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COARSE GRID APPROXIMATION GOVERNED BY LOCAL FOURIER ANALYSIS

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Abstract. *Solving discrete boundary value problems with the help of an appropriate multigrid method [1, 4, 5, 6] necessitates the construction of a sequence of coarse grids with corresponding coarse grid approximations for the given fine grid discretization. Popular choices in this context are the Galerkin coarse grid approximation (GCA) and the use of the same discretization on the coarser grids as on the fine grid with properly adjusted mesh sizes (DCA).*

In this paper we propose an alternative strategy to select the required coarse grid discretizations within a multigrid solution method. It can be applied to fine grid operators that can be locally represented by a stencil. The coarse grid approximations are constructed by minimizing a certain low-frequency L^2-norm. More precisely, a coarse grid discretization is chosen in such a way, that its Fourier symbol is a best approximation (w.r.t. low frequencies) of the Fourier symbol of the fine grid operator. This strategy is abbreviated by FCA since the design of coarse grid approximations is based on local Fourier analysis [5, 7]. The entries of the coarse grid stencils are simply given by linear combinations of the fine grid entries. As a consequence, FCA can be considered as a black-box method to construct coarse grid operators. This method has been successfully applied to (anisotropic) diffusion equations, operators with mixed derivatives, problems with dominant convection, and operators involving jumping coefficients. For nicely elliptic examples, FCA resembles the DCA approach, whereas for more difficult applications (w.r.t. an efficient multigrid treatment) it behaves similarly to GCA based on operator-dependent transfers.

1 INTRODUCTION

We consider a two-dimensional discrete (elliptic) boundary value problem with eliminated boundary conditions

$$L_h u_h(\boldsymbol{x}) = f_h(\boldsymbol{x}) \quad (\boldsymbol{x} \in \Omega_h) \tag{1}$$

resulting from a vertex-centered discretization of its continuous counterpart. The discrete domain is given as the intersection $\Omega_h = \Omega \cap G_h$ of the problem domain $\Omega \in \mathbb{R}^2$ with a vertex-centered infinite grid of mesh size h

$$G_h := \left\{ \boldsymbol{x} = (x, y)^T = h\,(k_x, k_y)^T \mid k_x, k_y \in \mathbb{Z} \right\}. \tag{2}$$

Suppose that L_h can be locally represented by a stencil $\left[\ell^h_{\boldsymbol{\kappa}}(\boldsymbol{x})\right]_h$, i.e.,

$$L_h u_h(\boldsymbol{x}) = \sum_{\boldsymbol{\kappa} \in J_h} \ell^h_{\boldsymbol{\kappa}}(\boldsymbol{x})\, u_h(\boldsymbol{x} + \boldsymbol{\kappa} h) \quad \text{with} \quad \boldsymbol{x} \in \Omega_h,$$

stencil entries $\ell^h_{\boldsymbol{\kappa}}(\boldsymbol{x}) \in \mathbb{R}$, and a certain index set $J_h \in \mathbb{Z}^2$ containing $(0, 0)$.

Multigrid methods for the solution of (1) are based on two main principles:

1. Many classical iterative methods have a strong smoothing effect on the error if they are applied to an elliptic boundary value problem like (1).

2. A smooth error term can be well approximated on a discrete domain with coarser resolution $H > h$.

These two principles suggest the following well-known structure of a two-grid method [1, 4, 5, 6]: Perform ν_1 steps of an iterative relaxation method S_h on the fine grid, compute the defect of the current fine grid approximation, restrict the defect to the coarse grid using a restriction operator R^H_h, solve the coarse grid defect equation, interpolate the correction using a prolongation operator P^h_H to the fine grid, add the interpolated correction to the current fine grid approximation, perform ν_2 steps of an iterative relaxation method on the fine grid. The resulting two-grid operator $M_{h,H}$ is given by

$$M_{h,H} := S_h^{\nu_2} \left(I_h - P^h_H L_H^{-1} R^H_h L_h \right) S_h^{\nu^1}.$$

