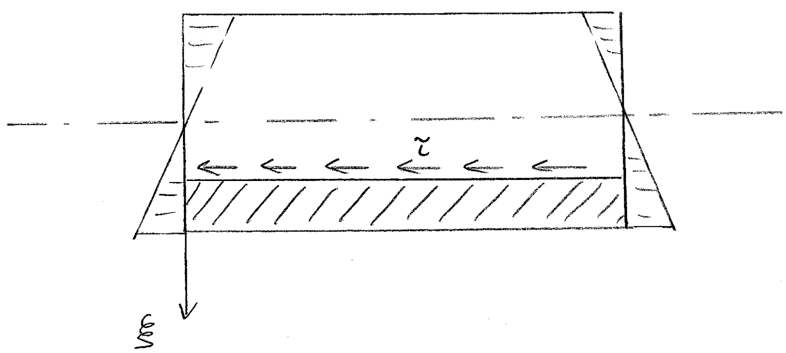
$\Rightarrow$ strain: $$\varepsilon(x,\xi,t) = \xi / R \qquad \triangleleft$$

stress: $$\sigma(x,\xi,t) = E \cdot \xi / R \qquad \triangleleft$$

bending moment: $\quad M = EI/R \quad \triangleleft$

transverse force: $\quad D = \dfrac{\partial M}{\partial x} - N_0 \dfrac{\partial w}{\partial x} \quad \triangleleft$

longitudinal shear force and shear stress:



$$-\int_{\xi}^{\xi_{max}} \sigma \cdot dA \quad - \tau \cdot dx \cdot b \quad + \int_{\xi}^{\xi_{max}} (\sigma + d\sigma)\, dA = 0$$

where: $\quad b$ = width of beam at ξ

$$dA = dy \cdot dz$$

$$\Rightarrow \tau = \frac{1}{b} \int_{\xi}^{\xi_{max}} \frac{\partial \sigma}{\partial x}\, dA$$

$$\left.\begin{aligned} \sigma &= E \cdot \xi / R \\ M &= EI / R \end{aligned}\right\} \Rightarrow \quad \sigma = \frac{M}{I}\xi$$

$$\frac{\partial \sigma}{\partial x} = \frac{\xi}{I}\frac{\partial M}{\partial x} = \frac{\xi D}{I}$$

$$\Rightarrow \quad \tau = \frac{D}{b \cdot I}\int_{\xi}^{\xi_{max}} \xi \cdot dA \qquad \Delta$$

"shear formula".

e.g. rectangular beam with height h:

τ_{max} occurs at $\xi = 0 \quad \Rightarrow \tau_{max} = \frac{3}{2} \cdot \frac{D}{A}$

solid circular beam with radius R:

τ_{max} occurs at $\xi = 0 \quad \Rightarrow \tau_{max} = \frac{4}{3}\frac{D}{A}$.

cf. Ref. Gere and Timoshenko, chapters 4, 5 and 9.

Vanishing bending stiffness.

We derived:

$$\rho A dx \cdot \frac{\partial^2 w_{NL}}{\partial t^2} = -EI \cdot \frac{\partial^4 w_{NL}}{\partial x^4} dx + N_0 \frac{\partial^2 w_{NL}}{\partial x^2} dx + f^e \cdot dx$$

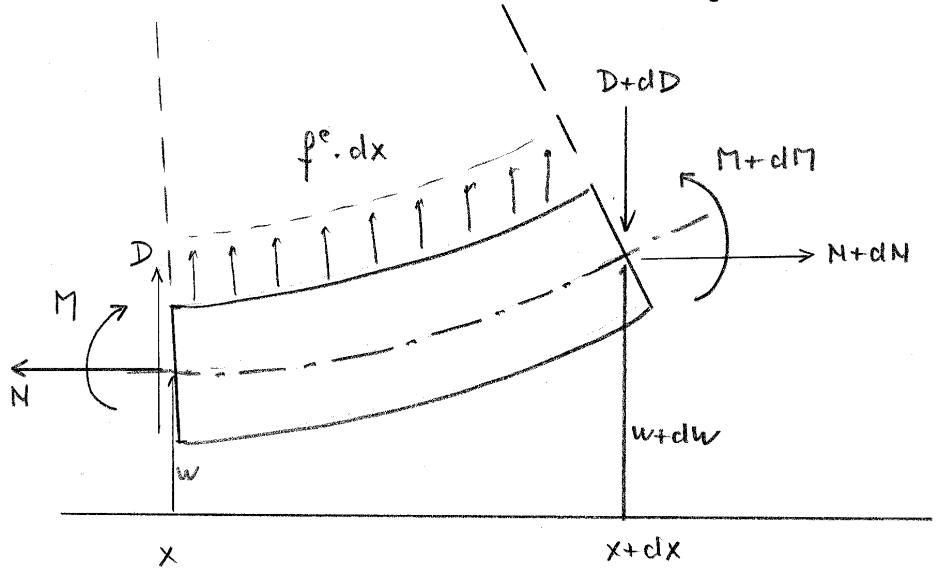
EI = "bending stiffness".

Take limit $EI \to 0$

Hence: $$\rho A dx \cdot \frac{\partial^2 w_{NL}}{\partial t^2} = N_0 \frac{\partial^2 w_{NL}}{\partial x^2} \cdot dx + f^e dx$$

Compare with the dynamics equation for a string!

Executive Summary



neutral line NL

fiber at distance ξ from neutral line

Newton, vertical
Euler, rotational (ignore angular acceleration)
Euler, beam geometry

$$\Rightarrow (\rho \cdot A \cdot dx) \cdot \frac{\partial^2 w}{\partial t^2} = -EI \cdot \frac{\partial^4 w}{\partial x^4} \cdot dx + N_0 \cdot \frac{\partial^2 w}{\partial x^2} dx + f^e \cdot dx$$

$$w \begin{cases} \varepsilon \rightarrow \sigma \\ D, M, \tau. \end{cases}$$

(p.m. Newton, horizontal: N = constant = N_0).

12. SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION

Problem statement

String: $\rho A dx \cdot \frac{\partial^2 w}{\partial t^2} = S \cdot \frac{\partial^2 w}{\partial x^2} dx + f^e dx$

Rod : $\rho A dx. \frac{\partial^2 u}{\partial t^2} = EA . \frac{\partial^2 u}{\partial x^2} dx + f^e dx$

Shaft : $\rho . I_p . dx. \frac{\partial^2 \theta}{\partial t^2} = \gamma . I_p . \frac{\partial^2 \theta}{\partial x^2} dx + f^e . dx$

These are special cases of the canonical equation:

$$\boxed{\rho A dx . \frac{\partial^2 u}{\partial t^2} = H . \frac{\partial^2 u}{\partial x^2} . dx + f^e dx}$$

where H is a positive constant.

State boundary conditions!

State initial conditions $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$. !

$\Rightarrow$ solve for $u(x,t)$.

$\boxed{\text{Free motion } (p \equiv 0)}$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}} \quad \text{where } c \triangleq \sqrt{\frac{H}{\rho A}}$$

(a velocity)

Separation of variables:

seek a solution in the form $\boxed{u(x,t) = U(x)\, q(t)}$

Substitute: $U\ddot{q} = c^2 U'' q$

where $(\ )^{\cdot\cdot} \triangleq \frac{\partial^2}{\partial t^2}$ and $(\ )'' \triangleq \frac{\partial^2}{\partial x^2}$

$$\frac{\ddot{q}}{q} = c^2 \cdot \frac{U''}{U} \doteq \text{constant} \triangleq \lambda.$$

<u>Prove:</u> $\lambda \le 0$.

Proof.

$$\begin{cases} \int_0^L U'' \cdot U\, dx = U' \cdot U \Big|_0^L - \int_0^L U'^2\, dx \\ \int_0^L U'' U\, dx = \int_0^L \left(\frac{\lambda}{c^2} U\right) U\, dx \end{cases}$$

Hence: $\frac{\lambda}{c^2} \int_0^L U^2 \cdot dx = \left[U'_{(L)} \cdot U_{(L)} - U'_{(0)} U_{(0)} \right] +$

$$- \int_0^L U'^2 \cdot dx$$

Hence : if $U'U = 0$ for $x = 0$ and for $x = L$

then $\lambda \leq 0$.

Define: $\lambda = -\omega^2$

Then:

(i) $\ddot{q} + \omega^2 q = 0$

$$\Rightarrow q = A \cdot \sin\omega t + B \cdot \cos\omega t \quad \triangleleft$$

(ii) $U'' + \left(\frac{\omega}{c}\right)^2 U = 0$

$$\Rightarrow \bar{U} = a \cdot \sin\left(\frac{\omega}{c} x\right) + b \cdot \cos\left(\frac{\omega}{c} x\right) \quad \triangleleft$$

(special case: $\omega = 0$; rigid body motion).

Solution: $u(x,t) = \left\{ a \cdot \sin\left(\frac{\omega}{c} x\right) + b \cdot \cos\left(\frac{\omega}{c} x\right) \right\} \cdot$

$$\cdot \left\{ A \cdot \sin\omega t + B \cos\omega t \right\} \quad \triangleleft$$

Apply boundary conditions

$\Rightarrow \omega_j \quad (j = 1, 2, 3, \ldots)$

eigenfrequencies

$$q_j = A_j \cdot \sin \omega_j t + B_j \cdot \cos \omega_j t$$

$$\Rightarrow \quad U_j = a_j \cdot \sin\left(\frac{\omega_j}{c} x\right) + b_j \cdot \cos\left(\frac{\omega_j}{c} x\right)$$

General solution:

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} U_j(x) \cdot q_j(t)}$$

ω_j = "eigenfrequency"

U_j = "mode shape" associated with ω_j

q_j = generalized (or modal) coordinate.

Ordering: $0 \leq \omega_1 < \omega_2 < \omega_3 < \omega_4 \; ----$

Properties of mode shapes

$$\left\{\begin{array}{l} \displaystyle\int_0^L U_i'' U_j \, dx = U_i' U_j \Big|_0^L - \int_0^L U_i' U_j' \, dx \\ \displaystyle\int_0^L U_j'' \cdot U_i \, dx = U_j' U_i \Big|_0^L - \int_0^L U_j' U_i' \, dx \end{array}\right.$$

Subtract, and substitute $U_i'' = -\left(\frac{\omega_i}{c}\right)^2 U_i$

$$\Rightarrow \left\{ \left(\frac{\omega_i}{c}\right)^2 - \left(\frac{\omega_j}{c}\right)^2 \right\} \int_0^L U_i \cdot U_j \cdot dx = \left[U_i' U_j - U_j' U_i \right]_0^L$$

Hence: <u>if</u> $\left[U_i' U_j - U_j' U_i \right]_0^L = 0$

<u>then</u> $\left\{ \left(\frac{\omega_i}{c}\right)^2 - \left(\frac{\omega_j}{c}\right)^2 \right\} \int_0^L U_i \cdot U_j \, dx = 0$

Consider $i \neq j$.

if $\omega_i \neq \omega_j \quad \Rightarrow \quad \boxed{\int_0^L U_i \cdot U_j \cdot dx = 0}$

"orthogonality" of mode shapes.

Forced motion

Consider the general case:

$$\rho A \cdot dx \cdot \frac{\partial^2 u}{\partial t^2} = H \cdot \frac{\partial^2 u}{\partial x^2} \cdot dx + f^e dx$$

where $f^e = f^e(x, t)$.

Consider a solution in the form of the "modal expansion"

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} U_j(x)\, Q_j(t)}$$

where $U_j(x)$ is the mode shape associated with free motion

and $Q_j(t)$ is a generalized coordinate yet to be determined.

Substitute:

$$(\rho A \cdot dx)\left(\sum_j U_j \ddot{Q}_j\right) = H\left(\sum_j U_j'' Q_j\right) dx + f^e \cdot dx$$

$$= -H \cdot \left\{\sum_j \left(\frac{\omega_j}{c}\right)^2 U_j Q_j\right\} dx + f^e dx$$

Pre-multiply both sides by U_i and integrate:

$$\rho A \cdot \int_0^L U_i \left\{ \sum_j U_j \ddot{Q}_j \right\} dx =$$

$$= -H \int_0^L U_i \left\{ \sum_j \left(\frac{\omega_j}{c}\right)^2 U_j Q_j \right\} dx + \int_0^L U_i \left\{ f^e dx \right\}$$

Now: $\int_0^L U_i \left\{ \sum_j U_j \ddot{Q}_j \right\} dx = \int_0^L U_i U_1 dx \, \ddot{Q}_1 +$

$$+ \int_0^L U_i U_2 dx \cdot \ddot{Q}_2 + \cdots + \int_0^L U_i \cdot U_i \cdot dx \cdot \ddot{Q}_i + \cdots$$

$\stackrel{D}{=} \int_0^L U_i^2 dx \; \ddot{Q}_i$ (exploiting orthogonality property)

$$\Rightarrow \rho A \cdot \int_0^L U_i^2 dx \; \ddot{Q}_i = -H \left(\frac{\omega_i}{c}\right)^2 \int_0^L U_i^2 dx \, Q_i + \int_0^L U_i f^e dx$$

but: $c^2 = \dfrac{H}{\rho A}$

$$\Rightarrow \boxed{\ddot{Q}_i + \omega_i^2 \cdot Q_i = \frac{1}{\rho A \cdot \int_0^L U_i^2 \cdot dx} \cdot \int_0^L U_i f^e \cdot dx}$$

The equations for $Q_i(t)$ $(i = 1, 2, 3, \ldots \infty)$ are decoupled.

They are of the type:

$$\boxed{\ddot{Q}_i + \omega_i^2 Q_i = f_i(t)}$$

General solution = complementary solution

+ particular solution

$$Q_i(t) = q_i^{compl.}(t) + q_i^{part.}(t)$$

Note that the complementary solution has already been established (free motion solution)

Finally:

$$\boxed{u(x,t) = \sum_{j=1}^{\infty} U_j(x) \left\{ q_j^{compl.}(t) + q_j^{partic.}(t) \right\}}$$

Initial conditions

Given:
$$\boxed{\begin{aligned} u(x,0) &= F(x) \\ \frac{\partial u}{\partial t}(x,0) &= G(x) \end{aligned}}$$

$\Rightarrow$ determine integration constants.

From application of the boundary conditions and subsequent normalisation of $U_j(x)$, the coefficients a_j and b_j are determined.

It remains to determine A_j and B_j.

Consider:

$$u(x,0) = \sum_{j=1}^{\infty} U_j(x) \cdot \left\{ B_j + q_{j(0)}^{part} \right\} = F(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{j=1}^{\infty} U_j(x) \left\{ A_j \cdot \omega_j + \dot{q}_{j(0)}^{part} \right\} = G(x)$$

Pre-multiply by $U_i(x)$ and integrate:

$$\int_0^L U_i(x) \left[\; - - - \; \right] dx$$

$$\Rightarrow \quad \int_0^L U_i^2 \, dx \cdot \left\{ B_i + q_{i(0)}^{part} \right\} = \int_0^L U_i \cdot F \cdot dx \quad \Rightarrow B_i \qquad \triangleleft$$

and
$$\int_0^L U_i^2 \cdot dx \cdot \left\{ A_i \omega_i + \dot{q}_{i(0)}^{part} \right\} = \int_0^L U_i \cdot G \cdot dx \Rightarrow A_i \omega_i \Rightarrow A_i \triangleleft$$

Executive Summary

Canonical form:

$$(\rho \cdot A \cdot dx) \cdot \frac{\partial^2 u}{\partial t^2} = H \cdot \frac{\partial^2 u}{\partial x^2} \cdot dx + f^e \cdot dx$$

- boundary conditions
- initial conditions.

<u>Homogeneous equation</u> (free motion)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where} \quad c \triangleq \sqrt{\frac{H}{\rho A}}$$

Separation of variables: $u(x,t) = U(x)\, q(t)$

$$\ddot{q} + \omega^2 q = 0$$

$$U'' + \left(\frac{\omega}{c}\right)^2 U = 0$$

(special case: $\omega = 0$; rigid body motion)

Apply boundary conditions $\Rightarrow \omega_i \quad (i = 1, 2, \ldots, \infty)$

$$\ddot{q}_i + \omega_i^2 q_i = 0 \quad \Rightarrow q_i$$

$$U_i'' + \left(\frac{\omega_i}{c}\right)^2 U_i = 0 \quad \Rightarrow U_i$$

Then: $u(x,t) = \sum_{i=1}^{\infty} U_i(x)\, q_i(t)$.

Orthogonality property: $\int_0^L U_i \cdot U_j \cdot dx = 0$ for $i \neq j$.

Non-homogeneous equation (forced motion)

$$(\rho \cdot A \cdot dx) \cdot \frac{\partial^2 u}{\partial t^2} = H \cdot \frac{\partial^2 u}{\partial x^2} \cdot dx + f^e \cdot dx$$

Modal expansion: $u(x,t) = \sum_{i=1}^{\infty} U_i(x)\, Q_i(t) \Rightarrow Q_i$?

$$\Rightarrow \quad \ddot{Q}_i + \omega_i^2 Q_i = \frac{1}{\int_0^L U_i^2 dm} \cdot \int_0^L U_i \cdot f^e \cdot dx$$

$$\hat{=} \ q_i(t).$$

$$\Rightarrow \quad Q_i(t) = q_i(t) + Q_i^{\text{particular}}(t).$$

$$\Rightarrow \quad u(x,t) = \sum_{i=1}^{\infty} U_i(x) \left\{ q_i(t) + Q_i^{\text{particular}}(t) \right\}.$$

Initial conditions:

$$u(x,0) = F(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = G(x)$$

$$\int_0^L U_i \cdot F \cdot dx = \left(\int_0^L U_i^2 \cdot dx\right) \cdot \left\{ q_i(0) + Q_i^{\text{partic.}}(0) \right\}$$

$$\int_0^L U_i \cdot G \cdot dx = \left(\int_0^L U_i^2 \cdot dx\right) \cdot \left\{ \dot{q}_i(0) + \dot{Q}_i^{\text{partic}}(0) \right\}$$

$\Rightarrow$ solve for integration constants in q_i.

13. FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATION

$$\boxed{\text{Problem statement.}}$$

Beam: $$\boxed{\rho \cdot A \cdot dx \cdot \frac{\partial^2 w}{\partial t^2} = -EI \frac{\partial^4 w}{\partial x^4} \cdot dx + f^e \, dx}$$

for the case $N_o = 0$.

