A geometric approach to nonlinear dissipative balanced reduction: Continuous and sampled-time

Proefschrift

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Yo no estudio para escribir, 
ni menos para enseñar, 
que fuera en mí desmedida soberbia, 
sino solo para ver si con estudiar 
ignoro menos.

Y no estimo tesoros ni riquezas; 
y así, siempre me causa más contento 
poner riquezas en mi pensamiento, 
que no mi pensamiento en las riquezas.

Juana Inés de Asbaje y Ramírez, 
Mexican writer and Poetess, (1651-1695).

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Chapter 2

Introduction

The subject of this dissertation is structure-preserving model reduction for nonlinear dissipative control systems, using a differential-geometric approach.

2.1 Motivation

Whether we deal with the transient evolvement of physical, chemical, geophysical, climatic, biologic or econometric phenomena, dynamical models provide us frequently with a concise and efficient description of our scientific knowledge. This may include model realizations in ordinary-differential equations (ODE) or in partial-differential equations (PDE). Indeed, dynamical models establish abstract quantitative relationships among the observable or manifest variables and time, imposing restrictions of causality and cause-effect among such variables by classifying them as independent or dependent signals and sometimes as inputs and outputs.

2.1.1 The space of signals and the inner model structure

Besides the set of time $T$, two other mathematical concepts are involved with the previous abstract model. One concept is the space of signals $\mathcal{W}$, which provides a mathematical representation space supporting the temporal evolution of the whole set of manifest variables. Examples of spaces of signals for manifest variables or the input-output signals are Laplace and Fourier spaces, spaces supported by $\mathbb{R}^n$ or by smooth manifolds, prolongation spaces, etc. In
addition, the subset of temporal trajectories defined by the evolution of the external or manifest variables $\mathcal{B} \subset \mathcal{W}^T$ is called the behavior in the system-theoretical framework called behavioral approach, [223, 211, 171]. Sometimes these spaces of signals are furnished with additional structural properties in order to become metric spaces, normed spaces, inner-product spaces, Hilbert spaces, compact spaces, spaces of functions, etc.

The other concept involved with the abstract model is the inner structure of the concrete model realization\(^1\), i.e. the model morphology in which the model quantitative relationships are constructed. Such inner structure is very useful when the system-theoretical properties of closure under composition (or interconnection), closure under factorization and closure under restriction or truncation are required from the model. Frequently the inner structure of the model requires an auxiliary space of signals $\mathcal{M}$ named latent, endogenous or internal variables in order to define such structure (viz., the state-space variables of linear or nonlinear first-order ODE’s or PDE’s).

### 2.1.2 Model classes and their defining properties

In this work we call class of models\(^2\) to the set of models $\mathcal{C}$ with the same defining properties which distinguish its morphology or model structure from other classes of models. The defining properties of a class of models $\mathcal{C}$ must be coordinate independent, implying that there exists a structure-preserving isomorphism $\phi$ transforming one realization into another realization within the same class. Therefore, we denote briefly such class of models by the pair $(\mathcal{C}, \phi)$. Examples of defining properties of the class are linearity, conservativeness, passivity [30, 205], dissipativity [219, 224], etc. Sometimes it is necessary to reconsider the selected set of manifest variables in order to detect these defining properties in the inner structure. These defining properties can be used

---

1. The inclusion of the inner structure into the definition of an abstract model does not contradict the system-theoretic definition by Willems’s behavioral approach [223, 211], where a system is defined by the triad $(\mathcal{T}, \mathcal{W}, \mathcal{B})$, see notation in Def. 4.1 in Chap. 4 of this work. We are including the structure of the abstract model as long as we assume it constitutes the systemic realization of the behavior $\mathcal{B}$. Furthermore, the model reduction methods discussed here deal with the inner structure of the model realizations always under such assumption.

2. Formal definitions on equivalent systems, equivalent model realizations and isomorphic systems in $(\mathcal{C}, \phi)$ can be found later in Chapter 4, Def. 4.4.1, Def. 4.60 and Def. 4.59. Some of the terms coined in this Motivation section are composed by the author from the introduction in [107], pp. 6-13, the modelling vocabulary and the most powerful unfalsified model concept by Willems in [220] and the structural formalization for nonlinear model reduction by Elkin [33].
over several model structures, for instance transfer function matrices, linear or nonlinear first-order (state-space), second order or higher order ODE’s realizations, port-Hamiltonian realizations, finite or infinite dimensional (PDE) realizations, etc. Examples of structure-preserving isomorphisms are unimodular matrices or similarity transformations for linear systems, transformations preserving the skew-symmetry of port-Hamiltonian systems, or in general for nonlinear systems structure-preserving diffeomorphisms.

2.1.3 The compromise between model complexity and accuracy

In most cases, the user of a dynamical model is faced with the fact that the same phenomenon can be described by several model realizations, with explicit limitations of validity within the space of signals chosen, with different levels of sophistication and complexity and frequently one model realization cannot be made equivalent to another. Furthermore, in most practical cases the quantitative relationships provided by dynamical models have a degree of unfitness with respect to the phenomenon it is modeling, which may be due to unmodeled dynamics (in the model in \( \mathcal{C} \) or in the supporting spaces of signals \( \mathcal{W} \) or \( \mathcal{M} \)) or the existence of a non-deterministic\(^3\) uncertainty which in both cases results in an overall approximation error or disagreement between the model and the phenomenon.

In principle, the properties that give advantage to one model realization over any other realization depend on the purpose of the model in the mind of the user. In particular, the properties of the best model realization may be associated to the degree of model complexity, the class of model used, the space of signals, its efficiency of numerical/analytical resolution or its inner model structure. As expressed by Willems in [220], pg. 675:

“The main shortcoming [...] is that in most applications the lack of fit between data and model is not in the first place due to randomness or measurement noise but to the fact that one consciously uses a model whose structure is unable to capture the complexity of the phenomenon which one is observing.”

Take as example the phenomenon of turbulence in fluid dynamics. Although some coherent structures can be partially captured as principal modes, the whole complexity of the phenomenon has not been captured completely with

\(^{3}\) Since the approach of this work is deterministic, we are not providing any systematic treatment of stochastic uncertainties or unknown stochastic inputs. Therefore we are assuming implicitly that such approximation errors are only due to unmodeled dynamics, discarding any uncertainty of non-deterministic nature.
the available class of models used today, see [77]. Nevertheless, in most cases the best model should result in a concise realization, whose estimates provide the highest attainable fidelity with the phenomena it is modeling (within the class of models selected and within the space of signals considered) and, whose inner model structure can be subject to rational analysis by the user.

The possibility of performing rational analysis to the inner model structure is important, because in view of the utilitarian ability of dynamical models of being deterministic and predictive, it has been the fundamental impulse for the user to develop feedback control principles and theories using such inner model structure. To name a few, control theory counts these days with well developed techniques for finite-dimensional systems, viz. for the class of linear systems [62, 228], for the class of passive systems [158], for the class of port-Hamiltonian systems [205, 159], for the class of input affine nonlinear systems [151, 82, 71, 205], etc.

Though, since increasing the model sophistication assumes decreasing model unfitness, seemingly, the user may believe a priori that choosing the most sophisticated model should lead the user to an improved control system design. This latter statement may fail to be true.

On one hand, improving the accuracy and detail of the model description frequently demands a more complex inner model structure (viz., increasing the order the ODE's used, using nonlinear ODE's instead of linear ones, the use of PDE's instead of ODE's, etc.) and therefore including further properties to the space of signals (viz., instead of using \( \mathbb{R}^n \) to consider smooth manifolds or even to consider signals spatially distributed in infinite-dimensional spaces).

On the other hand, complex, nonlinear large-dimensional systems are frequently harder to analyze for control purposes. Even in the linear case, the inclusion of a control system in closed-loop with a high-order dynamical model, introduces additional difficulties, since advanced control system design methods tend to supply controllers with an order comparable to the order of the plant, resulting therefore in high-order controllers, [154].

2.1.4 Model approximation and model reduction

Every model is an approximation abstracted from reality. The process of representing a given behavior \( \mathcal{B} \) with a concrete model realization is the concern of realization theory, see also [220], pg. 686. In contrast, the problem of model approximation entails the imposition of a notion of distance with a metric in the spaces of signals \( \mathcal{W} \) and \( \mathcal{M} \), measuring the disagreement between the original behavior \( \mathcal{B} \) and an approximated behavior \( \mathcal{B}_a \). Thus, model approximation is the process of obtaining a concrete approximated model realization
2.1 Motivation

whose approximated behavior $B_a$ keeps a minimal distance with the original behavior $B$. Under this notion, the basic problem of state-space system identification [107, 221] may be considered as a problem of model approximation (in agreement with [220]), since given a finite set of trajectories in $B$, and a class of models with a model structure (linear state-space models supported by $\mathbb{R}^n$), the basic problem of system identification consists in finding the realization with the best description, in the terms of the least squares of the approximation error.

From a purely system-theoretic viewpoint, model reduction is a closed operation performed within members in a class of models $(\mathcal{C}, \phi)$, since it is the consequence of two closed algebraic operations performed on the original concrete model: factorization (into two subsystems) and restriction (or truncation by elimination of the dynamics of one subsystem). But viewed as a subproblem of model approximation, the problem of model reduction implies the dimensional reduction of the supporting space of $B$ to obtain a reduced behavior $B_{\text{red}}$ from which the reduced-order model is resolved by factorization and restriction (truncation). This view for model approximation by dimensional reduction is closed for classes $(\mathcal{C}, \phi)$ supported on finite-dimensional spaces of signals $W$ or $M$. When the class $(\mathcal{C}, \phi)$ consists of infinite-dimensional models (expressed by systems of PDE's), there are several alternatives for model approximation. When the approximation method results in a system with a reduced number of infinite-dimensional states, it is also referred to as model reduction, because the reduced system of PDE's belongs to the same class $(\mathcal{C}, \phi)$, for instance, the Saint Venant Equations are simplified PDE's from the complete equations of fluid dynamics [24]. Alternatively, departing again from infinite-dimensional models, the approximation method may yield a system of lower-order ODE's [39, 26], which clearly does not belong to the same class. For instance, the pure transport delay from an advection PDE can be approximated by a rational transfer function using the finite-dimensional Padé approximation method, in the linear class. Furthermore, in some of such approximation methods, the approximated model may consist of a finite set of uniform components such that every component consists of a system of lower-order ODE's whose dynamic properties or structure is recognizably self-similar to the dynamic properties or structure of the original system, a process frequently referred as lumping the distributed parameters, for instance, a ladder-chain of LC circuits approximating an ideal transmission line [122], or the Saint Venant Equations of fluid
dynamics [164, 178]. Actually, most of the methods to solve PDE’s with a computer, require some finite-dimensional approximation.

2.1.5 Dimensional reduction for dynamical systems

The mathematical sciences have been concerned for a long time on dimensional reduction methods for dynamical systems. In particular, Jacobi, Liouville and Cartan developed differential geometric methods of model reduction using first integrals and symmetries in symplectic manifolds [132, 1], etc., and the method of symmetries is a well known reduction method for Hamiltonian systems [132]. Nevertheless, most of these reduction methods have shown to be inappropriate to characterize the input-output invariant properties required in reduced-order control systems.

While the classical scientific view of modeling mainly concentrates on behavioral observation and model-based description of discovered phenomena [221], control models are distinguished by the prescriptive use of the model inputs and their inner model structure, in order to influence the behavior of the phenomenon (described by the model) and with this to obtain a desired performance, see [16, 221] for a longer discussion on descriptive vs. prescriptive sciences. This prescriptive use of control models justifies the need of specific model approximation methods intended specifically for control systems, alternative to the model reduction methods used in descriptive sciences.

2.1.6 Justifications for dimensional reduction of control systems

Model order-reduction (MOR) methods for control systems arise from several practical reasons:

1. Computer implementation: Today, just like in the early days, model reduction is necessary in order to be able to down-size the complexity of the numerical algorithms implemented on a computer, speeding up their execution. Frequently, models are expected to perform in real-time, requiring with this a sophisticated real-time software architecture and expensive computing infrastructure, if model reduction is not considered [114, 115].

4 One exception is the method of characteristics which transforms without approximation a hyperbolic PDE into a family of ODE’s, but the solutions provided by the method are only valid in the intersections of the solutions of the family of ODE’s, see [118] and references therein, for a method of solution for distributed port-Hamiltonian fluid equations.

5 In a wide geometric sense, we assert that a mathematical object has a symmetry if it has a transformation that preserves the object’s structure (i.e. a structure-invariant transformation). Noether proved that conservation laws (viz. conservation of energy) are associated with a differentiable symmetry of a physical system, see [152].
2. To increase the efficiency during the simulation of model realizations: The property of system interconnection for control systems has been a great advantage to perform transient simulations for networks of systems. Unfortunately, system interconnection also leads to the simulation of interconnected systems of undesirably large dimensions which may include the simulation of useless structures within the network of interconnected systems. In contraposition, model reduction aims at reducing such simulation efforts by discarding systematically such useless structures from the simulations. Although a refined description may require frequently higher-order models (e.g. distributed parameter models), the order of control variables is, in contrast, usually much lower. For instance, while petroleum reservoir models may have several thousands of state variables, the number of control variables is usually in the order of several tens. Thus, for control purposes, a precise model description is not necessarily paired with model efficiency.

3. Control system design improvement: Dynamical models are useful for prediction and estimation, especially in closed-loop. The problem of model reduction for feedback control systems consists in the design of a reduced-order controller from a given (full-order) model of a plant. This class of problems are more complex than just reduction of descriptive models because the controller is part of the closed loop and the reduced-order controller should be designed in such a way that the individual (input-output) interaction of the controller and the plant model within the closed loop system is not lost and simultaneously the closed loop stability and performance are not deteriorated. This problem is usually approached in two ways: In the first approach, the problem consists in obtaining a reduced-order model from the full-order plant model, then designing the controller based on the structure of such reduced-order model and finally verifying the preservation of the performance of the controller and the full-order plant model in closed-loop. In the second approach, the problem consists in first designing the (full-order) controller based on the structure of the full-order plant model, then obtaining a reduced-order controller which preserves the structure and the stabilization properties of the full-order controller and finally verifying the preservation of the performance of the reduced-order controller and the full-order plant model in closed-loop.

4. The need to keep up the analyst’s intuition about the model: The analyst user may prefer some model structure e.g. first-order or second-order linear or input-affine nonlinear structure, external differential representations, etc. Moreover, especially in the nonlinear realm, there are certain
model properties helping to keep the analyst’s intuition about the model and to perform control design methods, e.g. mass conservation, energy conservation, passivity, dissipativity, etc., [205]. Other formal properties of fundamental importance in control design are the stability, controllability and observability of the system, [73, 151].

The relevance of models for applied science and engineering is such, that huge budgets are invested yearly by universities and research institutes to obtain ever faster super-computers to comply with the ever growing need of expensive simulation software to simulate complex dynamical models.

With the wide availability of software for multi-physics simulation and the ever-increasing computing capacity of present-day computers, it could be believed naively that model reduction methods will become unnecessary in the near future. It is in fact the opposite, model reduction methods are called to become just another stage for control system analysis and design, because the increase in the size and the mixed nature of such large-dimensional models have made harder to the human analyst the application of conventional tools for feedback control and dynamic optimization.

Needless to say, the increase of the portability of model-based solutions after reduced-order model technology, will result in savings on the budgets for computer simulation by optimizing the use of computer infrastructure, just like algorithms for data compression did for digital music.

2.1.7 Empirical methods for model reduction

Both for theoretical and practical reasons, several approaches for model reduction of control systems have been developed over the years. Of particular relevance for this work, problems of practical orientation have originated the development of empirical methods for model reduction, using essentially the same technique, with different names depending on the context.

Such technique is called principal component analysis (PCA). Although the most recognized antecedent to PCA as a technique for data approximation is [166], the most recognized reference of PCA in the realm of statistics is [78], where the author coined the term principal components of variance for a multivariable random variable. More than ten years later, in the realm of probability and stochastic processes was developed a theory, credited to independent work by K. Karhunen [92] and M. Loeve [108], about a theorem of proper orthogonal decomposition of second-order random functions supported by a Hilbert space, see Chapter 10 (pg. 478) in [109].

In order to determine numerically the cited orthogonal decomposition, a ma-
matrix decomposition known as singular value decomposition (SVD), must be obtained. According to [193], the algorithmic evolvement of SVD has a much earlier history than PCA, situating Beltrami and Jordan as progenitors of SVD. By 1829 Cauchy already had provided a proof that the eigenvalues of a symmetric matrix are real and that the corresponding quadratic form can be transformed into a sum of square terms by an orthogonal substitution, [69]. For real, square, nonsingular matrices having distinct singular values, the existence of SVD and an algorithm to find it is early proved by Beltrami in [12]. Independently, Jordan provided a simpler proof of existence of a bilinear form reduced to a diagonal form by orthogonal substitutions [85, 86, 87]. Similarly Sylvester in [198] describes an iterative algorithm for reducing a quadratic form to diagonal form (he later extended it to bilinear forms), being valid for real, square matrices with distinct eigenvalues. By 1883, the Gram-Schmidt orthonormalization process and the Gramian matrix/determinant were already known [55, 188]. The extension of SVD for general matrices was given by Schmidt [188] and Weyl [216]. Schmidt in his treatment of integral equations with unsymmetric kernels [188], introduced the generalization of SVD to the domain of integral equations in infinite dimensional spaces of functions, and provided his approximation theorem about the best lower rank approximation of a matrix [193]. The approximation theorem by [188] was extended by Weyl [216] with his general perturbation theory to find the rank of a matrix in the presence of error. More recently Mirsky in [139] showed that the approximate matrix is a minimizing matrix in any unitarily invariant norm [193]. Eckart and Young rediscovered Schmidt’s approximation theorem in [31] and extended SVD to rectangular matrices in [32]. Therefore PCA is sometimes also referred as the Schmidt-Mirsky or (incorrectly) Eckart-Young (approximation) theorem.

In the realm of model reduction for fluid dynamics PCA is known as Karhunen-Loève expansion/decomposition [77, 99], as Proper Orthogonal Decomposition (POD) [176, 175, 217] or as Empirical Eigenfunction Decomposition [191]. In geophysical signal processing it is known as the Karhunen-Loève Transform (KLT) [72], and in the realm of Meteorology and Oceanography it is known as Empirical Orthogonal Functions (EOF) [130, 172], etc.

2.1.8 The search for structural invariants in dynamical systems

Probably the most revealing aspect of PCA in the analysis of dynamical systems lies in suggesting the existence of structural invariants along the temporal evolution of such systems. These invariants, called (in the jargon of PCA) principal components, principal modes, or sometimes coherent structures [191, 77],
can be interpreted as geometric objects (viz., eigenvalues, eigenvectors, hyper surfaces) associated to the geometric properties of a (properly defined) self-adjoint operator and have been used to propose reduced-order models. One of the features of PCA lies in providing several degrees of model approximation, according to the number and influence of the principal modes included in each reduced model. Furthermore, despite the rather empirical nature of the method, the resulting models have shown to be extremely efficient, and it is even possible to provide structures useful for control system design [217, 8].

In control systems theory, it is known that the set of input-output structural invariants of a linear multivariable system with state-space realization \((A, B, C, D)\) supported by \(\mathbb{R}^n\), consists of the input and output Kronecker indices and a canonical permutation \([15, 144]\). Such Kronecker indices are invariants under a group of transformations and include a subset of the Markov parameters\(^6\) in the set of invariants, see details in \([15]\). Furthermore, the state-space realization is minimal if \(n\) is minimal, which occurs if and only if \((A, B)\) is controllable and \((C, A)\) is observable \([91]\).

Since such structural invariants are intimately related to the most essential (or characteristic) properties of dynamical systems, the correct characterization (and interpretation) of such structural invariants leads to appropriate methods of model reduction which inherit a subset of such properties in the reduced model, according some order of relevance.

### 2.1.9 Principal components and model reduction for linear systems

The paradigmatic work *Principal Component Analysis in Linear Systems*, by Moore \([143, 142]\), set forth the advantages of SVD as an efficient numerical tool for (linear, continuous-time) structure-preserving model reduction for a state-space realization \((A, B, C, D)\) supported by \(\mathbb{R}^n\). Moreover, after introducing the balanced truncation method, it was possible to show many control relevant properties associated to the resulting minimal balanced realizations, namely the preservation of geometric concepts like the controllability and observability subspaces, the controllability and observability Gramians, etc. In the work by Moore \([143]\) there is no claim of optimality in any sense for the reduced-order models obtained by this method. Even though balanced realizations were reported earlier in \([145]\), after Moore’s approach several works

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\(^6\) The Markov parameters \(\{h_i \overset{\text{def}}{=} CA^iB, i = 1, 2, ..., \infty\}\) are the impulse response values of (continuous or discrete-time) linear time-invariant systems and are well known invariants under similarity transformations. The Markov parameters appear as anti-diagonal entry coefficients in the Hankel matrix associated to the realization \((A, B, C, D)\), \([90]\).
followed along the line of *linear balanced reduction* also for discrete-time systems, see e.g., [167]. Balanced reduction by the singular perturbation method is discussed in [34] and an additional Gramian, the so-called *Cross-Gramian* is introduced in [36] and further developed in [35, 37, 38]. See [52, 95, 102] for details on numerical balancing algorithms.

Meanwhile, in [2], after showing several analytic properties of the *Schmidt pairs*\(^7\) for a Hankel operator, the authors provided an explicit formula for the (optimal Hankel-norm) approximation of an infinite Hankel matrix by a Hankel matrix of lower rank in terms of its singular vectors. Based on [2], Glover [49] established how several optimal approximation problems for linear continuous-time state space systems could be used to characterize the infinity norm bounds for all the optimal Hankel-norm approximations, resulting in *a priori upper error bounds* for the reduced-order models. In the case of linear discrete-time systems, error bounds and weighting techniques, were discussed in [4, 5].

Balancing of state-space doubly normalized coprime representations are discussed in [148]. While balancing of the normalized left coprime factorization is discussed in [153], balancing of the normalized right coprime factorization is discussed in [135]. LQG balanced reduction for passive systems was proposed in [84, 157]. The approach of LQG balancing in [157] was generalized in the behavioral-dissipative approach in [211], where the author provides an attractive extension to the balancing concept by using a dissipativity framework which preserves the dissipative structure in the linear lower-order model. In [137] was presented an approach for balancing based on the behavioral framework for linear systems in [171].

### 2.1.10 Principal components and model reduction for nonlinear systems

The extension of PCA and their properties for nonlinear systems evolved somehow differently.

On the practical side, while the prevalence of PCA methods for model reduction cannot be underestimated, in most methods of PCA for nonlinear systems, such structural invariants are found empirically with the use of the so-called

\(^7\) The linear Schmidt pairs in linear systems theory, are a consequence of a representation theorem, in the theory of functional analysis, for orthonormal sets in pre-Hilbert selfadjoint positive compact operators, due to E. Schmidt, originally defined for integral equations in infinite dimensional spaces of functions in [188] pg. 461.
Method of Snapshots. For instance, nonlinearities can be added to the cluster of time series from the method of snapshots, and the linear SVD procedure is performed in order to provide the qualitative identification of the nonlinearity and the quantitative determination of the nonlinearity in the time series, see [199]. Mainly in the statistics literature [28] but also in the so-called Intelligent Computing [53], there are several nonlinear ad hoc generalizations to PCA, where instead of summarizing data with a straight line, a smooth curve is set forth and the concept of principal curve of a probability distribution is introduced along with an algorithm in [67, 68] and (sometimes in combination with neural networks) the concepts of principal surfaces or principal manifolds are proposed as nonlinear generalizations, see [68, 98] and references therein. Furthermore, curvature has been proposed as a measure for nonlinearity, see [189, 11].

Nevertheless, apart from the aforementioned developments, there does not exist a unified, formal, mathematical or system-theoretic formulation neither for nonlinear principal component analysis nor for nonlinear singular value decomposition.

Regarding the use of PCA for reduction of nonlinear systems, several ad hoc methods were developed. To name a few, in [48, 99, 65] and other works, the use of empirical Gramians and an empirical balanced truncation method for nonlinear systems is reported. In such method, the invariance of singular values no longer holds [99]. See [7, 6] for model reduction methods for large-scale dynamical systems based on POD (combining SVD and Krylov methods) and others. In fluid dynamics PCA methods are prevalent [77, 176, 175, 218, 217, 8] and there is a large amount of research activity addressing the use of PCA methods for important problems in industry,—especially in the Oil & Gas Exploration and Production Industry,— demanding for improved methods of reduction for large dimensional dynamical models, viz. for feedback control of petroleum reservoirs [113, 70, 206, 131, 22, 23].

Thus, there is the need for a firmer theoretical framework, extending Moore’s balancing and Hankel-operator based balancing to the current nonlinear systems theory.

On the theoretical side, the development of the nonlinear balancing theory emerged by a series of papers discussing several extensions to their linear coun-

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8 Essentially, in Sirovich’ method of snapshots [77], such snapshots consist of clusters of time series (or collections of samples) of the system states, input and output trajectories. The principal components are found after a singular value decomposition of the data clusters. Though, the resulting singular values are not input-output invariants and they depend on the selection of snapshots.
terpart as isolated problems for continuous-time nonlinear systems expressed by

\[ \Sigma : \dot{x}(t) = f(x(t)) + g(x(t)) u(t), \]
\[ y(t) = h(x(t)), \]  

(2.1)

where \( x \in \mathbb{R}^n \) are local coordinates for a \( C^\infty \) state space manifold \( \mathcal{M} \), \( f \) and \( g_1, \ldots, g_p \) are \( C^\infty \) in \( \mathcal{M} \) where \( g = (g_1, \ldots, g_p) \), \( u = (u_1, \ldots, u_p) \), \( u \in \mathcal{U} \approx \mathbb{R}^p \) and \( y = (y_1, \ldots, y_q)^T \), \( y \in \mathcal{Y} \approx \mathbb{R}^q \), where the map \( h : \mathcal{M} \to \mathcal{Y} \), \( h = (h_1, \ldots, h_q)^T \) is \( C^\infty \) in \( \mathcal{Y} \).

Such balancing problems can be categorized as follows:

- **State-space balancing for stable systems:** In [180, 181] a nonlinear extension to Moore’s balancing method is presented after the definition of the observability and controllability energy functions. The main result is a method of balancing for the nonlinear system (2.1) assuming it is asymptotically stable. In this method, justified by Morse’s critical point theory [136], the author performs a change of coordinates to express the system into input normal form such that the controllability energy function is one half of the sum of squares of the state coordinates. After a second change of coordinates that preserves the input normal form and diagonalizes the observability energy function, the set of state-dependent diagonal entries, are called singular value functions. Furthermore two Hamilton-Jacobi-Bellman Equations (HJBE) involving the energy functions are proposed as nonlinear generalizations for the two Lyapunov equations proposed by Moore to find the Gramians.

- **LQG balancing:** In order to overcome the requirement of asymptotic stability of the previous extension, in [187] the authors presented two techniques that extend to nonlinear unstable systems, namely LQG balancing (for the linear case see [157]) and the method of balancing normalized representations (for the linear case see [148, 153, 135]). In particular, the authors provide a nonlinear extension to the concepts of (left and right) normalized coprime factorizations using the concepts of inner and co-inner nonlinear systems. Their balancing method is also posed in terms of Hamilton-Jacobi-Bellman equations.

- **\( H_\infty \)-balancing:** In [182] (see also [181]) the author developed the \( H_\infty \)-balancing problem by viewing it as an extension to the LQG balancing problem, and with the use of energy functions modified accordingly.

Further progress on the nonlinear extensions of balancing methods for continuous-time nonlinear systems, was mainly performed with the controllability and
observability functions and their corresponding singular-values [181, 180, 43], but also with alternative past and future energy functions [60, 187, 182] and for port-Hamiltonian Systems [127]. Local minimal realizations have been discussed earlier in [185]. In [165], the $\mathcal{H}_\infty$-balanced truncation approach by Mustafa and Glover [146] for linear plants is extended to input affine nonlinear plants using the nonlinear balancing approach by [182]. Furthermore in [149] several practical difficulties about the calculation of the energy functions used for balancing are discussed.

Nevertheless, some problems with this approach emerged, mainly associated to the implications that in the theoretical framework by [180, 181] the singular-value functions are coordinate-dependent, in contrast to the linear case, reducing their results to a local validity [59]. Since the reduction technique in [180, 181] is not intrinsic, the singular value functions are not unique (after [59]), and the resulting reduced order system depends on the changes of coordinates that are used to synthesize it and such coordinates are not unique either [97].

Some of such problems were discussed and solved in several publications, see e.g. [57, 185, 59]. In [44] the concept of principal axis-balancing was introduced as a generalization of the linear concept, and furthermore, it is shown that the nonlinear balancing problem can be solved with the solution of a nonlinear eigenvalue problem. In the same paper [44] the authors showed the existence of a normal form, but according to [97], such normal form is not unique, and moreover, in [97] is proposed a new normal form for the controllability and observability functions of a nonlinear control system such that such functions in the resulting reduced order model are almost the restrictions of the original functions of the full order model.

Nonlinear Hilbert adjoint systems for balanced reduction have been discussed previously in [58, 46, 186]. Hankel operators and nonlinear Gramians are presented in [56]. Singular value functions have been discussed in [183, 184, 61]. One nonlinear generalization of the linear Hankel balancing theory was presented earlier in [46]. Several results for the eigenvalue problem of the nonlinear Hankel operator have been reported earlier, see e.g. [44] and references therein. Schmidt pairs for the nonlinear balancing problem are proposed in [61].

Based on the differential eigenstructure of a nonlinear Hankel operator extension, nonlinear input-normal realizations were analyzed in [44, 43]. In [45] the authors showed the advantages of singular value analysis for balanced realizations based on the controllability and observability functions in both continuous and discrete-time nonlinear systems. Recently, the method of sin-
2.1 Motivation

Gular value analysis was successfully applied to nonlinear symmetric systems in [80].

2.1.11 Balanced model reduction for nonlinear dissipative systems

Meanwhile, another stream of research for nonlinear balanced reduction, that later became the program of research for this dissertation, was inaugurated in the early paper [127] with the first assumption on system (2.1) being dissipative, i.e. there exists some function $S : \mathcal{M} \to \mathbb{R}^+$ called storage function such that the inequality

$$ S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} r(w(t)) \, dt \quad (2.2) $$

is preserved for all trajectories $x(t) \in \mathcal{M}$, where $w(t) \overset{\text{def}}{=} (u(t), y(t)) \in \mathcal{W}$, $t \in \mathbb{R}$. The set of external variables$^9$ $\mathcal{W} \approx \mathbb{R}^\omega$, $p + q = \omega$, includes $u \in \mathcal{U} \approx \mathbb{R}^p$ and $y \in \mathcal{Y} \approx \mathbb{R}^q$ as subsets. From general dissipative systems theory [219], it is known that associated to the system (2.1) are the storage functions called required supply, $S_r : \mathcal{M} \to \mathbb{R}^+$, defined as

$$ S_r(x^0, r_r) \overset{\text{def}}{=} \inf_{u(\cdot) \in \mathcal{U} \subset \mathcal{W}} \int_{-T}^{0} r_r(w(t)) \, dt, \quad x^0 = x(0) \in \mathcal{M}, T \geq 0 $$

and the available storage, $S_a : \mathcal{M} \to \mathbb{R}^+$, defined as

$$ S_a(x^0, r_a) \overset{\text{def}}{=} \sup_{u(\cdot) \in \mathcal{U} \subset \mathcal{W}} - \int_{0}^{T} \! r_a(w(t)) \, dt, \quad x^0 = x(0) \in \mathcal{M}, T \geq 0 $$

where $r(w(t))$, $r : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$, is the supply rate (relative to $S_r$ or $S_a$), expressed by the quadratic function $r : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ satisfying $r(w(t)) = w^T(t)Zw(t) \geq 0, \forall t \in \mathbb{T}, w \in \mathcal{W}$, [75].

Consider the following remarks from [127], about the energy functions in [180, 181, 187, 60] for nonlinear balancing theory:

- By defining as supply rate for the required supply $r(t) = u^T(t)u(t)$ and $r(t) = y^T(t)y(t)$ for the available storage, the controllability and (natural) observability functions $L_c(x_0)$ and $L_o^N(x_0)$ respectively, can be obtained for continuous [180, 60] and discrete-time systems [121].

$^9$ In the notation of Ch. 4, $\mathcal{W} \approx \mathbb{R}^\omega$ means that $\mathcal{W}$ has the same cardinality (Hamel dimension) of a homeomorphic $\mathbb{R}^\omega$. 
If \( r_a(u, y) = r_r(u, y) = \|y\|^2 + \|u\|^2 \) is used to conform \( S_a \) and \( S_r \), then this parallels the treatment of *past and future energy functions* \( K^- \) and \( K^+ \) presented in [181, 187] for balancing unstable nonlinear systems, [127].

If \( r_r(u, y) = (1 - \frac{1}{\gamma^2})\|y\|^2 + \|u\|^2 \) is used for \( S_r(r_r) \) and \( r_a(u, y) = \|y\|^2 + (\frac{\gamma^2}{\gamma^2 - 1})\|u\|^2, \gamma > 1 \) for \( S_a(r_a) \), then this parallels the treatment of \( \mathcal{H}_\infty \)-past and \( \mathcal{H}_\infty \)-future energy functions \( Q^-_\gamma \) and \( Q^+_\gamma \) presented in [181, 182] for nonlinear \( \mathcal{H}_\infty \)-balancing, [127].

In view of the fact that most of the energy functions used in nonlinear balancing theory can be expressed as *storage functions* of the theory of dissipative systems, in the early paper [127], a plan for a framework for nonlinear balancing of dissipative systems was outlined as follows:

1. By replacing previous assumptions\(^{10}\) on system (2.1), by the assumption of being dissipative for the supply rates \( r_a \) and \( r_r \), and existence of \( S_a(x_0, r_a) \) and \( S_r(x_0, r_r) \) around a point \( x(0) = x^0, (S_r(x_0, r_r) \neq 0) \), a *principal gain* was defined by the so-called *storage quotient* defined by\(^{11}\)

\[
|\Sigma|_S^2 = \sup_{x^0 = x(0) \in \mathcal{M}} \left[ \frac{S_a(x^0, r_a)}{S_r(x^0, r_r)} \right]. \tag{2.3}
\]

2. In dissipative systems theory, a dynamical system is conceptualized as a mathematical object which maps inputs into outputs, via the state which summarizes the influence of past inputs [219]. This parallels the interpretation of the Hankel operator as a map from past inputs into future outputs.

Regarding point 1, the geometric problem of characterizing the eigenvalue problem of the quotient (2.3) consisting of finding the *principal directions* where (2.3) attains stationary values was presented using classical curvature theory in [110].

Regarding point 2, in [110], the behavioral operator was defined as another nonlinear generalization to the Hankel operator.

The results presented in [110] and further developed in [116, 117], mark a

\(^{10}\) In particular, we discard the assumption of asymptotic stability of system (2.1). Since we assume that the storage functions exist around a critical point, whether we use Morse theory on manifolds from Milnor [136] or from Palais et. al. [160, 162, 163], in order to guarantee existence of such critical point, we must include the assumption of compactness of the supporting manifolds. Furthermore, we assume that such Hilbert manifolds admit partitions of unity.

\(^{11}\) As observed by one member of the committee, existence of the quotient (2.3) may require additional assumptions as \( r_a \neq r_r \). For more precise existence conditions on this quotient and further references, see the text below Def. 3.6.
point of departure from the theoretical framework of nonlinear balancing developed until that moment in [44, 43], and provide an independent theoretical framework purely based on the assumption of the dissipativity of the dynamical system and standard differential-geometric concepts like Morse theory on Hilbert manifolds and curvature theory. Furthermore, other concepts like nonlinear Gramians, Schmidt pairs, Schmidt decomposition, etc., could be explained within the same geometric context [126, 111, 110, 125].

Thus, while the explicit use of dissipativity theory for balancing of linear systems was firstly presented in [211], the first paper, —where being a nonlinear dissipative system is the departing hypothesis—, to define the problem of nonlinear behavioral or dissipative balancing was firstly proposed in [127]. In this approach the balancing problem is viewed as a whole, instead of dealing with the variety of energy functions used for nonlinear balancing for continuous and discrete-time systems. After this approach, the energy functions for balancing in discrete-time nonlinear systems, originally presented in [121], were recasted into the generalized dissipativity approach in [126]. Moreover, in [123] an extension is proposed to the theory of dissipative systems in order to define proper storage functionals towards a theory of dissipative balancing for distributed-parameter systems. Alternative references to dissipative distributed systems are [224, 169].

The approach of dissipative balancing was also discussed in the context of the balancing method by Prof. Scherpen and co-workers. In particular in [79, 81] the theory of nonlinear balancing described in [44, 181] is combined with dissipativity theory in order to attack several problems involving positive-real, bounded-real and symmetric systems. Furthermore, the approach by Prof. Scherpen and co-workers establishes relationships between the nonlinear cross-Gramian and Gradients systems [80]. Similarly to the nonlinear balancing theory from [181], in their approach, the storage functions are characterized as solutions of Hamilton-Jacobi-Bellman equations.

Consider now the realm of lumped-approximation of distributed parameter models. A problem of current interest is the development of model approximation methods preserving the port-Hamiltonian structure. One of the reasons of this interest for control theorists is due in part to the existence of a well-developed finite-dimensional control theory for this class of systems [159]. Therefore from the early development of the distributed port-Hamiltonian theory [133, 203] there has been a lot of interest in the development of appropriate structure-preserving model approximation methods, e.g. using geometric methods based on finite elements [50, 51] or finite differences [122] also for
fluid dynamics [26, 164] and more recently with finite elements [210, 209, 10] and computational geometric methods [190].

Other effort of industrial orientation is the method of Hamiltonian characteristics for model approximation of a one-dimensional port-Hamiltonian fluid dynamical model, introduced in [118]. Such model is embedded in a boundary-feedback Luenberger-type estimator for detection of leaks in pipelines and is simulated successfully in a test rig facility in [119, 120].

2.2 Problem formulation

Consider an object called a nonlinear dynamical control system $\Sigma$ in Eq. (2.1), consisting of a system of ordinary differential equations (ODE) of order $n \in \mathbb{Z}^+$, with own structural properties of stability, reachability and observability, belonging to a class of models $(\mathcal{C},\phi)$ whose behavior is supported by the triad $(\mathcal{T}, \mathcal{W}, \mathcal{B})$ and whose internal state-space trajectories are supported by $\mathcal{M}^n$ and satisfy the defining property of dissipativity (i.e. they satisfy the dissipation inequality (2.2)). The problem of this dissertation consist in the development of an analytical method to obtain a family of sub-objects called reduced order models $\Sigma_r$, $r = 1, \ldots, n-1$; $r, n \in \mathbb{Z}^+$ belonging to the same class of models $(\mathcal{C},\phi)$ but supported by a reduced-order space of state trajectories $\mathcal{M}_r^r, r = 1, \ldots, n-1$; $r, n \in \mathbb{Z}^+$ (respectively), obtained with a structure-preserving isomorphism $\phi$ to perform the operations of factorization and restriction (or truncation), with behavior supported by $(\mathcal{T}, \mathcal{W}, \mathcal{B})$ (or by $(\mathcal{T}, \mathcal{W}_{\text{red}}, \mathcal{B}_{\text{red}})$ if it is model approximation) such that the structural properties of stability, reachability and observability and the property of dissipativity are preserved in each sub-object $\Sigma_r$, $r = 1, \ldots, n-1$; $r, n \in \mathbb{Z}^+$. The previous statement is called the problem of structure-preserving model reduction for nonlinear input-affine dissipative control systems.

2.3 Methodology

The problem of structure-preserving model reduction for nonlinear input-affine dissipative control systems is discussed in this dissertation, using a nonlinear interpretation, –in the language of differential geometry–, to the balanced reduction problem based on the theory of dissipative systems and the so-called behavioral approach.

Consider the set of temporal trajectories defined by the evolution of the external or manifest variables expressed briefly by the set $\{w(t) \overset{\text{def}}{=} (u(t), y(t)) | t \in$
2.3 Methodology

This set of external variables (under an appropriate framework) is called the behavior, denoted by $\mathcal{B}$, and is the central topic of the system-theoretical framework called behavioral approach (see [223, 211, 171] and references therein).

In this work we assume throughout that system (2.1) is dissipative, i.e. there exists some function $S(x(t))$ called storage function such that the inequality (2.2) is preserved for all trajectories $x(t) \in \mathcal{M}$ and $w(t) \in \mathcal{B}$. Furthermore the storage function called required supply with supply rate $r_r(w(t))$ is used to characterize past semi-trajectories and the storage function called available storage with supply rate $r_a(w(t))$ is used to characterize future semi-trajectories.

As expressed in [127], using general dissipative systems theory, several problems of nonlinear balancing can be posed in a unifying format summarized in Table 2.1. Furthermore, in order to avoid all the coordinate-dependencies concerning the results presented in the first works published about nonlinear balancing, in this work a geometric approach is considered using differential geometry and critical point theory in Hilbert manifolds.

The essence of the geometric approach consists of developing most of the mathematical support in coordinate-free form [9], separating structural questions from computational ones. It consists of the construction and characterization of intrinsic (coordinate-free) mathematical concepts (e.g. invariants, subspaces, quotient spaces, hyper surfaces, distributions or submanifolds and relationships between them) in connection with the behavior of dynamical systems under feedback, regulation and tracking problems. Thus, there are no coordinate transformations to consider since all the geometric objects must be characterized independently of the coordinates used.

Control theory is pervaded by geometric techniques because they provide fairly general and straight solvability conditions to the most fundamental control

<table>
<thead>
<tr>
<th>Problem</th>
<th>$L_2$-gain</th>
<th>Passivity</th>
<th>Hankel</th>
<th>LQG</th>
<th>$H_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_{past}$</td>
<td>$\begin{bmatrix} \frac{1}{2} \gamma^2 I_u &amp; 0 \ 0 &amp; -\frac{1}{2} I_y \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; \frac{1}{2} I_u \ \frac{1}{2} I_u &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} \frac{1}{2} I_u &amp; 0 \ 0 &amp; \frac{1}{2} I_y \end{bmatrix}$</td>
<td>$\begin{bmatrix} \frac{1}{2} I_u &amp; 0 \ 0 &amp; \frac{1}{2} I_y \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$Z_{future}$</td>
<td>$\begin{bmatrix} -\frac{1}{2} \gamma^2 I_u &amp; 0 \ 0 &amp; \frac{1}{2} I_y \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; \frac{1}{2} I_u \ \frac{1}{2} I_u &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; \frac{1}{2} I_y \ \frac{1}{2} I_y &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -\frac{1}{2} \gamma^2 I_u &amp; 0 \ 0 &amp; -\frac{1}{2} I_y \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1. Definition of $Z$ matrices for dissipative balancing problems with supply rate $r(w(t)) \overset{\text{def}}{=} w^T(t)Zw(t)$ (see Assumption 4.21 in Chapter 4 for notation).
problems, viz., linear [9, 225, 144] and nonlinear reachability and observability in continuous [73, 195] and discrete-time [83, 150] multivariable control systems, solvability of disturbance decoupling problems [225, 151, 128], linear and nonlinear fault detectability conditions [134, 129, 168], existence and uniqueness of minimal realization of nonlinear systems [197] etc., to name a few.

In this research, (semi-) trajectories are the geometric objects most frequently used. Therefore the elegant theory of Gauss’ curvature crops up as a unifying concept for this generalized balancing theory. For the historic development of this theory see e.g. [194]. Inherited by this theory, the terms principal gains, principal curvatures, principal directions, principal eigenvector fields, principal frame, principal eigenfunctions and principal (orthogonal) hyper surfaces can be integrated naturally to our technical parlance in the context of this work. See Figure 2.1 illustrating the curvature of a trajectory.

Even though Milnor’s exposition on Morse’s critical point theory [136] is a well-established and fundamental source, in this work the exposition of Morse theory on Hilbert manifolds by [160, 162, 163] resulted more appropriate for

\[ \begin{align*}
\text{Fig. 2.1.} & \quad \text{Illustration of the curvature of a trajectory } x(t) \in \mathcal{M}^2 \text{ with normal vector field } \eta \\
& \quad \text{and tangent vector field } \xi \text{ in the hyper plane } \Pi_M \text{ intersecting (by the half) the hyper sphere } S \subset T.M \text{ with radius } r \text{ (for the notation see Section 5.3.2 in Chapter 5).}
\end{align*} \]
our purposes due to their discussion on positive semigroups. Furthermore, besides the use of curvature theory, the use of isometries on Hilbert manifolds for nonlinear systems contributed to the development of the work presented in this document where linear structural concepts for Hilbert spaces like *duality*, *adjointness* and *orthogonality*, were reinterpreted for our differential geometric framework.

Although the work by Elkin [33] does not deal with balanced reduction, it influenced this work with his structural differential-geometric formalization for nonlinear model reduction. In particular model reduction is viewed as an operation performed within the members of a *class of models*. Such class has a *particular structure* (linear, bilinear, nonlinear, with a Dirac structure etc.) and *characteristic properties* (e.g. passive, $L_2$-gain, Hamiltonian). Furthermore such class is *closed under composition and factorization* and is furnished with a structure-preserving isomorphism.

The process of model reduction consists of a sequence of steps: the determination of an *isomorphic system*, *factorization into a quotient system* and a *restriction into a subsystem*, [33]. In the context of balanced reduction, we provide a factorization method based on two arguments: the first one is based on a *group extension* of two semi-group actions and the second is based on the *orthogonal separability* of integral invariant functions.

Separability is a classic method for integration by quadratures for Hamilton-Jacobi equations studied since the time of Levi-Civita [104], see [173] for a recent account. While our approach for orthogonal separability in differential manifolds is based on [13], the most crucial answers to complete details about critical point theory in Hilbert manifolds and nonlinear eigenvalue problems, were provided by Palais and Terng in [163].

The line of research of this dissertation, outlined in the early paper [127] under the name of *nonlinear behavioral or dissipative balanced reduction*, and further developed in [116, 117, 125], intends to provide a more general theoretical framework for nonlinear balancing theory to preserve the dissipative structure of nonlinear systems during a balanced reduction procedure. It was inspired by the linear approach to dissipative balanced reduction by Weiland [211, 179] along with the influential paper by Willems [219]. In [127], we proposed a dissipativity approach as a guideline-framework for structure-preserving nonlinear model reduction which includes passive, $L_2$-gain properties etc. as particular cases.

This line of research results in three streams for the present dissertation:

1. The establishment of algorithms to find past and future storage functions [121, 125] (as alternative solutions to Hamilton-Jacobi-Bellman equations)
and a framework for sampled-time systems where the invariant trajectories could be determined [112].

2. The review of nonlinear continuous-time theory in order to find a geometric interpretation to the invariant trajectories of some nonlinear operator associated to the original nonlinear system. The resulting interpretation leads inevitably to a differential-geometric framework [116, 117].

3. The proposed geometric framework should show to be essentially confluent to the present day nonlinear theories for nonlinear balancing.

In particular, we propose local isometric operators as a nonlinear generalization to linear all-pass operators, which serve to sustain appropriate definitions of adjoint and self-adjoint operators in Hilbert manifolds within our geometric framework\(^\text{12}\). As developed throughout our dissipative balancing theory, our definition of self-adjoint operator for nonlinear systems, is known in curvature theory as the shape operator\(^\text{13}\). Thus, based on curvature theory, we provide a decomposition in eigenvector fields, eigencovector fields and eigenfunctions, similarly to the linear SVD procedure for linear self-adjoint operators. Therefore, the results presented in this work admit a, not empirical, system-theoretic interpretation of the nonlinear generalization of principal components analysis (PCA) for nonlinear systems.

Notice though that in our framework we do not require the properties associated to the nonlinear self-adjoint operators developed in [46], since in our framework the shape operator is defined in the tangent space of a Hilbert manifold and even for nonlinear systems, it is invariably a linear operator\(^\text{14}\).

The essential confluence of the differential-geometric approach presented in this dissertation with the present-day theories of linear and nonlinear balancing are shown by examples throughout the text. Nevertheless, it is important to stress that in this dissertation the following topics of current research in nonlinear balancing are not discussed or require further analysis:

- Singular value analysis for balanced realizations, e.g. in the sense of [45].
- The relationships between the nonlinear cross-Gramian, gradient systems and symmetric systems, e.g. in the sense of [80].
- Balancing in terms of solutions of Hamilton-Jacobi-Bellman equations.

\(^{12}\) See our definition of local isometries in Chapter 4, Def 4.12 and its relation with linear all-pass systems in Example 4.30. Our consequent definitions of adjoint and self-adjoint operators are provided in Defs. 4.32 and 4.33.

\(^{13}\) Also known as Weingarten map after [213, 214, 215]; see Def. 4.41 in Chapter 4.

\(^{14}\) For a proof of self-adjointness and linearity of the shape operator see e.g. [163].
2.4 Contributions

The following statements outlined below are claimed as contributions of this work:

- The formalization of a differential-geometric Hilbert manifold framework in order to pose the nonlinear dissipative balancing problem as a problem of *trajectory approximation in a metric space*, see [116, 117].
- A formal, differential-geometric, coordinate-independent characterization of the invariants associated to the problem of structure-preserving model reduction for nonlinear dissipative (smooth) control systems, see [116, 117].
- The characterization of the invariants of a past to future trajectory operator called *behavioral operator*, which for specific past and future storage functions (namely the controllability and observability functions) are a nonlinear generalization equivalent to the combined Hankel and Hankel adjoint operators, see [117].
- Some equivalent results for nonlinear sampled-time systems, in particular algorithmic procedures to find the past and future storage functions, needed for balancing, but also needed to determine the past-future invariant trajectories of a nonlinear past-future operator, see [125, 112].
- Using classical curvature theory, a geometric interpretation of balancing, as a decomposition of a nonlinear isometric operator into principal curvatures, with associated principal eigenvector fields and a decomposition into principal eigenfunctions, see [116, 117].
- A novel geometric characterization of the nonlinear past and future Gramians [117, 111, 126].
- A formal, nonlinear generalization of Principal Component Analysis (PCA) for nonlinear Lie-group operators, see [116, 117].
- A connection between the traditional mathematical methods of reduction based on *symmetries and first integrals* and the theoretical framework presented in this work based on *isometries*, see [116, 117].
- In the realm of lumped approximation of distributed port-Hamiltonian systems, the development of an alternative geometric discretization of Stokes-Dirac structures which can be made to preserve the port-Hamiltonian structure. Besides its simplicity, this heuristic method is one of the first methods proposed for this purpose, and even today is the only one based on the paradigm of finite differences, see [122].
2.5 Outline of this dissertation

This dissertation consists of a collection of selected papers organized by chapters with a common list of references. Chapters 4, 5 and 6 were originally published in the *Journal Mathematics of Control, Signals and Systems* and Chapters 3 and 7 are conference papers.

2.5.1 Chapter 3: Energy-storage balanced reduction of Port-Hamiltonian systems

Supported by the framework of dissipativity theory, a procedure based on physical energy to balance and reduce port-Hamiltonian systems with collocated inputs and outputs is presented. Additionally, some relations with the methods of nonlinear balanced reduction are exposed. Finally a structure-preserving reduction method based on singular perturbations is shown.

2.5.2 Chapter 4: A geometric approach to nonlinear dissipative balanced reduction I: Exogenous signals.

In a typical PCA analysis, the structural invariants are found empirically with the application of SVD to a collection of the snapshots, or samples of the behavior \( w(t) \equiv (u(t), y(t)) \) and the system trajectories \( x(t) \) for \( t \in \mathbb{R}^+ \). If such arbitrary trajectories \( w(t) \) are produced by dissipative systems, then the dissipation inequality (2.2) is satisfied along such trajectories. Instead of considering arbitrary trajectories as in the typical PCA analysis, throughout Chapters 4 and 5, we perform a two-stage hypothetical exercise\(^{15}\) of controlled excitation-relaxation on a continuous-time dissipative dynamical system along with a theoretical analysis of the resulting behavior, in order to find the invariant trajectories of such exercise. In the notation of Chapter 4, the purpose of this exercise is to verify the following principle:

Consider a concrete realization of a dynamical model \((\mathcal{C}, \phi)\) representing faithfully the behavior \( \mathcal{B} \subseteq \mathcal{W}^T \) of a dynamical system. Furthermore consider a partition of time \( \mathbb{T} \) on past and future subintervals inducing two half-spaces of past semi-trajectories in the past

\(^{15}\) This expository device is actually recurrent in the foundational work by Willems [222], see Fig. 7 and Section 4.7.5 (Minimality and Reduction, pg. 255) and Section V (Controllability, pg. 267) for Figs. 2 and Section VII (State models, pg. 269) for Figs. 3 in [223]. See also [211]. Furthermore, the Markov parameters for linear time invariant systems could also be found if in this excitation-relaxation experiment an impulse is defined as input.
behavior \( \mathcal{B}^- \) and future semi-trajectories in the future behavior \( \mathcal{B}^+ \). Define a metric for past semi trajectories in \( \mathcal{B}^- \) and a metric for future semi trajectories in \( \mathcal{B}^+ \). Then, based in such dynamical model, construct an isometric operator \( \tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+ \) such that during the excitation stage past semi-trajectories with metric defined in \( \mathcal{B}^- \) are connected at null-time with future semi-trajectories with metric defined in \( \mathcal{B}^+ \) during the relaxation stage. We are interested in finding the whole set of invariant semi-trajectories under such isometric operator. Furthermore if the whole set of invariants is used as a basis to span the subset of the space of signals \( \mathcal{W} \), as a result from this exercise we would obtain a basis to support the whole behavior \( \mathcal{B} \subset \mathcal{W}^\Sigma \).

In order to perform this exercise of excitation-relaxation, we furnish a metric space with differential geometric structures: Hilbert manifolds and Lie-semigroups. Furthermore, we define the metrics precisely based on the storage functions defined in Willems’ theory of dissipative systems, see also [211]. As a result, in this chapter we provide a unified geometric theory for nonlinear dissipative and Hankel balanced reduction. In particular, we present a theory for the invariants of the behavior, in terms of exogenous signals, using classical Gauss’ curvature theory (see Figure 2.2 illustrating the action of the behavioral operator on past to future semi-trajectories). Furthermore, known concepts of the theory of balanced reduction like the Hankel operator, Schmidt decomposition, etc., can be understood under this geometric perspective.

**2.5.3 Chapter 5: A geometric approach to nonlinear dissipative balanced reduction II: Internal isometries.**

Derived from the same hypothetical exercise of Chapter 4, the invariants of the behavior of a dynamical system are independent of the system realization, in this chapter we investigate how these invariants can be expressed in terms of the endogenous variables.

In the notation of Chapter 5, the purpose of this exercise is to verify the following principle:

Consider the same past and future metrics but now represented in endogenous (state-space) coordinates in \( \mathcal{M} \). In particular, consider in the state-space semi-trajectories satisfying the past metrics in \( \mathcal{M}^\Sigma \) during the excitation stage to reach some initial condition \( x_0 \in \mathcal{M} \) and the state-space semi-trajectories satisfying the future
metrics in $\mathcal{M}_+^t$ during the relaxation stage. Since the state-space dynamical model realization $(\mathcal{C},\phi)$ has a coordinate independent representation and the properties of reachability and observability are coordinate independent properties inherent to such dynamical system, then the invariants of an input output map constructed from such a dynamical model are also invariants of the inner structure of the concrete realization of the dynamical model, and such realization is minimal by construction. Further reduction as an approximation of the behavior can be obtained if the state space $\mathcal{M}$ is correctly partitioned in two invariant submanifolds $\mathcal{M} = \mathcal{M}_a \oplus \mathcal{M}_b$. Consider the decomposition of an internal isometric operator $\Upsilon^t : t \times M_0 \rightarrow M_0$ in principal curvatures $\kappa_i$, $i = 1, \ldots, n$ with a principal frame such that $T_p\mathcal{M}_+^t = \text{span}\{\xi_1^+, \xi_2^+, \ldots, \xi_n^+\}$, $\xi_i^+ \in T_p\mathcal{M}_+^t$, and a principal coframe such that $T_p\mathcal{M}_-^t = \text{span}\{\gamma_1^-, \gamma_2^-, \ldots, \gamma_n^+\}$, $\gamma_i^- \in T_p\mathcal{M}_-^t$. Then the submanifold $\mathcal{M}_a$, such that $T_p\mathcal{M}_a = \text{span}\{\xi_1^+, \xi_2^+, \ldots, \xi_r^+\}$, $r < n$ is associated to the higher principal curvatures $\kappa_i$, $i = 1, \ldots, r$ and the remaining hyper-surfaces lie on $\mathcal{M}_b$. Finally, after factorization and restriction (from a quotient system supported by the $r$-dimensional submanifold $\mathcal{M}_a \subset \mathcal{M}$) of the model realization $(\mathcal{C},\phi)$,
a reduced-order model with approximate behavior $\mathcal{B}_r \subset \mathcal{B}$, can be obtained.

Again, the space of endogenous (state-space) variables is furnished with a metric space in a differential-geometric framework based on dissipativity theory, Lie-semigroups and submanifold Hilbert theory. Accordingly, we define the metrics precisely based on the storage functions defined in Willems’ theory of dissipative systems, but expressed in state-space coordinates. As a result, this chapter provides a novel characterization of the nonlinear Gramians along with an alternative view of some concepts of the theory of nonlinear balanced reduction and the Hankel operator, namely eigenvalue problems and eigenfunction decomposition, etc. along with structural aspects of the internal signals and operators used in this unified geometric theory for nonlinear dissipative and Hankel balanced reduction.

2.5.4 Chapter 6: Energy functions for dissipativity-based balancing of discrete-time nonlinear systems.

In this chapter we focus mainly on the description of the behavior at discrete instants of time. In contrast to the continuous-time situation, there are few, though important, phenomena whose behavior is by nature a discrete-time sequence of signals, e.g. those associated to radar, sonar and geophysical seismic signal processing. Therefore a theory of nonlinear discrete-time behaviors is mostly justified by the need of sampling in time continuous-time dynamical models, in order to reconstruct their behavior by computer algorithms [112]. Most of the energy functions used in nonlinear balancing theory can be expressed as storage functions in the framework of dissipativity theory. By defining a framework of discrete-time dissipative systems, this chapter presents existence conditions for their discrete-time energy functions along with algorithms to find them based on dynamic optimization problems. Furthermore, the important case of the nonlinear discrete-time versions of the controllability and observability functions, its properties and algorithms to find them are presented. The algorithms are illustrated with linear and nonlinear examples.

2.5.5 Chapter 7: Lumped approximation of a transmission line with an alternative geometric discretization.

In this chapter a geometric method for finite-dimensional, structure-preserving, lumped model approximation of distributed port-Hamiltonian systems is discussed.
In the collaborative work [26], the authors present a procedure for discretization of Stokes-Dirac structures that does not preserve the port-Hamiltonian structure after discretization. The procedure presented in this chapter, provides with an example some improvements to [26], essentially inspired on the finite difference paradigm, such that the resulting lumped-parameter model preserves the collocated port-Hamiltonian structure along with some other desirable conditions for interconnection. In particular, in the example presented, an electromagnetic one-dimensional transmission line represented in a distributed port-Hamiltonian form, is lumped into a chain of subsystems which preserve the port-Hamiltonian structure with inputs and outputs in collocated form. The simulation results are compared with those presented previously in [50].

