Mesh Association by Projection along Smoothed-Normal-Vector Fields: Association of Closed Manifolds

E.H. van Brummelen
E.H.vanBrummelen@tudelft.nl

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E.H. van Brummelen
Delft University of Technology, Faculty of Aerospace Engineering
P.O. Box 5058, 2600 GB Delft, The Netherlands

ABSTRACT
The necessity to associate two geometrically distinct meshes arises in many engineering applications. Current mesh-association algorithms are generally unsuitable for the high-order geometry representations associated with high-order finite-element discretizations. In the present work we therefore propose a mesh-association method for high-order geometry representations. The associative map defines the image of a point on a mesh as its projection along a so-called smoothed-normal-vector field onto the other mesh. The smoothed-normal-vector field is defined by the solution of a modified Helmholtz equation with right-hand-side data corresponding to the normal-vector field. Classical regularity theory conveys that the smoothed-normal-vector field is continuously differentiable, which renders it well suited for a projection-based association. Moreover, the regularity of the smoothed-normal-vector field increases with the regularity of the normal-vector field and, hence, the smoothness of the association increases with the smoothness of the geometry representations. The proposed association method thus accommodates the higher regularity that can be provided by high-order geometry representations. The elementary properties of smoothed-normal-projection association are established by analysis and by numerical experiments on closed manifolds.

Keywords and Phrases: Non-matching meshes, mesh association, mesh incompatibility, high-order finite-element methods, multiphysics, fluid-structure interaction.

1. Introduction
The construction of an associative map between two distinct meshes is a common conundrum in many engineering and scientific disciplines, e.g., multiphysics computational science, biomedical imaging, and computer graphics. In multiphysics problems such as fluid-structure interaction, the solutions pertaining to the different physical subsystems generally possess disparate regularity properties. These disparate regularity properties prompt distinct approximation spaces, both in mesh size and in order of approximation. However, this implies that the approximation spaces are incompatible at their interface, and an association between the meshes is required to transfer data between the subsystems. In biomedical imaging, system properties are frequently measured at different geometric realizations of the system. For instance, measurements of the electric field or composition of the cardiovascular system typically provide data at different instants of the cardiac cycle. If the data is represented on a mesh, then an association between geometrically distinct meshes is required to enable a comparison of the data. In computer graphics, mesh association occurs in, for instance, morphing. The emphasis in this paper is on multiphysics applications, although many of the considerations extend to other applications as well.

A fundamental problem of mesh incompatibility is the concomitant incompatibility of the geometric realizations, i.e., the sets represented by the meshes are noncoincident, unless both meshes allow an exact representation of the geometry; see [8] for geometrically exact representations of non-trivial geometric objects. An elementary premise for defining a proper association between distinct meshes, is that their geometric realizations are homeomorphic, i.e., there must exist a continuous and continuously invertible bijection between the geometric realizations. In multiphysics applications, this premise is generally satisfied, because both meshes represent the same geometric object, e.g., the interface between a fluid and a structure. A proper association between meshes must comply
with several conditions. First, the association must be a homeomorphism, i.e., the association must provide a continuous and continuously invertible one-to-one map. Second, the map should be close to the identity or, equivalently, the distance between any one point on a geometric realization and its associate in the other geometric realization should be minimal. Third, the association should be sufficiently smooth, so that functions retain their regularity under the transformation induced by the map. Fourth, the association must be generic. In particular, it must in principle be applicable to geometric realization of arbitrary order and, moreover, it should be essentially dimension independent. The latter implies that the association should not be contingent on primitives which are particular for a certain spatial dimension, and which do not generalize to other dimensions. Finally, the association must be constructive and must allow an efficient and robust implementation.

Currently available mesh association algorithms do not comply with the aforementioned conditions. For instance, the conventional nearest-neighbour algorithm violates the homeomorphism property, as it is not generally one-to-one. Denoting by \( A \) and \( B \) the geometric realizations of two meshes, the nearest-neighbour algorithm associates with each point \( a \) on set \( A \) the point \( b \) on \( B \) which minimizes the distance to \( a \). However, as this does not imply that \( a \) is the nearest point on \( A \) to \( b \), nearest-neighbour association is not one-to-one. Moreover, one can simply infer that this map is only injective, and not generally surjective. For instance, near boundaries there can be points in \( B \) which do not have an original in \( A \). Hence, nearest-neighbour association does not provide a homeomorphism. The normal-projection (or orthogonal-projection) association in [15] defines the image of a point \( a \) in \( A \) as its projection onto \( B \) along the normal vector on \( A \) at \( a \). However, unless the set \( A \) is continuously differentiable, the normal vector is not continuous on \( A \) and, accordingly, the normal-projection association does not provide a continuous map. Moreover, one can infer that the association is not necessarily injective, nor surjective. Therefore, normal-projection association does not provide a homeomorphism. On the other hand, both the nearest-neighbour association and the normal-projection association allow very efficient implementations, e.g., through the vine-search algorithm [9] or extensions of the techniques in [14] and, moreover, they are close to the identity. This renders them acceptable for multiphysics computations with low-order discretizations, for which the smoothness of the map is not so relevant because the regularity of the approximation functions is low, and for which small discontinuities and small violations of the one-to-one correspondence do not essentially interfere with the accuracy of the approximation.

A homeomorphic mesh association for surfaces is provided by the mesh-overlay method in [10, 11]. The mesh-overlay method constructs a common refinement of the meshes, viz., a mesh which consists of the combinatorial union of both meshes, i.e., the union in the sense of mesh topology. The association is then made in this common refinement. The existence of such a common refinement is ascertained by the theorem that two simplicial complexes that triangulate a surface have simplicially isomorphic subdivisions; see, for instance, Ref. [5, Th.3.4.5]. The mesh-overlay method is based on a nearly-orthogonal projection, viz., the projection along the vector field that is formed by the interpolation of the average normal vectors in the mesh vertices. The inverse of the association is defined accordingly. As the interpolated average-normal-vector field is continuous, the association and its inverse are continuous. Moreover, under nonrestrictive conditions the association is one-to-one in the interior of the domain. Near boundaries and sharp features, special treatments must be invoked [9,12]. The mesh-overlay method yields an association that is homeomorphic and close to the identity, and it allows an efficient implementation. Moreover, most primitives of the association, e.g., the nearly-orthogonal projection, are essentially dimension independent. However, the method does not generalize to high-order geometry representations and, moreover, it is not generally smooth. The construction of the nearly-orthogonal vector field by means of a piecewise-linear interpolation of the average normal vectors is particular for piecewise-linear geometry representations. Moreover, the nearly-orthogonal vector field is only continuous and, hence, the association is not generally smooth.

For multiphysics applications, mesh association forms a primitive, which in conjunction with an appropriate interpolation method can be used to transfer data between the boundaries of the domains pertaining to the physical subsystems. An alternative means of transferring data in multiphysics applications, that bypasses the complications of mesh association, is by means of extension, e.g., by
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finite-interpolation elements [3] or radial basis functions [4], or by simply evaluating finite-element interpolation functions outside the corresponding elements. See also [14] for an extension approach. Considering two nearly contiguous domains $A$ and $B$, the section of the boundary of $B$ that is identified with the interface will partly intersect with the interior of $A$, and will partly be exterior to $A$. On the intersection, data is trivially transferred from the interior of $A$ to the boundary of $B$. The data on the part that is exterior to $A$ is obtained by extending the solution on $A$ beyond its boundaries, for instance, by means of radial basis functions. For finite-volume and finite-difference methods, the data between the grid points in the interior of $A$ can be defined similarly. It may be noted that the extension approach obviates an association of the interface meshes. However, the extension approach has two important disadvantages. First, the method violates trace-space relations. In brief, a trace-space relation defines the manner in which a function is to be evaluated at the boundary. This is in general not simply the function value as the boundary is approached, which is the underlying assumption in the extension approach. Instead, it depends sensitively on the manner in which boundary conditions are enforced. Violation of the trace-space relations generally results in loss of accuracy and loss of conservation. Second, if a solution exhibits large gradients near the boundary, e.g., in the case of boundary layers or sharp features, the extension and the evaluation away from the boundary can result in large errors.

In the present article we present a mesh-association method that is suitable for high-order finite-element discretizations, with corresponding high-order geometry representations. The presented mesh association is homeomorphic, it is close to the identity, it is sufficiently smooth, it is generic, and it admits an efficient implementation. Similar to [10, 11], we define the associates of points in one geometric realization as their projection along a nearly-orthogonal vector field onto the other geometric realization. However, we define the vector field by the solution of a modified Helmholtz equation on the geometric realization, with right-hand-side data corresponding to the normal-vector field. The solution to this problem is approximated by means of finite elements of the same order as the geometric realization. Classical regularity theory for elliptic boundary-value problems conveys that the solution of the modified Helmholtz equation, referred to as the smoothed-normal-vector field, is at least continuously differentiable. This renders it well suited for a projection-based association. Moreover, the smoothness of the smoothed-normal-vector field increases with the smoothness of the geometric realization. The associative map therefore adapts to the higher regularity that can be provided by high-order geometry representations. In the present paper, we present the fundamentals of the smoothed-normal-projection association. To do so, we restrict our considerations to boundaryless manifolds. The extension to manifolds with boundaries will be treated elsewhere.

The contents of this paper are organized as follows: In section 2 we attend to a precise problem statement. Section 3 presents the fundamentals of the smoothed-normal-projection association for boundaryless manifolds. Based on the regularity properties of the smoothed-normal-vector field, this section establishes the suitability of the smoothed-normal-vector field for a projection-based association. In section 4 we examine several elementary properties of the smoothed-normal-vector field for a projection-based association, such as an upper bound on the distance between a point and its image for which uniqueness of the association can be ascertained. Section 5 illustrates the theory by means of representative numerical experiments for 2D and 3D settings. Finally, section 6 contains concluding remarks.

2. Problem statement

In this paper we are concerned with a methodology for associating two geometrically disparate meshes. To provide a precise specification of the mesh-association problem, this section first presents a general setting for the problem. Next, we provide details of the Lagrange elements used in this paper. The Lagrange basis is convenient for mesh-association problems by virtue of its nodal character. However, the presented analyses extend to other polynomial bases without modifications. Finally, we specify the mesh-association problem.
2.1 Problem setting

To furnish a setting for the problem, let \( X \) represent a Euclidean space of dimension \( d (d \in \{2, 3, 4\}) \), and let \( \{e_{(i)}\}_{i=1}^{d} \) represent an orthonormal basis of \( X \). We remark that the case \( d = 4 \) bears practical relevance for problems in space/time. Any element \( \mathbf{x} \in X \) can be characterized by its coordinates \( (x_1, \ldots, x_d) \) with respect to the basis according to \( \mathbf{x} = x_1 e_{(1)} + \cdots + x_d e_{(d)} \). Thus, we obtain an isomorphism between \( X \) and \( \mathbb{R}^d \), which enables us to identify \( X \) and \( \mathbb{R}^d \).

