A MODIFIED VELOCITY DEFECT LAW FOR TURBULENT BOUNDARY LAYERS WITH INJECTION

by

T. N. Stevenson
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by

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SUMMARY

It is shown by comparison with experimental results, that a modified velocity defect law for turbulent boundary layers with injection is

\[
\frac{2u_\tau}{v_w} \left\{ \left( 1 + \frac{v_w u}{u_r^2} \right)^{\frac{1}{2}} - \left( 1 + \frac{v_w u}{u_r^2} \right)^{\frac{1}{2}} \right\} = F \left( \frac{y}{b} \right)
\]

where \( F \) is a universal function.

The variation of skin friction with Reynolds number and the shear distribution are evaluated with the aid of this relation, and are found to be in good agreement with experimental results.
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<td>Local skin friction coefficient</td>
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<td>( u_r )</td>
<td>Friction velocity, ( u_r = \sqrt{\frac{c_f}{2}} = \sqrt{\frac{f_w}{\rho}} )</td>
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<td>( v )</td>
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<td>( x, y )</td>
<td>Coordinates along and normal to the wall</td>
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<td>( \delta )</td>
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<td>( \delta_p )</td>
<td>The value of ( y ) at which ( F = 1.0 )</td>
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<td>Shear stress ( \left( = \mu \frac{\partial u}{\partial y} - \rho u_r v \right) )</td>
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<td>( \chi )</td>
<td>( \frac{u_r}{u_t} )</td>
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LIST OF SYMBOLS

Superscripts

* values which were obtained from the 'law of the wall equation with suction or injection' as described in ref. 1.

' differentiation with respect to \( \zeta \)

Subscripts

1 free stream conditions

w conditions at the wall
1. Introduction

A law of the wall for turbulent boundary layers with suction or injection is given in reference 1. This is now extended to the outer part of the turbulent boundary layer, and a modified velocity defect law equation is obtained, which is found to be in good agreement with the available experimental data. The modified defect law is used to calculate the distribution of shear stress across the boundary layer and the variation of skin friction with Reynolds number.

The form of the modified velocity defect law equation is also derived by considering dimensional similarity.

2. The modified velocity defect equation

For a turbulent boundary layer, over an impermeable surface with zero pressure gradient, von Kármán showed, that the equation for the mean velocity distribution in the inner and outer region, is given by

\[ \frac{u}{u_T} = A \log_{10} \left( \frac{yu_T}{\nu} \right) + \phi \left( \frac{y}{\delta} \right) \]  

where \( \phi \left( \frac{y}{\delta} \right) \) is a universal function, having the constant value \( \phi(0) \) throughout the inner region.

It follows that

\[ \frac{u_1 - u}{u_T} = - A \log_{10} \left( \frac{y}{\delta} \right) + \phi(1) - \phi \left( \frac{y}{\delta} \right) = f \left( \frac{y}{\delta} \right) \]  

is the velocity defect law in the outer region of the boundary layer, whereas

\[ \frac{u}{u_T} = A \log_{10} \left( \frac{yu_T}{nu} \right) + \phi(0) \]

is the law of the wall in the inner region.

\( f \left( \frac{y}{\delta} \right) \) is a universal function.

In the case of a permeable wall, these laws are modified as a result of the finite transpiration velocity at the wall. When the external pressure gradient is zero, it is found that the 'law of the wall with suction or injection' (ref.1) is

\[ 2 \frac{u_T}{v_w} \left\{ \left( 1 + \frac{v_w u}{u_T} \right)^{\frac{1}{2}} - 1 \right\} = A \log_{10} \left( \frac{yu_T}{\nu} \right) + B \]

where \( B \) is in general a function of \( \frac{v_w}{u_T} \). (In ref.1, B was shown to take the same values as for the case \( v_w = 0 \) over the range of transpiration velocities, which result in a measurable skin friction.)
The equation for the inner and outer regions is now written as

\[ \frac{2u_r}{v_w} \left( \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} - 1 \right) = A \log \left( \frac{y u_r}{v} \right) + \Phi (\gamma_0) \quad (4) \]

where \( B \) in equation 3 is equal to \( \Phi (0) \).

