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The Uniqueness of Certain Flows in a Channel with Arbitrary Cross Section

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THE UNIQUENESS OF CERTAIN FLOWS IN A CHANNEL WITH ARBITRARY CROSS SECTION

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Abstract

This report presents a proof of the uniqueness of a parallel three-dimensional shear flow in a channel with arbitrary cross section where the speed of the flow is not less than the highest critical speed. The investigation also includes a two-dimensional analysis in which it is assumed that while the flow velocity varies with the depth, the density also depends on the depth; and for this case the development leads to a formula which gives a good approximation to the highest critical speed.

1. Introduction

Consider an incompressible, inviscid liquid contained in a horizontal, infinitely long straight channel whose cross section is arbitrary. Suppose that a gravitational force is the only body force which acts on the liquid. The nonlinear hydrodynamical equations given below in Section 2 show that a uniform parallel flow is a possible steady motion. This kind of flow is defined to be such that the only non-zero velocity component is the component v_1 in the axial direction of the channel; and v_1, although assumed to be independent of the longitudinal coordinate, may be a function of the lateral coordinates of the channel. If the equations are linearized with respect to a certain parallel flow the resulting linear equations also admit a similar flow and in particular the uniform parallel flow in which the axial velocity is constant. However, according to the linear theory, this is not the only possible motion if the speed of the liquid at infinity is less than one of a possible set of critical values. For example, if the cross section of the channel is a rectangle with depth h and if the speed of the liquid at infinity is less than the critical speed $\sqrt{gh}$, where g is the acceleration due to gravity, then the linear equations predict that a progressing wave motion is possible.

A discussion of critical speeds is necessary for the analysis of several hydrodynamical problems concerned with channel flow. They arise in the study of the motion due to a surface pressure distrubance which moves in the direction of the channel with fixed speed either when this problem is regarded as a steady state problem or when it is regarded as a Newtonian initial value problem. In the steady state analysis of the problem critical speeds arise not only with respect to the uniqueness of the solution but also with respect to the admissibility of the linearization. In the Newtonian approach based on an initial value problem for the linear theory it turns out, as Stoker [1] showed, that at a critical spped the velocity components of the flow become unbounded as time elapses. The nonlinear theory of a gravitating fluid in a channel leads to the interpretation of critical speeds as bifurcation values at which cnoidal and solitary waves may appear as well as parallel flows. These examples point to the fact that critical speeds can be defined in different ways. A discussion of the various definitions can be found in a paper by Peters and Stoker [2].

During conversations with the author and about problems similar to those mentioned above J. J. Stoker raised the following uniqueness question. If a gravitating liquid confined to a rectangular channel is in a state of parallel flow with a finite speed not less than the highest critical speed does the linear theory show that this flow is the only possible steady motion which is bounded? In the sections which follow we show that the answer to this question is in the affirmative. We show this under the assumption that the density of the liquid varies with depth and that the liquid is subject to a shear in velocity. Our method is based of course on an eigenvalue problem which possesses only the trivial solution provided that a parameter of the problem is not less than a certain value.

Weinstein [3] showed that if $\phi(x,y)$ is a potential function which is required to satisfy

1. $$\phi_{xx}(x,y) + \phi_{yy} = 0 , \qquad -\infty < x < \infty, \quad 0 < y < 1$$

2. $$\phi_y(x,0) = 0 ,$$

3. $$\phi_y(x,1) = p\phi(x,1) , \qquad p > 0$$

and if λ_o is the unique positive root of

$$\lambda_o \tanh \lambda_o = p ,$$

where p is a constant, then

$$\phi(x,y) = [a_o \cos \lambda_o + b_o \sin \lambda_o x]\cosh \lambda_o y$$

is the only bounded function which satisfies the above conditions. Weinstein's proof of this is based on a completeness theorem. In Section 3 of this paper we use Weinstein's method to analyze the eigenvalue problem which we derive in order to discuss the two-dimensional flow of a gravitating liquid with <u>non-constant</u>

density and velocity each varying with depth. In the course of the analysis we find a formula which gives an approximation to the highest critical speed.

Section 4 is devoted to an analysis of a three-dimensional motion of a gravitating liquid of constant density in which the velocity depends on the coordinates orthogonal to the direction of the containing channel which is assumed to have an arbitrary cross section. The character of the eigenvalue problem which we formulate for this case is different from that presented for the two-dimensional case. As a consequence, instead of seeking a method based on a completeness theorem, we base the analysis on the generalized Fourier transform theorem which incidentally can also be used for the case of Section 3 in lieu of Weinstein's procedure.

2. Formulation

Let a gravitating, incompressible, and inviscid liquid with density ρ be confined to an infinitely long horizontal channel whose cross section is constant. Suppose that the equilibrium free surface of the liquid is planar and that it coincides with the horizontal x_1,x_3-plane of a cartesian reference frame whose x_1-axis is taken parallel to the rigid cylinder which forms the channel. With the positive direction of the x_2-axis taken to be upward, let the channel wall be defined by

$$x_2 = Q(x_3)$$

and let the free surface be given by

$$x_2 = F(x_1, x_3, t) \ .$$

Let g denote the gravitational acceleration, let π denote the pressure, and let us use v_1, v_2, v_3 to denote the velocity components of a liquid particle, while t stands for time. In terms of these quantities the elementary theory of hydrodynamics predicts that if the gravitational force is the only force acting then the motion of the liquid is defined by the continuity equation

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \ , \tag{2.1}$$

the incompressibility condition

$$v_1 \frac{\partial \rho}{\partial x_1} + v_2 \frac{\partial \rho}{\partial x_2} + v_3 \frac{\partial \rho}{\partial x_3} = 0 \ , \tag{2.2}$$

the momentum equations

$$\begin{aligned} \rho\left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}\right) &= - \frac{\partial \pi}{\partial x_1} \ , \\ \rho\left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}\right) &= - \rho g - \frac{\partial \pi}{\partial x_2} \ , \\ \rho\left(\frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}\right) &= - \frac{\partial \pi}{\partial x_3} \ , \end{aligned} \tag{2.3}$$

the kinematic boundary conditions

$$(2.4) \qquad v_2 = v_3 \frac{\partial Q}{\partial x_3} ,$$

$$(2.5) \qquad v_2 = v_1 \frac{\partial F}{\partial x_1} + v_3 \frac{\partial F}{\partial x_3} + \frac{\partial F}{\partial t} ,$$

the dynamic free surface condition

$$(2.6) \qquad \pi(x_1, F, x_3, t) = 0 ,$$

plus initial conditions at $t = 0$, and conditions which specify the behavior of the liquid at distances arbitrarily far from the origin.