Consider a manifold \( M \) of dimension \( d := d - 1 \) embedded in \( \mathbb{R}^d \). For instance, \( M \) can model the interface between a solid and a fluid in 3D space. Below, we are concerned with approximations of \( M \). We restrict our considerations to simplicial approximations, i.e., approximations based on line segments in 2D, triangles in 3D, tetrahedrons in 4D, etc., because any other polytope can be subdivided into simplices. More precisely, we presume that the approximations are based on simplicial complexes; see, e.g., [5]. This implies that the simplices in the complex are connected in a one-to-one manner at their faces. The simplicial complexes generated by conventional meshing algorithms comply with this premise.

Denoting by \( A \) a simplicial complex in \( \mathbb{R}^d \), the underlying space of \( A \) is the subset \( |A| \subset \mathbb{R}^d \) consisting of the union of all the simplices in \( A \). If \( |A| \) is intended as an approximation to \( M \), then we assume that \( |A| \) is homeomorphic to \( M \), i.e., that there exists a continuous and continuously invertible one-to-one map from \( |A| \) to \( M \). In particular, for surfaces, the simplicial complex \( A \) (in conjunction with the homeomorphism \( |A| \rightarrow M \)) is referred to as a triangulation of \( M \). The underlying space \( |A| \) of a simplicial complex \( A \) represents a piecewise linear approximation of \( M \). Each simplex \( \sigma \) approximates a part of \( M \) according to

\[
\sigma = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{i=1}^{d} a_i N_i(\lambda_1, \ldots, \lambda_d), 0 \leq \lambda_i, \sum_{i=1}^{d} \lambda_i = 1 \}, \tag{2.1}
\]

where \( a_i \in \mathbb{R}^d \) are the vertices of the simplex, \( \lambda_1, \ldots, \lambda_d \) are barycentric coordinates, and \( N_i = \lambda_i \) are the linear base functions on the simplex. To define higher-order approximations, we conceive of the underlying space \( |A| \) as the domain of a finite-element-approximation space \( \mathcal{A}(|A|) \) of functions from \( |A| \) into \( \mathbb{R}^d \). Accordingly, the simplices are identified with finite elements. Considering a function \( \mathbf{u} : |A| \rightarrow \mathbb{R}^d \), the sets \( \mathbf{u}(|A|) \) and \( |A| \) and, hence, \( M \) are homeomorphic only if \( \mathbf{u} \) is continuous. Hence, we can limit ourselves to continuous finite-element spaces. For any \( \mathbf{u} \in \mathcal{A}(|A|) \), the set \( \mathbf{u}(|A|) \) is called the geometric realization of \( \mathbf{u} \). Note that if \( \mathcal{A}(|A|) \) contains the piecewise linear functions, then there exists a \( \mathbf{u} \in \mathcal{A}(|A|) \) such that the geometric realization \( \mathbf{u}(|A|) \) of \( \mathbf{u} \) coincides with the underlying space \( |A| \).

2.2 Lagrange elements on simplices

In particular, in this paper we opt for finite-element spaces based on classical Lagrange elements. Consider a generic simplex \( \Delta \subset \mathbb{R}^d \) with vertices \( e_{(1)}, e_{(2)}, \ldots, e_{(d)} \); see figure 1 for an illustration. Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \) represent barycentric coordinates on \( \Delta \). To facilitate the notation, we introduce the multi-index \( \mathbf{i} = (i_1, \ldots, i_d) \) and the index set

\[
I^v = \{ \mathbf{i} \in [0, 1, \ldots, v]^d : i_1 + \cdots + i_d = v \}, \tag{2.2}
\]

where \( v \) is a positive integer. The Lagrange base functions of order \( v+1 \) (degree \( v \)) are the polynomials \( \psi^v_i(\lambda) (i \in I^v) \) defined as

\[
\psi^v_{i_1, \ldots, i_d}(\lambda_1, \ldots, \lambda_d) = \prod_{k=1}^{d} \prod_{j=1}^{i_k} \left( \frac{v \lambda_k + 1}{j} - 1 \right). \tag{2.3}
\]

Introducing the nodal positions \( \mathbf{a}_i = v^{-1}(i_1 e_{(1)} + \cdots + i_d e_{(d)}) \), it holds that

\[
\psi^v_i(\mathbf{a}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases} \tag{2.4}
\]
Hence, the base functions can be associated with the nodes. We denote by \( \mathcal{P}^\nu(\Delta) \) the span of the Lagrange polynomials of degree \( \nu \). This space coincides with the span of all polynomials in \( \lambda_1, \ldots, \lambda_\Delta \) of degree at most \( \nu \); see, e.g., Brenner & Scott [6]. Furthermore, we indicate by \( \mathcal{P}^\nu(\Delta, \mathbb{R}^d) \) the vector-valued functions from \( \Delta \) into \( \mathbb{R}^d \) which reside component-wise in \( \mathcal{P}^\nu(\Delta) \). The global finite-element-approximation space \( \mathcal{A}^\nu([A]) \) pertaining to the Lagrange polynomials of degree \( \nu \) is now defined as the aggregate of all continuous functions from \( [A] \) into \( \mathbb{R}^d \) of which the restriction to a simplex \( \sigma \in A \) belongs to \( \mathcal{P}^\nu(\sigma)^d \):

\[
\mathcal{A}^\nu([A]) = \left\{ \mathbf{u} \in C^0([A], \mathbb{R}^d) : \mathbf{u}|_\sigma \in [\mathcal{P}^\nu(\sigma)]^d \right\}. \tag{2.5}
\]

For simplicity we have assumed in (2.5) that \( \nu \) is uniform on the entire simplicial complex \( A \), but this assumption is nonessential.

Any function \( \mathbf{u} \in \mathcal{A}^\nu([A]) \) can be expressed as:

\[
\mathbf{u}(x) = \sum_{i \in I^\nu} \hat{u}_i^\nu(x) \psi_i^\nu(\lambda(x)). \tag{2.6}
\]

The maps \( x \mapsto \sigma(x) \) and \( x \mapsto \lambda(x) \) associate with each \( x \in [A] \) a simplex \( \sigma \) such that \( x \in \sigma \), and the barycentric coordinate \( \lambda \) corresponding to the position of \( x \) in \( \sigma \), respectively. If \( x \in \text{int} \sigma \), then the association \( x \mapsto \sigma \) is unique; see, for instance, [5, Lemma 3.3.4]. However, if \( x \) resides on the boundary of a simplex, then \( x \) can belong to multiple simplices. On account of the continuity of \( \mathbf{u} \), however, this ambiguity can be resolved by any conclusive rule, and we can proceed under the assumption that the relation \( x \mapsto (\sigma, \lambda) \) is invertible. From (2.4) it follows that the coefficients \( \hat{u}_i^\nu \in \mathbb{R}^d \) can be conceived as nodal positions corresponding to the geometric realization of \( \mathbf{u} \). Moreover, continuity of \( \mathbf{u} \) is ascertained by identifying the nodal positions at the boundaries of contiguous simplices, viz., \( \hat{u}_i^\nu = \hat{u}_j^\nu \) for all simplex/index pairs \( (\sigma, i) \) and \( (\hat{\sigma}, \hat{i}) \) such that \( a_i^\nu 1 + \cdots + a_i^\nu d = a_j^\nu 1 + \cdots + a_j^\nu d \), with \( a_i^\nu \) the vertices of the simplex \( \sigma \). We remark that if the degree \( \nu \) is nonuniform on the simplices, then continuity of \( \mathbf{u} \) between contiguous simplices can be maintained similarly, by specifying appropriate relations between the nodal positions at the mutual boundary.

In summary, an approximation of a manifold \( M \subset \mathbb{R}^d \) by means of Lagrange polynomials on a complex of \( d \)-dimensional simplices can be constructed as follows: First, an appropriate vertex set \( \{a_i\}_{i=1}^N \) is constructed, by positioning points \( a_i \) on \( M \). This vertex set serves as the basis of a simplicial complex \( A \), such that the underlying space \( [A] \) is homeomorphic to \( M \) and, moreover, \( [A] \) is close to \( M \), e.g., in terms of the Hausdorff distance. Second, each simplex \( \sigma \in A \) is enhanced with \( (d + \nu)!(\nu d!) \) nodes \( a_i^\nu = v^{-1}(a_i^\nu 1 + \cdots + a_i^\nu d) \), and coincident nodes of contiguous simplices are identified. Finally, these nodes are appropriately repositioned relative to \( M \). For instance, the nodes can be positioned on \( M \). In that case, it is to be noted that a node with \( i_k = \nu \) and, necessarily,
Figure 2: Approximation of the unit sphere with piecewise Lagrange polynomials for $\nu = 1$ (left) and $\nu = 4$ (right). The lines indicate the boundaries of the simplices. The nodes associated with the Lagrange polynomials are indicated by dots.

$i_j = 0$ for all $j \neq k$ coincides with the vertex $a_k$. Because the vertices are already on $M$, such nodes need not be repositioned.

To illustrate the approximation of a manifold by means of piecewise Lagrange polynomials, figure 2 presents the approximation of the unit sphere in $\mathbb{R}^3$ on a complex $A$ of 12 simplices in a cuboid constellation for $\nu = 1$ (left) and $\nu = 4$ (right). The nodes have been positioned on the sphere through a gnomonic projection (sometimes referred to as a radial projection). The underlying space $|A|$ coincides with the geometric realization in the left figure, viz., the cube with vertices $(\pm 1, \pm 1, \pm 1)/\sqrt{3}$.