It follows that

\[ \frac{2u_r}{v_w} \left( \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} - 1 \right) = -A \log_{10} \left( \frac{\gamma_0}{\delta} \right) + \Phi (1) - \Phi (\gamma_0) \]

\[ = F \left( \frac{\gamma_0}{\delta} \right) \quad (5) \]

This is the 'modified velocity defect law with injection and zero pressure gradient'. It is shown how this law can be derived by dimensional similarity in the appendix.

Townsend (ref. 2) plots \( \frac{u_r - u}{u_r} \) against \( \frac{\gamma_0}{\delta} \) for various values of \( x \) for the case of zero blowing, and verifies that \( f \left( \frac{\gamma_0}{\delta} \right) \) is a universal function when there is no pressure gradient. \( \delta_0 \) is defined as the value of \( y \) at which \( \frac{u_r - u}{u_r} = 1 \).

When there is injection, the term \( \delta_0 \) will be defined as the value of \( y \) at which

\[ \frac{2u_r}{v_w} \left( \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} - \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} \right) = 1 \]

A comparison with experimental results will now be made to show the form of the function \( F \left( \frac{\gamma_0}{\delta_0} \right) \).

It can be seen from equation 5 that experimental results should fall onto one curve when

\[ \frac{2u_r}{v_w} \left( \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} - \left( \frac{1 + \nu_w u}{u_r^2} \right)^{\frac{1}{2}} \right) \]

is plotted against \( \frac{\gamma_0}{\delta_0} \).

The experimental results obtained by the author (ref. 3) are presented in fig. 1. The curve for no blowing which is presented in ref. 2, is also shown in fig. 1. The experiments in ref. 3 are for axisymmetric flow and are therefore not sufficient proof that \( F \left( \frac{\gamma_0}{\delta_0} \right) \) is universal for a flat plate.

Mickley and Davis' results (ref. 4) for flow over a flat plate, were analysed in some detail in ref. 1 using the 'law of the wall equation with suction or injection'. The values of \( u_r^* \), which were derived in ref. 1 were compared with Mickley and Davis' experimental values and found to be in good agreement. \( u_r^0 \) is the value of \( u_r^* \) which is obtained by plotting the experimental velocity profile on the law of the wall profiles, as described in detail in ref. 1. These values of \( u_r^0 \) are now used to calculate \( F \), which is presented in fig. 2.
The logarithmic plot of the velocity defect curve is shown in fig. 3.

These results collectively show that $F(\bar{V}/\delta)$ is a universal function. This is discussed in more detail in section 5.

3. The shear distribution across the boundary layer

If the continuity and momentum equations for a turbulent boundary layer are integrated from $o$ to $y$ then

$$v = - \int_0^y \frac{\partial u}{\partial x} \cdot dy + v_w$$

and

$$\frac{\partial}{\partial x} \int_0^y u^2 \ dy + uv = \frac{du}{dx} \ y + \frac{r}{\rho} \ rac{rv}{\rho}.$$  \(7\)

$r$ is the shear stress $\left( = \mu \frac{\partial u}{\partial y} - \rho u v \right)$. Variations of the mean square turbulent velocity components $u^2/\rho$ and $v^2/\rho$ with $x$ are neglected.

Thus, when equation (6) is substituted into (7)

$$\frac{r}{\rho} - \frac{rv}{\rho} = \frac{v_w u - u_t \frac{du}{dx}}{y} + \frac{\partial}{\partial x} \int_0^y \frac{u^2 dy}{u} - \frac{u_t}{\partial x} \int_0^y \frac{u dy}{u}.$$  \(8\)

This equation is now written in a form which includes the momentum defect term

$$\frac{r}{\rho} - \frac{rv}{\rho} = v_w u + \frac{du}{dx} \int_0^y (u - u_t) \ dy + (u_t - u) \frac{\partial}{\partial x} \int_0^y u \ dy + \frac{\partial}{\partial x} \int_0^y (u^2 - uu_t) \ dy.$$  \(9\)

The modified velocity defect law equation (5) is

$$\frac{2u_t}{u_w} \left( \left( 1 + \frac{v_w u}{u_t^2} \right) \right)^{1/2} - \left( 1 + \frac{v_w u}{u_t^2} \right)^{1/2} = F.$$  \(10\)

where $F$ is the universal function of $\bar{V}/\delta$.