State boundary conditions

State initial conditions. $w(x,0)$ and $\frac{\partial w}{\partial t}(x,0)$

$\Rightarrow$ solve for $w(x,t)$

$$\boxed{\text{Free motion } (p \equiv 0)}$$

$$\rho \cdot A \cdot dx \cdot \frac{\partial^2 w}{\partial t^2} = - E \cdot I \cdot \frac{\partial^4 w}{\partial x^4} \cdot dx$$

$$\Rightarrow \boxed{\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A} \cdot \frac{\partial^4 w}{\partial x^4} = 0}$$

Separation of variables:

seek a solution in the form $\boxed{w(x,t) = W(x) \cdot q(t)}$

Substitute: $W \cdot \ddot{q} + \frac{EI}{\rho A} \cdot \overset{(4)}{W} q = 0$

where $(\ )^{\cdot\cdot} \triangleq \frac{\partial^2}{\partial t^2}$ and $\overset{(4)}{(\ )} \triangleq \frac{\partial^4}{\partial x^4}$

$$\Rightarrow \frac{\ddot{q}}{q} = - \frac{EI}{\rho A} \cdot \frac{\overset{(4)}{W}}{W} \doteq \text{constant} \triangleq \lambda$$

Prove: $\lambda \leq 0$

Proof.

$$\left\{\begin{aligned} \int_0^L \overset{(4)}{W} \cdot W \cdot dx &= \overset{(3)}{W} W \Big|_0^L - \int_0^L \overset{(3)}{W} W' \, dx \\ &= \left[\overset{(3)}{W} W - \overset{(2)}{W} W'\right]_0^L + \int_0^L (W'')^2 \, dx \\ \int_0^L \overset{(4)}{W} W \cdot dx &= \int_0^L \left(-\lambda \cdot \frac{\rho A}{EI} \cdot W\right) W \cdot dx \end{aligned}\right.$$

$$\Rightarrow -\lambda \cdot \left(\frac{\rho A}{EI}\right) \int_0^L W^2 \cdot dx = \left[\overset{(3)}{W} W - \overset{(2)}{W} W'\right]_0^L + \int_0^L (W'')^2 \cdot dx$$

Hence: if $\left[\overset{(3)}{W} W - \overset{(2)}{W} W'\right]_0^L = 0$

then $\lambda \leq 0$

Define: $\lambda = -\omega^2$.

Then:

(i) $\ddot{q} + \omega^2 q = 0$

$$\Rightarrow q(t) = A \cdot \sin(\omega t) + B \cdot \cos(\omega t) \qquad \triangleleft$$

(ii) $\overset{(4)}{W} = \mu^4 \cdot W \qquad$ where $\mu \triangleq \left(\frac{\rho A}{EI} \cdot \omega^2 \right)^{1/4}$

$$\Rightarrow W(x) = a \cdot \sin(\mu x) + b \cdot \cos(\mu x) + $$
$$+ c \cdot \sinh(\mu x) + d \cdot \cosh(\mu x) \qquad \triangleleft$$

(Special case: $\mu = 0 \Rightarrow \omega = 0$; rigid body motion.)

Solution:

$$w(x,t) = \{ a \cdot \sin(\mu x) + b \cdot \cos(\mu x) + $$
$$+ c \cdot \sinh(\mu x) + d \cdot \cosh(\mu x) \} \cdot$$
$$\cdot \{ A \cdot \sin(\omega t) + B \cdot \cos(\omega t) \} \qquad \triangleleft.$$

Apply boundary conditions

$$\Rightarrow \mu_j \Rightarrow \omega_j$$

Then: $q_j(t) = A_j \cdot \sin(\omega_j t) + B_j \cdot \cos(\omega_j t)$

$$W_j(x) = a_j \cdot \sin(\mu_j x) + b_j \cdot \cos(\mu_j x) +$$
$$+ c_j \cdot \sinh(\mu_j x) + d_j \cdot \cosh(\mu_j x)$$

General solution:

$$\boxed{w(x,t) = \sum_{j=1}^{\infty} W_j(x) \cdot q_j(t)}$$

ω_j = "eigen frequency"

W_j = "mode shape" associated with ω_j

q_j = generalised (or modal) coordinate.

Ordering: $\omega_1 < \omega_2 < \omega_3 < \omega_4 \cdots$

Properties of mode shapes

$$\int_0^L \overset{(4)}{W_i}\, W_j \cdot dx = \overset{(3)}{W_i}\, W_j \Big|_0^L - \int_0^L \overset{(3)}{W_i}\, W_j'\, dx$$

$$= \left[\overset{(3)}{W_i}\, W_j - \overset{(2)}{W_i}\, W_j'\right]_0^L + \int_0^L \overset{(2)}{W_i}\, \overset{(2)}{W_j}\, dx$$

Similarly:

$$\int_0^L \overset{(4)}{W_j}\, W_i\, dx = \left[\overset{(3)}{W_j}\, W_i - \overset{(2)}{W_j}\, W_i'\right]_0^L + \int_0^L \overset{(2)}{W_i}\, \overset{(2)}{W_j}\, dx$$

Subtract, and substitute $\overset{(4)}{W_i} = +\mu_i^4\, W_i$

$$(\mu_i^4 - \mu_j^4) \int_0^L W_i \cdot W_j\, dx = \left[\left(\overset{(3)}{W_i}\, W_j - \overset{(2)}{W_i}\, W_j'\right) - \left(\overset{(3)}{W_j}\, W_i - \overset{(2)}{W_j}\, W_i'\right)\right]_0^L$$

Hence, <u>if</u> $\left[\overset{(3)}{W_i}\, W_j - \overset{(2)}{W_i}\, W_j' + W_i'\, \overset{(2)}{W_j} - W_i\, \overset{(3)}{W_j}\right]_0^L = 0.$

<u>then</u> $(\mu_i^4 - \mu_j^4) \int_0^L W_i \cdot W_j \cdot dx = 0$

Consider $i \neq j$

if $\mu_i \neq \mu_j$ (and hence $\omega_i \neq \omega_j$):

"orthogonality of mode shapes". $\Rightarrow \boxed{\int_0^L W_i \cdot W_j \cdot dx = 0}$

Forced motion

Consider the general case:

$$\rho \cdot A \cdot dx \cdot \frac{\partial^2 w}{\partial t^2} = -EI \cdot \frac{\partial^4 w}{\partial x^4} \cdot dx + f^e \, dx$$

where $f^e = f^e(x,t)$.

Consider a solution in the form of the "modal expansion":

$$\boxed{w(x,t) = \sum_{j=1}^{\infty} W_j(x) \cdot Q_j(t)}$$

where $W_j(x)$ is the mode shape associated with free motion

and $Q_j(t)$ is a generalised coordinate, yet to be determined.

Substitute:

$$(\rho \cdot A \cdot dx)\left(\sum_j W_j \ddot{Q}_j\right) = -EI\left(\sum_j W_j^{(4)} Q_j\right) dx + f^e \, dx$$

$$= -EI \cdot \left(\sum_j \mu_j^4 \cdot W_j \, Q_j\right) dx + f^e \, dx$$

$$= -\rho A \left(\sum_j \omega_j^2 \cdot W_j \, Q_j\right) dx + f^e \, dx$$

Multiply by $W_{i(x)}$ and integrate (add):

$$\rho A \int_0^L W_i \left(\sum_j W_j \ddot{Q}_j\right) dx =$$

$$= -\rho A \int_0^L W_i \left(\sum_j \omega_j^2 W_j Q_j\right) dx + \int_0^L W_i \cdot f^e \, dx$$

Now: $\int_0^L W_i \left(\sum_j W_j \ddot{Q}_j\right) dx = \int_0^L W_i \, W_1 \, dx \cdot \ddot{Q}_1 +$

$$+ \int_0^L W_i \cdot W_2 \, dx \cdot \ddot{Q}_2 + \cdots + \int_0^L W_i \cdot W_i \cdot dx \cdot \ddot{Q}_i + \cdots$$

$$\overset{D}{\underset{O}{=}} \int_0^L W_i^2 dx \cdot \ddot{Q}_i$$ (we exploited orthogonality property)

Similar with the second integral.

$$\Rightarrow \rho A \int_0^L W_i^2 dx \cdot \ddot{Q}_i = -\rho A \, \omega_i^2 \int_0^L W_i^2 dx \cdot Q_i + \int_0^L W_i \cdot f^e \cdot dx$$

$$\Rightarrow \boxed{\ddot{Q}_i + \omega_i^2 Q_i = \frac{1}{\rho A \cdot \int_0^L W_i^2 \cdot dx} \cdot \int_0^L W_i \cdot f^e \, dx}$$

The equations for $\bar{Q}_i(t)$ $(i = 1, 2, 3, \ldots)$ are decoupled.

They are of the type:

$$\boxed{\ddot{Q}_i + \omega_i^2 \cdot Q_i = f_i(t)}$$

General solution = complementary solution

+ particular solution.

$$Q_i(t) = q_{i}^{compl.}(t) + q_{i}^{part.}(t)$$

Note that the complementary solution has already been established (free motion solution).

Finally:

$$\boxed{w(x,t) = \sum_{j=1}^{\infty} W_j(x) \cdot \left\{ q_{j}^{compl.}(t) + q_{j}^{part.}(t) \right\}}$$

Initial conditions

Given:

$$\boxed{\begin{aligned} w(x,0) &= F(x) \\ \frac{\partial w}{\partial t}(x,0) &= G(x) \end{aligned}}$$

$\Rightarrow$ determine integration constants.

From application of the boundary conditions and subsequent normalisation of $W_j(x)$, the coefficients a_j, b_j, c_j and d_j are determined.

It remains to determine A_j and B_j.

Consider:

$$w(x,0) = \sum_{j=1}^{\infty} W_j(x) \left\{ B_j + q_{j(0)}^{part.} \right\} = F(x)$$

$$\frac{\partial w}{\partial t}(x,0) = \sum_{j=1}^{\infty} W_j(x) \left\{ A_j \cdot \omega_j + \dot{q}_{j(0)}^{part} \right\} = G(x)$$

Multiply by $W_i(x)$ and integrate:

$$\int_0^L W_i [\; \cdots \;] \, dx$$

$$\Rightarrow \int_0^L W_i^2 \cdot dx \cdot \left\{ B_i + q_{i(0)}^{part} \right\} = \int_0^L W_i \cdot F \cdot dx \Rightarrow B_i$$

$$\text{and } \int_0^L W_i^2 \cdot dx \cdot \left\{ A_i \omega_i + \dot{q}_{i(0)}^{part} \right\} = \int_0^L W_i \cdot G \cdot dx$$

$$\Rightarrow A_i \omega_i \Rightarrow A_i$$

More properties of mode shapes

The modeshapes are determined by the fourth-order differential equation

$$\frac{d^4 W_j}{dx^4} = \mu_j^4 W_j$$

together with the (four) appropriate boundary conditions.

In the main text we showed:

$$(\mu_i^4 - \mu_j^4)\int_0^L W_i \cdot W_j \cdot dx = \left[W_i^{(3)} W_j - W_i^{(2)} W_j' + W_i' W_j^{(2)} - W_i W_j^{(3)} \right]_0^L$$

We shall consider the case in which the right hand side is zero.

Then, for $i \neq j$ and $\mu_i \neq \mu_j$:

$$\boxed{\int_0^L W_i \cdot W_j \cdot dx = 0.}$$

Next, consider the case of a <u>clamped-free</u> beam:

$$
\begin{cases}
w(0,t) = 0 & \Rightarrow \quad W_j(0) = 0 \qquad \triangleleft \\[2ex]
\dfrac{\partial w}{\partial x}(0,t) = 0 & \Rightarrow \quad W_j'(0) = 0 \qquad \triangleleft \\[2ex]
M(L,t) = EI \cdot \dfrac{\partial^2 w}{\partial x^2}(L,t) = 0 & \Rightarrow \quad W_j^{(2)}(L) = 0 \qquad \triangleleft \\[2ex]
D(L,t) = EI \cdot \dfrac{\partial^3 w}{\partial x^3}(L,t) = 0 & \Rightarrow \quad W_j^{(3)}(L) = 0 \qquad \triangleleft
\end{cases}
$$

General solution:

$$W_j(x) = a_j \cdot \sin(\mu_j x) + b_j \cdot \cos(\mu_j x) +$$
$$+ c_j \cdot \sinh(\mu_j x) + d_j \cdot \cosh(\mu_j x)$$

Application of boundary conditions gives:

$$\cos(\mu_j L) \cdot \cosh(\mu_j L) = -1. \qquad \Rightarrow \mu_j \Rightarrow \omega_j$$

$$\boxed{W_j(x) = C_j \left[(\cosh \mu_j x - \cos \mu_j x) + - \left(\frac{\sinh \mu_j L - \sin \mu_j L}{\cosh \mu_j L + \cos \mu_j L} \right) \cdot (\sinh \mu_j x - \sin \mu_j x) \right]}$$

where c_j is a normalization factor.

Consider $W_j(L)$

$$\Rightarrow W_j(L) = c_j \cdot \frac{2 \cdot \sinh\mu_j L \cdot \sin\mu_j L}{\cosh\mu_j L + \cos\mu_j L}$$

$$W_j(L) = -2 \cdot c_j \cdot \frac{\sinh(\mu_j L)}{\tan(\mu_j L)}$$

Note: $\sinh(\mu_j L) > 0$

$\tan(\mu_j L)$ alternates sign. with increasing j.

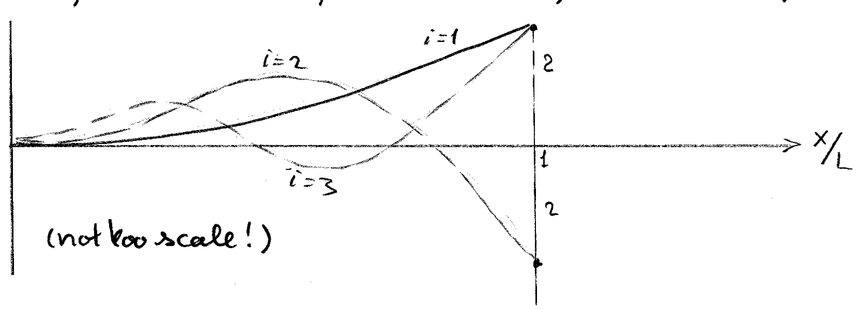
$\Rightarrow$ sinh/tan alternates sign.

$$\left(\frac{\sinh}{\tan}\right)^2 = \frac{(\cosh^2 - 1)\cdot\cos^2}{\sin^2} = \frac{+1 - \cos^2}{\sin^2} = 1.$$

Hence: $\boxed{|W_j(L)| = 2 \cdot |c_j|}$

E.g. if one would normalize W_j with $c_j = 1$, then:

$W_1(L) = 2, \quad W_2(L) = -2, \quad W_3(L) = 2, \quad W_4(L) = -2, \; \ldots$



Prove: $\boxed{\int_0^L W_j \cdot dx = c_j \cdot 2 \cdot \frac{\sigma_j}{\lambda_j}}$ $(c_j = 1)$

where $\sigma_j \triangleq \frac{\sinh \mu_j L - \sin \mu_j L}{\cosh \mu_j L + \cos \mu_j L}$.

Proof: $\int_0^L W_j \cdot dx = \int_0^L c_j \left[(\cosh \mu_j x - \cos \mu_j x) + \right.$
$\left. - \sigma_j (\sinh \mu_j x - \sin \mu_j x) \right] dx$

and the integration is straightforward.

Prove: $\boxed{\int_0^L W_j^2 \cdot dx = L . c_j^2}$ $(c_j = 1)$

The proof is not that easy. In Ref. Timoshenko and Young (1955) there is obtained.

$$4 \mu_j^4 \cdot \int_0^L W_j^2 \cdot dx = \left[3 . W_j . W_j^{(3)} + \mu_j^4 \cdot x . W_j^2 + \right.$$

$$\left. - 2 . x . W_j' \, W_j^{(3)} - W_j' \, W_j^{(2)} + x . (W_j^{(2)})^2 \right]_0^L$$

for arbitrary boundary conditions.

(Note: the derivative is with respect to x; in Ref. Timoshenko-Young 1955 it is with respect to $\mu_j x$).

For the clamped-free beam this reduces to:

$$\int_0^L W_j^2 \cdot dx = L \cdot C_j^2 \qquad (C_j = 1).$$

Results for many additional integrals of mode shapes, for a variety of boundary conditions, can be found in Ref. Blevins, App. C.

<u>Prove:</u> $\boxed{\int_0^L \overset{(2)}{W_i}\,\overset{(2)}{W_j} \cdot dx = 0 \quad \text{for} \quad i \neq j.}$

Proof.

Use the earlier result:

$$\int_0^L \overset{(4)}{W_i}\, W_j \cdot dx = \left[\overset{(3)}{W_i}\, W_j - \overset{(2)}{W_i}\,\overset{(1)}{W_j}\right]_0^L + \int_0^L \overset{(2)}{W_i} \cdot \overset{(2)}{W_j} \cdot dx$$

$$\underbrace{\lambda_i^4 \int_0^L \overset{(4)}{W_i} \cdot W_j \cdot dx}_{=0 \text{ for } i \neq j} = [\;\text{"}\; - \;\text{"}\;]_0^L + \int_0^L \text{"}\;.$$

Hence: if $\left[\overset{(3)}{W_i} W_j - \overset{(2)}{W_i} \overset{(1)}{W_j} \right]_0^L = 0$

then $\int_0^L \overset{(2)}{W_i} \overset{(2)}{W_j} \cdot dx = 0.$

Executive Summary

Canonical form:

$$(\rho \cdot A \cdot dx) \cdot \frac{\partial^2 w}{\partial t^2} = -EI \cdot \frac{\partial^4 w}{\partial x^4}\, dx + p^e \cdot dx$$

(for case $N = 0$)

- boundary conditions
- initial conditions.

Homogeneous equation (free motion).

$$\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A} \cdot \frac{\partial^4 w}{\partial x^4} = 0$$

Separation of variables: $w(x,t) = W(x) \cdot q(t)$

$$\ddot{q} + \omega^2 q = 0$$

$$\overset{(4)}{W} = \mu^4 W \quad \text{where } \mu \triangleq \left(\frac{\rho A}{EI} \cdot \omega^2 \right)^{1/4}$$

(special case: $\omega = 0$; rigid body motion).

Apply boundary conditions: $\Rightarrow \mu_i \Rightarrow \omega_i$

$(i = 1, 2, \ldots, \infty)$

$$\ddot{q}_i + \omega_i^2 q_i = 0 \quad \Rightarrow q_i$$

$$\overset{(4)}{W_i} = \mu_i^4 W_i \quad \Rightarrow W_i$$

Then: $w(x,t) = \sum_{i=1}^{\infty} W_{i}(x) \cdot q_i(t)$

Orthogonality property: $\int_0^L W_i \cdot W_j \cdot dx = 0$ for $i \neq j$.