2.6 List of publications

The following works were published during the course of this research:

Journal papers:


Conference papers:

1. Lopezlena R., ”Computer implementation of a boundary feedback leak detector and estimator for pipelines I: Transient Model”, In Mems. 16°
2.6 List of publications

Congreso Latinoamericano de Control Automatico (CLCA 2014), Cancun Q.R., Mexico, Octubre 14-17, 2014.


Chapter 3

Energy-storage balanced reduction of Port-Hamiltonian systems


Abstract: Supported by the framework of dissipativity theory, a procedure based on physical energy to balance and reduce port-Hamiltonian systems with collocated inputs and outputs is presented. Additionally, some relations with the methods of nonlinear balanced reduction are exposed. Finally a structure-preserving reduction method based on singular perturbations is proposed.

Keywords: Nonlinear systems, model approximation, model reduction.

The reduction of the order of physical dynamic models has been a subject of discussion and research for a long time now in the physics and the engineering literature as well. Reduction in classical mechanics counts with a large and rather well known history mainly based on symmetries within a differential geometric framework for Hamiltonian systems [132]. These systems are an important paradigm for modeling, analysis and control. However, when a reduction procedure is considered for these systems, and the resulting model is intended for control and systems analysis, besides the mere preservation of the Hamiltonian structure, the properties of controllability and observability are known to be important for an adequate input-output behavior. Such properties have been studied in [201, 151, 205]. Roughly speaking, Hamiltonian systems have a certain balance regarding observability and controllability [200].
The main purpose of this paper is to present a procedure to reduce the dimension of the state space of port-Hamiltonian systems (PHS) according to its capacity of storing energy but preserving its structure and moreover, preserving its input-output properties. Furthermore it is argued that when the influence of the inputs, outputs and dissipation is small, the reduction procedure is, at least locally, equivalent to a reduction based on the elimination of the less energy storing elements of the Hamiltonian. Although the aforementioned characteristics for the reduced system are conceptually independent, they can be combined harmonically in one framework. The theory of dissipative systems provides a firm groundwork for this purpose, as will be seen further on.

Different approaches to reduce this class of systems have been presented previously. In [204] for linear Hamiltonian systems, a reduction procedure is outlined by the use of its associated gradient system, being only valid for conservative or weakly damped systems. This latter approach was generalized for the class of nonlinear simple Hamiltonian systems (with positive energy) in [181]. A procedure for balancing linear systems with the dissipativity theory has been described in [212].

The paper is organized as follows. In Section 3.1 the reduction of Hamiltonian systems with ports is performed with a symmetries-inspired procedure for autonomous systems with the purpose of motivating the inclusion of the effect of ports in Section 3.2 within a balanced reduction framework based on dissipativity theory. Additionally some arguments are presented in order to clarify the relation of the input-output procedure with the autonomous one. Finally in Section 3.3 a singular perturbations method which preserves the port-Hamiltonian structure is presented, to conclude with some remarks.

3.1 Reduction of Hamiltonian Systems

Consider the following input-affine port-Hamiltonian system,

\[
\dot{x}(t) = [J(x) - R(x)]\frac{\partial H(x)}{\partial x} + G(x)u(t) \\
y(t) = G^T(x)\frac{\partial H(x)}{\partial x},
\]

(3.1)

where we assume that \( x \in X \), \( J(x) = -J(x)^T \), \( R(x) = R^T(x) > 0 \) with a Hamiltonian function \( H(x) \in C^\infty \) such that \( H(0) = 0 \) and \( \frac{\partial H}{\partial x}(0) = 0 \). By the fundamental theorem of integral calculus, it is possible to express such function \( H(x) \) on a convex neighborhood of 0 as a quadratic form \( H(x) = x^TE(x)x \), \( E(x) = E^T(x) \) with functions in each entry. There exist several examples of nonlinear systems that have such quadratic Hamiltonian structure.
3.1 Reduction of Hamiltonian Systems

Reduction based on EFD of the Hamiltonian

Denote $GL(n, C^\infty(\mathcal{X}))$ the set of $n \times n$ matrices with components in $C^\infty(\mathcal{X})$ and denote $SO(n, C^\infty(\mathcal{X}))$, the special orthogonal group of unimodular transformations as

$$ SO(n, C^\infty(\mathcal{X})) := \{ g \in GL(n, C^\infty(\mathcal{X})) \mid gg^T = I, \det g = 1 \}. $$

Remark 3.1. Consider a matrix of functions $E(x) \in GL(n, C^\infty(\mathcal{X}))$, $x \in \mathcal{X}$. In the particular case when $E(x) = E^T(x) \geq 0$, it may be expressed as

$$ E(x) = U(x) \Sigma(x) U^T(x) \quad (3.2) $$

where $\Sigma(x) = \text{diag}(\tau_1(x), \ldots, \tau_r(x), 0_{r+1}, \ldots, 0_n)$ s.t. $\tau_1(x) \geq \tau_2(x) \geq \cdots \geq \tau_r(x) > 0$ are the eigenvalues of $E(x)$ and $U(x) \in SO(n, C^\infty(\mathcal{X}))$. In the rest of the paper we will refer to it as eigenvalue function decomposition (EFD), [180].

Proposition 3.2. Consider the EFD of the Hamiltonian of (3.1),

$$ H(x) = x^T U(x) \Sigma(x) U^T(x) x, \quad U(x) \in SO(n, C^\infty(\mathcal{X})) $$

in a neighborhood defined as

$$ D = \left\{ x \in \mathcal{X} \mid \text{s.t. } \frac{\partial U^T(x)}{\partial x} = 0 \right\}. $$

Using the coordinate transformation $w = U(x)x$ around $x \in D$, an approximated reduced subsystem $(J_r, R_r, G_r, H_r)$ can be found representing the most energy-storing dynamics and preserving the port-Hamiltonian structure.

Proof. Define as the new coordinates $w$ and define $T(x) = U^T(x)x$ yielding $w = U^T(x)x$ or $x = U(x)(w)$. For $x \in D$, $\frac{\partial T(x)}{\partial x} = U^T(x)$. This yields $\tilde{H}(w) = w^T \Sigma(x) w$, $\frac{\partial w}{\partial x} = \frac{\partial T(x)}{\partial x}$, $\frac{\partial x}{\partial w} = \frac{\partial T^{-1}(x)}{\partial w}$ and also $\dot{w}(t) = \frac{\partial T(x)}{\partial x} \dot{x}(t)$ and $\dot{x}(t) = \frac{\partial T^{-1}(w)}{\partial w} \dot{w}(t)$. Since $\partial H(x)/\partial x$ can be written as

$$ \frac{\partial H}{\partial x} = \left[ \frac{\partial T H}{\partial w} \frac{\partial w}{\partial x} \right]^T = \frac{\partial T T(x)}{\partial x} \frac{\partial H}{\partial w} $$

denote $M(x) = J(x) - R(x)$ then the system (3.1) may be written as the triple $(\tilde{M}(x), \tilde{G}(x), H(w))$ where
Remark 3.3. Despite the possibility of finding an EFD of $E(x)$ for any $x \in \mathcal{X}$ the reduction procedure presented in this section is valid only for $x \in D \subset \mathcal{X}$.

3.2 Dissipativity theory framework

In this section a more general framework for nonlinear balancing theory for dissipative systems is presented. Consider the following continuous-time nonlinear system

$$
\tilde{M}(x) = \frac{\partial T(x)}{\partial x}M(x)\frac{\partial T(x)}{\partial x} = T_xM(x)T_x^T
$$

$$
\tilde{G}(x) = \frac{\partial T(x)}{\partial x}G(x) = T_xG(x)
$$

After transformation $\tilde{M}$ preserves its properties since transformation of $J(x)$ results in $T_xJ(x)T_x^T = T_xJ^T(x)T_x^T = -T_xJ(x)T_x^T$ and skew-symmetry of $J(x)$ is preserved. On the other hand, after transforming $R$ yields $\tilde{R} = T_xRT_x^T$ symmetric. It is possible to rewrite the system as $(\tilde{M}(x), \tilde{G}(x), \tilde{H}(w)) = (U^T(x)M(x)U(x), U^T(x)G(x), \tilde{H}(w))$ and the Hamiltonian takes the form $\tilde{H}(w, x) = w^T \Sigma(x)w$ which for a partition of the state $w = (w_1, w_2)$ may be decomposed as $\tilde{H}(w) = \Sigma(x^1)w_1 + \Sigma(x^2)w_2$. Furthermore, the whole system can be written as

$$
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix} = \begin{bmatrix}
  \tilde{M}_1 \tilde{M}_2^T \\
  \tilde{M}_2 \tilde{M}_1^T
\end{bmatrix} \begin{bmatrix}
  \frac{\partial \tilde{H}(w)}{\partial w_1} \\
  \frac{\partial \tilde{H}(w)}{\partial w_2}
\end{bmatrix} + \begin{bmatrix}
  \tilde{G}_1(w_1, w_2) \\
  \tilde{G}_2(w_1, w_2)
\end{bmatrix} u
$$

$$
y = \begin{bmatrix}
  \tilde{G}_1(w_1, w_2) \\
  \tilde{G}_2(w_1, w_2)
\end{bmatrix} \begin{bmatrix}
  \frac{\partial \tilde{H}(w)}{\partial w_1} \\
  \frac{\partial \tilde{H}(w)}{\partial w_2}
\end{bmatrix}
$$

If the subsystem associated to the least amount of stored energy is truncated, then the reduced system $(\tilde{M}_1^r(x), \tilde{G}_1^r(x), \tilde{H}_1^r(x))$ can be inversely transformed around $x$ with $T_r^{-1} = U_r^T(x)$ (i.e. adapted to the reduced coordinates) yielding the reduced model

$$(M_1^r(x^1), G_1^r(x^1), H_1^r(x^1)) = (M_1^r(x^1), G_r(x^1), H_r(x^1))$$

for each $x$. Such model preserves the Hamiltonian structure modulo

$$
\tilde{M}_2 \frac{\partial \tilde{H}(w)}{\partial w_1} + \tilde{M}_2 \frac{\partial \tilde{H}(w)}{\partial w_2} + \tilde{G}_2(w_1, w_2) u = 0.
$$

Instead of truncation, a singular perturbation method can be used as can be seen in Section 3.3.
\[ \Sigma : \dot{x}(t) = f(x(t), u(t)), \]
\[ y(t) = h(x), \]
(3.3)

where \( x \in \mathbb{R}^n \) are local coordinates for a \( C^\infty \) state space manifold \( \mathcal{X} \), \( F \) and \( h \) are \( C^\infty \), \( u \in \mathcal{U} \) and \( y \in \mathcal{Y} \). Assume that \( F \) and \( h \) are Lipschitz continuous in \( x \) and \( u \) and additionally \( x \) and \( y \) are locally square integrable. From now on it will be assumed that this system is dissipative and that there is no source of energy within the system. From general dissipative systems theory \([219]\) it is known that associated to the system \((5.1)\) are the storage functions called required supply, \( S_r : \mathcal{X} \rightarrow \mathbb{R}^+ \), defined as

\[ S_r(x_0, r) = \inf_{u(\cdot) \in \mathcal{U}} \int_{-T}^{0} r(u(t), y(t))dt, \quad x_0 = x, T \geq 0 \]

and the available storage, \( S_a : \mathcal{X} \rightarrow \mathbb{R}^+ \), defined as

\[ S_a(x_0, r) = -\inf_{u(\cdot) \in \mathcal{U}} \int_{0}^{T} r(u(t), y(t))dt, \quad x_0 = x, T \geq 0 \]

where \( r(u(t), y(t)) \) is the supply rate. \( r : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R} \), is the supply rate.

3.2.1 The input-output storage quotient

In dissipative systems theory, a dynamical system is conceptualized as a mathematical object which maps inputs into outputs, via the state which summarizes the influence of past inputs \([219]\). This parallels the interpretation of the Hankel operator as a map from past inputs into future outputs. In previous works \([180, 121]\) this operator has been the basic tool for nonlinear balancing as it is argued in the following.

Remark 3.4. By defining as supply rate for the required supply \( r(t) = u^T(t)u(t) \) and \( r(t) = y^T(t)y(t) \) for the available storage, the controllability and (natural) observability functions \( L_c(x_0) \) and \( L_o^N(x_0) \) respectively, can be obtained for continuous \([180, 60]\) and discrete-time systems \([121]\).

Remark 3.5. If \( r_a(u, y) = r_r(u, y) = \|y\|^2 + \|u\|^2 \) is used to conform \( S_a \) and \( S_r \) then this parallels the treatment of past and future energy functions \( K^- \) and \( K^+ \) presented in \([187]\) for balancing unstable nonlinear systems.

If \( r_r(u, y) = (1 - \frac{1}{\gamma^2})\|y\|^2 + \|u\|^2 \) is used for \( S_r(r_r) \) and \( r_a(u, y) = \|y\|^2 + \)
\[(\gamma_2^2 - \gamma_1^2)\|u\|^2; \gamma > 1 \text{ for } S_a(r_a) \] then this parallels the treatment of $\mathcal{H}_\infty$-past and $\mathcal{H}_\infty$-future energy functions $Q^-_\gamma$ and $Q^+_\gamma$ presented in [182] for nonlinear $\mathcal{H}_\infty$-balancing.

By defining a ratio between the input storage (required supply) and the output storage (available storage), for two determined supply rates, it is possible to have a measure of the storage capacity. Thus by using a transformation that changes the system accordingly, a balancing procedure shall be proposed.

**Definition 3.6 (Input-output storage quotient).** For the system (5.1) being dissipative for the supply rates $r_a$ and $r_r$, assuming existence of $S_a(x_0, r_a)$ and $S_r(x_0, r_r)$ around a point $x(0) = x_0$, $(S_r(x_0, r_r) \neq 0)$, define the input-output storage quotient as

\[
|\Sigma|_S = \sup_{x(0) \in \mathcal{X}} \left[ \frac{S_a(x_0, r_a)}{S_r(x_0, r_r)} \right]^{\frac{1}{2}}.
\]  

(3.4)

Depending on $r_a$ and $r_r$, $|\Sigma|_S$ may not be an induced norm. The existence of this quotient is restricted to the existence conditions of $S_a$ and $S_r$ namely reachability and zero-state observability of system (5.1). This quotient is an extension of the nonlinear Hankel norm concept. When restricting this quotient to be comprised of quadratic forms in $S_a$ and $S_r$ around a local equilibrium point, it parallels\(^1\) the treatment given in [180, 121] to the Hankel-type norm defined for nonlinear systems. By assuming that the energy functions exist around a critical point, and if the number of distinct eigenvalues is constant everywhere in a certain neighborhood $D$ [93], the existence of nonlinear transformations that allow for the balancing of such system on a neighborhood is guaranteed, as it was stated in the original theory of nonlinear balancing [180]. As a trivial property, if $\Sigma$ is dissipative with $r_a = r_r$, since $0 \leq S_a \leq S \leq S_r$ then $|\Sigma|_S \leq 1$, see [219].

### 3.2.2 Collocated port-Hamiltonian systems

Within the class of dissipative systems is the subclass of nonlinear conservative systems known as port-Hamiltonian systems (PHS) [205], represented as in eq. (3.1). In this section the class of PHS is restricted to have **collocated** inputs and outputs such that it is always possible to form **input-output** pairs of (power transferring) signals at or from the ports, as it is elementary shown in Figure 3.1.

\(^1\) As known in optimal control, some additional restrictions are required in order to admit an infinite horizon.
3.2 Dissipativity theory framework

The collocated representation will be explicitly written as

\[
\dot{x} = M(x) \frac{\partial H}{\partial x} + \begin{bmatrix} G^1(x) & G^2(x) \end{bmatrix} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}
\]

(3.5)

with \( M(x) = J(x) - R(x) \) and where, as usual, the system is dissipative for a supply rate \( r = y^T u \). Define as energy functions \( S^*_r(x_0, r) = S_r(x_0, y_1^T u_1) \) and \( S^*_a(x_0, r) = S_a(x_0, -y_2^T u_2) \) which can be recognized as the physical energy supplied to the system, \( \langle u_1, y_1 \rangle_{L_2} \), by its input two-port, and the physical energy deliverable by the system, \( \langle -u_2, y_2 \rangle_{L_2} \), through its output two-port.

Thus its associated input-output energy storage quotient can be expressed as

\[
|\Sigma| = \sup_{x(0) \in X} \left[ S^*_a(x_0) \right]^\frac{1}{2} = \sup_{x(0) \in X} \left[ \frac{\langle -u_2, y_2 \rangle_{L_2}}{\langle u_1, y_1 \rangle_{L_2}} \right]^\frac{1}{2}.
\]

**Proposition 3.7.** Given the (collocated) PHS (3.5), for vectors partitioned as \( u = (u_1^T, -u_2^T)^T \) and \( y = (y_1^T, y_2^T)^T \), where the input energy is associated to \( \langle y_1, u_1 \rangle_{L_2} \) and the output energy to \( \langle y_2, -u_2 \rangle_{L_2} \), then \( S_r(x) \) and \( S_a(x) \) can be written as

\[
S_r(x) = H(x) + \int_{-T}^{0} \frac{\partial T}{\partial x} R(x) \frac{\partial H}{\partial x} dt + \langle u_2, y_2 \rangle_{L_2}
\]

\[
S_a(x) = H(x) - \int_{0}^{T} \frac{\partial T}{\partial x} R(x) \frac{\partial H}{\partial x} dt + \langle u_1, y_1 \rangle_{L_2}
\]

and \( H(x) \) is the Hamiltonian function of the system. Furthermore, \( S_r(x) \) with \( u_2 = 0 \), and \( S_a(x) \) with \( y_1 = 0 \), can be found as the solution of the following Hamilton-Jacobi-Bellman (HJB) equations

\[
\nabla_x^T S_r \left[ M(x) - G^1(x)G^1(x) \right] \nabla_x H + \nabla_x^T S_r G^1(x)G^1(x)\nabla_x S_r = 0, \quad (3.6)
\]

\[
\nabla_x^T S_a \left[ M(x) - G^2(x)G^2(x) \right] \nabla_x H + \nabla_x^T H G^2(x)G^2(x)\nabla_x H = 0. \quad (3.7)
\]
Proof. The system (3.5) can be decomposed as two separate systems \((M, G^1, H)\) and \((M, G^2, H)\). In such conditions

\[
\begin{align*}
y^T_1 u_1 &= \frac{\partial T H}{\partial x} G^1 u_1 = \frac{\partial T H}{\partial x} \left[ \dot{x} - M(x) \frac{\partial H}{\partial x} + G^2 u_2 \right] \\
&= \frac{dH}{dt} + \frac{\partial T H}{\partial x} R(x) \frac{\partial H}{\partial x} + y^T_2 u_2, \\
y^T_2 u_2 &= \frac{\partial T H}{\partial x} G^2 u_2 = -\frac{dH}{dt} - \frac{\partial T H}{\partial x} R(x) \frac{\partial H}{\partial x} + y^T_1 u_1.
\end{align*}
\]

After integration, the result follows. Regarding Eqs. (3.6)-(3.7), since

\[
\nabla^T_x S_r \left( [J(x) - R(x)] \nabla_x H + G^1(x) u_1 \right) = y^T_1 u_1,
\]

\[
\nabla^T_x S_a \left( [J(x) - R(x)] \nabla_x H + G^2(x) u_2 \right) = y^T_2 u_2,
\]

for the following inputs \(u_1 = G^1 T(x) \nabla_x S_r(x)\) and \(u_2 = G^2 T(x) \nabla_x H(x)\), results in Eqs. (3.6)-(3.7) respectively. Nevertheless in can be shown that \(u_i\) is not unique.

In the remaining of this section, it will be assumed that \(S_r(x)\) is determined for \(u_2 = 0\), and \(S_a(x)\) for \(y_1 = 0\).

### 3.2.3 Balanced reduction as a more general paradigm

For our purposes we would like to clarify in this framework to what extent the reduction procedure presented in Section 3.1 may offer similar results to the procedure presented in Section 3.2 and in which sense the latter can be seen as a generalization of the former.

**Proposition 3.8.** Locally, for conservative, strongly accessible port-Hamiltonian systems, the reduction based on EFD of the Hamiltonian is equivalent to input-output energy-storage balancing.

**Proof.** \(S_r(r_r)\) and \(S_a(r_a)\) can be found from the solution of the HJE eqs. (3.6) and (3.7) respectively. A system is said to be **internally balanced** when \(S_r(r_r) = S_a(r_a)\). Consider the case \(R = 0\) (conservative) then \(S_r = S_a = H\), and apply this to eq. (3.6) and (3.7) resulting in the same equation for both cases and which simplifies to the known identity \(\nabla^T_x H J \nabla_x H = 0\).
3.2.4 Balanced truncation of PHS

Balancing and the related model reduction method, which is called balanced truncation, for nonlinear systems was introduced in [180]. We can also directly use the techniques used in the recent results on balanced truncation [42], where it was proven under certain assumptions that for any two positive definite scalar functions $L_c(x)$ and $L_o(x)$ there exists a coordinate transformation $z = \Phi(x)$ such that

$$z_i = 0 \iff \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \iff \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0$$

(3.8)

holds. Using this fact, we can prove the following model reduction procedure which also preserves the structure of PHS in some cases.

**Proposition 3.9.** Suppose that the dissipated energy in $S_r$ and $S_a$ are equal, that is, $S_r(x) + S_a(x) = 2H(x)$ holds. Then the input-output energy storage balanced truncation in the balanced coordinates in the sense of (3.8) with respect to the two storage functions $S_r(x)$ and $S_a(x)$ preserves the structure of PHS.

**Proof.** Since the structure of PHS is invariant under any coordinate transformation, the dynamics of the PHS in the balanced coordinate $z = (z_1, z_2) = \Phi^{-1}(x)$ can be represented by a PHS

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} J_{11} - R_{11} & J_{12} - R_{12} \\ J_{21} - R_{21} & J_{22} - R_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \frac{\partial H}{\partial z_2} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} u.$$  

(3.9)

Let us perform the balanced truncation by neglecting the $z_2$ dynamics and substituting $z_2 = 0$. Then, due to the property (3.8), we obtain

$$\dot{z}_1 = \left[ J_{11} - R_{11} \right] \frac{\partial H}{\partial z_1} + g_1 u$$

$$= (J_{11} - R_{11}) \frac{\partial H}{\partial z_1} + g_1 u$$

which is a PHS indeed. This completes the proof.

For special class of PHS systems, the balanced truncation technique can preserve its structure and the related properties. However, the structure is not always preserved since the assumption required in this proposition does not hold in general.
3.3 Singular Perturbations in PHS

The last step in the reduction procedures previously presented consists on the elimination of the remaining dynamics. This can be performed by truncation or by singular perturbations as follows. The following proposition, adapted from [202], provides a more general result than in the previous sections.

**Proposition 3.10.** [202] Let the PHS

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = M(x) \begin{bmatrix}
\frac{\partial H}{\partial x_1} \\
\frac{\partial H}{\partial x_2}
\end{bmatrix} + \begin{bmatrix}
g_1(x_1, x_2) \\
g_2(x_1, x_2)
\end{bmatrix} u
\] (3.10)

\[
y = \begin{bmatrix}
g_1(x_1, x_2) \\
g_2(x_1, x_2)
\end{bmatrix} \begin{bmatrix}
\frac{\partial H}{\partial x_1} \\
\frac{\partial H}{\partial x_2}
\end{bmatrix}
\] (3.11)

with a Hamiltonian given by \(H(x_1, x_2)\), where

\[
M(x) = \begin{bmatrix}
J_{11}(x) - R_{11}(x) & J_{12}(x) \\
J_{21}(x) & J_{22}(x) - R_{22}(x)
\end{bmatrix}
\]

and assume that \(\partial^2 H/\partial x_2^2\) has full rank and \(\det(J_{21}(x) - R_{22}(x)) \neq 0\). If the stored energy associated to states \(x_2\) is neglectable, then the state trajectories of the system lie in the submanifold defined as

\[
N = \left\{ (x_1, x_2, u) \mid J_{21}(x) \frac{\partial H}{\partial x_1}(x) + (J_{22}(x) - R_{22}(x)) \frac{\partial H}{\partial x_2}(x) + g_b(x)u = 0 \right\},
\]

and the dynamics of the system can be represented in a reduced form as follows

\[
\dot{x}_1(t) = M_r(x_1) \frac{\partial H^*}{\partial x_1}(x_1) + G_r(x_1)u,
\]

\[
y(t) = G^T_r(x_1) \frac{\partial H^*}{\partial x_1}(x_1).
\] (3.12)

with \(M_r(x)\) and \(G_r(x)\) given by

\[
M_r(x) = J_{11}(x) - R_{11}(x) - J_{12}(x)(J_{22}(x) - R_{22}(x))^{-1}J_{21}(x),
\]

\[
G_r(x) = g_1(x) - J_{12}(x)(J_{22}(x) - R_{22}(x))^{-1}g_2(x).
\]

**Proof.** Since by assumption \(\partial^2 H/\partial x_2^2\) has full rank, define the partial Legendre transform of \(H\) as follows

\[
H^*(x_1, z_2) = H(x_1, x_2) - z^T_2 x_2,
\]
with \( z_2 = \partial H(x_2, x_2)/\partial x_2 \). Immediately two relations result from this

\[
x_2 = -\frac{\partial H^*}{\partial z_2}(x); \quad \frac{\partial H^*}{\partial x_1}(x) = \frac{\partial H}{\partial x_1}(x).
\]

In terms of this Legendre transform the submanifold \( N \) can be re-expressed as

\[
N = \{(x_1, z_2, u) \mid J_{21}(x) \frac{\partial H^*}{\partial x_1}(x) + (J_{22}(x) - R_{22}(x))z_2 + g_2(x)u = 0\},
\]

where assuming that \( \det(J_{21}(x) - R_{22}(x)) \neq 0 \), \( z_2 \) is given by

\[
-(J_{22}(x) - R_{22}(x))^{-1} \left[ J_{21}(x) \frac{\partial H^*}{\partial x_1}(x) + g_2(x)u \right].
\]

In terms of this Legendre transform and (3.13), the PHS (3.11) can be re-expressed as follows

\[
\left[
\begin{array}{c}
\dot{x}_1 \\
-\frac{4}{dt} \frac{\partial H^*}{\partial z_2}(x)
\end{array}
\right] = M(x) \left[
\begin{array}{c}
\frac{\partial H^*}{\partial x_1} \\
\frac{\partial H^*}{\partial z_2}
\end{array}
\right] + \left[
\begin{array}{c}
g_1(x_1, x_2) \\
g_2(x_1, x_2)
\end{array}
\right] u
\]

\[
y = \left[
\begin{array}{c}
g_1(x_1, x_2) \\
g_2(x_1, x_2)
\end{array}
\right] \left[
\begin{array}{c}
\frac{\partial H^*}{\partial x_1} \\
\frac{\partial H^*}{\partial z_2}
\end{array}
\right]
\]

Thus the reduced dynamics for this system can be expressed as in eq. (3.12).

**Remark 3.11.** Proposition 3.10 can be interpreted in terms of separation of fast and slow dynamics in the Hamiltonian where the submanifold \( N \) plays the role of the state space of the slow dynamics.

**Example**

The series DC (universal) motor has the following PHS description

\[
\left[
\begin{array}{c}
\dot{h} \\
\dot{\phi}
\end{array}
\right] = \left[
\begin{array}{cc}
-B & \kappa T
\end{array}
\right] \left[
\begin{array}{c}
\frac{\partial H}{\partial h} \\
\frac{\partial H}{\partial \phi}
\end{array}
\right] + \left[
\begin{array}{c}
1 \\
0
\end{array}
\right] \left[
\begin{array}{c}
-\tau
\end{array}
\right]
\]

\[
\left[
\begin{array}{c}
\omega \\
I
\end{array}
\right] = \left[
\begin{array}{c}
1 \\
0
\end{array}
\right] \left[
\begin{array}{c}
\frac{\partial H}{\partial h} \\
\frac{\partial H}{\partial \phi}
\end{array}
\right]
\]

with total stored energy of the system given by \( H = \frac{1}{2J} \dot{h}^2 + \frac{1}{2L} \dot{\phi}^2 \) where \( I = \dot{\cdot}, q \omega = \dot{\theta}, L = L_a + L_f, \kappa = K, K_f, \phi = Lq \) and \( h = J\dot{\theta} \). This system is dissipative.
for \( r = y^T u = V_t I - \tau \omega \). Let \( r = V_t I \) (input power) for the required supply and \( r = -\tau \omega \) (output power) for the available storage. Thus one may state that \( S_r \) and \( S_a \) are given by

\[
S_r = \frac{1}{2L} \phi_0^2 + \frac{1}{2J} h_0^2 + \int_{-T}^{0} \left( R t^2 + B \omega^2 \right) dt + \langle \tau, \omega \rangle,
\]
\[
S_a = \frac{1}{2L} \phi_0^2 + \frac{1}{2J} h_0^2 - \int_{0}^{T} \left( R t^2 + B \omega^2 \right) dt + \langle V_t, I \rangle.
\]

With the purpose of using the method presented in Section 3.3, define \((x_1, x_2) = (h, \phi)\) and then by taking \( z_2 = \phi/L \), the following Legendre transform is obtained \( H^* = H - z_2 \phi = \frac{h^2}{2J} - \frac{\phi^2}{2L} \) since \( J_{22} - R_{22} = -R \) is invertible then the reduced system can be expressed as

\[
\dot{h} = -R_d \frac{\partial H^*}{\partial h} + u,
\]
\[
\omega = \frac{\partial H^*}{\partial h},
\]

with a dissipative term \( R_d = JB + \frac{J}{RL} \left( \frac{\kappa \phi}{L} \right)^2 \) a new defined input \( u = \frac{\kappa \phi}{RL} V_t \) and \( \frac{\partial H^*}{\partial h} = h/J = \omega \), and such equation remains valid as long as the system remains around a submanifold defined as

\[
N = \left\{ (h, I, V_t) \, | \, V_t = \frac{\phi}{L} \left( \frac{h}{J} + R \right) \right\}.
\]

### 3.4 Conclusions

In this paper a procedure of balancing collocated port-Hamiltonian systems is presented by submersing the nonlinear balancing procedures in the framework of dissipativity theory. This approach shows to be advantageous by exposing certain regularities with input-output balanced and autonomous reduction. Finally both procedures were enhanced with a method of singular perturbations-type which preserves the port-Hamiltonian structure.

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Chapter 4

A geometric approach to nonlinear dissipative balanced reduction I: Exogenous signals


Abstract: This paper is divided in two parts and provides a unified geometric theory for nonlinear dissipative and Hankel balanced reduction with the help of a framework based on differential geometry, dissipativity theory, Lie-semigroups and sub manifold Hilbert theory. Part I presents a theory for the invariants of the behavior using classical Gauss’ curvature theory. Furthermore, known concepts of the theory of balanced reduction like the Hankel operator, Schmidt decomposition, etc., can be understood in a proper and general perspective. Keywords: Nonlinear systems, Dissipative systems, Geometric approach, Invariants, Eigenvalue problems, System order reduction, Hankel operator, model approximation, Model Reduction.

4.1 Introduction

Dynamic models have become priceless assets, especially in the Oil & Gas Exploration and Production industry, due to their capacity of characterizing in a compact format the complete behavior of the systems they model and due to their predictive capacity in describing their future behavior.
The unprecedented increase of the dimensional size and complexity of the dynamical models used today in industry, has boosted the need of the user for improved methods for model reduction in order to simplify their analysis and to profit from their predictions.

This paper deals with a nonlinear interpretation to the balanced reduction problem based on the theory of dissipative systems and the so-called behavioral approach. The behavior, i.e. the set of temporal trajectories defined by the evolution of the external or manifest variables, is the central topic of the system-theoretical framework called behavioral approach (see [223, 211, 171] and references therein). Instead of prioritizing the interaction of inputs and outputs on dynamical systems as the starting point for the analysis of dynamic control models, it prioritizes the analysis of the behavior and places at a second stage the dynamical system as generator of such behavioral trajectories for systems analysis, interconnection and decomposition.

Nonlinear generalizations of the concept of balancing have been developed over the years, see e.g. [44, 61]. Regarding balancing based on the theory of dissipative nonlinear systems, some local concepts were introduced in [127]. The unifying approach presented here is preeminent after our reinterpretation of the balanced reduction problem for dissipative systems into a problem of characterization of the invariants of an isometric operator, resulting in a systematic and general reduction procedure of broader theoretical impact, confluent with the classical methods based on symmetries.

Model approximation using the behavioral approach is not new for linear systems. The theory for linear balanced reduction and linear model approximation based on the behavioral approach presented in e.g. [211, 174] is a satisfactory antecedent and can be considered an alternative enhancement to the linear balancing theory early pioneered by Moore [143].

Structure and behavior are complementary concepts in systems theory: while the former defines the inner constitution of a system, the latter defines its outer manifestation. The behavior is the first available notion of the system dynamics. Although it may consist of a set of external variables, it may include also a set of auxiliary or internal variables defined along the modeling process, resulting then in a dynamical system with hybrid variables (external and internal).

Structure is a necessary stage for a modeling process. The synthesis of a dynamic model within a defined class of models such that the external trajectories are reproduced, is a problem of behavioral reconstruction. If such class of models only may replicate approximately such behavior, it is a problem of behavioral approximation. Since the equivalence of any two models can be de-
fined by having the same behavior, a problem of *model approximation* consist in the synthesis of a dynamic model whose behavior is similar to the behavior generated by another model.

In all these cases, a notion of closeness of two behaviors must be defined in a measurable manner. Therefore the behavioral approach to balanced reduction is seen in this paper as a problem of *trajectory approximation in a metric space* and the aim of the research presented in this dissertation is to provide a sufficiently general mathematical framework for the systematic reconstruction of such behavioral trajectories within a wide set of classes of dynamical models (linear, affine nonlinear, etc.). We use the metric space to determine error bounds on the model approximation.

The framework for behavioral trajectory approximation considered in Part I of this work pursues the characterization of the invariant properties of behavioral trajectories preserved during temporal evolution of a properly defined behavioral operator. The concept of a *behavioral operator*, understood as a map processing past external variables into future external variables, along with its applicability for model reduction, can be traced back to early works of Willems [219], pp. 323, 331, although Willems never defined it in the way we do it here. Such operator may play a similar role that the Hankel operator has played for linear balancing theory and provides a useful generalization, to the linear Hankel operator, [127].

In the geometric framework introduced in this paper we furnish the past behavior and the future behavior with Riemannian metric spaces. Using such metrics, we may provide a notion of distance from each trajectory generated by the dynamic model to the behavioral trajectory. Furthermore, the fundamental concept of *isometries* in this work plays the same role that *symmetries* have played in classical reduction methods. The use of a differential-geometric approach assists in disregarding coordinate-dependent difficulties and to concentrate on the invariant properties associated with the problem. The overall contribution of this paper is to provide a differential-geometric framework for an alternative viewpoint to dimensional reduction of nonlinear dynamic models inspired by the concept of balancing in the behavioral approach due to Weiland in [211]. Therefore the approach presented here is called *nonlinear behavioral or dissipative balanced reduction* as in [127].

The problem of nonlinear balancing can be viewed in two manners: by characterizing the space that supports the trajectories of external signals over time or by characterizing the dynamical systems that reconstruct such trajectories. Part I concentrates on exogenous signals and is organized as follows.
In Sect. 4.2 elemental concepts of behavioral systems supported on finite-dimensional Hilbert manifolds is introduced. In Sect. 4.3 behavioral balancing is approached using the framework of submanifold Hilbert theory, in terms of external signals and metricized by storage functions. This framework is used to show that the singular values in linear balancing are adequately generalized by the principal curvatures of the Riemannian manifold supporting the system trajectories, and moreover, the balanced reduction problem is in fact a mean curvature approximation problem. In Sect. 4.4 issues related to balanced reduction and the properties of minimality of a balanced system are discussed along with some bounds of the error of approximation.

4.2 The geometric framework for exogenous signals

The notation used follows standard references e.g. [151, 33, 82]. Other known and simple concepts in semigroup theory like semi-interval, semi-trajectories, half-spaces, etc. are introduced intuitively along the text. Formal definitions can be consulted in standard references, e.g. [25, 76, 74, 155]. The notation for the behavioral approach has been mainly adopted from [211, 212, 171].

4.2.1 The behavioral approach and the behavioral operator

One essential feature of the behavioral approach for dynamical systems lies in deemphasizing the role of the input and output variables as a starting point for the description of dynamic phenomena, centering its attention to the behavior of the system [211]. In this approach a system is conceived as an exclusion law which discards any outcome outside the subset of time-trajectories that define the behavior of the dynamical system [219, 171]. The environment influences such behavior by a set of external or manifest variables that define such interaction with the system [211].

**Definition 4.1 (Behavioral system).** A dynamical system $\Sigma$ is a triad $(\mathbb{T}, \mathcal{W}, \mathfrak{B})$ where $\mathbb{T} \subseteq \mathbb{R}^1$ is the set of time, $\mathcal{W}$ is the set of external signals and the set of time-trajectories $\mathfrak{B} \subset \mathcal{W}^\mathbb{T}$ is called the behavior of the system [223, 211]. The set $\mathcal{W}^\mathbb{T} \overset{\text{def}}{=} \mathbb{U}$ is called universum.

The properties of $\mathcal{W}$ are defined later in Remark 4.2. Dynamical systems in control theory evolve in only one direction: strictly increasing for a forward-time evolution or strictly decreasing for a backward-time evolution. Denote the conventional forward-time evolution by the interval $\mathfrak{t} = \{t \mid t \in \mathbb{T}\}$ and a backward-time evolution by $\tau = \{\tau \mid \tau = -t, t \in \mathbb{T}\}$. Moreover, consider two
half-spaces $W^t \supset \mathcal{B}^+$ and $W^* \tau \supset \mathcal{B}^-$, where $W$ and $W^*$ are dual spaces joined at $t = 0$ by duality at their boundary or edge $W_0$ and $W_0^*$ respectively. The behavioral trajectories can be generated by an associated nonlinear dynamical system expressed by an external differential representation (see e.g. [151], pg. 125) in terms of the set of $\omega$ higher-order differential equations (denoted by $R_i$, $i = 1, 2, \ldots, \omega$)

$$
\Sigma^+ : R_i\left(w_i, \frac{dw_i}{dt}, \frac{d^2w_i}{dt^2}, \ldots, \frac{d^\omega w_i}{dt^\omega}\right) = 0, \quad \begin{cases} 
  i = 1, 2, \ldots, \omega, \\
  w(t) \in \mathcal{B}^+, \\
  t \geq 0,
\end{cases}
$$

where $w(t) \in C^\infty(W)$ are local coordinates for a $C^\infty$ manifold of external variables $W \approx \mathbb{R}^\omega$, $p + q = \omega$ which includes as subsets the set of inputs $u \in U \subset \mathbb{R}^p$ and the set of outputs $y \in Y \subset \mathbb{R}^q$.

Furthermore, since for a backward-time evolution $\tau = -t$ and $\frac{d^i}{d\tau^i}w = (-1)^i \frac{d^i}{dt^i}w$, we are thus obliged to consider additionally the following set of higher-order differential equations (denoted by $\hat{R}_i$, $i = 1, 2, \ldots, \omega$)

$$
\Sigma^- : \hat{R}_i\left(\hat{w}_i, -\frac{\hat{w}_i}{dt}, \frac{d^2\hat{w}_i}{dt^2}, \ldots, (-1)^\omega \frac{d^\omega \hat{w}_i}{dt^\omega}\right) = 0, \quad \begin{cases} 
  i = 1, 2, \ldots, \omega, \\
  \hat{w}(\tau) \in \mathcal{B}^-, \\
  \tau \leq 0,
\end{cases}
$$

we assume $\hat{w}(\tau) \in C^\infty(W^*)$ on a dual manifold of external variables $W^* \approx \mathbb{R}^\omega$ which includes $\hat{u} \in U^*$ and $\hat{y} \in Y^*$ as subsets.

**Remark 4.2.** The solutions of Eqs. (4.1)-(4.2) (assuming they exist) consist of $\omega$-differentiable functions $w_i(t)$ naturally supported by a finite-prolongation (extended) space $\mathfrak{M}$ containing all the derivatives up to the maximal order in each $w_i(t)$. For the discussion of this paper, we do not deal with such $\mathfrak{M}$ and we assume that such functions $w(t)$ can be properly adapted (approximated) to an unextended manifold $C^\infty(W)$. Furthermore, we assume that such solutions exist and are equivalent to the solutions resulting from the proper state-space realization discussed in Part II of this paper.

**Remark 4.3.** Throughout this work it is assumed that the initial condition $w(0) \overset{\text{def}}{=} (w_1, \ldots, w_\omega)_0 \in C^\infty(W)$ of Eq. (4.1) such that $w_i(0) \overset{\text{def}}{=} (w_i, \dot{w}_i, \ddot{w}_i, \ldots, w_i^{(\omega-1)})_0$ is equivalent to the initial condition $x(0) \in C^\infty(M)$ for a proper state-space realization discussed in Part II of this work, see [29] for the necessary and sufficient conditions for existence of a smooth generalized coordinate transformation to such state-space realizations. An equivalent assumption holds for $\hat{w}(0) \in C^\infty(W^*)$ and $\hat{x}(0) \in C^\infty(M^*)$. 

Throughout the paper we consider the evolution of $\Sigma^+$ on one space $W$ and the evolution of $\Sigma^-$ on the dual space $W^*$. Since both differential equations (4.1) and (4.2) must have a common initial condition uniquely defined at $t = 0$ as $w(0) \in W$ (at the edge), the trajectory $w(t)$ is unique for a unique set of external variables $W$, as defined by duality relationships between both spaces characterizing $w(t) \in B$ uniquely.

Example 4.4. Let us consider the class of linear time-invariant behavioral systems $R \left( \frac{d}{dt} \right) w(t) = 0$, $w(t) \in B^+$, $t \geq 0$ where $R \in \mathbb{R}^{\omega \times q}[\xi]$ is a polynomial matrix, see [171]. Their latent variables evolve on the space $W = \mathbb{R}^{\omega}$, along with a backward-time system (for $\tau \text{ def} = -t$), $R \left( \frac{d}{d\tau} \right) \hat{w}(\tau) = 0$, $\hat{w}(\tau) \in B^-$, $\tau \leq 0$ evolving on the dual space $W^* = \mathbb{R}^{\omega}$. □

By extending the notation of the linear behavioral approach [171], Eqs. (4.1)-(4.2) are regarded hereafter as the forward (resp. backward) behavioral equations.

Example 4.5. The universal motor from [125] is a nonlinear system with forward-time behavioral equations

$$L \frac{dI}{dt} + RI + \zeta \omega I - V_I = 0, \quad J \frac{d\omega}{dt} + B\omega - \zeta I^2 - \tau_L = 0,$$

with constant parameters $L, R, \zeta, J, B \in \mathbb{R}^+$ and local coordinates $(V_I, \tau_L, I, \omega)$ of the space of exogenous variables $W$, where $(V_I, \tau_L)$ are inputs and $(I, \omega)$ are outputs. The backward-time behavioral equations are given by

$$L \frac{d\hat{I}}{d\tau} - R\hat{I} - \zeta \hat{\omega} \hat{I} + \hat{V}_I = 0, \quad J \frac{d\hat{\omega}}{d\tau} - B\hat{\omega} + \zeta \hat{I}^2 - \tau_L = 0.$$ □

Definition 4.6 (Behavior). Regarding system (4.1)-(4.2), the set of points on $W$ defines the trajectories $w: t \rightarrow W$ and the set of these trajectories is called the behavior $\mathcal{B}_\Sigma$.

Remark 4.7. The space of external signals is supported on $W \supset U \oplus Y$ which admits a dual space on $W^* \supset U^* \oplus Y^*$. Although the behavior is uniquely defined by $\mathcal{B} \subset W^T$, the time partition on past and future subintervals induces two half-spaces of (past and future) semi-trajectories. Thus the future behavior is supported on $\mathcal{B}^+ \subset W^T \text{ def} = U^+$, and there is a dual past behavior on $\mathcal{B}^- \subset W^* \tau \text{ def} = U^-$. Dualization of $u(t) = \hat{u}(\tau)$ and $y(t) = \hat{y}(\tau), t = -\tau$, completes the missing semi-trajectories. This past-future duality induces the space of past signals on $W^- \supset U^* \oplus Y^*$ with a dual space of future signals on $W^+ \supset U \oplus Y$ supporting such semi-trajectories.

Denote by $\mathcal{L}_\omega(a, b)$ the equivalence class of Lebesgue-measurable, (square) integrable functions mapping the interval $(a, b)$ into $\mathbb{R}^{\omega}$. In systems theory, input-output maps $F: U \times \mathbb{R}^1 \rightarrow Y$ are such that (for each initial condition)
there is a map \( y(t) = F(u(t); 0 \leq t \leq T), t \geq 0 \) for separated spaces \( u \in \mathcal{U} \) and \( y \in \mathcal{Y} \). Given a point \( (u(0), y(0)) \overset{\text{def}}{=} w(0) \in \mathcal{B}_0 \), the following operator maps simultaneously past trajectories in \( \mathcal{B}^- \) into future trajectories in \( \mathcal{B}^+ \):

**Definition 4.8 (Behavioral operator).** Given a behavioral system \( \Sigma \) with a point \( (u(0), y(0)) \overset{\text{def}}{=} w(0) \in \mathcal{B}_0 \), and past and future subsets of the behavior \( \mathcal{B}^- \) and \( \mathcal{B}^+ \) defined in Remark 4.7, the behavioral operator is the map \( \tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+ \) defined by

\[
\begin{bmatrix}
  u^+(t)
  \\
  y^+(t)
\end{bmatrix}_{\mathcal{B}^+} \overset{\text{def}}{=} \begin{bmatrix}
  I^\dagger \circ y^-(t) \\
  \Gamma \circ u^-(t)
\end{bmatrix}, \quad t \in \mathbf{t},
\]

(4.3)

The procedure of building the behavioral operator takes us to consider some \( \mathcal{C} \)-class of model structure (formally defined in Definition 4.59) with equivalent behavior to the external representations in Eqs. (4.1)-(4.2).

**Example 4.9.** For a linear time-invariant realization \( (A, B, C, D) \), the behavioral operator in Definition 4.8 is given by

\[
\begin{bmatrix}
  u^+(\tau)
  \\
  y^+(t)
\end{bmatrix}_{\mathcal{B}^+} = \begin{bmatrix}
  \int_0^T G^\dagger(\tau, t) y^-(\tau) \, d\tau \\
  \int_0^T G(t, \tau) u^-(t) \, dt
\end{bmatrix}, \quad \tau \in \mathbf{\tau}, \quad t \in \mathbf{t}
\]

where \( G(t, \tau) = C\Phi(t, \tau)B + D\delta(t-\tau), t \geq \tau \) and \( G^\dagger(\tau, t) = -B^\dagger\Theta(\tau, t)C^\dagger(\tau) + D^\dagger\delta(\tau-t), \tau \geq t \), with state and co-state transition matrices \( \Phi(t, \tau) \) and \( \Theta(\tau, t) \) respectively, see [62], pg.86.

In the remainder of the section some concepts of submanifold Hilbert theory is provided in order to formalize the ambient space of external signals as past and future behavior.

### 4.2.2 Structures on Hilbert manifolds

The framework of Hilbert manifolds is formalized in this subsection. By a **Hilbert manifold structure** we refer to a pair of metric spaces with the Hilbert space structure adapted to smooth Riemannian manifolds and dualized by an appropriate relationship, formalized below. Two Hilbert manifold structures will be used in this Part I to characterize the behavior.

There is a natural Hilbert manifold structure for the coordinates of behavioral trajectories. Let us support such coordinates on a compact, differentiable manifold \( (\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}}) \). Associated to each trajectory are functionals which are supported by another manifold \( (\mathcal{W}^*, \langle \cdot, \cdot \rangle_{T^*\mathcal{W}}) \). These manifolds can be naturally dualized (in differential-geometric terms) by the metrics.
Let us partition the support of each behavioral trajectory by two half-spaces, namely the past half-space $\mathcal{B}^-$ and future half-space $\mathcal{B}^+$ such that each initial condition on $\mathcal{B}_0$ defines a common point of each past and future semi-trajectory, providing with this the duality in time evolution of both half-spaces. The behavioral past semi-trajectories influence the future semi-trajectories in a forward-time evolution and vice versa for a backward-time evolution.

Throughout we assume each manifold $\mathcal{W}$ compact. Consider a smooth function $F(w(t))$ parameterized by a trajectory $w(t) \in \mathcal{W}$. The support of $F$, is defined by $\text{supp}\{F\} = \text{cl}\{w \in \mathcal{W} \mid F(w) \neq 0\}$. While $w(t)$ is on $\mathcal{W}$, linear functionals (0-forms) $F$ are on a dual space $\mathcal{W}^*$ with 1-forms $\alpha = dF(w) \in T^*_F(w)\mathcal{W}$. With the tangent space $T_w\mathcal{W}$ and cotangent spaces $T^*_F(w)\mathcal{W}$ defined as usual, the contraction of $\xi = \xi^i(w) \frac{\partial}{\partial w^i}, \xi \in T_w\mathcal{W}$ and $\alpha = \alpha_i(w) dw^i, \alpha \in T^*_F(w)\mathcal{W}$ expressed as $i_\xi \alpha = \alpha_i \xi^i = \sum_i \frac{dw^i}{dt} \frac{\partial F}{\partial w^i} = \frac{dF}{dt}$ is well defined and yields scalar functions $f_i$ whose value at each point does not depend on the coordinate system used.

Assume that the space of functions $F(w(t))$ is on $\mathcal{L}^2_2(a,b)$:

**Proposition 4.10.** The space of functions $F(w(t)) \subset \mathcal{L}^2_2(a,b)$ with the inner product defined on the interval $(a,b)$ by (the Lebesgue $\mu$-integral)

$$\langle f_i(w(t)), f_j(w(t)) \rangle = \int_a^b f_i(w(t)) f_j(w(t)) \, d\mu(t) = h_{i,j}^2, \quad \begin{cases} i, j \in \mathbb{Z}^+, \\ h_{i,j} \in \mathbb{R}^1 \end{cases}$$

denoted $(F, \langle \cdot, \cdot \rangle)$ is a complete, separable, Hilbert space at each $p \in \mathcal{W}$. Furthermore, any two functions $f_i(w(t)), f_j(w(t)), i \neq j$ of an ensemble of functions $\{f_i, i = 1, \ldots, F \in \mathcal{L}^2_2(a,b)\}$ are orthogonal with respect to the inner product (4.4) iff $h_{i,j}^2 = 0$. If $i = j$, the $L_2$-norm of $f_i$ equals to $|h_{i,i}|$.

(For a proof of Proposition 4.10, see e.g. [208, 227]). In the geometric framework we are constructing, Proposition 4.10 must be properly adapted in differential geometric terms for Riemannian manifolds. A Riemannian manifold is a (differentiable) manifold endowed with a Riemannian metric, i.e., a tensor $g_{\mathcal{W}}$ such that for all $w_1, w_2 \in \mathcal{W}$, $g_{\mathcal{W}}(w_1, w_2)$ is symmetric and positive definite. A Riemannian Hilbert manifold $\mathcal{W}$ is a differentiable manifold locally modeled on a separable Hilbert space [163, 100]. Being $\mathcal{W}$ Riemannian, it has an inner product $g_{\mathcal{W}}$ for $T_w\mathcal{W}$ equivalent to the inner product $\langle \cdot, \cdot \rangle$ in $(F, \langle \cdot, \cdot \rangle)$ in Proposition 4.10 for all $w \in \mathcal{W}$. A formal definition follows (for further details see [160, 162, 21]).

**Definition 4.11 ($C^{k+1}$-Hilbert manifold).** The manifold $\mathcal{W}$ is a $C^{k+1}$-Hilbert manifold, $k \geq 0$ if $\mathcal{W}$ is a $C^{k+1}$-manifold and for each $p \in \mathcal{W}$, the
4.2 The geometric framework for exogenous signals

The geometric framework for exogenous signals is a separable Hilbert space. For each \( p \in \mathcal{W} \) an admissible inner product \( \langle \cdot, \cdot \rangle_p \) in \( T_p \mathcal{W} \) is a positive definite, symmetric, bilinear form on \( T_p \mathcal{W} \) such that the topology of \( T_p \mathcal{W} \) is defined by the natural norm \( \| \xi \|_p = \sqrt{\langle \xi, \xi \rangle_p} \) [160].

Throughout this work the concept of isometry is fundamental:

**Definition 4.12 (Local isometry).** Let \((\mathcal{W}, \langle \cdot, \cdot \rangle_{\mathcal{T} \mathcal{W}})\) and \((\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{T} \mathcal{V}})\) be two Riemannian manifolds. A local isometry is a metric-preserving bijective map \( \Lambda : \mathcal{W} \to \mathcal{V} \) s.t. \( \langle \xi, \zeta \rangle_{\mathcal{T} \mathcal{W}} = \langle \Lambda^{*} \xi, \Lambda^{*} \zeta \rangle_{\mathcal{T} \mathcal{V}} \) for \( \xi, \zeta \in T \mathcal{W} \).

Local isometries are structure preserving: lengths of curves, areas of regions and angles between curves remain undistorted by them. Based on the concept of isometry, in this work we provide appropriate definitions of duality pairings, adjoint and self-adjoint differential operators for Hilbert manifolds as it will be seen throughout the remainder of the section:

**Definition 4.13 (Duality pairing in Hilbert Manifolds).** Let the spaces \((\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{T} \mathcal{M}})\) and \((\mathcal{M}^{*}, \langle \cdot, \cdot \rangle_{\mathcal{T}^{*} \mathcal{M}})\) be Riemannian manifolds. A well defined abstract duality pairing \( \langle \cdot, \cdot \rangle_{\mathcal{T}^{*} \mathcal{M} \times \mathcal{T} \mathcal{M}} : \mathcal{T}^{*} \mathcal{M} \times \mathcal{T} \mathcal{M} \to \mathbb{R}^1 \) is such that for each element \( \alpha \in \mathcal{T}^{*} \mathcal{M} \) there is a unique element \( \hat{\alpha} \in \mathcal{T} \mathcal{M} \) and an isometric isomorphism \( \Gamma : \mathcal{M} \to \mathcal{M}^{*} \) satisfying \( \langle \hat{\alpha}, \zeta \rangle_{\mathcal{T} \mathcal{M}} = \langle \alpha, \zeta \rangle_{\mathcal{T}^{*} \mathcal{M} \times \mathcal{T} \mathcal{M}} = \langle \alpha, \Gamma^{*} \zeta \rangle_{\mathcal{T}^{*} \mathcal{M}} \), \( \forall \zeta \in \mathcal{T} \mathcal{M} \) and \( \| \hat{\alpha} \|_{\mathcal{T} \mathcal{M}} = \| \alpha \|_{\mathcal{T}^{*} \mathcal{M}} \).

The dual pairing in Definition 4.13 is abstract because duality has been endowed only by unique (by Riesz Representation Theorem), continuous linear functionals. A concrete dual Hilbert manifold \( \mathcal{N} \) is characterized by:

**Definition 4.14.** The abstract duality pairing in Definition 4.13 is said to provide a concrete characterization of the dual manifold, if there exist an auxiliary Hilbert manifold \((\mathcal{N}, \langle \cdot, \cdot \rangle_{\mathcal{T} \mathcal{N}})\) along with an isometry \( \gamma : \mathcal{N} \to \mathcal{M}^{*} \) satisfying the commutative diagram:
such that \( \langle \xi, \alpha \rangle_{T\mathcal{M} \times T^*\mathcal{M}} = \gamma \alpha(\xi), \xi \in \mathcal{M} \) and \( \alpha \in \mathcal{M}^* \). In such case \( \mathcal{N} \) is called a realization for the abstract manifold \( \mathcal{M}^* \).

### 4.2.3 Hilbert manifold structure for \( \mathcal{W} \)

**Proposition 4.15 (Hilbert manifold structure of \( \mathcal{W} \)).** Let the external signals defined by \( w = (u, y) \), \( w \in \mathcal{W} = \mathcal{U} \oplus \mathcal{Y} \) with a dual \( \hat{w} = (\hat{u}, \hat{y}) \), \( \hat{w} \in \mathcal{W}^* = \mathcal{U}^* \oplus \mathcal{Y}^* \) be supported on the compact, differentiable, Riemannian manifolds \( (\mathcal{W}, g_\mathcal{W}) \) and \( (\mathcal{W}^*, g_{\mathcal{W}^*}) \) respectively, with metrics \( g_\mathcal{W}, g_{\mathcal{W}^*} \) and natural norms \( \| \cdot \|_{T\mathcal{W}}, \| \cdot \|_{T^*\mathcal{W}} \) defined after the inner products \( \langle \cdot, \cdot \rangle_{T\mathcal{W}}, \langle \cdot, \cdot \rangle_{T^*\mathcal{W}} \) on \( T\mathcal{W} \) and \( T^*\mathcal{W} \), as defined in Table 4.1. Then \( (\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}}) \) and \( (\mathcal{W}^*, \langle \cdot, \cdot \rangle_{T^*\mathcal{W}}) \) are dual Hilbert manifolds with duality identified by a (well defined) duality pairing \( \langle \cdot, \cdot \rangle_{T\mathcal{W} \times T^*\mathcal{W}} \) whose resulting scalar functions are coordinate invariant.

The Hilbert manifold structure in Proposition 4.15 keeps the natural duality between external signals and linear functionals on \( \mathcal{W} \) and \( \mathcal{W}^* \) along with their tangent and cotangent spaces \( T\mathcal{W} \) and \( T^*\mathcal{W} \).

**Remark 4.16.** In particular, duality can be naturally identified via the metrics since the covariant 2\textsuperscript{nd} rank metric tensor \( g_\mathcal{W} \) on \( (\mathcal{W}, g_\mathcal{W}) \) is related to the dual contravariant 2\textsuperscript{nd} rank metric tensor \( g_{\mathcal{W}^*} \) on \( (\mathcal{W}^*, g_{\mathcal{W}^*}) \), by \[ [g_{ij}] = [g^{ij}]^{-1} \]
(here brackets denote a matrix representation of the tensor), see e.g. [41].