A more convenient form of (9) can be derived if we define the functions, $\zeta$, $S$, $\eta$, $C_n$, $\theta$ and $N$ (with $u_t$ and $v_w$ constant) as

$$\zeta(x) = \frac{u_t}{u_t(x)}$$  \(11a\)

$$S(x) = \left( 1 + \frac{v_w u}{u_t(x)} \right)^{1/2}$$  \(11b\)

$$\eta(x) = \frac{\bar{V}}{\delta(x)}$$  \(11c\)
Thus, for zero pressure gradient, equation (9) reduces to
\[
\frac{\tau}{\rho u_1^2} = \left( \frac{u_1}{u_1} \right)^2 + \frac{v_w}{u_1} \cdot \frac{d u}{u_1} + \left\{ \left( \frac{1}{u_1} \right) N' - \theta' \right\}
\]
(15)

where primes denote differentiation with respect to \( \zeta \)

When \( \eta = 1 \) equation 15 reduces to
\[
\theta = \left( \frac{u_1}{u_1} \right)^2 + \frac{v_w}{u_1} - \frac{d \theta}{dx} \theta_1' \tag{16}
\]

The subscript \( 1 \) is used to indicate values at \( \eta = 1 \)

Thus
\[
\frac{d \theta}{dx} = \left( \frac{1}{\zeta^2} + \frac{v_w}{u_1} \right)/\theta_1' \tag{17}
\]

When \( \eta = 1 \) the velocity distribution, equation (4), becomes
\[
\frac{2 u_T}{v_w} (S - 1) = \frac{1}{K} \log_e \left( \frac{u_T}{v} \right) + \Phi \tag{18}
\]

or
\[
\delta = \frac{v}{u_T} \exp M \tag{19}
\]

where
\[
M = \frac{2}{v} \cdot \frac{u_1}{v_w} (S - 1) - \Phi \tag{20}
\]

Rearrangement of the modified velocity defect equation (10) gives:
\[
u = u_1 - SF u_T + \frac{\rho^2 v_w}{4}
\]

The velocity distribution of equation (21) is substituted into the integrals for \( \theta \) and \( N \) and the integrals are evaluated. The errors involved in assuming that \( F(Y/\delta) \) is universal in the sublayer region, are negligible when \( Y/\delta \) is greater than about 0.08

Hence
\[
\theta = \frac{v}{u_1} e^M \left( \frac{\zeta S}{C_1} - \frac{v_w C_2 S^2 C_2}{4 u_1} + \frac{v_w C_2 S}{2 u_1 \zeta} - \frac{C_4 v_w}{16 u_1^2} \right) \tag{22}
\]
This is differentiated with respect to \( \zeta \) to give

\[
\theta' = \left\{ \left( \frac{C_5 V_W}{u_1} - \frac{5}{4} \frac{C_2 V_W}{u_1} + \frac{C_3}{2} \frac{C_5 V_W^2 \zeta}{2u_1^2} - \frac{C_4 V_W^2}{16 u_1^2} \right) + \right. \\
\left. \left( \frac{S C_5}{\zeta} - \frac{5}{4} \frac{C_2 V_W}{u_1} + \frac{S C_3 V_W}{2u_1^2} - \frac{C_4 V_W^2 \zeta}{16 u_1^2} \right) \right\} \frac{\nu}{u_1} \exp M
\]

(23)

where

\[
M' = \frac{2K}{S} - \frac{2Ku_1}{\zeta V_W} (S - 1)
\]

(24)