The above equations can be written in dimensionless form if we introduce a typical length in the vertical direction, say h, and the dimensionless quantities

$$x = x_1 h^{-1} , \quad y = x_2 h^{-1} , \quad z = x_3 h^{-1} ,$$

$$u_1 = v_1 (gh)^{-1/2} , \quad u_2 = v_2 (gh)^{-1/2} , \quad u_3 = v_3 (gh)^{-1/2} ,$$

$$\pi_1 = \pi(\tilde{\rho} gh)^{-1} , \quad f_1 = F h^{-1} , \quad q = Q h^{-1} ,$$

$$\tau = t(g/h)^{1/2} ,$$

where $\tilde{\rho}$ is some fixed quantity with the dimensions of density.

In terms of these quantities the equation of the channel wall is

$$y = q(z) ,$$

the equation of the free surface is

$$y = f_1(x,z,\tau) ,$$

and the basic hydrodynamical equations are

(2.7) $$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0 ,$$

(2.8) $$u_1 \frac{\partial \rho}{\partial x} + u_2 \frac{\partial \rho}{\partial y} + u_3 \frac{\partial \rho}{\partial z} = 0 ,$$

(2.9) $$\begin{aligned} \rho(\frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z}) &= - \tilde{\rho} \frac{\partial \pi_1}{\partial x} , \\ \rho(\frac{\partial u_2}{\partial \tau} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z}) &= - \rho - \tilde{\rho} \frac{\partial \pi_1}{\partial y} , \\ \rho(\frac{\partial u_3}{\partial \tau} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z}) &= - \tilde{\rho} \frac{\partial \pi_1}{\partial z} , \end{aligned}$$

with the boundary conditions

(2.10) $$u_2 = u_3 \frac{\partial q}{\partial z}$$

(2.11) $$u_2 = u_1 \frac{\partial f_1}{\partial x} + u_3 \frac{\partial f_1}{\partial z} + \frac{\partial f_1}{\partial \tau} ,$$

and

(2.12) $$\pi_1(x,f_1,z,\tau) = 0 .$$

This system is satisfied by the quantities

$$
(2.13) \qquad \begin{cases} u_1 = u_o(y,z) = \dfrac{\gamma + v_o(y,z)}{\sqrt{gh}} \ , \\[2ex] u_2 = 0 \ , \quad u_3 = 0 \ , \quad f_1 = 0 \ , \\[2ex] \rho = \tilde{\rho}\rho_o(y) \ , \quad \pi_1 = \displaystyle\int_y^0 \rho_o(\eta)d\eta \ , \end{cases}
$$

where γ is constant and v_o is a continuous non-negative function. They define a steady parallel motion in the channel and we will refer to this flow as the equilibrium flow. The velocity $v_o(y,z)$ gives the transverse shear in the axial velocity component and it is also a measure of the departure of the flow from a uniform state defined by the velocity γ. The function $\rho_o(y)$ measures the variation in density with the depth and we suppose that it does not decrease as the depth increases so that with respect to our coordinate system the derivative $\rho_o'(y)$ if it exists satisfies

$$
\frac{d\rho_o(y)}{dy} \leq 0 \ .
$$

Let us proceed to linearize the equations (2.7)-(2.12) with respect to the flow given by (2.13). That is, let us write

$$
(2.14)\qquad
\begin{array}{ll}
u_1 = u_o(y,z) + u \,, & \tilde{\rho} = \rho_o(y) + \sigma \,, \\[2ex]
u_2 = v \,, & \pi_1 = \int_y^0 \rho_o(\eta)\,d\eta + p \,, \\[2ex]
u_3 = w \,, & f_1 = f(x,z) \,,
\end{array}
$$

and assume steady motion. Let us substitute these quantities in equations (2.7)-(2.12) and neglect terms which involve products of two or more factors from the set u, v, w, σ, p and f. The result of the linearization of the equations (2.7)-(2.9) is

$$
(2.15)\qquad u_x + v_y + w_z = 0
$$

$$
(2.16)\qquad u_o\, {}_x + v\rho_{oy} = 0
$$

$$
(2.17)\qquad
\begin{cases}
\rho_o(u_o u_x + u_{oy} v + u_{oz} w) = -\, p_x \\[1ex]
\rho_o u_o v_x = -\,\sigma - p_y \,. \\[1ex]
\rho_o u_o w_x = -\, p_z
\end{cases}
$$

The condition at the channel wall is

$$
(2.18)\qquad v = w\,\frac{\partial q}{\partial z}\,.
$$

With respect to the free surface conditions (2.11), (2.12) they become conditions to be satisfied at y = 0. In place of (2.11) we have

$$v(x,0,z) = u_o(0,z)f_x(x,z) \quad , \tag{2.19}$$

and from (2.12) we have

$$\int_f^0 \rho_o(\eta)d\eta + p(x,f,z) = 0$$

which after differentiation and removal of second order terms becomes

$$-\rho_o(0)f_x(x,z) + p_x(x,0,z) = 0 \ . \tag{2.20}$$

Notice that if $v = 0$, $w = 0$, the linearized equations are again satisfied by a flow of the type (2.13).

Our object now is to show that if the speed γ is not less than a certain highest critical value then the only possible bounded solution of the problem formulated by the equations (2.15)-(2.20) is the one which defines an equilibrium flow, (2.13).

3. Two-dimensional Motion. Rectangular Channel

If

$$u_o = u_o(y) = \frac{\gamma + v_o(y)}{\sqrt{gh}} \ ;$$

where v_o is continuous; if

$$\rho_o = \rho_o(y) \ ,$$

$$w = 0 \qquad ;$$

and if the remaining quantities in the equations (2.15)-(2.20) are independent of z, then these equations define a two-dimensional motion which may be interpreted as a two-dimensional flow in a rectangular channel. For this case the basic linearized equations are

(3.1) $$u_x + v_y = 0$$

(3.2) $$u_o \sigma_x + v\rho_{oy} = 0$$

(3.3) $$\rho_o(u_o u_x + u_{oy} v) = - p_x$$

(3.4) $$\rho_o u_o v_x = - \sigma - p_y \ .$$

If the depth of the channel is h the equation of the bottom in our dimensionless variables is $y = -1$ and since the vertical velocity component must vanish there we must have

(3.5) $$v(x,-1) = 0 \ .$$

Corresponding to the free surface

(3.6) $$y = f(x)$$

the linearized free surface conditions

$$v(x,0) = u_o(0)f_x(x) \tag{3.7}$$

$$-\rho_o(0)f_x + p_x(x,0) = 0 \tag{3.8}$$

must be satisfied.