Instead of explicitly inverting L_H, the coarse grid defect equation can be solved by a recursive application of an appropriate two-grid method yielding a multigrid method. Hence we need a sequence of discrete domains with coarser resolution and corresponding coarse grid approximations of the fine grid discretization L_h. Here we only consider standard coarsening, where a sequence of coarse grids is obtained by doubling the mesh size in each space direction, giving us $\Omega_H = \Omega_{2h} = \Omega \cap G_{2h}$, $\Omega_{4h} = \Omega \cap G_{4h}$, and so forth.

A straight-forward way to obtain coarse grid discretizations L_{2h}, L_{4h}, etc. is to apply the same discretization technique as on the fine grid Ω_h but to replace h by $2h, 4h$, etc. Following [6], we call this approach discretization coarse grid approximation (DCA). It yields proper coarse grid approximations for nicely elliptic operators like the Laplacian. For more difficult applications (for example problems involving dominant convection or jumping coefficients) one usually has to switch to more sophisticated coarse grid operators [4, 5, 6]. The most prominent choice in this context is the Galerkin approach, setting $L_{2h} = R_h^{2h} L_h P_{2h}^h$ and so forth. Note that in the vertex-centered case one often needs operator-dependent restriction and prolongation in order to obtain proper coarse grid approximations, compare with [2, 3, 5, 6, 9]. A drawback of the Galerkin coarse grid approximation (GCA) is that the computation of the coarse grid operators is often quite costly and the size of the corresponding stencils might increase on coarser grids.

We propose an alternative strategy abbreviated by FCA to automatically select coarse grid discretizations within a multigrid solution method with the help of local Fourier analysis [1, 5, 7]. The coarse grid operators are constructed in such a way that the related Fourier symbols (compare with (3)) are best approximations w.r.t. smooth error components of the Fourier symbol of the fine grid discretization. The corresponding stencil entries $\ell_{\boldsymbol{\kappa}}^{2h}, \ell_{\boldsymbol{\kappa}}^{4h}$, etc. can be easily computed as they are simply given by linear combinations of the fine grid stencil entries $\ell_{\boldsymbol{\kappa}}^{h}$. Moreover, the quality of the coarse grid approximations and the computational work for determining the coarse grid stencils can be controlled by prescribing the stencil patterns J_{2h}, J_{4h}, etc. Following this approach, we recover well-established coarse grid operators based on DCA whenever they lead to efficient multigrid solvers like, e.g., for the Laplace operator. Considering more difficult applications, FCA yields improved operators compared to DCA and behaves similarly to the GCA approach based on operator-dependent prolongation and restriction [2, 3, 9].

The explicit construction of coarse grid approximations governed by local Fourier analysis (FCA) is detailed in section 2. Several examples of resulting coarse grid operators are discussed in section 3. Finally, some conclusions are drawn in section 4.

2 CONSTRUCTION OF COARSE GRID OPERATORS

2.1 Basic principles of local Fourier analysis

For the application of local Fourier analysis it is necessary to consider locally frozen operators with constant coefficients ($L_h \stackrel{\wedge}{=} \left[\ell_{\boldsymbol{\kappa}}^{h}\right]_h$) which are extended to an infinite grid G_h. The corresponding eigenfunctions (or Fourier components) and the related eigenvalues (or Fourier symbols) read

$$\varphi_h(\boldsymbol{x},\boldsymbol{\theta}) := e^{i\,\boldsymbol{\theta}\boldsymbol{x}/h} \quad (\boldsymbol{\theta} \in [-\pi,\pi]^2), \qquad \widetilde{L}_h(\boldsymbol{\theta}) := \sum_{\boldsymbol{\kappa}\in J_h} \ell_{\boldsymbol{\kappa}}^{h} e^{i\,\boldsymbol{\theta}\boldsymbol{\kappa}}. \tag{3}$$