<u>Non-homogeneous equation</u> (forced motion)

$$(\rho \cdot A \cdot dx) \frac{\partial^2 w}{\partial t^2} = -E.I. \frac{\partial^4 w}{\partial x^4} \cdot dx + f^e \cdot dx$$

Modal expansion: $w(x,t) = \sum_{i=1}^{\infty} W_i(x) \cdot Q_i(t) \Rightarrow Q_i$?

$$\Rightarrow \ddot{Q}_i + \omega_i^2 \cdot Q_i = \frac{1}{\int_0^L W_i^2 \cdot dm} \int_0^L W_i \cdot f^e \cdot dx$$

$$\hat{=} \; q_i(t).$$

$$\Rightarrow \quad Q_i(t) = q_i(t) + Q_i^{particular}(t)$$

$$\Rightarrow \quad w(x,t) = \sum_{i=1}^{\infty} W_i(x) \cdot \left\{ q_i(t) + Q_i^{particular}(t) \right\}$$

Initial conditions:

$$w(x,0) = F(x) \qquad \frac{\partial w}{\partial t}(x,0) = G(x).$$

$$\int_0^L W_i \cdot F \cdot dx = \left(\int_0^L W_i^2 \cdot dx \right) \cdot \left\{ q_i(0) + Q_i^{\text{particular}}(0) \right\}$$

$$\int_0^L W_i \cdot G \cdot dx = \left(\int_0^L W_i^2 \cdot dx \right) \cdot \left\{ \dot{q}_i(0) + \dot{Q}_i^{\text{particular}}(0) \right\}$$

$\Rightarrow$ solve for integration constants in q_i.

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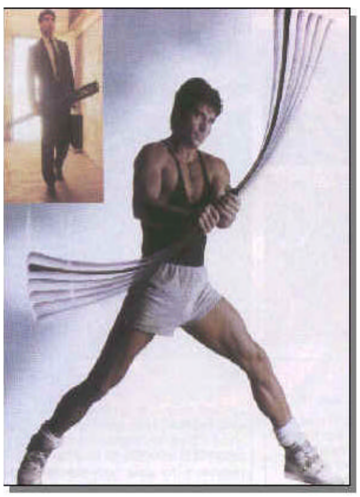
Notes on Linear Vibration Theory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Compiled by : P.Th.L.M. van Woerkom
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* Body Blade picture taken from: http://www.starsystems.com.au

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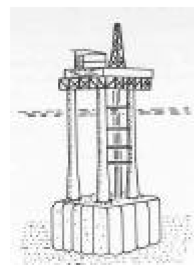
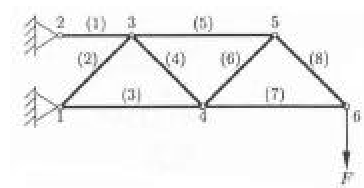


PART THREE: FINITE ELEMENT MODELLING



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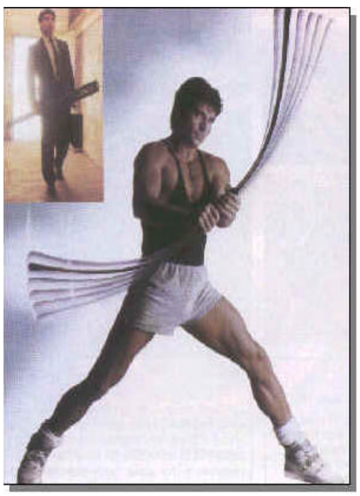
Notes on Linear Vibration Theory



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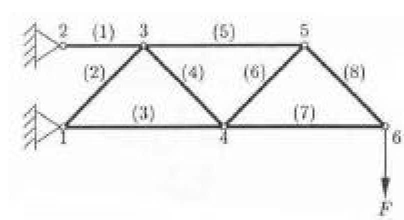
PART THREE:

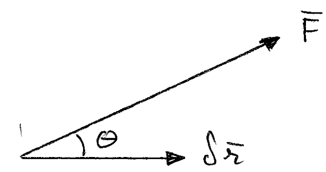
FINITE ELEMENT MODELLING



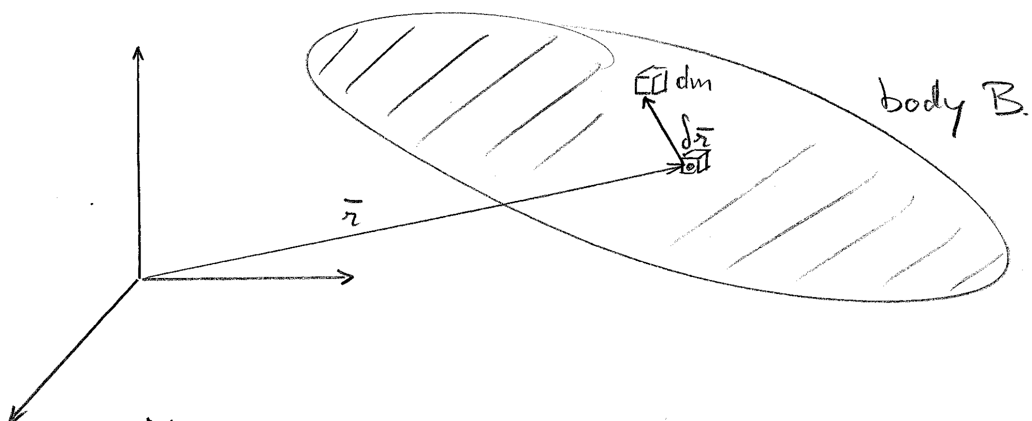
14. PRINCIPLE OF VIRTUAL WORK

The virtual work formalism.

Virtual work δw:



$$\delta w \triangleq \bar{F}^T \delta \bar{r} = (\delta \bar{r})^T \bar{F} = \|\bar{F}\| \cdot \|\delta \bar{r}\| \cdot \cos\theta .$$



Newton, for infinitesimal mass element:

$$\ddot{\bar{r}}\, dm = d\bar{f} \quad \text{(net force on dm)} \qquad \triangleleft$$

d'Alembert's Principle:

$$\underbrace{(-\ddot{\bar{r}}\, dm)}_{\text{"inertia force"}} + d\bar{f} = \bar{0} \qquad \triangleleft$$

Equilibrium of forces. ("static" treatment).

$(\delta \bar{r})^T(-\ddot{\bar{r}}\, dm)$ = virtual work produced by inertia force.

$(\delta \bar{r})^T d\bar{f}$ = virtual work produced by net applied force.

Note: $(\delta \bar{r})^T \{-\ddot{\bar{r}}\, dm + d\bar{f}\} = 0.$

Requirements on $\delta \bar{r}$:

- compatible with the kinematic constraints, but otherwise arbitrary
- instantaneous
- vanishingly small.

For a single body B_i:

$$\int_B (\delta \bar{r})^T (-\ddot{\bar{r}}\, dm + d\bar{f}) = 0 \qquad \triangleleft$$

For a system of n bodies B:

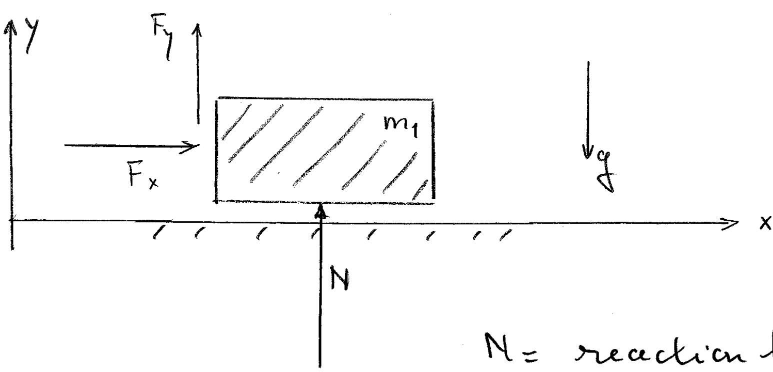
$$\sum_{i=1}^{n} \int_{B_i} (\delta \bar{r})^T (-\ddot{\bar{r}}\, dm + d\bar{f}) = 0 \qquad \triangleleft$$

"Lagrange form of d'Alembert's Principle:"

$$\boxed{\int_{\text{system}} (\delta \bar{r})^T (-\ddot{\bar{r}}\, dm + d\bar{f}) = 0}$$

This formalism is convenient, as the constraint loads disappear. Indeed:

(i) Block on a surface:



N = reaction force (constraint force).

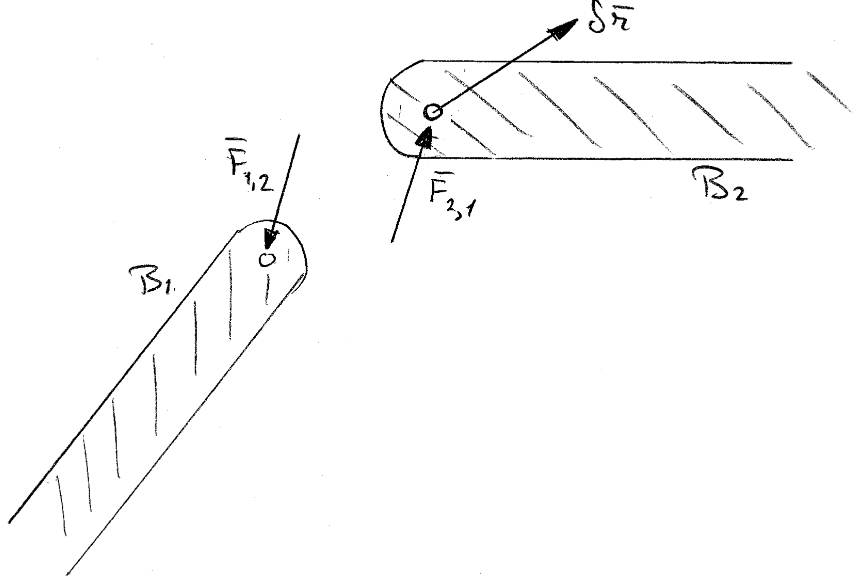
$$\delta x \cdot (-m_1 \ddot{x} + F_x) + \delta y \cdot (-m_1 \ddot{y} + F_y + N - m_1 g) = 0.$$

$$\begin{cases} \delta x \text{ arbitrary but small} \\ \delta y = 0 \end{cases}$$

$$\Rightarrow \quad -m_1 \ddot{x} + F_x = 0.$$

Observe: virtual work formalism leads directly to Newton's equation of motion in the kinematically allowable direction.

(iii) Two bodies connected by a rotary joint



Virtual work produced by these constraint loads:

$$\delta w = (\delta \bar{r}_{joint})^T \, \bar{F}_{1,2} + (\delta \bar{r}_{joint})^T \, \bar{F}_{2,1}$$

$$= (\delta \bar{r}_{joint})^T \underbrace{(\bar{F}_{1,2} + \bar{F}_{2,1})}_{= \bar{0} \text{ (Newton)}}$$

$\Rightarrow \bar{F}_{1,2}$ and $\bar{F}_{2,1}$ drop out of the expression!

By summing the contributions to virtual work produced by all forces in and on all system elements, the constraint loads disappear.

For multi-body system, the derivation of the equations of motion now becomes much more simple.

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Dynamics in second-order canonical form

Generalized coordinates: $q_1 \ldots q_n \Rightarrow \bar{q}$

Hence: $\bar{R} = \bar{R}(\bar{q})$

$$\Rightarrow \quad \dot{\bar{R}} = \sum_j \frac{\partial \bar{R}}{\partial q_j} \dot{q}_j = \frac{\partial \bar{R}}{\partial \bar{q}} \dot{\bar{q}} \qquad \left(\text{Hence: } \frac{\partial \dot{\bar{R}}}{\partial \dot{q}_j} = \frac{\partial \bar{R}}{\partial q_j} \right)$$

$$\ddot{\bar{R}} = \sum_i \left(\sum_j \frac{\partial^2 \bar{R}}{\partial q_i \partial q_j} \dot{q}_j \right) \dot{q}_i + \sum_j \frac{\partial \bar{R}}{\partial q_j} \ddot{q}_j$$

$$= \underbrace{\bar{\nu}(\dot{q}_i \dot{q}_j \text{ products})}_{\text{higher order (delete in linearized analysis)}} + \frac{\partial \bar{R}}{\partial \bar{q}} \ddot{\bar{q}}$$

$$\delta \bar{R} = \frac{\partial \bar{R}}{\partial \bar{q}} \delta \bar{q}$$

Substitute:

$$\int_{\text{system}} \left[\frac{\partial \bar{R}}{\partial \bar{q}} \delta \bar{q} \right]^T \left[- \left(\frac{\partial \bar{R}}{\partial \bar{q}} \ddot{\bar{q}} \right) dm + d\bar{f}^{\,int} + d\bar{f}^{\,appl} \right] = 0$$

$$(\delta \bar{q})^T \int_{\text{system}} \left(\frac{\partial \bar{R}}{\partial \bar{q}} \right)^T \left[- \left(\frac{\partial \bar{R}}{\partial \bar{q}} \ddot{\bar{q}} \right) dm + d\bar{f}^{\,int} + d\bar{f}^{\,appl} \right] = 0$$

δq_j arbitrary (but small)

$$\Rightarrow \int_{\text{system}} \qquad = 0$$

$$\boxed{\underbrace{\left[\int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right)^T \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right) dm\right]}_{\hat{=} M} \ddot{\bar{q}} = \int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right)^T d\bar{f}^{int} + \int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right) d\bar{f}^{appl.}}$$

Potential energy:

$$E_P = -\int (d\bar{R})^T d\bar{f}^{conserv.} \qquad \text{(conservative forces)}$$

$$= E_P(\bar{q})$$

Near equilibrium $\bar{q} = \bar{0}$:

$$E_P = \underbrace{E_P(\bar{0})}_{\text{irrelevant}} + \sum_j \underbrace{\frac{\partial E_P}{\partial q_j}}_{=0 \text{ at equil.}} q_j + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 E_P}{\partial q_i \partial q_j} q_i q_j + \cdots$$

$$E_P = \frac{1}{2} \bar{q}^T K \bar{q}$$

Structurally elastic medium: $K \geq 0$

$$\delta E_p = -\int (\delta \bar{R})^T d\bar{f}^{cons.} = -(\delta \bar{q})^T \int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right)^T d\bar{f}^{cons.}$$

$$\delta E_p = (\delta \bar{q})^T K \bar{q}$$

$$\Rightarrow -\int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right)^T d\bar{f}^{cons.} = K\bar{q}$$

Hence: $$\boxed{M\ddot{\bar{q}} + K\bar{q} = \bar{F}}$$

nonconservative forces, torques

Note: $M = M^T$
$K = K^T$

Euler-Lagrange formalism

We derived:

$$\sum_j \delta q_j \int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T \left(-\ddot{\bar{R}}\, dm + d\bar{f}^{\,int.} + d\bar{f}^{\,appl.}\right) = 0$$

Rewrite:

$$\int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T \ddot{\bar{R}}\, dm = \frac{d}{dt}\left[\int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T \dot{\bar{R}}\, dm\right] - \int \left\{\frac{d}{dt}\left(\frac{\partial \bar{R}}{\partial q_j}\right)\right\}^T \dot{\bar{R}}\, dm$$

where:
$$\frac{d}{dt}\left(\frac{\partial \bar{R}}{\partial q_j}\right) = \sum_i \frac{\partial^2 \bar{R}}{\partial q_i \partial q_j}\, \dot{q}_i$$

$$= \frac{\partial}{\partial q_j}\left\{\sum_i \frac{\partial \bar{R}}{\partial q_i}\, \dot{q}_i\right\} = \frac{\partial \dot{\bar{R}}}{\partial q_j}$$

$$\Rightarrow \int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T \ddot{\bar{R}}\, dm = \frac{d}{dt}\left[\int \left(\frac{\partial \dot{\bar{R}}}{\partial \dot{q}_j}\right)^T \dot{\bar{R}}\, dm\right] - \int \left(\frac{\partial \dot{\bar{R}}}{\partial q_j}\right)^T \dot{\bar{R}}\, dm$$

Define: kinetic energy $E_K \triangleq \frac{1}{2}\int \dot{\bar{R}}^T \dot{\bar{R}}\, dm$

Note: $E_K = \frac{1}{2}\int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\, \dot{\bar{q}}\right)^T \left(\frac{\partial \bar{R}}{\partial \bar{q}}\, \dot{\bar{q}}\right) dm$

$$= \frac{1}{2}\, \dot{\bar{q}}^T \int \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right)^T \left(\frac{\partial \bar{R}}{\partial \bar{q}}\right) dm\, \dot{\bar{q}} \overset{!}{=} \frac{1}{2}\, \dot{\bar{q}}^T M \dot{\bar{q}}$$

$$\Rightarrow \sum_j \delta q_j \left[-\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{q}_j}\right) + \frac{\partial E_K}{\partial q_j} - \frac{\partial E_P}{\partial q_j} + \right.$$

$$\left. + \int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T d\bar{f}^{\,\text{nonconserv.}} \right] = 0$$

Define Lagrangian: $L \triangleq E_K - E_P$

$$\sum_j \delta q_j \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial q_j} + \int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T d\bar{f}^{\,\text{nonconserv.}} \right] = 0$$

δq_j arbitrary (but small) $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \int \left(\frac{\partial \bar{R}}{\partial q_j}\right)^T d\bar{f}^{\,\text{nonconserv.}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{q}}}\right)^T - \left(\frac{\partial L}{\partial \bar{q}}\right)^T = \bar{F}^{\,\text{nonconserv.}}}$$

Euler–Lagrange eqs. of motion.

Note: $L \overset{!}{=} \frac{1}{2}\dot{\bar{q}}^T M \dot{\bar{q}} - \frac{1}{2}\bar{q}^T K \bar{q}$

$$\Rightarrow M\ddot{\bar{q}} + K\bar{q} = \bar{F}^{\,\text{nonconserv.}}$$

same as before.