**Remark 4.17.** The Hilbert manifold structure of Proposition 4.15 provides locally a notion of orthogonality for each Hilbert space. Furthermore, orthogonality of functions parameterized by time-trajectories as in Proposition 4.10 can be asserted with the duality pairing \( \langle \xi, \alpha \rangle_{T\mathcal{W} \times T^*\mathcal{W}} \) under the following geometric interpretation: let \( \phi^a, \phi^b \in \mathcal{W}^* \) have cotangent differentiable 1-forms \( \alpha^a, \alpha^b \in T^*\mathcal{W} \) being the duals of \( \hat{\alpha}^a, \hat{\alpha}^b \in T\mathcal{W} \). In particular the relation \( \langle \xi, \hat{\alpha}^a \rangle_{T\mathcal{W}} = 0, \xi \in T\mathcal{W} \) implies that the functional \( \phi^a \) is an integral invariant of \( \xi \), i.e. \( \xi \phi^a = \xi^i(w) \partial \phi^a / \partial w^i = 0 \). In consequence \( \phi^a, \phi^b \) are orthogonal iff \( \langle \alpha^a, \hat{\alpha}^b \rangle_{T^*\mathcal{W} \times T\mathcal{W}} = 0 = \langle \alpha^b, \hat{\alpha}^a \rangle_{T^*\mathcal{W} \times T\mathcal{W}} \) is satisfied.

**Definition 4.18.** A frame for \( \mathcal{W} \) (respectively, a coframe for \( \mathcal{W}^* \)), is an ordered set of vector fields \( \{\zeta_1(w), \zeta_2(w), \ldots, \zeta_\omega(w)\} \) (resp. 1-forms \( \{\beta^1(w), \ldots, \beta^\omega(w)\} \)
4.2 The geometric framework for exogenous signals

Table 4.1. Hilbert manifold structure for $\mathcal{W}$

<table>
<thead>
<tr>
<th>Structure</th>
<th>Primal space $(\mathcal{W}, (\cdot, \cdot)_{T\mathcal{W}})$ $w(t) \in \mathcal{W}$; $\xi, \zeta \in T\mathcal{W}$</th>
<th>Dual space $(\mathcal{W}^<em>, (\cdot, \cdot)^{T\mathcal{W}}_</em>)$ $\tilde{w}(\tau) \in \mathcal{W}^<em>$; $\alpha, \beta \in T^</em>\mathcal{W}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner product $^6$</td>
<td>$(\xi, \zeta)<em>{T\mathcal{W}} = \int</em>{0}^{\infty} (\xi, \zeta) , d\mu(t)$</td>
<td>$(\alpha, \beta)^{T\mathcal{W}} = \int_{0}^{\infty} (\alpha, \beta) , d\mu(\tau)$</td>
</tr>
<tr>
<td>Duality pairing $\xi \in T\mathcal{W}, \alpha \in T^*\mathcal{W}$</td>
<td>$(\xi, \alpha)<em>{T\mathcal{W} \times T^*\mathcal{W}} = \int</em>{0}^{\infty} \xi \alpha , d\mu(t) = \phi(w(t))</td>
<td>\mathbb{R}^\omega$</td>
</tr>
<tr>
<td>Metrics</td>
<td>$g_{\mathcal{W}} : T\mathcal{W} \times T\mathcal{W} \to \mathbb{R}^1$; $g_{\mathcal{W}} = g_{ij} , dw^i \otimes dw^j$</td>
<td>$g_{\mathcal{W}^<em>} : T^</em>\mathcal{W} \times T^<em>\mathcal{W} \to \mathbb{R}^1$; $g_{\mathcal{W}^</em>} = g^{ij} , \partial_i \otimes \partial_j$</td>
</tr>
<tr>
<td></td>
<td>$g_{\mathcal{W}}(\xi, \zeta) = (\xi - \zeta, \xi - \zeta)_{T\mathcal{W}}$</td>
<td>$g_{\mathcal{W}^<em>}(\alpha, \beta) = (\alpha - \beta, \alpha - \beta)_{T^</em>\mathcal{W}}$</td>
</tr>
<tr>
<td>Norms</td>
<td>$|\xi|<em>{T\mathcal{W}} = \sqrt{(\xi, \xi)</em>{T\mathcal{W}}}$</td>
<td>$|\alpha|<em>{T^*\mathcal{W}} = \sqrt{(\alpha, \alpha)</em>{T^*\mathcal{W}}}$</td>
</tr>
</tbody>
</table>

$^6 (\xi, \zeta) \overset{\text{def}}{=} \sum_{i=1}^{\omega} \xi^i \zeta^i = \xi^1 \zeta^1 + \cdots + \xi^\omega \zeta^\omega.$

$\beta^2(w), \ldots, \beta^\omega(w) \}$ such that at each $w \in \mathcal{W}$, it provides a basis for its tangent space $T_w\mathcal{W}$ (cotangent space $T^*_w(\mathcal{W})$), see e.g. [156]. Such frames are globally defined for parallelizable manifolds (i.e. s.t. $T\mathcal{W} \simeq \mathcal{W} \times \mathbb{R}^\omega$) only, otherwise such frames are defined locally.

**Remark 4.19.** Let

$$T_w\mathcal{W} = \text{span}\{\zeta_1(w), \zeta_2(w), \ldots, \zeta_\omega(w)\}$$

$$T^*_w(\mathcal{W}) = \text{span}\{\beta^1(w), \beta^2(w), \ldots, \beta^\omega(w)\}.$$  

If they satisfy $\langle \zeta_i, \zeta_j \rangle_{T\mathcal{W}} = \delta^i_j$ and $\langle \beta^i, \beta^j \rangle_{T^*\mathcal{W}} = \delta^i_j$ for $i, j = 1, \ldots, \omega$ respectively, such frames are orthonormal. Furthermore by Remarks 4.17, 4.16, each $\zeta_i \in T_w\mathcal{W}$ is dual to $\beta^i \in T^*_w(\mathcal{W})$, $i = 1, \ldots, \omega$.

**Remark 4.20.** Throughout this paper we deal with Hilbert manifolds of finite dimensions, understanding by dimension of a Hilbert manifold the cardinality (Hamel dimension of the homeomorphic $\mathbb{R}^\omega$) of any orthonormal frame in the Hilbert manifold. Thus, $\mathcal{W}$, $\mathcal{M}$ and their duals $\mathcal{W}^*$, $\mathcal{M}^*$ are finite-dimensional.
4.2.4 Hilbert manifold structure for $\mathcal{B}$

There is the need of another structure keeping duality of past $\mathcal{B}^-$ and future $\mathcal{B}^+$ behaviors. Consider the behavioral system $\Sigma = (t, \mathcal{W}, \mathcal{B})$ and a partition of the time interval $t = (-\infty, \infty)$ into a past subinterval $t^- = (-\infty, 0]$ and a future subinterval $t^+ = [0, \infty)$ such that $t = t^- \oplus t^+$ partitioning the behavior into half-spaces $\mathcal{B}^- \oplus \mathcal{B}^+$ with a common edge $\mathcal{B}^0$.

The following assumption conditions the property of dissipativeness to the direction of evolution.

Assumption 4.21 (Dissipativity condition). Throughout this Part I assume that the system $\Sigma^+$ is dissipative, i.e., there is a self-adjoint linear operator $Z \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ with associated quadratic function $r : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}^1$ satisfying $r(w(t)) = w^T(t)Zw(t) \geq 0, \forall t, w \in \mathcal{W}$ [75].

In particular, let $w(t) = (u(t), y(t))$, then the matrices $Z_{L_2}$ and $Z_{\text{passive}}$ defined by

$$Z_{L_2} = \begin{bmatrix} \frac{1}{2} \gamma^2 I & 0 \\ 0 & -\frac{1}{2} I \end{bmatrix}, \quad Z_{\text{passive}} = \begin{bmatrix} 0 & \frac{1}{2} I \\ \frac{1}{2} I & 0 \end{bmatrix},$$

can be associated to an $L_2$-gain system (where $\gamma$ is the $L_2$-gain) and a passive system, respectively.

Example 4.22. Recall the universal motor from Example 4.5 with coordinates $w = (V_t, \tau L, I, \omega)$. Time integration of the function $r(w(t)) = w^T(t)Z_{\text{passive}}w = V_t I + \tau L \omega \geq 0, \forall t \in t$, yields the function $S(w(t)) = \int V_t I dt + \int \tau L \omega dt$ (i.e. the electrical energy plus the rotational energy).

The function $r(w(t))$ is expressed as an (exact) differential 1-form $\partial_w r(w) dw$, $r(w(t)) = (\partial_w r, \cdot)_0$ such that it is completely integrable into a 0-form on $\mathcal{W}^*$ denoted by $S(w(t)) = \int V_t I dt + \int \tau L \omega dt$ (i.e. the electrical energy plus the rotational energy).

Assumption 4.23. The Hilbert manifolds $(\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}})$ and $(\mathcal{W}^*, \langle \cdot, \cdot \rangle_{T\mathcal{W}^*})$ admit partitions of unity.

When Hilbert vector spaces admit partitions of unity (see e.g. Prop 1.1 in [100] or Theorem 3.3 in [161]), they admit alternate Riemannian metrics. In particular, let $\psi_i \in \mathcal{L}_a^\omega(a, b)$ and define functions $\Psi = \sum_{i=1}^\omega \Psi_i$, $F_i = \Psi_i(\psi_i)/\Psi$, then since $\sum_{i=1}^\omega F_i = 1$ a partition of unity $\{U_i, F_i\}$ is well defined and then $ds^2 = \sum_{i=1}^\omega F_i ds_i^2$ is a well defined Riemannian metric. The following functions are well known in the Theory of dissipative systems, see [219].
4.2 The geometric framework for exogenous signals

**Definition 4.24 (Required supply and available storage[219]).** Assuming that the following functions associated to system $\Sigma^+$ in Eq. (4.1) exist, the storage functions called the required supply, $S_r : \mathcal{W} \to \mathbb{R}^+$, is defined by

$$S_r(w^0, r_r) \overset{\text{def}}{=} \inf_{u(\cdot) \in \mathcal{U} \subset \mathcal{W}} \int_{-\infty}^{0} r_r(w(t)) \, d\mu(t),$$  \hfill (4.5) 

and the available storage, $S_a : \mathcal{W} \to \mathbb{R}^+$, is defined by

$$S_a(w^0, r_a) \overset{\text{def}}{=} \sup_{u(\cdot) \in \mathcal{U} \subset \mathcal{W}} \int_{0}^{\infty} r_a(w(t)) \, d\mu(t)$$

$$= - \inf_{u(\cdot) \in \mathcal{U} \subset \mathcal{W}} \int_{0}^{\infty} r_a(w(t)) \, d\mu(t).$$  \hfill (4.6) 

where the function of external signals $w : t \to \mathcal{W}$, defined by $r : \mathcal{W} \to \mathbb{R}^1$, $r(w(t))$, is called supply rate relative to $S_r$ or $S_a$ respectively.

**Remark 4.25 (Backward required supply $S_r^*$).** The required supply $S_r(w^0, r_r)$ can be posed in terms of the backward-time system $\Sigma^-$ in (4.2) as the function

$$S_r^*(\hat{w}^0, r_r) \overset{\text{def}}{=} - \sup_{\hat{u}(\cdot) \in \mathcal{U}^* \subset \mathcal{W}^*} \int_{0}^{\infty} r_r(\hat{w}(\tau)) \, d\mu(\tau),$$  \hfill (4.7) 

Express by the sets

$$w^-_\tau \overset{\text{def}}{=} \{ \hat{w}^- = (\hat{u}, \hat{y}) | S_r^*(\hat{w}^0, r_r) \text{ satisfies Eq. (4.7) for } \tau \in \tau, \hat{u} \in \mathcal{U}^*, \hat{y} \in \mathcal{Y}^* \}$$  \hfill (4.8) 

$$w^+_t \overset{\text{def}}{=} \{ w^+ = (u, y) | S_a(w^0, r_a) \text{ satisfies Eq. (4.6) for } t \in t, u \in \mathcal{U}, y \in \mathcal{Y} \}$$  \hfill (4.9) 

the semi-trajectories solution of the optimal control problems Eq. (4.5) and Eq. (4.6) respectively. The operation of concatenating a past semi-trajectory $\hat{w}^-(\tau) \in \mathcal{W}^-$, $\tau \in \tau$ with a future semi-trajectory $w^+(t) \in \mathcal{W}^+$, $t \in t$ at $w^0$ is defined by $\hat{w}^-(\tau) \land w^+(t) \overset{\text{def}}{=} \{ \bar{w} \in \mathcal{W} | \bar{w} = \hat{w}^-(\tau); \tau \in \tau \& \bar{w} = w^+(t); t \in t \}$, see e.g. [212, 174].
**Definition 4.26.** For each initial condition $w^0 \in W$, the time-concatenation of trajectories solution of (4.7) and (4.6) define trajectories in $U$ and $Y$ for $t \in (-\infty, \infty)$. Under this conditions, based on Definition 4.6 the behavior $\mathcal{B}$ consists of the trajectories $w : t \rightarrow W$, $U \subset W$.

Since storage functions characterize the past and future influence of the environment on the system behavior $\mathcal{B}$ and conversely, the following assumption is crucial along the forthcoming sections (see also Table 4.2):

**Assumption 4.27.** The functionals $S^*_r(\hat{w}^0, r)$ and $S_a(w^0, r_a)$ are past and future induced metrics on the Hilbert manifold structure of Proposition 4.15 such that

$$S^*_r(\hat{w}^0, r) = \langle \alpha^- \rangle_{\mathcal{B}^-}, \quad \mathcal{B}^- \subset U^- (4.10)$$

$$S_a(w^0, r_a) = \langle \xi^+, \xi^+ \rangle_{\mathcal{B}^+}, \quad \mathcal{B}^+ \subset U^+. (4.11)$$

**Proposition 4.28 (Hilbert manifold structure of $\mathcal{B}$).** Let the past behavioral semi-trajectories $\hat{w}^-(\tau)$ on $W^- = U^* \oplus Y^*$ be supported by the Riemannian manifold $(\mathcal{B}^-, \langle \cdot, \cdot \rangle_{\mathcal{B}^-})$ and the future behavioral semi-trajectories $w^+(t)$ on $W^+ = U \oplus Y$ by the Riemannian manifold $(\mathcal{B}^+, \langle \cdot, \cdot \rangle_{\mathcal{B}^+})$ for metrics $g_{\mathcal{B}^-}$, $g_{\mathcal{B}^+}$ defined after the inner products $\langle \alpha_1^-, \alpha_2^- \rangle_{\mathcal{B}^-}$, $\langle \xi_1^+, \xi_2^+ \rangle_{\mathcal{B}^+}$ defined in Table 4.2. Under Assumption 4.21, 4.23 and 4.27, then $(\mathcal{B}^-, \langle \cdot, \cdot \rangle_{\mathcal{B}^-})$ and $(\mathcal{B}^+, \langle \cdot, \cdot \rangle_{\mathcal{B}^+})$ are dual Hilbert manifolds with duality identified by the pairing $\langle \xi^+, \alpha^- \rangle_{\mathcal{B}^- \times \mathcal{B}^+}$ for some isometry $\tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+$ s.t. $\|\alpha^-\|_{\mathcal{B}^-} = \|\xi^+\|_{\mathcal{B}^+}$ is satisfied.

**Assumption 4.29.** $\tilde{\Gamma}$ is a bijective isometry (and thus an isometric isomorphism) s.t. $\tilde{\Gamma}^\dagger : \mathcal{B}^+ \rightarrow \mathcal{B}^-$ in the duality pairing

$$\langle \xi^+, \alpha^- \rangle_{\mathcal{B}^+ \times \mathcal{B}^-} \overset{\text{def}}{=} \int_0^\infty \langle \alpha^-, \tilde{\Gamma}_\ast \xi^+ \rangle_0 d\mu(\tau), \quad \left\{ \begin{array}{l} \xi^+ \in T\mathcal{B}^+, \\ \alpha^- \in T\mathcal{B}^- \end{array} \right. (4.12)$$

is equivalent to $\langle \xi^+, \alpha^- \rangle_{\mathcal{B}^- \times \mathcal{B}^+}$ and satisfies $\|\alpha^-\|_{\mathcal{B}^-} = \|\xi^+\|_{\mathcal{B}^+}$.

Using this Hilbert manifold structure, several problems of balancing can be posed in a unifying format for different definitions of storage functions, namely $L_2$-gain, passive, etc.

**Example 4.30.** The $\mathcal{L}$-class input/state/output representation $(A, B, C)$ with state space $\mathbb{R}^n$ can be identified at any point with its tangent space, i.e. $\mathbb{R}^n = T_p\mathbb{R}^n$ implying that Hilbert manifolds simplify to Hilbert spaces and Lie groups.
4.2 The geometric framework for exogenous signals

Table 4.2. Hilbert manifold structure for \( \mathcal{B} \)

<table>
<thead>
<tr>
<th>Structure</th>
<th>Future behavior ( (\mathcal{B}^+, \langle \cdot, \cdot \rangle_{T\mathcal{B}^+}) )</th>
<th>Past behavior ( (\mathcal{B}^-, \langle \cdot, \cdot \rangle_{T\mathcal{B}^-}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w^+_1(t), w^+_2(t) \in \mathcal{B}^+; \xi^+_1, \xi^+_2 \in T\mathcal{B}^+ )</td>
<td>( \hat{w}^-(\tau), \hat{w}^-_2(\tau) \in \mathcal{B}^-; \alpha^_1, \alpha^_2 \in T\mathcal{B}^- )</td>
<td></td>
</tr>
</tbody>
</table>

| Inner product\(^a\) | \( \langle \xi^+_1, \xi^+_2 \rangle_{T\mathcal{B}^+} \overset{\text{def}}{=} \int_0^\infty \langle \xi^+_1, \xi^+_2 \rangle_0 \, d\mu(t) \) | \( \langle \alpha^_1, \alpha^_2 \rangle_{T\mathcal{B}^-} \overset{\text{def}}{=} \int_0^\infty \langle \alpha^_1, \alpha^_2 \rangle_0 \, d\mu(\tau) \) |
| Duality pairing | \( \langle \xi^+, \alpha^- \rangle_{T\mathcal{B}^+ \times T\mathcal{B}^-} \overset{\text{def}}{=} \int_0^\infty \langle \xi^+, \Gamma^* \alpha^- \rangle_0 \, d\mu(t) \) | \( \langle \xi^+, \alpha^- \rangle_{T\mathcal{B}^+ \times T\mathcal{B}^-} \overset{\text{def}}{=} \int_0^\infty \langle \xi^+, \alpha^- \rangle_0 \, d\mu(t) \) |
| Metrics | \( g^+_\mathcal{B} : T\mathcal{B}^+ \times T\mathcal{B}^+ \to \mathbb{R}^1 \) | \( g^-\mathcal{B} : T\mathcal{B}^- \times T\mathcal{B}^- \to \mathbb{R}^1 \) |
| | \( g^+_\mathcal{B}(w^+_1, w^+_2) = \langle \xi^+_1 - \xi^+_2, \xi^+_1 - \xi^+_2 \rangle_{T\mathcal{B}^+} \) | \( g^-\mathcal{B}(\hat{w}^-_1, \hat{w}^-_2) = \langle \alpha^_1 - \alpha^_2, \alpha^_1 - \alpha^_2 \rangle_{T\mathcal{B}^-} \) |
| Norms | \( \|\xi^+\|_{T\mathcal{B}^+} = \sqrt{\langle \xi^+, \xi^+ \rangle_{T\mathcal{B}^+}} \overset{\text{def}}{=} \sqrt{S_n(w^0, r_n)} \) | \( \|\alpha^-\|_{T\mathcal{B}^-} = \sqrt{\langle \alpha^-, \alpha^- \rangle_{T\mathcal{B}^-}} \overset{\text{def}}{=} \sqrt{S^*_\tau(\hat{w}^0, r_r)} \) |

\(^a\) Where \( \langle \cdot, \cdot \rangle_0 = \int_0^T \langle \cdot, \cdot \rangle \, dt \) and \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^\omega \).

are linear maps. Moreover, nonlinear isometric operators from Definition 4.12 reduce in the linear case to the known all pass systems in systems theory and lossless systems in network theory, see e.g. [62], pg. 86.

Embedded in the Hilbert manifold structure of Definition 4.28 there is an induced Hilbert manifold structure for two intertwined dual manifolds supporting the future system trajectories on \( \mathcal{M^k} \) for a forward-time evolution and the past system trajectories on \( \mathcal{M^*} \) for a backward-time evolution. The internal description of these spaces is formally stated in Part II of this paper.

### 4.2.5 Some eigenproblems for differential and integral operators

The behavioral operator \( \tilde{\Gamma} : \mathcal{B}^- \to \mathcal{B}^+ \) is naturally represented by generalized convolution integrals, mapping the space of past semi-trajectories \( \hat{w}^-(\tau) \in \mathcal{B}^- \) to the space of future semi-trajectories \( w^+(t) \in \mathcal{B}^+ \) for a \( \mu \)-integral operator

\[
\begin{align*}
  w^+(t) = \tilde{\Gamma} \circ \hat{w}^-(\tau) & \overset{\text{def}}{=} \int_0^\infty \cdots \int_0^\infty H(t, \varsigma) \hat{w}^-_1(\varsigma) \cdots \hat{w}^-_\omega(\varsigma) \, d\mu(\varsigma) \cdots d\mu(\varsigma), \\
  & \quad (4.13)
\end{align*}
\]
with kernel \( H(t, \tau) \) and \( \hat{w}^-(\tau) \in \mathcal{B}^- \). Nonlinear realization theory based on this approach has been early developed, see e.g. [177, 151, 82] and references therein. Furthermore, the theory of eigenvalue problems for this class of integral operators is well known, see e.g. [170]. The properties of the behavioral operator can be characterized from this theory by supporting the set of behavioral trajectories solution of (4.13), after Remark 4.2, on a differentiable Riemannian manifold and by expressing the integral operator (4.13) in a differential-geometric form.

**Assumption 4.31.** Eqs. (4.1)-(4.2) admit a first-order differentiable structure, i.e. whenever the behavioral operator \( \tilde{\Gamma} \) exists, there is a differentiable structure on its tangent map \( \tilde{\Gamma} : T\mathcal{B}^- \to T\mathcal{B}^+ \) for \( \tilde{\Gamma} \).

After Assumption 4.31, a differential-geometric treatment to \( \tilde{\Gamma} \) is provided in this section in order to characterize the properties of \( \tilde{\Gamma} \) along with its associated eigenvalue problems. For this purpose, it is necessary to introduce appropriate definitions for adjoint and self-adjoint operators on Hilbert manifolds based on the concept of isometry from Definition 4.12, providing an alternative geometric generalization to the traditional definitions used in linear operator theory, cfr. e.g. [226].

**Definition 4.32 (Adjoint operator).** Consider the Hilbert Manifold \((\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}})\) with dual \((\mathcal{W}^*, \langle \cdot, \cdot \rangle_{T^*\mathcal{W}})\). A differential operator \( \Xi : \mathcal{W} \to \mathcal{W}^* \) (with tangent map \( \Xi_* : T\mathcal{W} \to T^*\mathcal{W} \)) is said to have an adjoint differential operator \( \Xi^\dagger : \mathcal{W}^* \to \mathcal{W} \) (with tangent map \( \Xi^\dagger_* : T^*\mathcal{W} \to T\mathcal{W} \)) if it is such that

\[
\langle \Xi^\dagger_* \alpha, \xi \rangle_{T\mathcal{W}} = \langle \alpha, \Xi_* \xi \rangle_{T^*\mathcal{W}}, \quad \xi \in T\mathcal{W}, \ \alpha \in T^*\mathcal{W} \quad (4.14)
\]

for inner products defined in Table 4.1.

**Definition 4.33 (Self-adjoint operator).** Consider a Hilbert manifold \((\mathcal{W}, \langle \cdot, \cdot \rangle_{T\mathcal{W}})\). An operator \( \Omega : \mathcal{W} \to \mathcal{W} \) (with tangent map \( \Omega_* : T\mathcal{W} \to T\mathcal{W} \)) is self-adjoint if it is such that

\[
\langle \xi, \Omega_* \zeta \rangle_{T\mathcal{W}} = \langle \Omega_* \xi, \zeta \rangle_{T\mathcal{W}}, \quad \xi, \zeta \in T\mathcal{W}.
\]

**Example 4.34.** For linear time-invariant systems (the \( \mathcal{L} \)-class), Defs. 4.32 for adjoint and 4.33 for selfadjoint differential operators simplify to the respective known definitions for linear operators, see e.g. [62], pg.85. \( \square \)
Proposition 4.35. Let system $\Sigma = (t, W, \mathcal{B})$ satisfy Assumptions 4.21 and 4.27 and its behavior be supported by the Hilbert manifold structure in Proposition 4.28. The following can be asserted:

1. The isomorphism $\tilde{\Gamma} : \mathcal{B}^− \rightarrow \mathcal{B}^+$ (behavioral operator) is isometric satisfying the commutative diagram

$$
\begin{array}{ccc}
\mathcal{B}^− & \xrightarrow{\tilde{\Gamma}} & \mathcal{B}^+ \\
\pi_\mathcal{B} & & \pi_\mathcal{B}
\end{array}
$$

with $\langle \xi, \zeta \rangle_{T\mathcal{B}−} = \langle \tilde{\Gamma}_∗\xi, \tilde{\Gamma}_∗\zeta \rangle_{T\mathcal{B}+}$.

2. $\tilde{\Gamma}$ admits (in the sense of Definition 4.32) an adjoint operator $\Gamma^\dagger : L_2[0, \infty) \rightarrow L_2(-\infty, 0]$ given by the map $\tilde{\Gamma}^\dagger : \mathcal{B}^+ \rightarrow \mathcal{B}^−$, such that

$$
\begin{bmatrix}
\hat{u}^−(\tau) \\
\hat{y}^−(\tau)
\end{bmatrix}_{\mathcal{B}−} =
\begin{bmatrix}
\Gamma^\dagger \circ \hat{y}^+(\tau) \\
\Gamma \circ \hat{u}^+(\tau)
\end{bmatrix},
\quad \tau \in \tau,
(\hat{u}(\tau), \hat{y}(\tau)) \overset{\text{def}}{=} \hat{w}(\tau) \in \mathcal{B}−.
$$

3. The automorphisms $\tilde{\Gamma}^\dagger \circ \tilde{\Gamma} : \mathcal{B}^− \rightarrow \mathcal{B}^−$ and $\tilde{\Gamma} \circ \tilde{\Gamma}^\dagger : \mathcal{B}^+ \rightarrow \mathcal{B}^+$ are selfadjoint in the sense of Definition 4.33.

4. The behavioral operator in Definition 4.8 is an automorphism for $\mathcal{B} = \mathcal{B}^− \oplus \mathcal{B}^+$ and selfadjoint (in the sense of Definition 4.33) for this space.

5. The input-output sub operators $\Gamma : \mathcal{U} \rightarrow \mathcal{Y}$ and $\Gamma^\dagger : \mathcal{Y}^* \rightarrow \mathcal{U}^*$ satisfy

$$
\langle \Gamma^\dagger_∗\alpha_y, \xi_u \rangle_{TW|_u} = \langle \alpha_y, \Gamma_∗\xi_u \rangle_{TW|_y},
\alpha_y \in TW|_y, \xi_u \in TW|_u \text{ and thus are adjoint (in the sense of Definition 4.32).}
$$

To conclude this section, further standard concepts about the invariant eigenvalue problems for nonlinear map operators in the Hilbert manifold framework are presented (for a proof of Proposition 4.36, see e.g. [33, 155]).

Proposition 4.36 (Invariants). Let $x(t)$ be supported by a manifold $\mathcal{M}$. The invariant integral functions $f(x)$ of a vector field $\xi(x)$ take constant values on the integral trajectories $x(t)$ of $\xi(x)$ and thus $\xi f = 0$, equivalently $\xi(x)$ has an invariant $k$-form $\alpha$ iff $L\xi\alpha = 0$. A function $f(x)$ is an invariant of a 1-parameter (semi-) group of diffeomorphisms $\Phi^t : t \times \mathcal{M} \rightarrow \mathcal{M}$ if it satisfies $f(\Phi^t(x)) = f(x) = 0$, $\forall x \in \mathcal{M}$. Such $f$ are the integrals of the generator $\xi(x)$ of the (semi-) group $\Phi^t$, implying that $\xi(x)$ is invariant to the 1-parameter (semi-) group it generates $\Phi^t_\ast(x)|_x \xi(x) = \xi(x) = 0$, $\forall x \in \mathcal{M}$.

With 1-parameter (semi-) groups in mind, based on Proposition 4.36 the following eigenvalue problems for nonlinear operators are well defined in differential-geometric terms.
Proposition 4.37. Regarding $\lambda \in \mathbb{C}^1$ as eigenvalues, $\xi \in T^*W$ eigencovector fields, $\alpha \in T^*W$ eigencovector fields (eigenforms), the following eigenproblems are well defined:

1. Associated to a (differentiable map) operator $\Upsilon : W \to N$ for differentiable curves $\varrho(t) \in W$ there is an eigenvalue problem $\Upsilon(\varrho) - \lambda \varrho = 0$ and associated to its tangent map $\Upsilon_* : TW \to TN$ such problem is defined by $\Upsilon_*|_x \xi - \lambda \xi = 0$ at each $x \in T_pW$.

2. Dually, associated to $\Upsilon : N^* \to W^*$ for differentiable functionals $f(x) \in W^*$ is the eigenvalue problem $f(\Upsilon(x)) - \lambda f(x) = 0$ and associated to its differential map $\Upsilon^*|_x : T^*N \to T^*W$ such problem is defined by $\alpha \Upsilon^*|_x - \lambda \alpha = 0$ at each $x \in T^*_pN$.

A particular case of these eigenproblems is useful later for our purposes.

Remark 4.38. Consider the maps: $A : W \to \mathcal{V}$, $B : \mathcal{V} \to W$ with eigenvalue problems of Prop 4.36(1) $Aw(t) - \sigma v(t) = 0$, $Bv(t) - \sigma w(t) = 0$, $\sigma \in \mathbb{C}^1$ satisfied by one trajectory $w(t) \in W$ or $v(t) \in \mathcal{V}$. The problem can be posed as $B \circ A \circ w(t) - \sigma^2 w(t) = 0$ or by $A \circ B \circ v(t) - \sigma^2 v(t) = 0$.

Proposition 4.39. Assume existence of smooth solutions to the integral operator $\tilde{\Gamma} : \mathcal{B}^- \to \mathcal{B}^+$ in Eq. (4.13). Denote by $\tilde{\Gamma}_* : T\mathcal{B}^- \to T\mathcal{B}^+$ its tangent map. Let the $i$th-eigensemitrajectories, $\hat{w}_i^-(\tau) \in \mathcal{B}^-$ for past and $w_i^+(t) \in \mathcal{B}^+$ for future, have tangent eigenvector fields $\beta_i^-(\hat{x}) \in T\mathcal{B}^-$ and $\zeta_i^+ \in T\mathcal{B}^+$ respectively, and $\lambda_i \in \mathbb{R}^1$, $i = 1, \ldots$. The following is asserted:

1. Eq. (4.13) can be alternatively expressed in geometric terms by the $\mu$-integral

$$w^+(t) \overset{\text{def}}{=} \hat{w}^-(0) + \int_0^\infty \bar{\tilde{\Gamma}}^*(t, \tau) \beta_-(\hat{w}^-(\tau)) \, d\mu(\tau). \quad (4.17)$$

2. The eigenvalue problem $\tilde{\Gamma} \circ \hat{w}_i^- (\tau) - \lambda_i w_i^+(t) = 0$ of Eq. (4.17) is given by the 1st-kind Fredholm equations

$$\hat{w}_i^-(0) + \int_0^\infty \bar{\tilde{\Gamma}}^*(t, \tau) \beta_i^-(\hat{w}_i^- (\tau)) \, d\mu(\tau) = \lambda_i w_i^+(t). \quad (4.18)$$

3. At the tangent space it admits an associated eigenproblem

$$\tilde{\Gamma}^* \beta_i^-(\hat{w}_i^-) - \lambda_i \zeta_i^+(w_i^+) = 0.$$
4.3 Curvature of the behavior and balancing

4.3.1 Why curvature theory?

In any geometric study, the *invariants* of a particular transformation group are fundamental. In particular, the characterization of the invariants of Riemannian submanifolds under isometric transformations is the main topic in the elegant *Classical theory of Curvature* due (among others) to Euler and Gauss, see [103] for an introduction and [41, 163] for further reading.

Essentially, this theory characterizes the invariants when a hypersurface $S^m$ supported by a Riemannian manifold $\mathcal{W}^n$ is mapped by an isometry $\Lambda : \mathcal{W} \to \mathcal{V}$ (as defined in Definition 4.12) into another Riemannian manifold $\mathcal{V}^n$.

Let $(\mathcal{W}, \langle \cdot, \cdot \rangle_{\mathcal{T}\mathcal{W}})$ be a Riemannian manifold supporting a hypersurface $S^m \subset \mathcal{W}$ and distinguish two types of invariants:

**Definition 4.40.** The set of *isomorphic invariants* $\kappa(S^m) = \{\kappa_i, i \in m\}$ are local properties of $S^m$ preserved under isomorphic transformations. The *intrinsinc invariants* $K(S^m)$ are local properties on $S^m$ that are preserved globally by local isometries, and are said to be intrinsic to $S^m$.

Associated to the hypersurface $S^m$ there is a set of *principal curvatures* $\{\kappa_i, i = 1, \ldots, m\}$ on each $\mathcal{W}$ and $\mathcal{V}$. The principal curvatures are preserved under (isomorphic) transformations which do not modify the local form of the hypersurface. Isomorphic invariants frequently result from equivalence relations and thus define a (finite) set of *equivalence classes*. Examples of these are *congruence relationships* and the principal curvatures of a space: while a 2D plane has principal curvatures $\kappa_1 = \kappa_2 = 0$, a sphere of radius $r$ has principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$, etc.

Remarkably, the product $K = \Pi_i^m \kappa_i$ called *total or Gaussian curvature* is an invariant of $S^m$ under the isometry $\Lambda$. Gaussian or total curvature is intrinsic to $S^m$: while a 2D plane has total curvature zero, a sphere of radius $r$ has total curvature $\frac{1}{r^2}$, etc. These concepts are formalized later in this section.

At a conceptual level, in this paper we look for the invariant properties of the submanifold $\mathcal{B}$ supporting the behavioral trajectories of a dynamical system when this behavior is mapped by an isometric isomorphism (the behavioral operator) which preserves both invariants on Definition 4.40, namely the principal curvatures and hence their product. Thus, in systems theory the characterization of $\mathcal{B}$ in terms of these invariants is an interesting application for curvature theory.
Definition 4.41 (Shape operator). Let $\mathfrak{B} \subset \mathfrak{U}$ and a normal vector field $\eta \in (T\mathfrak{B})^\perp$ in the normal bundle. The Shape operator or Weingarten map $A_\eta^{\mathfrak{B}} : T\mathfrak{B} \to T\mathfrak{B}$ is defined by
\[
A_\eta^{\mathfrak{B}}(\xi) := -(\nabla_\xi \eta(w_0))^T, \quad w_0 \in \mathfrak{U}, \xi \in T\mathfrak{B}, \eta \in (T\mathfrak{B})^\perp
\] (4.19)
where (the covariant derivative) $\nabla$ is the unique Levi-Civita connection such that $\nabla_\xi \zeta - \nabla_\zeta \xi = [\xi, \zeta]$ and compatible with the metric $g_\mathfrak{U}$, i.e., for any $\varsigma, \zeta, \xi \in T\mathfrak{U}$, $\varsigma \langle \zeta, \xi \rangle_{TU} = \langle \nabla_\varsigma \zeta, \xi \rangle_{TU} + \langle \zeta, \nabla_\varsigma \xi \rangle_{TU}$.

The Shape operator satisfies $g_\mathfrak{U}(A_\eta^{\mathfrak{B}}(\xi), \zeta) = g_\mathfrak{U}(\xi, A_\eta^{\mathfrak{B}}(\zeta))$ and thus is self-adjoint (for a proof see e.g. [163]). In this section the curvature $K(\cdot)$ of the behavior $\mathfrak{B}$ is defined in terms of the Shape operator. Uniqueness of a Shape operator for a given $\mathfrak{B} \subset \mathfrak{U}$ poses two alternate ways to approach the problem of nonlinear balancing. One way to approach the balanced reduction problem of $\Sigma$ is presented in Section 4.3 and can be expressed in terms of approximating the curvature $K(\cdot)$ of the behavior $\mathfrak{B}$ (and hence approximating the behavior $\mathfrak{B}_\Sigma$) by reducing the support of the behavioral trajectories on $\mathfrak{U}$. Another way, presented in Part II of this paper, is to begin with the definition of a self-adjoint operator $A(\xi)$ which has an associated integral submanifold $\mathfrak{B} \subset \mathfrak{U}$ with particular properties. Two systematic ways to define a linear self-adjoint operator are presented in Part II of this paper: the behavioral and the Hankel operators. This framework does not need of properties associated to nonlinear self-adjoint operators, since the eigenvalue problem of the normal curvature is defined in the tangent space and even for nonlinear systems, the Shape operator (4.19) is invariably a linear operator.

Consider two subsets in the space of external variables $\mathcal{W}$ defined by the support of the past behavior $\text{supp}\mathfrak{B}^- = \mathcal{W}^- \subseteq \mathcal{W}$ and the support of the future behavior $\text{supp}\mathfrak{B}^+ = \mathcal{W}^+ \subseteq \mathcal{W}$ of the behavioral system $\Sigma = (\mathfrak{t}, \mathcal{W}, \mathfrak{B})$. Furthermore assume valid the Hilbert structure in Proposition 4.28. The problem of nonlinear balancing can be approached in terms of balancing the external signals on $\mathcal{W}$ or in terms of the properties of a self-adjoint operator.

This section is concerned with external signals and is focused in characterizing the curvature properties of a submanifold $\mathfrak{B} \subset \mathfrak{U}$ in terms of properly defined metrics. Each definition of metric $g_\mathfrak{B}$ in $\mathfrak{B}$ defines uniquely the problem since the total curvature is only dependent on $g_\mathfrak{B}$ and its derivatives.

4.3.2 The behavior and its curvature

In this subsection we adapt classical concepts of Gaussian curvature to characterize the behavioral trajectories on $\mathfrak{B}$. It is an unabridged version of preliminary results presented in [110].
Definition 4.42 (Fundamental forms and normal curvature of \( \mathcal{B} \)). Let Assumptions 4.21, 4.23 and 4.27 be satisfied for the system \( \Sigma = (t, \mathcal{W}, \mathcal{B}) \) whose behavior is supported by the Hilbert manifold structure in Proposition 4.28. As a submanifold of the universe, the behavior \( \mathcal{B} \subset \mathcal{U} \) has an associated Shape operator (4.19) denoted by \( A^\mathcal{B}_\eta(\xi), \xi \in (T\mathcal{B}), \eta \in (T\mathcal{B})^\perp \). The first fundamental form of \( \mathcal{B} \), is the restriction of \( \langle \cdot, \cdot \rangle_{T\mathcal{U}} \) to \( \mathcal{B} \), denoted by \( I_B \) as

\[
I_B(\xi, \zeta) = \langle \xi, \zeta \rangle_{T\mathcal{B}}, \quad \xi, \zeta \in T\mathcal{B}
\]  

(4.20)

the second fundamental form of \( \mathcal{B} \), denoted by \( II_B \) is defined by

\[
II_B(\xi, \zeta) = \langle A^\mathcal{B}_\eta(\xi), \zeta \rangle_{T\mathcal{B}}, \quad \xi, \zeta \in T\mathcal{B}
\]  

(4.21)

The normal curvature of \( \mathcal{B} \) is the quotient defined by

\[
K(\xi) = \frac{II_B(\xi, \xi)}{I_B(\xi, \xi)}, \quad \xi \in T\mathcal{B}.
\]  

(4.22)

The eigenvalue problem of the quotient (4.22) consist in finding the principal directions \( \xi \) along \( T\mathcal{B} \) where \( K(\xi) \) attains stationary values \( \kappa \) called principal normal curvatures. The mean and total curvatures are defined as \( \sum_i \kappa_i \) and \( \prod_i \kappa_i \) respectively.

Remark 4.43. Given the operator \( \tilde{\Gamma} : \mathcal{B}^- \to \mathcal{B}^+ \), for each trajectory \( w \text{ def } = \hat{w}^-(\tau) \wedge w^+(t) \in \mathcal{B} \) with tangent vector fields \( \alpha_- \in T_p\mathcal{B}^- \) and \( \hat{\alpha}_+ \in T_p\mathcal{B}^+ \), its future metric \( g_{\mathcal{B}^+}(\hat{\alpha}_+, 0) = \langle \hat{\alpha}_+, \hat{\alpha}_+ \rangle_{T\mathcal{B}^+} \) can be induced to the past by substitution of \( \hat{\alpha}_+ = \hat{\Gamma}_* \alpha_- \) on \( \langle \hat{\alpha}_+, \hat{\alpha}_+ \rangle_{T\mathcal{B}^+} = \langle \hat{\Gamma}_* \hat{\Gamma}^\dagger \alpha_-, \alpha_- \rangle_{T\mathcal{B}^+} \) and after division by the past metric \( g_{\mathcal{B}^-}(\hat{\alpha}_-, 0) \text{ def } = I_B(\alpha_-, \alpha_-) \) yields

\[
K(\alpha_-) = \frac{\langle (\hat{\Gamma}^\dagger \circ \hat{\Gamma}^\dagger \alpha_-, \alpha_- \rangle_{T\mathcal{B}^-}}{\langle \alpha_-, \alpha_- \rangle_{T\mathcal{B}^-}} \text{ def } \frac{I_B(\alpha_-, \alpha_-)}{I_B(\alpha_-, \alpha_-)}
\]  

(4.23)

providing a notion of future evolution gain with respect to the past metric.

The past evolution gain w.r.t. the future metric is given by

\[
K^{-1}(\xi^+) = \frac{\langle (\hat{\Gamma} \circ \hat{\Gamma}^\dagger) \xi_+, \xi_+ \rangle_{T\mathcal{B}^+}}{\langle \xi_+, \xi_+ \rangle_{T\mathcal{B}^+}} \text{ def } \frac{II_B(\xi_+, \xi_+)}{II_B(\xi_+, \xi_+)}
\]  

(4.24)

Let \( \mathcal{U} \oplus \mathcal{Y} = \mathcal{W} \) and \( \alpha^u_- \in T\mathcal{U}, \xi^+_y \in T\mathcal{Y} \) tangent vector fields restricted to \( \mathcal{U} \subset \mathcal{W}, \mathcal{Y} \subset \mathcal{W} \) respectively. An equivalent quotient for the adjoint sub operators \( \Gamma : \mathcal{U} \to \mathcal{Y} \) and \( \Gamma^\dagger : \mathcal{Y}^* \to \mathcal{U}^* \) is expressed by
Remark 4.45. By Theorem 4.44 (3), the set of eigenforms of a representation of the linear map

\[ A \]

Consider the submanifold \( B \) defined, see also Definition 4.18 and Remarks 4.17 and 4.19.

The following can be asserted:

1. The associated Shape operator (4.19) is given by \( A^\mathfrak{B}_\eta(\alpha_-) = \tilde{\Gamma}_i^j \tilde{\Gamma}^* \alpha_-, \alpha_- \in T_\mathfrak{B}^- \), \( \eta \in (T_\mathfrak{B}^-)\perp \) such that \( I_\mathfrak{B}(\alpha_-, \alpha_-) = S_r^* (\hat{w}^0, r_r) \) and \( II_\mathfrak{B}(\alpha_-, \alpha_-) = \langle A^\mathfrak{B}_\eta(\alpha_-), \alpha_- \rangle_{T_\mathfrak{B}^-} \).

2. A 1-form \( \beta_- \in T_\mathfrak{B}^- \), \( \langle \beta_- , \beta_- \rangle_{T_\mathfrak{B}^-} = 1 \) is solution to the eigenvalue problem associated to \( K(\beta_-) \) in (4.23) iff \( \beta_- \) is an eigenform of \( A^\mathfrak{B}_\eta \).

3. The set of eigenforms of \( A^\mathfrak{B}_\eta \), \( \{ \beta^i_- | i = 1, \ldots, \omega; \beta^i_- \in T_\mathfrak{B}^- \} \), defines an orthonormal coframe of \( T_\mathfrak{B}^- \). Furthermore \( T_\mathfrak{B}^- \) can be locally spanned by a partition of eigencodistributions \( W_1^* \oplus \cdots \oplus W_\omega^* \).

Remark 4.45. By Theorem 4.44 (3), the set of eigenforms of \( A^\mathfrak{B}_\eta \), \( \{ \beta^1_-, \beta^2_-, \ldots, \beta^\omega_- \} \) defines a coframe for \( T_w^\mathfrak{B}^+ \) at each \( w^0 \in \mathfrak{B} \) and by dualization

\[ \left\langle \beta^i_-, \xi^j_+ \right\rangle_{T_\mathfrak{B}^- \times T_\mathfrak{B}^+} = \delta^i_j, \quad i, j = 1, \ldots, \omega, \]

an equivalent frame \( \{ \zeta^1_+, \zeta^2_+, \ldots, \zeta^\omega_+ \} \) for the dual space \( T_w^\mathfrak{B}^+ \) is automatically defined, see also Definition 4.18 and Remarks 4.17 and 4.19.

Consider the submanifold \( \mathfrak{B}^+ \subset t \times \mathcal{W} \) and a frame \( \{ \zeta^1_+(p), \zeta^2_+(p), \ldots, \zeta^\omega_+(p) \} \) for its tangent space \( T_p^\mathfrak{B}^+ \). Furthermore let the dual submanifold \( \mathfrak{B}^- \subset \tau \times \mathcal{W} \) with a coframe \( \{ \beta^1_- , \beta^2_- , \ldots, \beta^\omega_- \} \) for its cotangent space \( T_\mathfrak{B}^- \). In view of the past \( g_{\mathfrak{B}^+} = g_{ij}^- \partial_i \otimes \partial_j \) and future metrics \( g_{\mathfrak{B}^-} = g_{ij}^+ dw^i \otimes dw^j \) defined after Assumption 4.27, dualization of \( \zeta^+ \in T_w^\mathfrak{B}^+ \) into \( \beta_- \in T_\mathfrak{B}^- \) can be performed by \( \beta^i_- = \sum_\omega a^i_j \beta^j_- \) and vice versa. The matrix representation of the linear map \( A^\mathfrak{B}_\eta : \mathfrak{T} \mathfrak{B} \rightarrow T \mathfrak{B} \) in (4.19) be denoted by \( [a^i_j] \)

Such that \( A^\mathfrak{B}_\eta (\beta^i_-) = \sum_\omega a^i_j \beta^j_- \), furthermore the metric (4.20) has the metric tensor \( G = [g^i_j] \) and let us express the associated tensor of (4.21) by \( Q = [g_{ij}^+] \).

Proposition 4.46. Under Assumption 4.21 and 4.27, \( A^\mathfrak{B}_\eta = Q G^{-1} \) and \( K(\beta_-) = \det Q / \det G, \forall \beta_- \in T_w^\mathfrak{B}^- \).
4.3 Curvature of the behavior and balancing

The issue of trajectory approximation is closely related to the possibility of projecting the components of the vector field of a trajectory on each Hilbert submanifold on the orthogonal (co-) frames in Remark 4.45. The following result provides a local orthogonal projection on the Hilbert manifold structure for the behavior.

**Proposition 4.47.** Consider orthonormal frames s.t.

\[ \text{span}\{\zeta_1^+, \ldots, \zeta_\omega^+\} = T_w B^+, \]
\[ \text{span}\{\beta_1^-, \ldots, \beta_\omega^-\} = T_w B^-, \]

for the Hilbert manifolds \((B^-, \langle \cdot, \cdot \rangle_{T(B^-)})\), \((B^+, \langle \cdot, \cdot \rangle_{T(B^+)})\). Then in terms of such frames, the orthogonal projection of any vector field and 1-form on a submanifold \(B_{\text{red}} \subseteq B\), \(\dim \text{supp} B_{\text{red}} = \varpi \leq \omega\), can be expressed in local coordinates by

\[ \xi_{op}^+ = \sum_{i=1}^\varpi \langle \xi^+, \zeta_i^+ \rangle_{T(B^+)} \zeta_i^+, \quad \xi^+ \in T_w B^+ \]  \hfill (4.27)
\[ \alpha_{op}^- = \sum_{i=1}^\varpi \langle \alpha^-, \beta_i^- \rangle_{T(B^-)} \beta_i^-, \quad \alpha^- \in T_w B^- \]  \hfill (4.28)

The following theorem characterizes a decomposition of \(K(\alpha_-)\) (adaptation of the classical result due to Euler).

**Theorem 4.48 (Euler’s formula).** Let \((U, \langle \cdot, \cdot \rangle_{TU})\) be a Hilbert manifold and \(\text{supp} B_{\Sigma} = \mathcal{W}\), \(\dim \mathcal{W} = \omega\) let the principal curvature directions \(\{\beta_1^-, \beta_2^-, \ldots, \beta_\omega^-\}\) be an orthonormal frame of \(T_w B^-\) with principal curvatures \(\{\kappa_1(w), \ldots, \kappa_\omega(w)\}\). Then the normal curvature \(K(\alpha_-)\) in the direction \(\alpha_- \in T(B^-), \langle \alpha_-, \alpha_- \rangle_{T(B^-)} = 1\), is given by

\[ K(\alpha_-) = \sum_{i=1}^\omega \kappa_i(w) \langle \alpha_-, \beta_i^- \rangle_{T(B^-)}^2 \]
\[ = \sum_{i=1}^\omega \kappa_i(w) \cos^2(\theta_i), \quad w \in B^-; \alpha_-, \beta_i^- \in T(B^-) \]  \hfill (4.29)

where \(\theta_i\) is the angle between 1-forms \(\alpha_-\) and \(\beta_i^-\).

**Remark 4.49.** By Gauss Theorema Egregium, the total curvature \(\prod_i^\omega \kappa_i\) is an intrinsic quantity (i.e. preserved by local isometries) of the submanifold of external signals \((B, \langle \cdot, \cdot \rangle_{T(B)})\), \(B \subset U\). \(K(\zeta)\) provides a measure of the speed of
change of $\varrho(t) \in \mathcal{B}$ as the behavior of the dynamical system $\Sigma$ evolves on $W_t$ along the direction defined by $\zeta \in T\mathcal{B}$. The normal curvature $K(\zeta)$ can be uniquely characterized in terms of the associated linear self-adjoint operator $A_\eta^\mathcal{B} : T\mathcal{B} \rightarrow T\mathcal{B}$ in Proposition 4.44.

Alternatively for balanced reduction purposes, instead of characterizing $\mathcal{B}$ by its metrics it may be easier to characterize a self-adjoint operator $A_\eta^\mathcal{B}$, as it will be seen later in this Part I.

### 4.3.3 Nonlinear Schmidt decomposition

**Proposition 4.50 (Schmidt pair).** For the operator $\tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+$ defined in Eq. (4.3), consider the eigenvalue problem of Proposition 4.37(1) $\tilde{\Gamma}w_i^- = \sigma_i w_i^+(t)$ otherwise written in terms of its sub-operators $\Gamma$ and $\Gamma^\dagger$ as

\[
\Gamma \circ u_i^- = \sigma_i y_i^+(t), \\
\Gamma^\dagger \circ y_i^+ = \sigma_i u_i^-(t),
\]

for invariant behavioral trajectories $w_i(t) = (u_i(t), y_i(t))$, $w_i(t) \in \mathcal{B}$, $i = 1, \ldots, \omega$, $\lambda_i = \sigma_i^2 \in \mathbb{R}^1$. The following can be asserted:

1. The singular value problems associated to the operators $\tilde{\Gamma}^\dagger \circ \tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^- \quad \text{and} \quad \tilde{\Gamma} \circ \tilde{\Gamma}^\dagger : \mathcal{B}^+ \rightarrow \mathcal{B}^+$

\[
\tilde{\Gamma}^\dagger \circ \tilde{\Gamma} \circ w_i^- - \lambda_i w_i^- = 0, \\
\tilde{\Gamma} \circ \tilde{\Gamma}^\dagger \circ w_i^+ - \lambda_i w_i^+ = 0.
\]

have as solution the set of eigen-trajectories $w_i^+ \in \mathcal{B}^+$, $w_i^- \in \mathcal{B}^-$ with eigenvalues $\lambda_i$, for $i = 1, \ldots, \omega$.

2. The i-solutions at the (co-)tangent spaces $T_w\mathcal{B}^-$, $T_w\mathcal{B}^+$ of Eqs. (4.32)-(4.33) yield the i-components of the orthogonal (co-)frames $\{\beta_i^\dagger\}$, $\{\xi_i^+\}$ defined on Remark 4.45.

3. The tangent map of $\tilde{\Gamma}$ on Eq. (4.42) $\tilde{\Gamma}^* : T\mathcal{B}^- \rightarrow T\mathcal{B}^+$ admits the (Schmidt) decomposition:

\[
\Gamma^* \alpha_- = \sum_{i=1}^\epsilon \lambda_i \langle \alpha_-, \beta_i^\dagger \rangle_{T\mathcal{B}^-} \xi_i^+, \quad \alpha_- \in T\mathcal{B}^- \\
\Gamma^\dagger_\ast \xi^+ = \sum_{i=1}^\epsilon \lambda_i \langle \xi^+, \xi_i^+ \rangle_{T\mathcal{B}^+} \beta_i^-, \quad \xi^+ \in T\mathcal{B}^+
\]

for orthonormal (co-)frames for $T_w\mathcal{B}^+$ and $T_w\mathcal{B}^-$ defined by Remark 4.45.
4.3 Curvature of the behavior and balancing

Example 4.51. Return to the linear time-invariant behavioral $\mathcal{L}$-class. The solution to the eigenproblem $\tilde{\Gamma}(w) = \sigma w$, $w \in W$, $\sigma^2 = \lambda$, is the Schmidt pair $w = (u(t), \hat{y}(\tau))$. Furthermore, the principal directions and normal curvatures are solution to the eigenproblem $\tilde{\Gamma}^\dagger \tilde{\Gamma}(w) = \lambda w$, $w \in W$, see e.g. [62, 211, 174].

Remark 4.52. The Hankel operator $\Gamma u : L_2(-\infty, 0] \to L_2[0, \infty)$ is a particular case of the behavioral operator under the following additional assumptions:

1. The past induced metric $g_{\mathcal{B}^-}$ on $\mathcal{B}^- \subset \mathcal{W}^\tau$ on the semi-interval $(-\infty, 0]$, depends only on the inputs, $\mathcal{U} \subseteq \mathcal{B}^-$.  
2. The future induced metric $g_{\mathcal{B}^+}$ on $\mathcal{B}^+ \subset \mathcal{W}^t$ on the semi-interval $[0, \infty)$, depends only on the outputs, $\mathcal{Y} \subseteq \mathcal{B}^+$, for $u \overset{\text{def}}{=} 0$, $u \in \mathcal{U} \subseteq \mathcal{B}^+$.  
3. System (4.1) is asymptotically stable.

From Proposition 4.50, the Schmidt pair for the Hankel operator follows from Proposition 4.50 by defining $\mathcal{U} = \mathcal{W}^-$ and $\mathcal{Y} = \mathcal{W}^+$ implying $u_i(t) \in \mathcal{B}^-$ and $y_i(t) \in \mathcal{B}^+$.

An alternative theory using different inner products for nonlinear Schmidt pairs for Hankel-based balancing has been discussed in [61].

4.3.4 Separability and eigenfunction decomposition

Up to now in this section we have shown that along the set of points where the curvature of the behavior $\mathcal{B}$ attains stationary values $\kappa_i$, there is a set of cotangent eigenforms derived from the Shape operator that spans the cotangent space. With this approach, we could show that the behavioral operator and the behavioral trajectories are projected on a set of orthogonal submanifolds. This section is concluded with the dual problem: we focus on a decomposition of the storage functionals into a set of orthogonal eigenfunctions on the space of integral functionals, leading us to the generalization of the concept known in linear balancing theory as a principal axis-balanced realization. For nonlinear systems, principal axis-balancing has been discussed previously in [44] on a different theoretical context. In the context of this dissertation such generalization is referred as principal frame-balancing. This name is more appropriate since such frame consist precisely of the set of principal directions presented in Section 4.3.2.

Separability of integral functions is a concept useful to understand the structure of a principal frame-balanced realization and for the factorization of the supporting submanifold as a previous step towards dimensional reduction. The following definitions are required.
Definition 4.53 (Separability). Let \( S(w) : W \to \mathbb{R} \) be a sufficiently smooth function supported by an \( \omega \)-dimensional \( C^\infty \)-manifold \( W \) on the metric space \((W, g_W)\). Such \( S(w) \) is said to be partially separated if it can be expressed as \( S(w) = \sum_{i=1}^{m} S_i(w, c^i) \) where \( m < \omega \) and each function \( S_i(w, c^i) \) depends on a disjoint subset of the \( \omega \)-vector \( w \) and a separation constant \( c^i \in \mathbb{R}^1 \). Furthermore, it is said to be completely separated if in such frame \( S(w) \) can be expressed as

\[
S(w) \overset{\text{def}}{=} \sum_{i=1}^{\omega} S_i(w^i, c^i) \tag{4.36}
\]

where \( S_i(w^i, c^i) \) denotes the (functionally independent) \( i^{\text{th}} \)-function depending on a single \( i^{\text{th}} \)-external variable and a separation constant \( c^i \in \mathbb{R}^1 \). Finally the function is said to be orthogonally separated if it is separated and the (Riemannian) tensor metric \( g_{ij} \) is diagonal, see e.g. [14].

Proposition 4.54. Let the smooth function \( S(w) \) be on a Riemannian manifold \((W, g_W)\).

1. The necessary and sufficient conditions for \( S(w) \) to be orthogonally separated are that the metric tensor \([g^{ij}]\) is diagonal and the matrix of the Hessian \([S]_{xx}\) has smooth functions along its diagonal and zero elsewhere:

\[
\nabla_w^2 S(w) = \text{diag}\{ \partial_w^2 S_i(w^i), i = 1, \ldots, \omega \} . \tag{4.37}
\]

2. The separation property is invariant under separated transformations, i.e. those whose Jacobian is diagonal.

Proofs of 1) and 2) in of Proposition 4.54 are omitted. See references in [104] and [13]. Associated to each \( \text{supp}(S_i(w)) \) there is an (open) complementary set \( \{ \eta \in W | \eta \notin \text{supp}(S_i(w^i)) \} \). Define the invariant set \( \mathcal{N} \) of a function \( S(w) : W \to \mathbb{R} \) as the largest submanifold in \( W \) such that for all \( \eta \in \mathcal{N} \subset W \), \( S(w + \eta) = S(w), w \in W \). This is formalized as follows:

Definition 4.55 (External orthogonal web). Let \((W, g)\) be a Riemannian manifold. The family \( \{ \mathcal{N}_i \} \overset{\text{def}}{=} \{ \mathcal{N}_1, \ldots, \mathcal{N}_\omega \} \) of \( \omega \)-orthogonal foliations of hyper surfaces –i.e. \((\omega - 1)\)-dimensional submanifolds–, defined on all \( W \) except a singular set \( \psi \), is called an orthogonal web, [13]. A parameterization of an orthogonal web consist of a set of functions \( \{ S_i(w^i) \} = \{ S_1(w^1), \ldots, S_\omega(w^\omega) \} \) on \( W \) s.t. \( dS_i(w)|_{\mathcal{N}_i} = 0 \) and \( dS_i(w)|_{\mathcal{N}_j} \neq 0 \), \( \forall i \neq j \). A parameterization \( \{ S_i \} \) adapted to the orthogonal web on \( W \) defines an orthogonal frame for \( W \), [14].
The following definition, associated to the classical topic of orthogonal separability in dynamical systems [104], is related to the separability of the metric functions imposed in Assumption 4.27, resulting in the following balancing and principal frame-balancing conditions:

**Definition 4.56 (External balancing conditions).** A realization of $\Sigma$ satisfying Assumption 4.29 is said to be balanced if it is such that

$$\left[ \nabla^2_{\hat{w}} S^*_r(\hat{w}, r_r) \right]^{-1} = \nabla^2_{\hat{w}} S_a(w, r_a).$$

(4.38)

Furthermore, if in such frame

$$\left[ \partial^2_{\hat{w}^i} S^*_r(\hat{w}^i, r_r) \right]^{-1} = \partial^2_{\hat{w}^i} S_a(w^i, r_a), \quad i = 1, \ldots, \omega$$

(4.39)

then such realization is said to be principal frame-balanced.