The Fourier symbols for $L_{2h} \stackrel{\wedge}{=} \left[\ell_{\boldsymbol{\kappa}}^{2h}\right]_{2h}$ are obviously given by

$$\widetilde{L}_{2h}(\boldsymbol{\theta}) := \sum_{\boldsymbol{\kappa}\in J_{2h}} \ell_{\boldsymbol{\kappa}}^{2h} e^{i\,\boldsymbol{\theta}2\boldsymbol{\kappa}}.$$

Hence, Fourier components $\varphi_h(\boldsymbol{x},\boldsymbol{\theta})$ with $\boldsymbol{\theta} \notin \Theta_{\text{low}} := [-\pi/2,\pi/2]^2$ can not be represented on G_{2h} as they coincide with certain Fourier components $\varphi_h(\boldsymbol{x},\boldsymbol{\theta}^*)$

with $\boldsymbol{\theta}^* \in \Theta_{\mathrm{low}}$ due to the periodicity of the exponential function. This observation is known as aliasing and frequencies $\boldsymbol{\theta} \in \Theta_{\mathrm{low}}$ are called low frequencies.

2.2 Optimal coarse grid operator w.r.t. smooth error components

For ease of presentation we assume that the stencil elements of the fine grid operator sum up to zero, i.e.,

$$\sum_{\boldsymbol{\kappa}\in J_h} \ell^h_{\boldsymbol{\kappa}} = 0 \qquad \Leftrightarrow \qquad \ell^h_{(0,0)} = - \sum_{\boldsymbol{\kappa}\in J_h, \boldsymbol{\kappa}\neq(0,0)} \ell^h_{\boldsymbol{\kappa}}. \tag{4}$$

Note that (4) holds for consistent discretizations of those differential operators L on an infinite grid which contain only derivatives of u. If this assumption is violated (for example for partial differential equations involving not only derivatives of u but u itself like the Helmholtz equation) the presented approach can be easily modified.

Due to the aliasing of Fourier components, the coarse grid discretizations should be good approximations of L_h especially w.r.t. the (very) low frequencies. In order to satisfy this requirement, the coarse grid approximations are constructed by the minimization of a certain low-frequency L^2-norm. More precisely, the coarse grid discretization L^{FCA}_{2h} is derived in such a way that its Fourier symbol $\widetilde{L}^{\mathrm{FCA}}_{2h}$ is a best approximation (w.r.t. low frequencies) of the Fourier symbol $\widetilde{L}_h$ of the fine grid operator. For the mathematical formulation of this approximation problem we consider the function space

$$L^2_{\mathrm{low}} := \left\{ v \;:\; \Theta_{\mathrm{low}} \to \mathbb{C} \quad \text{with} \quad \left(\int_{\Theta_{\mathrm{low}}} |v(\boldsymbol{\theta})|^2 \, d\boldsymbol{\theta} \right)^{1/2} < \infty \right\}$$

with corresponding inner product and norm, respectively:

$$\langle v, w \rangle_{\mathrm{low}} := \int_{\Theta_{\mathrm{low}}} v(\boldsymbol{\theta})\overline{w(\boldsymbol{\theta})} \, d\boldsymbol{\theta}, \qquad \|v\|_{\mathrm{low}} := \sqrt{\langle v, v \rangle_{\mathrm{low}}} \qquad \left(v, w \in L^2_{\mathrm{low}}\right).$$

L^2_{low} equipped with $\langle\, ,\, \rangle_{\mathrm{low}}$ yields a Hilbert space. For the derivation of L^{FCA}_{2h} we are looking for the optimal (w.r.t. $\|\,.\,\|_{\mathrm{low}}$) approximation of $\widetilde{L}_h \in L^2_{\mathrm{low}}$ in the following subspace

$$F_{2h} := \mathrm{span}\left\{ e^{i\,2\boldsymbol{\theta}\boldsymbol{\kappa}} - 1 \;:\; \boldsymbol{\kappa} \in J_{2h} \setminus \{(0,0)\} \right\} \subset L^2_{\mathrm{low}}. \tag{5}$$