Similarly,

\[
N' = \frac{\nu}{u_1} \left\{ - \frac{C_5 V_W}{u_1 S} \frac{V_W C_2}{4u_1} - \frac{S C_5 M'}{4u_1} + \frac{V_W C_2 \zeta M'}{4u_1} + \frac{M' \nu \zeta + \eta}{\exp M} \right\}
\]

(25)

All the terms in the expression for the shear stress \( \tau \) (equ. 15), are now known in terms of the universal profile parameters, \( F, C_1, C_2, C_3 \), and \( C_4 \). The profile parameters were evaluated using the universal function \( F(\eta) \) and equation 12. They are presented in fig. 4. In order to calculate the shear distribution across the boundary layer, only the skin friction, blowing velocity, \( V_W \), and the free stream velocity need be specified. No experimental results are required.

The shear distributions which have been calculated are discussed in Section 5.

4. The skin friction \( \sim \) Reynolds number variation

The equation relating \( R_\theta \) to the skin friction for particular values of \( \frac{V_W}{u_1} \) is obtained by multiplying equation (22) by \( \frac{u_1}{\nu} \) and letting \( \eta = 1 \). Thus

\[
R_\theta = \frac{\theta_1 u_1}{\nu} = \left( S(C_1) - \frac{V_W(C_2)}{4u_1} - \frac{S(C_5)}{\zeta} + \frac{V_W(C_2) S}{2u_1} - \frac{S(C_4) V_W^2 \zeta}{16 u_1^2} \right) \exp M
\]

(26)

\( R_\theta \) is presented in figs. 5 and 6 for different values of \( \frac{V_W}{u_1} \).

The equation for \( R_x \) is obtained by integrating equation (17):

\[
R_x = \frac{\frac{\nu}{u_1} \int_0^\zeta \frac{\theta_1'}{\left( \frac{1}{\zeta^2} + \frac{V_W}{u_1} \right)} d\zeta}{\nu}
\]

(27)

This was integrated numerically and the resulting curves of \( R_x \sim C_f \) are presented in fig. 7.
5. Discussion

The experimental results of Mickley and Davis (ref. 4) and of the author (ref. 3), as presented in figs. 1 and 2, show that the function $F$ in the modified velocity defect law equation with injection (equation 5), is independent of $v_w$ and $u_r$ in the outer region of the boundary layer.

Figure 3 shows how $\frac{2u_r}{v_w} \left[ \left( 1 + \frac{v_w u_t}{u_r^2} \right)^{\frac{1}{2}} - \left( 1 + \frac{v_w u_t}{u_r^2} \right) \right]$ varies with $\eta$ in the inner turbulent region and shows some scatter in the value of $\phi(1)$. However, this scatter is also noted in the case of no blowing by Coles (ref. 7). In the calculations $\phi(1)$ has been assumed constant. The actual values used for the constants differ from those used by Coles, but the final $C_f^{*}$-Reynolds number curves when $v_w = 0$ are almost the same. The constants are compared in Table 1.

Rubesin (ref. 5) presented a theory for injection and evaluated the variations of $C_f$ with $R_x$ by assuming that the law of the wall region extended to the edge of the boundary layer (see ref. 6). The constants in the law of the wall are changed from their no-blowing experimental values, to values which give the correct $C_f \sim R_x$ variation when $v_w = 0$. These constants were then used for the case with blowing, together with the assumption that $\frac{u}{u_r}$ at the edge of the sublayer is independent of $v_w$. The latter assumption is considerably different from that implied by the law of the wall equation with suction or injection (equation 3). The variations of $C_f$ with $R_x$ which were obtained by Rubesin and those presented in this report are compared in fig. 7. The curves are different because of the different assumptions which were made.

The distribution of shear, $T = \mu \frac{\partial u}{\partial y} - \rho \overline{u'v'}$, has been evaluated using equation (15), when $v_w = .003$ and when $v_w = 0$ (see fig. 8). There are large changes in the shear distribution with increase in the blowing velocity. The velocity defect curve, even when $v_w = 0$ does not remain universal in the region near the wall and the shear stress calculation must be extrapolated into this region. This is indicated by the dotted line in fig. 8.