A comparison of these definitions with the concept of principal axis-balancing presented previously by [44] is postponed to Part II where appropriate concepts are developed. Nevertheless, the rationale behind Definition 4.56 is the following: by condition (4.38) every past trajectory has a unique associated future trajectory and by condition (4.37) in Proposition 4.54 expressed otherwise as condition (4.39), both functionals $S^*_r(\hat{w}^0, r_r)$ and $S_a(w^0, r_a)$ defined by Eqs. (4.5)-(4.6) are orthogonally separated. These arguments are consequence of the following results:

**Proposition 4.57.** Let the system realization $\Sigma$ be supported by the Hilbert manifold structure of Proposition 4.28 satisfying Assumption 4.27, namely, with past and future metrics defined by the storage functions $S^*_r(\hat{w}^0, r_r) \in C^\infty(W^*)$ and $S_a(w^0, r_a) \in C^\infty(W)$ from Eqs. (4.5)-(4.6). Furthermore, assume that the principal frame-balancing conditions (4.38)-(4.39) are satisfied. The following is asserted:

1. The orthogonal web for $S^*_r(\hat{w}^0, r_r)$ is given by the set of annihilators $\{\Delta_i\}$ of the distributions $\Delta_i = \text{span}\{\zeta_i^+\}$. Furthermore the orthogonal web for $S_a(w^0, r_a)$ is given by the set of annihilators $\{\Omega_i\}$ of the codistributions $\Omega_i = \text{span}\{\beta_i^-\}$.

2. A parameterization of the orthogonal webs on $B^-$ and $B^+$ is given by the components $\{S^i_r\}$ and $\{S^i_a\}$, $i = 1, \ldots, \omega$ of each separated metric. Moreover, each component $S^i_r$ (resp. $S^i_a$) is an integral invariant of the trajectories on the associated (co-) distribution $\Delta_i^\perp$ (resp. $\Omega_i^\perp$).

Although it is not shown, it is apparent that the family of system realizations $\Sigma$ satisfying condition (4.39) is unique up to separated transformations.
Lemma 4.58. The functionals \( S^*_r(w^0, r_r) \) and \( S_a(w^0, r_a) \) in Eqs. (4.5)-(4.6) are well defined Riemannian metrics for the Hilbert manifolds \((\mathcal{B}^{-}, \langle \cdot, \cdot \rangle_{T\mathcal{B}^{-}})\), \((\mathcal{B}^{+}, \langle \cdot, \cdot \rangle_{T\mathcal{B}^{+}})\).

In the following section orthogonal separability is used for classification of behavioral subsystems.

4.4 Geometric balanced reduction

In this section the method of reduction of dynamical systems in manifolds is briefly discussed using concepts from the behavioral approach. Given a system \( \Sigma = (t, \mathcal{W}, \mathcal{B}) \) whose behavioral trajectories lie on a \( \omega \)-dimensional manifold \( \mathcal{W} \), the problem of behavioral reduction consists in the synthesis of another system \( \Sigma_{\text{red}} = (t, \mathcal{V}, \mathcal{B}_{\text{red}}) \) whose behavior evolves on an \( r \)-dimensional submanifold \( \mathcal{V} \subset \mathcal{W} \), \( r < \omega \). If in such submanifold \( \mathcal{V} \subset \mathcal{W} \) the external behavior is the same, \( \mathcal{B} = \mathcal{B}_{\text{red}} \) and the minimal dimension where this condition is satisfied is \( r \), the system realization \( \Sigma_{\text{red}} \) is said to have a minimal support. We would write \( \Sigma \equiv \Sigma_{\text{red}} \mod \mathcal{B}_\Sigma \). Further reduction of a realization minimally supported implies necessarily an approximation problem on a metric space in order to measure the closeness of both behavioral trajectories \( \mathcal{B} \) and \( \mathcal{B}_{\text{red}} \).

In any case, a subset of the manifold \( \mathcal{W} \) supporting \( \mathcal{B} \) is said to be a reduced support for \( \mathcal{B} \) if there is a regular equivalence relation characterizing such subset as a submanifold \( \mathcal{V} \subset \mathcal{W} \) allowing thereby for a partition or factorization of \( \mathcal{W} \) along with a restriction of \( \mathcal{W} \) to \( \mathcal{V} \). The projection from the original space into the isomorphic reduced submanifold serves to find the reduced system [33, 155].

This reduction procedure is formalized throughout this section.

4.4.1 Equivalence of behaviors

By classification of a behavioral system we refer to the use of an appropriate equivalence relation in order to characterize the equivalence classes of behavioral subsystems. Although the set of differential-algebraic Eqs. (4.1)-(4.2) defines naturally an equivalent relation for behavioral trajectories \( w(t) \in \mathcal{B} \), there are not enough structural properties in order to assert on their properties of invariance, regularity or uniqueness. Thereby a behavioral operator with controlled inputs defined by Eqs. (4.7) and (4.6) has been defined and their invariant properties have been found in previous sections. There remains to define the equivalence relation \( w(t) \mathcal{R} w(t), t \in \mathcal{T} \) for behavioral trajectories
4.4 Geometric balanced reduction

In view of Proposition 4.57(2) their principal curvatures $\kappa_i$ serve to define classes of equivalence of $\mathcal{B}$. In this subsection we use this facts in order to assert on the properties of a reduced order system inherited by the original system whose $\mathcal{C}$-class is the property of being linear, affine nonlinear, etc.

**Definition 4.59 (Isomorphic systems in $(\mathcal{C}, \phi)$).** A $\mathcal{C}$-class $(\mathcal{C}, \phi)$ is defined by a set $\mathcal{C}$ of dynamical models with a particular structure and characteristic properties, such that $\mathcal{C}$ is closed under composition and factorization, along with a structure-preserving isomorphism $\phi$, such that closure is exhibited when $\phi$ acts as a binary operator in the set $\mathcal{C}$.

Thus, $\Sigma_1, \Sigma_2 \in \mathcal{C}$, are said to be isomorphic if they preserve the same defining properties of $\mathcal{C}$ and there is an isomorphism $\phi$ transforming one system to the other.

**Definition 4.60 (Equivalent realizations).** A set of admissible realizations $\mathcal{F}_B = \{\Sigma_i | i = 1, \ldots \}$ of the behavior $\mathcal{B}$ is said to define a family $(\mathcal{F}_B, \phi)$ if any two realizations $\Sigma_i$ and $\Sigma_j$ in $\mathcal{F}_B$ are isomorphic for an isomorphism $\phi$ s.t. $\phi: \Sigma_i \to \Sigma_j$, $\phi^{-1}: \Sigma_j \to \Sigma_i$. In such case we say $\Sigma_i$ is equivalent to $\Sigma_j$ modulo $\mathcal{B}$.

The invariants of the isometric isomorphism $\Gamma$ (the behavioral operator) serve to provide a membership criterion to the family $(\mathcal{F}_B, \phi)$:

**Definition 4.61 (Approximate realization).** Consider a family of equivalent realizations $(\mathcal{F}_B, \phi)$ with behavioral operator $\Gamma$. The set $(\mathcal{F}_B^\text{red}, \phi')$ is said to be a family of isomorphic approximate realizations of $\mathcal{B}$ if there is a restriction to a submanifold $\mathcal{V} \subseteq \mathcal{W}$ such that both families are $\mathcal{B}^\text{red}$-equivalent and the set of invariants of $\Gamma'$ is a subset of the set of invariants of $\Gamma$.

The latter definition provides a measure of error approximation in terms of the degree similarity of the set of invariants. If a dynamical system $\Sigma$ is a realization of the behavior $\mathcal{B}$, based on Definition 4.61 an associated isometric isomorphism $\tilde{\Gamma}$ determines all the invariants in order to assess on its equivalence.

### 4.4.2 Reduction of the external balanced realization

Model reduction of dynamical systems is an operation performed within the members of the $\mathcal{C}$-class. Typical steps for every reduction method include determination of an isomorphic system, factorization into a quotient system, and restriction into a subsystem, [33]. With the tools developed so far, the classical method of order reduction in quotient manifolds [155, 33] adapted
to the behavioral balanced reduction method, can be presented in algorithmic form as follows:

**Proposition 4.62 (Geometric balanced reduction method).** Let Assumptions 4.21, 4.23 and 4.27 be satisfied for the \((\mathcal{C}, \phi)\)-system \(\Sigma = (t, \mathcal{W}, \mathcal{B})\) whose behavioral trajectories lie on a \(\omega\)-dimensional manifold \(\mathcal{W}\), whose \(S_a : \mathbb{R}^\omega \to \mathbb{R}^1\) and \(S_r : \mathbb{R}^\omega \to \mathbb{R}^1\) exist and are smooth on (the compact) \(\mathcal{W}\). An equivalent \((\mathcal{C}, \phi)\)-system \(\Sigma_{\text{red}} = (t, \mathcal{V}, \mathcal{B}_{\text{red}})\) whose behavior evolves on a reduced-order \(r\)-dimensional submanifold \(\mathcal{V} \subset \mathcal{W}\), \(r < \omega\) can be synthesized by the following steps:

1. **Semigroup characterization:** The maximal semi flow \(\{\Phi^t(w)\}_{t \geq 0}\) and the maximal semi flow \(\{\Theta^\tau(\hat{w})\}_{\tau \geq 0}\) generated by the vector fields \(\xi_G = -\nabla_w S_a\) and \(\xi_J = -\nabla_{\hat{w}} S_r\) are a positive and negative semigroups respectively; meaning that for all \(t \in t^+\), \(G^t(w)\) (resp. for all \(\tau \in \tau^-\), \(J^\tau(w)\)) are defined on all \(\mathcal{W}\).

2. **Group extension:** Whenever \([\Theta^\tau(\hat{w}^-(\tau))]^{-1} = \Phi^t(w^+(t))\) and the balancing condition \([G^\tau(\hat{w}(\tau))]^{-1} = Q^t(w(t))\) are regular on a subset \(\mathcal{W}_m \subseteq \mathcal{W}\), there is a group extension, regular on \(\mathcal{W}_m\), for the semigroups \(\{\Phi^t(w)\}_{t \geq 0}\) and \(\{\Theta^\tau(\hat{w})\}_{\tau \geq 0}\). If the subset \(\mathcal{W}_m\) is a submanifold s.t. \(\dim \mathcal{W}_m = \dim \mathcal{W}\) it is a minimal support.

3. **Equivalence:** The local group extension of diffeomorphisms \(\{\Phi^t(w)\}\) acting on the smooth manifold \(\mathcal{W}\) defines an equivalence relation with the smooth map \(\pi : \mathcal{W} \to \mathcal{W}/\Phi\) (the natural projection) which associates to each \(w \in \mathcal{W}\) its equivalence class \(\pi(w) \in \mathcal{W}/\Phi\) (i.e. the set of orbits of \(\Phi\) passing by \(w\) as the set of equivalence classes \(\mathcal{W}/\Phi\)).

4. **Factorization:** Whenever the frame-balancing condition (4.39) is satisfied, \(\Phi\) can be factorized by the group invariant orbits of a family of subgroups \(\Phi_i\). Furthermore \(\mathcal{W}\) can be arbitrary partitioned in two invariant submanifolds \(\mathcal{W} = \mathcal{W}_a \oplus \mathcal{W}_b\) where the submanifolds on \(\mathcal{W}_a\) have the associated higher principal curvatures \(\kappa_i, i = 1, \ldots, r\) and the remaining submanifolds are on \(\mathcal{W}_b\).

5. **Restriction:** The natural projection \(\pi_b : \mathcal{W} \to \mathcal{W}/\Phi_b\) defines a restriction from \(\mathcal{W}\) to \(\mathcal{V}\) and a quotient system such that \(\mathcal{B}\) is equivalent to \(\mathcal{B}_{\text{red}}\) within a submanifold \(\mathcal{V} \subset \mathcal{W}\).

**Remark 4.63 (Mean curvature approximation of \(\mathcal{B}\)).** Based on Definition 4.42, the difference between the Gaussian mean curvature of the original system and the corresponding of the reduced system is given by: \(\sum_{i=1}^\omega \kappa_i - \sum_{i=1}^r \kappa_i = \sum_{i=r+1}^\omega \kappa_i\) providing from Remark 4.49 the justification to interpret the problem.
of nonlinear balanced reduction as a problem of approximation of the Gaussian mean curvature of the behavior $\mathcal{B}$.

### 4.4.3 Bounds on trajectory error approximation

The objective of this subsection lies in finding a bound on the error incurred when the behavior $\mathcal{B}$ is approximately generated by a reduced-order system. Since the metrics $\langle \cdot, \cdot \rangle_{T\mathcal{B}}$ in Table 4.2 are well defined (namely satisfy properties of symmetry, separation and triangle inequality) in the metric space of trajectories $(\mathcal{B}, \langle \cdot, \cdot \rangle_{T\mathcal{B}})$, the distance of two trajectories is naturally measured in terms of its (induced) metric.

**Definition 4.64 (Distance between trajectories on $\mathcal{B}$).** Let $\sigma(t), \rho(t) \in \mathcal{B}$ be on $(\mathcal{B}, \langle \cdot, \cdot \rangle_{T\mathcal{B}})$. Their distance is given by their metrics $g_{\mathcal{B}}(\sigma(t), \rho(t))$ defined in terms of the natural norm

$$\|\sigma(t) - \rho(t)\|_{\mathcal{B}} = \sqrt{\langle \sigma(t) - \rho(t), \sigma(t) - \rho(t) \rangle_{T\mathcal{B}}}.$$  

**Definition 4.65 ($L_2$-gain of $\tilde{\Gamma}$).** The $L_2$-gain of the behavioral operator $\tilde{\Gamma}$ is the norm defined by the largest principal curvature (singular value) $\bar{\kappa}(\tilde{\Gamma})$.

Assume that two systems $\Sigma = (t, \mathcal{W}, \mathcal{B})$ and $\Sigma_{\text{red}} = (t, \mathcal{W}_{\text{red}}, \mathcal{B}_{\text{red}})$ are compared by the invariants of their respective behavioral operators. Then for the same unitary past behavior, the distance of their future behavioral trajectories determines the error of the reduced system.

**Proposition 4.66.** With the same Assumptions of Proposition 4.62, let the minimally supported $(\mathcal{C}, \phi)$-system realization $\Sigma = (t, \mathcal{W}, \mathcal{B})$ with behavioral operator $\tilde{\Gamma}$ be approximated by a reduced-order $(\mathcal{C}, \phi)$-system $\Sigma_r = (t, \mathcal{W}_{\text{red}}, \mathcal{B}_{\text{red}})$ with behavioral operator $\tilde{\Gamma}_{\text{red}}$ s.t. $\mathcal{B}_{\text{red}} \subset \mathcal{B}$, determined by the procedure described in Proposition 4.62. The local lower bound on the $L_2$-norm on the approximation error is given by

$$\|\tilde{\Gamma} - \tilde{\Gamma}_{\text{red}}\|_{\mathcal{B}^+} \geq \kappa_{r+1}(w).$$  

(4.40)

### 4.5 Conclusions

The preeminence of the geometric view to the problem of balanced reduction for nonlinear dissipative systems was shown in this Part I, by reinterpreting it as a problem of characterization of the invariants of an isometric operator: the behavioral operator. Furthermore, using Curvature Theory, the invariants
of such operator are shown to be associated to an orthogonal frame of tangent vector fields and concepts like shape operator, orthogonal projection, Schmidt decomposition, and balanced reduction were part of the discussion. One remarkable result of this approach consists in showing that the structure of the nonlinear balancing problems admit an orthogonal decomposition into separable invariant functions for nonlinear operators, which can be considered the nonlinear equivalent to the singular value decomposition for linear operators. Such operator decomposition consists of an orthogonal frame of tangent vector fields, a set of normalized separable functions and an orthogonal coframe of cotangent covector fields.

In Part I, the concept of balancing was shown to be actually a condition for the dualization of two Hilbert manifolds. With the introduction of additional tools, namely adjoint and semigroup Lie operators, etc., the Part II of this paper provides a sharp geometric characterization of internal balancing as a condition for group extension of Lie semigroups. Furthermore, the storage functions are shown to be generating functions of Lie group actions and issues of minimality of balanced internal realizations are also discussed. To conclude, numerical methods to find the invariant trajectories of the behavioral operator can be adapted from the numerical algorithms presented in [126].

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4.6 Appendix

Proof of Proposition 4.15: In the differential geometric framework, duality of these spaces follows from the known contraction operation $i_\xi\alpha$ between vector fields and differential 1-forms. Since the inner products in Table 4.1 are positive definite, symmetric, bilinear forms on (the space of vector fields) $T_p^\mathcal{W}$ (on the space of 1-forms $T_p^*\mathcal{W}$) such products are admissible. The remaining details can be seen in [160] and in the more general proof of Proposition 5, pg. 55 assuming $\mathcal{W}$ an infinite-dimensional Sobolev space (without differential geometry) in [226]. From Definition 4.13, the duality product $\langle \xi, \alpha \rangle_{T\mathcal{W} \times T^*\mathcal{W}}$ is well defined since for a $\phi \in \mathcal{W}^*$ the dual of $d\phi \in T^*\mathcal{W}$ is given by (the gradient) $\nabla \phi \in T\mathcal{W}$ satisfying $\phi(w) = \langle \nabla \phi, \dot{w} \rangle_{T\mathcal{W}}, \dot{w} \in T\mathcal{W}$. Coordinate invariance of $\langle \xi, \alpha \rangle_{T\mathcal{W} \times T^*\mathcal{W}}$ follows from known facts of differential geometry (see e.g. [33]).

Proof of Proposition 4.28: Since by construction, the inner products satisfy Assumption 4.21 on dissipativity, semi positive definiteness, symmetry and boundedness are satisfied and such inner products are admissible. The remaining Hilbert manifold properties are inherited to this structure from \((W, \langle \cdot, \cdot \rangle_{T^*W})\) and \((W^*, \langle \cdot, \cdot \rangle_{T^*W})\). Duality of \(\alpha_\cdot\) with \(\xi^+\) is identified by the abstract duality pairing \(\langle \xi^+, \alpha_- \rangle_{TB^- \times TB^+}\) for an isometry \(\tilde{\Gamma}\) s.t. \(\xi^+ = \tilde{\Gamma}^* \alpha_\cdot\), with \(\xi^+\) dual to \(\alpha_+\), which according to Definition 4.13 must satisfy \(\langle \alpha_-, \alpha_- \rangle_{TB^-} = \langle \xi^+, \alpha_- \rangle_{TB^- \times TB^+} = \langle \xi^+, \tilde{\Gamma}^* \alpha_- \rangle_{TB^+}\), which is equivalent to \(\|\alpha_-\|_{TB^-} = \|\xi^+\|_{TB^+}\).

Proof of Proposition 4.35: 1) This result is a structural consequence of the Definition 4.13 for the duality pairing \(\langle \xi^+, \alpha_- \rangle_{TB^- \times TB^+}\). By Definition 4.13, duality of \(\alpha_-\) with \(\xi^+\) requires the preservation of the relationship

\[
\langle \alpha_-, \alpha_- \rangle_{TB^-} = \langle \xi^+, \alpha_- \rangle_{TB^- \times TB^+} = \langle \xi^+, \tilde{\Gamma}^* \alpha_- \rangle_{TB^+}, \quad \forall \alpha_- \in B^-, \quad \xi^+ \in B^+
\]

(4.41)

for inner products defined as in Proposition 4.28, for an isometric isomorphism \(\tilde{\Gamma}: B^- \rightarrow B^+\) defined by Eq. (4.41), which from Definition 4.12 is seen to satisfy commutativity in diagram (4.15) with the following structure at \(TB^-\)

\[
\begin{bmatrix}
\xi^+(w) \\
\xi^-(w)
\end{bmatrix}_{TB^+} = \begin{bmatrix}
0 & \Gamma(w)|^* \\
\Gamma(w)| & 0
\end{bmatrix}
\begin{bmatrix}
\xi^+(w) \\
\xi^-(w)
\end{bmatrix}_{TB^-}
\]

(4.42)

where the map \(\Gamma: U \rightarrow Y\) has an adjoint map \(\Gamma^*: Y^* \rightarrow U^*\).

2) Define by \(\tilde{\Gamma}^*: B^+ \rightarrow B^-\) an associated operator to the behavioral operator in Definition 4.8 with the following structure at \(TB^-\)

\[
\begin{bmatrix}
\alpha^+(\tilde{w}) \\
\alpha^-(\tilde{w})
\end{bmatrix}_{TB^-} = \begin{bmatrix}
0 & \Gamma^*(\tilde{w})|^* \\
\Gamma^*(\tilde{w})| & 0
\end{bmatrix}
\begin{bmatrix}
\alpha^+(\tilde{w}) \\
\alpha^-(\tilde{w})
\end{bmatrix}_{TB^+}
\]

(4.43)

such operator is such that \(\langle \xi, \Gamma_* \zeta \rangle_{TB^-} = \langle \Gamma_* \xi, \zeta \rangle_{TB^+}\), \(\xi, \zeta \in T^+\) satisfying Definition 4.33 and therefore is selfadjoint. Since \(\tilde{\Gamma}\) is isometric, it satisfies \(\langle \xi^-, \zeta^- \rangle_{TB^-} = \langle \tilde{\Gamma}_* \xi^-, \tilde{\Gamma}_* \zeta^- \rangle_{TB^+} \equiv \langle \xi^+, \zeta^+ \rangle_{TB^+}, \quad \xi^-, \zeta^- \in T^-, \quad \xi^+, \zeta^+ \in T^+\). Let the map \(\tilde{\Gamma}^*: B^+ \rightarrow B^-\) satisfy \(\langle \xi^+, \zeta^+ \rangle_{TB^+} = \langle \tilde{\Gamma}^* \xi^+, \tilde{\Gamma}^* \zeta^+ \rangle_{TB^-}\) hence isometric, with \(\xi^+ \equiv \tilde{\Gamma}^* \xi^+\). Then Definition 4.32 is verified since \(\langle \tilde{\Gamma}^* \xi^+, \zeta^- \rangle_{TB^-} = \langle \xi^+, \tilde{\Gamma}_* \zeta^- \rangle_{TB^+}\).

3) Since we may write \(\langle \xi^-, \zeta^- \rangle_{TB^-} = \langle \tilde{\Gamma}^* \circ \tilde{\Gamma}_* \xi^-, \zeta^- \rangle_{TB^-} = \langle \xi^-, \tilde{\Gamma}^* \circ \tilde{\Gamma}_* \zeta^- \rangle_{TB^-}\) and \(\langle \xi^+, \zeta^+ \rangle_{TB^+} = \langle \xi^+, \tilde{\Gamma}^* \circ \tilde{\Gamma}_* \zeta^+ \rangle_{TB^+}\),

\[
\langle \xi^-, \zeta^- \rangle_{TB^-} = \langle \xi^+, \tilde{\Gamma}^* \circ \tilde{\Gamma}_* \zeta^+ \rangle_{TB^+}.
\]
the operators $\tilde{\Gamma}^\dagger \circ \tilde{\Gamma}(\cdot)$ and $\tilde{\Gamma} \circ \tilde{\Gamma}^\dagger(\cdot)$ are self-adjoint on $\mathfrak{B}^-$ and $\mathfrak{B}^+$ respectively, satisfying Definition 4.33.

4) Direct comparison of operators $\tilde{\Gamma}$ and $\tilde{\Gamma}^\dagger$ in Eqs. (4.3) and (4.16) shows that both are equivalent and their tangent maps Eqs. (4.42) and (4.43) satisfy Definition 4.33.

5) Denote by $\mathcal{W}|_U$, $\mathcal{W}|_Y$ the restriction of a manifold $\mathcal{W} \supset U \oplus Y$ by their disjoint subsets and consider the restricted maps $\Gamma : \mathfrak{B}^-|_U \to \mathfrak{B}^+|_Y$, $\Gamma^\dagger : \mathfrak{B}^-|_y \to \mathfrak{B}^+|_u$. Since we may write $\langle \xi^-, \zeta^- \rangle_{T\mathfrak{B}^-|_U} = \langle (\Gamma^\dagger \circ \Gamma)^* \xi^-, \zeta^- \rangle_{T\mathfrak{B}^-|_U} = \langle \xi^-, (\Gamma^\dagger \circ \Gamma)^* \zeta^- \rangle_{T\mathfrak{B}^-|_U}$ and $\langle \xi^+, \zeta^+ \rangle_{T\mathfrak{B}^+|_Y} = \langle \xi^+, (\Gamma \circ \Gamma^\dagger)^* \zeta^+ \rangle_{T\mathfrak{B}^+|_Y}$, the operators $\Gamma^\dagger \circ \Gamma(\cdot)$ and $\Gamma \circ \Gamma^\dagger(\cdot)$ are self-adjoint on $\mathfrak{B}^-|_u$ and $\mathfrak{B}^+|_y$ respectively, satisfying Definition 4.33. 

\textbf{Proof of Proposition 4.37:} 1) Consider the map $\Upsilon^t : \mathbb{R}^1 \times \mathcal{W} \to \mathcal{W}$ for the first eigenproblem $\Upsilon(\varrho(t)) = \lambda \varrho(t)$ and some eigenvalue $\lambda$ along the path $\varrho(t) \in \mathcal{W}$. Such path is the integral trajectory of a vector field satisfying $\frac{d\varrho(t)}{dt} = \xi(\varrho(t))$. On the other side, time derivation at both sides of the equation defining the eigenvalue problem yields $\frac{\partial \Upsilon}{\partial x} \frac{d\varrho(t)}{dt} = \lambda \frac{d\varrho(t)}{dt}$ which is precisely $\Upsilon_*|_x \xi = \lambda \xi$ (a linear eigenvalue problem) and then the problem is well defined. The inverse implication is straightforward. Since a generator $\xi$ of $\Upsilon^t$ acts on differentiable functions as $\xi f = \frac{d}{dt} f(\Upsilon^t(x))|_{t=0}$ a similar reasoning follows for the proof of Proposition 4.37(2).

\textbf{Proof of Proposition 4.39:} 1) Evident from the differentiable structure of $\tilde{\Gamma}$.

2) Express the map $\tilde{\Gamma}_*$ as a matrix with components $G^j_i = [\partial w_i \tilde{\Gamma}_i]$ and rewrite the eigenproblem (4.18) for each entry as $\tilde{\omega}_i^j(0) + \int_0^\infty \sum_j G^j_i \beta_i^j(w^-(t)) \ d\mu(\tau)$ which defines a system of 1st-kind Fredholm equations (for generic definitions see [170]).

3) Using Proposition 4.37(1) the eigenvalue problem in terms of $T\mathfrak{B}$ with tangent map $\Gamma_* : T\mathfrak{B} \to T\mathfrak{B}$ yields the eigenproblem presented above with tangent eigenvector field $\xi_i \in T\mathfrak{B}$.

\textbf{Proof of Theorem 4.44:} 1) Since by Assumption 4.27, $\mathfrak{g}_{\mathfrak{B}^-}(w^-, 0) \overset{\text{def}}{=} S^*_a(\hat{w}^0, r_r) = \langle \alpha_-, \alpha_- \rangle_{T\mathfrak{B}^-}$, $\mathfrak{g}_{\mathfrak{B}^+}(w^+, 0) \overset{\text{def}}{=} S_a(w^0, r_a) = \langle \xi^+, \xi^+ \rangle_{T\mathfrak{B}^+}$ which by the isometric isomorphism $\xi^+ = \tilde{\Gamma}_* \alpha_-$ transforms the future metrics in terms of the past metrics by $\langle \xi^+, \xi^+ \rangle_{T\mathfrak{B}^+} = \langle \tilde{\Gamma}_* \alpha_-, \tilde{\Gamma}_* \alpha_- \rangle_{T\mathfrak{B}^-}$, and therefore $A^\mathfrak{B} = \tilde{\Gamma}_* \tilde{\Gamma}^*$. 

2) Since the quotient (4.23) is such that \( K(\varepsilon) = K(\alpha) \) for any scalar \( 0 \neq \varepsilon \in \mathbb{R}^1 \) and \( \alpha \in T_p \mathcal{B}^- \), this implies that any tangent 1-form candidate solution to the eigenvalue problem satisfies \( \langle \alpha, \alpha_\varepsilon \rangle_{T_p \mathcal{B}^-} = 1 \). Therefore denote by \( \beta_- \) the elements of a sphere \( \mathcal{S} \subset T \mathcal{B}^- \) defined by \( \mathcal{S} = \{ \beta_- \in T \mathcal{B}^- | \langle \beta_-, \beta_- \rangle_{T_p \mathcal{B}^-} = 1 \} \). By the Stone-Weierstrass Theorem (e.g. [226]) any continuous function supported on \( \mathcal{S} \) (being a compact subset) attains in there its maximum and minimum. Express the numerator of (4.23) by \( \phi : \mathcal{S} \rightarrow \mathbb{R}^1 \), i.e. \( \phi(\beta_-) = II_2(\beta_-) = \langle A^2_\eta \beta_-, \beta_- \rangle_{T_p \mathcal{B}^-} \). A point \( \beta_0 \in T \mathcal{B}^- \) is a critical point in \( \mathcal{S} \) if for any curve \( \varrho : [-1, 1] \rightarrow \mathcal{S}, \varrho(0) = \beta_0 \), satisfies \( d\phi(\varrho)/d\tau|_{\tau=0} = 0 \). Performing this operation yields,

\[
\frac{d}{d\tau} \langle A^\eta_\varrho(\tau), \varrho(\tau) \rangle_{T_p \mathcal{B}^-} \bigg|_{\tau=0} = \langle A^\eta_\varrho(0), \varrho(0) \rangle_{T_p \mathcal{B}^-} + \langle A^\eta_\varrho(0), \dot{\varrho}(0) \rangle_{T_p \mathcal{B}^-} = \langle A^\eta_\varrho(0), \beta_0 \rangle_{T_p \mathcal{B}^-} + \langle A^\eta_\beta_0, \dot{\varrho}(0) \rangle_{T_p \mathcal{B}^-} = 2 \langle A^\eta_\beta_0, \dot{\varrho}(0) \rangle_{T_p \mathcal{B}^-}
\]

being the last equality due to selfadjointness of \( A^\eta_\beta \). This result implies that \( \beta_0 \) is a critical point of \( \phi(\beta_-) \) on \( \mathcal{S} \) if \( A^\eta_\beta \) is orthogonal to any tangent vector field \( \dot{\varrho}(0) \in TS \). Due to the geometry of the sphere \( \mathcal{S} \), any tangent vector field on \( TS \) is orthogonal to a (possibly scaled) normal vector field on \( \mathcal{S} \). Thus if \( \beta_0 \) is an eigenform of \( A^\eta_\beta \) there is a \( \kappa \) scaling \( \beta_0 \) normal in \( \mathcal{S} \), implying \( \beta_- \) is a critical point of \( \phi \), concluding the proof.

3) By induction. Let \( \dim \mathcal{W} = \omega \) for \( \omega = 1 \) the result is trivial. Assume it is true for \( \omega = k \). Let \( \omega = k + 1 \), from Theorem 4.44(2), there exist at least a unitary 1-form \( \beta_1^\omega \in T_p \mathcal{B}^- \) of \( A^\omega_\eta \). By Remark 4.17, define locally an orthogonal complementary codistribution by \( W_\omega^* = \{ \alpha_- | \langle \beta_1^\omega, \alpha_- \rangle_{T_p \mathcal{B}^-} = 0, \alpha_- \in T_p \mathcal{B}^- \} \) then since \( \langle A^\eta_\alpha, \beta_1^\omega \rangle_{T_p \mathcal{B}^-} = \langle \alpha, A^\omega_\beta \beta_1^\omega \rangle_{T_p \mathcal{B}^-} = \langle \alpha, \lambda_1 \beta_1^\omega \rangle_{T_p \mathcal{B}^-} = \lambda_1 \langle \alpha, \beta_1^\omega \rangle_{T_p \mathcal{B}^-} = 0, \) locally \( W_1^* \) is an \( A^\omega_\beta \)-invariant eigencodistribution, i.e. \( A^\omega_\beta W_1^* \subseteq W_1^* \). It is easy to see that the restriction of \( A^\omega_\eta \) to \( W_1^* \), \( A^\omega_\eta \mid W_1^* \) is also self-adjoint. Since \( \dim W_1^* = \dim \mathcal{W} - 1 = k \) there are \( k \)-basis 1-forms \( \{ \beta_1^\omega, \ldots, \beta_k^\omega \} \) for \( W_1^* \) that are eigenforms of \( A^\omega_\eta \mid W_1^* \). Though, each eigenform of \( A^\omega_\eta \mid W_1^* \) must be an eigenform of \( A^\omega_\beta \). Conclude that the eigenforms of \( A^\omega_\beta \), \( \{ \beta_1^\omega, \beta_2^\omega, \ldots, \beta_k^\omega \} \in T_p \mathcal{B}^- \) must be an orthonormal basis for \( T_p \mathcal{B}^- \) at each \( p \in \mathcal{B} \) and thereby \( T_p \mathcal{B}^- \) can be partitioned locally in eigencodistributions as \( W_1^* \oplus \ldots \oplus W_\omega^* \).

**Proof of Proposition 4.46:** Each component of \( \mathcal{G} \) can be calculated from \( g_{ij} = \langle \beta_i, \beta_j \rangle_{T_p \mathcal{B}^-} \). Since \( \langle \beta_j, \eta \rangle_{T_p \mathcal{B}^-} = 0 \) for \( \beta_j \in T \mathcal{B}^- \), \( \eta \in (T \mathcal{B}^-)^\perp \), then after derivation \( \partial_\beta_i \langle \beta_j, \eta \rangle_{T_p \mathcal{B}^-} = \langle \partial_\beta_i \beta_j, \eta \rangle_{T_p \mathcal{B}^-} + \langle \beta_j, \partial_\beta_i \eta \rangle_{T_p \mathcal{B}^-} = 0 \).
yields \( \langle \partial_i, \partial_j, \eta \rangle_{T\mathcal{B}} = \langle \beta_j, -\partial_i, \eta \rangle_{T\mathcal{B}} = \langle \beta_j, -\nabla_i \beta_i \rangle_{T\mathcal{B}} \). This implies that

\[
II(\beta_i, \beta_j) := \langle A_{\eta}^\mathcal{B} \beta_i, \beta_j \rangle_{T\mathcal{B}} = \langle \beta_j, \beta_i, \eta \rangle_{T\mathcal{B}}.
\]

In consequence the components of \( \mathcal{Q} \) can be calculated from \( q_{ij} = \langle \partial_i, \partial_j, \eta \rangle_{T\mathcal{B}} \). Now since

\[
q_{ij} = \langle A_{\eta}^\mathcal{B} \beta_i, \beta_j \rangle_{T\mathcal{B}} = \left( \sum_{k=1}^{\omega} a_{ik} \beta_k \right)_{T\mathcal{B}} = \sum_{k=1}^{\omega} a_{ik} \beta_k = \sum_{k=1}^{\omega} a_{ik} g_{kj}
\]

(4.44)

conclude that \( \mathcal{Q} = A_{\eta}^\mathcal{B} \mathcal{G} \). Since \( \mathcal{G} \) is a positive definite matrix then it is invertible and multiplication by its inverse yields the desired result. By linear algebra arguments \( K(\beta) = \det \mathcal{Q}/\det \mathcal{G} \), is another expression for Gauss (total) curvature.

**Proof of Proposition 4.47:** Any vector field \( \xi^+ \in T_w \mathcal{B}^+ \) (1-form \( \alpha_- \in T_w \mathcal{B}^- \)) can be expressed in local coordinates by a combination of their (co-) frame elements \( \xi^+ = \sum_{i=1}^{\omega} a_i \xi_i^+ \) (resp. \( \alpha_- = \sum_{i=1}^{\omega} b_i \beta_i^- \)) for scalar functions \( a_i, b_i \in C^\infty(\mathcal{W}) \), see e.g. Chapter 8 in [156]. In particular, let \( \xi^+ \in T_w \mathcal{B}^+ \) have an orthogonal projection \( \xi_{op}^+ \in \mathcal{B}_{red}^+ \subseteq \mathcal{B}^+ \) on a submanifold satisfying \( T\mathcal{B}_{red}^+ \equiv \text{span}\{\xi_1^+, \xi_2^+, \ldots, \xi_{\omega}^+\}, \omega \leq \omega \). Then their difference \( \xi^+ - \xi_0^+ \) must be orthogonal to each element of the frame (by duality, see Remark 4.19), i.e. \( \langle \xi^+ - \xi_{op}^+, \xi_i^+ \rangle_{T\mathcal{B}^+} = 0, i = 1, \ldots, \omega \) where orthogonality can be defined in terms of the duality pairing \( \langle \xi^+, \alpha_- \rangle_{T\mathcal{B}^+ \times T\mathcal{B}^-} \), see Remark 4.17 and Definition 4.18. Express \( \xi_{op}^+ \) by the \( \omega \)-elements of the frame. The \( a_i \) satisfying \( \langle \xi^+ - \sum_{i=1}^{\omega} a_i \xi_i^+, \xi_i^+ \rangle_{T\mathcal{B}^+} = 0 \) is equal to the projection of each vector field on the elements of the frame, i.e. \( a_i = \langle \xi^+, \xi_i^+ \rangle_{T\mathcal{B}^+} \). An equivalent reasoning for \( \alpha_- \in T_w \mathcal{B}^- \), results in \( b_i = \langle \alpha_-, \beta_i^- \rangle_{T\mathcal{B}^-} \), concluding the proof.

**Proof of Theorem 4.48:** Under the orthonormal frame \( \{\beta_i^i\} \) of Proposition 4.47 any \( \alpha_- \in T_w \mathcal{B}^- \), \( ||\alpha_-|| = 1 \), can be expressed as \( \alpha_- = \sum_{i=1}^{\omega} \langle \alpha_-, \beta_i^- \rangle_{T\mathcal{B}^-} \beta_i^- \). Since \( ||\alpha_-|| = 1 \) then \( K(\alpha_-) = \langle A_{\eta}^\mathcal{B} (\alpha_-), \alpha_- \rangle_{T\mathcal{B}^-} \) can be expressed as

\[
K(\alpha_-) = \left( A_{\eta}^\mathcal{B} \sum_{i=1}^{\omega} \langle \alpha_-, \beta_i^- \rangle_{T\mathcal{B}^-} \beta_i^i, \sum_{i=1}^{\omega} \langle \alpha_-, \beta_i^i \rangle_{T\mathcal{B}^-} \beta_i^- \right)_{T\mathcal{B}^-}
\]

\[
= \left( \sum_{i=1}^{\omega} \kappa_i(w) \langle \alpha_-, \beta_i^- \rangle_{T\mathcal{B}^-} \beta_i^i, \sum_{i=1}^{\omega} \langle \alpha_-, \beta_i^i \rangle_{T\mathcal{B}^-} \beta_i^- \right)_{T\mathcal{B}^-}
\]

notice that \( \langle \beta_i^i, \beta_i^- \rangle_{T\mathcal{B}^-} = 1 \) (by orthogonality) and we may write
From Proposition 4.47 the orthogonal projection of eigenproblem (4.46).\[
\Gamma \in (4.31) \text{ yields } \{4.44(1). \text{ By Theorem 4.44(2)-(3) the elements of the coframe span precisely the eigenvalue problem for the selfadjoint Shape operator in Theorem which (by commutativity of composition under the differential operation) is Eq. (4.27). Since } \langle \xi \rangle = \langle \alpha_-, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-}, \text{ yields (4.29).} \]

Proof of Proposition 4.50: 1) From Eq. (4.30) \( w_i^- = \frac{1}{\sigma_i} \Gamma \circ \hat{w}_i(\tau) \) substitution in (4.31) yields \( \tilde{\Gamma}^\dagger \circ \Gamma \circ w_i^- = \sigma_i^2 w_i^- \) with \( \lambda_i = \sigma_i^2 \) we obtain Eq. (4.32). Substitution of Eqs. (4.31) in (4.30) and the same steps verifies Eq. (4.33). 2) At the tangent space, Eqs. (4.32) and (4.33) are written as \[
(\tilde{\Gamma}^\dagger \circ \tilde{\Gamma}) |^* \alpha_-^i - \lambda_i \alpha_-^i = 0, \quad i = 1, \ldots, \omega, \quad (4.45) \\
(\tilde{\Gamma} \circ \tilde{\Gamma}^\dagger) |^* \xi^+_i - \lambda_i \xi^+_i = 0, \quad i = 1, \ldots, \omega. \quad (4.46)
\]
which (by commutativity of composition under the differential operation) is precisely the eigenvalue problem for the selfadjoint Shape operator in Theorem 4.44(1). By Theorem 4.44(2)-(3) the elements of the coframe span \( \{\beta_-^1, \ldots, \beta_-^\omega\} = T_w \mathfrak{B}^- \) are solution to the eigenproblem (4.45). Notice that since \( \xi^+_i = \tilde{\Gamma}^\dagger \beta_-^i \) the elements of the frame span \( \{\xi^+_1, \ldots, \xi^+_\omega\} = T_w \mathfrak{B}^+ \) are solution of the eigenproblem (4.46). 3) From Proposition 4.47 the orthogonal projection of \( \xi^+ \in T_w \mathfrak{B}^+ \) is given by Eq. (4.27). Since \( \xi^+ = \tilde{\Gamma}^\dagger \alpha_- \) is on \( T_w \mathfrak{B}^+ \), we obtain

\[
\tilde{\Gamma}^\dagger \alpha_- = \sum_{i=1}^{\omega} \langle \tilde{\Gamma}^\dagger \alpha_-^i, \xi^+_i \rangle \Gamma_{\mathfrak{B}^+} \xi^+_i
\]
where after dualization from Remark 4.19, the inner product can be written as \( \langle \tilde{\Gamma}^\dagger \alpha_-^i, \tilde{\Gamma}^\dagger \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} \times \Gamma_{\mathfrak{B}^+} = \langle \tilde{\Gamma}^\dagger \tilde{\Gamma}^\dagger \alpha_-^i, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} \). Since by Theorem 4.44(3), \( A_{\eta} \alpha_- = \tilde{\Gamma}^\dagger \tilde{\Gamma}^\dagger \alpha_- \), we may write

\[
\tilde{\Gamma}^\dagger \alpha_- = \sum_{i=1}^{\omega} \langle \lambda_i \alpha_-^i, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} \xi^+_i = \sum_{i=1}^{\omega} \lambda_i \langle \alpha_-^i, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} \xi^+_i,
\]
i.e. Eq. (4.34). Departing from the orthogonal projection of \( \alpha_- = \tilde{\Gamma}^\dagger \xi^+ \) from Eq. (4.28) \( \tilde{\Gamma}^\dagger \xi^+ = \sum_{i=1}^{\omega} \langle \Gamma^\dagger \xi^+, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} \beta_-^i, \alpha_- \in T_w \mathfrak{B}^- \). Since \( \langle \Gamma^\dagger \xi^+, \beta_-^i \rangle \Gamma_{\mathfrak{B}^-} = \langle \xi^+, \Gamma^\dagger \beta_-^i \rangle \Gamma_{\mathfrak{B}^+} \) implying \( \langle \xi^+, \lambda_i \xi^+_i \rangle \Gamma_{\mathfrak{B}^+} = \lambda_i \langle \xi^+, \xi^+_i \rangle \Gamma_{\mathfrak{B}^+} \) and Eq. (4.35) is obtained. ■
Proof of Proposition 4.57: 1) By condition (4.39) \( S_r^i(\tilde{w}^0, r) \) and \( S_a(w^0, r_a) \) are orthogonally invariant on their respective spaces. Hence, associated to each integral invariant function \( S_r^i \) is an (exact) differential 1-form \( \zeta_i^+ \) defining a distribution \( \Delta_i \) with annihilator \( \Delta_i^+ \overset{\text{def}}{=} \{ \alpha_+ \in T_p\mathfrak{B}^- \mid \langle \alpha_+, \zeta_i^+ \rangle_{T\mathfrak{B}^- \times T\mathfrak{B}^+} = 0, \zeta_i^+ \in \Delta_i \} \) (dual to the distribution obtained in Proposition 4.44(3)). From Definition 4.55, the set of annihilators \( \{ \Delta_i^+, \ldots, \Delta_j^+ \} \) is an orthogonal web. Similarly, associated to each integral invariant function \( S_a^i \) is an exact differential 1-form \( \beta_i^- \) and the family of annihilators \( \{ \Omega_i^-, \ldots, \Omega_j^- \} \), defined by \( \Omega_i^- \overset{\text{def}}{=} \{ \xi^- \in T_p\mathfrak{B}^+ \mid \langle \beta_i^-, \xi^- \rangle_{T\mathfrak{B}^- \times T\mathfrak{B}^+} = 0, \beta_i^- \in \Omega_i \} \) defining the orthogonal web for \( S_a(w^0, r_a) \).

2) Since by virtue of Theorem 4.44 each element in the set of orthogonal differential 1-forms \( \{ \beta_i^- \} \in T^*\mathcal{W} \) has an associated integral functional \( \hat{S}_i \in \mathcal{W} \), from the proof of Theorem 4.44(2), each principal direction \( \beta_i^- \) is normal to an orthogonal hypersurface \( \hat{N}_i \) defined by a (locally) integrable distribution (i.e. its annihilator) whose integral invariant function is precisely \( S_i \in \mathcal{W} \). By duality, the set of tangent vector fields \( \{ \zeta_i^+ \} \in T^*\mathcal{W} \) is normal to an orthogonal hypersurface \( \hat{N}_i \) defined by a (locally) integrable codistribution (i.e. its annihilator) whose integral invariant function is precisely \( S_i \in \mathcal{W} \).  

Proof of Lem. 4.58: Given a completely separated function \( S(w) \) on \( \mathcal{W} \), a Riemannian metric for \( \mathcal{W} \) can be defined by \( ds^2 = \sum_{i=1}^n S_i(w) \, ds^2_i \), and \( \{ (\mathcal{W}_i, S_i) \} \) defines a partition of unity. For a principal frame-balanced realization \( S_r^i(\tilde{w}^0, r) \) and \( S_a(w^0, r) \) can be expressed as in Eq. (4.36) and metrics defined by \( ds^2_p = \sum_{i=1}^\omega S_i^p(w) \, ds^2_i \) and \( ds^2_f = \sum_{i=1}^\omega S_i^f(w) \, ds^2_i \), with partitions of unity \( \{ (\mathcal{W}_i, S_i^p) \} \) and \( \{ (\mathcal{W}_i, S_i^f) \} \) are well defined Riemannian metrics since each \( S_i ds^2_i \) makes sense on all \( \mathcal{W} \) and \( S_i = 0 \) outside each \( \mathcal{W}_i \).

Proof of Proposition 4.62: Steps 1) and 2) are slight modifications of the corresponding arguments presented in Part II of this work (Props. 5.18, 5.20 and 5.31 for internal signals) and therefore their proofs are omitted. For step 3), we depart from an extended group \( \{ \Phi^i(w) \}_t \), regular by assuming Eq. (4.38) satisfied. We are looking for a bijection \( \mathcal{R}(w) \in C^\infty(\mathcal{W}) \) defining a regular equivalence relation compatible with the group i.e. \( w^1 \mathcal{R} w^2, \Phi \in \{ \Phi^i(w) \}_t, \Rightarrow \Phi(w^1) \mathcal{R} \Phi(w^2), \) where \( \mathcal{R}_i(w) \in C^\infty(\mathcal{W}), \, i = 1 \ldots \omega \) are components of a vector function \( \mathcal{R}(w) : \mathbb{R}^\omega \to \mathbb{R}^{\omega-r}, \) \( w \in \mathcal{W} \) s.t. \( \det[\partial \mathcal{R}(w)/\partial w] \neq 0, \forall w \in \mathcal{W}. \) Since the orbit of \( \{ \Phi^i(w) \}_t \geq 0 \) is given by the set \( \mathcal{O} \overset{\text{def}}{=} \{ w \in \mathcal{O} | \Phi^i(w) \in \mathcal{O}, \} \) the natural equivalence relation for all trajectories on the orbit \( \mathcal{O} \) is given precisely by the natural projection \( \pi : \mathcal{W} \to \mathcal{W}/\mathcal{R} \) since if two points \( w^1, w^2 \in \mathcal{W} \) lie on
the same orbit of $\Phi \in \{\Phi^i(w)\}_t$ such that $\Phi \cdot w \in \mathcal{W}$, the induced equivalence relation among them is given by $\pi(w^1) = \pi(w^2)$ s.t. $\pi(\Phi \cdot w) = \pi(w)$. In such regular equivalence relation, every equivalence class in the quotient set $\mathcal{W}/\Phi$ is an $r$-dimensional submanifold $\mathcal{R}_i(w) = k_i$, $i = 1, \ldots, r$, for constants $k_i \in \mathbb{R}^1$, [33, 155] (see also [196]). It is a bijection since if $w \in \mathcal{W}/\Phi$, then its corresponding orbit in $\mathcal{W}$ is given by $\pi^{-1}\{w\}$. Finally, since by assumption $\{\Phi_i^i(w)\}_t$ acts regularly on $\mathcal{W}$ and $\pi$ is open in $\mathcal{W}/\Phi$, then $\mathcal{W}/\Phi$ is endowed with the structure of a smooth manifold [155, 33].

4) Associated to each separated component function $S_i$ with the structure of a smooth manifold [155, 33].

Associated to each component function $S_i^i(w)$ and $S_a^i(w)$ is a subgroup $\Phi^i_t(w) = -\nabla_w S^i_a(w)$ whose orbit is defined on their corresponding orthogonal web as in Definition 4.55. This is natural since each generating function is an integral invariant of the groups it generates. There is thus a partition of the space s.t.

$$
\mathcal{W}_a \overset{def}{=} \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_r,
$$

$$
\mathcal{W}_b \overset{def}{=} \mathcal{W}_{r+1} \oplus \mathcal{W}_{r+2} \oplus \cdots \oplus \mathcal{W}_\omega, \quad r \leq \omega,
$$

(4.47)

Since associated to each subgroup there is a submanifold $\mathcal{W}_i \subset \mathcal{W}$ and a particular principal curvature $\kappa_i$, a local partial order for the submanifolds $\mathcal{W}_i$ can be provided by inequality relations for the principal curvatures (singular values): $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_\omega$.

5) Consider the submanifold partition in (4.47). By orthogonality a quotient manifold $\mathcal{W}/\mathcal{W}_b$ can be defined always and the reduced subsystem can be obtained with the use of the natural projection $\pi_b : \mathcal{W} \rightarrow \mathcal{W}/\mathcal{W}_b$, see [155]. In view of Proposition 4.47, the generator vector field $\xi_{op}$ restricted to the quotient space $\mathcal{W}/\mathcal{W}_b$ defines uniquely the reduced-order (quotient) system.

Proof of Proposition 4.66: Consider the set of past trajectories $\dot{w} \in \mathcal{B}^{-}_{\text{red}} \subset \mathcal{B}^-$ adapted to the behavior $\mathcal{B}^-$. From Eq. (4.28) any such adapted past trajectory has an adapted tangent vector field written as $\alpha_- = \sum_{i=1}^r b_i \beta_i^-$, $b_i \in C^\infty(\mathcal{W})$, $\alpha_- \in \mathcal{B}^-$. Consider set of input vector fields given by $\ddot{\alpha}_- = \alpha_- + b_{r+1} \beta_-^{r+1}$ for $\|\tilde{\Gamma} - \tilde{\Gamma}_{\text{red}}\|_{\mathcal{B}}$. Since by construction 0 $\neq b_{r+1} \in C^\infty(\mathcal{W})$ can be selected such that $\ddot{\alpha}_- \in \text{Ker} \tilde{\Gamma}_{\text{red}}$, $\|(\tilde{\Gamma} - \tilde{\Gamma}_{\text{red}}) \circ \dot{\omega}_-\|_{\mathcal{B}}^2 = \|\tilde{\Gamma} \circ \dot{\omega}_-\|_{\mathcal{B}}^2$ using Eq. (4.34) and Euler’s formula (4.29) the squared norm becomes $\langle \tilde{\Gamma}^* \ddot{\alpha}_-, \tilde{\Gamma}^* \ddot{\alpha}_- \rangle_{\mathcal{B}} = \sum_{i=1}^{r+1} \kappa_i(w)(\alpha_-^i, \beta_i^-)^2_{\mathcal{B}} = \sum_{i=1}^{r+1} \kappa_i(w) b_i^2 \geq \kappa_{r+1}(w) \sum_{i=1}^{r+1} b_i^2 = \kappa_{r+1}(w) \|w\|^2_{\mathcal{B}}$, concluding the proof.
Chapter 5

Abstract: This Part II provides structural aspects of the internal signals and operators used in the unified geometric theory for nonlinear dissipative and Hankel balanced reduction using the differential-geometric framework based on dissipativity theory, Lie-semigroups and submanifold Hilbert theory. In particular, a novel characterization of the nonlinear Gramians is presented along with an alternative view of some concepts of the theory of nonlinear balanced reduction and the Hankel operator, namely eigenvalue problems and eigenfunction decomposition, etc.

Keywords: Nonlinear systems, Dissipative systems, Geometric approach, Invariants, Eigenvalue problems, System order reduction, Hankel operator, model approximation, Model Reduction.

5.1 Introduction

While the analytic activity of modeling, –as an approximated description of real phenomena–, is natural for scientific purposes, the simplification of complex models is natural for engineering purposes. A recurrent need during the analysis of complex control systems, is model simplification on different levels of approximation, such that essential structural...
properties are preserved in the simplified model and useless structures are selectively discarded under a uniform, predefined criterion. Such structural properties can be those associated to the stability, controllability and observability of the original control system. But the criteria to reach different orders of approximation, must be associated to some measure on model performance, e.g. approximation error, dissipativity or the $L_2$-gain of the system.

In this regard, the paradigmatic method of balanced reduction [143] for linear state-space systems, successfully provides reduced-order models whose properties of controllability and observability are balanced, using as reduction criterion the spectral properties of the Hankel operator.

Hankel-type nonlinear balanced reduction has been discussed nowadays in several publications. Probably the most recognizable analytical nonlinear generalization of Moore’s approach is the one presented in [180] and is based on energy functions, keeping certain similarity with the approach of behavioral balanced reduction for linear systems due to [212]. In [44] it is shown that the nonlinear balancing problem can be solved with the solution of a nonlinear eigenvalue problem. Nonlinear adjoint operators for this purpose are presented in [46]. Schmidt pairs for the nonlinear balancing problem are proposed in [61].

Though, reduction procedures for dynamical systems have a long history in the mathematical sciences [155, 33]. The main classical approach of reduction is fundamentally based on symmetries, i.e. the invariant properties preserved during the evolution of a system isolated from its environment. In contrast, the approach taken in Part I, pursues the preservation of the invariant properties of an isometric operator from past external signals into future external signals supported on Hilbert manifold structures, i.e. the behavioral operator, framing nonlinear dissipative balanced reduction in the perspective of the classical reduction procedures.

In this Part II, the rich structure of the dissipative balanced reduction problem for latent or internal variables is discussed. In particular, this Part II concentrates on the past and future behavior internally parameterized by dynamical systems with state-space variables. The paper is organized as follows: In Sect. 5.2 additional concepts of behavioral systems on finite-dimensional Hilbert manifolds is introduced along with necessary concepts of evolutionary operators. While in Part I balanced reduction for dissipative systems was approached in terms of external signals only, this Part II is dedicated mostly to the internal system trajectories. Therefore Sect. 5.2 provides the framework of Hilbert submanifolds to support the internal trajectories and some Lie-semigroup theory to deal with nonlinear maps. In Sect. 5.3 the concepts previously discussed for
the space of external signals about the balancing condition, orthogonal separability and Curvature Theory, are discussed for the space of internal signals along with a study of the internal structure of the nonlinear Gramians and the behavioral and Hankel operators using Lie-semigroups. In Sect. 5.4 issues related to balanced reduction and the properties of minimality of a balanced realization are discussed.

5.2 The geometric framework for internal trajectories

To describe internal trajectories properly, we preserve the notation used in Part I to describe the backward-time and forward-time evolution of dynamical control systems. Supporting such internal trajectories are two half-spaces $\mathcal{M}^t$ and $\mathcal{M}^\tau$, where $\mathcal{M}$ and $\mathcal{M}^*$ are dual spaces joined at $t = 0$ by duality at their edge $\mathcal{M}_0$ and $\mathcal{M}_0^*$ respectively, formalized in terms of a framework of Hilbert manifolds similar to Part I. The reader is asked to keep in mind the definitions for Riemannian Hilbert manifolds, local isometries and duality pairing presented earlier in Part I. All the proofs are collected in the Appendix.

5.2.1 Evolutionary operators and Hilbert manifold structures

A useful concept in the description of dynamical systems are Lie semigroups of diffeomorphisms, briefly recalled in this section in order to characterize evolutionary operators. We need the following:

**Definition 5.1.** A family $\{\Phi(x,t), t \in \mathbb{R}, x \in D \subset \mathcal{M}\}$ in a class of bounded operators in $\mathcal{M}$ is called a 1-parameter semigroup if it is such that the mapping $\Phi : \mathbb{R}^+ \times D \to D$, $\Phi(t, x) = \phi^t(x)$ depends smoothly on $t \in \mathbb{R}^+$; $\phi^0(x) = x$ and $\phi^{t_2} \circ \phi^{t_1}(x) = \phi^{t_2+t_1}(x)$. Such semigroup is called strongly continuous (or $C_0$-semigroup) if $t \to \phi^t(x)$ is continuous on $[0, \infty)$ for every $x \in \mathcal{M}$ [25]. If a semigroup carries the structure of an $n$-dimensional smooth manifold $\mathcal{M}$ satisfying the semigroup property, it is called a Lie-semigroup. A Lie semigroup denoted $S_G$ is called differentiable if each operation $\circ : S_G \times S_G \to S_G$ yields a differentiable map [54]. If a Lie-semigroup $G$ satisfies the inversion operation $i : G \to G$, with smooth maps between manifolds, $i(g) = g^{-1}$, $g \in G$ then $G$ is called a Lie-group.