For the analysis of the two-dimensional equations we will work with the dependent variable v/u_o which we denote by $\chi(x,y)$. The elimination of u, p and σ from these equations shows that

$$\chi(x,y) = \frac{v(x,y)}{u_o(y)}$$

must satisfy

$$\frac{\partial}{\partial x}(\rho_o u_o^2 \chi_x) + \frac{\partial}{\partial y}(\rho_o u_o^2 \chi_y) - \rho_{oy}\chi = 0 \,, \qquad -1 < y - 0 \,, \quad -\infty < x < \infty \tag{3.9}$$

with the boundary conditions

$$\chi(x,-1) = 0 \tag{3.10}$$

$$u_o^2(0)\ \chi_y(x,0) - \chi(x,0) = 0\,. \tag{3.11}$$

We turn now to the method of Weinstein [3] and replace

$$\frac{\partial}{\partial x}(\rho_o u_o^2 \chi_x)$$

in equation (3.9) with $-\lambda^2\rho_o u_o^2\psi(y)$; while we replace χ_y and χ respectively with ψ_y and ψ. This formulates and introduces the following standard eigenvalue problem:

(3.12) $$\frac{\partial}{\partial y}(\rho_o u_o^2 \psi_y) - \rho_{oy}\psi(y) = \lambda^2 \rho_o u_o^2 \psi , \qquad -1 < y < 0$$

(3.13) $$\psi(-1) = 0$$

(3.14) $$u_o^2(0)\psi_y(0) - \psi(0) = 0 .$$

Here, the eigenvalues depend on the magnitude of γ and the eigenfunctions must satisfy the orthogonality relation

$$\int_{-1}^{0} \rho_o(y)u_o^2(y)\psi_m(y)\psi_n(y)dy = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

where ψ_n and ψ_m are the eigenfunctions which correspond respectively to the eigenvalues λ_n and λ_m. It is easy to see that if λ_n is an eigenvalue then so is $-\lambda_n$. Besides this, it is not difficult to verify that the eigenvalues λ_n are either real or pure imaginary numbers. It is known that there can be only a finite number of eigenvalues in any bounded domain of the complex λ-plane. It is also well known that the above second order system defines a complete set of eigenfunctions $\{\psi_n(y)\}$ such that a twice differentiable function $G(y)$ can be expanded in the Fourier series

$$G(y) = \sum_{n=0}^{\infty} \alpha_n \psi_n(y) .$$

It follows from the above remarks that for any fixed value of x, $\chi(x,y)$ has the unique expansion

$$\chi(x,y) = \frac{v(x,y)}{u_o(y)} = \sum_{n=0}^{\infty} \alpha_n(x)\psi_n(y) \ , \qquad -1 \le y \le 0 \ .$$

Here, the coefficient $\alpha_n(x)$ is

$$\alpha_n(x) = \int_{-1}^{0} \rho_o(y)u_o^2(y)\chi(x,y)\psi_n(y)dy$$

and as a function of x it must satisfy

$$\frac{d^2\alpha_n(x)}{dx^2} = \int_{-1}^{0} \rho_o u_o^2 \chi_{xx}\psi_n dy$$

$$= -\int_{-1}^{0} [\frac{\partial}{\partial y} \rho_o u_o^2 \chi_y - \rho_{oy}\chi]\psi_n dy$$

$$= -\lambda_n^2 \int_{-1}^{0} \rho_o u_o^2 \chi \psi_n dy \ .$$

Hence the coefficient $\alpha_n(x)$ must satisfy

$$\frac{d^2\alpha_n(x)}{dx^2} + \lambda_n^2\alpha_n(x) = 0$$

from which

(3.15)
$$\alpha_n(x) = a_n \cos \lambda_n x + b_n \sin \lambda_n(x) \ , \qquad \lambda_n \ne 0$$
$$\alpha_o(x) = a_o + b_o x \ , \qquad \lambda_o = 0 \ .$$

This shows that if $\chi(x,y)$ is to be bounded everywhere then if α_j does not correspond to a real eigenvalue we must take $\alpha_j = 0$. If $\lambda_o = 0$ is an eigenvalue the corresponding term in the development of $\chi = v/u_o$ is

$$(a_o + b_o x)\psi_o(y) \ .$$

Since

$$u_x = -v_y$$

the corresponding term in the development of u is

$$-(a_o x + b_o \frac{x^2}{2})\psi_{oy} + f_o(y)$$

but this is unbounded as $x \to \infty$ and therefore for a bounded motion we must take $\alpha_o = 0$. It is now apparent that our problem is reduced to a study of how γ affects the disposition of the eigenvalues of the system of equations (3.12)-(3.14).

In order to study these eigenvalues let us convert the equation (3.12) namely

$$\frac{\partial}{\partial y} \rho_o u_o^2 \psi_y(y) - \rho_{oy}\psi = \lambda^2 \rho_o u_o^2 \psi \ , \qquad -1 < y < 0$$

into an integral equation. If we integrate (3.12) from y to zero we find

$$\rho_o(0)u_o^2(0)\psi_y(0) - \rho_o(y)u_o^2(y)\psi_y(y)$$

$$= \rho_o(0)\psi(0) - \rho_o(y)\psi(y) - \int_y^0 \rho_o(\eta)\psi_\eta(\eta)d\eta + \lambda^2 \int_y^0 \rho_o(\eta)u_o^2(\eta)\psi(\eta)d\eta$$

from which, by using $u_o^2(0)\psi_y(0) - \psi(0) = 0$, we obtain

$$\rho_o(y)u_o^2(y)\psi_y(y) = \rho_o(y)\psi(y) + \int_y^0 \rho_o(\eta)\psi_\eta(\eta)d\eta - \lambda^2 \int_y^0 \rho_o(\eta)u_o^2(\eta)\psi(\eta)d\eta \; .$$

Since $\psi(-1) = 0$, the last equation can be written

$$\rho_o(y)u_o^2(y)\psi_y(y) = \rho_o(y) \int_{-1}^y \psi_\eta(\eta)d\eta + \int_y^0 \rho_o(\eta)\psi_\eta(\eta)d\eta - \lambda^2 \int_y^0 \rho_o(\eta)u_o^2(\eta) \int_{-1}^\eta \psi_\xi(\xi)d\xi d\eta$$

which after an integration by parts yields

$$\rho_o(y)u_o^2(y)\psi_y(y) = \rho_o(y) \int_{-1}^y \psi_\eta(\eta)d\eta + \int_y^0 \rho_o(\eta)\psi_\eta(\eta)d\eta - \lambda^2 \left\{ \int_y^0 \rho_o(\xi)u_o^2(\xi)d\xi \cdot \int_{-1}^0 \psi_\xi(\xi)d\xi - \int_0^y \psi_\eta(\eta) \int_y^\eta \rho_o(\xi)u_o^2(\xi)d\xi d\eta \right\}$$

or, after a rearrangement,

$$\rho_o(y)u_o^2(y)\psi_y(y) = \rho_o(y)\int_{-1}^{y}\psi_\eta(\eta)d\eta + \int_{y}^{0}\rho_o(\eta)\psi_\eta(\eta)d\eta$$

$$- \lambda^2 \left\{ \begin{aligned} &\int_{y}^{0}\rho_o(\xi)u_o^2(\xi)d\xi \cdot \int_{-1}^{y}\psi_\eta(\eta)d\eta \\ &+ \int_{y}^{0}\psi_\eta(\eta)\cdot\int_{\eta}^{0}\rho_o(\xi)u_o^2(\xi)d\xi d\eta \end{aligned} \right\} .$$