Leidon (ref. 8) substituted some of Mickley and Davis' experimental results directly into equation (8) and integrated numerically. The shear distribution which Leidon obtained is shown in fig. 9. The distribution is of the same form as that of equation (15).

It is not surprising to find a likeness between the shear distribution with blowing, and that for a boundary layer in an adverse pressure gradient (ref. 9).

A similarity solution is presented in the appendix. Similar profiles are assumed to exist of the form

$$f \left( \frac{u}{u_r}, \frac{v_w}{u_r} \right) = g \left( \frac{y u_r}{v} \right)$$

in the inner region,
and \[ F\left(\frac{u}{u_T}, \frac{u_T}{u_T}, \frac{v_w}{u_T}\right) = G\left(\frac{y_b}{f}\right) \]

in the outer region.

It is shown that an overlap region exists if

\[ f\left(\frac{u}{u_T}, \frac{v_w}{u_T}\right) = \frac{1}{K} \log_e \left(\frac{y_{ur}}{v}\right) + C_3 \]

in the overlap region. \( K \) and \( C_3 \) are constants.

The equation for the outer region is shown to be

\[ f\left(\frac{u}{u_T}, \frac{v_w}{u_T}\right) - f\left(\frac{u}{u_T}, \frac{v_w}{u_T} \right) = \phi(1) - \psi\left(\frac{y_b}{f}\right) - \frac{1}{K} \log_e \left(\frac{y_b}{f}\right) \]

where \( \psi\left(\frac{y_b}{f}\right) \) is a constant in the overlap region. This is of the same form as that obtained in Section 2.

Equation (28) is significant because it shows that the outer region is
governed by a function of the form \( f(u_t - u) \), and not \( f(u_t - u) \). This is discussed in more detail in ref. (10).

At the higher blowing velocities the modified velocity defect equation with injection reduces to

\[ 2\left(\frac{u_T}{v_w}\right)^\frac{1}{2} \left(1 - \left(\frac{u}{u_T}\right)^\frac{1}{2}\right) = F\left(\frac{y_b}{f}\right) \]  

(29)

Two velocity profiles of Mickley and Davis (ref. 4) for \( \frac{v_w}{u_T} = .009 \) are shown in fig. 10. The curves again fall close to the zero pressure gradient no blowing curve.

### 6. Conclusions

The modified velocity defect law, for turbulent boundary layers with injection is

\[ \frac{2u_T}{v_w} \left(1 + \frac{v_w u_t}{u_T^2}\right)^\frac{1}{2} - \left(1 + \frac{v_w u}{u_T^2}\right)^\frac{1}{2} = F\left(\frac{y_b}{f}\right) \]

where \( F\left(\frac{y_b}{f}\right) \) is universal.

The skin friction variations with Reynolds number, and also the shear distribution across a layer with injection, have been evaluated.
References


2. Townsend, A.A. The Structure of Turbulent Shear Flow Cambridge Univ. Press 1956. pp. 242-246

3. Stevenson, T.N. Experiments on injection into an incompressible turbulent boundary layer. To be published.


Acknowledgements

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A dimensional similarity derivation of the modified velocity defect law with injection

The method follows that of Millikan (ref. 11) but now includes a dimensionless parameter, \( P \). In the region near the wall the mean velocity distribution is assumed independent of the boundary layer thickness, \( \delta \), and in the outer region the mean velocity measured relative to the free stream velocity is assumed to depend on \( \delta \) and not on the viscosity, \( \nu \). It is assumed that similar velocity profiles exist of the form:

\[
\begin{align*}
\text{Inner region:} & & f\left( \frac{u}{u_r}, P \right) = g\left( \frac{\nu u_r}{\nu} \right) \quad \text{A.1} \\
\text{Outer region:} & & F_1 \left( \frac{u_r - u}{u_r}, \frac{u}{u_r}, P \right) = G\left( \frac{\gamma}{\delta} \right) \quad \text{A.2} \\
& & \text{or} \quad F_2 \left( \frac{u}{u_r}, \frac{u_1}{u_r}, P \right) = G\left( \frac{\gamma}{\delta} \right) \quad \text{A.3}
\end{align*}
\]

\( P \) must be independent of \( y \).