When the semigroup represents a dynamical system, we would prefer to call it evolutionary operator:

**Definition 5.2 (Dynamical system).** A dynamical system is the triad $(t, \mathcal{M}, \Phi^t)$ in forward-time or the triad $(\tau, \mathcal{M}^*, \Theta^\tau)$ in backward-time where
$t$ (resp. $\tau$) is a semi-interval of evolution, $\mathcal{M}$ (resp. $\mathcal{M}^*$) defines the (co-) state-space and the semigroup $\Phi^t$ (resp. $\Theta^\tau$) is called an evolutionary operator.

This classical definition disregards the effect of exogenous variables. Along this section Definition 5.2 will be complemented with an operator derived from concepts of the behavioral approach in order to provide a theory useful for balanced reduction of nonlinear control systems.

Consider the following continuous-time, input-affine nonlinear system (the $\mathfrak{N}$-class, after Definition 4.59 in Part I) defined by

$$\Sigma^+ : \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), \\ y(t) = h(x(t)), \end{cases} \quad (5.1)$$

where $x \in \mathbb{R}^n$ are local coordinates for a $C^\infty$ state space manifold $\mathcal{M}$ and $f$ and $g$ are $C^\infty$ in $\mathcal{M}$. The set of external variables $\mathcal{W} \approx \mathbb{R}^\omega$, $p + q = \omega$, includes $u \in \mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathcal{Y} \subset \mathbb{R}^q$ as subsets. The map $h : \mathcal{M} \rightarrow \mathcal{Y}$ is $C^\infty$ in $\mathcal{Y}$ and locally Lipschitz continuous in $x$. We additionally assume $x$ and $y$ locally square integrable. To guarantee (local) existence and uniqueness of solutions, we assume that the map $u : t \rightarrow \mathcal{U}$ consist of piecewise constant control inputs $u(t), u \in \mathcal{U}$ and $t \in (t_a, t_b), t_a, t_b \in t$ (a sufficiently small interval), and assume that $f$ is (locally) Lipschitz continuous in $x$. Associated to the (smooth) time varying vector field $x \rightarrow F(x, u(t))$, $F_u \overset{\text{def}}{=} F(x, u(t)) \overset{\text{def}}{=} f(x(t)) + g(x(t))u(t)$, there is a family of vector fields denoted by $\mathcal{F}_u = \{F_u : u \in \mathcal{U}\}$. The (phase) trajectories of this system are continuous curves $g(t)$ in $\mathcal{M}$ that define on an interval $[0, T]$, integral curves of the family $\mathcal{F}_u$ for some partition $0 = t_0 \leq t_1 \leq \cdots \leq t_m = T$ and associated vector fields $\xi_1, \ldots, \xi_m \in \mathcal{F}_u$ such that the restriction of $g(t)$ to each open interval $(t_i, t_{i+1}), i = 0, \ldots, m$, is differentiable and such that $dg(t)/dt = \xi_i(g(t)), g(t_i) = g^i, t \in (t_i, t_{i+1}), i = 0, \ldots, m$.

When the system $\Sigma$ evolves backwards in time, the dynamical relation in (5.1) is expressed as

$$\Sigma^- : \begin{cases} \dot{x}(\tau) = -f(\hat{x}(\tau)) - g(\hat{x}(\tau)) \hat{u}(\tau), \\ \hat{y}(\tau) = h(\hat{x}(\tau)), \end{cases} \quad (5.2)$$

where, for reasons that will be clear later, we assume $\hat{x}$ (the co-state) are local coordinates for a $C^\infty$ dual manifold $\mathcal{M}^*$ and $f$, $g$ are $C^\infty$ in $\mathcal{M}^*$. The set of dual external variables $\mathcal{W}^*$ includes $\hat{u} \in \mathcal{U}^*$ and $\hat{y} \in \mathcal{Y}^*$ as subsets. The (locally Lipschitz continuous in $\hat{x}$) map $h : \mathcal{M}^* \rightarrow \mathcal{Y}^*$ is $C^\infty$ in $\mathcal{Y}^*$. Here it is also defined a (smooth) time varying vector field $\hat{x} \rightarrow -F(\hat{x}, \hat{u}(\tau)), F_{\hat{u}} \overset{\text{def}}{=} -F(\hat{x}, \hat{u}(\tau)) = -f(\hat{x}(\tau)) - g(\hat{x}(\tau)) \hat{u}(\tau)$ where the (piecewise constant)
5.2 The geometric framework for internal trajectories 91

control input \( \dot{u}(\tau) \) is a map \( \dot{u} : \tau \to U^* \) and there is a family of vector fields \( \hat{F}_u = \{ F_{\dot{u}} : \dot{u} \in U^* \} \) associated to the (backward-time) phase trajectories.

Remark 5.3. The initial conditions \( x^0 \triangleq x(0) \in C^\infty(M), \hat{x}^0 \triangleq \hat{x}(0) \in C^\infty(M^*) \) for the proper state-space realizations (5.1)-(5.2) are defined in accordance to Remark 4.3 in Part I.

Example 5.4. Let us assume the behavior is internally reconstructed by linear time-invariant systems (the \( \mathcal{L} \)-class, after Definition 4.59 in Part I) defined by

\[
\Sigma^+ : \begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t),
\end{cases} \quad t \geq 0. \tag{5.3}
\]

whose states evolve on the state space \( M = \mathbb{R}^n \). Then, with a slight variation on the approach taken by [212], its backward-time system (with \( \tau := -t \))

\[
\Sigma^- : \begin{cases}
\dot{\hat{x}}(\tau) = -A\hat{x}(\tau) - B\hat{u}(\tau), \\
\hat{y}(\tau) = C\hat{x}(\tau), \quad \tau \geq 0.
\end{cases} \tag{5.4}
\]

evolves on the co-state space \( M^* = \mathbb{R}^n \). \( \square \)

Remark 5.5. The internal realizations \( \Sigma^+, \Sigma^- \) in Eqs. (5.1)-(5.2) can be obtained from the behavioral equations (higher-order external differential representations), see Part I (Sec. 4.2.1), using a smooth generalized coordinate transformation, see e.g. [29] for the existence conditions. For the inverse realization problem, \( \Sigma^+, \Sigma^- \) can be expressed equivalently as behavioral equations, whenever such realizations are constant-dimensional, see e.g. [151], pp. 125 for an algorithm.

Example 5.6. Consider the universal motor from [125] with a state-space \( (\phi, \alpha_m) \) (the flux linkage and angular momentum, resp.) resulting in the forward-time equations

\[
\frac{d}{dt}\phi = -\frac{R}{L}\phi - \frac{\zeta}{L}\alpha_m\phi + V_t, \quad \frac{d}{dt}\alpha_m = -\frac{B}{J}\alpha_m + \frac{\zeta}{L^2}\phi^2 - \tau_L
\]

with output equations \( \omega = \alpha_m/J, \ I = \phi/L \) and constant parameters \( L, R, \zeta, J, B \in \mathbb{R}^+ \). The local coordinates \((V_t, \tau_L, I, \omega)\) of the space of exogenous variables \( W \), where \((V_t, \tau_L)\) are inputs and \((I, \omega)\) are outputs. The backward-time equations are given by

\[
\frac{d}{d\tau}\phi = \frac{R}{L}\phi + \frac{\zeta}{L}\alpha_m\phi - V_t, \quad \frac{d}{d\tau}\alpha_m = \frac{B}{J}\alpha_m - \frac{\zeta}{L^2}\phi^2 + \tau_L.
\]

\( \square \)

Though, there is no reason to believe \( a \ priori \) that the integral trajectories of a tangent vector field \( \xi \) are defined for both positive \( t \in \mathbb{R}^+ \) and negative \( \tau \in \mathbb{R}^- \) evolving times.
Definition 5.7 (Complete vector field, symmetric system). A tangent vector field $\xi$ is called a complete vector field if the integral trajectories of $\xi$ are defined for all times (positive and negative). The system $\Sigma$ is called symmetric if, whenever a vector field $\xi$, e.g. $\xi = F(x, u)$, belongs to the family $\mathcal{F}_u$, then it follows that $-\xi$ also belongs to $\mathcal{F}_u$, in particular $\exists \hat{u}$ s.t. $F(x, u) = -F(x, \hat{u})$.

For instance, the integral trajectories defined by the transformations $g^t : x \to x/(1 - tx)$ on $M = \mathbb{R}^1$ are generated locally by the vector field $\xi(x) = x^2 \frac{\partial}{\partial x}$ which is not complete on any submanifold of $\mathbb{R}^1\{0\}$, since the mapping $M(t, x) = x/(1 - tx)$ is only defined on the set $D = \{(t, x) : 1 - tx > 0\}$, [33, 3].

Remark 5.8. The general description of arbitrary vector fields $\xi(x, u)$ with dependent and independent variables requires of the prolongation of the basic space $M \times U$ to a space that accounts for their several partial derivatives [155]. Nevertheless, whenever systems (5.1) and (5.2) are rendered to satisfy two optimal control problems (see Eqs. (5.7) and (5.8) below) for admissible state feedback inputs $u : M \times U \times t \to M$, $u(t) \in U$ and $\hat{u} : M \times U \times \tau \to M$, $\hat{u}(\tau) \in U^*$, the resulting closed-loop system depends only on the initial condition. Though, we may assert that a vector field is generator of a group action only when the associated vector fields $\xi(x) \in \mathcal{F}_u$ and $\hat{\xi}(\hat{x}) \in \hat{\mathcal{F}}_u$ are of bounded length. This is discussed later on Sec. 5.3.1.

Whenever the boundedness condition mentioned in Definition 5.1 and Remark 5.8 is satisfied for an adequate control input $u(x(t))$ defined in terms of the state $x(t)$, the images of all the 1-parameter semigroups $\Phi^t(x) : \mathbb{R}^1 \to SG$, $t \to \exp(t\xi)$, $\xi \in \mathcal{F}_u$ generate a Lie semi-group as a closed sub semigroup $SG$ of the Lie Group $G$, see [74]. Moreover, the system semi-trajectories of (5.1) on the space $t \times M$ can be expressed in terms of semigroups of diffeomorphisms of the forward-time solution of Eq. (5.1) in the form

$$\Phi^t(x^0) = \exp t_k \xi_k \cdot \exp t_{k-1} \xi_{k-1} \cdots \exp t_i \xi_i \cdots \exp t_2 \xi_2 \cdot \exp t_1 \xi_1 x^0, \quad (5.5)$$

such that $t_i \geq 0$ for $i = 1, \ldots, k$. Denote by $S^+_u$ the set of the (positive) semigroups $\{\Phi(t, x(t))\}_{t \geq 0}$ solution of (5.1). The action of one of such elements is denoted as $x(t) = \Phi(t; t_0, x_0, u(t))$, $u(t) \in U$.

When the solution of Eq. (5.2) is bounded for an appropriate piecewise $\hat{u}(\hat{x}(\tau))$ defined in terms of the co-state $\hat{x}(\tau)$, such solution can be defined in terms of an evolution operator map (semigroup of diffeomorphisms) whose backward-time evolution as semigroup on $M^{*\tau}$ is expressed as

$$\Theta^\tau(\hat{x}^0) = \exp \tau_k \alpha_k \cdots \exp \tau_i \alpha_i \cdots \exp \tau_2 \alpha^2 \cdot \exp \tau_1 \alpha^1 \hat{x}^0, \quad (5.6)$$
with $\tau_i \leq 0$ for $i = 0, \ldots, k - 1, k$, denoted by $S_{\hat{F}_u}^-$. The action of one of such elements is denoted as $\hat{x}(\tau) = \Theta(\tau; \tau_0, \hat{x}_0, \hat{u}(\tau))$, $\hat{u}(\tau) \in U^*$.

The vector field $\xi(x)$ is the generator of the (forward-time) Lie semigroup of diffeomorphisms $S^+_F u = \{ \Phi(x, t) \}_{t \in t^+}$ if the limit $\xi(x) = \lim_{t \to 0^+} (\Phi(x, t) - x) / t$ exists in a domain $D(x)$. Similarly, the vector field $\alpha(\hat{x})$ generates the negative semigroup $S^-_{\hat{F}_u} = \{ \Theta(\hat{x}, \tau) \}_{\tau \in \tau^-}$ whenever $\alpha(\hat{x}) = \lim_{\tau \to 0^-} (\Theta(\hat{x}, \tau) - \hat{x}) / \tau$ exists in a domain $\hat{D}(\hat{x})$. After Definition 5.7, completeness of a tangent vector field $\xi(x) = \alpha(\hat{x})$ on a domain properly defined, implies that in such domain their integral semi trajectories are connected, defining (complete) trajectories for positive and negative times.

**Definition 5.9 (System trajectories).** The collection of (complete) trajectories $\varrho(t) \in \mathcal{M}$ induced by the dynamical system (5.1)-(5.2), is named the system trajectories.

The behavior is manifest in the dynamical system $\Sigma$ by influencing the set of state-space system trajectories on $\mathcal{M}$ where all its possible trajectories (orbits) evolve. Their local coordinates, called latent or internal variables in the behavioral approach, are auxiliary variables used by the model governing laws or for subsystem interconnection [171].

With the inclusion of a response map [192] $h : \mathcal{M} \to \mathcal{W}$, the behavior can be internally reproduced by $(t, \mathcal{W}, \mathcal{M}, h)$. Moreover, we associate as generator of such trajectories a particular class $\mathcal{E}$ of dynamical systems $\Sigma$ (see Definition 4.59 in Part I, namely linear, nonlinear, affine, port-Hamiltonian, etc., such that the number of state space variables is fixed as the system evolves in time. The procedure of internally building the behavioral operator takes us to consider some class of model structure. In particular in this Part II we will concentrate in the nonlinear class defined in Eqs. (5.1)-(5.2).

In the remainder of the section the ambient space of the system trajectories is formalized in terms of Hilbert manifold structures.

Similarly to Part I, throughout we assume each compact manifold $\mathcal{M}$ supports trajectories $x(t) \in \mathcal{M}$ that parameterize a space of smooth linear functionals $F(x(t))$ on a dual space $\mathcal{M}^*$ whose differential $dF(x)$, $x \in \mathcal{M}$ lies on $T^*_x \mathcal{M}$. Moreover, the well defined contraction $i_\xi \alpha = \alpha_i \xi^i = \sum_i \frac{dx^i}{dt} \frac{\partial F}{\partial x^i} = \frac{dF}{dt}$ yields (coordinate independent) scalar functions $f_i$. Such space $L^2_\mathcal{M}(a, b)$ on $\mathcal{M}$ consists of (the equivalence class of) Lebesgue-measurable, (square) integrable functions mapping the interval $(a, b)$ into $\mathbb{R}^n$, and the space of functions $(F, \langle \cdot, \cdot \rangle)$ is a complete, separable, Hilbert space at each $p \in \mathcal{M}$ with an appropriate notion of orthogonality as defined in Part I.

A Hilbert manifold structure consist of a given pair of Hilbert manifolds
(\mathcal{M}, \langle \cdot, \cdot \rangle_{T\mathcal{M}}) \text{ and } (\mathcal{M}^\ast, \langle \cdot, \cdot \rangle_{T^\ast\mathcal{M}}) \text{ dualized by an abstract dual pairing denoted as } \langle \cdot, \cdot \rangle_{T^\ast\mathcal{M} \times T\mathcal{M}} : T^\ast\mathcal{M} \times T\mathcal{M} \to \mathbb{R}^1. \text{ such that for each element } \alpha \in T^\ast\mathcal{M} \text{ there is a unique element } \hat{\alpha} \in T\mathcal{M} \text{ and an isometric isomorphism } 
abla : \mathcal{M} \to \mathcal{M}^\ast \text{ satisfying } \langle \hat{\alpha}, \zeta \rangle_{T\mathcal{M}} = \langle \alpha, \zeta \rangle_{T^\ast\mathcal{M} \times T\mathcal{M}} = \langle \alpha, \lambda_\ast \zeta \rangle_{T^\ast\mathcal{M}}, \forall \zeta \in T\mathcal{M} \text{ and } \|\hat{\alpha}\|_{T\mathcal{M}} = \|\alpha\|_{T^\ast\mathcal{M}}, \text{ see Part I (Definition 4.13).}

Two Hilbert manifold structures will be used throughout this Part II. The first one characterizes the natural duality of the state and co-state spaces of the compact, differentiable manifolds \((\mathcal{M}, \langle \cdot, \cdot \rangle_{T\mathcal{M}})\) and \((\mathcal{M}^\ast, \langle \cdot, \cdot \rangle_{T^\ast\mathcal{M}})\) where the system trajectories and associated functions are supported. A second Hilbert structure characterizes the past and future state semi-trajectories on the past \(\mathcal{M}^\tau\) and future \(\mathcal{M}^\delta\) half-spaces.

### 5.2.2 Hilbert manifold structure for \(\mathcal{M}\)

The following result needs no proof since it is based on the same arguments of the equivalent structure in Part I (Proposition 4.15):

**Proposition 5.10 (Hilbert manifold structure of \(\mathcal{M}\)).** Let the state and co-state be supported on the compact, differentiable, Riemannian manifolds \((\mathcal{M}, g_\mathcal{M})\) and \((\mathcal{M}^\ast, g_{\mathcal{M}^\ast})\) respectively, with metrics \(g_\mathcal{M}, g_{\mathcal{M}^\ast}\) and natural norms \(\| \cdot \|_{T\mathcal{M}}, \| \cdot \|_{T^\ast\mathcal{M}}\) defined after the inner products \(\langle \cdot, \cdot \rangle_{T\mathcal{M}}, \langle \cdot, \cdot \rangle_{T^\ast\mathcal{M}}\) locally isomorphic to Hilbert space on \(T\mathcal{M}\) and \(T^\ast\mathcal{M}\), as defined in Table 5.1. Then \((\mathcal{M}, \langle \cdot, \cdot \rangle_{T\mathcal{M}})\) and \((\mathcal{M}^\ast, \langle \cdot, \cdot \rangle_{T^\ast\mathcal{M}})\) are dual Hilbert manifolds with duality identified by a (well defined) duality pairing \(\langle \cdot, \cdot \rangle_{T\mathcal{M} \times T^\ast\mathcal{M}}\) whose resulting scalar functions are coordinate invariant.

The Hilbert manifold structure in Proposition 5.10 keeps the natural duality between state and co-state spaces on \(\mathcal{M}\) and \(\mathcal{M}^\ast\) along with their tangent and cotangent spaces \(T\mathcal{M}\) and \(T^\ast\mathcal{M}\). As in Remark 4.16 (Part I), duality can be naturally identified via the metrics since the covariant 2\(^{\text{nd}}\) rank metric tensor \(g_\mathcal{M}\) on \((\mathcal{M}, g_\mathcal{M})\) is related to the dual contravariant 2\(^{\text{nd}}\) rank metric tensor \(g_{\mathcal{M}^\ast}\) on \((\mathcal{M}^\ast, g_{\mathcal{M}^\ast})\), by \([g_{ij}] = [g^{ij}]^{-1}\). Also from Remark 4.17 (Part I), the Hilbert manifold structure of Proposition 5.10 preserves locally the property of orthogonality in the Hilbert space of functions.

**Example 5.11.** In terms of class complexity, linear time-invariant systems (\(\mathcal{L}\)-class) is structurally the simplest, since the (Euclidean) state space \(\mathbb{R}^n\) can be identified at any point with its tangent space, i.e. \(\mathbb{R}^n = T_p \mathbb{R}^n\), Hilbert manifolds simplify to Euclidean Hilbert spaces and Lie group actions consist of linear maps. □
5.2.3 Hilbert manifold structure for system trajectories

The Hilbert manifold structure of the behavior $\mathcal{B} \subset \mathcal{W}_t$ has an induced Hilbert manifold structure for the future system trajectories on $\mathcal{M}_t$ and the past system trajectories on $\mathcal{M}^*_t$. In terms of the backward-time interval $\tau = \{\tau | \tau = -t, t \in t\}$, such spaces can be alternatively interpreted as two intertwined dual manifolds with forward-time evolution on $\mathcal{M}_t$ and backward-time evolution on $\mathcal{M}^*_\tau$. With a slight abuse of notation, by $\mathcal{M}_t^+$ we express the half space supporting forward evolution on $\mathcal{M}_t$ and by $\mathcal{M}^{*}\tau_-$ the half space supporting backward evolution on $\mathcal{M}^*_\tau$.

Associated to the Hilbert manifold structure of external variables $\mathcal{W}$ supporting inputs and outputs, two known functions are used as alternative metrics for the internal system Hilbert structures in time, serving as a bridge between $\mathcal{W}$ and $\mathcal{M}$. Consider again the storage functions defined on Part I but now supported by subsets of the state and co-state spaces $\mathcal{M}$ and $\mathcal{M}^*$:
\[ S_a(x^0, r_a) = \sup_{u(\cdot) \in U} \left. \int_0^\infty r_a(w^+(t)) \, dt \right|_{x^0 = x \in M, t \geq 0} \ \ \ \ \ (5.7) \]

\[ S_r^*(\hat{x}^0, r_r) = -\sup_{\hat{u}(\cdot) \in U^*} \left. \int_0^\infty r_r(\hat{w}^-(\tau)) \, d\tau \right|_{\hat{x}^0 = \hat{x} \in M^*, \tau \geq 0} \ \ \ \ \ (5.8) \]

An assumption made throughout the forthcoming sections follows:

**Assumption 5.12.** The metrics on the past and future Riemannian manifolds \((M^* \tau, \langle \cdot, \cdot \rangle_{T^* M^\tau})\) and \((M^t, \langle \cdot, \cdot \rangle_{T M^t})\), are defined by \(S_r^*(\hat{x}^0, r_r)\) and \(S_a(x^0, r_a)\) respectively, such that

\[ S_r^*(\hat{x}^0, r_r) = \langle \alpha^-, \alpha^- \rangle_{T^* M^-} = g_{B^-}(\hat{w}^-, 0), \quad B^- \subset \tau \times \mathcal{W}^* \]

\[ S_a(x^0, r_a) = \langle \xi^+, \xi^+ \rangle_{T M^t} = g_{B^+}(w^+, 0), \quad B^+ \subset t \times \mathcal{W}. \]

Moreover Assumption 4.23 from Part I stands valid throughout Part II implying that Hilbert vector spaces admit alternate Riemannian metrics since they admit partitions of unity.

**Example 5.13.** For linear time-invariant (L-class) dissipative systems \(\Sigma^+\) and \(\Sigma^-\), their storage functions (5.8)-(5.7), have the form

\[ S_r^*(\hat{x}^0, r_r) = \frac{1}{2} \hat{x}_0^T P \hat{x}_0 \]

\[ S_a(x^0, r_a) = \frac{1}{2} x_0^T Q x_0, \quad P, Q \in \mathcal{L}(M, M^*). \]

\[ \square \]

**Definition 5.14 (System semi-trajectories).** Assuming that the required supply \(S_r^* : M^* \rightarrow \mathbb{R}^+\) and the available storage \(S_a : \mathcal{W} \rightarrow \mathbb{R}^+\), functions associated to system (5.1) exist as defined in Eqs. (5.7) and (5.8), the pairs expressed by

\[ (x_\ast^-, \hat{x}_\ast^-) \overset{def}{=} \{(x, \hat{x}) | S_r^*(\hat{x}^0, r_r) = S_r^*(\hat{w}^0, r_r), \ \forall x^0 \in M, \ \hat{x}^0 \in M^*, \ w^0 \in \mathcal{W}^*, \ t \in \tau \} \]

\[ (x_\ast^+, \hat{x}_\ast^+) \overset{def}{=} \{(x, \hat{x}) | S_a(x^0, r_a) = S_a(w^0, r_a), \ \forall x^0 \in M, \ \hat{x}^0 \in M^*, \ w^0 \in \mathcal{W}, \ t \in t \} \]

are called **system semi-trajectories**.

Using the wedge (\(\wedge\)) notation for semi-trajectory concatenation of Part I, consider the following:
**Definition 5.15.** For each initial condition \( x^0 \in \mathcal{M} \), the time-concatenation of the semi-trajectories (5.11) and (5.12) defines trajectories in \( \mathcal{M} \) and \( \mathcal{M}^* \) for \( t \in (-\infty, \infty) \) characterized by \( (x, \hat{x}) = (x_\ast \wedge x_\ast^+, \hat{x}_\ast \wedge \hat{x}_\ast^+) \). Denote by \( \varphi(x^0) \)-trajectory the set of points \( (x, \hat{x}), x \in \mathcal{M}, \hat{x} \in \mathcal{M}^* \) given by \( x = x_\ast \wedge x_\ast^+, \hat{x} = \hat{x}_\ast \wedge \hat{x}_\ast^+ \). The system trajectories are the sets \( \{ \varphi(x^0) | x^0 \in \mathcal{M} \} \).

The following result is similar to the one sketched for Proposition 4.28 in Part I, and a proof is not needed:

**Proposition 5.16 (Hilbert manifold structure of system trajectories).**

Let the system trajectories \( x(t) \in \mathcal{M}, \hat{x}(\tau) \in \mathcal{M}^* \) with tangent vector fields \( \xi^+ \in T \mathcal{M}_t, \alpha^- \in T^* \mathcal{M} \) be supported on the Riemannian manifolds \( \mathcal{M}_t^\tau \) and \( \mathcal{M}_t^\tau \) for metrics and norms defined after the inner products \( \langle \cdot, \cdot \rangle_{T \mathcal{M}_t^\tau} \) and \( \langle \cdot, \cdot \rangle_{T^* \mathcal{M}_t^\tau} \), as defined in Table 5.2. Then \( (\mathcal{M}_t^\tau, \langle \cdot, \cdot \rangle_{T^* \mathcal{M}_t^\tau}) \) and \( (\mathcal{M}_t^\tau, \langle \cdot, \cdot \rangle_{T^* \mathcal{M}_t^\tau}) \) are dual Hilbert manifolds with duality identified by the abstract duality pairing \( \langle \cdot, \cdot \rangle_{T^* \mathcal{M}_t^\tau \times T \mathcal{M}_t^\tau} \) for a surjective isometry \( \Xi : \mathcal{M}_t^\tau \to \mathcal{M}_t^\tau \) s.t. \( \| \alpha^- \|_{T^* \mathcal{M}_t^\tau} = \| \xi^+ \|_{T \mathcal{M}_t^\tau} \) is satisfied.

**Remark 5.17.** The duality pairing can be defined for an alternative surjective isometry \( \Xi^\dagger : \mathcal{M}_t^\tau \to \mathcal{M}_t^\tau \) by \( \langle \xi^+, \alpha^- \rangle_{T^* \mathcal{M}_t^\tau \times T \mathcal{M}_t^\tau} \), in Table 5.2.

The duality pairing \( \langle \xi^+, \alpha^- \rangle_{T^* \mathcal{M}_t^\tau \times T \mathcal{M}_t^\tau} \) satisfies

\[
\langle \alpha^-, \alpha^- \rangle_{T \mathcal{M}_t^\tau} = \langle \xi^+, \alpha^- \rangle_{T^* \mathcal{M}_t^\tau \times T \mathcal{M}_t^\tau} = \langle \xi^+, \Xi \alpha^- \rangle_{T \mathcal{M}_t^\tau},
\]

\( \forall \xi^+ \in T \mathcal{M}_t^\tau \) for an associated isometric isomorphism \( \Xi : \mathcal{M}_t^\tau \to \mathcal{M}_t^\tau \) satisfying the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_t^\tau & \xrightarrow{\Xi} & \mathcal{M}_t^\tau \\
\pi_{\mathcal{M}_t^\tau} \downarrow & & \uparrow \pi_{\mathcal{M}_t^\tau} \\
T \mathcal{M}_t^\tau & \xleftarrow{\Xi} & T^* \mathcal{M}_t^\tau
\end{array}
\]

With equivalent results to those of Part I, Assumption 5.12 for the Hilbert Manifolds structures in Props. 5.10 and 5.16 turn out to be fundamental for a nonlinear balancing principle based on Curvature Theory in order to define an isometric isomorphism \( \Upsilon : \mathcal{M}_0 \to \mathcal{M}_0 \) on dual spaces \( \mathcal{M}_0^\tau \) and \( \mathcal{M}_0 \) along with a minimal frame supporting the invariant trajectories of such isometry.
5.3 Isometric invariants and the balancing condition

The problem of nonlinear balancing was approached in Part I in terms of the exogenous signal trajectories on $\mathfrak{B}$ along with the properties of an isometric self-adjoint operator. Since the same metrics for past and future behavior is used for the past and future system trajectories, an equivalent problem of nonlinear balancing can be posed for the internal system trajectories associated to the dynamical systems $(t, M, \Phi^t)$ and $(\tau, M^*, \Theta^\tau)$.

This section is focused on characterizing the curvature properties of a submanifold $M_0 \subset M$ in terms of properly defined metrics. The characterization of the metric $g_{M_0}$ on $M_0$ defines uniquely the problem since the total curvature is only dependent on $g_{M_0}$ and its derivatives. Moreover, we are interested in identifying how the invariants of the behavioral operator $\tilde{\Gamma} : \mathfrak{B}^- \to \mathfrak{B}^+$ are inherited to the nonlinear map operator $\Upsilon : M_0 \to M_0$ on the space of...
5.3 Isometric invariants and the balancing condition

5.3.1 Generating functions, group extension and nonlinear Gramians

We have asserted that a vector field \( \xi(x) \) is a generator of the (positive) Lie semigroup of diffeomorphisms \( S_G = \{ \Phi(x, t) \}_{t \in \mathbb{T}} \) if the limit \( \xi(x) = \lim_{t \to 0^+} (\Phi(x, t) - x)/t \) exist in a domain (denoted \( D(\xi) \)). If additionally there is an \( \alpha(\hat{x}) \) dual to \( \xi(x) \) satisfying

\[
\alpha(\hat{x}) = \lim_{\tau \to 0^-} (\Theta(\hat{x}, \tau) - \hat{x})/\tau,
\]

\( \Theta(\hat{x}, \tau), \tau \in \mathbb{T} \) such that \( \Phi(x, t) = [\Theta(\hat{x}, \tau)]^{-1} \) then \( \alpha(\hat{x}), \xi(x) \) is a complete vector field and \( G = \{ \Phi(x, t) \}_{t \in \mathbb{T}} \) is the unique group extension, see e.g. [76]. Throughout this Part II we assume that \( G \) is connected and the tangent wedge of \( S_G \) (the closed convex cone defined by \( \mathcal{L}(S_G) := \{ \xi \in \mathcal{L}(G) | \exp(t\xi) \subseteq S_G \} \)) is a Lie-semi algebra, see [138].

By Corollary 9.1.5 in [163], the smooth vector field \( \xi \in T\mathcal{M} \) of bounded length generates the 1-parameter group of diffeomorphisms. In this subsection we use semigroups of diffeomorphisms on Hilbert manifolds to describe some interesting properties of the storage functions defined in Eqs. (5.11) and (5.12).

**Proposition 5.18.** Assume that \( S_a : \mathbb{R}^n \to \mathbb{R}^1 \) exists and is smooth on (the compact) \( \mathcal{M} \). The (maximal) flow generated by the vector field \( \xi^+(x) = -\nabla S_a(x) \) defined by

\[
\{ \Phi^t(x) \}_{t \geq 0} = \{ \phi : t \to \exp(t\xi^+)x | \xi^+(x) = -\nabla x^T S_a(x), x \in \mathcal{M}_0, \xi^+ \in T\mathcal{M}, t \in \mathbb{T} \} \tag{5.14}
\]

is a positive semigroup; meaning that for all \( t \in \mathbb{T}^+ \), \( \Phi^t(x) \) is defined on all \( \mathcal{M} \). Moreover, for any \( x \in \mathcal{M} \), \( \{ \Phi^t(x) \}_{t \geq 0} \), has at least one critical point of \( S_a \) as a limit point as \( t \to \infty \).

**Example 5.19.** For input-affine nonlinear systems (\( \mathfrak{N} \)-class), the observability function \( L_o(x^0) = \frac{1}{2} \int_0^\infty y^T y \, dt \), \( x(0) = x^0, u(t) = 0, 0 \leq t < \infty \), see e.g. [180], may fail to generate a semigroup action since it may be unbounded, thereby the need of assuming stability of \( \Sigma \). \( \square \)

**Proposition 5.20.** Assume that \( S_r^* : \mathbb{R}^n \to \mathbb{R}^1 \) exists and is smooth on (the compact) \( \mathcal{M}^* \). The (maximal) flow generated by the vector field \( \alpha_-(\hat{x}) = -\nabla S_r(\hat{x}) \) defined by
\[ \{ \Theta^\tau(\hat{x}) \}_{\tau \geq 0} = \{ \theta : \tau \to \exp(\tau \alpha_-(\hat{x}) | \alpha_-(\hat{x}) = -\nabla^T_{\tau} S^*_r(\hat{x}), \hat{x} \in M_0^*, \alpha_- \in T^* M, \tau \in \tau \} \]  
\hspace{1cm} (5.15)

is a negative semigroup (a positive semigroup on \( \tau \times M^* \)); meaning that for all \( \tau = -t \in \tau, \Theta^\tau(\hat{x}) \) is defined on all \( M^* \). Moreover, for any \( \hat{x} \in M^* \), \( \{ \Theta^\tau(\hat{x}) \}_{\tau \geq 0} \) has at least one critical point of \( S^*_r \) as a limit point as \( \tau \to -\infty \).

Since the storage functions decrease monotonically along the evolution of the semigroups \( \{ \Phi^t(x) \}_{t \geq 0} \) and \( \{ \Theta^\tau(\hat{x}) \}_{\tau \geq 0} \) (see proofs of Props. 5.18 and 5.20 in the Appendix), \( S_a \) and \( S_r \) are not integral invariants for these semigroups. Those maps whose integral functions \( S_a \) and \( S_r \) are invariant, are fundamental because they define isometries and appropriate names are given by:

**Definition 5.21 (Future Gramian).** The nonlinear map homeomorphism \( Q^t : M_0^t \to M_0^*, \hat{x}^0 = Q(x^0) \overset{\text{def}}{=} \nabla^T S^*_a(x^0, r_a), x^0 \in M_0, \hat{x}^0 \in M_0^* \) is an isometric isomorphism called the nonlinear future Gramian such that for \( \xi^+(x^0) \in TM_+ \) and \( \alpha_-(\hat{x}^0) \in T^* M_-, \) satisfy \( \langle \xi^+(x^0), \xi^+(x^0) \rangle_{T M^*_+} = \langle \alpha_-(\hat{x}^0), Q\xi^+(x^0) \rangle_{T^* M^*} \), where \( Q = [g_{ij}^+] \in \bigwedge^2, g_{ij}^+ = \partial^2 S_a(x, r_a) / \partial x_i \partial x_j \) is the future Hessian metric tensor.

**Definition 5.22 (Past Gramian).** The nonlinear map homeomorphism \( P^\tau : M_0^* \to M_0^t, x^0 = P(\hat{x}^0) \overset{\text{def}}{=} \nabla^T S^*_r(\hat{x}^0, r_r), x^0 \in M_0, \hat{x}^0 \in M_0^* \) is an isometric isomorphism called the nonlinear past Gramian such that for \( \xi^-(x^0) \in TM_- \) and \( \alpha_-(\hat{x}^0) \in T^* M_-, \) satisfy \( \langle \xi^-(x^0), \alpha_-(\hat{x}^0) \rangle_{T M^*_+} = \langle \alpha_-(\hat{x}), \alpha_-(\hat{x}) \rangle_{T^* M^*} \), where \( P = [g_{ij}^-] \in \bigwedge^2, g_{ij}^- = \partial^2 S^*_r(\hat{x}, r_r) / \partial \hat{x}_i \partial \hat{x}_j \) is the past Hessian metric tensor.

The internal structure of the Gramians is postponed to Proposition 5.46. Nevertheless in order to help understanding Definition 5.21 and Definition 5.22, a conceptual interpretation is provided. Consider the set of all trajectories \( x(t) \in M_+^t, t \in t \) (with initial condition \( x^0 \in M_0 \)) characterized by some vector field \( \xi^+ \in TM_+ \) with metrics \( \| \xi^+ \|_{T M_+} \). If there exist a dual (exact) differential \( \xi^- \in T^* M_- \) of \( S_a(x(t)) \) s.t. \( \| \xi^- \|_{T^* M_-} = \| \xi^+ \|_{T M_+} \), then \( Q^t \) is the map relating each primal element with its dual

\[ \hat{\xi}^-(\hat{x}^0) = \bigwedge Q \xi^+(x^0), \hat{x}^0 \in M^*_+, x^0 \in M_+. \]  
\hspace{1cm} (5.16)

Using a dual reasoning for \( S^*_r(\hat{x}, r_r) \) one obtains

\[ \hat{\alpha}^+(x^0) = \bigwedge P \alpha_-(\hat{x}^0), \hat{x}^0 \in M^*_+, x^0 \in M_+. \]  
\hspace{1cm} (5.17)
From Eqs. (5.16)-(5.17) we conclude that the vector field $\hat{\alpha}^+ \in TM_+$ has a dual covector (differential 1-form) $\alpha^- \in T^*M_-$, identified by its metrics $\hat{\alpha}_j = \sum_i P_{ij} \alpha^i$ and vice versa $\alpha^i = \sum_j Q^{ij} \hat{\alpha}_j$.

The initial conditions are dualized similarly. Since $\hat{x}^-(\tau) = \hat{x}_0^- + \int_0^\infty \hat{\xi}_-(x^-) \, dt$, $\hat{\xi}_- \in T^*M_-$, using standard arguments, one may write after Eq. (5.16), $\hat{x}^-(\tau) = \hat{x}_0^- + \int_0^\infty Q \xi^+ \, dt$, for $\xi^+ \in TM_+$. Moreover, the differential equation $d/dt Q^t(x(t)) = \partial_x Q^t(x(t)) \xi^+(x(t)) = Q \xi^+(x(t))$ with initial condition $x(0) = x^0 \in M$, can be expressed as the following integral equation:

$$ Q^t \circ x^+(t) \overset{\text{def}}{=} \hat{x}_0^0 + \int_0^\infty Q \xi^+ \, d\mu(t), \quad x^+(t) \in M^t_+, \xi^+ \in TM^t_+. \quad (5.18) $$

With an equivalent reasoning, Eq. (5.17) can be alternatively expressed by the integral equation

$$ P^\tau \circ \hat{x}^-(\tau) \overset{\text{def}}{=} x^0 + \int_0^\infty P \alpha_- \, d\mu(\tau), \quad \hat{x}^-(\tau) \in M^\tau_-, \alpha_- \in T^*M^-_. \quad (5.19) $$

**Example 5.23.** For linear time-invariant ($L$-class) dissipative systems $\Sigma^+$ and $\Sigma^-$, with general supply rates $r = r_a$, the Gramians $P$ and $Q$ are the extreme solutions of a Riccati equation, [212]. Recalling Example 5.13, $P \hat{x} = \partial_x^T S^*_x(\hat{x}, r_r)$ and $Q x = \partial_x^T S_a(x, r_a)$ and trivially $P = \partial^2 S^*_x(\hat{x}, r_r)/\partial \hat{x}^2$ and $Q = \partial^2 S_a(x, r_a)/\partial x^2$.

By duality (see Remark 4.16 in Part I the metric tensors can be made to satisfy $P_{ij} \overset{\text{def}}{=} [Q^{ij}]^{-1}$ such that the past and future metric tensors $P$ and $Q$ satisfy the Hilbert manifold structure of Proposition 5.16. This justifies the following definition.

**Definition 5.24 (Nonlinear internal balanced realization).** An internal system realization $\Sigma$ satisfying

$$ [P^\tau(\hat{x}(\tau))]^{-1} = Q^t(x(t)), \quad [P]^{-1} = Q. \quad (5.20) $$

is called balanced.

**Example 5.25.** For linear time-invariant ($L$-class) dissipative systems $\Sigma^+$ and $\Sigma^-$, by the balancing condition (5.20), $[P \hat{x}]^{-1} = Q x$, there is a group extension and a unique flow is defined. Furthermore, in this class, assuming asymptotic stability, with $r = u^T u$ and $r_a = y^T y$ ($u(t) = 0, 0 \leq t \in t^+$), $P$ and $Q$ are solution of two known Lyapunov equations, [143, 27].
Remark 5.26. In the Hilbert manifold framework of Proposition 5.16, a past semi-trajectory \( \hat{x}(t) \in \mathcal{M}^{*\tau} \) is identified with its dual future semi-trajectory \( x(t) \in \mathcal{M}_t^\tau \) by the relation in Eq. (5.20) namely \( [P^\tau(\hat{x}(\tau))]^{-1} = Q^t(x(t)), \) \( \hat{x}(0) = x(0). \) In particular, using the notation of Def 5.14, \( x_+^* \) is dual to \( x_+ \) and \( x_-^*(0) = x_+^*(0) = x^0. \)

Notice that it can be defined an associated automorphism in \( \Upsilon^t : t \times \mathcal{M}_0 \to \mathcal{M}_0^t \) given by

\[
\Upsilon^t \circ x(t) = P^\tau \circ Q^t \circ x(t), \quad x(t) \in \mathcal{M}_0^t
\]

with associated singular value problem associated to (5.21), with \( \sigma^2 \in \mathbb{R}^+ \) defined by

\[
\Upsilon^t \circ x(t) - \sigma^2 x(t) = 0, \quad x(t) \in \mathcal{M}_0^t.
\]

In the following result is shown that the balancing condition (5.20) serves to define the isometric isomorphism associated to a duality pairing, i.e., duality is provided with an isometric bijective map with inverse map such that both are structure-preserving:

**Proposition 5.27.** Let the system trajectories of \( \Sigma \) be supported by the manifolds \( (\mathcal{M}_t^\pi, \langle \cdot, \cdot \rangle_{T\mathcal{M}_t^\pi}) \) and \( (\mathcal{M}^{*\tau}, \langle \cdot, \cdot \rangle_{T^* \mathcal{M}^{*\tau}}) \) on the Hilbert Manifold structure of Proposition 5.16. The following can be asserted:

1. Let us define \( \Xi(\hat{x}^0(\tau)) \overset{\text{def}}{=} P^\tau(\hat{x}^0(\tau)) \) (Eq. (5.19)) in the duality pairing in Table 5.2, \( \langle \cdot, \cdot \rangle_{T^* \mathcal{M}^{*\tau} \times T \mathcal{M}_t^\pi} : T^* \mathcal{M}^{*\tau} \times T \mathcal{M}_t^\pi \to \mathbb{R}^1. \) Then duality can be identified by \( \langle \cdot, \cdot \rangle_{T^* \mathcal{M}^{*\tau} \times T \mathcal{M}_t^\pi} \) and \( \Xi(\hat{x}^0(\tau)) \) is an isometry.

2. Let us define \( \Xi^\dagger(x(t)) \overset{\text{def}}{=} Q^t(x(t)) \) (Eq. (5.18)) in the duality pairing in Table 5.2, \( \langle \cdot, \cdot \rangle_{T \mathcal{M}_t^\pi \times T^* \mathcal{M}^{*\tau}} : T \mathcal{M}_t^\pi \times T^* \mathcal{M}^{*\tau} \to \mathbb{R}^1. \) Then duality can be identified by \( \langle \cdot, \cdot \rangle_{T \mathcal{M}_t^\pi \times T^* \mathcal{M}^{*\tau}} \) and \( \Xi^\dagger(x(t)) \) is an isometry.

3. Assume the balancing condition (5.20) is satisfied. Then both duality pairings in Table 5.2 are equivalent and the nonlinear past Gramian \( P^\tau \) is adjoint to the future Gramian \( Q^t \) satisfying:

\[
\langle \alpha_-, Q^t \xi^+ \rangle_{T^* \mathcal{M}^{*\tau}} = \langle P^\tau \alpha_-, \xi^+ \rangle_{T \mathcal{M}_t^\pi}, \quad \xi^+ \in T \mathcal{M}_t^\pi, \quad \alpha_- \in T^* \mathcal{M}^{*\tau}. \tag{5.23}
\]

4. The composition maps \( \Upsilon \overset{\text{def}}{=} P^\tau \circ Q^t : \mathcal{M}_t^\pi \to \mathcal{M}_t^\pi \) and \( \Upsilon^\dagger \overset{\text{def}}{=} Q^t \circ P^\tau : \mathcal{M}^{*\tau} \to \mathcal{M}^{*\tau} \) satisfy

\[
\langle \xi^+, \Upsilon_* \xi^+ \rangle_{T \mathcal{M}_t^\pi} = \langle \Upsilon_* \xi^+, \xi^+ \rangle_{T \mathcal{M}_t^\pi} , \tag{5.24}
\]

\[
\langle \alpha_-, \Upsilon^\dagger \beta_- \rangle_{T^* \mathcal{M}^{*\tau}} = \langle \Upsilon^\dagger \alpha_-, \beta_- \rangle_{T^* \mathcal{M}^{*\tau}} , \tag{5.25}
\]
In Part I, Gauss’ Curvature Theory was used to characterize the invariants of behavior $\mathfrak{B} \subset \mathcal{W}$, and thus are selfadjoint auto morphisms in this sense.

5. Assume the balancing condition (5.20) is satisfied. Then on the Hilbert manifold structure of Proposition 5.10 the same operators $P^T$, $Q^t$, $\Upsilon$ and $\Upsilon^t$ of Proposition 5.27 (3,4) satisfy the following

\begin{equation}
\langle \alpha, Q\xi \rangle_{T^*M_0} = \langle \mathcal{P}_\alpha, \xi \rangle_{TM_0}, \quad \xi \in TM_0, \quad \alpha \in T^*M_0 \quad (5.26)
\end{equation}

\begin{equation}
\langle \xi, \Upsilon_\alpha \xi \rangle_{TM_0} = \langle \Upsilon^t_\alpha \xi, \xi \rangle_{TM_0} \quad (5.27)
\end{equation}

\begin{equation}
\langle \alpha, \Upsilon^t_\alpha \beta \rangle_{T^*M_0} = \langle \Upsilon^t_\alpha \beta, \alpha \rangle_{T^*M_0}, \quad \alpha, \beta \in T^*M_0. \quad (5.28)
\end{equation}

5.3.2 Curvature of the space of internal signals

In Part I, Gauss’ Curvature Theory was used to characterize the invariants of behavior $\mathfrak{B} \subset \mathcal{W}$. Similar results can be obtained for the space of internal signals on $M_0 \subset M$.

**Theorem 5.28.** Let the system trajectories of $\Sigma$ be supported by the Riemannian Hilbert manifolds $(M^t, \langle \cdot, \cdot \rangle_{T^*M^t})$ and $(M^+, \langle \cdot, \cdot \rangle_{T^*M^+})$ endowed with the inner products and duality identified from Table 5.2. Furthermore assume that Eqs. (4.10)-(4.11) in Part I determine the past and future metrics by $g^-_M(x^-,0) \overset{def}{=} g^-_M(\hat{w}^-,0)$ and $g^+_M(x^+,0) \overset{def}{=} g^+_M(\hat{w}^+,0)$ respectively. The following can be asserted:

1. The curvature can be expressed by

\begin{equation}
K(\alpha_-) \overset{def}{=} \frac{II_M(\alpha_-, \alpha_-)}{I_M(\alpha_-, \alpha_-)} = \frac{\langle \Upsilon^t_\alpha \alpha_-, \alpha_- \rangle_{T^*M^+}}{\langle \alpha_-, \alpha_- \rangle_{T^*M^+}}, \quad (5.29)
\end{equation}

s.t. $I_M(\alpha_-, \alpha_-) = S^*(\hat{x}^0, r)$ and $II_M(\alpha_-, \alpha_-) = \langle A^M_\eta(\alpha_-), \alpha_- \rangle_{T^*M^-}$, with Shape operator given by $A^M_\eta(\alpha_-) = \Upsilon^t_\alpha \alpha_-= (Q^t \circ P^t)|^* \alpha_- = Q^t_\alpha P^t*, \alpha_- \in T.\mathcal{M}^\tau, \eta \in (T.\mathcal{M}^\tau)^\perp$.

2. A 1-form $\beta_- \in T.\mathcal{M}^\tau, (\beta_-, \beta_-)_{T.\mathcal{M}^\tau} = 1$ is solution to the eigenvalue problem associated to $K(\beta_-)$ in (5.29) iff $\beta_-$ is an eigenform of $A^M_\eta$.

3. The set of eigenforms of $A^M_\eta$, \{ $\beta^i_- \mid i = 1, \ldots, n; \beta^i_- \in T_p\mathcal{M}^\tau$ \}, defines an orthonormal coframe of $T_p\mathcal{M}^\tau$. Furthermore $T_p\mathcal{M}^\tau$ can be locally spanned by a partition of eigencodistributions $M^t_1 \oplus \cdots \oplus M^t_n$.

4. The frame $\{ \zeta^+_1, \zeta^+_2, \ldots, \zeta^+_n \}$ defined by dualization of the coframe in Thm 5.28(3) spans $T_p\mathcal{M}^t_+.$

5. The Shape operator is related to the metric tensors $Q$ and $P$ defined in Defs. 5.21 and 5.22 by $A^M_\eta = Q^t P^{-1}, \forall x^0 \in T_p\mathcal{M}_0$. 


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6. Given a vector field \( \xi^+ \in T_p M^+_t \), its local orthogonal projection \( \xi^+_\text{op} \subset T_p M^+_\text{+red} \) on a submanifold \( M^+_\text{+red} \subset M^+_t \), s.t. \( T_p M^+_\text{+red} = \text{span}\{\zeta^+_i; i \in m, m \leq n\} \) is given by

\[
\xi^+_\text{op} = \sum_{i=1}^m \langle \xi^+, \zeta^+_i \rangle_{T_p M^+_t} \zeta^+_i, \quad \xi^+ \in T_p M^+_t
\]

(5.30)

\[
\alpha^-_\text{op} = \sum_{i=1}^m \langle \alpha_-, \beta^-_i \rangle_{T_p M_-^\tau} \beta^-_i, \quad \alpha_- \in T_p M_-^\tau.
\]

(5.31)

7. The operators \( \Upsilon, \Upsilon^\dagger \) admit a (spectral) decomposition

\[
\Upsilon^* \xi^+ = \sum_{i=1}^n \lambda_i \langle \xi^+, \zeta^+_i \rangle_{T_p M^+_t} \zeta^+_i, \quad \xi^+ \in T_p M^+_t,
\]

(5.32)

\[
\Upsilon^\dagger^* \alpha_- = \sum_{i=1}^n \lambda_i \langle \alpha_-, \beta^-_i \rangle_{T_p M_-^\tau} \beta^-_i, \quad \alpha_- \in T_p M_-^\tau.
\]

(5.33)

5.3.3 The nonlinear eigenproblem for evolutionary operators

In Part I, Proposition 4.37, several eigenproblems for nonlinear map operators in a Hilbert manifold framework for external signals were presented. Such nonlinear maps are semigroups of diffeomorphisms whose invariants are characterized as in Proposition 4.36 in terms of integral invariant functions and generating vector fields. The following nonlinear eigenvalue problem will be needed throughout several sections of this Part II:

**Proposition 5.29 (Nonlinear eigenproblem for internal functions).** Consider the Hilbert manifold structure defined in Proposition 5.16 under Assumption 5.12 supporting the storage functions \( S_a, S^*_r \in L^2_b[0, \infty), S_a : M \to \mathbb{R}^1, S^*_r : M^* \to \mathbb{R}^1 \) and the nonlinear maps \( \nabla S_a : M \to M^* \) and \( \nabla S^*_r : M^* \to M \). The following is asserted:

1. The eigenproblem defined by \( \nabla S_a(x) = \lambda \nabla S^*_r(\hat{x}) \) for eigenvalues \( \lambda \in \mathbb{C}^1 \) and eigentrajectories \( 0 \neq x(t) \in M^+_t, 0 \neq \hat{x}(\tau) \in M^*_- \) is well posed.

2. The eigenproblem in Theorem 5.28(1) in Eq. (5.29) is equivalent to the nonlinear eigenproblem

\[
\nabla_x S_a(x(t), r) - \lambda_i \nabla_{\hat{x}} S^*_r(\hat{x}(\tau), r) = 0,
\]

(5.34)

for \( \lambda_i \in \mathbb{R}^1, x_i(t) \in M^t, \hat{x}_i(\tau) \in M^*_- \), in the sense that both have the same set of eigentrajectories \( \{\rho_i(x^0) \in M_0| i = 1, \ldots, n\} \) along the principal
directions \( \{ \zeta^+_i \in T^*_M, \beta^+_i \in T^*M \} \) with principal curvatures \( \{ \kappa_i(\beta^+_i) = \lambda_i \} \).

The nonlinear eigenvalue problem in Eq. (5.34) has been known for a while and their properties were analyzed earlier, see [18, 17, 19, 47, 147] and references in [163]. For nonlinear balancing theory, it was proposed firstly in [44] as a solution for nonlinear balancing problems. The proof of Proposition 5.29 provides the geometric interpretation to the nonlinear balancing problem in the context of the Hilbert manifold structures previously presented and for future use. Since with evident changes the same Proposition 5.29 serves to explain the balancing situation of Props. 4.37 in Part I, for this Part II we provide the relevant results only for internal signals.

Example 5.30. In the case of Hankel type balancing where \( S_a(x^0, y^T y) = L_o(x^0) \) and \( S_r(x^0, u^Tu) = L_c(x^0) \) are the observability and controllability functions respectively associated to the asymptotically stable system (5.1), and whenever the balancing condition (5.20) is satisfied, Eq. (5.34) can be written as \( \nabla_x L_o(x(t)) - \lambda_i \nabla_x L_c(x(t)) = 0 \) and such equation was also firstly proposed for nonlinear balancing in [44] being the nonlinear version of the generalized eigenvalue problem.

5.3.4 Past-future invariance and the Legendre transform

An intrinsic property of a mathematical object is called a symmetry if it is invariant under a class of transformations. A symmetry leads to a conserved quantity, for instance a symmetry in time leads to the conservation of some form of energy in physical systems. Throughout this paper we dealt with isometries whose invariant property is evidently the distance defined by well-defined metrics. Thus there is a metric which is a functional invariant of the space of trajectories preserved throughout the past and future evolution of the system. The following result, (from [126] with corrections) specifies this invariant:

**Proposition 5.31 (Duality Legendre transform).** The following statements are equivalent:

1. The storage functions \( S_a(x(t), r_a), x(t) \in M, t \geq 0 \) and \( S^*_r(\dot{x}(\tau), r_r), \dot{x}(\tau) \in M^*, \tau \leq 0 \) satisfy the following:

\[
L(x(t), \dot{x}(\tau)) = S_a(x(t), r_a) + S^*_r(\dot{x}(\tau), r_r) - \langle \xi^+(x), \alpha_-(\dot{x}) \rangle_{T^*M \times T^*M} = 0,
\]

where \( \dot{x}(t) = \xi^+(x(t)) \) and \( \dot{x}(\tau) = \alpha_-(\dot{x}(\tau)) \) and \( \langle \cdot, \cdot \rangle_{T^*M \times T^*M} \) is defined in Table 5.1.
2. Invariance of the Legendre transform (5.35) implies:

\[
\dot{x}(\tau) = \nabla^T S_a(x(t), r_a),
\]

\[
x(t) = \nabla^T S_r^*(\dot{x}(\tau), r_r).
\]

3. Assuming that \(\nabla^T S_a(x(t), r)\) has a regular inverse, (5.36) and (5.37) are related by

\[
[\nabla^T S_a(x(t), r_a)]^{-1} = \nabla^T S_r^*(\dot{x}(\tau), r_r).
\]

Although arbitrary trajectories along the evolution of a dissipative system are such that the metric of a past semi-trajectory is generically larger than the metric of a future semi-trajectory, there is an exceptional set of trajectories (with tangent vector fields invariant under the isometry \(\Upsilon: \mathcal{M}_0 \rightarrow \mathcal{M}_0\)) such that the dissipation inequality remains an equality with stationary storage functions related by singular functions. The isometry is evident since along each of such trajectories the duality product preserves invariant both storage functions.

5.3.5 Orthogonal separability of functions

Along the set of points where the curvature of the space of internal signals \(\mathcal{M}_0\) attains stationary values \(\kappa_i\), there is a set of tangent eigenvector fields that span its tangent space. The internal trajectories have a projection on every orthogonal eigenspace and therefore each of this projections have an associated invariant function along its evolution. The dual problem consist in the decomposition of the storage functionals into a set of orthogonal eigenfunctions on the space of integral functionals, i.e. the principal frame-balancing problem discussed in Sec. 4.3.4, Part I. Concepts like partial, complete and orthogonal separability are also important for the discussion of this section and therefore the reader is advised to review the definitions presented therein. We concentrate on smooth storage functions \(S(x): \mathcal{M}_0 \rightarrow \mathbb{R}\) on the \(n\)-dimensional Riemannian \(C^\infty\)-manifold \((\mathcal{M}_0, g_{\mathcal{M}_0})\), expressed as

\[
S(x) \overset{\text{def}}{=} \sum_{i=1}^{n} S_i(x^i, c^i), \quad x \in \mathcal{M}_0
\]

where \(S_i(x^i, c^i)\) denotes a (functionally independent) \(i^{th}\)-function depending on a single \(i^{th}\)-external variable and a separation constant \(c^i \in \mathbb{R}^1\). Such function is said to be orthogonally separated if it is separated and the (Riemannian)
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tensor metric $g_{ij}$ is diagonal, see e.g. [14]. In such conditions it has been stated that: (a) $S(x)$ is orthogonally separated iff the metric tensor $[g^{ij}]$ is diagonal and the matrix of the Hessian $[S]_{**}$ has smooth functions along its diagonal and zero elsewhere:

$$\nabla^2_x S(x) = \text{diag} \{ \partial^2_{x_i} S_i(x^i), i = 1, \ldots, n \}. \quad (5.40)$$

(b) The separation property is invariant under separated transformations. Associated to each $\text{supp}(S_i(x))$ there is an (open) complementary set $\{ \eta \in M_0 | \eta \notin \text{supp}(S_i(x^i)) \}$. Define the invariant set $N$ of a function $S(x) : M_0 \rightarrow \mathbb{R}$ as the largest submanifold on $M_0$ such that for all $\eta \in N \subset M_0$, $S(x + \eta) = S(x), x \in M_0$.

**Definition 5.32 (Internal orthogonal web).** Let $(M_0, g_{M_0})$ be a Riemannian manifold. The family $\{N_i\} \overset{\text{def}}{=} \{N_1, \ldots, N_n\}$ of $n$-orthogonal foliations of hypersurfaces, i.e. $(n-1)$-dimensional submanifolds, defined on all $M_0 \setminus \psi$ (with the exception of a singular set $\psi$), is called an orthogonal web, [13]. A parameterization of an orthogonal web consists of a set of functions $\{S_i(x^i)\} = \{S_1(x^1), \ldots, S_n(x^n)\}$ on $M_0 \setminus \psi$ s.t. $dS_i(x)|_{N_i} = 0$ and $dS_i(x)|_{N_j} \neq 0, \forall i \neq j$. A parameterization $\{S_i\}$ adapted to the orthogonal web on $M_0$ defines an orthogonal frame for $M_0$, [14].