If we set

$$\phi = \sqrt{\rho_o(y)}\, u_o(y)\psi_y(y)$$

the last equation becomes

$$\phi(y) = \frac{\sqrt{\rho_o(y)}}{u_o(y)}\int_{-1}^{y}\frac{\phi(\eta)d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}} + \frac{1}{\sqrt{\rho_o(y)}u_o(y)}\int_{y}^{0}\frac{\sqrt{\rho_o(\eta)}\phi(\eta)d\eta}{u_o(\eta)}$$

$$- \lambda^2 \left\{ \begin{aligned} &\frac{\int_{y}^{0}\rho_o(\xi)u_o^2(\xi)d\xi}{\sqrt{\rho_o(y)}u_o(y)} \cdot \int_{-1}^{y}\frac{\phi(\eta)d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}} \\ &+ \frac{1}{\sqrt{\rho_o(y)}u_o(y)}\cdot\int_{y}^{0}\frac{\phi(\eta)}{u_o(\eta)\sqrt{\rho_o(\eta)}}\int_{\eta}^{0}\rho_o(\xi)u_o^2(\xi)d\xi d\eta \end{aligned} \right\} .$$

Now if we introduce the symmetric kernels

$$k_1(y,\eta;\gamma) = \begin{cases} \dfrac{\sqrt{\rho_o(y)}}{u_o(y)\sqrt{\rho_o(\eta)}u_o(\eta)}, & -1 < \eta < y \\[2ex] \dfrac{\sqrt{\rho_o(\eta)}}{u_o(y)\sqrt{\rho_o(y)}u_o(\eta)}, & y < \eta < 0 \end{cases}$$

$$k_2(y,\eta;\gamma) = \begin{cases} \dfrac{\int_y^0 \rho_o(\xi)u_o^2(\xi)d\xi}{u_o(y)u_o(\eta)\sqrt{\rho_o(y)\rho_o(\eta)}}, & -1 < \eta < y \\[2ex] \dfrac{\int_\eta^0 \rho_o(\xi)u_o^2(\xi)d\xi}{u_o(y)u_o(\eta)\sqrt{\rho_o(y)\rho_o(\eta)}}, & y < \eta < 0 \end{cases}$$

we find the integral equation

$$(3.16) \qquad \phi(y) = \int_{-1}^{0} k_1(y,\eta;\gamma)\phi(\eta)d\eta - \lambda^2 \int_{-1}^{0} k_2(y,\eta;\gamma)\phi(\eta)d\eta$$

as one which is equivalent to the differential system (3.12)-(3.14).

There is no loss of generality in assuming that

$$\int_{-1}^{0} \phi^2(y)dy = 1 \; .$$

Then if we multiply (3.16) by $\phi(y)$ and integrate from -1 to 0 we have

$$1+\lambda^2\int_{-1}^{0}\phi(y)\int_{-1}^{0}k_2(y,\eta)\phi(\eta)d\eta dy=\int_{-1}^{0}\phi(y)\int_{-1}^{0}k_1(y,\eta)\phi(\eta)d\eta dy \ .$$

The integral on the left hand side of the last equation is equal to

$$\int_{-1}^{0}\rho_o(y)u_o^2(y)\left[\int_{-1}^{y}\frac{\phi(\eta)d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}}\right]^2 dy$$

so that

$$1+\lambda^2\int_{-1}^{0}\rho_o(y)u_o^2(y)\left[\int_{-1}^{y}\frac{\phi(\eta)d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}}\right]^2 dy$$

$$=\int_{-1}^{0}\phi(y)\int_{-1}^{0}k_1(y,\eta)\phi(\eta)d\eta dy \ .$$

An application of Schwartz's inequality gives

$$1+\lambda^2\int_{-1}^{0}\rho_o(y)u_o^2(y)\left[\int_{-1}^{y}\frac{\phi(\eta)d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}}\right]^2 dy$$

$$\leq\sqrt{\int_{-1}^{0}\int_{-1}^{0}k_1^2(y,\eta)d\eta dy} \ .$$

The integral of $k_1^2(y,\eta)$ is

$$2\int_{-1}^{0}\frac{\rho_o(y)}{u_o^2(y)}\int_{-1}^{y}\frac{d\eta}{\rho_o(\eta)u_o^2(\eta)}dy \ .$$

Hence we see that

$$\lambda^2 \int_{-1}^{0} \rho_o(y) u_o^2(y) \left[\int_{-1}^{y} \frac{\phi(\eta)\,d\eta}{u_o(\eta)\sqrt{\rho_o(\eta)}} \right]^2 dy$$

$$(3.17) \qquad \leq \sqrt{2 \int_{-1}^{0} \frac{\rho_o(y)}{u_o^2(y)} \int_{-1}^{y} \frac{d\eta}{\rho_o(\eta) u_o^2(\eta)}\, dy - 1} \; .$$

The inequality (3.17) shows that there can be only a finite number of real eigenvalues because it requires these to lie in a bounded line segment which, as we noted above, cannot contain an infinite number of eigenvalues. The inequality (3.17) also shows that if γ in

$$u_o(y) = \frac{\gamma + v_o(y)}{\sqrt{gh}}$$

is taken sufficiently large then the absolute magnitude can be made as small as we please. In addition, (3.17) shows that if γ is so large that

$$(3.18) \qquad \sqrt{2 \int_{-1}^{0} \frac{\rho_o(y)}{u_o^2(y)} \int_{-1}^{y} \frac{d\eta}{\rho_o(\eta) u_o^2(\eta)}\, dy} \leq 1$$

then the eigenvalues cannot be real and non-zero. From what has been noted above, this means that when (3.18) holds the only bounded solution of (3.9)-(3.11) is the trivial one $\chi(x,y) = v(x,y)/u_o(y) \equiv 0$. With the vertical velocity component v equal to zero everywhere the only solution of the original equations (3.1)-(3.8) for the two-dimensional flow is one of the type (2.13). In other words, the linear theory implies that the flow

$$u_1 = \frac{\gamma + v_o(y)}{\sqrt{gh}} , \qquad u_2 = 0 ,$$

$$f_1 = 0 ,$$

$$\rho = \tilde{\rho}\rho_o(y) \qquad , \qquad \pi_1 = \int_y^0 \rho(\eta)d\eta$$

is a unique steady two-dimensional flow if (3.18) holds.

In the foregoing we have made the tacit assumption, which we retain, that if m and M are respectively the minimum and maximum values of the non-negative continuous function $v_o(y)$ then γ does not have a value between -M and -m. Without such an assumption the integral

$$\int_{-1}^{y} \frac{d\eta}{\rho_o(\eta)u_o^2(\eta)}$$

would fail to exist.