2.3 Mesh association
The above exposition enables us to give a precise specification of the problem considered in this paper. Let us consider a manifold $M \subset \mathbb{R}^d$ and two distinct approximations of $M$, viz., an approximation based on a simplicial complex $A$ with a finite-element space $\mathcal{A}_\nu(|A|)$ and an approximation based on a simplicial complex $B$ with a finite-element space $\mathcal{A}_\varphi(|B|)$. For example, $M$ can represent a surface modeling the interface between a fluid and a solid subdomain in $\mathbb{R}^3$, $A$ and $B$ are surface meshes at the interface corresponding to finite-element partitions of the subdomains, and $\mathcal{A}_\nu(|A|)$ and $\mathcal{A}_\varphi(|B|)$ are the position trace spaces pertaining to the subsystems, viz., the aggregates of interface positions in accordance with the finite-element spaces in the fluid and the solid. The approximations to $M$ are furnished by the geometric realizations of a function $u$ in $\mathcal{A}_\nu(|A|)$ and a function $v$ in $\mathcal{A}_\varphi(|B|)$. Disparity of the finite-element spaces $\mathcal{A}_\nu(|A|)$ and $\mathcal{A}_\varphi(|B|)$ then generally implies that the geometric realizations $u(|A|)$ and $v(|B|)$ are noncoincident, i.e., the sets $u(|A|)$ and $v(|B|)$ do not match. Our objective is to construct a proper associative map $u(|A|) \rightarrow v(|B|)$ for the purpose of transferring functions between $u(|A|)$ and $v(|B|)$.

3. Smoothed-normal-projection association of boundaryless manifolds
The mesh-association method presented in this paper is based on projection along a so-called smoothed-normal-vector field, viz., the solution of a modified Helmholtz equation with right-hand-side data corresponding to the normal-vector field. In section 3.1 we give a dimension-independent definition of the normal-vector field on simplicial complexes. Section 3.2 provides the notational conventions pertaining to the functional setting of the smoothing operation, i.e., the map between the normal-vector field and the solution of the modified Helmholtz equation. The smoothing operation is specified
in section 3.3. In section 3.4 we attend to the smoothness properties of the smoothed-normal-vector field on the basis of classical theory on the interior regularity of solutions of elliptic equations. Finally, section 3.5 presents the smoothed-normal-projection association.

3.1 Normal-vector fields on geometric realizations

Let us consider a simplicial complex $A$ with a boundaryless underlying space $|A|$ (i.e., $|A|$ is a closed manifold), an approximation space $\mathcal{A}^s(|A|)$, and the geometric realization $\mathbf{u}(|A|)$ pertaining to a particular $\mathbf{u}$ in the approximation space. The geometric realization $\mathbf{u}(|A|)$ admits a simple-ex wise parameterization:

$$\mathbf{u}(|A|) = \bigcup_{\sigma \in A} \{ x \in \mathbb{R}^d : x = \mathbf{u}^\sigma(\lambda), 0 \leq \lambda_1, \lambda_1 + \cdots + \lambda_d = 1 \}.$$  \hspace{1cm} (3.1)

To accommodate the partition-of-unity property of the barycentric coordinates, we define the hyper-surface coordinates (or coordinate chart) $\xi \in \mathbb{R}^d$, and the map $\xi \mapsto \lambda(\xi)$:

$$\lambda = (\xi_1, \ldots, \xi_d), \quad \xi \mapsto \lambda(\xi) = (\xi_1, 1 - |\xi|),$$  \hspace{1cm} (3.2)

where $|\xi| = \xi_1 + \cdots + \xi_d$. The vectors $\mathbf{e}_i = \partial_i \mathbf{u}^\sigma(\lambda(\xi))$ ($\partial_i := \partial_{\xi_i}$, $i = 1, \ldots, d$) provide a basis of the tangent space to $\mathbf{u}(|A|)$. It is to be noted that

$$\mathbf{e}_i = \partial_i \mathbf{u}^\sigma(\lambda(\xi)) = D_i \mathbf{u}^\sigma - D_d \mathbf{u}^\sigma,$$  \hspace{1cm} (3.3)

where $D_i$ denotes differentiation with respect to the $i$-th argument. Indicating the unit normal vector to $\mathbf{u}(|A|)$ by $\mathbf{e}^d$, the vectors $\{ \mathbf{e}_i \}_{i=1}^{d-1}$ provide a basis of $\mathbb{R}^d$. The reciprocal basis, $\{ \mathbf{e}^i \}_{i=1}^{d-1}$, is defined by $\{ \mathbf{e}_i, \mathbf{e}^j \} = \delta^j_i$, where $\delta^j_i$ stands for the Kronecker delta and $\{ \cdot, \cdot \}$ generically represents the contraction of tensors. The vectors $\mathbf{e}_i$ and $\mathbf{e}^i$ ($i = 1, \ldots, d$) are referred to as covariant and contravariant base vectors, respectively. On account of $\{ \mathbf{e}_i, \mathbf{e}^d \} = \delta^d_i$ and $|\mathbf{e}^d| = 1$ with $| \cdot |$ the vector norm in $\mathbb{R}^d$, it holds that $\mathbf{e}_d = \mathbf{e}^d$. Upon collecting the components of $\{ \mathbf{e}^i \}_{i=1}^{d-1}$ with respect to the basis $\{ \mathbf{e}_i \}_{i=1}^{d}$ in a matrix $T^\sigma = \mathbf{e}^i = (\mathbf{e}_i, \mathbf{e}^d)$, the components of the unit normal vector follow from $T^\sigma \mathbf{e}_d = \delta^d_i$. Throughout, the summation convention applies to repeated indices, unless explicitly stated otherwise. Application of Cramer’s rule yields

$$\mathbf{e}_i^d = (-1)^{i+d} g_{\sigma}^{-1/2} M_{id}^\sigma =: \mathbf{u}_i^\sigma(\lambda(\xi)), \hspace{1cm} (3.4a)$$

with

$$g_{\sigma} = (M_{1d}^\sigma)^2 + \cdots + (M_{dd}^\sigma)^2,$$  \hspace{1cm} (3.4b)

and $M_{\sigma}^\sigma$ the minors of $T^\sigma$, i.e., the determinant of the matrix that is obtained by removing the $i$-th row and $j$-th column of $T^\sigma$. The normal-vector field $\mathbf{n}$ on $\mathbf{u}(|A|)$ is now defined as the piecewise-continuous vector-valued function from $\mathbf{u}(|A|)$ into $\mathbb{R}^d$ of which the restriction to any simplex $\sigma$ complies with (3.4). We assume throughout that $\mathbf{n}$ is directed outward.

3.2 Functional setting

To provide an appropriate functional setting for the smoothing operation, we regard a generic manifold $M$. Let $C^\infty(M)$ represent the space of infinitely-differentiable functions on $M$. For all nonnegative integers $m$ and all real $p \geq 1$, the Sobolev space $H^{k,p}(M)$ is defined as the closure of $C^\infty(M)$ with respect to the norm

$$\| u \|_{H^{k,p}(M)} = \sum_{l=0}^{k} \| \nabla^l_x u \|_{L^p(M)}, \quad \| U \|_{L^p(M)} = \begin{cases} \left\{ \int_M |U|^p \, d\mu_M \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{x \in M} |U(x)| & \text{if } p = \infty, \end{cases}$$  \hspace{1cm} (3.5)

with $\mu_M$ the measure on $M$ and $|U|$ the local norm of the tensor $U$, i.e., $|U|^2 = \langle U, U \rangle$. The operator $\nabla_x$ designates the gradient on the manifold $M$. If $M$ admits a (local) parameterization,

$$M = \{ x \in \mathbb{R}^d : x = \mathbf{u}(\xi_1, \ldots, \xi_d) \},$$  \hspace{1cm} (3.6)
then $\nabla_s w = \epsilon^{(1)} \partial_1 w + \cdots + \epsilon^{(d)} \partial_d w$, where $\epsilon^{(i)}$ denote the reciprocal-base vectors corresponding to $\epsilon_{(i)} = \partial_i u$ and the normal vector to $M$. For $p = 2$, we use the condensed notation $H^k(M)$. The space $H^k(M)$ is a Hilbert space under the inner product

$$
(u,v)_{H^k(M)} = \sum_{i=0}^{k} \int_{M} \langle \nabla^i u, \nabla^i v \rangle \, d\mu_M.
$$

(3.7)

The definition of the Sobolev spaces is extended to negative integer numbers by identifying $H^{-k}(M)$ with the dual of $H^k(M)$ with $L^2$ as pivot space; see, e.g., [13, 17]. The above definitions are extended to vector-valued functions in the usual manner. The vector-valued functions $M \to \mathbb{R}^d$ that reside component-wise in $H^{k,p}(M)$ are indicated by $H^{k,p}(M, \mathbb{R}^d)$ or $[H^{k,p}(M)]^d$.

3.3 Smoothing of normal-vector fields

For a general geometric realization, the normal-vector field is only piecewise smooth, i.e., the normal-vector field is smooth on each simplex, but it can be discontinuous across edges. On account of this irregularity, the normal-vector field is unsuitable for a projection-based association. Therefore, we propose to use instead a smoothed-normal-vector field, viz., the $H^1$ representation of the normal-vector field.

We define the smoothed-normal-vector field corresponding to the normal-vector field $n$ as the vector field $m \in H^1(u([A]), \mathbb{R}^d)$ according to

$$
\int_{M} \langle m, w \rangle + \gamma \langle \nabla_s m, \nabla_s w \rangle \, d\mu_M = \int_{M} \langle n, w \rangle \, d\mu_M \quad \forall w \in H^1(M, \mathbb{R}^d),
$$

(3.8)

with $M = u([A])$. The positive parameter $\gamma$ is referred to as the smoothing parameter. Let us remark that equation (3.8) represents a weak formulation of the modified Helmholtz equation,

$$
m - \gamma \nabla^2_m n = 0,
$$

(3.9)

with $\nabla^2$ the Laplace-Beltrami operator on the manifold $u([A])$.