Equations A.1 and A.3 are differentiated to give:

\[
\begin{align*}
\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial u} \left( \frac{u}{u_r}, P \right) & = \frac{\nu u_r}{\nu} \cdot g'\left( \frac{\nu u_r}{\nu} \right) \quad \text{A.4} \\
\text{and } \frac{\partial F}{\partial y} \cdot \frac{\partial F}{\partial u} \left( \frac{u}{u_r}, \frac{u_1}{u_r}, P \right) & = \frac{\gamma}{\delta} \cdot G'\left( \frac{\gamma}{\delta} \right) \quad \text{A.5}
\end{align*}
\]

\( g' \) is \( \frac{d g}{d \left( \frac{\nu u_r}{\nu} \right)} \) and \( G' \) is \( \frac{d G}{d \left( \frac{\gamma}{\delta} \right)} \).

In an overlap region in which equations A.1 and A.2 are valid, the velocity gradient given by the two equations must be the same.

\[
\begin{align*}
\frac{\partial u}{\partial y} & = \frac{\nu u_r}{\nu} \cdot g'\left( \frac{\nu u_r}{\nu} \right) = \frac{\gamma}{\delta} \cdot G'\left( \frac{\gamma}{\delta} \right) \quad \text{A.6}
\end{align*}
\]

Now \( \frac{\nu u_r}{\nu} \) and \( \frac{\gamma}{\delta} \) are formally independent, and \( g'\left( \frac{\nu u_r}{\nu} \right) \) and \( G'\left( \frac{\gamma}{\delta} \right) \) are not functions of \( P \).
If \( f_1 \) and \( F_z \) are simply related i.e. the form of \( u \) in both equations is the same, then the gradients of the curves (\( f_1 \) against \( \frac{y u_r}{\nu} \) and \( F_z \) against \( \frac{y l_b}{\delta} \)) with respect to \( y \) will be related by the equation

\[
\frac{\partial f}{\partial y} = C_1 \frac{\partial F_z}{\partial y}
\]

where \( C_1 \) is a constant.

This is discussed in more detail in reference 10.

\( F_z \) is therefore of the form

\[
F_z \left( \frac{u}{u_r}, \frac{u_1}{u_r}, P \right) = F_z \left( \frac{u}{u_r}, P \right) + F_4 \left( \frac{u_1}{u_r}, P \right)
\]

Equations A.6 are now written

\[
\frac{\partial u}{\partial y} \frac{\partial f}{\partial u} = C_2
\]

and

\[
y \frac{\partial u}{\partial y} \frac{\partial F_z}{\partial u} = \frac{C_2}{C_1}
\]

where \( C_2 \) is a constant.

Therefore

\[
f \left( \frac{u}{u_r}, P \right) = C_2 \log_e \left( \frac{y u_r}{\nu} \right) + C_3
\]

and

\[
F_z \left( \frac{u}{u_r}, \frac{u_1}{u_r}, P \right) = \frac{C_2}{C_1} \log_e \left( \frac{y l_b}{\delta} \right) + C_4
\]

\( C_2, C_3 \), and \( C_4 \) are constants in view of equations A.1 and A.3.

The equation for \( F_z \), (or \( F_3 + F_4 \)) throughout the outer region may be written

\[
F_z = F_3 \left( \frac{u}{u_r}, P \right) + F_4 \left( \frac{u_1}{u_r}, P \right) = \frac{1}{C_1} \left\{ C_2 \log_e \left( \frac{y l_b}{\delta} \right) + \phi \left( \frac{y l_b}{\delta} \right) \right\}
\]

where \( \frac{1}{C_1} \phi \left( \frac{y l_b}{\delta} \right) = C_4 \) in the overlap region.
When \( u = u_1 \), this equation reduces to

\[
F_3\left( \frac{u_1}{u_T}, P \right) + F_4\left( \frac{u_1}{u_T}, P \right) = \frac{1}{C_1} \Phi(1) \quad \text{A.14}
\]