We are now in a position to define a critical speed as a speed $\gamma = c_n$ which corresponds to a transition from real values of λ to pure imaginary values. This occurs when λ passes to the zero value and as we can see from the integral equation (3.16) this takes place when $\gamma = c_n$ is an eigenvalue of the equation

(3.19) $$\phi(y) = \int_{-1}^{0} k_1(y,\eta;\gamma)\phi(\eta)d\eta \ .$$

It can be seen from (3.15) that c_n is the limit speed of a wave motion whose wave length becomes infinite. If $\gamma = c_o$ is the highest critical speed there is no real value of λ, say λ_r, corresponding to a γ value $\gamma = \gamma_r$ such that $\gamma_r > c_o$. If there were the inequality (3.17) would show that we could force γ_r to zero by <u>increasing</u> γ_r to some value γ_r'. This would produce a critical speed higher than c_o contrary to the assumption that c_o is the highest critical speed. In other words there is no bounded flow other than the equilibrium flow if $\gamma \geq c_o$. It is evident from (3.19) and (3.17) that the formula

$$(3.20) \qquad gh \sqrt{2 \int_{-1}^{0} \frac{\rho_o(y)}{[c + v_o(y)]^2} \int_{-1}^{y} \frac{d\eta}{\rho_o(\eta)[c + v_o(\eta)]^2} \, dy} = 1$$

provides an estimate for the highest critical speed. This estimate in general is such that $c_o \leq c$.

Under some circumstances the formula (3.20) actually yields the highest critical speed. If the density is constant (3.20) gives

$$gh \sqrt{2 \int_{-1}^{0} \frac{1}{[c + v_o(y)]^2} \int_{-1}^{y} \frac{d\eta}{[c + v_o(\eta)]^2} \, dy} = 1$$

which, after an integration, is

$$(3.21) \qquad gh \int_{-1}^{0} \frac{dy}{[c + v_o(y)]^2} = 1 .$$

This is a known formula for the critical speeds, $c_o = c$, where the density is constant. For a discussion of this formula and other ways of deriving it see, for example, Burns [4], or Peters [5]. If we set $v_o(y) = 0$ in (3.21) we find the well known result $c^2 = gh$ for the critical speed in a rectangular channel when the density is constant and the equilibrium flow is without vorticity.

If the density is not constant then the assumption (in order to have stability) is that the density does not increase as y increases. Hence we see from (3.20) that

$$1 = gh\sqrt{2\int_{-1}^{0}\frac{\rho_o(y)}{[c+v_o(y)]^2}\int_{-1}^{y}\frac{d\eta}{\rho_o(\eta)[c+v_o(\eta)]^2}\,dy}$$

$$\leq gh\sqrt{2\int_{-1}^{0}\frac{\rho_o(y)}{[c+v_o(y)]^2}\cdot\frac{1}{\rho_o(y)}\int_{-1}^{y}\frac{d\eta}{[c+v_o(\eta)]^2}\,dy}$$

$$1 \leq gh\int_{-1}^{0}\frac{dy}{[c+v_o(y)]^2}\quad .$$

This means that the highest critical speed for the case of variable density cannot be greater than the highest critical speed for constant density.

When the flow possesses no vorticity due to a velocity variation, i.e. $v_o(y) = 0$, the formula (3.20) reduces to

$$2\int_{-1}^{0}\rho_o(y)\int_{-1}^{y}\frac{d\eta}{\rho_o(\eta)}\,dy = \frac{c^4}{g^2h^2}\quad . \tag{3.22}$$

If the density variation is exponential, say

$$\rho_o(y) = e^{-2ky} ,$$

we find

$$2\int_{-1}^{0} e^{-2ky} \int_{-1}^{y} e^{2k\eta} d\eta dy = \frac{c^4}{g^2h^2}$$

$$\frac{1}{k} + \frac{e^{-2k}}{2k^2} - \frac{1}{2k^2} = \frac{c^4}{g^2h^2}$$

and when k is small

$$\frac{c^4}{g^2h^2} \sim 1 - \frac{2k}{3} + \frac{k^2}{3} - \frac{2k^3}{15} .$$

For k small, this agrees, up to and including second order terms, with the approximation to the highest critical speed given in Peters and Stoker [2], namely

$$\frac{c^2}{gh} = \frac{1}{1 + k/3} .$$

In the latter paper the critical speeds, c_n, are defined by

$$\tan s_n = \frac{2ks_n}{s_n^2 - k^2} \tag{3.23}$$

$$s_n^2 = 2k \cdot \frac{gh}{c_n^2} - k^2 .$$

Equations (3.23) can be found by explicitly solving (3.19).

It should be pointed out that our results hold for an equilibrium flow which is composed of homogeneous layers. At an interface where the density is discontinuous, sat at $y = -r$, the linear theory requires continuity in the vertical velocity component and the pressure. In terms of $\psi(y)$ the interface conditions are

$$\psi(-r-0) = \psi(-r+0)$$

$$\rho_o(-r-0)[u_o^2(-r)\psi_y(-r-0) - \psi(-r-0)]$$

$$= \rho_o(-r+0)[u_o^2(-r)\psi_y(-r+0) - \psi(-r+0)] \ .$$

It can be verified that these conditions are automatically satisfied by the integral equation (3.16). It is sufficient here to confine the discussion to the case of a medium with just two layers and $v_o = 0$. Suppose that the lower layer is defined by $-1-r < y < -r$ where the density ρ_o is $\rho_o = 1$, and that the upper layer is defined by $-r < y < 0$ where the density is $\rho_o = \rho_o < 1$. In terms of the original variables the depth of the lower layer is h and the depth of the upper layer is $H = rh$ so that r is the ratio of the depth of the upper layer to that of the lower layer and the corresponding density ratio is $\rho_o/1 = \rho_o < 1$. The integral equation for $\psi(y)$ is

$$\phi(y) = \int_{-1-r}^{0} k_1(y,\eta;\gamma)\phi(\eta)d\eta - \lambda^2 \int_{-1-r}^{0} k_2(y,\eta;\gamma)\phi(\eta)d\eta$$

where

$$\phi(y) = \sqrt{\rho_o(y)}\cdot u_o(y)\psi_y(y) .$$

The equation which prevails for $\lambda = 0$ and determines the critical speeds is

$$\phi(y) = \int_{-1-r}^{0} k_1(y,\eta;\gamma)\phi(\eta)d\eta$$

or

$$\phi(y) = \frac{\sqrt{\rho_o(y)}}{u_o(y)} \int_{-1-r}^{y} \frac{\phi(\eta)d\eta}{\sqrt{\rho_o(\eta)}u_o(\eta)} + \frac{1}{u_o(y)\sqrt{\rho_o(y)}} \int_{y}^{0} \frac{\sqrt{\rho_o(\eta)}\phi(\eta)d\eta}{u_o(\eta)} .$$

If $v_o = 0$ so that $u_o = \gamma/\sqrt{gh}$ we have

$$u_o^2\phi(y) = \sqrt{\rho_o(y)} \int_{-1-r}^{y} \frac{\phi(\eta)d\eta}{\sqrt{\rho_o(\eta)}} + \frac{1}{\sqrt{\rho_o(y)}} \int_{y}^{0} \sqrt{\rho_o(\eta)}d(\eta)d\eta .$$