The integrals in (3.8) can be separated into sums of contributions of the simplices. Upon introducing simplex-wise coordinate charts $\xi \mapsto x = u^\sigma(\lambda(\xi))$, the identity (3.8) can be cast into the form

$$
\int_{\Sigma} \left( \langle m, w \rangle + \gamma g^\sigma_{ij}(\partial_i m, \partial_j w) \right) \sqrt{g^\sigma} \, d\xi = \int_{\Sigma} \langle n^\sigma, w \rangle \sqrt{g^\sigma} \, d\xi,
$$

(3.10)

where $g^\sigma_{ij} = \langle \epsilon^{(i)}, \epsilon^{(j)} \rangle$ \,(i, j = 1, \ldots, d) is the contravariant metric tensor pertaining to the simplex $\sigma$ and $\Sigma$ represents the simplex $\Sigma = \{ \xi \in \mathbb{R}^d : 0 \leq \xi_1, \xi_2, \ldots + \xi_d \leq 1 \}$. One can infer that $\sqrt{g^\sigma}$ in accordance with (3.4b) represents the ratio of the measures $d\mu_{u^\sigma}/d\xi$. In tensor analysis, $g_\sigma$ is conventionally defined as the determinant of the covariant metric tensor, viz.,

$$
g_\sigma = \det(g_{\sigma ij}) \quad \text{with} \quad g_{\sigma ij} = \langle \epsilon_{(i)}, \epsilon_{(j)} \rangle \quad (i, j = 1, \ldots, d) \quad \text{and} \quad |\epsilon_{(i)}| = 1.
$$

However, $g_\sigma$ according to (3.4b) is identical to $\det^2(\epsilon_{(i)})$ \,(i, j = 1, \ldots, d). Moreover, by virtue of $\langle \epsilon_{(i)}, \epsilon_{(d)} \rangle = \delta_{id}$ and $|\epsilon_{(d)}| = 1$, it holds that

$$
\det(\epsilon_{(i)}, \epsilon_{(j)}) = \det(\epsilon_{(k)}, \epsilon_{(l)}) \quad (i, j, k, l = 1, \ldots, d). \quad \text{The product rule for determinants yields} \quad \det(\epsilon_{(i)}, \epsilon_{(j)}) = \det(\epsilon_{(i)}, \epsilon_{(j)}) \quad (i, j, k = 1, \ldots, d).
$$

Hence, the definition of $g_\sigma$ according to (3.4b) is equivalent to the conventional definition.

Of course, in an actual computation the smoothed-normal-vector field according to (3.8) is not determined explicitly. Instead, it is approximated by finite elements. We typically opt to use the same finite-element-approximation space for $m$ as for the geometric realization, i.e., we replace the Sobolev spaces $H^1(u([A]), \mathbb{R}^d)$ that furnish the setting of (3.8) by $A^u([A])$.

3.4 Regularity of the smoothed-normal-vector field

An important property of the smoothed-normal-vector field $m$ pertains to its regularity in relation to the regularity of the normal-vector field $n$. Equation (3.8) or, equivalently, (3.9) or (3.10) defines $m$
as the solution of a second-order elliptic equation with right-hand-side data \( n \). Hence, to establish the relation between the regularity of \( n \) and the regularity of \( m \) we can appeal to classical theory on the regularity of solutions of elliptic equations; see, for instance, [2, 7, 13, 16]. Here, we shall be concerned with interior regularity only, because we restrict our considerations to boundaryless manifolds. The case with boundaries, which is profoundly more complicated, will be treated elsewhere.

The most useful expression of the interior-regularity theorem for elliptic equations for our purposes is that according to [2, Theorem 3.54]: Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and \( A = a_i \nabla^j \) an elliptic linear operator of order \( 2m \) with \( C^\infty \) coefficients. If \( f \in H^{k,p}(\Omega) \) and \( u \) is the distributional solution of \( Au = f \), then \( u \) belongs locally to \( H^{k+2m,p}(\Omega) \). The latter implies that \( u \in H^{k+2m,p}(\omega) \) for all open subsets \( \omega \subset \Omega \) with compact closure in \( \Omega \).

The interior-regularity theorem conveys that under the stated premises, the solution is in fact more regular than is to be expected from the general setting. For a boundaryless manifold associated with a geometric realization \( u(|A|) \), the theorem implies that the smoothed-normal-vector field resides in \( H^{2,p}(u(|A|),\mathbb{R}^d) \) for any \( p \geq 1 \). The derivation of the regularity of the smoothed-normal-vector field is elaborated by the possible non-smoothness of manifolds pertaining to geometric realizations and, therefore, it is deferred to Appendix A. An essential attribute of the smoothed-normal-vector field is its continuous differentiability: By the Sobolev embedding theorem for compact manifolds [2, Theorems 2.10 and 2.20], for a \( d \)-dimensional manifold it holds that functions in \( H^{k,p}(M) \) are equivalent to functions in \( C^l_p(M) \) if \( (k-l)/d > 1/p \), where \( C^l_p(M) \) is the space of \( l \)-times continuously differentiable functions that are bounded in the \( \| \cdot \|_{H^{l,\infty}(M)} \)-norm. This implies that \( m \in C^l_p(M,\mathbb{R}^d) \) for all \( l < 2 - d/p, p \geq 1 \) and, in particular, that the smoothed-normal-vector field is bounded continuously differentiable. Therefore, the smoothed-normal-vector field is appropriate for a projection-based association.

The interior-regularity theorem moreover yields the important corollary that an increase in the smoothness of the geometric realization leads to higher regularity of the corresponding smoothed-normal-vector field. Hence, the smoothness of the projection-based association between geometric realizations increases with the smoothness of the geometric realizations.

### 3.5 Smoothed-normal-projection association

On account of its continuity, the smoothed-normal-vector field is suitable for a projection-based association. We denote by \( \ell(x) \) the line through \( x \in u(|A|) \) in the direction of the smoothed-normal vector at \( x \). Denoting by \( v(|B|) \) a geometric realization homeomorphic to \( u(|A|) \), the smoothed-normal-projection associates which each \( x \in u(|A|) \) the intersection of \( \ell(x) \) and \( v(|B|) \). If \( v(|B|) \) is sufficiently close to \( u(|A|) \), then the intersection is nonempty. However, the intersection generally contains multiple elements, as the line \( \ell(x) \) can have multiple intersections with the manifold \( v(|A|) \).

The associate \( y \in v(|B|) \) of \( x \in u(|A|) \) is therefore isolated as the element of the intersection set that is closest to \( x \):

\[
\text{arg inf}_{y \in v(|B|) \cap \ell(x)} |y - x| \quad \text{in} \quad v(|B|). \quad (3.11)
\]

The smoothed-normal-projection association according to (3.11) is indicated by \( \kappa \).

To retain the necessary one-to-one correspondence, the inverse \( \kappa^{-1} : v(|B|) \to u(|A|) \) is based on the smoothed-normal-vector field on \( u(|A|) \), i.e., \( x = \kappa^{-1}(y) \) is the closest point on \( u(|A|) \) for which \( y \) is located on the line \( \ell(x) \):

\[
\kappa^{-1}(y) = \text{arg inf}_{x \in u(|A|), y \in \ell(x)} |y - x|. \quad (3.12)
\]

This definition of the inverse is similar to that in [10, 11].

Because both \( \kappa \) and \( \kappa^{-1} \) utilize the smoothed-normal-vector field on \( u(|A|) \), the pair of associations \( (\kappa, \kappa^{-1}) \) possesses a so-called master/slave structure. If the role of \( u(|A|) \) and \( v(|B|) \) is reversed, then the association is based on the smoothed-normal-vector field on \( v(|B|) \). In general, the corresponding association \( v(|B|) \ni y \mapsto x \in u(|A|) \) is different from \( \kappa^{-1} \).
4. Properties of smoothed-normal-projection association

In this section we consider elementary properties of the smoothed-normal-vector field and of the smoothed-normal-projection association. In section 4.1 it is shown that the smoothed-normal-vector field is strictly outward, under some nonrestrictive conditions on the geometry. Based on this externality of the smoothed-normal-vector field, section 4.2 establishes a local-uniqueness condition, viz., a lower bound on the distance between a point and its image such that no nearby point possesses the same image.

4.1 Externality of the smoothed-normal-vector field

An elementary property of the smoothed-normal-vector field is that under the standing assumption that \( \mathbf{n} \) represents the outward normal-vector field, and some quite nonrestrictive conditions on the geometry which are elaborated below, it holds that \( \mathbf{m} \) is strictly outward, i.e., \( \langle \mathbf{m}, \mathbf{n} \rangle > 0 \). A precise verification of this property and the corresponding prerequisites is beyond the scope of this paper. In the analysis below we restrict ourselves to a rudimentary analysis. The externality of \( \mathbf{m} \) is further considered in section 5.

Let us consider a point \( \mathbf{x} \in M \) and a local normal coordinate system (see [2, p.7] and also Appendix A), i.e., a system of coordinates \( (\xi_1, \ldots, \xi_d) \) such that the components of the metric tensor at \( \mathbf{x} \) satisfy \( g_{ij} = \delta_{ij} \) and \( \partial_k g_{ij}(\mathbf{x}) = 0 \). With respect to the normal coordinates, (3.9) reduces to:

\[
\mathbf{m} - \gamma \nabla^2_\xi \mathbf{m} = \mathbf{m} - \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \mathbf{m}) = \mathbf{m} - \gamma \nabla^2_\xi \mathbf{m} = \mathbf{n},
\]

where \( \nabla^2_\xi \) is a condensed notation for the Laplace operator in \( \xi \)-coordinates, \( \nabla^2_\xi = \partial^2_\xi + \cdots + \partial^2_\xi \).

By means of the Green’s functions for the modified Helmholtz equation, the solution to (4.1) can be expressed as:

\[
\mathbf{m}(\xi) = \int \frac{1}{\sqrt{\gamma}} K \left( \frac{|\xi - \xi'|}{\sqrt{\gamma}} \right) n(\xi') d\xi',
\]

where the kernel \( K(\cdot) \) is given by

\[
K(r) = \frac{e^{-|r|}}{2}, \quad K(r) = \frac{K_0(|r|)}{2\pi}, \quad K(r) = \frac{e^{-|r|}}{4\pi |r|^2},
\]

for \( d = 1, 2, 3 \), respectively. In (4.2b), \( K_0(\cdot) \) represents the modified Bessel function of the second kind of order 0; see [1]. Equation (4.2) implies that \( \mathbf{m} \) is outward at \( \xi \) unless

\[
\langle \mathbf{m}(\xi), \mathbf{n}(\xi) \rangle = \int \frac{1}{\sqrt{\gamma}} K \left( \frac{|\xi - \xi'|}{\sqrt{\gamma}} \right) \langle \mathbf{n}(\xi'), \mathbf{n}(\xi) \rangle d\xi' \leq 0.
\]

The kernels (4.2b) have the following essential properties: First, they are strictly decreasing and vanish exponentially for large \( |r| \), i.e., \( K(r) > K(r') \) for all \( r > r' > 0 \) and \( 0 < K(r) \leq \exp(-|r|) \) as \( r \to \pm \infty \). Second, they are symmetric, i.e., \( K(-r) = K(r) \). Third, they are strictly positive, i.e., \( K(r) > 0 \) for all \( r \in \mathbb{R} \). On account of the strict decrease and strict positivity of the kernels, \( \langle \mathbf{n}(\xi'), \mathbf{n}(\xi) \rangle \) must be negative for \( \xi' \) near \( \xi \) for (4.3) to hold. This can only occur if \( u(|A|) \) is strongly curved, for instance, near kinks. Let us now consider a point \( \mathbf{x}(\xi) \) in the vicinity of an edge \( e \) where \( u(|A|) \) displays a strong kink; see figure 3 for an illustration. The inner product \( \langle \mathbf{n}(\xi'), \mathbf{n}(\xi) \rangle \) is positive if \( \mathbf{x}(\xi') \) resides in the simplex \( \sigma^+ \) and negative if it resides in \( \sigma^- \). However, for sufficiently small fixed \( |\xi - \xi'| \) the angle between \( \mathbf{n}(\xi) \) and \( \mathbf{n}(\xi') \) is smaller on \( \sigma^+ \) than on \( \sigma^- \). Hence, the negative contribution from \( \sigma^- \) is subordinate to the positive contribution from \( \sigma \), so that (4.3) cannot hold.