Hamilton's formalism

Lagrange form of d' Alembert's principle:

$$\delta w^{constr.} = - \int (\delta \bar{R})^T (- \ddot{\bar{R}}\, dm + d\bar{f}^{int} + d\bar{f}^{appl}) = 0$$

(i) write: δW = virtual work due to all internal forces and applied forces

$$= \int (\delta \bar{R})^T (d\bar{f}^{int} + d\bar{f}^{appl})$$

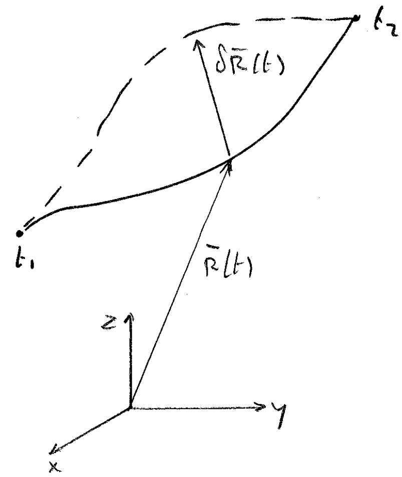
(ii) $$(\delta \bar{R})^T \ddot{\bar{R}}\, dm = \frac{d}{dt}\left[(\delta \bar{R})^T \dot{\bar{R}}\, dm\right] - \left\{\frac{d(\delta \bar{R})}{dt}\right\}^T \dot{\bar{R}}\, dm$$

$$\frac{d}{dt}(\delta \bar{R}) = \frac{d}{dt}\left\{\bar{R}(t) - \bar{R}_{nom}(t)\right\} = \dot{\bar{R}} - \dot{\bar{R}}_{nom} = \delta \dot{\bar{R}}$$

$$\Rightarrow \left\{\frac{d(\delta \bar{R})}{dt}\right\}^T \dot{\bar{R}} = (\delta \dot{\bar{R}})^T \dot{\bar{R}} = \frac{1}{2}\delta(\dot{\bar{R}}^T \dot{\bar{R}})$$

$$\Rightarrow \int_{t_1}^{t_2} (\delta \bar{R})^T \ddot{\bar{R}}\, dm\, dt = (\delta \bar{R})^T \dot{\bar{R}}\, dm \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta\left(\tfrac{1}{2} \dot{\bar{R}}^T \dot{\bar{R}}\, dm\right) dt$$

Choose $\delta\bar{R}$: $\qquad \delta\bar{R} = \bar{0}$ at t_1 and at t_2



$$\Rightarrow \int_{t_1}^{t_2} (\delta\bar{R})^T \ddot{\bar{R}}\, dm\, dt = - \int_{t_1}^{t_2} \delta\left(\tfrac{1}{2}\dot{\bar{R}}^T \dot{\bar{R}}\, dm\right) dt$$

Sum over all elementary masses:

$$\int_{t_1}^{t_2} \left[\int (\delta\bar{R})^T \ddot{\bar{R}}\, dm\right] dt = - \int_{t_1}^{t_2} \delta \left[\tfrac{1}{2}\int \dot{\bar{R}}^T \dot{\bar{R}}\, dm\right] dt$$

$$= - \int_{t_1}^{t_2} \delta E_K \, dt$$

$$\Rightarrow \boxed{\int_{t_1}^{t_2} \delta(E_K + W)\, dt = 0 \quad \text{with } \delta\bar{R}(t_1) = \bar{0} \text{ and } \delta\bar{R}(t_2) = \bar{0}}$$

"extended Hamilton's principle".

Executive Summary

Newton: $\ddot{\bar{R}}\, dm = d\bar{f}$

(infinitesimal element dm)

d'Alembert: $(-\ddot{\bar{R}}\, dm) + d\bar{f} = \bar{0}$

Lagrange form of d'Alembert's Principle:

$$\int_{system} (\delta\bar{R})^T (-\ddot{\bar{R}}\, dm + d\bar{f}) = 0$$

Virtual displacement $\delta\bar{R}$:

- vanishingly small
- compatible with kinematic constraints
- otherwise: arbitrary.

Advantage: all non-working loads (forces, torques) disappear

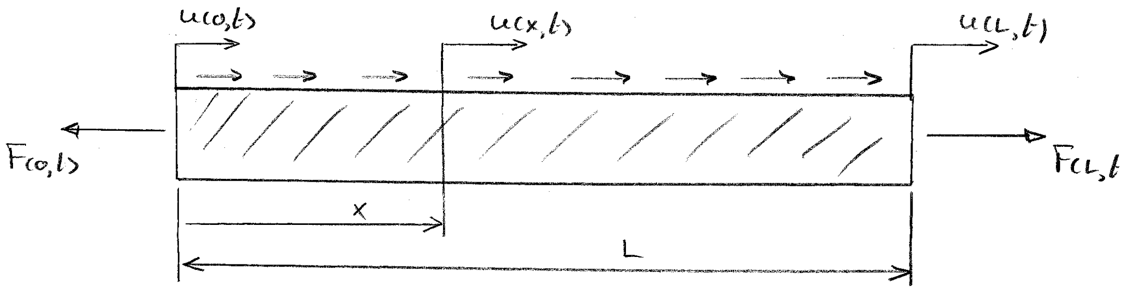
$$\Rightarrow \quad M\ddot{\bar{U}} + K\bar{U} = \bar{F}(t)$$

where $\bar{U}$ = vector of independent degrees-of-freedom.

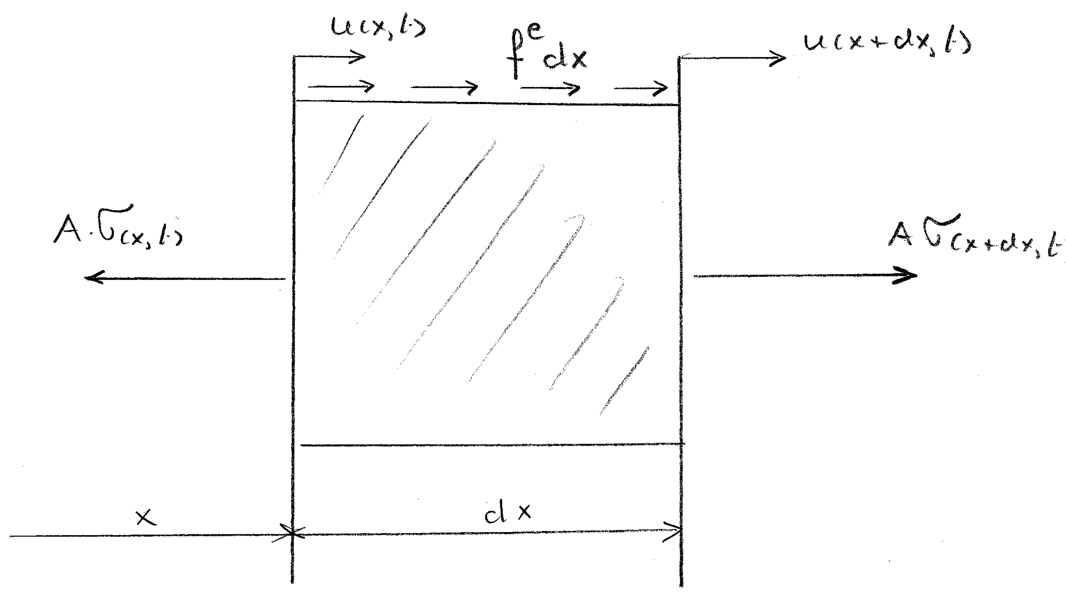
15. FINITE ELEMENT MODELLING: SINGLE ROD ELEMENT

Dynamics of an infinitesimal mass element.

Consider a single rod of finite length:



Consider an infinitesimally small mass element:



Newton:

$$dm \cdot \frac{\partial^2}{\partial t^2} u_{(x+dx/2,\,t)} = -A \cdot \sigma_{(x,t)} + A \cdot \sigma_{(x+dx,\,t)} + $$
$$+ f^e \, dx$$

$$dm \cdot \frac{\partial^2}{\partial t^2} \left\{ u_{(x,t)} + \frac{\partial u}{\partial x} \cdot \frac{dx}{2} \right\} = \cancel{- A \cdot \sigma_{(x,t)}} + $$
$$+ A \cdot \left\{ \cancel{\sigma_{(x,t)}} + \frac{\partial \sigma}{\partial x} \cdot dx \right\} + $$
$$+ f^e \cdot dx$$

$$dm \cdot \frac{\partial^2 u}{\partial t^2} = A \frac{\partial \sigma}{\partial x} \cdot dx + f^e \cdot dx$$

Reformulation (d'Alembert's Principle):

$$\underbrace{\left(- \frac{\partial^2 u}{\partial t^2} \cdot dm \right)}_{\text{inertia force}} + \underbrace{A \cdot \frac{\partial \sigma}{\partial x} \cdot dx}_{\text{net normal force on cross section}} + \underbrace{f^e \cdot dx}_{\text{surface force.}} = 0$$

Lagrange form of d' Alembert's Principle:

$$\int_B \delta u \cdot \left(-\frac{\partial^2 u}{\partial t^2} \cdot dm + A \cdot \frac{\partial \sigma}{\partial x} \cdot dx + f^e \cdot dx \right) = 0 \qquad \triangleleft$$

where $\qquad dm = \rho \cdot A \cdot dx$

Development of the Lagrange expression.

$$\int_B \delta u \left(-\frac{\partial^2 u}{\partial t^2} \cdot dm + A \cdot \frac{\partial \sigma}{\partial x} \cdot dx + f^e \cdot dx \right) = 0$$

For the rod we assume: A = constant.

Consider $\int_B \delta u \cdot \frac{\partial \sigma}{\partial x} \cdot dx = \int_B \delta u \cdot d\sigma$

Partial integration: $= \delta u \cdot \sigma \Big|_0^L - \int \sigma \cdot d(\delta u)$

Now: $\delta(du) = \delta \left[u_{(x+dx,t)} - u_{(x,t)} \right]$

$= \delta u_{(x+dx,t)} - \delta u_{(x,t)}$

$= d(\delta u)$

Hence:

$$\int_B \delta u \cdot \frac{\partial \sigma}{\partial x} \cdot dx = \delta u \cdot \sigma \Big|_0^L - \int \sigma \cdot \delta(du)$$

$$= \delta u \cdot \sigma \Big|_0^L - \int \sigma \cdot \delta \left(\frac{\partial u}{\partial x} \cdot dx \right)$$

$$= \delta u \cdot \sigma \Big|_0^L - \int \sigma \cdot \delta \varepsilon \cdot dx$$

Substitute:

$$\int_B \delta u \left(- \frac{\partial^2 u}{\partial t^2} dm \right) + \Big[\delta u(L,t) F(L,t) - \delta u(0,t) F(0,t) +$$

$$- \int_B \sigma \, \delta \varepsilon \cdot d\bar{V} \Big] + \int_B \delta u \cdot f^e dx = 0. \triangleleft$$

where $dV = A \cdot dx$

To interpret this result, rearrange the terms:

$$\int_B \sigma \cdot \delta \varepsilon \cdot dV = \int_B \delta u \left(- \frac{\partial^2 u}{\partial t^2} dm \right) +$$

$$+ \Big[- \delta u(0,t) F(0,t) + \delta u(L,t) F(L,t) \Big] +$$

$$+ \int_B \delta u \cdot f^e \cdot dx$$

Terms on the right:

- first term: virtual work produced by inertia forces

- second term: virtual work produced by the forces acting on the extreme sides of the rod.

- third term: virtual work produced by the forces acting along the cylindrical sides of the rod.

The three terms together represent the net virtual work acting on the rod.

The term on the left represents the virtual change in stored elastic energy (strain energy).

In the static case one has $\partial^2 u/\partial t^2 \equiv 0$. The resulting equation is familiar in finite element literature.
The only new term in the present course, is the term:

$$\int \delta u \left(-\frac{\partial^2 u}{\partial t^2}\, dm\right).$$

All other terms must already be familiar to the reader.

Assumed displacement field.

(i) Consider first the <u>static case</u>, with end-forces F but no surface load $f^e dx$.

The equation of motion now reads:

$$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{=0} . dm = A . \frac{\partial \sigma}{\partial x} . dx + \underbrace{f^e}_{=0} . dx$$

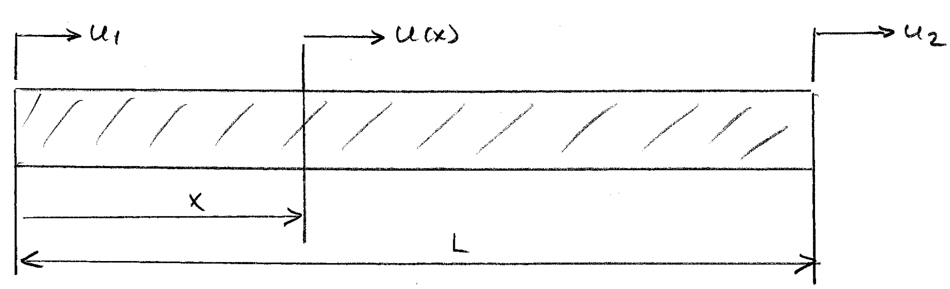
Hence: $\frac{\partial \sigma}{\partial x} = 0 \Rightarrow \sigma = \text{constant}$

From Hooke's law: $\sigma = E . \varepsilon \Rightarrow \varepsilon = \text{constant}$.

But, by definition, $\varepsilon \triangleq \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \text{constant}$.

$$\Rightarrow u(x) = a_0 + a_1 . x$$

Express a_0 and a_1 in terms of the displacements at the ends ("nodes"):



$$x = 0: \quad u_1 = a_0$$
$$x = L: \quad u_2 = a_0 + a_1 . L$$
$$\Rightarrow \quad a_0 = u_1, \quad a_1 = \frac{u_2 - u_1}{L}$$

Hence: $u(x) = u_1 + \left(\frac{u_2 - u_1}{L}\right) x$

$$u(x) = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$

Define: $\phi_1(x) \triangleq 1 - x/L$

$\phi_2(x) \triangleq x/L$

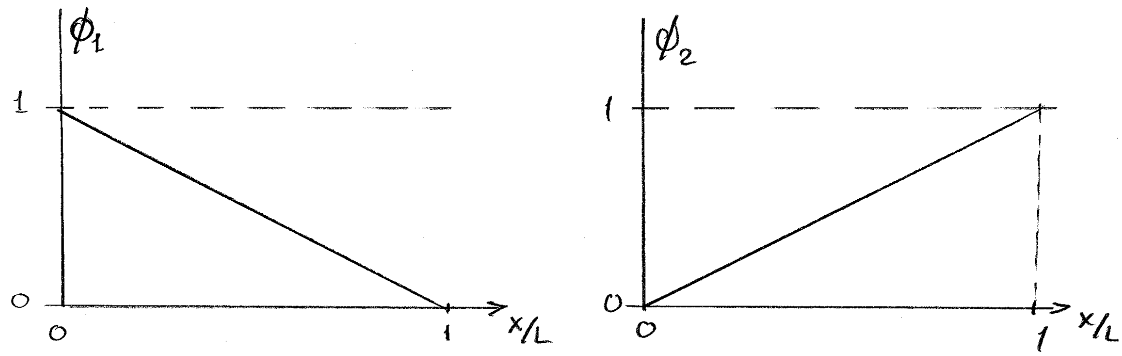
$\bar{\phi} \triangleq [\phi_1, \phi_2]^T$

$\bar{u}_e \triangleq [u_1, u_2]^T$ (vector of nodal displacements)

Hence: $u(x) = \bar{\phi}_{(x)}^T \; \bar{U}_e$ $\triangleleft$

"displacement field" (static).

$\bar{\phi}$ = vector of shape functions:



ϕ_i = shape function.

u_i = "amplification" factor.

Linear interpolation.

(ii) Next, consider the <u>dynamic case.</u>

Now: $\quad u_1 = u_1(t)$ and $u_2 = u_2(t)$

$$u = u(x,t)$$

Assume that the actual displacement field may be approximated by the static displacement field:

$$u(x,t) \cong \underbrace{\bar{\phi}^T_{(x)}}_{} \; \bar{u}_e(t) \qquad \triangleleft$$

↳ from static analysis!

First, approximate rod by one finite element. Analyze dynamic behavior.

Then, approximate rod by two finite elements. Analyze dynamic behavior.

And so on.

One obtains $0 \leq \omega_1 < \omega_2 < \omega_3 \ldots\ldots$

Decide in which frequency range the system dynamics is to be represented accurately.

If computed frequencies within that range have not converged sufficiently, increase the number of finite elements in the system under consideration.

Remark.
Rotorblade: in the static case, stress is not constant, but increases with distance x.
Hence, the static displacement field is not linear in x, but it is parabolic in x.
In this case, compute the appropriate shape functions. Or: introduce more elements in each rotor blade.

Evaluation of the Lagrange expression

Recall the Lagrange expression:

$$\int_B \delta u \left(-\frac{\partial^2 u}{\partial t^2}\, dm\right) - \int_B \sigma . \delta \varepsilon . dV +$$

$$+ \left[\delta u_{(l,t)} . F_{(l,t)} - \delta u_{(0,t)} F_{(0,t)}\right] + \int_B \delta u . f^e . dx = 0$$

Substitute: $u(x,t) = \bar{\phi}_{(x)}^T \bar{U}_e(t)$

Note: $\frac{\partial^2 u}{\partial t^2} = \bar{\phi}^T \ddot{\bar{u}}_e$ and $\delta u = \bar{\phi}^T \delta \bar{u}_e$.

(i) $\delta W^{inertia} \triangleq \int_B \delta u \left(-\frac{\partial^2 u}{\partial t^2}\, dm\right) =$

$$= -\int (\bar{\phi}^T \delta \bar{u}_e)\, (\bar{\phi}^T \ddot{\bar{u}}_e\, dm)$$

$$= -\int (\delta \bar{u}_e^T\, \bar{\phi})\, (\bar{\phi}^T \ddot{\bar{U}}_e\, dm)$$

$$\delta W^{inertia} = -(\delta \bar{U}_e)^T \int_B \bar{\phi}\, \bar{\phi}^T dm\ \ddot{\bar{U}}_e$$

$$= -(\delta \bar{u}_e)^T M \ddot{\bar{U}}_e$$

where $M \triangleq \int_B \bar{\phi}\, \bar{\phi}^T dm$ "generalized mass matrix."

(ii) $$\delta W^{strain} \triangleq \int_B \sigma . \delta\varepsilon . dV$$

Now: $$\varepsilon = \frac{\partial u}{\partial x} = \frac{d\bar{\phi}^T}{dx} \bar{u}_e$$

$$\sigma = E . \varepsilon = E . \frac{d\bar{\phi}^T}{dx} . \bar{u}_e$$

$$dV = A . dx$$

$$\delta W^{strain} = \int_B \left(E . \frac{d\bar{\phi}^T}{dx} \bar{u}_e \right) \left(\frac{d\bar{\phi}^T}{dx} . \delta \bar{u}_e \right) A . dx$$

$$= E . A . \int_B \left(\delta \bar{u}_e^T \frac{d\bar{\phi}}{dx} \right) \left(\frac{d\bar{\phi}^T}{dx} \bar{U}_e \right) dx$$

$$\delta W^{strain} = (\delta \bar{u}_e)^T \left\{ E.A. \int_B \frac{d\bar{\phi}}{dx} \frac{d\bar{\phi}^T}{dx} . dx \right\} \bar{U}_e$$

$$= (\delta \bar{u}_e)^T K \bar{U}_e$$

$$\text{where } K \triangleq EA \int_0^L \frac{d\bar{\phi}}{dx} \frac{d\bar{\phi}^T}{dx} dx$$

"generalised stiffness matrix."