This is a classical approximation problem which can be easily solved yielding the optimal (w.r.t. $\|\,.\,\|_{\mathrm{low}}$) coarse grid stencil entries $\ell^{2h}_{\boldsymbol{\kappa}}$ ($\boldsymbol{\kappa} \in J_{2h} \setminus \{(0,0)\}$). Note that all basis functions share the summand -1 which ensures a consistent coarse grid discretization, i.e.,

$$\ell^{2h}_{(0,0)} = - \sum_{\boldsymbol{\kappa}\in J_{2h}, \boldsymbol{\kappa}\neq(0,0)} \ell^{2h}_{\boldsymbol{\kappa}}.$$

2.3 A simple example

To illustrate the presented procedure we consider the construction of a coarse grid discretization given by a compact 5-point stencil:

$$L_{2h}^{\mathrm{FCA}} \triangleq \begin{bmatrix} & \ell^{2h}_{(0,1)} & \\ \ell^{2h}_{(-1,0)} & -\sum_{\kappa\neq(0,0)} \ell^{2h}_{\boldsymbol{\kappa}} & \ell^{2h}_{(1,0)} \\ & \ell^{2h}_{(0,-1)} & \end{bmatrix}_{2h}, \quad F_{2h} := \mathrm{span}\{\phi_1,\, \phi_2,\, \phi_3,\, \phi_4\}$$

$$\text{with} \quad \phi_1(\boldsymbol{\theta}) := e^{i\,2\boldsymbol{\theta}(-1,0)} - 1, \quad \phi_2(\boldsymbol{\theta}) := e^{i\,2\boldsymbol{\theta}(1,0)} - 1,$$
$$\phi_3(\boldsymbol{\theta}) := e^{i\,2\boldsymbol{\theta}(0,-1)} - 1, \quad \phi_4(\boldsymbol{\theta}) := e^{i\,2\boldsymbol{\theta}(0,1)} - 1 \qquad (\boldsymbol{\theta} \in \Theta_{\mathrm{low}}).$$

The optimal coarse grid stencil is then given by the solution of a small linear system:

$$\left(\langle \phi_i, \phi_j \rangle_{\mathrm{low}}\right)_{i,j=1,\ldots,4} \begin{pmatrix} \ell^{2h}_{(-1,0)} \\ \ell^{2h}_{(1,0)} \\ \ell^{2h}_{(0,-1)} \\ \ell^{2h}_{(0,1)} \end{pmatrix} = \left(\langle \widetilde{L}_h, \phi_i \rangle_{\mathrm{low}}\right)_{i=1,\ldots,4}.$$

We would like to emphasize that each inner product occurring in the above linear system is a linear combination of the following integrals which can be explicitly calculated

$$\int_{\Theta_{\mathrm{low}}} e^{i\,(\theta_1\mu_1+\theta_2\mu_2)}\, d\boldsymbol{\theta} = \begin{cases} 4\pi^2/n^2 & \text{for } \mu_1 = \mu_2 = 0 \\ 4\pi \sin\left(\pi\mu_2/n\right)/(\mu_2 n) & \text{for } \mu_1 = 0, \mu_2 \neq 0 \\ 4\pi \sin\left(\pi\mu_1/n\right)/(\mu_1 n) & \text{for } \mu_1 \neq 0, \mu_2 = 0 \\ -2[\cos\left(\pi\left(\mu_1+\mu_2\right)/n\right) & \text{for } \mu_1, \mu_2 \neq 0 \\ -\cos\left(\pi\left(\mu_1-\mu_2\right)/n\right)]/\left(\mu_1\mu_2\right) & \end{cases}$$