The above argument fails near cusp-like features and certain types of reentrant corners. In section 5 it will be shown that in the neighborhood of such features the smoothed-normal-vector field can indeed become non-outward. Near reentrant corners, however, the externality of the smoothed-normal-vector field can be restored by an appropriate choice of the smoothing parameter \( \gamma \): The asymptotic
behavior of the kernels (4.2b) implies that the contribution of points at distance \( |\xi - \xi'| \) is proportional to \( \exp(-|\xi - \xi'|/\gamma) \). Therefore, the smoothed-normal-vector field approaches the normal-vector field as \( \gamma \to 0 \) and, hence, for sufficiently small \( \gamma \) the smoothed-normal-vector field will be directed outwards. It is to be noted, however, that in this case the externality comes at the expense of smoothness, as the smoothness of \( m \), e.g., in the sense of \( \|m\|_{L^2(M)}/\|m\|_{H^1(M)} \), decreases with \( \gamma \).

4.2 Local-uniqueness of the association

The smoothed-normal-projection association enables a unique association between geometric realizations, provided that these realizations are sufficiently close. To elaborate the notion of closeness, let us consider a parameterized manifold \( M \subset \mathbb{R}^d \) in conformity with (3.6) and an arbitrary vector field \( m \in C^1(M, \mathbb{R}^d) \). Consider two arbitrary distinct points \( u(\xi_0) \) and \( u(\xi_1) \) on \( M \). If the lines through \( u(\xi_0) \) and \( u(\xi_1) \) intersect, then the association through projection along \( m \) is nonunique at the intersection. In particular, if \( u(\xi_0) + \theta_0 m(\xi_0) \) and \( u(\xi_1) + \theta_1 m(\xi_1) \) are the images of \( u(\xi_0) \) and \( u(\xi_1) \) under the associative map, respectively, then the association is nonunique if the images coincide:

\[
u(\xi_0) + \theta_0 m(\xi_0) = u(\xi_1) + \theta_1 m(\xi_1).
\]

Upon inserting \( \xi_1 = \xi_0 + \varepsilon \eta \) and \( \theta_1 = \theta_0 + \varepsilon \omega \) into (4.4) and taking the limit \( \varepsilon \to 0 \), we obtain

\[
u(\xi_0) + \theta_0 m(\xi_0) = u(\xi_0) + \varepsilon \eta_0 \partial_1 u(\xi_0) + \theta_0 m(\xi_0) + \varepsilon \theta_0 \eta_0 \partial_1 m(\xi_0) + \varepsilon \omega m(\xi_0) + o(\varepsilon),
\]

as \( \varepsilon \to 0 \), where the Landau symbols \( o \) and \( O \) (used below) represent terms such that \( o(\varepsilon)/\varepsilon = 0 \) and \( O(\varepsilon)/\varepsilon \leq \text{const} \) as \( \varepsilon \to 0 \). On account of the fact that terms of \( O(\varepsilon) \) must vanish separately as \( \varepsilon \to 0 \), it holds that \( \theta(\varepsilon = \theta_0) \) is determined by the generalized eigenvalue problem: Find \( (\theta, \eta, \omega) \) such that

\[
(\partial_1 u + \theta \partial_1 m)\eta_1 + \omega m = 0.
\]

To elucidate the structure of the generalized eigenvalue problem, we define the matrices \( E, H \in \mathbb{R}^{d \times d} \) and the vector \( h \in \mathbb{R}^d \) according to

\[
E = \begin{pmatrix}
\partial_1 u_1 & \cdots & \partial_2 u_1 & m_1 \\
& \ddots & \vdots & \vdots \\
\partial_1 u_d & \cdots & \partial_2 u_d & m_d
\end{pmatrix}, \quad
H = \begin{pmatrix}
\partial_1 m_1 & \cdots & \partial_2 m_1 & 0 \\
& \ddots & \vdots & \vdots \\
\partial_1 m_d & \cdots & \partial_2 m_d & 0
\end{pmatrix}, \quad
h = \begin{pmatrix}\eta_1 \\
\vdots \\
\eta_d \omega
\end{pmatrix}.
\]

With these definitions, equation (4.6) can be expressed in the form \((E_{ij} + \theta H_{ij})h_j = 0\). It is important to note that as \( \varepsilon \to 0 \) in (4.5), we restrict ourselves to local considerations. Specifically, the association of \( u(\xi) \) to \( u(\xi) + \theta m(\xi) \) is locally unique if \( \theta \) is distinct from the generalized eigenvalues according to (4.6). This does, however, not prevent a point on \( M \) at finite distance from \( u(\xi) \) from having the same image as \( u(\xi) \). Such a global violation of uniqueness is not included in the analysis.
The association of \( \mathbf{u}(\xi) \) to \( \mathbf{u}(\xi) + \vartheta \mathbf{m}(\xi) \) is locally unique if \( |\vartheta| < |\theta_{\min}| \), where \( \theta_{\min} \) represent the generalized eigenvalue of smallest modulus. This implies that a sufficient condition for local uniqueness of the association through projection along \( \mathbf{m} \) is that the distance between a point on \( M \) and its associate is less than \( |\theta_{\min}| |\mathbf{m}| \). We refer to this condition as the local-uniqueness condition on the association distance. Let us remark that if, specifically, \( \mathbf{m} \) represents the unit-normal-vector field on \( M \), then the eigenvalues \( \vartheta \) can be identified as the principal radii of curvature and, indeed, the local-uniqueness condition stipulates that the association distance is subordinate to the smallest principal radius of curvature.

To corroborate that the local-uniqueness condition depends on intrinsic properties of \( M \) and \( \mathbf{m} \) and that it is independent of the selected coordinate chart \( \xi_1, \ldots, \xi_d \), we recall the definition of the tangential base vectors \( \epsilon_{(i)} = \partial_i \mathbf{u} \) and, accordingly, we identify \( \eta \epsilon_{(i)} \) with a vector \( \eta \) in the tangential hyperplane \( \{ \mathbf{n} \}^\perp \). Thus, problem (4.6) can be reformulated as: Find \( (\vartheta, \eta, \omega) \in \mathbb{R} \times \{ \mathbf{n} \}^\perp \times \mathbb{R} \) such that

\[
\eta + \vartheta \eta \nabla_s \mathbf{m} + \omega \mathbf{m} = 0.
\]  

(4.8)

From (4.8) it is apparent that \( \vartheta \) depends only on \( \mathbf{m} \) via the vector in the tangent space \( \eta \in \{ \mathbf{n} \}^\perp \) and on the vector field \( \mathbf{m} \) via \( \mathbf{m} \) and \( \nabla_s \mathbf{m} \).

To construct a lower bound for the association distance \( |\theta_{\min}| |\mathbf{m}| \) corresponding to an arbitrary vector field \( \mathbf{m} \in C^1(M, \mathbb{R}^d) \), we cast the generalized eigenvalue problem (4.6) into the variational form: Find \( (\vartheta, \mathbf{h}) \in \mathbb{R} \times \mathbb{R}^d \) such that

\[
w_i E_{ij} h_j + \vartheta w_i H_{ij} h_j = 0 \quad \forall \mathbf{w} \in \mathbb{R}^d.
\]  

(4.9)

Let us now consider an eigenvector \( \mathbf{h} \) in accordance with (4.9). Observing that the identity in (4.9) must hold for all \( \mathbf{w} \in \mathbb{R}^d \), it follows for the corresponding eigenvalue \( \vartheta(\mathbf{h}) \) that

\[
|\vartheta(\mathbf{h})| \geq \sup_{\mathbf{w} \in \mathbb{R}^d} \frac{|w_i E_{ij} h_j|}{|w_i H_{ij} h_j|}.
\]  

(4.10)

By virtue of the fact that the minimum of \( |\vartheta(\mathbf{h})| \) over the subset of eigenvectors is at least equal to the minimum over all vectors in \( \mathbb{R}^d \), it holds that

\[
|\theta_{\min}| \geq \inf_{\mathbf{h} \in \mathbb{R}^d} \sup_{\mathbf{w} \in \mathbb{R}^d} \frac{|w_i E_{ij} h_j|}{|w_i H_{ij} h_j|}.
\]  

(4.11)

Upon expanding \( \mathbf{E}, \mathbf{H} \) and \( \mathbf{h} \) in accordance with (4.7) and identifying \( \eta_i \partial_i \mathbf{u} \) with a vector \( \eta \in \{ \mathbf{n} \}^\perp \), we obtain the following inequalities:

\[
|\theta_{\min}| \geq \inf_{(\eta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d} \sup_{\mathbf{w} \in \mathbb{R}^d} \frac{|w_i \eta_i \partial_i \mathbf{u} + \omega \langle \mathbf{w}, \mathbf{m} \rangle|}{|w_i \eta_i \partial_i \mathbf{m}|} \geq \inf_{(\eta, \omega) \in \{ \mathbf{n} \}^\perp \times \mathbb{R}^d} \sup_{\mathbf{w} \in \mathbb{R}^d} \frac{|\langle \mathbf{w}, \eta \rangle + \omega \langle \mathbf{w}, \mathbf{m} \rangle|}{|\langle \eta \mathbf{w}, \nabla_s \mathbf{m} \rangle|}.
\]  

(4.12a)