(iii) $\delta W^F \triangleq \delta u(L,t) F(L,t) - \delta u(0,t) F(0,t)$

$$= [\delta u_1, \delta u_2] \begin{bmatrix} -F(0,t) \\ F(L,t) \end{bmatrix}$$

$$= (\delta \bar{u}_e)^T \bar{\bar{F}}_T^e$$

$$\text{where } \bar{\bar{F}}_T^e \triangleq \begin{bmatrix} -F(0,t) \\ \\ +F(L,t) \end{bmatrix}$$

"generalized end force"

(iv) $\delta w^f \triangleq \int \delta u . \; f^e . dx$

$$= \int (\bar{\phi}^T \delta \bar{u}_e) . f^e . dx$$

$$\delta W^f = \int (\delta \bar{u}_e^T \bar{\phi}) f^e dx$$

$$= (\delta \bar{u}_e)^T \int_0^L \bar{\phi} \, f^e . dx$$

$$= (\delta \bar{u}_e)^T \bar{\bar{F}}_s^e$$

where $\bar{\bar{F}}_s^e \triangleq \int_0^L \bar{\phi} \, f^e . dx$

"generalized surface force".

Collect results:

$$- (\delta \bar{u}_e)^T M \ddot{\bar{u}}_e - (\delta \bar{u}_e)^T K \bar{u}_e + $$

$$+ (\delta \bar{u}_e)^T \bar{\bar{F}}_T^e + (\delta \bar{u}_e)^T \bar{\bar{F}}_s^e = 0$$

$$(\delta u_e)^T [- M . \ddot{\bar{u}}_e - K . \bar{u}_e + \bar{\bar{F}}_T^e + \bar{\bar{F}}_s^e] = 0$$

But: all elements of $\delta \bar{u}_e$ are small but arbitrary.
Hence:

$$[\cdots\cdots] = 0$$

$$M \ddot{\bar{u}}_e + K . \bar{u}_e = \bar{\bar{F}}_T^e + \bar{\bar{F}}_s^e \qquad \triangleleft$$

Equation of motion for a single rod element.

We shall now <u>evaluate M and K and $\bar{F}_s^e$</u>

(i) $$M \triangleq \int \bar{\phi}\,\bar{\phi}^T\, dm = \int \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} \left[1 - x/L ,\ x/L \right] (\rho . A . dx)$$

$$= \rho . A . \int_0^L \begin{bmatrix} (1 - x/L)^2 & (1-x/L) . x/L \\ x/L\,(1-x/L) & (x/L)^2 \end{bmatrix} dx$$

$$\Rightarrow \quad M = \frac{\rho . A . L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \triangleleft$$

(ii) $$K \triangleq E.A. \int_0^L \frac{d\bar{\phi}}{dx} . \frac{d\bar{\phi}^T}{dx} . dx = EA . \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} \left[-1/L ,\ 1/L \right] dx$$

$$= EA . \begin{bmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{bmatrix} \int_0^L dx$$

$$\Rightarrow \quad K = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \triangleleft$$

$$(iii)\ \bar{F}_s^e \triangleq \int_{x=0}^{L} \bar{\phi} \cdot f^e \cdot dx = \int_{x=0}^{L} \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} f^e \, dx$$

$$\Rightarrow \bar{F}_s^e = \int_0^L \begin{bmatrix} (1 - x/L) f^e \\ x/L \cdot f^e \end{bmatrix} dx. \qquad \triangleleft$$

where $f^e = f^e_{(x,t)}$ in general.

Design parameters.

Once the equations of motion have been established, one attempts to find $\bar{v}_e(t)$.

From the solution of the equations of motion one derives:

$$
\begin{cases}
\text{displacement field:} & u(x,t) = \bar{\phi}^T(x)\, \bar{v}_e(t) \\
\text{strain field:} & \varepsilon(x,t) = \dfrac{d\bar{\phi}^T}{dx}(x)\, \bar{v}_e(t) \\
\text{stress field:} & \sigma(x,t) = E \cdot \dfrac{d\bar{\phi}^T}{dx}(x)\, \bar{v}_e(t).
\end{cases}
$$

On Galerkin's method

For simplicity, consider the bar element without external loads.

$$dm \cdot \frac{\partial^2 u}{\partial t^2} = d\sigma \cdot A \quad \Rightarrow \quad \frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho} \cdot \frac{\partial^2 u}{\partial x^2} = 0$$

Approximate $u(x,t)$ by:

$$u(x,t) \cong u_a(x,t) = \bar{\mu}^T_{(x)} \, \bar{q}(t)$$

where μ consists of "comparison functions" ("trial functions").

Substitute $\Rightarrow$ $$\frac{\partial^2 u_a}{\partial t^2} - \frac{E}{\rho} \cdot \frac{\partial^2 u_a}{\partial x^2} = R_{(x,t)} \quad \text{(error)}$$

Then, construct:

$$\int_0^L \delta u_a \left[\frac{\partial^2 u_a}{\partial t^2} - \frac{E}{\rho} \frac{\partial^2 u_a}{\partial x^2} \right] dx = \int_0^L \delta u_a \cdot R \, dx$$

$$\int_0^L \left[\bar{\mu}^T \delta \bar{q} \right] \left[\bar{\mu}^T \ddot{\bar{q}} - \frac{E}{\rho} \bar{\mu}''^T \bar{q} \right] dx = \int \left[\bar{\mu}^T \delta \bar{q} \right] R \, dx$$

$$\delta \bar{q}^T \int_0^L \bar{\mu} \left[\bar{\mu}^T \ddot{\bar{q}} - \frac{E}{\rho} \bar{\mu}''^T \bar{q} \right] dx = \delta \bar{q}^T \int_0^L \bar{\mu} R \, dx$$

$\delta \bar{q}$ arbitrary, small $\Rightarrow$

$$\int_0^L \bar{\mu}\left[\bar{\mu}^T \ddot{\bar{q}} - \frac{E}{\rho}\,\bar{\mu}''^T \bar{q}\right] dx = \int_0^L \bar{\mu} R\, dx$$

Require: "weighted error" = zero.

$$\boxed{\int_0^L \bar{\mu} R\, dx \doteq \bar{0}}$$

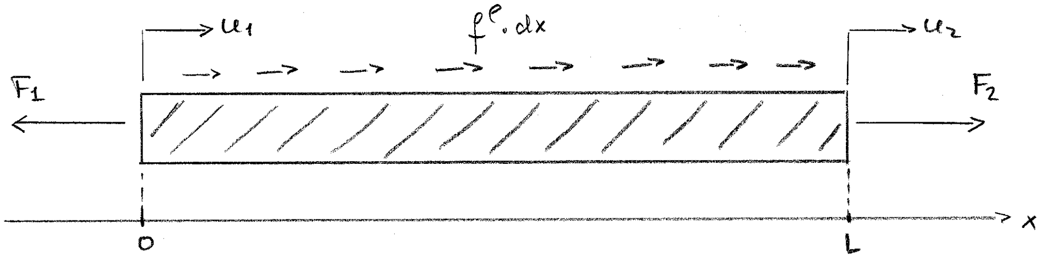
Result: $$\int_0^L \bar{\mu}\left[\bar{\mu}^T \ddot{\bar{q}} - \frac{E}{\rho}\,\bar{\mu}''^T \bar{q}\right] dx = 0$$

$$\underbrace{\left(\int_0^L \bar{\mu}\,\bar{\mu}^T dx\right)}_{\triangleq\, M_\mu} \ddot{\bar{q}} \;-\underbrace{\left(\frac{E}{\rho}\int_0^L \bar{\mu}\,\bar{\mu}''^T dx\right)}_{\triangleq\, -K_\mu} \bar{q} = 0$$

Show: $K_\mu = \frac{E}{\rho}\int_0^L \bar{\mu}'\,\bar{\mu}'^T dx$

If one chooses $\bar{\mu} = \bar{\phi}$, then the result obtained is identical to the one obtained through application of the Lagrange form of d'Alembert's principle.

Executive Summary



Lagrange form:

$$\int \delta u \left\{ -(\rho \cdot A \cdot dx)\cdot \frac{\partial^2 u}{\partial t^2} + (d\sigma \cdot A) + f^e \cdot dx \right\} = 0$$

$$\Rightarrow -\int \delta u \cdot \frac{\partial^2 u}{\partial t^2} \cdot dm - AE \cdot \int \delta\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial u}{\partial x}\cdot dx \quad +$$

$$- F_1 \cdot \delta u_1 + F_2 \cdot \delta u_2 + \int \delta u \cdot f^e \cdot dx = 0.$$

Assumed displacement field: static.

$$\Rightarrow u(x,t) = \left(1-\frac{x}{L}\right)\cdot u_1(t) + \frac{x}{L}\cdot u_2(t) = \bar{\phi}^T(x)\,\bar{U}_e(t)$$

Substitute

$$\Rightarrow \quad M\ddot{\bar{U}}_e + K\bar{U}_e = \bar{F}(t)$$

where:

$$\bar{U}_e \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ; \qquad M \triangleq \int \bar{\phi}\,\bar{\phi}^T dm$$

$$K \triangleq AE\int \bar{\phi}'\,\bar{\phi}'^T dx \; ; \; \bar{F} = \bar{F}_{ends} + \int \bar{\phi}\, f^e \cdot dx$$

Work out:

$$M = \frac{\rho \cdot A \cdot L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad ; \qquad K = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\bar{F} = \begin{bmatrix} -F_1 & + \int_0^L (1 - x/L) \cdot f^e \cdot dx \\ +F_2 & + \int_0^L \; x/L \cdot f^e \cdot dx \end{bmatrix}$$

Finally: $\bar{u}_e(t) \Rightarrow u(x,t) \Rightarrow \varepsilon(x,t) \Rightarrow \sigma(x,t)$.

16. FINITE ELEMENT MODELLING:

SINGLE BEAM ELEMENT

16.1 Dynamics of an infinitesimal mass element

16.2 Development of the Lagrange expression

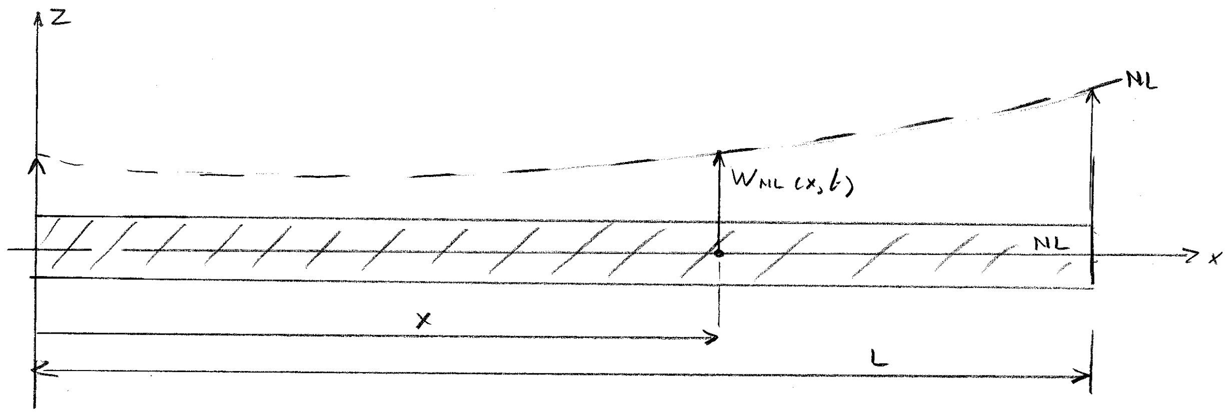
16.3 Assumed displacement field

16.4 Evaluation of the Lagrange expression

16.5 Design parameters

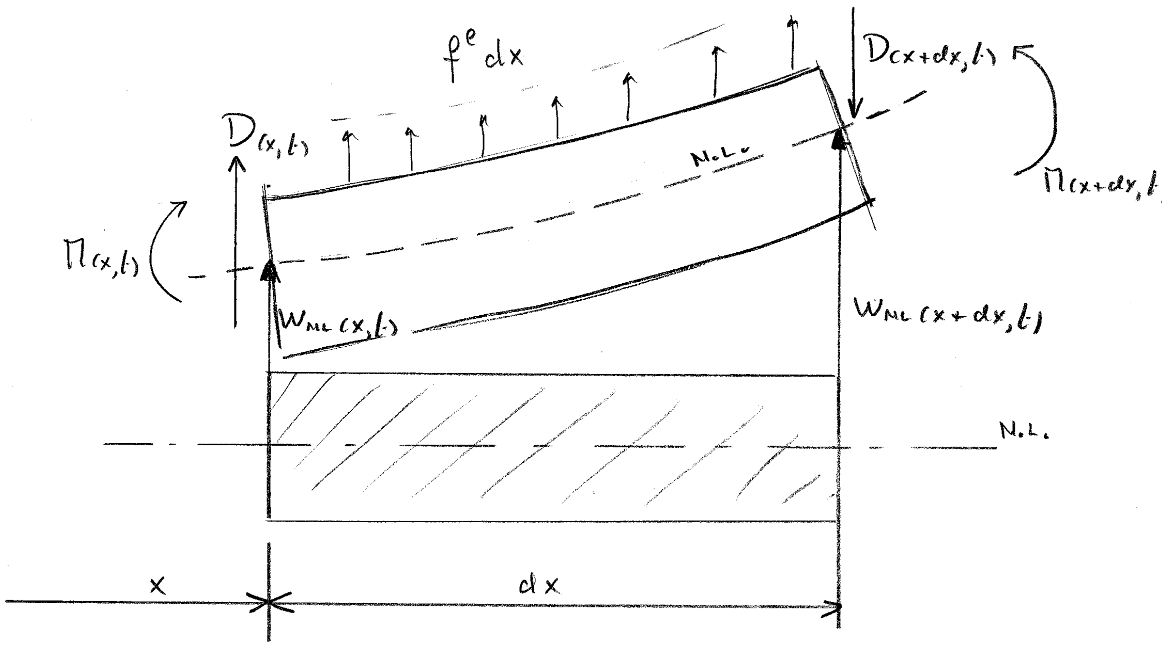
Dynamics of an infinitesimal mass element

Consider a single beam of finite length L:



NL = neutral line.

Consider an infinitesimally small beam element:



(i) Newton, vertical:

$$dm.\frac{\partial^2}{\partial t^2}\{W_{NL}(x+dx/2,t)\} = D(x,t) - D(x+dx,t) + f^e.dx$$

(ii) Ignoring angular acceleration; Euler:

$$-M(x,t) - D(x+dx,t) \cdot dx + M(x+dx,t) = 0$$

Expand:

(i) $$dm.\frac{\partial^2}{\partial t^2}\left\{W_{NL}(x,t) + \frac{\partial W_{NL}}{\partial x}.\frac{dx}{2}\right\} =$$

$$= \cancel{D(x,t)} - \left\{\cancel{D(x,t)} + \frac{\partial D}{\partial x}.dx\right\} + f^e.dx$$

$$\Rightarrow dm.\frac{\partial^2 W_{NL}}{\partial t^2} = -\frac{\partial D}{\partial x}.dx + f^e.dx$$

and:

(ii) $$-\cancel{M(x,t)} - \left\{D(x,t) + \frac{\partial D}{\partial x}.dx\right\}dx + \left\{\cancel{M(x,t)} + \frac{\partial M}{\partial x}.dx\right\} = 0$$

$$\Rightarrow D = \frac{\partial M}{\partial x}.$$

Reformulation (d'Alembert's Principle):

$$\boxed{\underbrace{\left(-\frac{\partial^2 w_{NL}}{\partial t^2}\,dm\right)}_{\text{inertia force}} \;\underbrace{-\frac{\partial D}{\partial x}\cdot dx}_{\substack{\text{net normal force}\\ \text{on cross section}}} + \underbrace{f^e\cdot dx}_{\text{surface force.}} = 0.}$$

Lagrange form of d'Alembert's Principle:

$$\boxed{\int_B \delta w_{NL}\left(-\frac{\partial^2 w_{NL}}{\partial t^2}\,dm - \frac{\partial D}{\partial x}\cdot dx + f^e\cdot dx\right) = 0} \tag{4}$$

where $dm = \rho . A . dx .$

$$\boxed{\text{Development of the Lagrange expression.}}$$

Recall:

$$\boxed{\int_B \delta w_{NL} \cdot \left(-\frac{\partial^2 w_{NL}}{\partial t^2}\, dm - \frac{\partial D}{\partial x} \cdot dx + f^e \cdot dx \right) = 0}$$

Hence forth, drop the index "NL".

<u>Consider</u> $\displaystyle \int_B \delta w \cdot \frac{\partial D}{\partial x} \cdot dx = \int_B \delta w \cdot dD$

Partial integration: $\displaystyle = \delta w . D \Big|_0^L - \int D . d(\delta w)$

$$= \delta w . D \Big|_0^L - \int D . \delta(dw)$$

Note: $\displaystyle \int D . \delta(dw) = \int \frac{\partial M}{\partial x} \cdot \delta\left(\frac{\partial w}{\partial x} \cdot dx \right)$

$$= \int \delta\left(\frac{\partial w}{\partial x} \right) \cdot \frac{\partial M}{\partial x} \cdot dx = \int \delta\left(\frac{\partial w}{\partial x} \right) dM$$

$$\int \delta\left(\frac{\partial w}{\partial x}\right) dM = \delta\left(\frac{\partial w}{\partial x}\right) \cdot M \Big|_0^L - \int M \, . \, d\left\{\delta\left(\frac{\partial w}{\partial x}\right)\right\}$$

$$= \delta\left(\frac{\partial w}{\partial x}\right) . M \Big|_0^L - \int M \, . \, \delta\left\{d\left(\frac{\partial w}{\partial x}\right)\right\}$$

$$= \delta\left(\frac{\partial w}{\partial x}\right) . M \Big|_0^L - \int M \, . \, \delta\left(\frac{\partial^2 w}{\partial x^2} dx\right)$$

Collecting results:

$$\int \delta w \cdot \frac{\partial D}{\partial x} . dx = \left[\delta w . D - \delta\left(\frac{\partial w}{\partial x}\right) . M\right]\Big|_0^L + \int_0^L M . \delta\left(\frac{\partial^2 w}{\partial x^2}\right) dx$$

Substitute: $M = E.I. \dfrac{\partial^2 w}{\partial x^2}$

$$\Rightarrow \int \delta w . \frac{\partial D}{\partial x} . dx = \left[\delta w . D - \delta\left(\frac{\partial w}{\partial x}\right) M\right]\Big|_0^L + EI \int_0^L \frac{\partial^2 w}{\partial x^2} . \delta\left(\frac{\partial^2 w}{\partial x^2}\right) . dx$$

Substitute this result into the Lagrange expression:

$$\boxed{\begin{aligned} & -\int \delta w \cdot \frac{\partial^2 w}{\partial t^2}\, dm \quad - E I \int_0^L \delta\left(\frac{\partial^2 w}{\partial x^2}\right) \cdot \left(\frac{\partial^2 w}{\partial x^2}\right) \cdot dx \; + \\ & \quad + \left[-\delta w \cdot D + \delta\left(\frac{\partial w}{\partial x}\right) \cdot M \right]_0^L \; + \int_0^L \delta w \cdot f^e \cdot dx = 0 \end{aligned}}$$

Assumed displacement field.