with $\Theta_{\mathrm{low}} := [-\pi/n, \pi/n]^2$. This means that the matrix $(\langle \phi_i, \phi_j \rangle_{\mathrm{low}})_{i,j=1,\ldots,m}$ with $m = \#J_{2h} - 1$ can be precomputed for a given coarse grid pattern J_{2h}. Hence, general formulas for the stencil entries $\ell^{2h}_{\boldsymbol{\kappa}}$ can be calculated in terms of $\ell^{h}_{\boldsymbol{\kappa}}$. The complete sequence of coarse grid operators (L_{2h}^{FCA}, L_{4h}^{FCA}, L_{6h}^{FCA}, …) is obtained by a repeated application of this strategy.

2.4 Choosing Θ_{low}

Analyzing this approach we made the following important observation: Choosing $\Theta_{\mathrm{low}} = [-\pi/2, \pi/2]^2$ – which seems natural in connection with standard coarsening – yields bad coarse grid approximations whereas considering only very low frequencies (i.e., $\Theta_{\mathrm{low}} = [-\pi/n, \pi/n]$ with $n \to \infty$) yields excellent coarse grid approximations. Moreover, the analytical formulas for the coarse grid stencil entries converge to fixed expressions for $n \to \infty$. For example, one obtains the following formulas for a 5-point coarse grid operator assuming a consistent 5-point fine grid stencil:

$$\begin{aligned}
&\ell^{2h}_{(-1,0)} = \frac{3}{8}\,\ell^{h}_{(-1,0)} - \frac{1}{8}\,\ell^{h}_{(1,0)}, \quad \ell^{2h}_{(1,0)} = -\frac{1}{8}\,\ell^{h}_{(-1,0)} + \frac{3}{8}\,\ell^{h}_{(1,0)},\\
&\ell^{2h}_{(0,-1)} = \frac{3}{8}\,\ell^{h}_{(0,-1)} - \frac{1}{8}\,\ell^{h}_{(0,1)}, \quad \ell^{2h}_{(0,1)} = -\frac{1}{8}\,\ell^{h}_{(0,-1)} + \frac{3}{8}\,\ell^{h}_{(0,1)}\\
&\text{and hence} \qquad \ell^{2h}_{(0,0)} = -\frac{1}{4}\left(\ell^{h}_{(-1,0)} + \ell^{h}_{(1,0)} + \ell^{h}_{(0,-1)} + \ell^{h}_{(0,1)}\right).
\end{aligned} \tag{6}$$

The derivation of such formulas (in particular the evaluation of the limit case $n \to \infty$) is a tedious task but can be very easily done with the help of a symbolic math package like *Maple*.

3 APPLICATIONS

In this section, we evaluate the quality of the FCA approach by comparing the resulting coarse grid operators with those operators based on DCA (and GCA).

3.1 Laplace operator

We start with the Laplace operator $Lu = -\Delta u = -u_{xx} - u_{yy}$. A second and fourth order discretization of the Laplacian is given by

$$L_h^{2o} \mathrel{\hat{=}} \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h \quad \text{and} \quad L_h^{4o} \mathrel{\hat{=}} \frac{1}{12h^2}\begin{bmatrix} & & 1 & & \\ & & -16 & & \\ 1 & -16 & 60 & -16 & 1 \\ & & -16 & & \\ & & 1 & & \end{bmatrix}_h, \tag{7}$$

with stencil patterns

$$J^5 = \{(0,0), (\pm 1, 0), (0, \pm 1)\} \ \text{ and } \ J^9 = \{(0,0), (\pm 1,0), (0,\pm 1), (\pm 2, 0), (0, \pm 2)\}\,,$$

respectively. Employing (6) to construct a 5-point coarse grid approximation for L_h^{2o} yields

$$L_{2h}^{\mathrm{FCA}} \mathrel{\hat{=}} \frac{1}{h^2}\begin{bmatrix} & -1/4 & \\ -1/4 & 1 & -1/4 \\ & -1/4 & \end{bmatrix}_{2h} = \frac{1}{(2h)^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{2h} \mathrel{\hat{=}} L_{2h}^{\mathrm{DCA}},$$

i.e., FCA and DCA lead to the same coarse grid operators, which are known to be reasonable in the multigrid context.