To continue the sequence of inequalities, we first restrict \( \mathbf{w} \) to \( \{ \mathbf{m} \}^\perp \), thus eliminating the term multiplied by \( \omega \) in the numerator. Next, we judiciously set \( \mathbf{w} = P_{\{\mathbf{m}\}^\perp} \eta \), i.e., we specify \( \mathbf{w} \) as the orthogonal projection of \( \eta \) onto the hyperplane \( \{ \mathbf{m} \}^\perp \):

\[
|\theta_{\min}| \geq \inf_{\eta \in \{ \mathbf{n} \}^\perp} \sup_{\mathbf{w} \in \{ \mathbf{m} \}^\perp} \frac{|\langle \mathbf{w}, \eta \rangle|}{|\langle \eta \mathbf{w}, \nabla_s \mathbf{m} \rangle|} \geq \inf_{\eta \in \{ \mathbf{n} \}^\perp} \frac{|\langle P_{\{\mathbf{m}\}^\perp} \eta, \eta \rangle|}{|\langle \eta P_{\{\mathbf{m}\}^\perp} \eta, \nabla_s \mathbf{m} \rangle|}.
\]  

(4.12b)

As \( \mathbf{w} = P_{\{\mathbf{m}\}^\perp} \eta \) minimizes the angle between \( \mathbf{w} \in \{ \mathbf{m} \}^\perp \) and \( \eta \in \{ \mathbf{n} \}^\perp \), it is anticipated that the second inequality in (4.12b) does not severely degrade the sharpness of the lower bound. Geometric considerations impart that for all \( \eta \in \{ \mathbf{n} \}^\perp \) it holds that

\[
\frac{|\langle P_{\{\mathbf{m}\}^\perp} \eta, \eta \rangle|}{|P_{\{\mathbf{m}\}^\perp} \eta| |\eta|} \geq \frac{|\langle \mathbf{m}, \mathbf{n} \rangle|}{|\mathbf{m}| |\mathbf{n}|} = \frac{|\mathbf{m}, \mathbf{n}|}{|\mathbf{m}|};
\]  

(4.13)
see Appendix B for proof. From (4.12), (4.13) it follows that

$$|\theta_{\min}| \|m\| \geq \inf_{\eta \in (n)^+} \frac{|P(m) \cdot \eta|}{|\eta|} \geq \frac{|\langle m, n \rangle|}{\|\nabla_s m\|},$$

(4.14)

with $|\nabla_s m| := \langle \nabla_s m, \nabla_s m \rangle^{1/2}$.

The lower bound for the association distance (4.14) implies that under the condition that the smoothed-normal-vector field $m$ is strictly outward, the smoothed-normal-projection association according to (3.11) admits a nonzero association distance, provided that $|\nabla_s m|$ is pointwise bounded. To establish that the latter stipulation is satisfied, we note that according to the second part of the Sobolev embedding theorem [2, Theorem 2.10], if $(k - l)/d \geq 1/p$ then $H^{l,p}(M) \subset C^k(M)$ and the injection from $H^{l,p}(M)$ into $C^k(M)$ is continuous, i.e., there exists a constant $C$ such that for all $u \in H^{l,p}(M)$ it holds that $\|u\|_{H^{l,p}(M)} \leq C \|u\|_{H^{l,p}(M)}$. Therefore, in particular, if $m \in H^{l,p}(M)$ for some $p > 1/d$ then it holds for all $x \in M$ that

$$|\nabla_s m(x)| \leq \|m\|_{H^{l,p}(M)} \leq C \|m\|_{H^{l,p}(M)}.$$  

(4.15)

The results from section 3.4 in turn convey that $|\nabla_s m|$ is indeed bounded.

5. Numerical Experiments

To illustrate the properties of the smoothed-normal-projection association, we present numerical experiments in 2D and 3D. Section 5.1 examines the properties of the smoothed-normal-vector field near kinks and cusps in 2D. In section 5.2 we consider smoothed-normal-projection association of high-order representations of the unit sphere in 3D. In this section, we also investigate the convergence behavior of the associative map under mesh refinement. Finally, section 5.3 presents numerical results for a complex geometry in 3D corresponding to a scanned object.

5.1 Properties of the smoothed-normal-vector field: 2D examples

To illustrate the properties of the smoothed-normal-vector field, we first consider $m$ according to (4.2) in a 2D setting ($d = 1$) for an infinite domain with a kink, viz., $M = M^- \cup M^+$ with

$$M^- = \{x \in \mathbb{R}^2 : x = (-\xi, 0), -\infty < \xi \leq 0\},$$

$$M^+ = \{x \in \mathbb{R}^2 : x = (\xi \cos(a), \xi \sin(a)), 0 \leq \xi < \infty\}.$$  

(5.1)

Figure 4 displays the smoothed-normal-vector field $m$ for smoothing parameters $\gamma = 1$ (top) and $\gamma = 50$ (bottom) and for kink angles of $a = \pi/2$ (left) and $a = \pi/8$ (right). The smoothed-normal-vector field has been extended to the interior, to illustrate the effects pertaining to inward association. The figure conveys that $m$ is indeed outward, even near sharp kinks. In particular, it follows from (4.2) that $m(0) = (n^+ + n^-)/2$ with $n^+ = (-\sin(a), \cos(a))$ and $n^- = (0, -1)$. This implies that $\langle m(\xi), n(\xi) \rangle$ is continuous at the kink and that $\langle m(0), n(0) \rangle = \sin^2(a/2)$. Moreover, it holds that $|m(0)| = \sin(a/2)$. Hence, the local norm $|m(0)|$ at the kink decreases with the kink angle $a$, but $|m(0)|$ remains strictly positive for all nonzero kink angles.

The positions marked by the symbol $\circ$ indicate the points closest to $M$ where local uniqueness of the association by projection along $m$ is violated; cf. section 4.2. Hence, the association distance is determined by the distance between $\circ$ and $M$ along $m$. Figure 4 (top, right) illustrates that the association distance can become excessively small if the smoothing is weak and the manifold $M$ contains a strong kink. However, figure 4 (bottom, right) conveys that the association distance can be effectively increased by increasing the smoothing parameter. It is to be noted that this result is in accordance with the lower bound (4.14), as an increase in the smoothing parameter leads to a reduction in $|\nabla_s m|$. Specifically, equation (4.2) yields $\nabla_s m(0) = (n^+ - n^-)/(2\sqrt{\gamma})$ and, therefore, $|\nabla_s m(0)| = \cos(a/2)/\sqrt{\gamma}$. This leads to the following lower bound for the association distance:

$$|\theta_{\min}| m \geq \frac{|\langle m, n \rangle|}{\|\nabla_s m\|} = \sqrt{\gamma} \frac{\sin^2(a/2)}{\cos(a/2)};$$  

(5.2)
Figure 4: Illustration of the smoothed-normal-vector field $m$ in 2D ($d = 1$) according to (4.2) for a kinked domain with smoothing parameters $\gamma = 1$ (top) and $\gamma = 50$ (bottom) and kink angles $a = \pi/2$ (left) and $a = \pi/8$ (right). The positions marked by $\circ$ indicate the points closest to $M$ where local uniqueness of the association by projection along $m$ is violated.
Mesh Association by Projection along Smoothed-Normal-Vector Fields

<table>
<thead>
<tr>
<th>$\alpha = \pi/2, \gamma = 1$</th>
<th>$\alpha = \pi/2, \gamma = 50$</th>
<th>$\alpha = \pi/8, \gamma = 1$</th>
<th>$\alpha = \pi/8, \gamma = 50$</th>
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</thead>
<tbody>
<tr>
<td>exact</td>
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<td>$5.000 \times 10^0$</td>
<td>$3.881 \times 10^{-2}$</td>
</tr>
<tr>
<td>lower bound</td>
<td>$7.071 \times 10^{-1}$</td>
<td>$5.000 \times 10^0$</td>
<td>$3.881 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 1: Comparison of the exact association distance and the lower bound (4.14) or, equivalently, (5.2) for the test cases in figure 4.

Figure 5: Illustration of the non-outwardness of the smoothed-normal-vector field near a cusp.

cf. equation (4.14). Table 1 compares the exact association distance to the lower bound according to (5.2). The table conveys that in the present case the lower bound in fact coincides with the actual association distance. This indicates that the simplifications that lead from (4.8) to (4.14) do not severely degrade the sharpness of the lower bound (4.14).

Next, we consider the effect of cusp-like features of the manifold on the smoothed-normal-vector field. To this end, we consider the smoothed-normal-vector field according to (4.2) associated with the manifold $M = M^- \cup M^+$ with

$$M^- = \{ x \in \mathbb{R}^2 : x = (-\xi, 0), -\infty < \xi \leq 0 \},$$
$$M^+ = \{ x \in \mathbb{R}^2 : x = (\sin(\alpha), 1 - \cos(\alpha)), 0 \leq \alpha < \pi/2 \} \cup \{ x \in \mathbb{R}^2 : x = (1, \xi), 1 \leq \xi < \infty \}. \quad (5.3)$$

The manifold (5.3) contains a cusp at $x = (0, 0)$. Figure 5 plots the smoothed-normal-vector field $m$ for $\gamma = 1$ near the cusp. The figure clearly illustrates that $m$ is non-outward in the neighborhood of the cusp. The non-outwardness of $m$ is caused by fact that the normal vectors $n^+$ on $M^+$ and $n^-$ on $M^-$ are oppositely directed near the cusp. Hence, their contributions to $m$ according to (4.2) cancel on account of the symmetry of the kernel. In the vicinity of the cusp, $m$ is then essentially determined by the normal-vector field away from the cusp, and this can cause $m$ to be non-outward. In particular, in figure 5 the infinite horizontal branch $M^-$ yields a dominant contribution in the direction $n^- = (0, -1)$ which causes $m$ to be inward on $M^+$. In view of the restricted importance of cusp-like features in practical applications, however, we will not pursue this issue further.

5.2 Association between approximations of the unit sphere

As an elementary example of smoothed-normal-projection association in 3D ($d = 2$), we consider the association between two distinct approximations of the unit sphere. One of the simplicial complexes consists of 4 simplices in a tetrahedral constellation, the other of 12 simplices in a cuboid constellation.
The underlying spaces pertaining to these simplicial complexes coincide with the tetrahedral and hexahedral geometric realization in figure 6 (top, left) and (top, right), respectively. The geometric realizations have been obtained by positioning the vertices on the unit sphere. Let us note that the geometric realizations in figure 6 (top) can be conceived as piecewise linear ($v = 1$) approximations of the unit sphere. In addition, figures 6 (bottom) depict the geometric realizations corresponding to cubic ($v = 3$) approximations of the sphere. These geometric realizations have been obtained by positioning the nodes of the Lagrange elements on the sphere through a gnomonic projection; see also section 2.2.