(i) Consider first the static case, with end-loads but no surface load $f^e \cdot dx$.

The equation of motion now reads:

$$\frac{\partial^2 w}{\partial t^2} dm = -\frac{\partial D}{\partial x} \cdot dx + f^e \cdot dx$$

with $D = \dfrac{\partial M}{\partial x}$.

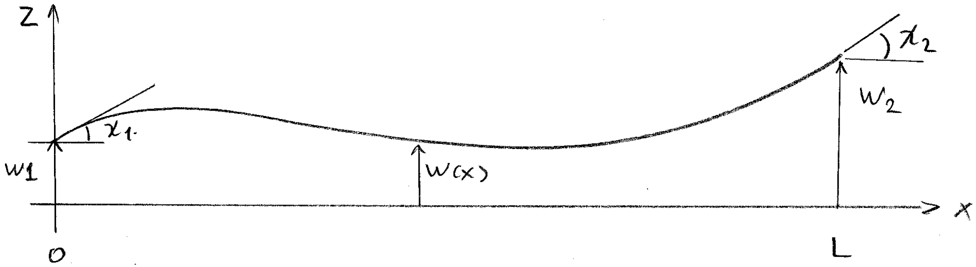
Hence: $\dfrac{\partial D}{\partial x} = 0 \quad \Rightarrow \quad \dfrac{\partial^2 M}{\partial x^2} = 0$

Recall: $M = EI \dfrac{\partial^2 w}{\partial x^2} \quad \Rightarrow \quad \dfrac{\partial^4 w}{\partial x^4} = 0$

General solution:

$$w(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$$

Express the coefficients $a_0 \dots a_3$ in terms of the displacements at the ends ("nodes"):



Notation: $\begin{cases} w_1 \triangleq W(0) \quad ; \quad w_2 \triangleq W(L) \\ \chi_1 \triangleq \dfrac{dw}{dx}(0) \quad ; \quad \chi_2 \triangleq \dfrac{dw}{dx}(0) \end{cases}$

One then has:

$$W(0) = a_0 \doteq w_1$$

$$W(L) = a_0 + a_1 . L + a_2 . L^2 + a_3 . L^3 \doteq w_2$$

$$\frac{dw}{dx}(0) = a_1 \doteq \chi_1$$

$$\frac{dw}{dx}(L) = a_1 + 2 a_2 L + 3 a_3 . L^2 \doteq \chi_2$$

Four scalar equations in $a_0 \dots a_3$. Solve!

One obtains:

$$w(x) = [\phi_1, \phi_2, \phi_3, \phi_4] \begin{bmatrix} w_1 \\ \chi_1 \cdot L \\ w_2 \\ \chi_2 \cdot L \end{bmatrix}$$

where:
$$\phi_1(x) = 1 - 3(x/L)^2 + 2(x/L)^3$$
$$\phi_2(x) = (x/L) - 2(x/L)^2 + (x/L)^3$$
$$\phi_3(x) = 3(x/L)^2 - 2(x/L)^3$$
$$\phi_4(x) = -(x/L)^2 + (x/L)^3$$

Define:
$$\bar{\phi}_{(x)} \hat{=} [\phi_1, \phi_2, \phi_3, \phi_4]^T$$

$$\bar{u}_e \hat{=} [w_1, \chi_1 L, w_2, \chi_2 L]^T$$

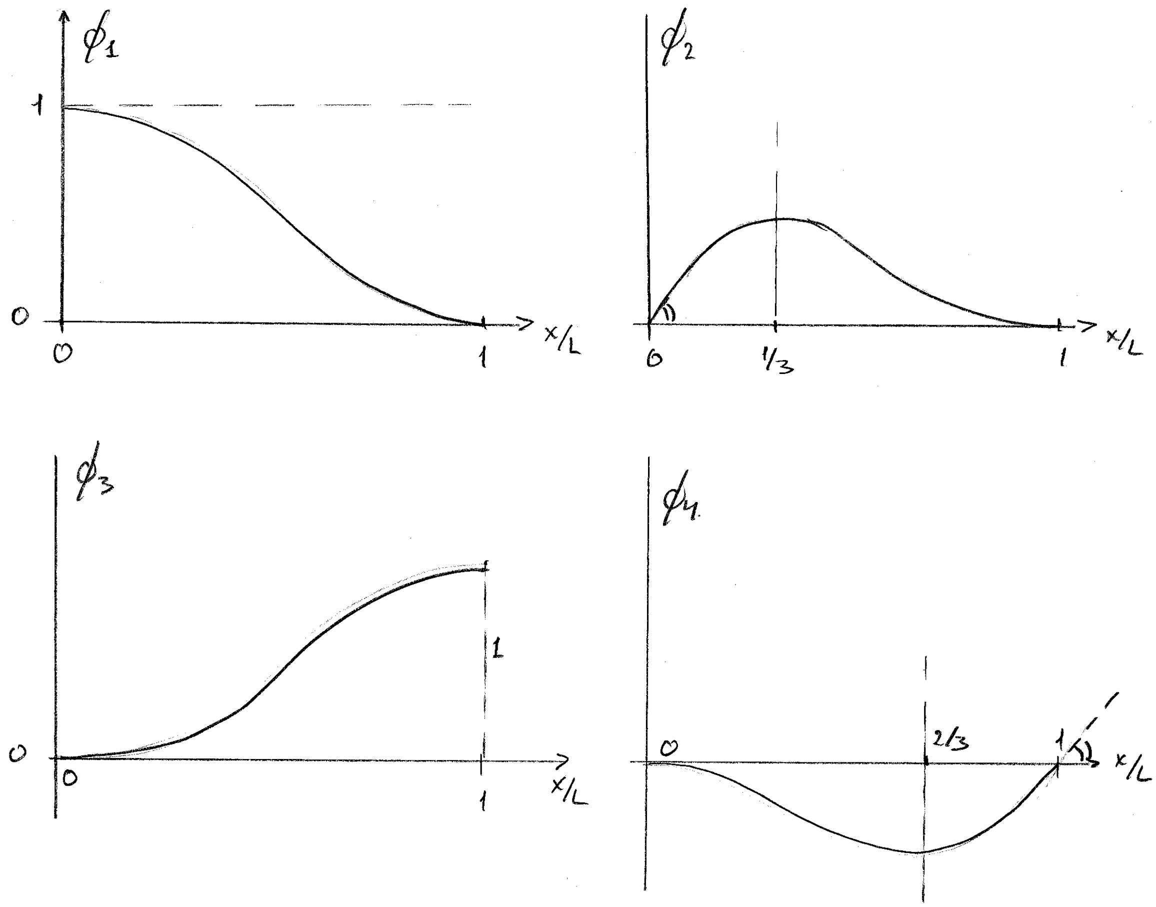
(vector of nodal displacements)

Hence:

$$w(x) = \bar{\phi}_{(x)}^T \, \bar{U}_e \qquad \Delta.$$

"displacement field" (statics)

$\vec{\phi}$ = vector of shape functions.



Linear interpolation, through the "amplification" factors w_1, $\theta_1 L$, w_2 and $\theta_2 L$.

(ii) Next, consider the <u>dynamic case</u>.

Now: $w_1 = w_1(t)$ $\qquad w_2 = w_2(t)$

$\chi_1 = \chi_1(t)$ $\qquad \chi_2 = \chi_2(t)$.

Assume that the actual displacement field may be approximated by the static displacement field:

$$\boxed{u(x,t) \cong \underbrace{\bar{\phi}^T_{(x)}}_{\text{from static analysis.}} \; \bar{U}_e(t)}$$

Good approximation if the length of the finite element is sufficiently small.

Evaluation of the Lagrange expression.

Recall:

$$\boxed{-\int \delta w \cdot \frac{\partial^2 w}{\partial t^2} \cdot dm \; - EI \cdot \int \delta\left(\frac{\partial^2 w}{\partial x^2}\right) \cdot \left(\frac{\partial^2 w}{\partial x^2}\right) \cdot dx \; + \left[-\delta w \cdot D + \delta\left(\frac{\partial w}{\partial x}\right) \cdot M\right]_0^L + \int \delta w \cdot f^e \cdot dx = 0.}$$

Substitute: $\quad w(x,t) = \bar{\phi}^T_{(x)} \; \bar{u}_{e(t)}$

Hence: $\quad \dfrac{\partial^2 w}{\partial t^2} = \bar{\phi}^T \ddot{\bar{u}}_e \;, \qquad \dfrac{\partial w}{\partial x} = \dfrac{d\bar{\phi}^T}{dx} \bar{u}_e$

$$\frac{\partial^2 w}{\partial x^2} = \frac{d^2\bar{\phi}^T}{dx^2} \bar{u}_e \;, \qquad \delta w = \bar{\phi}^T \delta \bar{u}_e.$$

$$\delta\left(\frac{\partial w}{\partial x}\right) = \frac{d\bar{\phi}^T}{dx} \delta \bar{u}_e$$

Substitute:

(i) $$\delta W^{inertia} \hat{=} - \int \delta w \cdot \frac{\partial^2 w}{\partial t^2}\, dm =$$

$$= - \int (\bar{\phi}^T \cdot \delta \bar{u}_e) \cdot (\bar{\phi}^T \ddot{\bar{u}}_e)\, dm$$

$$= - \int (\delta \bar{u}_e^T \bar{\phi}) \cdot (\bar{\phi}^T \ddot{\bar{u}}_e)\, dm$$

$$= - (\delta \bar{u}_e)^T \int \bar{\phi}\, \bar{\phi}^T dm \;\; \ddot{\bar{u}}_e$$

$$= - (\delta \bar{u}_e)^T M \bar{U}_e$$

where $M \hat{=} \int \bar{\phi}\, \bar{\phi}^T dm$

"generalized mass matrix".

(ii) $$\delta W^{strain} \hat{=} EI \int \delta \left(\frac{\partial^2 w}{\partial x^2} \right) \cdot \frac{\partial^2 w}{\partial x^2} \cdot dx$$

$$= EI \cdot \int \left(\frac{d^2 \bar{\phi}^T}{dx^2} \delta \bar{u}_e \right) \cdot \left(\frac{d^2 \bar{\phi}^T}{dx^2} \bar{u}_e \right) \cdot dx$$

$$= EI \int \left(\delta \bar{u}_e^T \frac{d^2 \bar{\phi}}{dx^2} \right) \cdot \left(\frac{d^2 \bar{\phi}^T}{dx^2} \bar{u}_e \right) \cdot dx$$

$$= (\delta \bar{u}_e)^T \quad EI \int \frac{d^2\bar{\phi}}{dx^2} \frac{d^2\bar{\phi}}{dx^2}^T dx \quad \bar{u}_e$$

$$= (\delta \bar{u}_e)^T K \bar{u}_e$$

$$\text{where} \quad K \triangleq EI \int_0^L \frac{d^2\bar{\phi}}{dx^2} \cdot \frac{d^2\bar{\phi}}{dx^2}^T dx$$

"generalized stiffness matrix".

(iii) $$\delta W^{end} \triangleq \left[-\delta w \cdot D + \delta\left(\frac{\partial w}{\partial x}\right) M \right]_0^L$$

$$= \left\{ -\delta w_2 . D(L,t) + \delta \chi_2 . M(L,t) \right\} +$$

$$- \left\{ -\delta w_1 \, D(0,t) + \delta \chi_1 . M(0,t) \right\}$$

$$= [\delta w_1, \; L \cdot \delta \chi_1, \; \delta w_2, \; L \cdot \delta \chi_2] \begin{bmatrix} D(0,t) \\ -\frac{1}{L} M(0,t) \\ -D(L,t) \\ +\frac{1}{L} M(L,t) \end{bmatrix}$$

$$= (\delta \bar{u}_e)^T \bar{F}_T^{\,e}$$

$$\text{where} \quad \bar{F}_T^{\,e} \triangleq \begin{bmatrix} D(0,t) \\ -\frac{1}{L} M(0,t) \\ -D(L,t) \\ +\frac{1}{L} M(L,t) \end{bmatrix}$$ generalised end force.

(iv) $\delta W^f \triangleq \int \delta w \cdot f^e \cdot dx$

$$= \int (\bar{\phi}^T \delta \bar{u}_e)\, f^e dx$$

$$= \int (\delta \bar{u}_e^T\, \bar{\phi})\, f^e dx$$

$$= (\delta \bar{u}_e)^T \int_0^L \bar{\phi}\, f^e dx$$

$$= (\delta \bar{u}_e)^T\, \bar{F}_s^e$$

where $\bar{F}_s^e \triangleq \int_0^L \bar{\phi}\, f^e \cdot dx$

"generalized surface force".

Collect results:

$$-(\delta \bar{u}_e)^T M \ddot{\bar{u}}_e - (\delta \bar{u}_e)^T K \bar{u}_e +$$

$$+ (\delta \bar{u}_e)^T \bar{F}_T^e + (\delta \bar{u}_e)^T \bar{F}_s^e = 0.$$

$$(\delta \bar{u}_e)^T [-M \ddot{\bar{u}}_e - K \bar{u}_e + \bar{F}_T^e + \bar{F}_s^e] = 0.$$

But: all elements of $\bar{U}e$ are small but arbitrary.
Hence:

$$\boxed{M\ddot{\bar{U}}e + K\bar{u}e = \bar{F}_T^e + \bar{F}_s^e} \qquad \triangleleft$$

Equation of motion for a single beam element.

<u>We shall now evaluate M and K and $\bar{F}_s^e$</u>

$$(i)\quad M \triangleq \int \bar{\phi}\,\bar{\phi}^T dm = \int \begin{bmatrix}\phi_1\\ \vdots \\ \phi_4\end{bmatrix}[\phi_1 \cdots \phi_4](\rho A\,dx)$$

$$= \rho\cdot A\cdot \int \begin{bmatrix} \phi_1^2 & \phi_1\cdot\phi_2 & \phi_1\cdot\phi_3 & \phi_1\cdot\phi_4 \\ \phi_2\cdot\phi_1 & - - - - & & \\ \phi_3\cdot\phi_1 & - - - & & \\ \phi_4\cdot\phi_4 & - - - & & \phi_4^2 \end{bmatrix} dx$$

$$\boxed{M = \frac{\rho\cdot A\cdot L}{420}\cdot \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{Symmetric} & & & 4 \end{bmatrix}} \qquad \triangleleft$$

(ii) $K \triangleq EI \int \bar{\phi}'' \bar{\phi}''^T dx = EI \int \begin{bmatrix} \phi_1'' \\ \vdots \\ \phi_4'' \end{bmatrix} [\phi_1'' \cdots \phi_4''] \, dx$

$$= EI \int \begin{bmatrix} (\phi_1'')^2 & \cdots & \phi_1'' \phi_4'' \\ \vdots & & \\ \phi_4'' \phi_1'' & \cdots & (\phi_4'')^2 \end{bmatrix} dx$$

$$\boxed{K = \frac{EI}{L^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ \text{Symmetric} & & & 4 \end{bmatrix}} \quad \triangleleft$$

(iii) Next, evaluate the generalized force vector $\bar{F}_s^e$.

$$\bar{F}_s^e \triangleq \int_0^L \bar{\phi} \, f^e \, dx = \int_0^L \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_4 \end{bmatrix} f^e dx.$$

$$\boxed{\bar{F}_s^e = \int_0^L \begin{bmatrix} \phi_1 \cdot f^e \\ \phi_2 \cdot f^e \\ \phi_3 \cdot f^e \\ \phi_4 \cdot f^e \end{bmatrix} dx} \quad \triangle.$$

where $f^e = f^e(x,t)$ in general.

Design parameters

Once the equations of motion have been established, one attempts to find $\bar{v}_e(t)$.

From the solution of the equations of motion one derives:

displacement field: $$w(x,t) = \bar{\phi}^T_{(x)}\, \bar{v}_e(t)$$

strain field: $$\varepsilon(x,t,\xi) = \xi \left\{ \bar{\phi}''^T_{(x)}\, \bar{v}_e(t) \right\}$$

stress field: $$\sigma(x,t,\xi) = E.\, \xi \left\{ \bar{\phi}''^T_{(x)}\, \bar{v}_e(t) \right\}$$

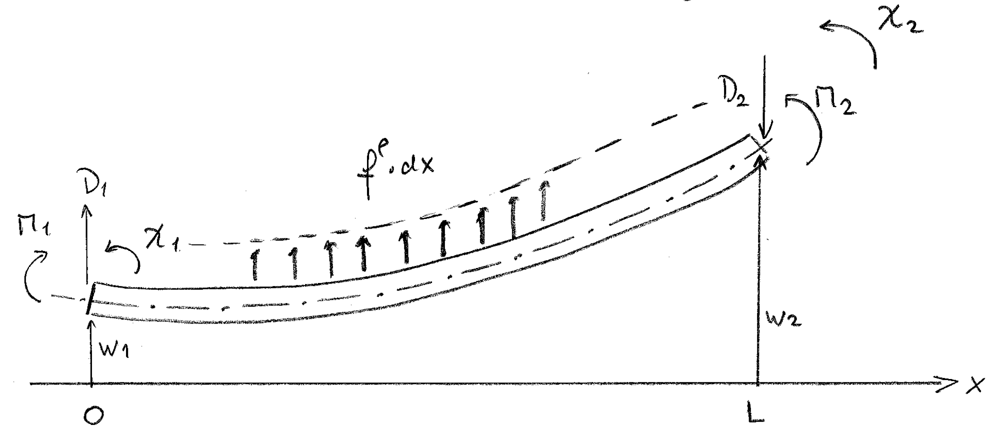
transverse force: $$D(x,t) = E.I. \left\{ \bar{\phi}'''^T_{(x)}\, \bar{v}_e(t) \right\}$$

transverse moment: $$M(x,t) = E\,I. \left\{ \bar{\phi}''^T_{(x)}\, \bar{v}_e(t) \right\}$$

shear in plane parallel to the neutral plane:

$$\tau(x,t,\xi) = \frac{E}{b} \int_{\xi}^{\xi_{max}} \xi . dA . \left\{ \bar{\phi}'''^T_{(x)}\, \bar{v}_e(t) \right\}.$$

Executive Summary



Lagrange form:

$$\int \delta w . \left\{ -(\rho . A . dx) . \frac{\partial^2 w}{\partial t^2} - dD + f^e . dx \right\} = 0$$

$$\Rightarrow \quad - \int \delta w . \frac{\partial^2 w}{\partial t^2} . dm - EI . \int \delta w^{(2)} . w^{(2)} . dx +$$

$$+ \left[-\delta w . D + \delta w^{(1)} . M \right]_0^L + \int \delta w . f^e . dx = 0.$$

Assumed displacement field: static

$$\Rightarrow \quad w(x,t) = \phi_1 . w_{1(t)} + \phi_2 . (L . \chi_{1(t)}) +$$

$$+ \phi_3 . w_{2(t)} + \phi_4 . (L . \chi_{2(t)})$$

$$= \bar{\phi}^T_{(x)} \, \bar{U}_e(t).$$

where $\phi_i(x)$ are cubic polynomials.