For a 9-point fine grid operator L_h with $J_h = J^9$, FCA yields the following

9-point coarse grid operator assuming the same stencil pattern, i.e., $J_{2h} = J^9$:

$$\ell^{2h}_{(-1,0)} = \frac{15}{32}\,\ell^h_{(-1,0)} - \frac{5}{32}\,\ell^h_{(1,0)} + \ell^h_{(-2,0)}, \quad \ell^{2h}_{(1,0)} = -\frac{5}{32}\,\ell^h_{(-1,0)} + \frac{15}{32}\,\ell^h_{(1,0)} + \ell^h_{(2,0)},$$
$$\ell^{2h}_{(0,-1)} = \frac{15}{32}\,\ell^h_{(0,-1)} - \frac{5}{32}\,\ell^h_{(0,1)} + \ell^h_{(0,-2)}, \quad \ell^{2h}_{(0,1)} = -\frac{5}{32}\,\ell^h_{(0,-1)} + \frac{15}{32}\,\ell^h_{(0,1)} + \ell^h_{(0,2)},$$
$$\ell^{2h}_{(-2,0)} = -\frac{5}{128}\,\ell^h_{(-1,0)} + \frac{3}{128}\,\ell^h_{(1,0)}, \quad \ell^{2h}_{(2,0)} = \frac{3}{128}\,\ell^h_{(-1,0)} - \frac{5}{128}\,\ell^h_{(1,0)},$$
$$\ell^{2h}_{(0,-2)} = -\frac{5}{128}\,\ell^h_{(0,-1)} + \frac{3}{128}\,\ell^h_{(0,1)}, \quad \ell^{2h}_{(0,2)} = \frac{3}{128}\,\ell^h_{(0,-1)} - \frac{5}{128}\,\ell^h_{(0,1)},$$
$$\ell^{2h}_{(0,0)} = -\frac{19}{64}\left(\ell^h_{(-1,0)} + \ell_{(1,0)} + \ell^h_{(0,-1)} + \ell^h_{(0,1)}\right) - \ell^h_{(-2,0)} - \ell_{(2,0)} - \ell^h_{(0,-2)} - \ell^h_{(0,2)}.$$

Using these formulas to construct a coarse grid operator for L_h^{4o}, we find – as for the second order approximation – $L_{2h}^{\text{FCA}} = L_{2h}^{\text{DCA}}$, which is again a reasonable choice w.r.t. an efficient multigrid treatment, see [5].

One major advantage of the FCA strategy is that the size and pattern of the coarse grid stencil can be varied. For example, one could try to approximate the fourth order discretization of the Laplacian based on the long stencil L_h^{4o} by a compact 5-point stencil on the coarse grids. Especially on very coarse grids this is advantageous because at grid points adjacent to boundary points the long stencil cannot be applied since it has entries which lie outside the discrete domain, whereas a compact 5-point stencil can be applied throughout the domain. For a given fine grid 9-point stencil with stencil pattern $J_h = J^9$ we obtain the following coarse grid compact 5-point stencil ($J_{2h} = J^5$) by the FCA strategy:

$$\begin{aligned} &\ell^{2h}_{(-1,0)} = \frac{3}{8}\,\ell^h_{(-1,0)} - \frac{1}{8}\,\ell^h_{(1,0)} + \ell^h_{(-2,0)}, \quad \ell^{2h}_{(1,0)} = -\frac{1}{8}\,\ell^h_{(-1,0)} + \frac{3}{8}\,\ell^h_{(1,0)} + \ell^h_{(2,0)}, \\ &\ell^{2h}_{(0,-1)} = \frac{3}{8}\,\ell^h_{(0,-1)} - \frac{1}{8}\,\ell^h_{(0,1)} + \ell^h_{(0,-2)}, \quad \ell^{2h}_{(0,1)} = -\frac{1}{8}\,\ell^h_{(0,-1)} + \frac{3}{8}\,\ell^h_{(0,1)} + \ell^h_{(0,2)}, \\ &\ell^{2h}_{(0,0)} = -\frac{1}{4}\left(\ell^h_{(-1,0)} + \ell^h_{(1,0)} + \ell^h_{(0,-1)} + \ell^h_{(0,1)}\right) - \ell^h_{(-2,0)} - \ell_{(2,0)} - \ell^h_{(0,-2)} - \ell^h_{(0,2)}. \end{aligned} \tag{8}$$

Inserting the particular stencil entries of L_h^{4o} into (8) yields

$$\ell^{2h}_{(\pm1,0)} = \ell^{2h}_{(0,\pm1)} = \frac{1}{12h^2}\left(\frac{-3\cdot 16}{8} + \frac{16}{8} + 1\right) = -\frac{3}{12h^2} = -\frac{1}{(2h)^2}, \quad \ell^{2h}_{(0,0)} = \frac{4}{(2h)^2},$$

recovering the second order approximation L_{2h}^{2o} of $-\Delta$ on Ω_{2h}, compare with (7). In section 5.4.1 of [5] it has been shown, that this second order discretization is a very good approximation of the fourth order discretization w.r.t. (very) low frequencies (see figure 5.16 in [5]). Moreover, in section 4.1.2 of [7] it has been demonstrated that a multigrid algorithm that applies the fourth order discretization on the fine grid and the second order discretization an all coarser grids leads to an improved V-cycle convergence compared to an algorithm that applies the fourth order discretization on all grids. This means that the coarse grid operator suggested by the FCA approach is a very good choice to be applied within a multigrid solution method.

3.2 Convection-diffusion

As a more difficult test case (w.r.t. an efficient multigrid treatment) we consider the convection diffusion operator

$$Lu = -\varepsilon\Delta + u_x + u_y$$

with dominant convection, i.e., $0 < \varepsilon \ll 1$. The diffusive part is dicretized by second order differences whereas the convective part is discretized by first order upwind differences yielding

$$L_h^{codi} \triangleq \frac{\varepsilon}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h + \frac{1}{h}\begin{bmatrix} & 0 & \\ -1 & 2 & 0 \\ & -1 & \end{bmatrix}_h .$$

It is well-known that applying DCA leads to multigrid algorithms with a limited two-grid convergence factor of 1/2 [5, 8] (unless a special smoothing method is applied with a downstream numbering of grid points). This results in a further deterioration of the convergence factor if more grids are involved. The coarse grid correction difficulty is due to the fact that the leading truncation term of the DCA coarse grid operator does not match the leading truncation term of the fine grid operator [8]. We explain this observation in some more detail for the one-dimensional case $Lu = -\varepsilon u'' + u'$. Taylor expansion applied to the first order upwind discretization L_h^{upw} of $u'(x)$ and to the corresponding DCA operator on Ω_{2h} yields:

$$\begin{aligned} L_h^{upw}u(x) &= \frac{1}{h}\left(u(x) - u(x-h)\right) = u'(x) - \frac{1}{2}hu''(x) + O\left(h^2\right), \\ L_{2h}^{\mathrm{DCA}}u(x) &= \frac{1}{2h}\left(u(x) - u(x-2h)\right) = u'(x) - hu''(x) + O(h^2) \end{aligned}$$

which immediately reveals the wrong scaling factor of the leading truncation term of L_{2h}^{DCA}. The straight-forward modification of (6) to the one-dimensional case with $J_h = J^3 = \{0, -1, 1\} = J_{2h}$ yields an improved coarse grid approximation with the same leading truncation term as L_h^{upw}:

$$L_{2h}^{\mathrm{FCA}}u(x) = \frac{1}{h}\left(-\frac{3}{8}u(x-2h) + \frac{2}{8}u(x) + \frac{1}{8}u(x+2h)\right) = u'(x) - \frac{1}{2}hu''(x) + O(h^2).$$