The following definition is related to the separability of the metric functions imposed in Assumption 4.27 in Part I, resulting in the following balancing and principal frame-balancing conditions:

**Definition 5.33 (Internal balancing conditions).** A realization of $\Sigma$ satisfying Assumption 4.29 (Part I) is said to be balanced if it is such that

$$[\nabla^2_x S_r^*(\hat{x}, r_r)]^{-1} = \nabla^2_x S_a(x, r_a). \quad (5.41)$$

is regular. Furthermore, if in such frame

$$[\partial^2_{x_i} S_{r_i}^*(\hat{x}^i, r_r)]^{-1} = \partial^2_{x_i} S_{a_i}(x^i, r_a), \quad i = 1, \ldots, n \quad (5.42)$$

then such realization is said to be principal frame-balanced.

Consider the framework by [212] for linear time-invariant systems (the $\mathcal{L}$-class). If the system is balanced and additionally $P = Q$ is diagonal and their vectors are ordered in terms of their ordered spectrum, it is called principal-axis balancing, [27]. One nonlinear generalization to this is presented in the following.
Remark 5.34. In [44] the concept of principal axis-balancing was introduced as a generalization of the linear concept. Although such work considered a differentiable structure in terms of differentiable operators, it should be remarked that in [44] it is missed the natural differential-geometric structure for the reconstruction of integral trajectories. One of the main contributions of the differential-geometric approach presented in this work is a structured conceptualization and the geometric characterization of every nonlinear operator used. Furthermore if $T\mathcal{M}^t \cong \mathcal{M} \times \mathbb{R}^n$ the space $\mathcal{M}^t$ is parallelizable (for a definition see Definition 4.18 in Part I and a global definition of the frame is attained. The principal frame-balancing, is a proper generalization of the linear principal-axis balancing since at every point $p \in \mathcal{M}$ the principal set of fixed-axis vectors is obtained.

By condition (5.41) every past trajectory has a unique associated future trajectory and by condition (5.40) in Proposition 4.54, Part I, expressed otherwise as condition (5.42), both functionals $S^*_r(\hat{x}^0, r)$ and $S_a(x^0, r)$ defined by Eqs. (5.8) and (5.7) are orthogonally separated. These arguments are evidenced from the following results:

Proposition 5.35. Let the system realization $\Sigma$ be supported by the Hilbert manifold structure of Proposition 5.16 satisfying Assumption 5.12, namely, with past and future metrics defined by the storage functions $S^*_r(\hat{x}^0, r) \in C^\infty(\mathcal{M}^*)$ and $S_a(x^0, r) \in C^\infty(\mathcal{M})$ from Eqs. (5.8)-(5.7). Furthermore, assume that the principal frame-balancing conditions (5.41)-(5.42) are satisfied. The following is asserted:

1. The orthogonal web for $S^*_r(\hat{x}^0, r_r)$ is given by the set of annihilators $\{\Delta^\perp_i\}$ of the distributions $\Delta_i = \text{span}\{\zeta^+_i\}$. Furthermore the orthogonal web for $S_a(x^0, r_a)$ is given by the set of annihilators $\{\Omega^\perp_i\}$ of the codistributions $\Omega_i = \text{span}\{\beta^-_i\}$.

2. A parameterization of the orthogonal webs on $\mathcal{M}^-_r$ and $\mathcal{M}^+_a$ is given by the components $\{S^+_i\}$ and $\{S^-_a\}$, $i = 1, \ldots, n$ of each separated metric. Moreover, each component $S^+_r$ (resp. $S^-_a$) is an integral invariant of the trajectories on the associated (co-) distribution $\Delta^\perp_i$ (resp. $\Omega^\perp_i$).

Although it is not shown, it is apparent that the family of system realizations $\Sigma$ satisfying condition (5.42) is unique up to separated transformations. The following result is completely equivalent to the proof of Lemma 4.58 (Part I):

Lemma 5.36. The functionals $S^*_r(\hat{x}^0, r)$ and $S_a(x^0, r)$ in Eqs. (5.8)-(5.7) are well defined Riemannian metrics for the Hilbert manifolds $(\mathcal{M}^-_r, \langle \cdot, \cdot \rangle_{T\mathcal{M}^-_r})$, $(\mathcal{M}^+_a, \langle \cdot, \cdot \rangle_{T\mathcal{M}^+_a})$. 
The concept of Lie-subgroups is implied on the role that orthogonal separability has on classification, factorization and reduction of nonlinear systems: Denote by $\mathcal{M}_i$ an immersed submanifold of a manifold $\mathcal{M}$. Let $G_i$ be a subgroup on $\mathcal{M}_i$, $G$ a Lie-Group on $\mathcal{M}$ and $\phi : G_i \to G$ a smooth map between them. We say that such $G_i$ is a Lie-subgroup on $\mathcal{M}$ if $G_i$ is itself a Lie-group on $\mathcal{M}$, $G_i = \phi (G_i)$ is the image of $\phi$, and $\phi$ is a Lie-group homomorphism. The subgroup $G_i$ is regular (a regular submanifold of $\mathcal{G}$) if it is closed [155].

Consider in particular a principal-frame balanced realization of $\Sigma$ implying that $S_a(x,r_a)$ is orthogonally separated in the form of Eq. (5.39). Associated to every orthogonal component $S_a^i(x^i,c^i)$ of the separated function is a subgroup $\Phi^t : \mathcal{M}_0 \to \mathcal{M}_0$ defined by:

$$\{ \Phi^t_i(x^0) \}_t = \{ \phi : t \to \exp(t\xi)x^0|\xi(x^0) = -\nabla^T_x S_a^i(x^0), x^0 \in \mathcal{M}_0, \xi \in T\mathcal{M}, t \in \mathbb{T} \}$$

(5.43)

for $i = 1, \ldots, n$, defining a Lie group isomorphism, i.e. a smooth map $\phi : \mathcal{M}_i \to G$ satisfying the group operations: $\phi(g \cdot \hat{g}) = \phi(g) \cdot \phi(\hat{g})$ with smooth inverse $\phi^{-1}$ for $g, \hat{g} \in G$, since $\Phi^t_i(x^0) = \exp(t\xi_{1+}) \cdots \cdot \exp(t\xi_{n+})x^0$. Thus, every component function is a generating function of a subgroup of the group of diffeomorphisms and the action of such subgroups lies on a submanifold $\mathcal{M}_i$ of the state-space, inducing a decomposition of the state-space manifold $\mathcal{M}$ in submanifolds, useful for classification and balanced reduction.

5.3.6 The adjoint system and duality

Until now we have paid lip service to the dynamical system associated to the adjoint sub operator $\Gamma^\dagger : \mathcal{U}^* \to \mathcal{Y}^*$ in the behavioral operator from Part I (Definition 4.8). The analysis of this adjoint system and the internal structure of the adjoint operators of the following Sec. 5.3.7 requires a backward-time evolution of $\Sigma^\dagger$ with reverted causality relationships on inputs and outputs. Adjoint systems for balanced reduction have been discussed previously in [46]. Nevertheless in this subsection we provide a geometric view adequate to the context of this dissertation. In particular we generalize the linear concept of pseudo inverse (limited to a local Euclidean metric) with the geometric concepts of inverse map and equivalence relations:

Consider the set of input trajectories $\{ u_i(t)|u_i(t) \in \mathcal{U}, t \in t^- \}$ on the set of admissible inputs $\mathcal{U}$ such that a point $x^0 \in \mathcal{M}_0 \subseteq \mathcal{M}$ can be reached from the origin following a trajectory $x(t)$. Define for this set the equivalence relation $u_iGu_j$ for a vector function $G : \mathcal{U} \to \mathcal{M}, G = g(u)$, such that $u_i$ is equivalent to $u_j$ if both produce the $x(t)$, i.e. $g(u_i) = g(u_j)$, resulting in the equivalence
After these definitions, we are prepared to the following:

**Definition 5.37.** The inverse map of \( h \), denoted by \( h^{-1} : \mathcal{Y}^* \rightarrow \mathcal{M}^* \) is the unique vector function in the set \( \{ h^{-1}(\hat{x}(t)), \hat{x}(t) \in [\hat{x}(t)] \} \) s.t. \( \hat{x}(t) \) minimizes the squared norm \( \| \hat{x}(t) \|^2_{T_{\mathcal{M}}} \) defined \( S_a(x^0, r_a) \). Furthermore, the inverse map of \( G \), denoted by \( g^{-1} : \mathcal{M}^* \rightarrow \mathcal{U}^* \) is the unique vector function in the set \( \{ g^{-1}(\hat{u}(t)), \hat{u}(t) \in [\hat{u}(t)] \} \) s.t. \( \hat{u}(t) \) minimizes the squared norm \( \| \hat{u}(t) \|^2_{T_{\mathcal{M}}} \) defined \( S_a(\hat{x}^0, r_r) \).

A regular equivalence relation may not be compatible with group actions, see e.g. [33], therefore we have to consider valid the following:

**Assumption 5.38.** The equivalence relations \( H \) and \( G \) are regular and compatible with the group actions \( \Phi \) and \( \Theta \), s.t. \( \Phi H \Phi \) and \( \Theta G \Theta \) are well defined.

After these definitions, we are prepared to the following:

**Definition 5.39 (Adjoint system \( \Sigma^+_+ \)).** The nonlinear system (\( \mathfrak{S} \)-class) adjoint to \( \Sigma^+ \) is given by:

\[
\Sigma^+_+ : \begin{cases} 
\dot{x}(t) = \hat{F}(\hat{x}(t), h^{-1} \circ \hat{y}(t)), & t \geq 0 \\
\dot{u}(t) = g^{-1}(\hat{x}(t)), & t \geq 0 \end{cases}
\]  

(5.44)

where \( \hat{x}(t) \in \mathcal{M}^* \), \( \hat{u}(t) \in \mathcal{U}^* \) and \( \hat{y}(t) \in \mathcal{Y}^* \). The adjoint system is determined as the inverse system (i.e. with reversed input/output causality) of the backward-time system \( \Sigma^- \) in Eq. (5.2) mapped into forward-time. Therefore the evolutionary operator (semigroup) \( \hat{x}(t) = \Theta(T, 0, \hat{x}^0, h^{-1} \circ \hat{y}(t)) \) describes the trajectories of \( \Sigma^+_+ = (t, \mathcal{M}^*, \Theta^t) \).

**Example 5.40.** For linear time-invariant (\( \mathfrak{L} \)-class) dissipative systems \( \Sigma^+ \) and \( \Sigma^- \), from Definition 5.39 and Eq. (5.4) yields the adjoint system

\[
\Sigma^+_+ : \begin{cases} 
\dot{x}(t) = -A^T \hat{x}(t) - C^T \hat{y}(t), & t \geq 0 \\
\dot{u}(t) = B^T \hat{x}(t), & t \geq 0 \end{cases}
\]  

(5.45)

The following definition is posed as complementary to the concept of a dissipative dynamical system:
Definition 5.41 (Backward-dissipative system). A system $\Sigma^+$ is called backward-dissipative, assimilative or absorptive if its associated backward-time system $\Sigma^-$ is dissipative.

In terms of the theory of dissipative systems [219], such assimilative system is a dissipative system with a negative dissipation rate that would be called then assimilation rate. Thus a dissipative system interconnected to an assimilative system would yield a conservative system if the dissipation rate of the former equals the assimilation rate of the latter. Based on the work developed throughout the previous sections some observations are in order.

Proposition 5.42. Support systems $\Sigma$ and $\Sigma^\dagger_+$ on the Hilbert manifold structure of Proposition 5.16. Then:

1. The adjoint system $\Sigma^\dagger_+$ is backward-dissipative.
2. Let the vector field $\alpha_+ \in T^*_p M^t$ be generator of the group action $\{\Theta^t\}_{t \geq 0}$ of $\Sigma^\dagger_+$. Then $\alpha_+ \in T^*_p M^t_+$ has an orthogonal projection given by

\[ \alpha^{op}_+ = \sum_{i=1}^m (\alpha_+ , \beta^i_+) T^*_p M^t_+ \beta^i_+ , \quad \alpha_+ \in T^*_p M^t_+ . \]

(5.46)

Since after the Hilbert manifold structure in Table 5.2, both co-adjoint systems share their past and future metrics, it is apparent that the past Gramian of $\Sigma^+$ is the future Gramian of $\Sigma^\dagger_+$ and the future Gramian of $\Sigma^+$ is the past Gramian of $\Sigma^\dagger_+$.

5.3.7 Structural relationships of internal and exogenous operators

We have seen that the invariants on the space of exogenous variables have a parallel set of invariants on the space of internal signals. In particular the nonlinear exogenous operators called behavioral operator and Hankel operator have an associated self-adjoint operator $A^M_\eta$ whose invariants determine the orthonormal principal frame. To conclude this section we establish further structural relationships for these operators.

Based on the fairly general concept of isometries, in Part I we introduced two particular definitions of adjoint and self-adjoint operators. In particular for the Hilbert Manifold structure of Proposition 5.16, it has been stated in Proposition 5.27 (3) that the nonlinear future Gramian $Q^t$ has as adjoint the past Gramian $P^t$. Moreover, from Proposition 5.27 (5) the internal isometric isomorphism $\Upsilon : M_0 \rightarrow M_0$ defined on the Hilbert manifold $(M_0, (\cdot, \cdot)_{T M_0})$ is self-adjoint with a tangent map $\Upsilon_* : T M_0 \rightarrow T M_0$ coincident with the Shape
operator $\Upsilon_\ast \overset{\text{def}}{=} A^M_\eta$.

The adequate decomposition of such self-adjoint operator provides us with a properly defined partition of the state-space into subsets, a criterion to discard unimportant dynamic structures and finally, the synthesis procedure for a reduced-order system, similarly to Proposition 4.62 in Part I.

As expressed in Sec. 5.2.1, the response $x(t) = \Phi(t; t_0, x_0, u(t))$, $u(t) \in \mathcal{U}$ in forward-time for system (5.1) and its backward-time response $\hat{x}(\tau) = \Theta(\tau; \tau_0, \hat{x}_0, h^{-1} \circ \hat{y}(\tau))$, $\hat{y}(\tau) \in \mathcal{Y}^*$ in terms of system (5.2) can be efficiently described using Lie semigroups, introduced in Definition 5.1, see also [74, 3] and [155] for details on exponentiation of vector fields.

These semigroup actions are interrelated as described in the following commutative diagram:

$$
\begin{array}{c}
\mathcal{M}_+ \xleftarrow{\Phi} \mathcal{M}_0 \xleftarrow{\Theta^{-1}} \mathcal{M}_-\\
\ast \ \ \ \ | \ \ \ | \ \ \ | \ast \\
\mathcal{M}_+^* \xrightarrow{\Phi^{-1}} \mathcal{M}_0^* \xrightarrow{\Theta} \mathcal{M}_-^*
\end{array}
$$

(5.47)

By $G(\mathcal{F}_u)$ we have expressed the (connected) Lie group of diffeomorphisms in $\mathcal{M}$ generated by the union of $\{\exp t \xi \mid t \in \mathcal{t}, \xi \in \mathcal{F}_u\}$. Assuming that $\mathcal{F}_u$ is composed of complete vector fields, each element $\Phi$ of $G(\mathcal{F}_u)$ is a diffeomorphism of $\mathcal{M}$ of the form of Eq. (5.5) for $t_i \in \mathcal{t}$ and vector fields $\xi_i \in \mathcal{F}_u$, $i = 1, \ldots, k$.

In the remainder of the section the action of semigroups of diffeomorphisms on different subintervals is discussed. In order to ease readability sometimes the temporal super index $t$ is replaced by the time subinterval $[t, t_0]$ where the group action is taking place. For instance, in this notation the semigroup $S^+_\mathcal{F}_u$ generated by the family $\xi \in \mathcal{F}_u$ is expressed by

$$
\Phi^{[t_k, t_0]}(x_0) = \exp t_k \xi_k \exp t_{k-1} \xi_{k-1} \cdots \exp t_2 \xi_2 \exp t_1 \xi_1 x^0.
$$

Assumption 5.43. The system $\Sigma$ is subject to appropriate inputs $u(x(t)) \in \mathcal{U}$, $\hat{u}(\hat{x}(\tau)) \in \mathcal{U}^*$ defined in terms of state or co-state variables as solution of the optimal control problems such that from Sec. 5.3.1 the corresponding solutions $\Phi, \Theta$ are well defined semigroups of diffeomorphisms.

Some previous definitions are necessary:

**Definition 5.44.** For system (5.1) the following associated operators $\Psi_p : \mathcal{L}_2[-T, 0] \rightarrow \mathbb{R}^n$, $\Psi_f : \mathbb{R}^n \rightarrow \mathcal{L}_2[0, T]$ are called the past and future behavior and defined by
5.3 Isometric invariants and the balancing condition

\[ \Psi_p \circ u(t) \overset{\text{def}}{=} \Theta^{-1}(0, -T, 0, g \circ u(t)), \quad \Theta \in S_{F_u}^- \]  

(5.48)

\[ \Psi_f \circ x^0 \overset{\text{def}}{=} h \circ \Phi(T, 0, x^0, g \circ u(t)). \quad \Phi \in S_{F_u}^+ \]  

(5.49)

Using Definition 5.37, their adjoint operators \( \Psi_p^\dagger : \mathbb{R}^n \rightarrow \mathcal{L}_2[-T, 0], \Psi_f^\dagger : \mathcal{L}_2[0, T] \rightarrow \mathbb{R}^n \) are given by

\[ \Psi_p^\dagger \circ x^0 \overset{\text{def}}{=} g^{-1} \circ \Theta(-T, 0, \hat{x}^0, h^{-1} \circ \hat{y}(\tau)), \quad \Theta \in S_{F_u}^- \]  

(5.50)

\[ \Psi_f^\dagger \circ \hat{y}(\tau) \overset{\text{def}}{=} \Phi^{-1*}(0, T, h^{-1} \circ \hat{y}(\tau)). \quad \Phi \in S_{F_u}^+ \]  

(5.51)

Furthermore, define the operator \( \Gamma : \mathcal{L}_2[-T, 0] \rightarrow \mathcal{L}_2[0, T] \) by \( \Gamma \overset{\text{def}}{=} \Psi_f \circ \Psi_p \circ u(t) \), i.e. the composition of (5.48) and (5.49):

\[ \Gamma \circ u \overset{\text{def}}{=} h \circ \Phi[T,0] \circ \Theta^{-1}(0, -T, 0, g \circ u(t)). \quad t \in \mathfrak{t} \]  

(5.52)

with adjoint operator \( \Gamma^\dagger : \mathcal{L}_2[0, T] \rightarrow \mathcal{L}_2[-T, 0] \) defined by \( \Gamma^\dagger \circ \hat{y} \overset{\text{def}}{=} \Psi_p^\dagger \circ \Psi_f^\dagger \circ \hat{y}(\tau) \), i.e. the composition of the operators (5.50) and (5.51):

\[ \Gamma^\dagger \circ \hat{y} \overset{\text{def}}{=} g^{-1} \circ \Theta[0,-T,0] \circ \Phi^{-1*}(0, T, h^{-1} \circ \hat{y}(\tau)). \quad \tau \in \mathfrak{r} \]  

(5.53)

The operators of Definition 5.44 are nonlinear generalizations of the linear balancing theory. Another nonlinear generalization of the linear Hankel balancing theory was presented earlier in [46]. Nevertheless our definitions distinguish from [46] on the use of Lie semigroups to define all the operators on our Hilbert manifold framework.

**Theorem 5.45 (Eigenvalue problem for the behavioral operator).**

Consider the nonlinear maps from Defs. 5.22 and 5.21 in the operator (5.21) with eigenvalue problem (5.22) defined by \( \Upsilon_t(\varrho(t)) = \sigma \varrho(t) = 0, \varrho(t) \in \mathcal{M} \). Then the resulting eigenvalues \( \lambda = \sigma^2 > 0 \) are the same eigenvalues of the eigenvalue problem \( \Gamma^\dagger \circ \Gamma(u(t)) = \lambda u(t) \) associated to the behavioral operator \( \tilde{\Gamma} \).

Some additional results about the structure of the past and future Gramians \( P^\tau \) and \( Q^t \) are presented:

**Proposition 5.46.** The past and future Gramians \( P^\tau : \mathcal{M}^{*\tau} \rightarrow \mathcal{M}_0 \) and \( Q^t : \mathcal{M}_0^t \rightarrow \mathcal{M}_0^* \) from Defs. 5.22 and 5.21 are such that:

1. They can be alternatively written as (respectively) \( P^\tau(\hat{x}^0) = \Psi_p \circ \Psi_p^\dagger(\hat{x}^0) \) and \( Q^t(x^0) = \Psi_f^\dagger \circ \Psi_f(x^0) \), i.e.
2. For $0 \neq \sigma_i^2 = \lambda_i \in \mathbb{R}^+$, $i = 1, \ldots, n$ the eigenproblem
\[
\begin{align*}
Q^t(x^0) &= \Psi^\dagger f \circ \Psi f(x^0) \\
&= \left[ \Phi^{-1^*}[0,T] \circ h^{-1} \circ h \circ \Phi^*[T,0](x^0) \right], \quad \Phi \in S^+\mathcal{F}_u.
\end{align*}
\] (5.55)

3. Let $P^t \circ x_i^+(t) \text{ def } = \left[ P^\tau \right]^{-1} \circ x_i^+(t)$ and $Q^\tau \circ \hat{x}_i^-(\tau) \text{ def } = \left[ Q^t \right]^{-1} \circ \hat{x}_i^-(\tau)$. Then (5.56) and (5.57) are equivalent to
\[
\begin{align*}
\begin{cases}
Q^t \circ x_i^+(t) = \lambda_i x_i^+(t), \\
P^\tau \circ \hat{x}_i^-(\tau) = \lambda_i \hat{x}_i^-(\tau).
\end{cases}
\end{align*}
\] (5.58)

Example 5.47. For linear systems ($\mathcal{L}$-class), for $\Upsilon \text{ def } = P^{-1}Q$ the eigenproblem (5.57) yields the principal directions and normal curvatures. Furthermore, there is a subspace partition $M = M^1 \oplus \ldots \oplus M^n$, [225].

Remark 5.48. A particular case of the present framework, is the nonlinear Hankel operator. Assume that the past induced metric $g^-_\mathcal{B}$ on $\mathcal{B}^- \subset \mathcal{W}^\tau$ depends only on the inputs, $\mathcal{U} \subseteq \mathcal{B}^-$. Furthermore, assume that the future induced metric $g^+_\mathcal{B}$ on $\mathcal{B}^+ \subset \mathcal{W}^t$ depends only on the outputs, $\mathcal{Y} \subseteq \mathcal{B}^+$, see also Remark 4.52 in Part I. To overcome the limitation expressed in Remark 5.19 the additional assumption of asymptotic stability of System (5.1) is required.

Definition 5.49 (Hankel operator). The Hankel operator $\Gamma u : \mathcal{L}_2(-\infty, 0] \to \mathcal{L}_2[0, \infty)$ associated to system (5.1) can be described in terms of semigroups by
\[
\Gamma u(t) = h \circ \Phi_{[\infty,0]} \circ \Theta^{-1}(0, -\infty, x_{-\infty}, u(t)), \quad t \in (-\infty, \infty)
\] (5.59)
being the composition of the following operators $\Psi_C : \mathcal{L}_2(-\infty, 0] \to \mathbb{R}^n$, $\Psi_{\varnothing} : \mathbb{R}^n \to \mathcal{L}_2[0, \infty)$ defined by
and called controllability and observability operators respectively.

As can be seen, the Hankel operator has a formulation that includes the complete flow associated to the homogeneous and complete solutions of the system (5.1). The construction of the adjoint operator requires a backward-time evolution of the operators (5.60) and (5.61) with inputs acting as outputs and vice versa. Definition 5.37 is invoked newly.

**Definition 5.50 (Adjoint Hankel operator).** The adjoint Hankel operator $\Gamma^\dagger : L^2[0,\infty) \to L^2(-\infty,0]$ associated to the complete flow of system (5.1) can be described in terms of semigroups by

$$
\Gamma^\dagger \hat{y}(\tau) = g^{-1} \circ \Theta[-\infty,0] \circ \Phi^{-1}(0,\infty,h^{-1}(\hat{y}(\tau))), \quad \tau \in (-\infty,\infty)
$$

(5.62)

being the composition of the following operators $\Psi^\dagger_C : \mathbb{R}^n \to L^2(-\infty,0]$, $\Psi^\dagger_O : L^2[0,\infty) \to \mathbb{R}^n$ defined by

$$
\Psi^\dagger_C(\hat{x}^0) \overset{\text{def}}{=} g^{-1} \circ \Theta(-\infty,0,\hat{x}^0,0), \quad \Theta \in S_{\mathcal{F}^{-}_u}
$$

(5.63)

$$
\Psi^\dagger_O \hat{y}(\tau) \overset{\text{def}}{=} \Phi^{-1}(0,\infty,h^{-1}(\hat{y}(\tau))), \quad \Phi \in S_{\mathcal{F}^{+}_u}
$$

(5.64)

and called adjoint controllability and adjoint observability operators respectively.

All these operators can be described efficiently in the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\Psi_C} & \mathcal{M}_0 & \xleftarrow{\Psi_C^\dagger} & \mathcal{U} \\
\ast & \xrightarrow{\ast} & \ast & \xleftarrow{\ast} & \ast \\
\mathcal{Y}^* & \xrightarrow{\Psi_O^\dagger} & \mathcal{M}_0^* & \xleftarrow{\Psi_C^\dagger} & \mathcal{U}^*
\end{array}
$$

(5.65)

**Remark 5.51.** Consider system (5.1) on the riemannian manifolds $(\mathcal{M}, \langle \cdot, \cdot \rangle_{T\mathcal{M}})$ and $(\mathcal{M}^*, \langle \cdot, \cdot \rangle_{T^*\mathcal{M}})$ endowed with the Hilbert Manifold structure of Theorem 5.10. The controllability and observability map diffeomorphisms $P^\tau : \mathcal{M}_0^* \to \mathcal{M}_0$, $Q^t : \mathcal{M}_0^t \to \mathcal{M}_0^*$ can be expressed in terms of the (adjoint) controllability and (adjoint) observability operators by:

$$
P^\tau(\hat{x}^0) = \Psi_C \circ \Psi_C^\dagger(\hat{x}^0) = [\Theta^{-1}][0,-\infty] \circ g \circ g^{-1} \circ \Theta[-\infty,0](\hat{x}^0), \quad \Theta \in S_{\mathcal{F}^{-}_u},
$$

(5.66)

$$
Q^t(x^0) = \Psi_O^\dagger \circ \Psi_O(x^0) = [\Phi^{-1}][0,\infty] \circ h^{-1} \circ h \circ \Phi[0,\infty](x^0), \quad \Phi \in S_{\mathcal{F}^{+}_u}
$$

(5.67)
With evident substitutions $\Psi_f \approx \Psi_O$, $\Psi_p \approx \Psi_C$ and using the same procedure of the proof in Theorem 5.45 the following consequence is set forth without proof:

**Corollary 5.52 (Eigenvalue problem for Hankel Operator).** Consider the nonlinear maps (5.66)-(5.67) on the behavioral operator $\tilde{\Gamma}$ with additional assumptions observed in Remark 4.52 (Part I) with associated eigenvalue problem in Proposition 4.37(1,2) in Part I defined by $\Upsilon \circ \varrho(t) - \lambda \varrho(t) = 0$, $\varrho(t) \in \mathcal{M}$. Then the resulting eigenvalues $\lambda = \sigma^2 > 0$ are the same eigenvalues of the operator $\Gamma^\dagger \circ \Gamma \circ \varrho(t)$.

Several results for the eigenvalue problem of the nonlinear Hankel operator have been reported earlier, see e.g. [44] and references therein.

### 5.4 Issues of minimality of state realizations

In Sec. 4.4 of Part I the geometric method of balanced reduction of dynamical systems in manifolds was discussed using concepts from the behavioral approach. After appropriate definitions of equivalence and approximation of behaviors (Part I,Defs. 4.59 and 4.60), the procedure was exposed as a sequence of steps: semigroup characterization, group extension, equivalence, factorization and finally restriction to a quotient system, see Part I (Proposition 4.62). Part II concentrates on state realizations of dynamical systems with latent variables and although the essential geometric balanced reduction method remains the same, the concepts of equivalence, approximation and factorization from Sec. 4.4.1, Part I are reviewed in order to confront our results on minimal realizations for state-space systems with well-known works on this topic.

The state-space realizations generating a behavior $\mathcal{B}$ are *classified* by the equivalence relation induced by the invariants of their isometric isomorphism $\Upsilon: \mathcal{M}_0 \to \mathcal{M}_0$. Following Defs. 4.59 and 4.60, any two state-space realizations $\Sigma_1$ and $\Sigma_2$ on a set of admissible state-space realizations $\mathcal{F}_\mathcal{B} = \{ \Sigma_i | i = 1, \ldots \}$ of the behavior $\mathcal{B}$ are *isomorphic* if for the same past and future exogenous variables they have the same principal curvatures. Finally after Definition 4.61 the set $(\mathcal{F}'_{\mathcal{B}_{red}}, \phi')$ is said to be an *isomorphic approximate realization* if there
is a restriction to a state submanifold $M'_0 \subset M_0$ where both systems are $B_{\text{red}}$-equivalent and the set of invariants of $\mathcal{R}'$ is a subset of the set of invariants of $\mathcal{R}$.

### 5.4.1 Reduction of the internal balanced realization

The following result is adapted from Part I (Proposition 4.62) to the state space and therefore a proof is not needed.

**Proposition 5.53 (Geometric balanced reduction method).** Assume that $S_a : \mathbb{R}^n \to \mathbb{R}^1$ and $S_r : \mathbb{R}^n \to \mathbb{R}^1$ exist and are smooth on (the compact) $M$ for the $(\mathcal{M}, \phi)$-system $\Sigma = (t, \mathcal{W}, \mathcal{B}_\Sigma) \overset{\text{def}}{=} (t, \mathcal{M}, \Phi^t)$ in Eq. (5.1) whose internal trajectories lie on an $n$-dimensional manifold $M$. An equivalent system $\Sigma_{\text{red}} = (t, \mathcal{V}, \mathcal{B}_{\text{red}}) \overset{\text{def}}{=} (t, \mathcal{M}', \Phi_{\text{red}}^t)$ whose behavior evolves on a reduced-order $r$-dimensional submanifold $M' \subset M$, $r < n$ can be synthesized by the following steps:

1. **Semigroup characterization:** The (maximal) semi flows $\{\Phi^t(x)\}_{t \geq 0}$ and $\{\Theta^\tau(\hat{x})\}_{\tau \geq 0}$ generated by the vector fields $\xi^+ = -\nabla_x S_a$ and $\alpha_- = -\nabla_{\hat{x}} S_r$ are a positive and negative semigroups respectively; meaning that for all $t \in t^+$, $\Phi^t(x)$ (resp. for all $\tau \in \tau^-$, $\Theta^\tau(x)$) are defined on all $M$.

2. **Group extension:** Whenever $[\Theta^\tau(\hat{x}^- (\tau))]^{-1} = \Phi^t(x^+ (t))$ and the balancing condition $[P^\tau(\hat{x} (\tau))]^{-1} = Q^t(x(t))$ are regular on a subset $M_m \subseteq M$, there is a group extension, regular on $M_m$, for the semigroups $\{\Phi^t(x)\}_{t \geq 0}$ and $\{\Theta^\tau(x)\}_{\tau \geq 0}$. If the subset $M_m$ is a submanifold s.t. $\dim M_m = \dim M$ it is a minimal support.

3. **Equivalence:** The local group extension of diffeomorphisms $\{\Phi^t(x)\}$ acting on the smooth manifold $M$ defines an equivalence relation with the smooth map $\pi : M \to M/\Phi$ (the natural projection) which associates to each $x \in M$ its equivalence class $\pi(x) \in M/\Phi$ (i.e. the set of orbits of $\Phi$ passing by $x$ as the set of equivalence classes $M/\Phi$).

4. **Factorization:** Whenever the principal frame-balancing condition (5.42) is satisfied regularly, $\Phi$ can be factorized by the group invariant orbits of a family of subgroups $\Phi_i$. Furthermore $M$ can be arbitrary partitioned in two invariant submanifolds $M \overset{\text{def}}{=} M_a \oplus M_b$

\[
M_a \overset{\text{def}}{=} M_1 \oplus M_2 \oplus \cdots \oplus M_r,
\]

\[
M_b \overset{\text{def}}{=} M_{r+1} \oplus M_{r+2} \oplus \cdots \oplus M_n,
\]

where the submanifolds on $M_a$ have the associated higher principal curvatures $\kappa_i$, $i = 1, \ldots, r$ and the remaining submanifolds are on $M_b$. 
Consider a local realization of the vector fields comprising the frame \( \{ \zeta_1^+, \zeta_2^+, \ldots, \zeta_n^+ \} = T_p M^t \) in Thm 5.28(4) and conform the columns of the nonlinear coordinate transformation \( z = \phi(x) \) where \( \phi(x) \overset{\text{def}}{=} [\zeta_1^+ | \zeta_2^+ | \cdots | \zeta_n^+] = [\phi_a(x) | \phi_b(x)] \) is a local diffeomorphism in the \((\mathfrak{X}, \phi)\)-class of system in Eq. (5.1). With the nonlinear coordinate transformation \( z = \phi(x) \) system (5.1) can be expressed in the new coordinates in factorized form as (e.g. [33], Ch.4):

\[
\begin{bmatrix}
\dot{z}_a(t) \\
\dot{z}_b(t)
\end{bmatrix} = \begin{bmatrix}
f_a(z_a(t)) \\
 f_b(z_b(t))
\end{bmatrix} u(t),
\]
\[
y(t) = h_a(z_a(t)) + h_b(z_b(t)).
\]

5. Restriction: The natural projection \( \pi_b : M \to M/\Phi_b \) defines a restriction from \( M \) to \( M' \) and a quotient system such that \( \mathfrak{B} \) is equivalent to \( \mathfrak{B}_{\text{red}} \) within a submanifold \( M' \subset M \). The quotient system has the form:

\[
\Sigma_{\text{red}}^+: \left\{ \begin{array}{l}
\dot{z}_a(t) = f_a(z_a(t)) + g_a(z_a(t))u(t), \\
y(t) = h_a(z_a(t)),
\end{array} \right.
\]

and the inverse coordinate transformation \( x = \tilde{\phi}_a^{-1}(z) \) (adapted to the quotient space \( M' \subset M \)) yields the desired reduced-order model.

5.4.2 Issues of minimality

This subsection deals with the implication that the nonlinear balanced reduction procedure has on the minimality of the resulting realization. As expressed in Sect. 5.3, given two semigroups \( S_{\mathcal{F}_u}^+ \) and \( S_{\mathcal{F}_u}^- \) the balancing condition (5.20) establishes the condition to define a group extension \( G(\mathcal{F}_u) = S_{\mathcal{F}_u}^+ \cup S_{\mathcal{F}_u}^- \), being \( G(\mathcal{F}_u) \) the smallest local group containing \( S_{\mathcal{F}_u}^+ \), see Cor. 3.6 in [76]. In this subsection we investigate on the properties that has a nonlinear balanced realization satisfying Eq. (5.20) in terms of minimality. The geometric theory of minimal realizations for nonlinear systems is satisfactorily covered in [197, 73] and essentially relies on Lie-semigroups, see e.g. [192, 88, 89, 195]. Notice that local minimal realizations have been discussed earlier in [185] without the use of Lie-semigroup theory, which is fundamental for our geometric treatment.

Definition 5.54 (Accessibility, reachable set). The accessible set \( \mathcal{A}(x^0, T) \) is the set of terminal points \( x(T) \) originated at \( x^0 \in M \) of the integral curves of the family of vector fields \( \mathcal{F}_u = \{ f_u : u \in \mathcal{U} \} \), generated by associated vector fields \( \xi_1, \ldots, \xi_k \in \mathcal{F}_u \) for a partition \( 0 = t_0 \leq t_1 \leq \cdots \leq t_k = T \) as...
expressed in Eq. (5.5). The system (5.1) is said to have the property of accessibility if for any given \( x^0 \in \mathcal{M} \), the accessible set has a nonempty interior (i.e. \( \text{Int}(\mathcal{A}(x^0, T)) \neq \emptyset \)). The set reachable from \( x^0 \) at time \( T \geq 0 \) of system (5.1) consist of the union of all \( \mathcal{A}(x^0, T) \) [88].

**Theorem 5.55 (Weak controllability).** System (5.1) is said to be weakly controllable if for every \( x \in \mathcal{M} \) and every neighborhood \( D \) of \( x^0 \), \( \mathcal{A}(x^0, T) \) has a nonempty interior, [73].

**Definition 5.56 (Observability).** Two points \( x_1, x_2 \in \mathcal{M} \) are indistinguishable, \( x_1 \sim x_2 \), if for every admissible input \( u \in \mathcal{U} \), their input output maps with initial conditions \( x_0 \) and \( x_1 \) satisfy the relation \( h(\Phi(t, t_0, x_1, u(t))) = h(\Phi(t, t_0, x_2, u(t))) \). If \( x_1 \sim x_2 \) implies \( x_1 = x_2 \) the system is observable [73].

**Theorem 5.57 (Uniqueness of a minimal nonlinear system [197]).** A nonlinear system \( \Sigma \), Eq. (5.1) which is observable, weakly controllable and either analytic or symmetric, is minimal and furthermore two minimal realizations \( \Sigma_1, \Sigma_2 \) with the same input-output map \( \Gamma \) differ only by a diffeomorphism of the state space \( \mathcal{M} \) (uniqueness).

For linear-time invariant systems (see Example 5.25) a balanced realization is minimal. To conclude this section, the following theorem discusses the equivalent question for a nonlinear realization:

**Theorem 5.58.** With the same assumptions of Theorem 5.28, consider the behavior \( \mathfrak{B} \) generated for each \( x^0 \in \mathcal{M} \) by a dynamical system \( \Sigma^+ \) in the state space realization (5.1) along with an associated adjoint system \( \Sigma^+_1 \), \( \dot{x}^0 \in \mathcal{M}^* \) in Eq. (5.44), well defined by assumption (i.e. after Definition 5.39 with \( h^{-1} : \mathcal{Y}^* \rightarrow \mathcal{M}^* \) and \( g^{-1} : \mathcal{M}^* \rightarrow \mathcal{U}^* \) satisfying Definition 5.37 and Assumption 5.38). If and only if the realization satisfies the balancing condition (5.20) for all \( x^0 \in \mathcal{M}, \dot{x}^0 \in \mathcal{M}^* \), then:

1. Both realizations \( \Sigma^+ \) and \( \Sigma^+_1 \) are minimal in the sense of Theorem 5.57.
2. Weak controllability of \( \Sigma^+ \) implies observability of \( \Sigma^+_1 \) and observability of \( \Sigma^+ \) implies weak controllability of \( \Sigma^+_1 \), and vice versa.

### 5.5 Conclusions

In this Part II the geometric viewpoint for nonlinear dissipative balanced reduction was further developed by considering internal (state-space) trajectories and using only standard geometric objects and concepts. Disregarding all
the differential-geometric machinery, the overall framework is an appropriate
ground field for the class of nonlinear dissipative systems, including $L_2$-gain
systems and the important class of passive systems (e.g. port-Hamiltonian sys-
tems). Another stream of this research lies on the development of numerical
algorithms, or those based on symbolic-algebra for Lie Groups, for nonlinear
balanced reduction using this approach. See e.g. [125] for some results in this
direction.

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5.6 Appendix

Proof of Proposition 5.18: The smooth vector field $\nabla S_a$ (dual to $dS_a$)
points to the direction of fastest increase of $S_a$. Let $\{\Phi^t(x)\}$ denote the max-
imal flow generated by $\xi^+(x) = -\nabla S_a(x)$, then we may write $\frac{d}{dt}\Phi^t(x) =
\xi^+(\Phi^t(x)) = -\nabla S_a(\Phi^t(x))$ and therefore $\frac{d}{dt}S_a(\Phi^t(x)) = \nabla S_a(-\nabla S_a(\Phi^t(x))) = -\|\nabla S_a(\Phi^t(x))\|^2$. Thus, $S_a(x)$ is monotonically decreasing along the forward-
time evolution of $\Phi^t(x)$. Since $S_a(x)$ exists, it is finite and then $S_a(\Phi^t(x))$ has
a limit as $t \to T_f$, $0 \leq t \leq T_f$. Since a smooth vector field $\xi$ on $\mathcal{M}$ of bounded
length $\|\xi\| \leq m < \infty$, generates a 1-parameter group of diffeomorphisms on
$\mathcal{M}$, see Corollary 9.1.5 in [163]. In particular, let $q(t) = S(\Phi^t(x))$ and since
$B_a \leq q(T) = q(0) + \int_0^T \dot{q}(t) dt = q(0) - \int_0^T \|\nabla S_a(\Phi^t(x))\|^2 dt$. Since this holds in $t \leq T$, for a finite time interval such integral is finite, since using Schwarz
inequality,

$$\int_0^T \|\nabla S_a(\Phi^t(x))\| dt \leq \sqrt{T} \left( \int_0^T \|\nabla S_a(\Phi^t(x))\|^2 dt \right)^{\frac{1}{2}}$$

and therefore is finite as long as $T < \infty$. Since we assumed that $\mathcal{M}$ is com-
 pact, then any sequence in $\mathcal{M}$ s.t. $|S(x_k)|$ is bounded and $\|dS(x_k)\| \to 0$ then
$\{x_k\}$ has a convergent subsequence $x_k \to p$, implying the existence of a critical
point $p$ as $t \to \infty$. ■

Proof of Proposition 5.20: Notice that similarly to Remark 4.25 (Part I),
$S_r(x, r_r)$ in Eq. (5.11) can be expressed on $\mathcal{M}^{* \tau}$ by $S^*_r(\hat{x}, r_r)$. Thus the proof
is quite similar to the proof of Proposition 5.18 and just dissimilar points are shown. Since for $\tau = -t \in \tau$, and $\{\Theta^\tau(\dot{x})\}$ as the maximal flow generated by $\alpha_-(\dot{x}) = -\nabla S_r(\dot{x})$, $\frac{d}{d\tau} \Theta^\tau(\dot{x}) = \alpha_-(\Theta^\tau(\dot{x})) = -\nabla S_r(\Theta^\tau(\dot{x}))$ implying that $\frac{d}{d\tau} S_r(\Theta^\tau(\dot{x})) = \nabla S_r(\nabla S_r(\Theta^\tau(\dot{x}))) = -\|\nabla S_r(\Theta^\tau(\dot{x}))\|^2$. Thus, $S_r^*(\dot{x})$ is monotonically decreasing along the (backward $t$-time) evolution of $\Theta^\tau(\dot{x})$. Since $S_r^*(\dot{x})$ exists, it is bounded from below by $B_r > -\infty$, then $S_r(\Theta^\tau(\dot{x}))$ has a limit as $\tau \to -T, -T \leq \tau \leq 0$. Let $p(\tau) = S(\Theta^\tau(\dot{x}))$, since $B_r \leq p(-T) = p(0) + \int_0^{-T} \dot{p}(\tau) d\tau = p(0) - \int_{-T}^0 \|\nabla S_r(\Theta^\tau(\dot{x}))\|^2 d\tau$. With similar arguments to the proof of Proposition 5.18 the proof is completed.

**Proof of Proposition 5.27:** For appropriate definitions of isometry and duality pairing, see Part I. Since an abstract dual pairing $\langle \cdot, \cdot \rangle_{T^* \mathcal{M}_\tau \times T \mathcal{M}_\tau^t}$ must be such that for each continuous linear functional $\alpha_- \in T^* \mathcal{M}_\tau^t$ there is a unique element $\hat{\alpha}^+ \in T \mathcal{M}_\tau^t$ satisfying $\|\hat{\alpha}^+\|_{T \mathcal{M}_\tau^t} = \|\alpha_-\|_{T^* \mathcal{M}_\tau^t}$ (by Riesz representation theorem).

1) In particular duality of $\alpha_-$ with $\hat{\alpha}^+$ implies that

$$\langle \alpha_-, \alpha_- \rangle_{T^* \mathcal{M}_\tau^t} = \langle \hat{\alpha}^+, \alpha_- \rangle_{T^* \mathcal{M}_\tau^t \times T \mathcal{M}_\tau^t} = \langle \hat{\alpha}^+, \mathbb{P} \alpha_- \rangle_{T \mathcal{M}_\tau^t} \quad (5.74)$$

for some $P^r : \mathcal{M}_{\tau}^t \to \mathcal{M}_{\tau}^t$ s.t. $\hat{\alpha}^+ = \mathbb{P} \alpha_-$. 

2) With a similar reasoning, duality of $\xi^+$ with $\hat{\xi}^-$ implies

$$\langle \xi^+, \xi^+ \rangle_{T \mathcal{M}_\tau^t} = \langle \hat{\xi}^+, \xi^+ \rangle_{T \mathcal{M}_\tau^t \times T^* \mathcal{M}_\tau^t} = \langle \hat{\xi}^+, \mathbb{Q} \xi^+ \rangle_{T^* \mathcal{M}_\tau^t}, \quad (5.75)$$

for some $Q^r : \mathcal{M}_{\tau}^t \to \mathcal{M}_{\tau}^t$ s.t. $\hat{\xi}^- = \mathbb{Q} \xi^+$.

3) The duality pairs in Table 5.2 are not necessarily equivalent: arbitrary values of unique $\alpha_-$ and $\xi^+$ yields different values in Eqs. (5.74) and (5.75). Now assume the balancing condition (5.20) is satisfied. From Eqs. (5.17) and (5.16) we get $\xi^+ \equiv \hat{\alpha}^+ = \mathbb{P} \alpha_-$ and $\hat{\xi}^- \equiv \alpha_- = \mathbb{Q} \xi^+$. In such conditions Eqs. (5.74) and (5.75) have the same value since substitution of $\xi^+ = [\mathbb{Q}]^{-1} \alpha_-$ yields relation (5.23) verified, implying that the duality pairs in Table 5.2 are equivalent. By Definition 4.32 (Part I) relation (5.23) implies adjointness of the past Gramian $P^r$ to the future Gramian $Q^r$.

4) From relation (5.23) let $\alpha_- \equiv \mathbb{Q} \xi^+$ implying on the right side $[P^r \circ Q^r]^{-1} \xi^+ \equiv \mathbb{Y}_* \xi^+$. Observe that $\langle \alpha_-, \mathbb{Q} \xi^+ \rangle_{T \mathcal{M}_\tau^t} = \langle \mathbb{P} \alpha_-, \mathbb{P} \hat{\xi}^- \rangle_{T \mathcal{M}_\tau^t}$ with $\hat{\xi}^- \equiv \mathbb{Q} \xi^+$ and condition (5.20) yielding $\langle \xi^+, \mathbb{Y}_* \xi^+ \rangle_{T \mathcal{M}_\tau^t}$ i.e. the left hand side of Eq. (5.24) and selfadjointness in the sense of Definition 4.33 (Part I). An equivalent argument for $[Q^r \circ P^r]^{\dagger}_* \beta_- \equiv \mathbb{Y}_*^{\dagger} \beta_-$ leads to satisfaction of (5.25).
5) Consider the inner products defined for the Hilbert manifold structure of Proposition 5.10, in Table 5.1. The integrand in \( \langle \mathbb{P} \alpha, \xi \rangle_{T \mathcal{M}} \) given by \( \sum_{j=1}^{n} \sum_{i=1}^{n} \xi_j \mathbb{P}_{ij} \alpha^i \) is equal (by dualization) to \( \sum_{j=1}^{n} \sum_{i=1}^{n} \hat{\alpha}_j \mathbb{P}_{ij} \xi^i \), with \( [\mathbb{P}_{ij}] = [\mathbb{P}^{-1}]_{ij} \) implying after dualization of \( \hat{\alpha} \) and \( \xi \) that \( \langle \alpha, [\mathbb{P}]^{-1} \xi \rangle_{T \mathcal{M}} = \langle \alpha, \mathbb{Q} \xi \rangle_{T \mathcal{M}} \) (by condition (5.20)) resulting in Eq. (5.26). Similarly, the integrand of \( \langle \xi, \Upsilon \rangle_{T \mathcal{M}} \) can be written as \( \sum_{j=1}^{n} \sum_{i=1}^{n} \xi_j \Upsilon_{ij} \zeta^i \) with \( \Upsilon^i_j = \sum_{\ell} \mathbb{P}_{\ell j} \mathbb{Q}^{i\ell} = \sum_{\ell} \mathbb{Q}_{\ell j} \mathbb{P}^{\ell i} \) and by symmetry of the metric tensors \( \Upsilon^i_j = \Upsilon^j_i \), Eq. (5.27) is satisfied. With similar arguments relations Eq. (5.28) is shown to be true.

**Proof of Theorem 5.28:** 1) Consider a system trajectory \( x = \hat{x}^{-}(\tau) \wedge x^{+}(t) \) with tangent vector fields \( \alpha_{-} \in T_{p} \mathcal{M}_{-}^{\tau} \) and \( \hat{\alpha}^{+} \in T_{p} \mathcal{M}_{+}^{\tau} \). Since by adjointness of the Gramians in Proposition 5.27 (3), \( \langle \alpha_{-}, \mathbb{Q} \hat{\alpha}^{+} \rangle_{T \mathcal{M}^{\tau}_{+}} = \langle \mathbb{P} \alpha_{-}, \hat{\alpha}^{+} \rangle_{T \mathcal{M}^{\tau}_{+}} \), i.e. Eq. (5.23), the future metric \( g_{+}^{\tau}(x^{+}, 0) = \langle \hat{\alpha}^{+}, \hat{\alpha}^{+} \rangle_{T \mathcal{M}^{\tau}_{+}} \) can be induced to the past from substitution of \( \hat{\alpha}^{+} = \mathbb{P} \alpha_{-} \), i.e. \( \langle \alpha_{-}, \mathbb{Q} \alpha_{-} \rangle_{T \mathcal{M}^{\tau}_{-}} = \langle \hat{\alpha}^{+}, \hat{\alpha}^{+} \rangle_{T \mathcal{M}^{\tau}_{+}} \). Dividing this latter result by the past metric \( g_{-}^{-}(\hat{x}^{-}, 0) \overset{\text{def}}{=} I_{\mathcal{M}}(\alpha_{-}, \alpha_{-}) \) yields the quotient (5.29) with the Shape operator indicated.

2), 3), 4) and 6) These results can be proved *mutatis mutandis* from the proofs in the theory developed in Part I (Theorem 4.44).

5) Similar to Part I (Proposition 4.46).

7) From Theorem 5.28 (6) \( \Upsilon_{*} \xi^{+} = \sum_{i=1}^{n} \langle \Upsilon_{*} \xi^{+}, \zeta^{+}_{i} \rangle_{T \mathcal{M}^{+}} \zeta^{+}_{i} \) and \( \Upsilon_{i}^{+} \alpha_{-} = \sum_{i=1}^{n} \langle \Upsilon_{i}^{+} \alpha_{-}, \beta^{+}_{j} \rangle_{T \mathcal{M}^{+}} \beta^{+}_{j} \) which (using (5.57) at the tangent space) may be written otherwise as Eqs. (5.32)-(5.33).

**Proof of Proposition 5.29:** 1) This eigenproblem is well posed in differential-geometric terms due to the following: Support \( S_{a}, S_{r}^{-} \) by the riemannian manifolds \( (\mathcal{M}, g_{\mathcal{M}}) \) and \( (\mathcal{M}^{*}, g_{\mathcal{M}^{*}}) \) on the Hilbert manifold structure defined by Proposition 5.16. Let \( c \) be a regular value of \( S_{r}^{*} \), i.e. s.t. \( d S_{r}^{*} \mid c \neq 0 \) defined by the \( c \)-level \( \mathcal{N} = \{ \eta \mid S_{r}^{*}(\eta) = c, \eta \in \mathcal{M} \} \) the sub manifold \( \mathcal{N} \subset \mathcal{M} \). Since \( d S_{r}^{*} \mid x \) is orthogonal to \( \nabla_{x} S_{r}^{*} \), this implies that \( T_{x} \mathcal{N} = \ker(d S_{r}^{*} \mid x) = [\nabla_{x} S_{r}^{*}]^{-1} \), implying that \( \nabla_{x} S_{r}^{*} \) spans \( T_{x} \mathcal{N} \). On the other side, express the restriction of \( S_{a} \) to \( \mathcal{N} \) by \( S_{a}^{\mathcal{N}} = S_{a} \mid \mathcal{N} \), then \( \nabla_{x} S_{a}^{\mathcal{N}} \) is the orthogonal projection onto \( T_{x} \mathcal{N} \) of \( \nabla_{x} S_{a} \) (since \( d S_{a}^{\mathcal{N}} \mid x = d S_{a} \mid T_{x} \mathcal{N} \) and it is orthogonal to \( \nabla_{x} S_{a}^{\mathcal{N}} \)). Thus, a critical point \( p \) of \( S_{a}^{\mathcal{N}} \) (i.e. s.t. \( d S_{a}^{\mathcal{N}} \mid p = 0 \)) must be such that \( \nabla_{p} S_{a}^{\mathcal{N}} \) is orthogonal to \( T_{p} \mathcal{N} \). Therefore at \( p \) we may express \( \nabla_{p} S_{a}^{\mathcal{N}} \) in terms of the space \( T_{x} \mathcal{N} \) spanned by \( \nabla_{x} S_{r}^{*} \), otherwise said, \( \nabla S_{a}(p) = \lambda \nabla S_{r}^{*}(p) \) for a real scalar \( \lambda \in \mathbb{R} \).

2) Consider an eigentrajectory \( \rho(x^{(t)}) = \hat{x}^{-}(\tau) \wedge x^{+}(t) \) where \( x^{+}(t) \in \mathcal{M}_{+}^{\tau} \) and \( \hat{x}^{-}(\tau) \in \mathcal{M}_{-}^{\tau} \) are semi-trajectories generated by the semigroups \( \{ \Phi^{t} \}_{t \geq 0} \) and
\{\Theta^T\}_{T \geq 0} in Eqs. (5.14) and (5.15) whose generating tangent vector fields by Props. 5.18 and 5.20 are given by \(\xi^+ = \frac{d\Phi(t,x)}{dt}\bigg|_{t=0}\) and \(\alpha^- = \frac{d\Theta(\tau,\hat{x})}{d\tau}\bigg|_{\tau = 0}\). By Assumption 5.12 the past and future metrics are \(\langle\alpha^-,\alpha^-\rangle_{T,\mathcal{M}^T} = S^*_a(\hat{x}(\tau))\) and \(\langle\xi^+,\xi^+\rangle_{T,\mathcal{M}^T} = S_a(x(t))\). Since the system is dissipative, it satisfies \(S_a(x(t)) = \lambda S^*_a(\hat{x}(\tau))\) for some \(\lambda \leq 1\). Such \(\lambda\) attains stationary values (is constant) only along the eigentrajectory \(\rho(x^0)\) implying that \(\nabla_x S_a(x(t)) = \lambda \nabla_{\hat{x}} S^*_a(\hat{x}(\tau))\) is constant along \(\rho(x^0)\).

**Proof of Proposition 5.31:**

1) Coordinate independence of (5.35) is inherited by the dual pairing in the Hilbert manifold structure of Table 5.1.

2) Since \(L(x, \hat{x})\) is invariant then \(\nabla L(x, \hat{x}) = 0\), i.e.

\[
\frac{\partial L(x(t), \hat{x}(\tau))}{\partial [x(t), \hat{x}(\tau)]} = \frac{\partial L(x(t), \hat{x}(\tau))}{\partial x(t)} \bigg|_{x(\tau) = \hat{x}^0} + \frac{\partial L(x(t), \hat{x}(\tau))}{\partial \hat{x}(\tau)} \bigg|_{x(t) = x^0} = 0,
\]

then using \(\langle\cdot,\cdot\rangle_{T^*,\mathcal{M} \times \mathcal{M}}\) in Table 5.1, clearly

\[
\frac{\partial}{\partial \hat{x}} \langle\xi^+(x), \alpha_-(\hat{x})\rangle_{T^*,\mathcal{M} \times \mathcal{M}} = \hat{x}
\]

and

\[
\frac{\partial}{\partial \hat{x}} \langle\xi^+(x), \alpha_-(\hat{x})\rangle_{T^*,\mathcal{M} \times \mathcal{M}} = x
\]

then \(\nabla^T_x L(x, \hat{x}) = \nabla^T S_a(x, r) - \hat{x} - \hat{x} = 0\) and \(\nabla^T_{\hat{x}} L(x, \hat{x}) = \nabla^T S^*_a(\hat{x}, r) - x(t) = 0\), which are precisely Eqs. (5.36) and (5.37).

3) Equation (5.38) asserts in fact that Eq. (5.36) is the inverse map of Eq. (5.37) and vice versa, therefore we proceed to prove this by verifying Eq. (5.37) using the inverse operation: Since at the right side of Eq. (5.37) \(dS^*_a(\hat{x}, r) = \sum_{i=1}^n \partial_i S^*_a(\hat{x}, r) d\hat{x}^i\) using the transform (5.35) we get \(dS^*_a(\hat{x}, r) = d[\langle x, \hat{x} \rangle - S_a(x, r)]\). Since \(dS_a(x, r) = \sum_{i=1}^n \partial_i S_a(x, r) dx^i\) we obtain \(dS^*_a(\hat{x}, r) = \sum_{i=1}^n \left[x^i d\hat{x}^i + \hat{x}^i dx^i - \partial_i S_a(x, r) dx^i\right]\) and this last result by Eq. (5.36) yields precisely \(dS^*_a(\hat{x}, r) = \sum_i x^i d\hat{x}^i\), verifying Eq. (5.37). An analog procedure verifies Eq. (5.36). Conclude that one transformation is the inverse of the other and thus (5.38) is obtained.

**Proof of Proposition 5.35:**

1) By condition (5.42) \(S^*_r(\hat{x}^0, r_r)\) and \(S_a(x^0, r_a)\) are orthogonally separated on their respective spaces. Hence, associated to each integral invariant function \(S^i_r\) is an (exact) differential 1-form \(\zeta^+_i\) defining
a distribution $\Delta_i$ with annihilator $\Delta_i^\perp \overset{\text{def}}{=} \{\alpha_- \in T_pM^- | \langle \alpha_-, \zeta_i^+ \rangle_{T_M^- \times T_M^+} = 0, \zeta_i^+ \in \Delta_i \}$ (dual to the distribution obtained in Proposition 4.44(3)). From Definition 5.32, the set of annihilators $\{\Delta_1^\perp, \ldots, \Delta_n^\perp\}$ is an orthogonal web. Similarly, associated to each integral invariant function $S_a^\perp$ is an exact differential 1-form $\beta_\perp$ and the family of annihilators $\{\Omega_1^\perp, \ldots, \Omega_n^\perp\}$, defined by $\Omega_i^\perp \overset{\text{def}}{=} \{\xi^+ \in T_pM^+_i | \langle \beta_\perp, \xi^+ \rangle_{T_M^- \times T_M^+} = 0, \beta_\perp \in \Omega_i \}$ defining the orthogonal web for $S_a(x^0, r_a)$.