This leads to the following. If $-1-r < y < -r$ we have

$$u_o^2\phi(y) = \int_{-1-r}^{-r} \phi(\eta)d\eta + \rho_o \int_{-r}^{0} \phi(\eta)d\eta$$

and after integration

$$u_o^2 \int_{-1-r}^{-r} \phi(y)dy = \int_{-1-r}^{-r} \phi(\eta)d\eta + \rho_o \int_{-r}^{0} \phi(\eta)d\eta . \tag{3.24}$$

If $-r < y < 0$ we have

$$u_o^2\phi(y) = \rho_o \int_{-1-r}^{-r} \phi(\eta)d\eta + \int_{-r}^{0} \phi(\eta)d\eta$$

or

$$(3.25) \qquad u_o^2 \int_{-r}^{0} \phi(y)dy = r\rho_o \int_{-1-r}^{-r} \phi(\eta)d\eta + r \int_{-r}^{0} \phi(\eta)d\eta \; .$$

We cannot have both of the integrals

$$\int_{-r}^{0} \phi(y)dy \quad \text{and} \quad \int_{-1-r}^{-r} \phi(y)dy$$

equal to zero because this would imply $\phi \equiv 0$ and $\psi \equiv 0$. Hence the determinant of the equations (3.24) and (3.25) must be zero. This gives

$$(3.26) \qquad u_o^4 - (1+r)u_o^2 + r(1-\rho_o) = 0 \; ,$$

an equation which defines two critical speeds. It is the same as that given by Peters and Stoker [2]. The higher critical speed is given by

$$\frac{c_o^2}{gh} = \frac{1+r+\sqrt{(1-r)^2+4\rho_o r}}{2}$$

$$\leq \sqrt{1+2\rho_o r + r^2} \; .$$

This should be compared with the estimate of the higher critical speed which comes from (3.20) namely

$$\frac{c^2}{gh} = \sqrt{2\int_{-1-r}^{0} \rho_o(y) \int_{-1-r}^{y} \frac{d\eta}{\rho_o(\eta)}\, dy}$$

$$= \sqrt{2\left\{\int_{-1-r}^{-r} \int_{-1-r}^{y} d\eta dy + \rho_o \int_{-r}^{0} \int_{-1-r}^{-r} d\eta dy + \int_{-r}^{0} \int_{-r}^{y} d\eta dy\right\}}$$

$$\frac{c^2}{gh} = \sqrt{1 + 2\rho_o r + r^2}$$

which confirms the fact that for the higher critical speed c_o, c_o is such that $c_o \leq c$.

4. Three-dimensional Motion. Channel with Arbitrary Cross Section

This part of the paper is concerned with a three-dimensional case of our problem in which a liquid of constant density is confined to a channel whose cross section is like that shown in Fig. 4.1.

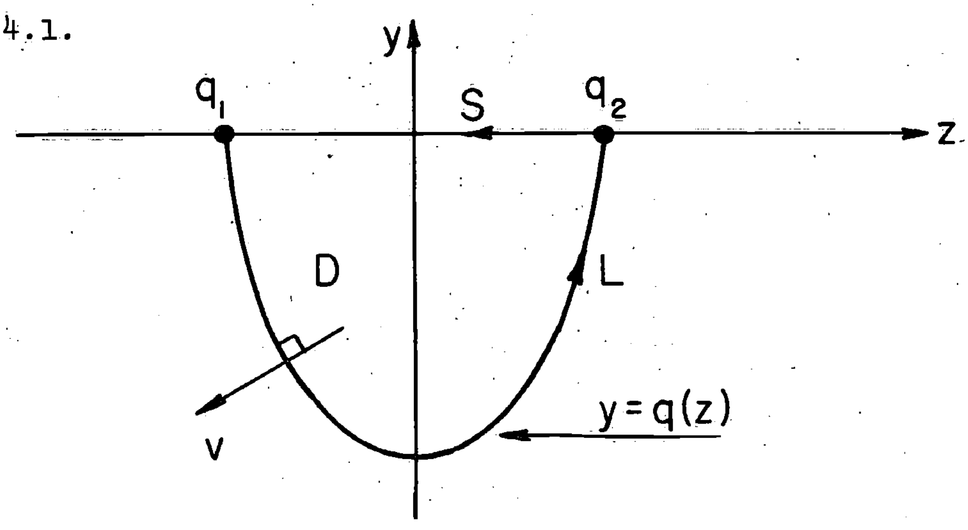

Figure 4.1

The linearized equations for the motion come from (2.15)-(2.20) by setting

$$\rho_o = 1$$

$$\sigma = 0$$

$$u_o = \frac{\gamma + v_o(y,z)}{\sqrt{gh}} .$$

They are

$$(4.1) \qquad \begin{cases} u_o u_x + u_{oy} v + u_{oz} w = - p_x \\ u_o v_x = - p_y \\ u_o w_x = - p_z \\ u_x + v_y + w_z = 0 . \end{cases}$$

At the channel wall given by $y = q(z)$ we must have

$$(4.2) \qquad v = w \frac{dq}{dz} .$$

If the equation of the free surface is

$$y = f(x,z)$$

the linearized free surface conditions, to be satisfied at $y = 0$, are

$$(4.3) \qquad \begin{cases} v(x,0,z) = u_o(0,z) f_x(x,z) \\ -f_x(x,z) + p_x(x,0,z) = 0 . \end{cases}$$

It seems that the easiest way to conduct an analysis of the above equations is to regard the dynamic pressure p as the fundamental dependent variable. For any point in the bounded domain D the pressure p must satisfy

$$\frac{\partial}{\partial x}\left(\frac{p_x}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{p_y}{u_o^2}\right) + \frac{\partial}{\partial z}\left(\frac{p_z}{u_o^2}\right) = 0 \; . \tag{4.4}$$

In addition the following boundary conditions must be satisfied.

$$\begin{aligned} &\text{On L:} \quad p_{\nu} = 0 \; . \\ &\text{On S:} \quad p_y(x,0,z) + u_o^2 p_{xx}(x,0,z) = 0 \; . \end{aligned} \tag{4.5}$$

We use ν to denote the unit outward normal to the boundary of D.

If we attempt to follow the method of Weinstein we are led to this eigenvalue problem. For D:

$$\frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) = \frac{\lambda^2 \psi(z,y)}{u_o^2} \; .$$

On L: $\psi_{\nu} = 0$.

On S: $\psi_y = u_o^2 \lambda^2 \psi$.

This, however, is not a standard eigenvalue problem because the eigenvalue parameter λ^2 appears in the boundary condition along S (with unusual sign) and S covers only part of the boundary of D. Instead of trying to establish the existence of a complete set of eigenfunctions for this case we will proceed by using an alternate method based on the generalized Fourier transform.