To exemplify the smoothed-normal-projection association, figure 6 displays the associates of edges (faces) of the simplices of the cuboid configuration in the tetrahedral configuration (left) under the map $\varkappa$ according to (3.11) and the associates of the edges of the tetrahedral configuration in the cuboid configuration (right) under the inverse map $\varkappa^{-1}$ according to (3.12). In particular, the dotted lines in figure 6 (left) represent the edges of simplices of the tetrahedron itself, whereas the solid lines represent the images of the edges of the hexahedron under the smoothed-normal-projection association. The same legend applies to figure 6 (right). For completeness, we mention that the smoothing parameter in (3.8) has been set to $\gamma = 1$ and that the cuboid configuration acts as master and the tetrahedral configuration as slave, i.e., the association between the geometric realizations is based on the smoothed-normal-vector field on the hexahedron; cf. section 3.5. Furthermore, it is to be remarked that the smoothed-normal-vector field according to (3.8) is approximated by means of finite-elements of the same order as the geometric realizations. As a digression, we note that this underlies the asymmetry of the association in figure 6 (top): Although both the tetrahedron and the regular hexahedron are symmetric with respect to the plane $x_2 = 0$, the association is not, on account of the fact that the finite-element configuration in the hexahedron does not possess this symmetry. For $v = 3$, this asymmetry is less pronounced by virtue of the improvement of the approximation of the smoothed-normal-vector field corresponding to the increase in the polynomial degree. Figure 6 (top) illustrates that the smoothed-normal-projection association yields a meaningful association, even if the discrepancy between the geometric realizations is large. For $v = 3$ (figure 6 (bottom)), the two geometric realizations are closer and, accordingly, also the distance between a point and its image under $\varkappa$ or $\varkappa^{-1}$ is smaller. This illustrates that the associative map converges to the identity.

To examine the approximation-to-the-identity properties of the association in more detail, we consider the convergence of $\varkappa(\cdot) - (\cdot)$ under refinement of the simplicial complexes for distinct polynomial degrees. The refinement consists in subdividing each simplex in the original simplicial complexes into $n^2$ ($n \in \mathbb{N}$) uniform simplices. Each of the simplices in the refined complex acts as a finite element of degree $v$. The geometric realizations are again obtained by positioning the nodes of the elements on the unit sphere via a gnomonic projection. Figure 7 displays the norms $\|\varkappa_n(\cdot) - (\cdot)\|_{L^2(M_n)}$ and $\|\varkappa_n(\cdot) - (\cdot)\|_{H^1(M_n)}$ for $n = 1, 2, \ldots, 12$ and $v = 1, 2, 3$, where $\{M_n\}$ represents the sequence of refinements of the cuboid constellation and $\{\varkappa_n\}$ is the sequence of associative maps corresponding to the sequence of refinements. For completeness, we mention that the integrals on $M_n$ that underly the norms have been computed element-wise by means of a 24-point Gauss-quadrature rule. Because both geometric realizations represent distinct polynomial approximations of degree $v$ of a $C^\infty$ manifold, interpolation theory (see, e.g., [17, §8.5]) conveys that the optimal convergence behavior is bounded by:

$$\|\varkappa_n(\cdot) - (\cdot)\|_{H^1(M_n)} \leq \text{const} \cdot h_n^{v+1-s}, \quad (5.4)$$

where $h_n$ designates the maximum of the diameters of the elements. Noting that on both geometric realizations $h_n$ is proportional to $n^{-1}$, it follows from figure 7 that the smoothed-normal-projection association indeed yields optimal convergence behavior in the $L^2$ and $H^1$ norms.

5.3 Association between approximations of a scanned object

To demonstrate the versatility of smoothed-normal-projection association, we finally consider the association of two distinct representations of a complex geometry associated with a scanned object, viz., a hippic figure; see figure 8. The fine and coarse representations comprise 6668 and 668 triangular el-
Figure 6: Smoothed-normal-projection association between approximations of the unit sphere in 3D by means of 4 elements in a tetrahedral constellation (left) and 12 elements in a hexahedral constellation (right) for polynomial degrees $\nu = 1$ (top) and $\nu = 3$ (bottom). The solid lines in the left figures represent the images of edges of the hexahedral constellation in the tetrahedral constellation under $\kappa$ according to (3.11). The solid lines in the right figures are the images of edges of the tetrahedral constellation in the hexahedral constellation under $\kappa^{-1}$ according to (3.12).
Figure 7: $\|\varkappa_n(\cdot) - (\cdot)\|_{\nu_n}$ versus $n$ for $\nu_n = H^1(M_n)$ (△) and $\nu_n = L^2(M_n)$ (○) and $\nu = 1 (\ldots)$, $\nu = 2 (\ldots)$, $\nu = 3 (\ldots)$. The triangles below the plot illustrate the refinement. The triangles adjacent to the curves indicate the slopes of the curves.
Mesh Association by Projection along Smoothed-Normal-Vector Fields

...ments, respectively. Figure 8 (top) displays the coarse representation and the associates of the edges of the fine representation under the smoothed-normal-projection association \( \kappa \). The bottom image depicts the fine representation and the associates of the edges of the coarse representation under \( \kappa^{-1} \). The smoothing parameter in (3.8) has been set to \( \gamma = 5 \times 10^{-4} \). This value has been selected on the basis of the size of the object (w1.8, d0.8, h1.5) and its geometric complexity. Furthermore, the fine representation acts as master, i.e., the association is based on the smoothed-normal-vector field on the fine representation. In addition, figure 9 zooms in on the head of the horse. The figures illustrate that the smoothed-normal-projection association yields a proper one-to-one association, even near sharp features such as the ears.

6. Conclusion
Motivated by the inappropriateness of concurrent mesh-association methods for high-order geometry representations, we developed a new mesh-association technique based on projection along a so-called smoothed-normal-vector field. The smoothed-normal-vector field consists of the solution of a modified Helmholtz equation on the geometric realization of a mesh, with right-hand-side data provided by the normal-vector field. The image of a point on one mesh is then defined as its projection along the smoothed-normal-vector field onto the other mesh. To retain the necessary one-to-one correspondence, the association possesses a master/slave structure, i.e., the association and its inverse are both based on the same smoothed-normal-vector field.

By means of regularity theory for elliptic partial-differential equations we established that the smoothed-normal-vector field is bounded continuously differentiable, which supports its suitability for a projection-based association. Furthermore, we showed that the smoothed-normal-vector field is in general strictly outward, under quite nonrestrictive conditions on the geometry, such as the absence of cusp-like features. We then derived a lower bound for the distance between a point and its image for which non-uniqueness of the association can occur, and we showed that this distance is bounded from below, provided that the smoothed-normal-vector field is strictly outward.

We illustrated the properties of the smoothed-normal-vector field in 2D by means of numerical experiments. The numerical experiments indicate that the lower bound on the association distance is sharp. Moreover, the results corroborate that the smoothed-normal-vector field can indeed become non-outward near cusps. Numerical experiments on distinct approximations of the unit sphere in 3D convey that the smoothed-normal-projection association displays optimal convergence in the \( H^1 \) norms under mesh refinement and under increase of the polynomial degree of the geometry representations. Finally, we tested the smoothed-normal-projection association on a complicated geometry in 3D. The numerical results confirm that the smoothed-normal-projection association yields a meaningful one-to-one correspondence, even near sharp features.

A. Regularity of smoothed-normal-vector fields on geometric realizations
Let us consider a simplicial complex \( A \) in \( \mathbb{R}^d \) with underlying space \( |A| \) in conjunction with a geometric realization \( \mathbf{u}(|A|) \), where \( \mathbf{u} \) is a member of a continuous, simplex-wise-polynomial space \( \mathcal{P}^p(|A|) \). The normal-vector field, simplex-wise defined by (3.4), serves as right member for the smoothed-normal-vector field \( \mathbf{m} \) according to (3.8). To facilitate the presentation, we use the notation \( M := \mathbf{u}(|A|) \) and \( [H^s(M)]^d := H^s(M, \mathbb{R}^d) \). Our objective is to show that if \( \mathbf{n} \in [H^{k,p}(M)]^d \) then \( \mathbf{m} \in [H^{k+2,p}(M)]^d \).

To enable application of the interior-regularity theorem for elliptic equations (see §3.4), we cover \( \mathbf{u}(|A|) \) with an atlas (see, e.g., [2]), viz., a collection of open charts \( (\Omega_i, \chi_i)_{i \in I} \) such that \( \cup_{i \in I} \Omega_i = M \). Clearly, it holds that \( \| \mathbf{m} \|_{[H^s(M)]^d}^2 \leq \sum_{i \in I} \| \mathbf{m}_i \|_{[H^s(\Omega_i)]^d}^2 \). By definition, \( \Omega := \chi_i \Omega_i \subset \mathbb{R}^d \). On account of (3.8), we have on each subset \( \Omega_i := \chi_i \Omega_i \subset \mathbb{R}^d \) with underlying space \( |\Omega_i| \), the following relation:

\[
\int_{\Omega_i} (\mathbf{m}, \mathbf{w}) + g^{jk} \langle \partial_j \mathbf{m}, \partial_k \mathbf{w} \rangle \sqrt{g} \, d\xi = \int_{\Omega_i} (\mathbf{n}, \mathbf{w}) \sqrt{g} \, d\xi \quad \forall \mathbf{w} \in [H^1(\Omega_i)]^d, \tag{A.1}
\]

\footnote{We acknowledge the use of the gts library (gts.sourceforge.net) for constructing the meshes of the hippoc figure, and of the original geometry file of the horse from the gts sample-file repository.}
Figure 8: Smoothed-normal-projection association between nontrivial geometries: Coarse representation of a hippic figure with the image of edges of the fine representation under $\kappa$ according to (3.11) (top) and fine representation of the figure with the image of edges of the coarse representation under $\kappa^{-1}$ according to (3.12).
where $g^{jk}$ is the contravariant metric tensor on $\Omega$, and $\langle \cdot , \cdot \rangle$ stands for the vector inner product in $\mathbb{R}^d$. Moreover, $[H^1(\Omega)]^d$ represents the subspace of functions in $[H^1(\Omega)]^d$ that vanish on the boundary, $\partial \Omega$, in the appropriate sense. For the derivatives of $g_{ij}$ and $\sqrt{g}$ with $g := \det(g_{ij})$ it holds that

$$
\partial_k g_{ij} = -g_{il} g_{jm} \partial_k g_{lm}, \quad \partial_k \sqrt{g} = \frac{1}{2\sqrt{g}} \partial_k \det(g_{ij}) = \frac{\sqrt{g}}{2} g^{ji} \partial_k g_{ij}.
$$

The identities (A.2) imply that $g_{ij} \in C^s(\omega)$ $\Leftrightarrow$ $g_{ij} \in C^s(\omega)$ $\Rightarrow$ $\sqrt{g} \in C^s(\omega)$ for any positive integer $s$ and any open bounded subset $\omega \subset \mathbb{R}^d$. Therefore, it follows from the interior-regularity theorem that if the covariant or contravariant metric tensor resides in $C^\infty(\Omega)$ and $n \in H^{k,p}(\Omega, \mathbb{R}^d)$, then $m \in H^{k+2,p}(\omega, \mathbb{R}^d)$ for any subset $\omega$ with compact closure in $\Omega$.