Substitute $\Rightarrow \quad M\ddot{\bar{U}}_e + K\bar{U}_e = \bar{F}_{(t)}$

where:

$$\bar{U}_e \triangleq \begin{bmatrix} W_1 \\ L\cdot\theta_1 \\ W_2 \\ L\cdot\theta_2 \end{bmatrix} \qquad M \triangleq \int \bar{\phi}\,\bar{\phi}^T\,dm$$

$$K \triangleq EI \int \bar{\phi}''\,\bar{\phi}''^T\,dx$$

$$\bar{F} = \bar{F}_{ends} + \int \bar{\phi}\cdot f^e\cdot dx$$

Work out:

$$M = \frac{\rho\cdot A\cdot L}{420}\begin{bmatrix} 156 & 22 & 54 & 13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{Symm.} & & & 4 \end{bmatrix}$$

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ \text{Symm.} & & & 4 \end{bmatrix}$$

$$F = \begin{bmatrix} D_1 & + & \int \phi_1 \cdot f^e \cdot dx \\ -M_1/L & + & \int \phi_2 \cdot f^e \cdot dx \\ -D_2 & + & \int \phi_3 \cdot f^e \cdot dx \\ M_2/L & + & \int \phi_4 \cdot f^e \cdot dx \end{bmatrix}$$

Finally:

$$\bar{v}_e(t) \Rightarrow w(x,t) \Rightarrow \varepsilon(x,\xi,t) \Rightarrow \sigma(x,\xi,t)$$

$$\Rightarrow M(x,t)$$

$$\Rightarrow D(x,t)$$

$$\Rightarrow \tau(x,\xi,t)$$

17. FINITE ELEMENT MODELLING:

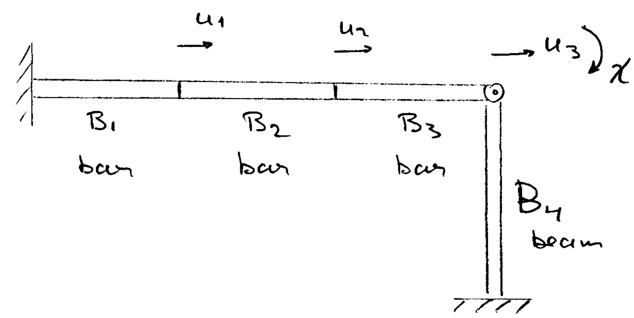
STRUCTURES WITH MULTIPLE ELEMENTS

17.1 Structures with multiple elements

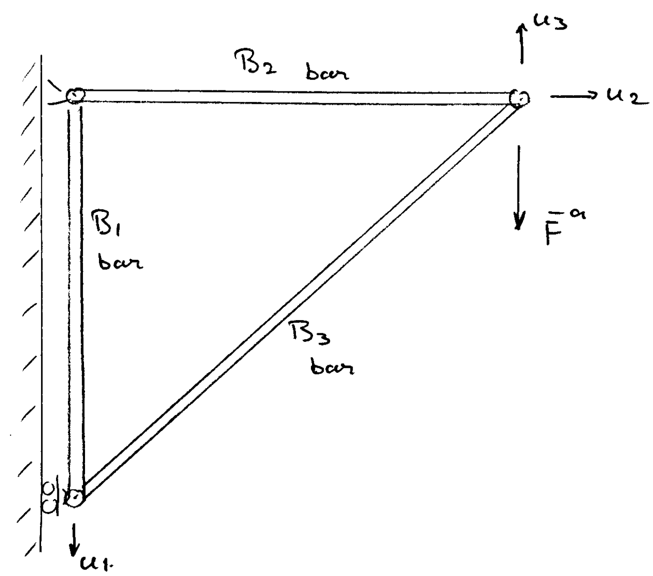
17.2 From local coordinates to global coordinates

Structures with multiple elements

Example:



Example:



element "nodal points".

Lagrange form of d'Alembert's principle:

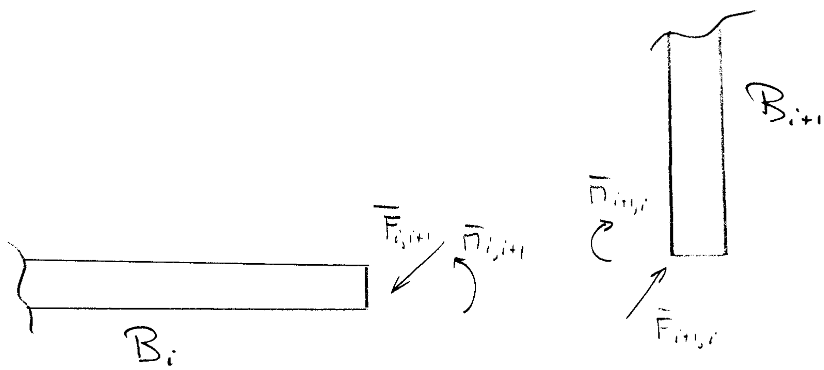
$$\int_{\text{system}} (\delta \bar{R})^T \left(-\ddot{\bar{R}}\, dm + d\bar{f}^{\,int} + d\bar{f}^{\,appl} \right) = 0$$

$$\boxed{\int_{\text{system}} (\delta \bar{R})^T (\cdots) = \sum_j \int_{B_j} (\delta \bar{R})^T (\cdots) = 0.}$$

Hence: determine $\int_{B_j}$ (excluding contributions from constraint loads)

$\Rightarrow$ add: $\displaystyle\int_{\text{system}} = \sum_j \int_{B_j} = 0.$

Note on constraint loads in a rigid joint:

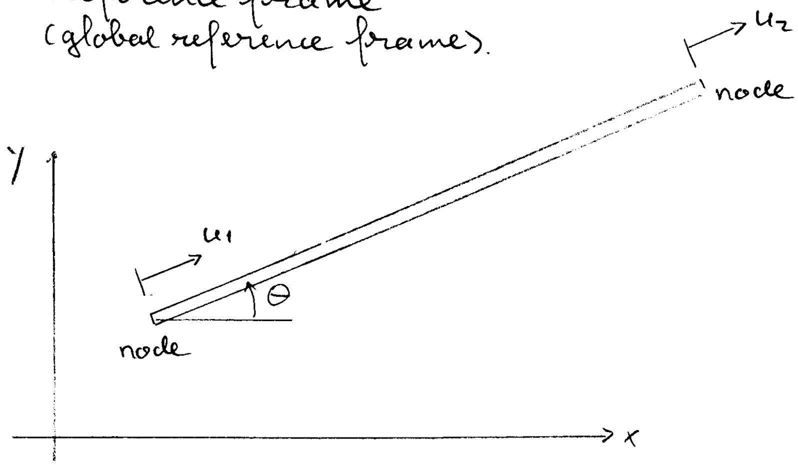


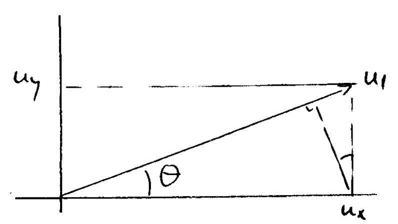
$$\delta W^{\text{constraint loads}} = \delta \bar{R}^T \bar{F}_{i,i+1} + \delta \bar{\Theta}^T \bar{M}_{i,i+1} +$$
$$+ \delta \bar{R}^T \bar{F}_{i+1,i} + \delta \bar{\Theta}^T \bar{M}_{i+1,i}$$
$$= \delta \bar{R}^T (\bar{F}_{i,i+1} + \bar{F}_{i+1,i}) +$$
$$+ \delta \bar{\Theta}^T (\bar{M}_{i,i+1} + \bar{M}_{i+1,i}) \overset{!}{=} 0$$

If the joint contains a free hinge axis, then $\delta \Theta_{hinge} M_{hinge} = 0$ again (as $M_{hinge} = 0$).

From local coordinates to global coordinates

Transform displacements of nodes of body B_i, to displacements along axes of system inertial reference frame
(global reference frame).





$$u_1 = u_{1,x} \cos\theta + u_{2,y} \sin\theta$$

$$u_2 = u_{2,x} \cdot \cos\theta + u_{2,y} \sin\theta$$

$$u_1 = [\cos\theta, \sin\theta] \begin{bmatrix} u_x \\ u_y \end{bmatrix}_1$$

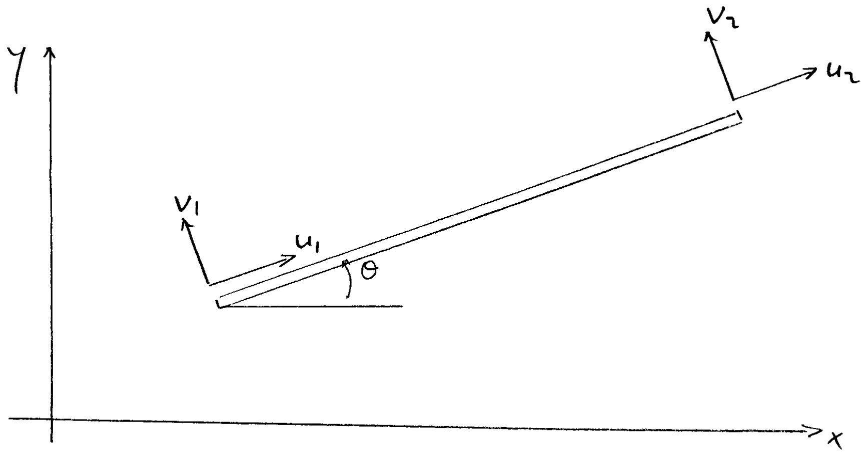
$$u_2 = [\cos\theta, \sin\theta] \begin{bmatrix} u_x \\ u_y \end{bmatrix}_2$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta \end{array}\right] \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \end{bmatrix}$$

displacements in local (body fixed) ref. frame

displacements in global ref. frame

General case: also transverse displacement



$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \left[\begin{array}{cc|cc} \cos\theta & \sin\theta & \multicolumn{2}{c}{O_{2\times 2}} \\ -\sin\theta & \cos\theta & & \\ \hline \multicolumn{2}{c|}{O_{2\times 2}} & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{array}\right] \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \end{bmatrix}$$

$$\bar{u} = T(\theta)\, \bar{u}_{\text{global ref. frame}}$$

Body B_i : $\boxed{\bar{u}_i = T_{\theta_i} \cdot \bar{u}_{g,i}}$

Then, define the vector of global system coordinates, $\bar{U}$ (containing the coordinates for all bodies B_i)

$$\boxed{\bar{u}_{g,i} = S_i\, \bar{U}}$$

Hence: $\quad \bar{u}_j = T_{\bar{\theta}_j} \, \bar{u}_{g,j} = T_{\bar{\theta}_j} \, S_j \, \bar{U}$

local coord. in local frame — local coord. in global frame — all coordinates of entire system, in global frame

$$\boxed{\bar{U}_j = T_j \, \bar{U}} \qquad \text{for body } B_j.$$

Now.

for a single element:

$$(\delta \bar{u})^T [-M \ddot{\bar{u}} - K \bar{u} + \bar{F}^a] = 0$$

for a structure consisting of multiple elements:

$$\sum_j (\delta \bar{u}_j)^T [-M_j \ddot{\bar{u}}_j - K \bar{u}_j + \bar{F}_j^a] = 0$$

$$\bar{u}_j = T_j \cdot \bar{U}$$

$$\Rightarrow \dot{\bar{u}}_j = \frac{dT_j}{dt} \bar{U} + T_j \dot{\bar{U}}$$

$$= \left[\frac{dT_j}{d\bar{\theta}_j} \frac{d\bar{\theta}_j}{d\bar{U}} \dot{\bar{u}} \right] \bar{U} + T_j \dot{\bar{U}}$$

$$= T_j \cdot \dot{\bar{U}} + \text{h.o.t.}$$

Similarly:

$$\delta \bar{u}_i = T_i \, \delta \bar{U} + \text{h.o.t.}$$

$$\ddot{\bar{u}}_i = T_i \, \ddot{\bar{U}} + \text{h.o.t.}$$

Substitute:

$$\sum_i (T_i \, \delta \bar{U})^T \left[-M_i (T_i \ddot{\bar{U}}) - K_i (T_i \bar{U}) + \bar{F}_i^a \right] = 0$$

$$(\delta \bar{U})^T \sum_i \left[-T_i^T M_i T_i \, \ddot{\bar{U}} - T_i^T K_i T_i \, \bar{U} + T_i^T \bar{F}_i^a \right] = 0$$

The equations can be rewritten as:

$$\boxed{M \ddot{\bar{U}} + K \bar{U} = \bar{F}^a}$$

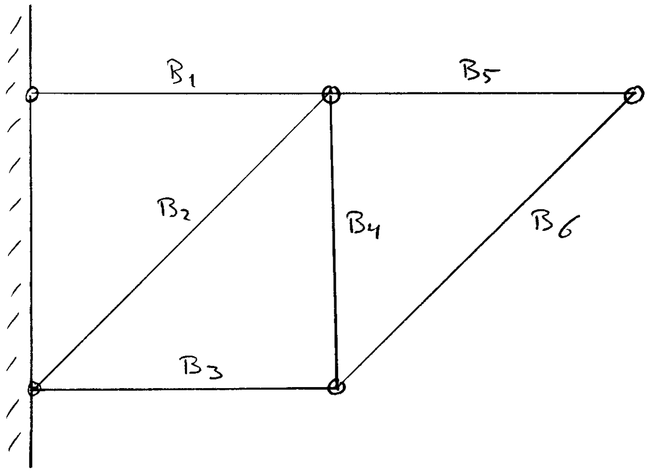
where:

$$M = \sum_i T_i^T M_i T_i$$

$$K = \sum_i T_i^T K_i T_i$$

$$\bar{F}^a = \sum_i T_i^T \bar{F}_i^a$$

Executive Summary



Elements B_i are connected at nodes.

$$\int_{system} (\delta \bar{R})^T (-\ddot{\bar{R}}\, dm + d\bar{f}) = \sum_{i=1}^{N} \int_{B_i} (\delta \bar{R})^T (\cdots) = 0$$

$$B_i : \quad M_i, \; K_i, \; \bar{\bar{F}}_i, \; \bar{U}_i$$

Relate local coordinates $\bar{u}_i$ to global coordinates $\bar{U}$:

$$\bar{U}_i = T_i \, \bar{U}$$

$$\Rightarrow \quad M \ddot{\bar{U}} + K \bar{U} = \bar{\bar{F}}_{(t)}$$

where:

$$M = \sum_i T_i^T M_i T_i \; ; \; K = \sum_i T_i^T K_i T_i$$

$$\bar{F} = \sum_i T_i^T \bar{\bar{F}}_i$$

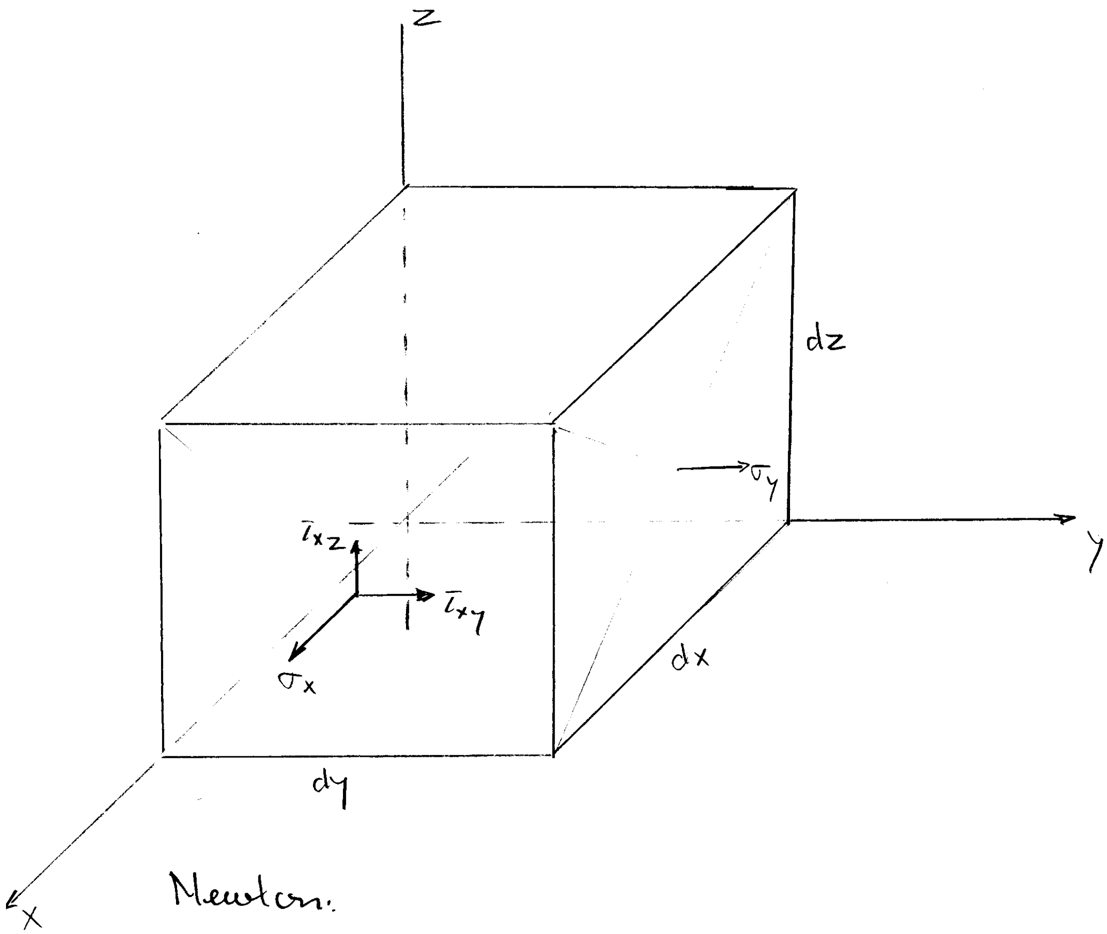
18. FINITE ELEMENT MODELLING:

THREE-DIMENSIONAL MASSIVE BODY

18.1 Dynamics of a three-dimensional, infinitesimal mass element

18.2 Structure with material damping

Dynamics of a three-dimensional, infinitesimal mass element.