Let the right hand transform of $p(x,y,z)$ be

$$\Phi(\lambda,y,z) = \int_0^{\infty} e^{i\lambda x}\, p(x,y,z)dx$$

where $\operatorname{Im} \lambda = a > 0$. Let the left hand transform of p be

$$\Phi_1(\lambda,y,z) = \int_{-\infty}^{0} e^{i\lambda x}\, p(x,y,z)dx$$

where $\operatorname{Im} \lambda = b < 0$. By taking the magnitudes of a and b sufficiently large these transforms exist for any p of exponential order. The recovery formula for p is

$$p(x,y,z) = \frac{1}{2\pi}\int_{-\infty+ia}^{\infty+ia} e^{-ix\lambda}\,\Phi d\lambda + \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} e^{-ix\lambda}\,\Phi_1 d\lambda \,.$$

The application of the right hand transform to (4.4) and (4.5) for $x > 0$ gives

$$(4.6) \qquad \frac{\partial}{\partial z}\left(\frac{\Phi_z}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{\Phi_y}{u_o^2}\right) = \frac{\lambda^2\Phi}{u_o^2} + \frac{p_x(0,z,y)}{u_o^2} - \frac{i\lambda p(0,z,y)}{u_o^2}$$

to be satisfied in D, subject to

$$(4.7) \qquad \Phi_\nu = 0$$

on L, and

$$(4.8) \qquad \Phi_y = u_o^2\lambda^2\Phi + p_x(0,z,0)u_o^2(0,z) - i\lambda p(0,z,0)u_o^2(0,z)$$

on S. Similarly the application of the left hand transform to (4.4) and (4.5) for $x < 0$ gives

$$\frac{\partial}{\partial z}\left(\frac{\Phi_{1z}}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{\Phi_{1y}}{u_o^2}\right) = \frac{\lambda^2\Phi_1}{u_o^2} - \frac{p_x(0,z,y)}{u_o^2} + \frac{i\lambda p(0,z,y)}{u_o^2}$$

to be satisfied in D, subject to

$$\Phi_{1\nu} = 0$$

on L, and

$$\Phi_{1y} = u_o^2\lambda^2\Phi_1 - p_x(0,z,0)u_o^2(0,z) + i\lambda p(0,z,0)u_o^2(0,z)$$

on S.

From the equations and boundary conditions which Φ and Φ_1 must satisfy, it is evident that

$$\Phi_1(\lambda,z,y) = -\Phi(\lambda,z,y) .$$

Therefore we see that

$$(4.9) \qquad p(x,y,z) = -\frac{1}{2\pi}\int_C e^{-ix\lambda}\,\Phi(\lambda,z,y)d\lambda$$

where C is the path $C = C_1 + C_2$ shown in Fig. 4.2. The lines C_1 and C_2 are parallel to the real axis in the λ-plane and their

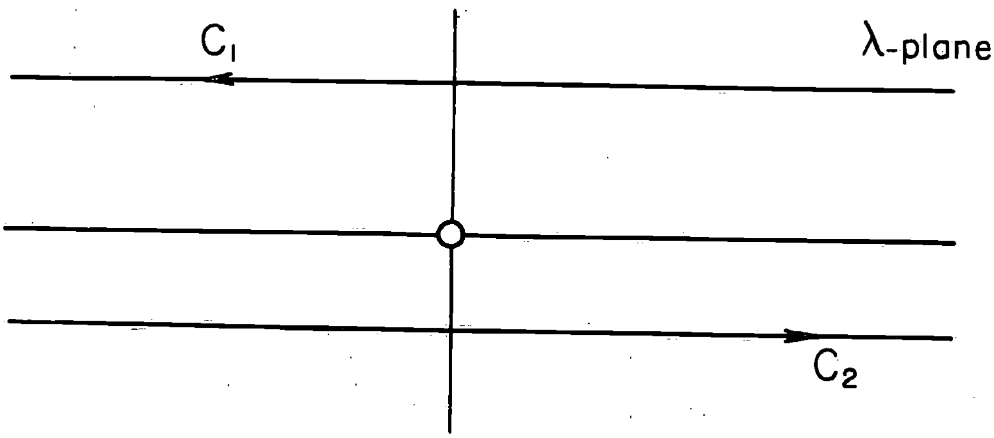

Figure 4.2

distances from the real axis can be adjusted to admit functions p of various exponential orders.

An integral equation formulation for the determination of Φ can be used to show that it is expressible as a ratio

$$\Phi = \frac{\psi(z,y;\lambda)}{\omega(\lambda)}$$

in which each of ψ and ω is an entire function of λ. If the disposition of the zeros of $\omega(\lambda)$ is known then the behavior of p with respect to x can be found from (4.9) by using the theory of residues. Also, if we require p to be bounded we must choose the path C in (4.9) so that it contains only <u>real</u> poles of ψ/ω and if necessary ψ must be modified so that these poles are poles of the first order. Now, the substitution of ψ/ω for Φ in (4.6)-(4.8) shows that the zeros of $\omega(\lambda)$ are just the eigenvalues of

(4.10) $$\frac{\partial}{\partial z}\left(\frac{\psi_z}{u_o^2}\right) + \frac{\partial}{\partial y}\left(\frac{\psi_y}{u_o^2}\right) = \frac{\lambda^2\psi}{u_o^2}$$

with

(4.11) $$\psi_\nu = 0$$

on L, and

(4.12) $$\psi_y = u_o^2\lambda^2\psi$$

on S. Hence our problem is again reduced to a study of eigenvalues. Here, however, we do not need to know anything about the completeness of the set of eigenfunctions.

For the operator

$$E \equiv \frac{\partial}{\partial z}\frac{1}{u_o^2}\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\frac{1}{u_o^2}\frac{\partial}{\partial y}$$

we have the following identities

(4.13) $$\iint_D \psi E(\theta) + \iint_D \left(\frac{\psi_z\theta_z + \psi_y\theta_y}{u_o^2}\right) = \int_{L+S} \frac{\psi\theta_\nu}{u_o^2}\, ds$$

(4.14) $$\iint_D \{\psi E(\theta) - \theta E(\psi)\} = \int_{L+S} \left\{\frac{\psi\theta_\nu - \theta\psi_\nu}{u_o^2}\right\} ds$$

where s is the arc length along the boundary of D. If $\overline{\lambda}$ and $\overline{\psi}$ are the respective conjugates of λ and ψ and if we identify the conjugates with θ in the above identities they show that

(4.15) $$(\overline{\lambda}^2 - \lambda^2)\left\{\int_{q_1}^{q_2} \psi\overline{\psi}dz - \iint_D \frac{\psi\overline{\psi}}{u_o^2}\right\} = 0$$

and

$$(4.16) \qquad \bar{\lambda}^2 \left\{ \int_{q_1}^{q_2} \psi\bar{\psi} dz - \iint_D \frac{\psi\bar{\psi}}{u_o^2} \right\} = \iint_D \left[\frac{\psi_z\bar{\psi}_z + \psi_y\bar{\psi}_y}{u_o^2} \right] .$$

From (4.15) and (4.16) it follows that

$$(4.17) \qquad (\bar{\lambda}^2 - \lambda^2) \iint_D \left[\frac{\psi_z\bar{\psi}_z + \psi_y\bar{\psi}_y}{u_o^2} \right] dzdy = 0 ,$$

and from this we conclude that the eigenvalues are either real or pure imaginary numbers. It should be noted that if λ is an eigenvalue then so is $-\lambda$. It should also be noted that $\lambda = 0$ is an eigenvalue of (4.10)-(4.12) and that the corresponding eigenfunction is ψ_o = const. $\neq 0$. We infer from the last two observations that $\lambda = 0$ is at least a triple zero of $\omega(\lambda)$ if the speed γ corresponds to a transition from real eigenvalues to pure imaginary eigenvalues.