2) Since by virtue of Theorem 4.44 each element in the set of orthogonal differential 1-forms $\{\beta_\perp \}$ in $T^*M^-$ has an associated integral functional $\hat{S}_i \in M$, from the proof of Theorem 4.44(2), each principal direction $\beta_\perp$ is normal to an orthogonal hypersurface $\hat{N}_i$ defined by a (locally) integrable distribution (i.e. its annihilator) whose integral invariant function is precisely $S_i \in M$. By duality, the set of tangent vector fields $\{\xi^+ \} \in TM^+_{\perp_i}$ is normal to an orthogonal hypersurface $N_i$ defined by a (locally) integrable codistribution (i.e. its annihilator) whose integral invariant function is precisely $S_i \in M$.

**Proof of Proposition 5.42:** 1) By Proposition 5.20, along the forward-time evolution of $\Sigma_i^\perp$, the storage function $S_r(\tau(x(t)), r_r)$ decreases monotonically and with the equivalent reasoning in the backward-time evolution, Proposition 5.18 states that $S_a(\hat{x}(\tau), r_a)$ decreases monotonically either.

2) Since the orthonormal base span $\{\beta_\perp(1), \ldots, \beta_\perp(\omega(x))\} = T_xM^-_{\perp_i}$ is independent of the direction of evolution, the orthogonal projection is given by Eq. (5.31).

**Proof of Theorem 5.45:** The eigenvalue problem of the behavioral operator can be stated as finding the eigenvalue $\lambda \neq 0$ and the eigenvector $0 \neq u(t) \in U$ such that $\Gamma^\perp \circ \Gamma \circ u(t) = \lambda u(t)$. Express such eigenvalue problem by

$$\Gamma^\perp \circ \Gamma \circ u(t) = \Psi_p^\perp \circ \Psi_f^\perp \circ \Psi_f \circ \Psi_p \circ u(t) = \lambda u(t)$$

(5.76)

which after being mapped by the nonlinear map $\Psi_p : U \rightarrow M, \varrho(t) = \Psi_p(u)$ for some state trajectory $\varrho(t) \in M$, yields

$$\Gamma^\perp \circ \Gamma \circ u(t) = \Psi_p \circ \Psi_p^\perp \circ \Psi_f^\perp \circ \Psi_f \circ \Psi_p \circ u(t) = P^\tau \circ Q^f(\varrho(t)) = \lambda_\varrho(t)$$

(5.77)

where $P^\tau(\varrho(t))$ and $Q^f(\varrho(t))$ are defined from (5.54) and (5.55) yielding $P^\tau \circ Q^f(\varrho(t)) - \lambda_\varrho(t) = 0$.

Assume now the eigenvalue $\lambda \neq 0$ and the state trajectory (eigenvector) $0 \neq \varrho(t) \in M$ are solution of $P^\tau \circ Q^f(\varrho(t)) - \lambda_\varrho(t) = 0$. Map the latter equation by $\Psi_p^\perp \circ Q^f : M \rightarrow U, u := \Psi_p^\perp \circ Q^f(\varrho(t))$, yielding $\Gamma^\perp \circ \Gamma \circ u(t) - \lambda u(t) = 0$. 

$$\Gamma^\perp \circ \Gamma \circ u(t) - \lambda u(t) = 0.$$
Proof of Proposition 5.46: 1) Consider the maps (5.48)-(5.51) expressed in the commutative diagram (5.78).

\[
\begin{array}{c}
\mathcal{Y} \xleftarrow{\psi_f} \mathcal{M}_0 \xleftarrow{\psi_p} \mathcal{U} \\
\star \quad \star \quad \star \\
\mathcal{Y}^* \xrightarrow{\psi^*_f} \mathcal{M}^*_0 \xrightarrow{\psi^*_p} \mathcal{U}^*
\end{array}
\]

(5.78)

It can be discerned that duality of $\mathcal{M}_0$ with $\mathcal{M}^*_0$ can be attained by the alternative maps presented.

2) From (5.56a), $\hat{x}_i^-(\tau) = \frac{1}{\sigma_i} Q^t \circ x_i^+(t)$, thus $P^r \circ Q^t \circ x_i^+(t) = \sigma_i^2 x_i^+(t)$, i.e. (5.57a). Take (5.56b), $x_i^+(t) = \frac{1}{\sigma_i} P^r \circ \hat{x}_i^- (\tau)$, thus $Q^t \circ P^r \circ \hat{x}_i^- (\tau) = \sigma_i^2 \hat{x}_i^- (\tau)$, i.e. (5.57b).

3) From (5.56a) $x_i^+(t) = \sigma_i Q^r \circ \hat{x}_i^- (\tau)$ in (5.56b) yields (5.58b). From (5.56b) $\hat{x}_i^- (\tau) = \sigma_i P^t \circ x_i^+(t)$ in (5.56a) yields (5.58a).

Proof of Theorem 5.58: 1) ($\Rightarrow$) : Let us verify the conditions in Theorem 5.57 after [197]: Since $\Sigma^+$ in (5.1) is dissipative by assumption, the optimal control problems implied in (5.11)-(5.12) exist with state $x^*$ and co-state $\hat{x}^*$ trajectories for each $x^0 \in \mathcal{M}$, in the notation of Definition 5.14. Since the collection of all $\mathcal{A}(x^0, T)$ defines the reachable set, then $Int(\mathcal{A}(x^0, T)) \neq \emptyset$, $\forall x^0 \in \mathcal{M}$ (possibly zero though) and after Definition 5.55, we may assert that the system is weakly controllable and the support of $S_r(x^0, r)$ is the reachable set. Since after the assumption that the balancing condition (5.20) is satisfied, as justified in Remark 5.26, each edge point of $x^*$ is connected to a dual edge point of $x_+^*$ by $x^*(0) = x_+^*(0) = x^0 \in \mathcal{M}_0$, then every trajectory $x_+^*$ begins from the $r$-reachable set. Now, in the notation of Definition 5.15, consider two system trajectories $\varrho(x^0)_1, \varrho(x^0)_2$. Assume that $\varrho(x^0)_1$ is state indistinguishable from $\varrho(x^0)_2$ i.e. $\varrho(x^0)_1 I \varrho(x^0)_2$. Since, due to duality, each $\varrho(x^0)_1$ is solution of one optimal control problem, $\varrho(x^0)_1 I \varrho(x^0)_2$ implies that this is only possible if $\varrho(x^0)_1 = \varrho(x^0)_2$. Though, from Definition 5.56 indistinguishability of the input-output map implies that $h(\varrho(x^0)_1) I h(\varrho(x^0)_2)$. Since from Definition 5.37 given an output $y$ we assumed that $h : \mathcal{M} \rightarrow \mathcal{Y}$ provides, over all the state trajectories $x(t)$ that define the output $y$, the only one that minimizes the squared norm $\|x(t)\|^2_{\mathcal{J}_M} \overset{\text{def}}{=} S_a(x^0, r_a)$. Thus the system is observable in the sense of Definition 5.56, verifying Theorem 5.57, and then $\Sigma^+$ is minimal. Consider now $\Sigma^+_1$ in (5.44). Since for every input trajectory $\hat{y}(t) \in \mathcal{Y}^*$ after $h^{-1} : \mathcal{Y}^* \rightarrow \mathcal{M}^*$ from Definition 5.37, there is only one co-state trajectory
\( \hat{x}(t) \in \mathcal{M}^* \) for some \( \hat{x}^0 \in \mathcal{M}^* \). The set denoted by \( \hat{\mathcal{A}}(\hat{x}^0, T) \) defines the reachable set and for every \( \hat{x}(t) \) in the set \( \{ g^{-1} | \hat{u}(t) = g^{-1}(\hat{x}(t)), \hat{u}(t) \in [\hat{u}(t)] \} \) there is only one output trajectory \( \hat{u}(t) \in \hat{U}^* \) minimizing the squared norm \( \| \hat{x}(t) \|^2_{\mathcal{I}} \overset{\text{def}}{=} \mathcal{S}_r(\hat{x}^0, r_r) \). Thus, any assertion for minimality of \( \Sigma^+ \) has a dual assertion of minimality of \( \Sigma^+_\dag \). This part of the proof is concluded by observing that the vector fields and the systems itself are by construction symmetric, see Definition 5.7, thus there is no need of assuming analyticity. Since the group property is coordinate invariant, there exist a class of diffeomorphisms, that allow for different realizations of a unique system. Conclude then that such realizations are minimal in the sense of [73]. \( \Leftarrow \) : By contradiction, now assume \( \Sigma^+ \) and \( \Sigma^+_\dag \) are not minimal and not balanced. Without assuming duality in the metric spaces defined by the Hilbert manifold structure of Proposition 5.16, still any point \( \hat{x}^0 \in \mathcal{M}^* \) must have a metric defined by optimal control problem (5.8) for \( S_r(\hat{x}^0, r_r) \) (and some trajectory \( x^-_r \)), and any point \( x^0 \in \mathcal{M} \) must have a metric defined by optimal control problem (5.10) for \( S_a(x^0, r_a) \) (and some trajectory \( x^+_a \)). Any other set of points are automatically discarded since they fall outside the support of such storage functions. Nevertheless there is no way to relate both initial conditions \( x^0 \) and \( \hat{x}^0 \) and thus, may exist incomplete semi trajectories \( x^r_- \) and \( x^r_+ \) since there is no duality to observe in Remark 5.26.

2) \( \Rightarrow \) : By duality in Remark 5.26, every semi trajectory \( x^r_+ \in \mathcal{M} \) solution of the system \( \Sigma^+ \) has an associated dual semi trajectory \( \hat{x}^r_- \in \mathcal{M}^* \) solution of the adjoint system \( \Sigma^+_\dag \) and moreover every weak controllable semi trajectory \( x^-_r \) of \( \Sigma \) is an observable semi trajectory \( x^-_- \) of \( \Sigma^\dag \) and vice versa. Therefore the adjoint realization \( \Sigma^+_\dag \) must preserve in dual form the weak controllability and observability properties of \( \Sigma^+ \). \( \Leftarrow \) : By contradiction, assuming Eqs. (5.20) or equivalently duality in Remark 5.26 is not satisfied, then there may exist incomplete semi trajectories \( x^r_- \) and \( x^r_+ \) with arbitrary initial conditions \( x^0 \) and \( \hat{x}^0 \), and no assessment can be made on the properties of system \( \Sigma^+ \) and \( \Sigma^+_\dag \). \( \blacksquare \)
Abstract: Most of the energy functions used in nonlinear balancing theory can be expressed as storage functions in the framework of dissipativity theory. By defining a framework of discrete-time dissipative systems, this paper presents existence conditions for their discrete-time energy functions along with algorithms to find them based on dynamic optimization problems. Furthermore, the important case of the nonlinear discrete-time versions of the controllability and observability functions, its properties and algorithms to find them are presented. These algorithms are illustrated with linear and nonlinear examples.

Keywords: Nonlinear systems, Dissipative systems, Discrete-time systems, Controllability, Observability.

6.1 Introduction

The study of systematic tools for model reduction of dynamic systems has been an early topic of interest in the systems and control fields. Model approximation based on the Hankel norm and the balancing method [49, 143] have shown to be useful tools for model reduction for linear systems.
Furthermore, singular values-based balancing, LQG balancing and $\mathcal{H}_\infty$ balancing are nowadays important tools for linear model reduction. With the use of the behavioral approach, in [212] a general balancing framework for model approximation and reduction is provided, which has Lyapunov, LQG, and $\mathcal{H}_\infty$ as special cases being valid for linear unstable systems [212]. These developments provide interesting paradigms for nonlinear generalizations.

In continuous-time nonlinear systems, there has been important progress on the nonlinear extensions of balancing methods, mainly based on the controllability and observability functions and their corresponding singular-values [181, 180, 43], but also with other energy functions [60] or for particular problems, namely LQG [187], $\mathcal{H}_\infty$ [182] or for port-Hamiltonian Systems [127]. The use of the theory of dissipative systems offers to a certain extent a generalized approach in order to deal with the variety of energy functions used for nonlinear balancing of continuous systems, see [127]. The explicit use of dissipativity theory for balancing of linear systems was firstly presented in [211]. Most of the efforts have been devoted to continuous-time systems. The prototypical case is precisely nonlinear balancing based on the controllability and observability functions. Roughly speaking, in the procedure presented in [180], a Hamilton-Jacobi equation and a Lyapunov-like partial differential equation have to be solved in order to determine the energy functions. Then with the use of a nonlinear coordinate transformation, the system is represented in a balanced form. After truncation of the less important dynamic subsystem and application of an inverse transformation a reduced system is obtained. The mathematical complexity in solving the partial differential equations associated to the energy functions has stimulated the search for alternative methods [149].

In this paper some aspects of the discrete-time framework for dissipativity theory for balancing nonlinear systems are introduced. Such framework relies on storage functions, in particular the required supply and the available storage, instead of the controllability and observability functions. Therefore, in order to find such storage functions, optimization algorithms are proposed. Furthermore, the energy functions for stable nonlinear discrete-time systems are discussed as important particular cases, extending the continuous-time theory presented in [181, 180]. Since the determination of such storage functions is a fundamental condition for nonlinear balancing and model reduction, in this paper attention is given to computer implementation of the theory. Furthermore, with applications in mind, some preliminary connections with continuous-time systems that are time-discretized are presented. This approach does not assume any linearization procedure at all, contrasting with [207]. In the pre-
liminary work [121] it was shown that once the energy functions are found, some procedures for the continuous-time balancing presented in [181, 43] can be followed in order to find a reduced system.

The paper is organized as follows. After fixing the notation used, in Section 6.2 relationships with time-discretized systems are presented. These concepts allow us to represent in a different form an optimal control problem that appears in the following Section 6.3, where several concepts of the discrete-time dissipativity theory are discussed. One important case of the storage functions presented in Section 6.3 are the discrete-time energy functions. In Section 6.4, the observability and controllability functions and their properties are discussed and algorithms to find them are presented. In order to illustrate these methods, linear and nonlinear examples are shown and briefly discussed. Finally, some conclusions are presented.

Notation: Despite the efforts of several experts [141, 140, 83], notation for nonlinear discrete-time systems is not standard. The notation in this paper tends to follow [105, 106]. The set of nonnegative and non positive integers are denoted as \( \mathbb{Z}^+ \) and \( \mathbb{Z}^- \) respectively. Time is denoted by \( t \in \mathbb{R}^+ \) and while \( T \in \mathbb{R}^+ \) denotes the endpoint of a certain finite period of time, \( T \in \mathbb{R}^+ \) denotes the sampling time, \( t = kT, k \in \mathbb{Z}^+ \). Discrete-time vector variables are denoted for instance as \( x_k \) or \( x(k) \). A continuous-time function is expressed as \( f(t) \), which is expressed after discretization by the discrete-time function \( F(k) \). Where convenient, for clarity of exposition a function of several variables \( F(x_k, u_k) \) may be denoted simply as \( F_k \). Given a function \( F_k \) its inverse function (map) is denoted as \( F_k^{-1} \). An optimal input variable at time \( k \) is denoted as \( v_k^\ast \). Finally, the solution of the system \( x_{k+1} = F(x_k, u_k) \) at the interval \([k, k + m]\) with initial condition \( x(k) = x_k \) and input \( u_k \in \ell_2(0, \infty) \) is denoted by \( x_{k+m} = \phi(k + m, k, x_k, u) \).

6.2 Some relationships between continuous and discrete-time systems

In this section we begin to study several relations of continuous-time and discrete-time systems (and other associated systems that can be derived from them) with the purpose of simplifying the interpretation of one optimal control problem that appears in the subsequent Section 6.3.

Consider the following continuous-time nonlinear system
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\[ \dot{x}(t) = f(x(t), u(t)), \]
\[ y(t) = h(x(t)), \]
\[ (6.1) \]

where \( x \in \mathbb{R}^n \) are local coordinates for a \( C^\infty \) state space manifold \( X \), \( f \) and \( h \) are \( C^\infty \), \( u \in U \subset \mathbb{R}^p \) and \( y \in Y \subset \mathbb{R}^q \). Assume throughout that \( u, x \) and \( y \) are locally square integrable. On the other side, consider the following discrete-time nonlinear system,

\[ x_{k+1} = F(x_k, u_k), \]
\[ y_k = h(x_k), \quad k \in \mathbb{Z} \]
\[ (6.3) \]

where \( u_k = (u_1, \ldots, u_p)_k \in U \subset \mathbb{R}^p \), \( y_k = (y_1, \ldots, y_q)_k \in Y \subset \mathbb{R}^q \) and \( x_k = (x_1, \ldots, x_n)_k \in \mathbb{R}^n \) are local coordinates for the smooth state space manifold \( X \). Moreover \( F \) and \( h \) are class \( C^\infty \) in a neighborhood \( D \subset \mathbb{R}^n \) around an equilibrium point in \( x = 0 \) such that \( F(0, 0) = 0 \) and \( h(0) = 0 \).

In discrete-time systems, evolution in reversed-time implies the invertibility of the map \( F(\cdot, u) \), which is only possible under certain conditions, discussed in [40] and [83]. Roughly speaking, any discrete-time nonlinear system that is causal can be described by a generically reversible dynamics [40], and when sampling or discretizing a system in the form (6.1) to obtain (6.3) the resulting dynamics is reversible, meaning that the Jacobian matrix \( [\partial F/\partial x, \partial F/\partial u] \) is generically nonsingular for all values of \( x \) and \( u \), [40, 83], and the system is said to be generically submersive [63].

Some problems in optimal control can be simplified when the time direction of evolution is reversed, like in the cases discussed in the following Sections 6.2 and 6.3 where we will assume that \( F(\cdot, u) \) is a diffeomorphism. If it is the case that the system (6.3) results from discretization of the continuous-time system (6.1) then this latter assumption is satisfied automatically, [40, 83]. There are several publications regarding procedures to obtain a discrete-time equivalent of the state equation (6.1), e.g. [140, 94, 96]. In particular, the method known as the Taylor-Lie series discretization [94], yields a system in the form of (6.3) for \( k \in \mathbb{Z}^+ \). The Taylor-Lie series discrete-time equivalent to (6.1) is given by

\[ x_{k+1} = x_k + \sum_{i=1}^{\infty} \frac{T^i}{i!} \left( \frac{d^i x}{dt^i} \right)_k, \]
\[ (6.5) \]

where \( T \) is the sampling-time and

\[ \frac{d^i x}{dt^i} = \left\{ \begin{array}{ll} f(x(t), u(t)), & \text{for } i = 1, \\ \frac{\partial^T}{\partial x} \left( \frac{d^{i-1} x}{dt^{i-1}} \right) f(x(t), u(t)), & \forall \ i > 1. \end{array} \right. \]
\[ (6.6) \]
This discretization procedure preserves several analytical properties like equilibrium and is a generalization of the exponential matrix discretization procedure for linear systems.

When needed, such procedure will be used to establish certain relationships between continuous and time-discretized systems, which allows us to count with alternative representations of certain time-reversed discrete-time nonlinear optimal control problems in the following sections. An exact discretization from Eq. (6.5) is assumed in the sense that the infinite series is considered without truncation during calculations.

Example 6.1. Consider the linear time invariant, stable, minimal, continuous-time system

\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t),
\]

(6.7)

which by the Taylor-Lie method (or by the step-invariant discretization procedure) results in the following discrete-time system

\[
x_{k+1} = Ax_k + Bu_k, \\
y_k = Cx_k,
\]

(6.8)

where \( u \in \mathbb{R}^p, y \in \mathbb{R}^q \) and \( x \in \mathbb{R}^n \), \( A = e^{AT} \) and \( B = \left( \sum_{i=1}^{\infty} \frac{(A)^{i-1}T^i}{i!} \right)B = \int_0^T e^{As}B \, ds \).

In order to distinguish stringently the direction of time evolution throughout the paper, define the past interval \( I_p = [-T_p, 0] \subset \mathbb{R}^1 \), \( T_p > 0 \) the future interval \( I_f = [0, T_f] \subset \mathbb{R}^1 \), \( T_f > 0 \) and the total interval \( I = [-T_p, T_f] \subset \mathbb{R}^1 \), \( T_f + T_p = T_I > 0 \). A forward-time evolution is denoted by \( t \in I \) (a strictly increasing variable from \( t_0 \in I \)) and a backward-time evolution with \( \tau \in I \) (a strictly decreasing variable from \( \tau_0 \in I \)). A sequence evolving in forward-time is denoted by \( k \in \mathbb{Z} \) such that \( -N_p \leq k \leq N_f \) from some initial \( k_0 \in \mathbb{Z} \) and a backward-time evolution by, \( \kappa \in \mathbb{Z} \) such that \( -N_p \leq \kappa \leq N_f \) from some initial \( \kappa_0 \in \mathbb{Z} \); \( N_p, N_f \in \mathbb{Z}^+ \).

Definition 6.2. Assume that system (6.1) evolves in \( t \in [-T_p, 0] \). Define an associated system by

\[
\frac{d}{dt}w(t) = -f(w(t), v(t)), \quad t \in [0, T_p],
\]

(6.9)

which can be obtained by performing two sequential operations on Eq. (6.1):
Backward-time: By rendering Eq.(6.1) evolve in regressive time i.e. \( \tau = -t \),
\[
x(-\tau) = x(t), \quad dx(-\tau)/d\tau = -dx(t)/dt \quad \text{and} \quad u(-\tau) = u(t).
\]

Flip-time: By defining \( w(t) \overset{\text{def}}{=} x(-\tau) \) and \( v(t) \overset{\text{def}}{=} u(-\tau) \).

In the same form an associated discrete-time system can be defined as follows:

**Definition 6.3.** Assume that departing from \( k = -N_p \), the system (6.3) evolves in \(-N_p \leq k \leq 0\). Define an associated system by

\[
w_{k+1} = F^{-1}(w_k, v_{k+1}), \quad 0 \leq k \leq N_p
\]

which departs from \( k = 0 \) and can be obtained by applying two sequential operations on Eq.(6.3):

Backward-time: Inverting the map in Eq.(6.3) replacing the time index by \( \kappa \) and, evolving, from 0, with the decreasing sequence \( \kappa \in \mathbb{Z}^- \).

Flip-time: By replacing the time index as \( k := -\kappa \), \( 0 \leq k \leq N_p \) and (mirrored with respect to 0) defining \( w_k \overset{\text{def}}{=} x_k \) (thus \( w_{k-1} := x_{k+1} \)) and \( v_k \overset{\text{def}}{=} u_k \).

The steps detailed in the continuous-time and discrete-time systems under Definitions 6.2 and 6.3, respectively, involve the realization of a sequence of associated systems, which are related to each other by a discretization procedure, as shown in the Commutative Diagram of Figure 6.1.

**Remark 6.4.** Commutativity of the diagram presented in Figure 6.1 depends on the fact that the discrete-time systems represented there, depart from continuous-time systems and therefore the map \( F(\cdot, u) \) is invertible [83]. For practical applications several ways to obtain the state of the time-discretized system can be used, including some algorithms of numerical integration.
Remark 6.5. In particular, the associated system (6.10) can be found alternatively by time-discretizing (6.1) and performing the operations in Definition 6.3, or alternatively by direct time-discretization of system (6.9), resulting in the following system

\[ w_{k+1} = w_k + \sum_{i=1}^{\infty} \frac{T^i}{i!} \left( \frac{d^i w}{dt^i} \right)_k \]  

(6.11)

with

\[
\frac{d^i w}{dt^i} = \begin{cases} -f(w(t),v(t)), & \text{for } i = 1 \\ (-1)^i \frac{\partial}{\partial w} \left( \frac{d^{i-1} w}{dt^{i-1}} \right) f(w(t),v(t)), & \forall \ i > 1, \end{cases}
\]

(6.12)

and defining \( v(k+1) \) as the time discretization of \( v(t) \), which circumvents the inversion of the map \( F(\cdot, u) \).

In the next section an optimal control problem will have a simpler solution due to the tools we have presented in this section. This section concludes with the linear system example to illustrate these concepts.

Example 6.6. Consider the system (6.8) in Example 6.1 which by Definition 6.3 it has an associated system given by

\[ w_{k+1} = A^{-1} w_k - A^{-1} B v_{k+1}, \]  

(6.13)

since \( A^{-1} B = e^{-AT} \int_0^T e^{As} B \, ds \) and by defining \( \xi = T - s, \ d\xi = -ds \), it can be alternatively written in terms of \((A, B)\) as

\[ w_{k+1} = e^{-AT} w_k - \int_0^T e^{-A\xi} B \, d\xi \, v_{k+1}. \]  

(6.14)

Consider now the use of the commutative diagram in Fig.6.1 along with Remark 6.5. Since the use of Definition 6.2 in system (6.7) provide us with the continuous associated system \( \dot{w}(t) = -A w(t) - B v(t), \ t \geq 0 \), the discretization of this system yields straightforwardly a system in the form of (6.8) with \( A = e^{-AT} \) and \( B = -\int_0^T e^{-A\xi} B \, d\xi \), which is Eq. (6.14).

6.3 Discrete-time dissipativity theory and storage functions

In [127] a framework based on dissipativity theory for balancing and nonlinear model reduction of continuous systems was presented. In this section, such
The framework is presented in its discrete-time form along with some results whose proofs (direct equivalence with the continuous-time, e.g. [205, 219]) provide the way to reinterpret some optimal control problems as dynamic optimization problems. Some concepts of the discrete-time theory of dissipative systems have been developed elsewhere [64, 106].

The system (6.3) is said to be dissipative with supply rate \( r(y_k, u_k), r : \mathcal{Y} \times \mathcal{U} \to \mathbb{R} \), if there exist a nonnegative function \( S : \mathbb{R}^n \to \mathbb{R} \), \( S(0) = 0 \) called storage function such that for all \( u_k \in \mathcal{U} \) and all \( k \in \mathbb{Z} \) [106],

\[
S(x_{k+1}, r) - S(x_k, r) \leq r(y_k, u_k),
\]

which for all \( k, m \in \mathbb{Z} \), with \( m \geq 0 \) is equivalent to

\[
S(x_{k+m}, r) - S(x_k, r) \leq \sum_{i=k}^{k+m-1} r(y_i, u_i).
\]

This latter relation is named the discrete-time dissipation inequality [106, 219].

**Theorem 6.7.** The system (6.3)-(6.4) is dissipative with supply rate \( r_a(y_k, u_k) \) if and only if the function called available storage, \( S_a : \mathcal{X} \to \mathbb{R}^+ \), defined as

\[
S_a(x^0, r_a) = \sup_{u(\cdot) \in \mathcal{U}, x(0)=x^0, N_f \in \mathbb{Z}^+} -\sum_{i=0}^{N_f} r_a(y_i, u_i)
\]

\[
= -\inf_{u(\cdot) \in \mathcal{U}, x(0)=x^0, N_f \in \mathbb{Z}^+} \sum_{i=0}^{N_f} r_a(y_i, u_i), \quad i \in \mathbb{Z}^+.
\]

is finite for all \( x^0 \in \mathcal{X} \). In such case \( S_a(x^0, r_a) \) is a storage function such that \( S_a(x^0, r_a) \leq S(x^0, r) \) for all \( x^0 \in \mathcal{X} \), for any other storage function \( S(x^0, r) \).

The proof Theorem 6.7 can be found in the Appendix.

**Lemma 6.8.** Assume that there exists an optimal sequence of inputs \( \{u^*_i|i = 0, 1, \ldots, N_f-1\} \) that fulfills (6.17). Then

\[
S_a(x^0, r_a) = -\sum_{i=0}^{N_f} r_a(y_i, u^*_i),
\]

(6.18)
and moreover, it can be found from the limit \( S_a(x_0, r_a) = \lim_{k \to N_f} S_a(x_{k+1}, r_a) \) where \( S_a(x_{k+1}, r_a) \) is the solution of the following recurrence equation

\[
S_a(x_{k+1}, r_a) = S_a(x_k, r_a) - r_a(y_k, u_k^*)
\]  

(6.19)

with initial condition \( S_a(x_0, r_a) = 0 \).

**Proof of Lemma 6.8:** The simple result of Lemma 6.8 can be shown by solving the iterative equation (6.19) with the initial condition provided. The iteration begins for \( S_a(x_0, r_a) = 0 \) and then

\[
S_a(x_1, r_a) = S_a(x_{k+1}, r_a) - r_a(y_1, u_1^*)
\]

\[
i = 2,
\]

\[
S_a(x_2, r_a) = S_a(x_1, r_a) - r_a(y_2, u_2^*)
\]

\[
i = 3,
\]

\[
\vdots
\]

\[
i = k + 1, S_a(x_{k+1}, r_a) = S_a(x_k, r_a) - r_a(y_k, u_k^*)
\]

in general we could write for \( i = N_f + 1 \)

\[
S_a(x_{N_f+1}, r_a) = - \sum_{i=0}^{N_f} r_a(y_i, u_i^*)
\]

which is Eq. (6.18).

Consider the second representation in Eq. (6.17). With the assumptions of Theorem 6.7, the optimal control problem (6.17) can be interpreted as a dynamic optimization algorithm as follows. Let \( \epsilon, N_f \in \mathbb{Z}^+ \) be such that \( \|x_{N_f}\| \leq \epsilon \) for \( \epsilon \) small enough. Assume that \( N_f \) is known and that the (closed) set of admissible inputs \( \{u|u \in U\} \) (each \( u \in U \) being locally square integrable) satisfy the usual regularity conditions of being convex with nonempty interior. Then \( S_a(x_0, r_a) \) as given by (6.17) restricted to system (6.3)-(6.4) with boundary conditions \( x_{N_f} = 0 \) and \( x_0 = x(0) \) can be posed as the solution of the following dynamic optimization problem

\[
\min_{\{u_i| i=0, \ldots, N_f - 1\}} S_a(x_{N_f}, r_a),
\]

(6.20)

with equality constraints

\[
\begin{align*}
x_{i+1} &= F(x_i, u_i), \\
S_a(x_{i+1}, r_a) &= S_a(x_i, r_a) + r_a(h(x_i), u_i), \\
x_{N_f} &= 0, \\
x_0 &= x^0,
\end{align*}
\]

(6.21)
with initial inputs \( \{ u^0 \mid u_j^0 \in U, \ j = 0, 1, ..., N_f - 1 \} \) and with \( S_a(x_0, r_a) = r_a(h(x^0), u^*_0) \). With the determination of \( u^* \), then \( S_a(x_k, r_a) = -S_a(x_k, r_a) \).

**Theorem 6.9.** Assume that (6.3) is reachable from \( x^* \in \mathcal{X} \), then the required supply, \( S_r : \mathcal{X} \to \mathbb{R}^+ \), defined as

\[
S_r(x^0, r_r) = \inf_{u(\cdot) \in \mathcal{U}} \sum_{i=-N_p}^{0} r_r(y_i, u_i), \quad i \in \mathbb{Z}^-, \quad (6.22)
\]

satisfies the dissipation inequality (6.16). Then (6.3) is dissipative if and only if \( S_r(x^0, r_r) \) is finite for all \( x \in \mathcal{X} \).

The proof of Theorem 6.9 can be found in the Appendix.

The definition of the associated system (6.10) provides a way to express the same optimal control problem defining \( S_r(x, r_r) \) but in forward time, convenient for the subsequent developments. During the rest of the paper from Theorem 6.9 it will be assumed that \( x^*_{-N_p} = 0 \).

**Remark 6.10.** Taking in consideration the system (6.10) from Definition 6.3, then \( S_r \) from Eq. (6.22), may be alternatively expressed as

\[
S_r(w^0, r_r) = \inf_{v(\cdot) \in \mathcal{U}} \sum_{i=0}^{N_p} r_r(y_i, v_i), \quad (6.23)
\]

for \( w_k \) and \( v_k \) as in Definition 6.3.

**Remark 6.11.** \( v_0 \) does not influence the new state \( w_1 \) in (6.10), where it results \( w_1 = F^{-1}(w_0, v_1) \). Therefore the value of \( v_0 \) which minimizes (6.23) is \( v_0^* = 0 \) and thus \( u_0^* = 0 \).

**Lemma 6.12.** Assume the existence of the optimal sequence \( v^* = \{ v_i^* \mid i = 0, 1, ... N_p \} \) such that it satisfies (6.23) and consider the following recursive equation

\[
S_r(w_{i+1}, r_r) = S_r(w_i, r_r) + r_r(y_i, v_i), \quad (6.24)
\]

for \( i = 0, 1, 2, ... \) and initial condition \( S_r(w^0, r_r) = r_r(y_0, v_0^*) \). Then \( S_r(w^0, r_r) \) can be found from the iterative solution of Eq. (6.24).
Proof of Lemma 6.12: Express (6.23) as,

\[ S_r(w^0, r_r) = r_r(y_0, v_0^*) + \sum_{i=0}^{N_p} r_r(y_i, v_i^*), \]  

(6.25)

which may be written as a recurrence equation with the initial condition

\[ S_r(w^0, r_r) = r_r(y_0, v_0^*) \]  

as consequence of Remark 6.11. By solving iteratively

(6.24), \( S_r(w^0, r_r) \) can be found as \( i \) tends to \( N_p \).

Assuming that the conditions of Theorem 6.9 hold, then the approximate solution of the optimal control problem (6.22) can be found by a reinterpretation of the problem as follows. Let \( \epsilon, N_p \in \mathbb{Z}^+ \) be such that \( \|w_{N_p}\| \leq \epsilon \) for \( \epsilon \) small enough. Assume that \( N_p \) is known and assume the following regularity condition is satisfied: the (closed) set of admissible inputs \( \{v \mid v \in \mathcal{V}\} \) is convex with nonempty interior. Then \( S_r(w_0, r_r) \) as given by (6.22) restricted to (6.10) with boundary conditions \( w_{N_p} = 0 \) and \( w^0 = w_k \) can be posed as the solution of the following dynamic optimization problem

\[
\min_{\{v_i| i=1,...,N_p\}} S_r(w_{N_p}, r_r),
\]  

(6.26)

with equality constraints

\[
\begin{cases}
  w_{i+1} = F^{-1}(w_i, v_{i+1}), \\
  S_r(w_{i+1}, r_r) = S_r(w_i, r_r) + r_r(h(w_i), v_i), \\
  w_{N_p+1} = 0, \\
  w^0 = w_0,
\end{cases}
\]  

(6.27)

with initial inputs \( \{v^0 \mid v^0_j \in \mathcal{V}, j = 1,...,N_p\} \) and with \( S_r(w^0, r_r) = r_r(h(w^0), 0) \), determining \( v_i^* \).

6.4 Discrete-time controllability and observability functions

In this section we restrict ourselves to the important case in the context of dissipative systems, with the required supply \( S_r \) with supply rate \( r_r = \frac{1}{2} u_k^T u_k \) and the available storage \( S_a \) with supply rate \( r_a = -\frac{1}{2} y_k^T y_k \).

Definition 6.13. The controllability and observability functions of the system (6.3) are defined respectively as,
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\[ L_c(x^0) = \inf_{u \in \ell_2(-\infty,0), \ x(-\infty)=0, \ x^0=x_0} \left\{ \frac{1}{2} \sum_{k=-\infty}^{0} \| u_k \|^2, \ k \in \mathbb{Z}^- \right\} \quad (6.28) \]

\[ L_o(x^0) = \frac{1}{2} \sum_{k=0}^{\infty} \| y_k \|^2, \ x^0 = x_0, \ u_k = 0, \ k \in \mathbb{Z}^+. \quad (6.29) \]

The energy functions (6.28) and (6.29) so defined are the discrete-time equivalents of the continuous versions presented in [180]. When these functions are defined for linear systems like Eq. (6.8) some known functions result in terms of Gramians [167].

**Corollary 6.14.** Consider the system (6.8). Then \( L_c \) and \( L_o \), as defined in Eqs. (6.28) and (6.29), are given by,

\[ L_c(x^0) = \frac{1}{2} x_0^T P^{-1} x_0, \quad (6.30) \]

\[ L_o(x^0) = \frac{1}{2} x_0^T Q x_0, \quad (6.31) \]

with Gramians \( P = \sum_{k=0}^{\infty} A^k B B^T A^k \) and \( Q = \sum_{k=0}^{\infty} A^T C^T C A^k \).

The proof of this result is presented in the examples throughout this section.

**Remark 6.15.** Assuming that system (6.3) is dissipative for a supply rate \( r_a = -\frac{1}{2} y_k^T y_k \) and \( N_f \to \infty \), the discrete-time version of the new or generalized observability energy function defined by Gray and Mesko [60]

\[ \hat{L}_o(x^0) = \sup_{u \in \ell_2(0,\infty), \| u \|_2 \leq \alpha, \ x_0=x^0, \ x(\infty)=0} \frac{1}{2} \sum_{k=0}^{\infty} \| y_k \|^2, \ \alpha \geq 0, \ k \in \mathbb{Z}^+ \]

can be considered a particular case of (6.17) and furthermore can be posed as a dynamic optimization algorithm in the form of (6.20) and (6.21).

During the rest of this section we restrict ourselves to the functions (6.28) and (6.29) resulting from system (6.3). From now on assume system (6.3) to be locally asymptotically stable at \( x^0 = x_0 \) for \( u_k = 0 \), in a neighborhood \( D \subset \mathbb{R}^n \).

### 6.4.1 Observability function

Since the definition of \( L_o \), Eq. (6.29) does not define an optimal control problem, in this subsection a recursive procedure to find the observability function
is provided along with some properties. Also a Lyapunov-like difference equation analog to that found in [181], is presented.

**Lemma 6.16.** Consider the following recursive equation

\[ L_o(x_{i+1}) = L_o(x_i) + \frac{1}{2} h^T [F(x_i, 0)] h[F(x_i, 0)], \]  

(6.32)

for \( i = 0, 1, 2 \ldots \) and \( L_o(x^0) = \frac{1}{2} h^T(x^0) h(x^0) \) as initial condition. Then \( L_o(x^0) \) can be found from the solution of (6.32) as follows

\[ L_o(x^0) = \lim_{i \to \infty} L_o(x_i). \]  

(6.33)

**Proof of Lemma 6.16:** By Eq. (6.3) and by definition of \( L_o \) in Eq. (6.29), one obtains

\[ L_o(x^0) = \frac{1}{2} \sum_{i=0}^{\infty} h^T(x_i) h(x_i) \]

\[ = \frac{1}{2} h^T(x^0) h(x^0) + \frac{1}{2} \sum_{i=0}^{\infty} h^T[F(x_i, 0)] h[F(x_i, 0)], \]  

(6.34)

for \( i \in \mathbb{Z}^+ \). Noting that

\[ L_o(x_{N_f+1}) = L_o(x^0) + \frac{1}{2} \sum_{i=0}^{N_f} h^T[F(x_i, 0)] h[F(x_i, 0)], \quad N_f \in \mathbb{Z}^+ \]

the result is obtained from Eq. (6.33) when \( N_f \to \infty \). \( \blacksquare \)

**Example 6.17.** An alternative proof of Corollary 6.14 to find \( L_o \) follows. Use recurrent Eq. (6.32) for the system (6.8) resulting in the following difference equation

\[ L_o(x_{i+1}) = L_o(x_i) + \frac{1}{2} x_i^T A^T C^T C A x_i, \]  

(6.35)

with initial condition \( L_o(x_0) = \frac{1}{2} x_0^T C^T C x_0 \). Then the solution of (6.35) yields

\[ L_o(x^0) = \lim_{i \to \infty} L_o(x_i) = \sum_{k=0}^{\infty} x_0^T A^k C^T C A^k x_0, \]

which is Eq. (6.31). \( \blacksquare \)
Definition 6.18. System (6.3)-(6.4) is said to be strongly locally observable at \( x^0 \) if at \( x^0 \in \mathcal{X} \) there is a neighborhood \( D \subset \mathcal{X} \) such that for any \( \bar{x} \in D \),
\[
h(F(\bar{x})) = h(F(x^0)) \quad \text{for} \quad k = 0, 1, \ldots, n - 1 \quad \text{implies} \quad \bar{x} = x^0 \quad [150].
\]
System (6.3)-(6.4) is locally zero-state detectable in a neighborhood \( D \) of \( x = 0 \) if for all \( x_k \in D \), \( u \equiv 0 \), \( y = h(\phi(k,0,x^0,0)) \) \( \equiv 0 \) implies that \( \lim_{k \to \infty} \phi(k,0,x^0,0) = 0 \), \( k \in \mathbb{Z}^+ \).

Note that according to the notation used \( x_k = \phi(k,0,x^0,0) = F_k \circ \cdots \circ F_0 \), where \( F_i = F(x_i,0), i \in \mathbb{Z}^+ \).

Theorem 6.19 ([150]). Consider the map \( \mathcal{O} : \mathcal{X}^n \mapsto (\mathbb{R}^q)^n \) defined by
\[
\mathcal{O}^{n-1}(x^0) = \begin{bmatrix}
  h(x^0) \\
  h \circ F_0 \\
  h \circ F_1 \circ F_0 \\
  \vdots \\
  h \circ F_{n-2} \circ \cdots \circ F_0
\end{bmatrix}
= \begin{bmatrix}
  h(x^0) \\
  h \circ \phi(1,0,x^0,0) \\
  h \circ \phi(2,0,x^0,0) \\
  \vdots \\
  h \circ \phi(n-2,0,x^0,0)
\end{bmatrix}.
\]
\[(6.36)\]
If the system (6.3)-(6.4), with \( u_k = 0 \), is such that it satisfies
\[
\text{rank} \left[ \frac{\partial}{\partial x^0} \mathcal{O}^{n-1}(x^0) \right]_{x^0=x(0)} = n, \quad x^0 \in \mathcal{X}
\]
then the system is strongly locally observable at \( x^0 \in \mathcal{X} \).

Theorem 6.20. If the system (6.3)-(6.4) satisfies (6.37), then the system is locally zero-state observable at \( x^0 \).

Proof of Theorem 6.20: The output nulling submanifold \( \mathcal{N} \subset \mathcal{X} \) (see [151]) associated to the output map (6.4) is defined by \( \mathcal{N} = \{ x | h(x) = 0, \ x \in \mathcal{X} \} \). If the system (6.3)-(6.4) is such that \( u_k = 0 \) and \( y_k = 0 \), then any state trajectory evolves on \( \mathcal{N} \). Any state in \( \mathcal{N} \) is unobservable, since any \( x \in \mathcal{N} \) (with neighborhood \( D \)) is indistinguishable from another \( \bar{x} \in D \subset \mathcal{N} \). If the rank condition (6.37) is satisfied, necessarily \( \mathcal{N} = \{ 0 \} \), implying \( x^0 = 0 \).

The previous conclusion can also be deduced from the discussion in [150]. The property of zero-state observability is important in order to assert positive definiteness of the observability function, as can be seen in the following result.

Theorem 6.21. Assume that (6.3) with \( F(\cdot,0) \) is asymptotically stable on a neighborhood \( D \) of \( x = 0 \). If the system is zero-state observable and \( L_0 \) exists and is smooth on \( \mathcal{X} \), then \( L_0(x^0) > 0, \ \forall \ x^0 \in \mathcal{X}, \ x^0 \neq 0 \).
Proof of Theorem 6.21: Recall Eq. (6.29), then, if \( x^0 \neq 0 \), zero state observability implies that for some \( \bar{K} \in \mathbb{Z}^+ \setminus \{0\} \) we have \( h(\phi(\bar{k}, 0, x^0, 0)) \neq 0 \) for some \( 0 \leq \bar{k} < \bar{K} \). Therefore if \( x^0 \neq 0 \), \( L_o(x^0) > 0 \).

Theorem 6.22 (Existence of \( L_o \)). If there exists a convergent \( \sum_{k=0}^{\infty} M_k \), \( M_k \in \mathbb{R} \), such that \( \|h(x_k)\|^2 \leq M_k \) for all \( x_k \in D \), then \( L_o \) exists as given by (6.33) and is a smooth solution of (6.32) for all \( x_0 \in D \).

Proof of Theorem 6.22: By Lemma 6.16, Eq. (6.33) is a solution of (6.32). Existence of the limit (6.33) for all \( x^0 \in D \) is necessary and sufficient for existence of \( L_o \). Since \( (\mathbb{R}^n, \| \cdot \|_2) \) is a complete normed space, by Weierstrass’ M-Test, the series of functions (6.34) converges uniformly and absolutely.

With the tools developed until now, the following result can be proved and serves to establish the connection with the dissipativity theory concepts presented in the previous section.

Proposition 6.23. Assume that the observability function \( L_o \) exists and is positive definite. Then \( L_o \) as defined in Eq. (6.29) is a Lyapunov function for system (6.3) and furthermore such system is dissipative with storage function \( L_o \) and supply rate \( \frac{1}{2} h^T(x_k)h(x_k) \).

Proof of Proposition 6.23: The proof uses similar proving techniques to those in [106] and [64]. In order to show that the difference \( L_o(x_{k+1}) - L_o(x_k) \) is negative semi-definite (and thus a Lyapunov function [101]), express \( L_o(x_k) \) for an arbitrary state \( x_k \) as,

\[
L_o(x_k) = \frac{1}{2} h^T(x_k)h(x_k) + \frac{1}{2} \sum_{i=k}^{\infty} h^T[F(x_i, 0)]h[F(x_i, 0)],
\]

doing the same for \( x_{k+1} \), and taking the difference then

\[
L_o(x_{k+1}) - L_o(x_k) = -\frac{1}{2} h^T(x_k)h(x_k), \quad (6.38)
\]

for \( k \in \mathbb{Z}^+ \), which is negative semidefinite. As can be seen, the discrete-time dissipation inequality (6.15) is preserved and since by assumption Theorem 6.22 is satisfied, there exist \( M_i \) such that \( L_o \) is bounded, then by Theorem 6.7, \( L_o \) is a storage function with supply rate \( \frac{1}{2} h^T(x_k)h(x_k) \).
Remark 6.24. Following the terminology used in [180], Eq.(6.38) can be called the discrete-time Lyapunov-like equation.

To conclude this subsection, when the assumptions of Proposition 6.23 are compared with those of Theorem 6.7, the asymptotic stability and zero-state observability of system (6.3)-(6.4) seem to be natural additional requirements due to the assumption of $u_k = 0$ from the definition of $L_o$. The following subsection deals with the controllability function for (6.3)-(6.4).

### 6.4.2 Controllability function

Before determining some properties of the controllability function (6.28) of (6.3), it is useful to transform the definition of $L_c$ into a more adequate representation with the help of Definition 6.3.

Remark 6.25. Consider the system (6.10). Then the definition of $L_c$ from Eq. (6.28), may be expressed as

$$L_c(w^0) = \inf_{v \in \ell_2(0,\infty), \ w(\infty)=0, \ w^0=w_0} \frac{1}{2} \sum_{k=0}^{\infty} v_k^T v_k, \quad (6.39)$$

for $w$ and $v$ defined in (6.10).

**Lemma 6.26.** Assume the existence of the optimal sequence $v^* = \{v_i^*|i = 0, 1, \ldots\}$ such that it satisfies (6.39) and consider the following recursive equation

$$L_c(w_{i+1}) = L_c(w_i) + \frac{1}{2} v_i^{*T} v_i^*, \quad (6.40)$$

for $i = 0, 1, 2, \ldots$ and initial condition $L_c(w^0) = 0$. Then $L_c(w^0)$ can be found from the solution of (6.40) as follows

$$L_c(w^0) = \lim_{i \to \infty} L_c(w_i). \quad (6.41)$$

**Proof of Lemma 6.26:** Express (6.39) as,

$$L_c(w^0) = \frac{1}{2} v_0^{*T} v_0^* + \sum_{i=0}^{\infty} v_{i+1}^{*T} v_{i+1}^*, \quad (6.42)$$

which may be written as a recurrence equation with the initial condition $L_c(w^0) = \frac{1}{2} v_0^{*T} v_0^* = 0$ as consequence of Remark 6.11. By solving iteratively
(6.40), $L_c(w^0)$ can be found as $i$ tends to infinity.

As can be seen, Remark 6.25 and Lemma 6.26 are particular cases of Remark 6.10 and Lemma 6.12.

Properties of $L_c$

Proposition 6.27. Assume that the system (6.3) is asymptotically stable on $D$, that there exist a solution $v^*$ to (6.39) and that the limit (6.41) exists. Then $L_c(w^0) > 0$ for $w^0 \in D$, $w^0 \neq 0$, if and only if the system

$$w_{k+1} = F^{-1}(w_k, v^*_{k+1}), \quad k \in \mathbb{Z}^+,$$

(6.43)

is asymptotically stable on $D$.

Proof of Proposition 6.27: Assume that there exists $w^0 \in D$, $w^0 \neq 0$ such that $L_c(w^0) = 0$. Since in Eq. (6.42) this is only possible if all $v^*_k = 0$, for $k = 0, ..., \infty$, the system (6.43) is equivalent to the unforced system $w_{k+1} = F^{-1}(w_k, 0)$, for $k \in \mathbb{Z}^+$, but this system cannot be stable, since this would imply that the associated system $w_{\kappa} = F(w_{\kappa+1}, 0)$, for $\kappa \in \mathbb{Z}^-$ is unstable, which contradicts the asymptotic stability of $F$.

Proposition 6.28. A necessary existence condition of $L_c(w_k)$ in Eq. (6.40), is that $v^*_k$ is the solution of the following two-point boundary value problem

$$\lambda_k = \left[ \frac{\partial}{\partial w_k} F^{-1}(w_k, v_{k+1}) \right]^T \lambda_{k+1},$$

(6.44)

$$v_{k+1} = -\left[ \frac{\partial}{\partial v_{k+1}} F^{-1}(w_k, v_{k+1}) \right]^T \lambda_{k+1},$$

(6.45)

subject to the boundary conditions $w(\infty) = 0$ and $w^0 = w(0)$.

Proof of Proposition 6.28: In order to find $L_c(w_k)$ given by Eq. (6.39), applying standard tools of the discrete optimal control theory (see for instance [105], [20]) results in the following Hamiltonian,

$$H_k = \frac{1}{2} v_{k+1}^T v_{k+1} + \lambda_{k+1}^T F^{-1}(w_k, v_{k+1}),$$

(6.46)

resulting in
\[
\begin{align*}
\frac{\partial H_k}{\partial w_k} &= \lambda_{k+1}^T \frac{\partial}{\partial w_k} F^{-1}(w_k, v_{k+1}) = \lambda_k^T, \\
\frac{\partial H_k}{\partial v_{k+1}} &= v_{k+1}^T + \lambda_{k+1}^T \frac{\partial}{\partial v_{k+1}} F^{-1}(w_k, v_{k+1}) = 0,
\end{align*}
\]

from which Eqs. (6.44) and (6.45) follow.

As can be observed from Eq. (6.45), the input \(v_{k+1}\) may appear implicitly. Therefore the analytical solution of this problem may be difficult to find in the nonlinear case. In the Appendix, this optimal control problem is presented in a sequential fashion in order to show the structure of this problem. In contrast for linear systems Eqs. (6.44) and (6.45) can be solved explicitly, as can be seen in the following example.

**Example 6.29.** An alternative proof of Corollary 6.14 to find \(L_c\) is provided here. Assume the existence of \(A^{-1}\) and consider the system from Definition 6.3 associated to Eq. (6.8) and provided in Eq. (6.13) whose general solution can be expressed as

\[
w_k = A^{-k} w^0 - \sum_{i=0}^{k-1} (A^{-1})^k B v_{i+1}.
\]

Using (6.44) and (6.45), results in

\[
\begin{align*}
\lambda_k &= A^{-T} \lambda_{k+1}, \quad (6.48) \\
v_{k+1} &= B^T A^{-T} \lambda_{k+1}. \quad (6.49)
\end{align*}
\]

Substitution of (6.49) in (6.13) yields,

\[
w_{k+1} = A^{-1} w_k - A^{-1} B B^T A^{-T} \lambda_{k+1}. \quad (6.50)
\]

Solving Eq. (6.48) explicitly in backward time, results in

\[
\lambda_k = (A^{-T})^{N_p-k} \lambda_{N_p}. \quad (6.51)
\]

Then the solution of (6.50) with input \(\lambda_{k+1}\) given by (6.51) is

\[
w_k = A^{-k} w_0 - \sum_{i=0}^{k-1} A^{i-k} B B^T (A^T)^{i-N_p} \lambda_{N_p}. \quad (6.52)
\]

For \(w_{N_p} = 0\), Eq. (6.52) implies that, \(w^0 = P(A^T)^{-N_p} \lambda_{N_p}\) where \(P = \sum_{i=0}^{N_p-1} A^i B B^T (A^T)^i\), which can be expressed as \(\lambda_{N_p} = (A^T)^{N_p} P^{-1} x^0\), which in Eq. (6.51) for \(\lambda_{k+1}\) and this result in Eq. (6.49), yields \(v_{k+1}^* = B^T (A^T)^k P^{-1} w^0\) which after substitution in Eq. (6.40) results in Eq. (6.30). \(\square\)
6.4 Discrete-time controllability and observability functions

**Theorem 6.30 (Existence of $L_c$).** Assume that $v^*$ satisfies Eq. (6.39) with $L_c(w^0)$ smooth for all $x^0 \in D$ and such that Eq. (6.43) is asymptotically stable. Let $\|v^*_i\|^2 \leq M_i$, $M_i \in \mathbb{R}$ such that $\sum_{i=0}^{\infty} M_i$ converges uniformly and absolutely. Then $L_c(w^0)$ exists as given by (6.41) and is a smooth solution of (6.40) for all $w^0 \in D$.

**Proof of Theorem 6.30:** By Remark 6.25 existence of $L_c(x^0)$ is equivalent to existence of $L_c(w^0)$. By Lemma 6.26, Eq. (6.41) is a solution of (6.40). $L_c(w^0)$ exists if the series of functions (6.41) converges. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is a complete normed space, by Weierstrass’ M-Theorem, the series (6.41) converges uniformly and absolutely.

The latter results serve to prove the following proposition, which establishes the connection with the concepts of dissipativity theory of the previous section.

**Proposition 6.31.** Assume that the conditions of Theorem 6.30 are satisfied, then the controllability function $L_c(w^0)$ as defined in Eq. (6.39) is a Lyapunov function for system (6.10). Furthermore the system (6.10) is dissipative and $L_c(w_k)$ is also a storage function, with supply rate $\frac{1}{2}v^*_k T v^*_k$.

**Proof of Proposition 6.31:** That $L_c(w_k)$ is a Lyapunov function for (6.10), can be shown noticing its nonnegative definiteness from Eq. (6.39). By Proposition 6.27 $L_c(w^0) > 0$ for $w^0 \in D$. In order to show that the difference $L_c(w_{k+1}) - L_c(w_k)$ is negative semi-definite, note that for an arbitrary state $w_k$, from (6.42), $L_c$ can be expressed as

$$L_c(w_k) = \frac{1}{2} v^*_k T v^*_k + \frac{1}{2} \sum_{i=k}^{\infty} v^*_{i+1} T v^*_{i+1}, \quad (6.53)$$

doing the same for $w_{k+1}$, and taking the difference yields,

$$L_c(w_{k+1}) - L_c(w_k) = -\frac{1}{2} v^*_k T v^*_k, \quad (6.54)$$

which is negative semidefinite. Since the discrete-time dissipation inequality (6.15) is preserved and by Theorem 6.30 $L_c(w_k)$ is bounded, then $L_c(w_k)$ is a storage function with supply rate $\frac{1}{2}v^*_k T v^*_k$.

Some comments about the latter results are pertinent. A comparison of the assumptions of Proposition 6.31 with those of Theorem 6.9 reveal that asymptotic stability of system (6.3) is a stronger assumption than just dissipativity.
However the same differences can be observed in the continuous-time case, see [181, 205].

The existence condition of Theorem 6.30 is useful in order to provide solvability conditions for the dynamic optimization problem (6.26)-(6.27) restricted to the supply rate \( r_r = u_k^T u_k \), which finally results in the optimization problem of the following section.

### 6.4.3 Optimization-based search of \( v_k^* \)

Define the finite set \( \{ v_i | i = 0, ..., N_p \} \subset \{ v_i | i = 0, ..., \infty \} \) such that Eq. (6.39) is satisfied. Then by using a dynamic optimization approach [20], the optimization problem takes the form

\[
\min_{\{ v_i | i = 1 \ldots N_p \}} \frac{1}{2} \sum_{i=0}^{N_p} v_i^T v_i,
\]

with equality constraints \( w_{N_p} = 0 \) and \( w^0 = w_k \), which is expressed in the form of the Mayer problem (see e.g. [20]).

Assume that the conditions of Theorem 6.30 are satisfied. Let \( \epsilon, N_p \in \mathbb{Z}^+ \) be such that \( \| w_{N_p} \| \leq \epsilon \) for \( \epsilon \) small enough. Assume that \( N_p \) is known and the (closed) set of admissible inputs \( \{ v | v \in \mathcal{V} \} \) is convex with nonempty interior. Then \( L_c(w_k) \) in Eq. (6.40) can be determined with the solution of the following optimization problem

\[
\min_{\{ v_i | i = 1, \ldots, N_p \}} L_c(w_{N_p}),
\]

with equality constraints

\[
\begin{align*}
   w_{i+1} &= F^{-1}(w_i, v_{i+1}), \\
   L_c(w_{i+1}) &= L_c(w_i) + \frac{1}{2} v_i^T v_i, \\
   w_{N_p} &= 0, \\
   w^0 &= w_k,
\end{align*}
\]

with initial inputs \( \{ v^0 | v_j^0 \in \mathcal{V}, j = 1, \ldots, N_p \} \) and with \( L_c(w^0) = 0 \), determining \( v^*_i \).

A drawback of this approach can be pointed out. Though for an asymptotically stable system \( N_p \) can be approximated to be finite, introducing some error in the result, the best value of \( N_p \) is unknown prior to the nonlinear optimization process.
6.4.4 Example: the storage functions from a continuous nonlinear system

In order to illustrate some advantages and limitations of the approach previously presented, we find the energy functions associated to a simple model of a series interconnected motor (also known as universal motor). When departing from a discrete-time system, invertibility of the discrete-time map $F(\cdot, u_k)$ is an assumption required for reversed-time evolution. A simple discrete-time example can be found in [121]. In this example we depart from a continuous-time system and therefore the map $F(\cdot, u_k)$ is a diffeomorphism [40]. In consequence, the controllability function can be found using Remark 6.4 circumventing the need of explicitly inverting such map. Consider the universal motor depicted in Figure 6.2 with specifications defined in Table 6.1.