Newton:

$$dm \cdot \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x^e \right) dV$$

$$dm \cdot \frac{\partial^2 v}{\partial t^2} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y^e \right) dV$$

$$dm \cdot \frac{\partial^2 w}{\partial t^2} = \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z^e \right) dV$$

Lagrange formulation:

$$\int \Big[\delta u \Big\{ - dm \cdot \frac{\partial^2 u}{\partial t^2} + \Big(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x^e \Big) dV \Big\} +$$

$$+ \delta v \{ \cdots \} + \delta w \{ \cdots \} \Big] = 0$$

This can be rewritten in the form:

$$\int \Big[\delta u \Big(- \frac{\partial^2 u}{\partial t^2} dm + f_x^e dV \Big) + \delta v (\cdots) + \delta w (\cdots) \Big] +$$

$$- \int \Big[\sigma_x . \delta \varepsilon_x + \sigma_y . \delta \varepsilon_y + \sigma_z . \delta \varepsilon_z +$$

$$+ \tau_{xy} . \delta \gamma_{xy} + \tau_{xz} . \delta \gamma_{xz} + \tau_{yz} . \delta \gamma_{yz} \Big] = 0$$

(i) Define: $\bar{u} \triangleq [u, v, w]^T$

$\bar{f}^e \triangleq [f_x^e, f_y^e, f_z^e]^T$

The integral on the first line then reads:

$$\int (\delta \bar{u})^T \big(- \ddot{\bar{u}}\, dm + \bar{f}^e dV \big) .$$

(ii) Define: $\bar{\sigma} \triangleq [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}]^T$

$$\bar{\varepsilon} \triangleq [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]^T$$

The second integral then reads:

$$-\int \bar{\sigma}^T \delta\bar{\varepsilon}\, dV.$$

$\Rightarrow$ Collecting results:

$$\boxed{\int (\delta\bar{u})^T \left(-\ddot{\bar{u}}\, dm + \bar{f}^e dV\right) - \int \bar{\sigma}^T \delta\bar{\varepsilon}\, dV = 0} \quad \triangleleft$$

Introduce: $\bar{u} = G\,\bar{v}_e$

$\quad$ system nodal coordinates

$$\bar{\sigma} = D\,\bar{\varepsilon} \quad \text{(generalized Hooke's law)}$$

$$\bar{\varepsilon} = B\,\bar{v}_e$$

where $B = B(x,y,z) = B(\bar{r})$

(i) $$\delta W^{inertial} = - \int (\delta \bar{u})^T \ddot{\bar{u}} \, dm$$

$$= - \int (G . \delta \bar{u}_e)^T (G . \ddot{\bar{U}}_e) \, dm$$

$$= - (\delta \bar{u}_e)^T \underbrace{\int G^T . G \, dm}_{\hat{=} M} \ddot{\bar{u}}_e$$

(ii) $$\delta W^{strain} = - \int \bar{\sigma}^T \delta \bar{\varepsilon} \, dV$$

$$= - \int (D . B . \bar{U}_e)^T (B . \delta \bar{u}_e) \, dV$$

$$= - \int (\delta \bar{u}_e^T B^T)(D B \bar{U}_e) \, dV$$

$$= - (\delta \bar{u}_e)^T \underbrace{\int B^T . D . B . dV}_{\hat{=} K} \bar{U}_e$$

(iii) $\delta W^{\text{ext. load}} \hat{=} \int (\delta \bar{u})^T \bar{f}^e \, dV$

$$= \int (G \cdot \delta \bar{u}_e)^T \bar{f}^e \, dV$$

$$= \int (\delta \bar{u}_e^T \, G^T) \bar{f}^e \, dV$$

$$= (\delta \bar{u}_e)^T \underbrace{\int G^T \bar{f}^e \, dV}_{\hat{=} \bar{F}^e}$$

Hence: $(\delta \bar{u}_e)^T [-M \ddot{\bar{v}}_e - K \bar{v}_e + \bar{F}^e] = 0$

$$\Rightarrow \boxed{M \ddot{\bar{v}}_e + K \cdot \bar{v}_e = \bar{F}^e.} \qquad \triangle$$

Exercise :

derive M, K for a single, finite beam element. (as a special case).

A concise reference for the three-dimensional case:

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Eindige Elementen Methode, Deel 2
Nijgh & Van Ditmar, Rijswijk, 1996.

$$\boxed{\text{Structure with material damping}}$$

Recall:

$$\boxed{\int (\delta \bar{u})^T \left(-\ddot{\bar{u}}\, dm + \bar{f}^e\, dV\right) - \int \bar{\sigma}^T \delta \bar{\varepsilon}\, dV = 0}$$

We then introduced:

$$\bar{u} = G\, \bar{U}_e$$

$$\bar{\sigma} = D\, \bar{\varepsilon}$$

$$\bar{\varepsilon} = B\, \bar{U}_e$$

To include damping, we use the following simple model:

$$\boxed{\bar{\sigma} = D_0\, \bar{\varepsilon} + D_1\, \dot{\bar{\varepsilon}}}$$

(generalized Voight model;
generalized Kelvin model).

(i) $\delta W^{\text{inertial}} = -\int (\delta \bar{u})^T \ddot{u}\, dm$

$$= -(\delta \bar{u}_e)^T M \ddot{\bar{U}}_e$$

as before.

(ii) $\delta W^{\text{external load}} = \int (\delta \bar{u})^T f^e\, dV$

$$= (\delta \bar{u}_e)^T \bar{F}^e$$ as before.

(iii) $\delta W^{\text{strain}} = \int \bar{\sigma}^T \delta \bar{\varepsilon}\, dV$

$$= \int (D_0 \bar{\varepsilon} + D_1 \dot{\bar{\varepsilon}})^T \delta \bar{\varepsilon}\, dV$$

$$= \int (D_0 B \bar{U}_e + D_1 B \dot{\bar{U}}_e)^T \delta (B \bar{U}_e)\, dV$$

$$= \int \{\delta (B \bar{U}_e)\}^T (D_0 B \bar{U}_e + D_1 B \dot{\bar{U}}_e)\, dV$$

$$= (\delta \bar{u}_e)^T \left[\left(\int B^T D_0 B\, dV \right) \bar{U}_e + \right.$$

$$\left. + \left(\int B^T D_1 B\, dV \right) \dot{\bar{U}}_e \right]$$

$$= (\delta \bar{u}_e)^T (K \bar{U}_e + C \dot{\bar{U}}_e)$$

C = "generalised damping matrix".

Collecting results:

$$(\delta \bar{u}_e)^T \left[-M \ddot{\bar{u}}_e + \bar{F}_s^e - (K \bar{u}_e + C \dot{\bar{u}}_e) \right] = 0$$

$$\Rightarrow \boxed{M \ddot{\bar{u}}_e + C \dot{\bar{u}}_e + K \bar{u}_e = \bar{F}_s^e}$$

<u>Remark</u> on the generalised Voight / Kelvin model.

Hooke's law:
$$\overline{\sigma}_{(t)} = D\,\overline{\varepsilon}_{(t)}$$

Hooke's law including time delay τ:
$$\overline{\sigma}_{(t)} = D\,\overline{\varepsilon}_{(t+\tau)}$$

Taylor expansion:
$$\overline{\varepsilon}_{(t+\tau)} = \overline{\varepsilon}_{(t)} + \dot{\overline{\varepsilon}}_{(t)}\,\tau + \mathcal{O}(\tau^2)$$

$$\Rightarrow \overline{\sigma}_{(t)} = D\,\overline{\varepsilon}_{(t)} + (\tau D)\,\dot{\overline{\varepsilon}}_{(t)} + \cdots$$

This model is a special case of the generalised Voight Kelvin model, with the constraint
$$D_1 = \tau D_0.$$

Executive Summary

Material not required for exam!

19. FINITE ELEMENT MODELLING: MODEL ORDER REDUCTION

Problem statement

Dynamics equations: $M\ddot{\bar{u}} + K\bar{u} = \bar{F}$

where $\dim \bar{u} = n$.

$\rightarrow$ eigen frequences $\omega_1 < \omega_2 \cdots < \omega_n$

Consider the case of large n.

n large $\Rightarrow$ large system of differential equations

Solution requires many operations per integration time step ($\bar{u}_k \rightarrow \bar{u}_{k+1}$).

n large $\Rightarrow$ ω_n large. (and not accurate, physically).

Numerical integration then requires reduction in time step. Hence, more operations per second real time.

To reduce computational effort, it is advantageous to reduce the dimension n of the dynamics equations

= "model-order reduction".

Static condensation (Guyan)

Let the coordinate vector consist of:

$\bar{U}_m$ "master coordinates", associated with appreciable local mass

$\bar{u}_s$ "slave coordinates", associated with negligible local mass.

$$\bar{u} = \begin{bmatrix} \bar{U}_m \\ \bar{U}_s \end{bmatrix}$$

The dynamics equations must then be re-ordered, to obtain

$$\begin{bmatrix} M_{mm} & M_{ms} \\ M_{ms}^T & M_{ss} \end{bmatrix} \begin{bmatrix} \bar{U}_m \\ \bar{U}_s \end{bmatrix}^{\cdot\cdot} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{ms}^T & K_{ss} \end{bmatrix} \begin{bmatrix} \bar{U}_m \\ \bar{U}_s \end{bmatrix} = \begin{bmatrix} \bar{F}_m \\ \bar{F}_s \end{bmatrix}$$

Consider the lower system of equations.

$$(M_{ms}^T \ddot{\bar{U}}_m + M_{ss} \ddot{\bar{U}}_s) + (K_{ms}^T \bar{U}_m + K_{ss} \bar{U}_s - \bar{F}_s) = \bar{0}$$

Approximation:

$$\underbrace{\| M_{ms}^T \ddot{\bar{U}}_m + M_{ss} \ddot{\bar{U}}_s \|}_{\text{dynamic correction term}} \ll \underbrace{\| K_{ms}^T \bar{U}_m + K_{ss} \bar{U}_s - F_s \|}_{\text{static}}$$

$$\Rightarrow \boxed{K_{ms}^T \cdot \bar{U}_m + K_{ss} \cdot \bar{U}_s - \bar{F}_s \hat{=} \bar{0}}$$

Solve: $\bar{U}_s = K_{ss}^{-1} (\bar{F}_s - K_{ms}^T \bar{U}_m)$

i.e.: if $\bar{U}_m(t)$ known, then $\bar{U}_s(t)$ known.

$$\bar{u} = \begin{bmatrix} \bar{U}_m \\ \bar{U}_s \end{bmatrix} = \begin{bmatrix} I_{m \times m} \\ -K_{ss}^{-1} K_{ms}^T \end{bmatrix} \bar{U}_m + \begin{bmatrix} O_{m \times n} \\ K_{ss}^{-1} \end{bmatrix} \bar{F}$$

$$\bar{U} = T_m \cdot \bar{U}_m + T_F \, \bar{F} \qquad \Delta$$

Lagrange form of d'Alembert principle gave:

$$(\delta \bar{u})^T (-M \ddot{\bar{U}} - K \bar{u} + \bar{F}) = 0$$

$$(T_m \delta \bar{u})^T \Big[-M \cdot (T_m \ddot{\bar{U}}_m + T_F \cdot \ddot{\bar{F}}) + $$
$$- K (T_m \bar{U}_m + T_F \cdot \bar{F}) + \bar{F} \Big] = 0$$

$$\Rightarrow \boxed{M_m \cdot \ddot{\bar{U}}_m + K_m \bar{U}_m = \bar{F}_m}$$

where:
$$\begin{cases} M_m \triangleq T_m^T M T_m \\ K_m \triangleq T_m^T K T_m \\ \bar{F}_m \triangleq -(T_m^T M T_F) \ddot{\bar{F}} - (T_m^T K T_F) \bar{F} + T_m^T \bar{F} \end{cases}$$

Note that $\dim \bar{U}_m < \dim \bar{U}$.

Working out these results gives:

$$M_m \triangleq T_m^T M T_m$$

$$= M_{mm} - M_{ms}(K_{ss}^{-1} K_{ms}^T) - \{M_{ms}(K_{ss}^{-1} K_{ms}^T)\}^T +$$

$$+ (K_{ss}^{-1} K_{ms}^T)^T M_{ss} (K_{ss}^{-1} K_{ms}^T) \qquad \triangleleft$$

$$K_m \triangleq T_m^T K T_m$$

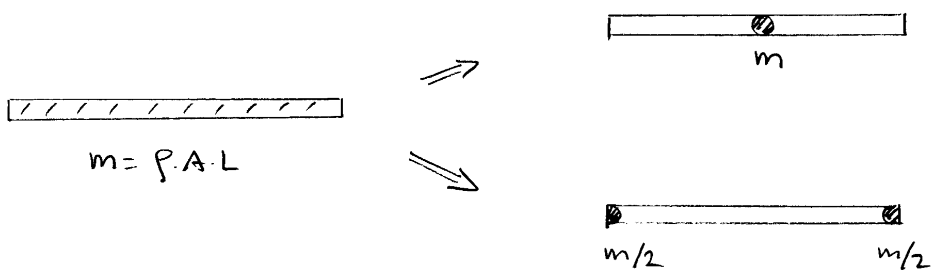
$$= K_{mm} - K_{ms} K_{ss}^{-1} K_{ms}^T \qquad \triangleleft$$

$$T_m^T M T_F = (M_{ms} - K_{ms} K_{ss}^{-1} M_{ss}) K_{ss}^{-1} \qquad \triangleleft$$

$$T_m^T K T_F = 0 \qquad \triangleleft$$

This procedure is also known as the Guyan / Irons reduction procedure.

Mass condensation.



Virtual work performed by inertia loads:
- contribution where mass is associated with linear displacement.
- no contribution where mass is associated with angular displacement.

Coordinates describing linear displacement: $\bar{u}_1$.

Coordinates describing angular displacement: $\bar{u}_2$

The dynamics equations are re-ordered, to obtain:

$$\begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}^{\cdot\cdot} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{U}_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix}$$

Consider the lower equation:

$$\boxed{K_{12}^T \bar{U}_1 + K_{22} \bar{U}_2 = \bar{F}_2}$$

Solve for $\bar{u}_2$:
$$\bar{U}_2 = K_{22}^{-1}\left(-K_{12}^T \bar{U}_1 + \bar{F}_2\right)$$

Substitute in upper equation:

$$M_{11}\ddot{\bar{U}}_1 + \left(K_{11} - K_{12}K_{22}^{-1}K_{12}^T\right)\bar{U}_1 =$$
$$= [F_1 \cdot - K_{12}K_{22}^{-1}\bar{F}_2)$$

$$\Rightarrow \boxed{M_1\ddot{\bar{U}}_1 + K_1\bar{U}_1 = \bar{F}_1^*}$$

where:
$$\begin{cases} M_1 \hat{=} M_{11} \\ K_1 \hat{=} K_{11} - K_{12}K_{22}^{-1}K_{12}^T \\ \bar{F}_1^* \hat{=} \bar{F}_1 - K_{12}.K_{22}^{-1}\bar{F}_2 \end{cases}$$

where $\dim \bar{u}_1 < \dim \bar{u}$

Solve for $\bar{u}_1(t) \Rightarrow$ find $\bar{u}_2(t)$

Finally: $\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{U}_2 \end{bmatrix}$.

Note: mass condensation is a special case of static (Guyan) condensation, by setting $M_{ms} = 0$ and $M_{ss} = 0$.

Executive Summary

Material not required for exam!

20. CONCLUDING REMARKS

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The present Course Notes have been compiled to address the following topics:
- vibrations of systems with a single degree-of-freedom
  (planar translation or rotation)
- vibrations of systems with multiple degrees-of-freedom
  (discrete systems in planar translation and/or rotation)
- vibrations of continuum systems (modelled exactly)
- vibrations of continuum systems (modelled as a collection of finite elements).

In all cases treated the resulting equations of motion are *linear*; an approximation usually valid for small amplitudes of displacements.

The present Course Notes have been compiled to provide the student with a compact summary of applicable mechanical and mathematical concepts. Its objectives are:
- to clarify or amplify mathematical concepts already presented in earlier courses on Linear Algebra and on Differential Equations;
- to clarify or amplify engineering mechanics concepts already presented in earlier courses on Statics, Elasticity Theory, and Dynamics;
- to clarify, amplify, and extend the material in the Course Textbook.

Educational experience has shown that the following activities are ESSENTIAL if the student is to master the subject:
- clear-minded attendance of the classes, actively writing comments in his own course notebook;
- thorough study of the relevant material in the Course Textbook;
- thorough study of the relevant material in the present Course Notes;
- energetic development of experience in solving recommended dynamics exercises.

The present document has the character of "Course Notes" - where it should be emphasized that these notes are quite informal and quite concise. They are quite informal indeed, for several reasons. They are meant only to provide the reader with material to quickly refresh his memory. And they have been compiled only rather recently and would therefore still need to undergo a considerably number of rewriting cycles before they might attain any status of authority. It follows that the teacher would welcome any comments from the users of this document. Please, send in any comments you might have!

As explained, the document does not replace textbooks. Textbooks are more verbose, they explain more, they explain better, they have nicer figures and graphs, and they have been tested on their didactic quality.

Nor does the possession of this document obviate the need to actually attend the classes where the teacher attempts to convey the essence of the material to the student. On the contrary! An assembly of a good teacher with good students is characterized by creative interaction, and on the basis of that interaction the teacher may amplify, modify, or even delete material that might appear too simple or possibly too advanced for those particular students.

Students are expected to study thoroughly with the aid of an appropriate textbook or several appropriate textbooks. Consider purchasing the recommended Course Textbook (the one by S.G. Kelly). Buy that textbook or buy another appropriate textbook that may be more to your liking. And consult good textbooks in your university libraries. Keep in mind that a professional engineer carefully builds up his professional library, starting with textbooks from his university courses!

Students generally study better when they also attend classes, as explained above. Attending classes increases the efficiency of study.

And students can generally truly master the course material only through dedicated and rigorous exercising, through additional exercising, and through even more exercising.

To master the material of this course on one of the true foundations of engineering one has to study efficiently and intensely. There is no other way. And it is a beneficial way.
Studying efficiently and intensely is beneficial because courses dealing with the true foundations of engineering are fascinating, challenging, and rewarding. They are rewarding because they open a vast horizon for further exploration, they sharpen the mind, and they increase one's competitiveness as a young engineer. Solidly mastering the foundations of engineering provides the young engineer with solid foundations for his future career - in engineering of any type or wherever it will lead him.

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