Suppose $\lambda = 0$ is a simple zero of $\omega(\lambda)$ and the only one in a strip $\sum$ which contains the real axis. Then, as we can see from (4.9), the only bounded solution for p is given by the residue of

$$e^{-ix\lambda} \Phi(z,y;\lambda) = \frac{e^{-ix\lambda} \psi(z,y;\lambda)}{\omega(\lambda)}$$

at $\lambda = 0$; that is, $p = p(z,y)$. However, if p does not depend on x the equations (4.4)-(4.5) show that p is constant. With p constant it follows from the equations (4.1)-(4.3) that the flow must be an equilibrium flow.

Suppose next that the speed γ is such that $\lambda = 0$ is a triple zero of $\varpi(\lambda)$ and the only one in $\sum$. For this case, the residue of

$$e^{-ix\lambda}\,\Phi(z,y;\lambda) = \frac{e^{-ix\lambda}\,\psi(z,y;\lambda)}{\varpi(\lambda)}$$

at $\lambda = 0$ would generate unbounded terms for p unless $\psi(z,y;\lambda) = 0$; or unless $\psi(z,y;\lambda)$ possesses a double zero at $\lambda = 0$. If we impose either of these conditions we find again that $p = p(z,y)$ and this, as we indicated above, implies an equilibrium flow. If $\lambda = 0$ is the only zero of $\varpi(\lambda)$ in $\sum$; and if it is a zero of odd multiplicity greater than three; an analysis similar to the above leads to the same result. In other words, we conclude from this paragraph and the last one that the equilibrium flow is the only bounded flow if all the eigenvalues are pure imaginaries including $\lambda = 0$.

With the real eigenvalue λ_r we can associate the real eigenfunction ψ. By setting $\theta = \psi$ in (4.13) we have

$$(4.18) \qquad \lambda_r^2 \int_{q_1}^{q_2} \psi^2 dz = \lambda_r^2 \iint_D \frac{\psi^2}{u_o^2} + \iint_D \frac{(\psi_z^2 + \psi_y^2)}{u_o^2} .$$

By setting $\theta = 1$ in (4.14) we have

$$(4.19) \qquad \int_{q_1}^{q_2} \psi dz = \iint_D \frac{\psi}{u_o^2} .$$

An application of Schwartz's inequality to the last equation gives

$$\left(\int_{q_1}^{q_2} \psi dz\right)^2 \leq \iint_D \frac{1}{u_o^2} \cdot \iint_D \frac{\psi}{u_o^2} .$$

With this, (4.18) leads to

$$(4.20) \qquad \lambda_r^2 \left\{\iint_D \frac{1}{u_o^2} \cdot \int_{q_1}^{q_2} \psi^2 dz - \left[\int_{q_1}^{q_2} \psi dz\right]^2\right\}$$

$$\geq \iint_D \frac{1}{u_o^2} \cdot \iint_D \left[\frac{\psi_z^2 + \psi_y^2}{u_o^2}\right] .$$

For the case of the rectangular channel ψ does not depend on z and the inequality (4.20) reads

$$(4.21) \qquad \lambda_r^2 \psi^2(0)\left\{\int_{-1}^{0} \frac{dy}{u_o^2} - 1\right\} \geq \int_{-1}^{0} \frac{dy}{u_o^2} \cdot \int_{-1}^{0} \frac{\psi_y^2 dy}{u_o^2} .$$

This shows, as we deduced in a different way in Section 3, that the critical speed is given by

$$\int_{-1}^{0} \frac{dy}{u_o^2} = 1$$

and that no real non-zero eigenvalue exists if γ is such that

$$(4.22) \qquad \int_{-1}^{0} \frac{dy}{u_o^2} \le 1 .$$

Hence, as we have shown above, for speeds γ which satisfy (4.22) the only bounded flow is the equilibrium flow.

When the cross section of the channel is arbitrary we can define the critical speed by requiring

$$(4.23) \qquad \iint_D \frac{1}{u_o^2} \cdot \int_{q_1}^{q_2} \psi^2 dz - \left[\int_{q_1}^{q_2} \psi dz\right]^2 = 0 .$$

This implies by virtue of (4.20) that

$$\iint_D \frac{\psi_z^2 + \psi_y^2}{u_o^2} = 0$$

or ψ = const. $\neq 0$. With this eigenfunction, (4.23) becomes

$$(4.24) \qquad \iint_D \frac{dzdy}{u_o^2} = b$$

where $b = q_2 - q_1$ is the dimensionless breadth of the channel. The critical speed defined by (4.24) is the value of γ which allows transit from real eigenvalues to pure imaginary values through $\lambda = 0$. A more detailed discussion of (4.24) can be found in Peters [5].

If $1/u_o^2$ were a negative parameter then all of the eigenvalues would have to be pure imaginaries as (4.16) would show. Since the eigenvalues λ depend continuously on $1/u_o^2$ we conclude that when

$1/u_o^2$ is small there must be a corresponding λ^* of least absolute magnitude which is either a pure imaginary or a real number which is small in absolute magnitude. If the magnitude of $1/u_o^2$ is sufficiently small we cannot satisfy (4.24) because then

$$\iint_D \frac{dzdy}{u_o^2} <<< b$$

and therefore λ^* must be a pure imaginary. Now let γ in

$$u_o = \frac{\gamma + v_o(y,z)}{\sqrt{gh}}$$

be decreased until γ satisfies

$$gh \iint_D \frac{dzdy}{(\gamma + v_o)^2} = b \ .$$

The eigenvalue λ^* must then pass through a continuum of imaginary values until the origin is reached. If we continue to decrease γ until

$$gh \iint_D \frac{dzdy}{(\gamma + v_o)^2} > b$$

the eigenvalue λ^* passes to real values.

We can now conclude from the above analysis that if the speed γ is such that

$$gh \iint_D \frac{dzdy}{[\gamma + v_o]^2} \leq b$$

then no real and non-zero eigenvalue exists and consequently the only possible bounded flow in the channel with arbitrary cross section is the equilibrium flow defined in Section 2.

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13. ABSTRACT

This report presents a proof of the uniqueness of a parallel three-dimensional shear flow in a channel with arbitrary cross section where the speed of the flow is not less than the highest critical speed. The investigation also includes a two-dimensional analysis in which it is assumed that while the flow velocity varies with the depth, the density also depends on the depth; and for this case the development leads to a formula which gives a good approximation to the highest critical speed.

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