Equation A.13 subtracted from equation A.14 gives

\[
F_3\left( \frac{u_1}{u_T}, P \right) - F_3\left( \frac{u}{u_T}, P \right) = \frac{1}{C_1} \left\{ \Phi(1) - \Phi\left( \frac{\gamma}{\delta} \right) - C_z \log_e \left( \frac{\gamma}{\delta} \right) \right\} \quad \text{A.15}
\]

and integration of A.7 gives

\[
f\left( \frac{u}{u_T}, P \right) = C_1 \left[ F_3\left( \frac{u_1}{u_T}, P \right) - F_3\left( \frac{u}{u_T}, P \right) \right]
\]

therefore

\[
f\left( \frac{u_1}{u_T}, P \right) - f\left( \frac{u}{u_T}, P \right) = C_1 \left\{ F_3\left( \frac{u_1}{u_T}, P \right) - F_3\left( \frac{u}{u_T}, P \right) \right\} = \Phi(1) - \Phi\left( \frac{\gamma}{\delta} \right) - C_z \log_e \left( \frac{\gamma}{\delta} \right) \quad \text{A.16}
\]

This is the equation for the outer region. It is significant because it shows that the outer region is governed by a function of the form \( f(u_1) - f(u) \) and not \( f(u_1 - u) \).

If \( P = \frac{v_w}{u_1} \), equation A.11 has the same form as equation (3), 'The law of the wall equation with suction or injection'. Therefore \( f \) has the form

\[
f\left( \frac{u}{u_T}, \frac{v_w}{u_T} \right) = 2u_T v_w \left\{ \left( 1 + \frac{v_wu}{u_T^2} \right)^{-\frac{1}{2}} - 1 \right\}
\]

The constants in equation A.11 do of course take the same values as in the case of zero injection when \( f \) reduces to \( \frac{u}{u_T} \). The expression for \( f \) is now substituted into equation A.17 to give

\[
2u_T v_w \left\{ \left( 1 + \frac{v_wu}{u_T^2} \right)^{-\frac{1}{2}} - \left( 1 + \frac{v_wu}{u_T^2} \right)^{-\frac{1}{2}} \right\} = \Phi(1) - \Phi\left( \frac{\gamma}{\delta} \right) - C_z \log_e \left( \frac{\gamma}{\delta} \right)
\]

This equation is the modified velocity defect law equation, and is the same as that derived in section 2. (equation 5).
TABLE 1. The constants used by different authors.

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**FIG. 1. THE VELOCITY DEFECT CURVE.**

\[ \frac{2u_x}{u_y} \left( \frac{u_x}{u_y} \right)^{1/2} \left( \frac{v_x}{v_y} \right)^{1/2} \]

**FIG. 2. THE VELOCITY DEFECT CURVE.**

\[ \frac{2u_x}{u_y} \left( \frac{u_x}{u_y} \right)^{1/2} \left( \frac{v_x}{v_y} \right)^{1/2} \]

**FIG. 3. THE VELOCITY DEFECT CURVE - LOGARITHMIC PLOT.**
FIG. 4. THE VARIATION OF THE PROFILE PARAMETERS $c_1$, $c_2$, $c_3$, $c_4$. (EQUATION 12)

FIG. 5. THE VARIATION OF SKIN FRICTION WITH REYNOLDS NUMBER, $R_\theta$. (EQUATION 26)
FIG. 6. THE VARIATION OF SKIN FRICTION WITH REYNOLDS NUMBER, $R_b$. (EQUATION 26)

FIG. 7. THE VARIATION OF SKIN FRICTION WITH REYNOLDS NUMBER, $R_x$.

FIG. 8. SHEAR DISTRIBUTION ACROSS THE BOUNDARY LAYER. (EQUATION 17)
FIG. 9. SHEAR DISTRIBUTION CALCULATED BY LEADON FROM MICKLEY & DAVIS' RESULTS.

FIG. 10. THE VELOCITY DEFECT CURVE.