![Universal motor](image)

**Fig. 6.2.** Universal motor

The dynamic behavior of this system may be described by

\[
\begin{align*}
L \frac{dI}{dt} &= V_t - RI - \zeta \omega I, \\
J \frac{d\omega}{dt} &= \zeta I^2 - B\omega,
\end{align*}
\]

with state defined by $x = (I, \omega)$, $x \in X$ with input $V_t$ and outputs $I$ and $\omega$. Considering that $K_m = K_t$, define $\zeta = K_t K_f$. Although this system is *locally accessible* it is not controllable, since $\dot{\theta} = \omega \geq 0 \ \forall \ x \in X$. The corresponding controllability function has a region where $L_c = 0$ in one half of the state space with a non-smooth transition to the other half. While such difficulties are essentially unimportant for the optimization approach, such energy function cannot be approximated adequately by analytic functions limiting the computer representation and manipulation of such functions. For instance for
Energy functions for dissipativity-based balancing of discrete-time nonlinear systems

Table 6.1. Specifications of the universal motor

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance</td>
<td>2.5</td>
<td>Ω</td>
</tr>
<tr>
<td>Inductance (field + armature)</td>
<td>0.08</td>
<td>H</td>
</tr>
<tr>
<td>Rotational Inertia</td>
<td>10</td>
<td>Kg·m^2·s/rad^2</td>
</tr>
<tr>
<td>Angular position</td>
<td>θ ∈ ℝ</td>
<td>rad</td>
</tr>
<tr>
<td>Angular velocity</td>
<td>Ω(θ, ω) ∈ Y</td>
<td>rad/s</td>
</tr>
<tr>
<td>Current</td>
<td>I(θ, ω) ∈ Y</td>
<td>A</td>
</tr>
<tr>
<td>Voltage at terminals</td>
<td>V_k</td>
<td>V</td>
</tr>
<tr>
<td>Rotational damping</td>
<td>0.50</td>
<td>N·m·s/rad</td>
</tr>
<tr>
<td>Constant (torque)</td>
<td>0.42</td>
<td>N·m/Wb·A</td>
</tr>
<tr>
<td>Constant (MMF)</td>
<td>0.42</td>
<td>V·s/Wb·rad</td>
</tr>
<tr>
<td>Constant (field)</td>
<td>0.53</td>
<td>Wb/A</td>
</tr>
</tbody>
</table>

a polynomial fit the approximation of such energy functions may be unsuccessful. For convenience define $a = \frac{-R}{L}$, $b = \frac{-\zeta}{L}$, $c = \frac{\zeta}{J}$, $d = \frac{-B}{J}$ and $e = \frac{1}{L}$. By the Taylor-Lie series, a discrete-time approximation of system (6.57) is given by

\[
\begin{bmatrix}
  I_{k+1} \\
  \omega_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  I_k \\
  \omega_k
\end{bmatrix} + T \begin{bmatrix}
  aI + b\omega + eV_t \\
  cI^2 + d\omega
\end{bmatrix}_k + \frac{T^2}{2} \begin{bmatrix}
  a + b\omega bI \\
  2cI \\
  d
\end{bmatrix}_k \begin{bmatrix}
  aI + b\omega + eV_t \\
  cI^2 + d\omega
\end{bmatrix}_k + \ldots
\] (6.58)

with outputs $I_k$ and $w_k$. Following the sequence of steps described in Definition 6.2, the associated system of (6.57) is in this case,

\[
\begin{align*}
L \frac{dI}{dt} &= RI + \zeta \omega I - V_t, \\
J \frac{d\omega}{dt} &= B\omega - \zeta I^2,
\end{align*}
\] (6.59)

and due to the commutativity in the diagram of Figure 6.1 the associated discrete-time system for a state $w_k = (I_k, \omega_k)$ is given by

\[
\begin{bmatrix}
  \bar{I}_{k+1} \\
  \bar{\omega}_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  \bar{I}_k \\
  \bar{\omega}_k
\end{bmatrix} + T \begin{bmatrix}
  -a\bar{I} - b\bar{I}\bar{\omega} - e\bar{V}_t \\
  -c\bar{I}^2 - d\bar{\omega}
\end{bmatrix}_k + \frac{T^2}{2} \begin{bmatrix}
  -a - b\bar{\omega} - b\bar{I} \\
  -2c\bar{I} \\
  -d
\end{bmatrix}_k \begin{bmatrix}
  -a\bar{I} - b\bar{I}\bar{\omega} - e\bar{V}_t \\
  -c\bar{I}^2 - d\bar{\omega}
\end{bmatrix}_k + \ldots
\] (6.60)

where the input is $\nu_k = \bar{V}_t(k+1)$ and the outputs are $\bar{I}_k$ and $\bar{\omega}_k$. In order to determine $L_c$ and $L_o$ the values of the parameters shown in Table 6.1 were
used. With the purpose of preserving the continuity of the concepts discussed so far, the numerical aspects of analytic approximation of the energy functions are presented in the Appendix. All the routines and graphics of this example were performed using Matlab.

**Observability function:** Consider the iterative solution of Eq.(6.32), as $i \to \infty$ for each initial state $x_0$ within the desired region to plot. The resulting observability function is given in Figure 6.3a. It can be seen from the graphic in Figure 6.3a that $L_o$ is more influenced by $I$ than by $\omega$.

**Controllability function:** By using the optimization approach of Subsection 6.4.3 and defining a finite set $\{v_i|i=1...N_p\}$, the optimization problem stated in Eq.(6.55)-(6.56) can be solved for each selected state $w$ of (6.60) and thus the results can be plotted resulting in Figure 6.3b. The region where $L_c = 0$ in one half of the state space which corresponds to the unreachable part where $\dot{\theta} = \omega \leq 0$ for $x \in X$ can be seen from the figure. Also it can be discerned that several points did not converge to the region where $L_c = 0$. The Optimization Toolbox of Matlab was used to find $v^*$ using an spiraled grid with the purpose of simplifying the optimization task until an adequate radius is obtained.

### 6.5 Conclusions

Dissipativity theory is a general framework useful to establish input-state-output relationships and with this, to pose several storage functions used in nonlinear balancing. Using properly-defined dynamic optimization problems, along with adequate nonlinear discretization algorithms− including those based on Taylor-Lie series [94] or numerical integration algorithms−, it is possible to provide a framework to find approximations of such storage functions. In particular, the discrete-time versions of the controllability and observability energy functions were discussed. Instead of looking for the solution of a Hamilton-Jacobi-Isaacs and a Lyapunov-like partial differential equations as in the continuous-time case, an optimization approach and an iterative algorithm were proposed to find $L_c$ and $L_o$ respectively. This approach was exemplified with linear and nonlinear discrete-time systems. In particular using this approach on the discrete-time equivalent model of a universal motor the approximated energy functions were found.

Topics of ongoing research are the further development of the nonlinear approach of dissipative balancing. Of particular interest for us is the case of port-Hamiltonian systems since the interaction of ports with the state can be naturally posed in this framework, establishing relations with the physical energy stored in the system [127].
6.6 Appendix

Proofs of Theorems 6.7 and 6.9
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**Proof of Theorem 6.7:** Suppose that $S_a(x^0, r_a)$ is finite. That $S_a(x^0, r_a) \geq 0$ can be verified by taking $N_f = 0$ in (6.17). Consider the value of $S_a(x^0, r_a)$ at two points $x_{k+m}$ and $x_k$ located at the trajectory defined by the optimal sequence of inputs $\{u_i^*| i = 0, 1, \ldots, N_f - 1\}$ that satisfy (6.18), then the difference can be expressed as $S_a(x_{k+m}, r_a) - S_a(x_k, r_a) = \sum_{i=k}^{k+m-1} r_a(y_i, u_i^*)$. In any other suboptimal trajectory we have $\sum_{i=k}^{k+m-1} r_a(y_i, u_i) \geq \sum_{i=k}^{k+m-1} r_a(y_i, u_i^*)$, resulting in $S_a(x_{k+m}, r_a) - S_a(x_k, r_a) \leq \sum_{i=k}^{k+m-1} r_a(y_i, u_i)$ satisfying (6.16).

Assume now that (6.3)-(6.4) is dissipative. Then there exist some $S$ that for any $u_k$ satisfies (6.16). Since reachability implies the possibility of steering $x_{k+m}$ located at the trajectory defined by the optimal sequence of inputs $\{u_i^*| i = 0, 1, \ldots, N_f - 1\}$ that satisfy (6.22) and departs from $x_{-N_p} = x^*_{-N_p}$ towards $x^0$. The difference $S_r(x_{k+m}, r_r) - S_r(x_k, r_r)$ is given by

\[
S_r(x_{k+m}, r_r) - S_r(x_k, r_r) = \left\{ \sum_{i=-N}^{k+m} r_r(y_i, u_i^*) \right\} - \sum_{i=-N}^{k} r_r(y_i, u_i^*)
\]

\[
= \sum_{i=-N}^{k} r_r(y_i, u_i^*) + \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*) - \sum_{i=-N}^{k} r_r(y_i, u_i^*)
\]

\[
= \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*),
\]

while for any other suboptimal trajectory

\[
\sum_{i=k+1}^{k+m} r_r(y_i, u_i) \geq \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*),
\]

and therefore $S_r(x_{k+m}, r_r) - S_r(x_k, r_r) \leq \sum_{i=k+1}^{k+m} r_r(y_i, u_i)$ satisfying (6.16). At the point $x^*$ the following relation holds $S_a(x^*, r_a) = \sup_x S_r(x, r_r)$, [205]. Since reachability implies the possibility of steering $x^*$ to $x$ in finite time, in order to have $S_a(x^*, r_a)$ finite, there must exist a bound $M$ for $S_r(x, r_r)$ such that $-\infty < M \leq S_r(x, r_r)$, concluding the proof. 

**Proof of Theorem 6.9:** Consider the value of $S_r(x^0, r_r)$ at two points $x_k$ and $x_{k+m}$ located at the trajectory defined by the optimal sequence of inputs $\{u_i^*| i = -N_p, \ldots, -1, 0\}$ that satisfy (6.22) and departs from $x_{-N_p} = x^*_{-N_p}$ towards $x^0$. The difference $S_r(x_{k+m}, r_r) - S_r(x_k, r_r)$ is given by

\[
S_r(x_{k+m}, r_r) - S_r(x_k, r_r) = \left\{ \sum_{i=-N}^{k+m} r_r(y_i, u_i^*) \right\} - \sum_{i=-N}^{k} r_r(y_i, u_i^*)
\]

\[
= \sum_{i=-N}^{k} r_r(y_i, u_i^*) + \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*) - \sum_{i=-N}^{k} r_r(y_i, u_i^*)
\]

\[
= \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*),
\]

while for any other suboptimal trajectory

\[
\sum_{i=k+1}^{k+m} r_r(y_i, u_i) \geq \sum_{i=k+1}^{k+m} r_r(y_i, u_i^*),
\]

and therefore $S_r(x_{k+m}, r_r) - S_r(x_k, r_r) \leq \sum_{i=k+1}^{k+m} r_r(y_i, u_i)$ satisfying (6.16). At the point $x^*$ the following relation holds $S_a(x^*, r_a) = \sup_x S_r(x, r_r)$, [205]. Since reachability implies the possibility of steering $x^*$ to $x$ in finite time, in order to have $S_a(x^*, r_a)$ finite, there must exist a bound $M$ for $S_r(x, r_r)$ such that $-\infty < M \leq S_r(x, r_r)$, concluding the proof.
About the structure of \( v_k^* \)

In this subsection, some difficulties of the optimal control discussed in Proposition 6.28 are reviewed. In order to study the structure of \( v_k^* \) in (6.44)-(6.45), the corresponding boundary value problem is addressed.

Some definitions are useful to simplify notation. A successive composition of the corresponding boundary value problem is addressed.

Given an initial \( \lambda \) transitions 6.28 are reviewed. In order to study the structure of \( v \), (6.63)

\[
F \equiv F_m \circ F_{m+1} \circ \ldots \circ F_n, \quad \text{with} \quad F_{[n,n]} \equiv F_n.
\]

With a slight abuse of notation, \( F_i^{-1} \equiv F^{-1}(x_i, v_{i+1}) \).

Define the following maps,

\[
\Psi_k = \frac{\partial}{\partial w_k} F^{-1}(w_k, v_{k+1}), \quad (6.61)
\]

\[
\Upsilon_k = -\frac{\partial}{\partial v_{k+1}} F^{-1}(w_k, v_{k+1}), \quad (6.62)
\]

and denote by \( \Psi_{[m,n]} \equiv \Psi_m \Psi_{m+1} \ldots \Psi_n \) the successive application of a step variant linear map \( \Psi_k \) for a discrete interval \( k \in [m, n] \). Then the solution of (6.44), given an initial \( \lambda_{N_p} \), with \( 0 \leq k \leq N_p \) can be expressed as,

\[
\lambda_k = \Psi^T_{[k,N_p-1]} \lambda_{N_p},
\]

and in consequence the possibly implicit input \( v_{k+1} \) can be obtained from the following expression,

\[
v_{k+1} = \Upsilon^T_k \Psi^T_{[k+1,N_p-1]} \lambda_{N_p}. \quad (6.63)
\]

Consider the following composition operations for the map \( F_{[i,N_p]} \equiv F_{i+1} \circ F_i \circ \ldots \circ F_n \), and for the inverse map \( F_{[i,0]}^{-1} \equiv F_{1}^{-1} \circ F_{i-1}^{-1} \circ \ldots \circ F_0^{-1} \).

Then both Eq. (6.10) and the system \( w_{\kappa} = F(w_{\kappa+1}, v_{\kappa+1}) \), \( \kappa \in \mathbb{Z}^- \), which evolves in backward-time, can be expressed in terms of equation (6.63) as,

\[
w_{\kappa+1} = F^{-1}(w_{\kappa}, \Upsilon^T_{\kappa} \Psi^T_{[\kappa+1,N_p-1]} \lambda_{N_p}), \quad \text{and} \quad w_{\kappa} = F(w_{\kappa+1}, \Upsilon^T_{\kappa} \Psi^T_{[\kappa+1,N_p-1]} \lambda_{N_p}).
\]

At the boundary for \( \kappa = 0 \), \( w(0) = w_0 \), \( w_0 = F(w_1, \Upsilon^T_{0} \Psi^T_{[1,N_p-1]} \lambda_{N_p}) = F_{[0,N_p]} \), and for \( \kappa = N_p \), \( w_{N_p} = 0 \),

\[
0 = F^{-1}(w_{N_p-1}, \Upsilon^T_{N_p-1} \Phi^T_{N_p-1} \lambda_{N_p}) = F_{[N_p,0]}^{-1}, \quad (6.64)
\]

and its inverse map is \( w_{N_p-1} = F(0, \Upsilon^T_{N_p-1} \Phi^T_{N_p-1} \lambda_{N_p}) = F_{[N_p,N_p]} \). In this last equation, we have a nonlinear relation between \( w_{N_p-1} \) and \( \lambda_{N_p} \). Notice also that \( \Upsilon_{N_p-1} = \Upsilon(w_{N_p-1}, v_{N_p}) \), with \( v_{N_p} \) inserted possibly in implicit form. In the linear case this never occurs and it is solvable, see Example 6.29. In the general case this problem is difficult to solve in closed form. However, as shown throughout the paper, dynamic optimization algorithms can be used in order
to solve it.

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Chapter 7

Lumped approximation of a transmission line with an alternative geometric discretization


Abstract: In this paper, an electromagnetic one-dimensional transmission line represented in a distributed port-Hamiltonian form is lumped into a chain of subsystems which preserve the port-Hamiltonian structure with inputs and outputs in collocated form. The procedure is essentially an adaptation of the procedure for discretization of Stokes-Dirac structures presented in [26], that does not preserve the port-Hamiltonian structure after discretization. With some modifications essentially inspired on the finite difference paradigm, the procedure now results in a system that preserves the collocated port-Hamiltonian structure along with some other desirable conditions for interconnection. The simulation results are compared with those presented previously in [50].

Keywords: Distributed-parameter systems, partial differential equations, electrical circuits, model approximation, models.

7.1 Introduction

The symbiosis of systems and control theory and classical mechanics has resulted in a fruitful field of research that has provided highly structured and systematic tools for a diversified class of physical systems. In par-
ticular, the class of port-Hamiltonian systems has interesting properties useful for modeling and control purposes. Since the introduction of distributed port-Hamiltonian systems in [133] and lately in [203], several important applications could be envisioned with the use of the so called Stokes-Dirac structures based on the success of the applications that have resulted from its finite dimensional counterpart, the Dirac structures.

At the heart of the interconnection of such infinite-dimensional models of conservation laws is precisely the Stokes-Dirac structure. For an applications point of view the adequate spatial discretization of this structure is fundamental for the preservation of such conservation and interconnection relations. Therefore, some efforts have been devoted to the systematic discretization of such structure [51, 26]. In order to deal with arbitrary dimensions a common framework using differential geometry concepts is used. Despite having such common ground, each procedure approaches the problem of approximation of the geometric objects in a different form.

On the one hand, in [51] their geometric procedure is based, roughly speaking, on the approximation of the differentiable n-forms into finite element objects. For instance 0-forms and 1-forms as linear splines, etc. With such tools at hand the Stokes-Dirac structure of the port-Hamiltonian model of a one-dimensional transmission line is discretized, simulated and compared with the exact solution of the model. This approach allows them to deal with uniform and non-uniform grids (for further details see [51]).

On the other hand, in our approach [26], such differentiable n-forms lie in a spatial domain which has associated a grid of points or nodes from where a finite-difference provides a way to approximately express each n-form. In this way a 0-form (a function) lies at the node, a 1-form is defined to lie on the midpoints between two collinear nodes, 2-forms lie at the center of the square, etc. The procedure was tested on the non-homentropic model of a one-dimensional pipeline, but no further analysis on the precision of the discretization was presented. In this paper the precision of the methods is tested for an improved version of the discretization presented in [26], which preserves the port-Hamiltonian structure, for the (simpler) model of the one-dimensional transmission line. The availability of an exact solution for a state variable at the boundary in [51] provides us a trustable benchmark on the precision of the method.

The paper is organized as follows. After briefly presenting in Section 7.2 some necessary adaptations of the procedure shown in [26] in the particular case of the Stokes-Dirac structure of the model of the one-dimensional transmission line, in Section 7.3 we present some structures in the collocated port-
7.2 The Stokes-Dirac modified discretization

Consider a 1-dimensional Riemannian manifold $\mathcal{M}$ with metric $g$ and 0–dimensional boundary $\partial\mathcal{M}$, consider two spaces $\mathcal{F}$ and $\mathcal{E}$ defined by

$$
\mathcal{F} = \mathcal{F}_q \times \mathcal{F}_\varphi \times \mathcal{F}_b = \bigwedge^1(\mathcal{M}) \times \bigwedge^1(\mathcal{M}) \times \bigwedge^0(\partial\mathcal{M}),
$$

and

$$
\mathcal{E} = \mathcal{E}_q \times \mathcal{E}_\varphi \times \mathcal{E}_b = \bigwedge^0(\mathcal{M}) \times \bigwedge^0(\mathcal{M}) \times \bigwedge^0(\partial\mathcal{M}),
$$

where the subindex $b$ stands for boundary variables. Throughout the section we denote

$$
\begin{align*}
\mathcal{F} & \ni \begin{cases} 
  f^i = (f^i_q, f^i_\varphi, f^i_b) \\
  e^i = (e^i_q, e^i_\varphi, e^i_b)
\end{cases} \\
\mathcal{E} & \ni \begin{cases} 
  f^i = (f^i_q, f^i_\varphi, f^i_b) \\
  e^i = (e^i_q, e^i_\varphi, e^i_b)
\end{cases}
\end{align*}
\tag{7.1}
$$

along with a product $\langle \langle (f^1, e^1), (f^2, e^2) \rangle \rangle$ in the space $\mathcal{F} \times \mathcal{E}$ defined as

$$
\int_{\mathcal{M}} \left\{ e^1_q \wedge f^2_q + e^1_\varphi \wedge f^2_\varphi + e^2_q \wedge f^1_q + e^2_\varphi \wedge f^1_\varphi \right\} + \int_{\partial\mathcal{M}} (e^1_b \wedge f^2_b + e^2_b \wedge f^1_b).
\tag{7.1}
$$

As described by [203] the subspace $D$ associated to the one-dimensional transmission line defined by

$$
\begin{bmatrix}
  f_q \\
  f_\varphi \\
  e_b \\
  f_b
\end{bmatrix} =
\begin{bmatrix}
  0 & -d & 0 & 0 \\
  -d & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  e_q \\
  e_\varphi \\
  e_q|_{\partial\mathcal{M}} \\
  e_\varphi|_{\partial\mathcal{M}}
\end{bmatrix},
\tag{7.2}
$$

defines a Stokes-Dirac structure on $\mathcal{F} \times \mathcal{E}$, where $q$ and $\varphi$ stand for charge and flux densities (one-forms) respectively.

As shown in [203], the distributed port-Hamiltonian representation of the one-dimensional transmission line is given by
\[
\begin{bmatrix}
\partial_t q \\
\partial_t \varphi \\
\varphi \\
q
\end{bmatrix} =
\begin{bmatrix}
0 & -d & 0 & 0 \\
-d & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_q H \\
\delta_\varphi H \\
\delta_q H|_{\partial Z} \\
\delta_\varphi H|_{\partial Z}
\end{bmatrix}
\] (7.3)

with energy stored in the Hamiltonian expressed by the functional

\[
\mathcal{H} = \int_\Omega \frac{1}{2} \left\{ \frac{q^2}{C} + \frac{\varphi^2}{L} \right\} d\Omega,
\] (7.4)

where \( \Omega \) denotes the boundary. Based on the procedure of discretization of Stokes-Dirac structures presented in [26], it is necessary first to identify the geometric objects involved in the particular structure used to model the transmission line, eq. (7.2), namely the one-forms and the exterior differential operator \( d \).

The Hamiltonian function \( H \) is defined by a sequence of nodes in a uniform grid. Given a Hamiltonian functional (7.4) and a finite grid defining \( n \)–finite sections, a finite dimensional Hamiltonian

\[
H_i = \int_{\mathcal{V}_i} \frac{1}{2} \left\{ \frac{q^2}{C} + \frac{\varphi^2}{L} \right\} d\mathcal{V}_i,
\] (7.5)

can be obtained for each \( i \)–volume \( \mathcal{V}_i \), resulting in a total Hamiltonian

\[
H = \sum_i^n H_i.
\] (7.6)

Since the integration takes place assuming that the energy is uniformly distributed in each grid-volume, a lumped Hamiltonian can be defined for each volume. For instance, for the one-dimensional case define the following lumped Hamiltonian 'per unit length',

\[
H = \frac{1}{\Delta(x_i)} H(X),
\] (7.7)

where the length \( \Delta(x_i) \) depends on the different block element used. Since both \( q \) and \( \varphi \) are one-forms, such forms can be placed at the middle of two adjacent nodes. Finally the approximation of the partial derivative \( \partial_x H \) can be performed in several forms namely backward \( \partial_x H \approx (H_{i+1} - H_i)/\Delta x \), forward \( \partial_x H \approx (H_{i+1} - H_i)/\Delta x \) or central differentiation \( \partial_x H \approx (H_{i+1} - H_{i-1})/2\Delta x \). Different selections of approximated differentiations, result in different structures which may fail to be skew symmetric.
7.3 Lumping the transmission line

The structure of the blocks is defined with port-Hamiltonian systems in a particular structure called **collocated port-Hamiltonian systems** [127] (see Figure 7.2) the essential characteristic of this arrangement is that the inputs and outputs are paired at each boundary such that they determine the energy transfer, i.e., at each boundary there exists a transfer of energy without a predefined assumption on the direction of transfer. For a given set of power defining variables at the boundary, some variables are defined by the system at one two-port and the complementary variables are defined at the opposite two-port (for further details see [127]). This does not modify at all the structure of the Port-Hamiltonian paradigm. The difference remains in the way of looking at interconnection of these structures: they can be chained in 1-D problems, grided in 2-D and assembled into polyhedral structures for 3-D cases. For more general structures including dissipation and feed forward terms see [124]). In particular, assume that $J(x) = -J^T(x)$. The collocated representation can be written as

---

**Fig. 7.1.** A schematic grid of nodes interconnection for the transmission line.

**Fig. 7.2.** A PHS with collocated inputs and outputs
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\[
\begin{bmatrix}
    \dot{x}_1 \\
    -y_1 \\
    y_2
\end{bmatrix}
= \begin{bmatrix}
    J_1^1 & J_1^2 & J_1^3 \\
    J_2^1 & 0 & 0 \\
    J_3^1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial H_1}{\partial x_1} \\
    u_1 \\
    u_2
\end{bmatrix}
\] (7.8)

With the purpose of series interconnection consider a second system with \( G(x) = -G^T(x) \) in the form

\[
\begin{bmatrix}
    \dot{x}_2 \\
    -z_1 \\
    z_2
\end{bmatrix}
= \begin{bmatrix}
    G_1^1 & G_1^2 & G_1^3 \\
    G_2^1 & 0 & 0 \\
    G_3^1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial H_2}{\partial x_2} \\
    v_1 \\
    v_2
\end{bmatrix}
\] (7.9)

Assuming that the interconnection is compatible, \( i.e. \ y_2 = v_1 \) and \( u_2 = z_1 \), then, as can be seen after some simple algebra, the interconnected system has the form :

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    -y_1 \\
    z_2
\end{bmatrix}
= \begin{bmatrix}
    J_1^1 & -J_1^3 & G_2^1 & J_1^2 & 0 \\
    G_1^2 & J_3^1 & G_1^1 & 0 & G_1^3 \\
    J_2^1 & 0 & 0 & 0 & 0 \\
    0 & G_3^1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial H_1}{\partial x_1} \\
    \frac{\partial H_2}{\partial x_2} \\
    u_1 \\
    v_2
\end{bmatrix}
\]

and it can be verified that the resulting structure is again skew symmetric.

\[\text{Fig. 7.3. A single block in A form.}\]

**Proposition 7.1.** The following system (block) associated to the lumped Hamiltonian in (7.7),

\[
(A) \begin{cases}
    \dot{q}_i \\
    \dot{\varphi}_i \\
    -\partial_\varphi H_i \\
    \partial_q H_i
\end{cases}
= \begin{bmatrix}
    0 & -1 & 0 & -1 \\
    1 & 0 & 1 & 0 \\
    0 & -1 & 0 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \partial_q H_i \\
    \partial_\varphi H_i \\
    -\partial_q H_{i+1} \\
    -\partial_\varphi H_{i-1}
\end{bmatrix}
\]
provides a space-discretization of Telegrapher’s equations in distributed port-
Hamiltonian form and is a finite-dimensional port-Hamiltonian system with
collocated inputs and outputs. Furthermore, the PHS (A) satisfies the energy
conservation equation

\[
\frac{dH_i}{dt} = y_i^T u_i \tag{7.10}
\]

and for a chain of \( n \)-blocks PHS, the total energy satisfies the energy balance

\[
\frac{dH}{dt} = \sum_{i=1}^{n} \frac{\partial H_i}{\partial t} = y^T u \tag{7.11}
\]

Proof. Consider the following discrete approximation

\[
d(\delta_\varphi H) \approx \Delta^{-1}(\delta_\varphi H_{i+1} - \delta_\varphi H_i) \quad \text{and} \quad d(\delta_q H) \approx \Delta^{-1}(\delta_q H_i - \delta_q H_{i-1})
\]

in the structure (7.3). The first statement can be proved straightforwardly. The second statement is proved as
follows: Consider the Hamiltonian function

\[
H_i = \frac{1}{2} \left( \frac{q_i^2}{C_i} + \varphi_i^2/L_i \right)
\]

and the product

\[
\frac{dH_i}{dt} = \partial_q H_i \partial_\varphi H_i - \partial_\varphi H_i \partial_q H_i = y_i^T u_i
\]

Since \( \partial_q H_i = V_i \) and \( \partial_\varphi H_i = I_i \), the product defines power which after inte-
gration reproduces the required supplied energy or delivered energy.

In order to prove the last statement consider the following: By induction, for
\( i = 1 \) it was already proved in the previous proposition. Consider the inter-
connection of two systems. In such case

\[
\frac{dH}{dt} = \frac{dH_1}{dt} + \frac{dH_2}{dt} = \partial_q H_1 \partial_\varphi H_0 - \partial_\varphi H_1 \partial_q H_2 + \partial_q H_1 \partial_\varphi H_0 - \partial_\varphi H_1 \partial_q H_2
\]

which reduces trivially to \( y^T u \). Assume it is true for \( i = 1 \cdots n \) such that at
\( i = m \) the total energy is

\[
\frac{dH}{dt} = \partial_q H_i \partial_\varphi H_{i-1} - \partial_\varphi H_i \partial_q H_{i+1}
\]

Since there is a cancelation of any individual intermediate product, it can be
seen that the only residual terms are those at the boundary, resulting in the
previous equation, which concludes the proof.
Consider the following alternative discrete-time approximation \( d(\delta \varphi H) \approx \Delta^{-1}(\delta \varphi H_i - \delta \varphi H_{i-1}) \) and \( d(\delta q H) \approx \Delta^{-1}(\delta q H_{i+1} - \delta q H_i) \) in the structure (7.3). Then an alternative form follows
\[
(B) \begin{cases}
\dot{q}_i \\
\dot{\varphi}_i \\
-\partial \varphi H_i \\
\partial q H_i
\end{cases} = \begin{bmatrix}
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \begin{cases}
\partial q H_i \\
\partial \varphi H_i \\
\partial q H_{i-1} \\
\partial \varphi H_{i+1}
\end{cases}
\]
and represented in collocated form in Figure 7.4. Since both lumped blocks A
\[\text{Fig. 7.4. A block of distributed approximation of DPHS in B form}\]
and B are based on a first-order discretization algorithms, one may conceive the use of a higher-order approximation in order to increase the precision or the resulting models. Consider finally the following discrete approximation \( d(\delta \varphi H) \approx (2\Delta)^{-1}(\delta \varphi H_{i+1} - \delta \varphi H_{i-1}) \) and \( d(\delta q H) \approx (2\Delta)^{-1}(\delta q H_{i+1} - \delta q H_{i-1}) \) in the structure (7.3). For a Hamiltonian defined as \( H_i = H_i/2\Delta_i \) one such block is presented as follows
\[
(C) \begin{cases}
\dot{q}_i \\
\dot{\varphi}_i \\
-\partial \varphi H_i \\
\partial q H_i
\end{cases} = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{cases}
\partial q H_i \\
\partial \varphi H_i \\
\partial q H_{i+1} \\
\partial \varphi H_{i+1} \\
\partial q H_{i-1} \\
\partial \varphi H_{i-1}
\end{cases}
\]
which is based on a central difference approximation and provides a second order error in approximating the derivative. Higher-difference approximations could be probably conceived, but in the end there does not seem to be of much benefit.
7.4 Simulation results and comparisons

The efficiency of the method is tested in the problem posed and described in [51] which consists of an ideal transmission line terminated by a lumped resistor, whose parameters $C(x)$ and $L(x)$ are varying along its position $x$ in the transmission line, in the interval $[0, \ell]$. Assuming initial conditions zero, an input voltage source $u = \sin(t)$ in this form results in a voltage distribution in the form $v(x, t) = \sin(t - ln(x + 1))$ which results in a voltage at the terminal of the resistor in the waveform $v(\ell, t) = \sin(t - 1)$. This last waveform is to be simulated in order to assert the precision of the lumping method. In their test [51], a fixed and a variable grid are used with the Runge-Kutta-4 integration technique and step size of 0.01 s. For our method two adequate lumping methods are available, the method A presented in the previous section, and the method C based on central differences. The latter method of discretization C has an error of second-order and therefore it could be expected to perform possibly better than A which is constructed from a combination of backward and forward differences. Figure 7.5 shows the different waveform obtained by the methods. While it can be seen that actually method A performs better than C, both results are still far from the exact solution as can be seen on the plots of their error, Figure 7.6. Even though method C was used for $n = 6$, (since its response is only comparably good for an even numbers of elements), the error of method C is still higher than that obtained by the A-method.

As can be seen in Table 7.2, three parameters were used in order to assert on the precision and performance of the method. Since the number of port-Hamiltonian blocks is typically associated to the precision, after considering the number of elements $n$, two parameters of error were considered: the peak error and the steady-state (or nominal) error. While the first one appears typically at the simulations around $t = 1$, the second parameter was considered as

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total length</td>
<td>$\ell$</td>
<td>$e - 1$</td>
</tr>
<tr>
<td>Spatial position</td>
<td>$x$</td>
<td>$x \in [0, \ell]$</td>
</tr>
<tr>
<td>Specific Capacitance</td>
<td>$C$</td>
<td>$1/(x + 1)$</td>
</tr>
<tr>
<td>Specific Inductance</td>
<td>$L$</td>
<td>$1/(x + 1)$</td>
</tr>
<tr>
<td>Lumped Resistance</td>
<td>$R$</td>
<td>1</td>
</tr>
<tr>
<td>Number of elements</td>
<td>$n$</td>
<td>5</td>
</tr>
<tr>
<td>Fixed Grid interval</td>
<td>$\Delta$</td>
<td>$\ell/n$</td>
</tr>
<tr>
<td>Variation of Grid interval</td>
<td>$\Delta_i$</td>
<td>$e^{\Delta} - e^{-\Delta}$</td>
</tr>
</tbody>
</table>

Table 7.1. Specifications of the transmission line
the extreme value (higher or lower) of the absolute error during the rest of the simulation (say, after \( t = 2 \)). Notice that in the curves plotted in our example, fig. 7.5, the approximation of the time delay is performed smoothly, while in the figures presented in [51] the error swings with a peak error relatively larger than the nominal error. Nevertheless in all their simulations its peak error is still smaller than the peak error of ours for the same number of elements. A slight increase of the order of our methods (\( n = 7 \) for A and \( n = 10 \) for C) seems to be helpful in order to obtain the same peak error. That is not the case for their remarkable nominal error at experiment 1, \( n = 5 \). In order to reach the same nominal error, we increase the number of our blocks. The resulting simulation required the order of \( n = 113 \) for method A and \( n = 166 \) for method C. Finally while trying to attain their nominal error of Experiment 1, \( n = 10 \), a chain of up to 500 elements were needed in our simulations.

\[ \text{Voltage at the end of transmission line (constant } \Delta \text{)} \]

**Fig. 7.5.** Output voltage.
7.5 Concluding remarks

In this paper, several improvements of the procedure of discretization of Stokes-Dirac structures presented in [26] were shown to be necessary in order to preserve the port-Hamiltonian structure of the system. Additional modifications were necessary in order to ensure the integrability by quadratures of the resulting system. The Stokes-Dirac approximation procedure, which is mainly based on a finite difference philosophy, was shown to be useful for the dynamic simulation of a one-dimensional electromagnetic transmission line and comparisons with another approach [51], mainly based on a finite element philosophy, were provided. The use of finite differences provides quite direct and simple structures, which are prone for further analysis with control purposes. But simplicity seems to bear the price of precision. The simulation results presented in this note show that in all cases they provide a higher peak and nominal errors compared with the lumping procedures presented in [51]. Despite having a higher total error in contrast to the charts presented in [51], in our results the transient peak error at \( t = 1 \) tends to be almost comparable.

Fig. 7.6. Absolute error.
Table 7.2. Comparison of simulation results

<table>
<thead>
<tr>
<th>Method</th>
<th>n</th>
<th>Peak error</th>
<th>Nominal error</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>0.07291</td>
<td>0.0673</td>
</tr>
<tr>
<td>A</td>
<td>7</td>
<td>0.05554</td>
<td>0.04943</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>0.04169</td>
<td>0.03534</td>
</tr>
<tr>
<td>A</td>
<td>113</td>
<td>0.00666</td>
<td>0.0033</td>
</tr>
<tr>
<td>A</td>
<td>500</td>
<td>0.002348</td>
<td>0.0008</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>0.585</td>
<td>0.585</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>0.0784</td>
<td>0.0772</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>0.0535</td>
<td>0.0525</td>
</tr>
<tr>
<td>C</td>
<td>114</td>
<td>0.00932</td>
<td>0.00474</td>
</tr>
<tr>
<td>C</td>
<td>166</td>
<td>0.00721</td>
<td>0.0033</td>
</tr>
<tr>
<td>Exp. 1*</td>
<td>5</td>
<td>-0.055, 0.051</td>
<td>0.0033</td>
</tr>
<tr>
<td>Exp. 2*</td>
<td>5</td>
<td>-0.055, 0.051</td>
<td>0.004138</td>
</tr>
<tr>
<td>Exp. 3*</td>
<td>5</td>
<td>-0.055, 0.051</td>
<td>0.00662</td>
</tr>
<tr>
<td>Exp. 4*</td>
<td>5</td>
<td>-0.06, 0.055</td>
<td>0.00828</td>
</tr>
<tr>
<td>Exp. 1*</td>
<td>10</td>
<td>?</td>
<td>0.00084</td>
</tr>
</tbody>
</table>

* From simulation experiments in [51].

to its associated nominal error, resulting in smoother transitions. According to [51], the accuracy of their method is conjectured to be of the order of \(1/n^2\), while in our procedure it can be asserted to follow an exponential law as seen in the chart of Figure 7.7. In [51] the case of a non-uniform grid was also considered. Since our approach assumes the existence of a uniform grid of points, and the finite difference method assumes uniformity of the grid, its application on a nonuniform grid destroys any precision that the method may have. There exists though finite difference methods especially designed for non-uniform grids. Such methods may provide some improved performance but certainly such result will not be better than those obtained with a fixed grid.

Nevertheless for simple applications like the one presented, the margins of error provided by this method can be considered acceptable with a still low computational effort. The simplicity of the method provides a way to analyze its stability and storage properties based on its constitutive blocks. These advantages are attractive for the model reduction procedures like the one presented in [127]. It could be argued that the search of reduction methods for distributed systems is unnecessary with the availability of procedures like [51] or the one presented here, which provide accurate operational low-order models from the model equations. This may be especially true for conservative systems. The justification of the additional use of reduction procedures can be found on the need of having a deeper understanding of such methods in terms
of control properties like stability, controllability, observability, passivity or dissipativity inherited to the reduced system.

**Acknowledgement** The first author appreciates the interesting talks and comments of Dr. G. Golo and Prof. A.J. van der Schaft regarding their discretization procedure in [51] during his short visit at Twente University.
Chapter 8

Conclusions

There is no chance, no destiny, no fate, that can circumvent or hinder or control the firm resolve of a determined soul.

-Ella Weeler Wilcox

In this work, a theory for structure-preserving model reduction for nonlinear dissipative control systems was presented using a differential-geometric approach. We provided a solution to the problem formulated in Section 2.2 as follows: Given an object called a nonlinear dynamical control system \( \Sigma \), with own structural properties of stability, reachability and observability, belonging to the class of dissipative models \( (\mathcal{C},\phi) \); the problem consists in developing an analytical method to obtain a family of sub-objects called reduced order models \( \Sigma_r \) of the same class of models \( (\mathcal{C},\phi) \) but supported by a reduced-order space of state trajectories, such that the structural properties of stability, reachability and observability and dissipativity are preserved in each sub-object \( \Sigma_r \).

The importance of using a differential-geometric approach lies in providing an analytical method based on coordinate-independent arguments and established concepts in differential geometry like curvature theory, isometries, etc. In particular Chapter 3 introduced the work where the approach of dissipative structure-preserving nonlinear model reduction was originally proposed. Furthermore, a procedure based on physical energy to balance and reduce port-Hamiltonian systems with collocated inputs and outputs is presented along with a structure-preserving reduction method based on singular perturbations. When restricted to linear Hamiltonian systems, both structure-preserving model reduction methods proposed in [127] are object of further discussion and comparison (by an alternate research group) in [66].

In Chapters 4 and 5 the construction of a unified geometric theory for continuous-time nonlinear dissipative balanced reduction is treated. In particular, in Chapter 4 several results about dissipative balanced reduction in terms of exogenous signals is presented. The geometric view to the
problem of balanced reduction for nonlinear dissipative systems is shown in
this chapter, by reinterpreting it as a problem of characterization of the in-
variants of an isometric operator: the behavioral operator. Furthermore, using
curvature theory, the invariants of such operator are shown to be associated
to an orthogonal frame of tangent vector fields and concepts like shape oper-
ator, orthogonal projection, Schmidt decomposition, and balanced reduction
are part of the discussion.
One remarkable result of Chapter 4 consists in showing that the structure of
the nonlinear balancing problems admits an orthogonal decomposition into sep-
arable invariant functions for nonlinear operators, which can be considered the
nonlinear equivalent to the singular value decomposition for linear operators.
Such operator decomposition consists of an orthogonal frame of tangent vector
fields, a set of normalized separable functions and an orthogonal coframe of
cotangent covectorfields. In this chapter, the concept of balancing is shown to
be actually a condition for the dualization of two Hilbert manifolds.
With the introduction of standard geometric objects and additional tools,
namely adjoint and semigroup Lie operators, in Chapter 5 the geometric view-
point for nonlinear dissipative balanced reduction is further developed for en-
dogenous (state-space) trajectories. In particular, Chapter 5 provides a sharp
geometric characterization of internal balancing as a condition for group ex-
tension of Lie semigroups. Furthermore, the storage functions are shown to be
generating functions of Lie group actions and issues of minimality of balanced
internal realizations are also discussed.
Another part of this research lies on the development of numerical algorithms,
for nonlinear balanced reduction using this approach. In Chapter 6 numeri-
cal methods to find the storage functions for dissipative balanced reduction
of nonlinear sampled-time systems are introduced. Using properly-defined dy-
namic optimization problems, along with adequate nonlinear discretization
algorithms,— including those based on Taylor-Lie series [94] or numerical inte-
gration algorithms—, it is possible to provide a framework to find approxima-
tions to such storage functions.
In particular, the discrete-time versions of the controllability and observabil-
ity energy functions are discussed. Instead of looking for the solution of a
Hamilton-Jacobi-Isaacs and a Lyapunov-like partial differential equations as in
the continuous-time case, an optimization approach and an iterative algorithm
are proposed to find $L_c$ and $L_o$ respectively. This approach is exemplified with
linear and nonlinear discrete-time systems. In particular using this approach
on the discrete-time equivalent model of a universal motor the approximated
energy functions are found.
In the last Chapter 7, we introduce an improved procedure for discretization of Stokes-Dirac structures presented in [26] which does not preserve the port-Hamiltonian structure after discretization. After some heuristic modifications essentially inspired by the finite difference paradigm, the procedure now results in a system that preserves the collocated port-Hamiltonian structure along with some other desirable conditions for interconnection. The procedure is exemplified by a lumped approximation of a transmission line with the alternative geometric discretization. Thus, the electromagnetic one-dimensional transmission line represented in a distributed port-Hamiltonian form is lumped into a chain of subsystems which preserve the port-Hamiltonian structure with inputs and outputs in collocated form.

The simulation results are compared with those presented previously in [50] (see also [51]). The use of finite differences provides quite direct and simple structures, which are prone for further analysis with control purposes. But simplicity seems to bear the price of precision. The simulation results presented in Chapter 7 show that in all cases, for the same number of nodal structures, our lumping method provides a higher peak and nominal errors compared with the lumping procedures presented in [51]. Its stronger advantage lies in its simplicity, in being one of the first methods proposed, and even today is the only one based on finite differences. This is true even when this method is placed in context with the recent research on the subject, namely finite elements [210, 209, 10] and computational geometric methods [190].

As remarked in the introduction, the following topics of current research in nonlinear balancing were not included (or only partially included) in the discussion of this dissertation:

- Singular value analysis for balanced realizations, e.g. in the sense of [45].
- The relationships between the nonlinear cross-Gramian, gradients systems and symmetric systems (in the sense of [80]).
- Balancing in terms of solutions of Hamilton-Jacobi-Bellman Equations.

These topics are the natural goal for future research using this differential-geometric approach.
References

References


A **MANIFOLD** \( M \) is a topological space such that at each point \( x \in M \) there exists a neighborhood \( U \) of \( x \) and a homeomorphism \( \phi : U \to \mathbb{R}^n \) mapping from \( U \) onto an open set of \( \mathbb{R}^n \). Sometimes, under a proper definition of dimension on a manifold \( M \) we denote its dimension by \( M^n \), with a finite \( n \in \mathbb{R}^+ \). A manifold \( M \) is said to be **differentiable** if \( M \) is Hausdorff and is covered by a union of sets \( \{ U_i \} \) in a countable collection of compatible charts \( \{(\phi_i, U_i)\} \).

For coordinate patches denoted by \( U_i \), consider a finite covering \( \{ U_i \} \) along with real-valued differentiable functions \( \psi_i : M \to \mathbb{R} \). A partition of unity subordinated to the covering \( \{ U_i \} \) on a manifold \( M \) is a collection \( \{(U_i, \psi_i)\} \) such that \( \{ U_i \} \) is a locally finite open covering of \( M \), \( \psi_i(x) \geq 0 \) for all \( x \in M, \ i = 1, \ldots, n \) where \( \psi_i \) has compact support such that \( \text{supp}(\psi_i) \subset U_i \), and that satisfies \( \sum_{i=1}^{n} \psi_i(x) = 1 \) for all \( x \in M \).

Consider a vector function \( \Psi(x) : \mathbb{R}^n \to \mathbb{R}^m, x \in M \) with components defined by \( \psi_i(x) \in C^\infty, \ i = 1 \ldots m \) such that \( \det[\partial \Psi(x)/\partial x] \neq 0, \forall x \in M \).

A regular equivalence relation \( R \) defined in \( M \) is such that for \( x, w \in M \), \( xRw \iff \Psi(x) = \Psi(w) \).

Denote by \( C(p), p \in M \), the space of differentiable curves \( \varrho \in M \) defined on open intervals \( I_i = (-\epsilon_i, \epsilon_i) \subset \mathbb{R} \) such that \( \varrho(0) = p \). An intrinsic definition of this space relies in an equivalence relation as follows: Two curves \( \varrho, \rho \in C(p) \) are equivalent, \( \varrho \sim \rho \) iff they are such that

\[
\left. \frac{d}{dt} \varrho(t) \right|_{t=0} = \left. \frac{d}{dt} \rho(t) \right|_{t=0}
\]

for any chart \( \{ \phi, D \} \) at \( p \). Let \( \varrho \in C(p) \). The tangent space \( T_pM \) is the set of all equivalence classes \([\varrho]\) of \( C(p) \). Each element of this space \( \xi \in T_pM \) is denoted in the standard basis \( e_i = \partial/\partial x^i, \ i = 1, \ldots, n \) as a differential operator

\[\xi(x) = \xi^i(x) \frac{\partial}{\partial x^i}\].

1 The symbol “\( \circ \)” denotes composition of two functions: \( (f \circ g)(x) = f(g(x)) \).
Denote by $F(p)$ the class of differentiable functions $f$ defined in a neighborhood of $p \in \mathcal{M}$ such that $f(p) = 0$. We define an equivalence class of these functions as follows. Two functions $f, g \in F(p)$ are said to be equivalent $f \sim g$ if they are such that $d((f \circ \phi^{-1})|_{x=\phi(p)} = d((g \circ \phi^{-1})|_{x=\phi(p)}$, for any local chart $(\phi, U)$ at $p \in \mathcal{M}$. Let $f \in F(p)$. The cotangent space $T^*_p \mathcal{M}$ is the vector space of equivalence classes $[f]$ of $F(p)$. An element $\alpha \in T^*_p \mathcal{M}$ is (a differential 1-form) denoted by $\alpha = \alpha_i(x) \, dx^i$.

The elements of $T^*_p \mathcal{M}$ are dual to those of $T_p \mathcal{M}$. For any differentiable function $f \in F(p)$ and any differentiable curve $\gamma \in C(p)$ the following pairing $\langle [f], [\gamma] \rangle = \frac{d}{dt} f \circ \gamma |_{t=0}$ is well defined and is bilinear.

A diffeomorphism is a bijective smooth mapping $f : \mathcal{M} \to \mathcal{N}$ with smooth inverse $f^{-1}$. A 1-parameter group of diffeomorphisms is a set of transformations $G = \{g^t | t \in \mathbb{R}^1 \} \subset \Omega(S)$ such that the mapping $M : \mathbb{R}^1 \times D \to D$, $M(t, x) = g^t(x)$ depends smoothly on $t \in \mathbb{R}^1$; $g^0(x) = x$, $g^{t_2}(g^{t_1}(x)) = g^{t_2+t_1}(x)$ and $g^{-t}(g^t(x)) = x$.

A vector field $\xi(x)$ is called a generator of a 1-parameter group of diffeomorphisms if it is such that $\xi(x) = [\partial M(t, x)/\partial t]|_{t=0}$. Conversely in a Lie Group, the 1-parameter group of diffeomorphisms with generator $\xi$ is expressed in the exponential map notation by $g^t(x) = \exp(t\xi)x$.

For a smooth map $\Upsilon : \mathcal{M}^n \to \mathcal{N}^m$ denote its differential tangent map by $\Upsilon_*|_{x_0} : T_x\mathcal{M} \to T_{\Upsilon(x)}\mathcal{N}$ acting (from the left) on vectorfields with matrix $[\partial \Upsilon/\partial x]|_{x_0}$. The same map $\Upsilon$ (being a diffeomorphism) has an associated mapping $\Upsilon^*|_{x_0} : T^*_{\Upsilon(x)}\mathcal{N} \to T^*_x\mathcal{M}$ called differential map acting (from the right) on 1-forms with matrix $[\partial \Upsilon/\partial x]|_{x_0}$.

Given a smooth real function $f : \mathcal{M} \to \mathbb{R}$, a point $p \in \mathcal{M}$ is a critical point if $df_p = 0$, otherwise it is a regular point of $f$. The support of $f$ is the closure of the set of points where the function is not zero-valued, $\text{supp}\{f\} = \text{cl}\{x \in \mathcal{M} | f(x) \neq 0\}$. Given a function $f \in F(p)$ with critical point $p$, the Hessian $f_{**}|_p$ on $T_p \mathcal{M}$ is a symmetric, bilinear functional $f_{**}(\xi, \zeta)$, $\xi, \zeta \in T_p \mathcal{M}$ with matrix $[\partial^2 f/\partial x^i \partial x^j]|_p$. A differentiable mapping $\Upsilon : \mathcal{M}^n \to \mathcal{N}^m$ (both differentiable manifolds) is called an immersion if $\text{rank}(\Upsilon_*|_{x_0}) = n$, $\forall x_0 \in \mathcal{M}$, being called a submersion if $\text{rank}(\Upsilon_*|_{x_0}) = m$. A subset $\mathcal{N}$ of the differentiable manifold $\mathcal{M}$ is called a submanifold of $\mathcal{M}$ if $\mathcal{N}$ is a differentiable manifold and the inclusion map $\iota : \mathcal{N} \to \mathcal{M}$ is an immersion.
In almost every field of applied science and engineering, including physics, chemistry, geophysics, climatology, biology or econometry, dynamical models are widely used as a professional tool to describe in a compact format the scientific knowledge we have about the phenomena we are analysing. Furthermore, such models are widely used for the estimation of variables, for optimization, for the detection of failures and even to control these phenomena in real-time. The relevance of models is such, that huge budgets are invested yearly by universities and research institutes to obtain ever faster super-computers to comply with the ever growing need of expensive simulation software to simulate complex dynamical models. Nevertheless, experience has shown that the dimension of the models used for simulation and control can be significantly reduced if techniques of model order reduction (MOR) are used. Given a full-order dynamical model (FOM), the problem of MOR consists in finding a reduced-order model (ROM) which keeps structural properties like stability, reachability/controllability and observability and other desirable characteristic properties of the FOM into the ROM. For this and other reasons, MOR is a topic of growing concern for every scientific and engineering field.

Although nowadays there are formal MOR methods for linear control systems, these methods have inherent limitations when the size the FOM is very large. This situation becomes worse when dealing with nonlinear systems, where the available methods yield approximated ROM models of local validity and empirical formulation. Since still there are not satisfactory solutions to these needs, a more formal approach is required.

This dissertation discusses a theory for structure-preserving MOR for nonlinear dissipative control systems. In particular, this thesis provides a theoretical framework for nonlinear balanced reduction for a rather large class of dynami-
Modeling models with the property of being dissipative. A widely used mathematical tool used to characterize nonlinear dynamical phenomena in control systems is differential geometry. With this tool along with other concepts like critical point theory, submanifold Hilbert theory, adjoint and semigroup Lie operators and curvature theory, in this dissertation we have constructed a framework for structure-preserving nonlinear MOR.

The advantage constructing a ROM realization theory using a differential-geometric approach lies in providing more formal and reliable ROM realizations, since they are imposed to be, by construction, independent of the coordinates used for their representation.

Associated to the problem of balanced reduction for nonlinear dissipative systems with endogenous (state-space) trajectories, we have proposed a nonlinear isometric operator, -the behavioral operator-, which has served to provide an original reinterpretation to the problem. In particular, using curvature theory, we characterized the invariants of such operator and showed that they are associated to an orthogonal decomposition into a set of normalized separable invariant functions, an orthogonal frame of tangent vector fields and an orthogonal coframe of cotangent covector fields. Therefore the decomposition of isometric operators provided here can be considered the nonlinear equivalent to the Singular Value Decomposition (SVD) for linear operators and therefore it constitutes a nonlinear generalization to Principal Component Analysis (NL-PCA). Furthermore, the concept of balancing is shown to be actually a condition for group extension of Lie semigroups and a condition for the dualization of two Hilbert manifolds, where the Gramians are components of two Riemannian metrics and duality suffices for a balanced realization and concepts like shape operator, orthogonal projection, Schmidt decomposition, and balanced reduction could be explained within this framework. Furthermore, the storage functions are shown to be generating functions of Lie group actions and issues of minimality of balanced internal realizations are also discussed.

Other related results are presented in this dissertation. In particular, we provide a procedure based on physical energy to balance and reduce port-Hamiltonian systems and a structure-preserving reduction method based on singular perturbations. Moreover, we provide numerical algorithms to find the storage functions for dissipative balanced reduction of nonlinear sampled-time systems. Finally, we introduce an improved procedure for discretization of Stokes-Dirac structures, exemplified by a lumped approximation of a transmission line, which after some heuristic modifications based on finite differences, it results in a system that preserves the port-Hamiltonian structure and is appropriate for system interconnection.
Samenvatting

In bijna elk gebied van de toegepaste wetenschap en techniek, – met inbegrip van fysica, chemie, geofysica, klimatologie, biologie of econometry –, worden dynamische modellen gebruikt als een professionele tool om te beschrijven in een compact formaat van de wetenschappelijke kennis die we hebben over de verschijnselen die we analyseren. Bovendien worden dergelijke modellen voor de schatting van variabelen voor optimalisatie, voor de detectie van fouten en zelfs deze fenomenen in real-time controle. De relevantie van de modellen is zodanig, dat het belegde enorme budgetten jaarlijks door universiteiten en onderzoeksinstellingen om steeds snellere supercomputers verkrijgen voor het voldoen aan de steeds groeiende behoefte van dure simulatiesoftware om complexe dynamische modellen te simuleren.

Toch heeft de ervaring geleerd dat de afmeting van de gebruikte voor de simulatie en monitoring modellen kunnen worden verminderd als significant technieken van model orde reductie (MOR) worden gebruikt. Bij een volledige orde dynamisch model (FOM), het probleem van MOR deze bestaat uit het vinden van een gereduceerde orde model (ROM) die blijft structurele eigenschappen zoals stabiliteit, bereikbaarheid / regelbaarheid en waarneembaarheid en andere gewenste karakteristieke eigenschappen van de FOM in de ROM. Redenen voor deze en andere, MOR is een onderwerp van groeiende zorg voor elke wetenschappelijke en technische veld.

Hoewel er tegenwoordig MOR formele methoden voor het regelen lineaire systemen, deze werkwijzen hebben beperkingen inherent als het formaat FOM is zeer groot. Deze situatie nog erger wordt wanneer omgaan met niet-lineaire systemen, waar de beschikbare methoden leveren benaderde ROM lokale modellen en empirische geldigheid van de formulering. Aangezien er nog steeds geen bevredigende oplossingen voor deze behoeften, op een formele benadering is meer nodig.
Dit proefschrift bespreekt een theorie voor structuur-behoud van MOR controle voor niet-lineaire dissipatief systeem. In het bijzonder, dit proefschrift een theoretisch kader voor een evenwichtige niet-lineaire verlaging voor een vrij grote klasse van dynamische modellen met de eigenschap dat dissipatief. Een grote schaal gebruikt wiskundige instrument dat wordt gebruikt om niet-lineaire dynamische systemen te karakteriseren controleert verschijnselen in differentiële meetkunde. Met deze tool samen met andere concepten zoals kritisch punt theorie, Hilbert deelvariëteit theorie, Lie geadjuinieerde exploitanten en semigroup theorie en de kromming theorie, hebben we in dit proefschrift wordt geconstrueerd van een kader voor niet-lineaire structuur behoud van MOR.

Het voordeel construeren van een theorie realisatie ROM met behulp van een differentiële-geometrische aanpak ligt in het leveren van meer formeel en betrouwbare ROM realisaties, omdat ze moeten worden opgelegd, door de bouw, onafhankelijk van de coördinaten die voor hun vertegenwoordiging.

Geassocieerd met het probleem van een evenwichtige vermindering voor niet-lineaire dissipatieve Systems met endogene (state-space) trajecten, Hebben Wij stelden een niet-lineaire operator isometrisch, –de gedrags-operator–, die gediend heeft een originele herinterpretatie voor het probleem. In het bijzonder, met behulp van kromming theorie, we Gekenmerkt invarianten van de exploitant en toonde aan dat dergelijke ze zijn gekoppeld aan een orthogonale decompositie in een reeks van genormaliseerde onveranderlijk gescheiden functies, een orthogonale kader van raakvector velden en een orthogonale coframe van cotangens covectorfields. Daarom is de descomposition van isometrische operatoren hier kan worden verstrekt beschouwd als de niet-lineaire gelijk aan de Singular waarden ontbinding (SVD) voor lineaire operatoren en daarom vormt een niet-lineaire generalisatie Hoofdcomponentenanalyse aan (NL-PCA). Bovendien is het concept van balancing aangetoond dat whos voorwaarde voor groep verlenging van Lie semigroepen en een voorwaarde voor de dualisering van twee Hilbert spruitstukken, waar de Gramians zijn onderdelen van twee Riemann metrieken en dualiteit is voldoende voor een evenwichtige realisatie en begrippen als vorm operator, orthogonale projectie, Schmidt afbraak, en evenwichtige vermindering esta kon worden verklaard binnen kader. Bovendien worden de opslag functies aangetoond dat genererende functies van Lie groepsacties en kwesties van minimaliteit interne realisaties in balans zijn ook besproken.

Andere verwante resultaten worden gepresenteerd in dit proefschrift. In het bijzonder, Wij bieden een procedure op basis van fysische energie in evenwicht en vermindert de poort-Hamiltoniaanse systemen en een structuur behoud
reductie methode op basis van unieke verstoringen. Bovendien zijn de numerieke algoritmes, wij bieden opslag om de functies te vinden voor dissipatief evenwichtige vermindering van de niet-lineaire bemonsterde-time systemen. Tot slot, introduceren we een verbeterde procedure voor de discretisatie van Stokes-Dirac structuren, geïllustreerd door een hoop gegooid benadering van een transmissielijn, die na een aantal heuristische Wijzigingen op basis van eindige verschillen, het resulteert in een systeem dat de poort-Hamiltoniaanse structuur behouden en is geschikt voor systeem-interconnectie.
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His professional interest lies in the research and development of advanced techniques of modeling, feedback control, estimation, supervision and optimization for its application in the area of Oil & Gas Exploration and Production.
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