Conversely, it holds for the norm of $m$ on any $\omega = \chi^{-1}_i \omega \subset \Omega$ that

$$
\|m\|_{H^s(\omega)^d}^2 = \sum_{l=0}^s \int_{\omega} g^{j_1 k_1} \cdots g^{j_l k_l} (m_{j_1 \cdots j_l}, m_{k_1 \cdots k_l}) \sqrt{g} \, d\xi,
$$

with $m_{j_1 \cdots j_l}$ the covariant derivatives of order $l$. In particular, for orders $1, 2$ and $3$ it holds that

$$
m_{.,i} = \partial_i m, \quad m_{.,ij} = \partial_{ij} m - \Gamma^k_{ij} \partial_k m, \quad m_{.,ijk} = \partial_{jk} (\partial_i m - \Gamma^m_{jk} \partial_m m) - \Gamma^l_{ij} \partial_{kl} m - \Gamma^m_{ik} \partial_{jm} m - \Gamma^m_{jl} \partial_{im} m,
$$

where $\Gamma^l_{ij} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})g^{kl}$ is the Christoffel symbol of the second kind. In general, $m_{.,i_1 \cdots i_l}$ ($l \geq 1$) contains derivatives of the covariant metric tensor $g_{ij}$ up to order $l - 1$. Hence, under the standing hypotheses on the metric tensor, there exists a constant $C$ such that $\|m\|_{H^s(\omega)^d} \leq C \|m\|_{H^s(\omega)^d}$.

It follows from the preceding theory that if there exists an atlas $(\Omega_i, \chi_i)_{i \in \mathcal{I}}$ of $M$ with the following properties:

(1) there exists a finite cover $\bigcup_{j \in \mathcal{J}} \omega_j = M$ such that each $\omega_j$ has compact closure in some $\Omega_i$,
\[ (2) \] in each subset \( \Omega = \chi(\Omega) \subset \mathbb{R}^d \) the covariant metric tensor \( g_{kl} \) resides in \( C^\infty(\Omega) \).

and, moreover, \( n \in H^{k,p}(M, \mathbb{R}^d) \) then \( m \in H^{k+2,p}(M, \mathbb{R}^d) \). Below, we prove the existence of an atlas with the aforementioned properties for \( d = 2 \) by construction. The proof generalizes mutatis mutandis to \( d > 2 \).

Let us consider a cover of the geometric realization \( M \) conforming to figure 10. More precisely, if \( A, E \) and \( V \) denote the sets of simplices, edges and vertices, respectively, then the atlas is defined by \( (\Omega, \chi)_{\sigma \in A} \cup (\Omega, \chi)_{e \in E} \cup (\Omega, \chi)_{v \in V} \), where \( \Omega_\sigma = u(\sigma) \). \( \Omega_e \) is a topological disc covering a neighborhood of \( M \) around a vertex \( v \) and \( \Omega_v \) is defined as an open subset of the union of the closure of the simplices adjacent to edge \( e \) such that \( u(e) \subset \Omega_{v_1} \cap \Omega_{v_2} \) for certain vertex neighborhoods \( \Omega_{v_1} \) and \( \Omega_{v_2} \).

The existence of a homeomorphism \( \chi : x \in \Omega_\sigma \mapsto \xi \in \Omega_\chi \subset \mathbb{R}^2 \) such that the covariant metric tensor \( g_{ij} = (\partial_i x, \partial_j x) \in C^\infty(\Omega_\chi) \) follows straightforwardly from the smoothness of \( u \) on the simplices.

For the edge sets \( \Omega_e \), a homeomorphism with covariant-metric-tensor components in \( C^\infty \) can be constructed as follows: Let \( \sigma e \) arbitrarily represent a simplex adjacent to edge \( e \) and let us consider a homeomorphism \( \xi \in \Delta \mapsto x \in u(\sigma e) \) with \( \Delta = \{ \xi \in \mathbb{R}^2 : 0 \leq \xi_1, \xi_2 \leq 1 \} \). The covariant metric tensor corresponding to this map can be expanded as \( g_{ij} = \partial_i x_k \partial_j x_k \), where \( \partial_i := \partial / \partial \xi_i \) and summation on the repeated index \( k \) is implied. Being the gradient of a scalar field, \( \partial_i x_k \) forms an irrotational vector field from \( \Delta \) into \( \mathbb{R}^2 \). There exist 2 linear combination of these vector fields, \( q_{ki} = h_{ki} \partial_i x_t \), such that \( \tilde{g}_{ij} = g_{kl} q_{kl} \). In particular, the covariant metric tensor can be identified with a symmetric positive-definite matrix, again denoted by \( \tilde{g}_{ij} \), which admits a Cholesky factorization according to \( \tilde{g}_{ij} = q_{ki} q_{kj} \). The vectors \( q_{ki} \) being irrotational, there exist scalar fields \( \xi_k : \Delta \to \mathbb{R} \) such that \( \partial_i \xi_k = q_{ki} \). In particular, \( \xi_k \) can be defined as the classical solution of the Poisson problem

\[
\frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_i} \left( \sqrt{g} g^{ij} \frac{\partial \xi_k}{\partial \xi_j} \right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_i} \left( \sqrt{g} g^{ij} q_{kj} \right),
\]

for \( k = 1, 2 \), subject to certain Dirichlet boundary conditions on \( \partial \Delta \). Denoting the covariant-metric-tensor components with respect to \( (\xi_1, \xi_2) \) by \( \tilde{g}_{ij} \), it holds that

\[
\tilde{g}_{ij} = \frac{\partial \xi_k}{\partial \xi_i} \frac{\partial \xi_j}{\partial \xi_k} g_{kl} = q_{ki} q_{kj} g_{kl},
\]

which implies that \( \tilde{g}_{ij} = \delta_{ij} \).

Let us mentioned that the coordinates \( (\xi_1, \xi_2) \) define a so-called normal coordinate system; see [2, p. 7]. The covariant-metric-tensor components \( g_{ij} \) with respect to the normal coordinates \( (\xi_1, \xi_2) \) reside in \( C^\infty(\Delta) \) and, by (A.2), so do \( g^{ij} \) and \( \sqrt{g} \). Analogously, we can define coordinates in the other simplex adjacent to edge \( e \). Moreover, we can select the boundary conditions

![Figure 10: Illustration of the covering of the geometric realization with simplex, edge, and vertex domains.](image-url)
in the Poisson problems such that the simplex-wise mappings are continuous across the edge $e$. The composite map yields a homeomorphism $\chi_e : x \in \Omega_e \mapsto \xi \in \Omega = \mathbb{R}^2$ with the desired properties.

For the vertex sets, a homeomorphism $\chi_v : x \in \Omega_v \mapsto \xi \in \Omega_v \subseteq \mathbb{R}^2$ such the covariant metric tensor with respect to the coordinate chart $(\xi_1, \xi_2)$ is of class $C^\infty$ can be constructed by separating the covering disc into $n_v$ sectors, where $n_v$ is the cardinality of set of simplices connected to the vertex $v$. On each sector we can define coordinates in a similar manner as for the edge sets above, and continuity of the composite map can again be accomplished through a suitable choice of the boundary conditions on the sector-wise Poisson problems.

### B. Proof of lower bound (4.13)

The objective here is to prove that for all $\eta \in \{ n \}^\perp \subset \mathbb{R}^d$, it holds that the orthogonal projection $P_{\{ m \}} : \eta$ onto $\{ m \}^\perp \subset \mathbb{R}^d$ satisfies (4.13). The orthogonal projection $\pi := P_{\{ m \}} : \eta$ is defined by the variational problem: Find $\pi \in \{ m \}^\perp$ such that

$$
\langle v, \pi \rangle = \langle v, \eta \rangle \quad \forall v \in \{ m \}^\perp
$$

where $\langle v, w \rangle := v^T \cdot w$ denotes the scalar product in $\mathbb{R}^d$. By means of the Lagrange-multiplier formalism, $\pi$ according to equation (B.1) can be equivalently defined as: Find $(\pi, \lambda) \in \{ m \}^\perp \times \mathbb{R}$ such that

$$
\langle v, \pi \rangle + \lambda \langle v, m \rangle = \langle v, \eta \rangle \quad \forall v \in \mathbb{R}^d.
$$

Upon inserting $v = \pi$ in (B.2), we obtain $||\pi||^2 = \langle \pi, \eta \rangle$. Hence, by the Cauchy-Schwarz inequality,

$$
\frac{(||\pi||)}{||\eta||} \geq \frac{||\pi||}{||\eta||} \geq \frac{||\pi||}{||\eta||}.
$$

Moreover, by inserting $v = n$ and $v = m$ we can extract from (B.2) the identities

$$
||\eta||^2 = \mu \langle n, \eta \rangle = \langle m, \eta \rangle.
$$

The first identity in (B.4) yields $||\eta||^2 = ||\eta||^2$. Furthermore, by the second identity and the Cauchy-Schwarz inequality we obtain $||\mu||^2 \leq ||\eta||^2$. Therefore, in continuation of (B.3),

$$
\frac{||\pi||}{||\eta||} = \mu \langle n, \eta \rangle = \frac{||\mu||^2}{||\eta||^2} \geq \frac{||\mu||^2}{||\mu||^2} = \frac{||\mu||^2}{||n||^2},
$$

and thus we have established the lower bound (4.13).

### References