A recursive application of the FCA approach gives a sequence of coarse grid stencils L_k^{FCA} $(k = 1, 2, ...)$ with stencil entries

$$\ell_{-1}^k = \frac{3}{8}\ell_{-1}^{k-1} - \frac{1}{8}\ell_1^{k-1}, \quad \ell_1^k = \frac{3}{8}\ell_1^{k-1} - \frac{1}{8}\ell_{-1}^{k-1}, \quad \ell_0^k = -\frac{1}{2}\left(\ell_{-1}^{k-1} + \ell_1^{k-1}\right) \qquad (9)$$

where the sub- and superscript k refers to the sequence of coarser mesh sizes $h_k = 2^k h$. Assuming

$$\ell_{-1}^h = \ell_{-1}^0 = -1, \quad \ell_0^h = \ell_0^0 = 1, \quad \text{and} \quad \ell_1^h = \ell_1^0 = 0,$$

it can be easily shown that for $k \to \infty$ the related stencils L_k^{FCA} tend to the limit case

$$\frac{1}{2h_k}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}_{h_k} \tag{10}$$

which represents the well-known central second order approxiamtion of u'. This is in good accordance with the observation from [8] that a proper coarse grid approximation for L_h^{upw} must become a second order approximation of u'. Unfortunately, the central discretization is not stable and should not be applied (at least not for dominant convection, see, for example, [5]). Since there is no other second order approximation of u' that can be represented by a 3-point stencil with $J_{h_k} = J^3$ one should switch to another stencil pattern. An obvious upwind-type choice is to use a nonsymmetric stencil pattern like

$$J^4 = \{-2, -1, 0, 1\}\,.$$

Applying this pattern the FCA approach gives us the following sequence of stencil entries:

$$\ell_{-2}^k = -\frac{1}{16}\ell_{-1}^{k-1} + \frac{1}{16}\ell_1^{k-1}, \quad \ell_{-1}^k = \frac{9}{16}\ell_{-1}^{k-1} - \frac{5}{16}\ell_1^{k-1} + \ell_{-2}^{k-1},$$
$$\ell_0^k = -\left(\frac{7}{16}\ell_{-1}^{k-1} + \frac{1}{16}\ell_1^{k-1} + \ell_{-2}^{k-1}\right), \quad \ell_1^k = -\frac{1}{16}\ell_{-1}^{k-1} + \frac{5}{16}\ell_1^{k-1}.$$

First of all note that on Ω_{2h} one obtains

$$\begin{aligned} L_{2h}^{\mathrm{FCA}} &= \frac{1}{h}\left(\frac{1}{16}u(x-4h) - \frac{9}{16}u(x-2h) + \frac{7}{16}u(x) + \frac{1}{16}u(x+2h)\right) \\ &= u'(x) - \frac{1}{2}hu''(x) + O\left(h^2\right) \end{aligned}$$

yielding the same leading truncation term as for L_h^{upw}. Secondly, for $k \to \infty$ the limit stencil

$$L_k^{\mathrm{FCA}} \triangleq \frac{1}{h_k}\begin{bmatrix}1/6 & -1 & 1/2 & 1/3 & 0\end{bmatrix}_{h_k}$$

results which represents a second order upwind discretization of u'. From these two observations we can conclude that the FCA strategy yields excellent and stable coarse grid approximations for L_h^{upw} if the stencil pattern J^4 is prescribed. Moreover, for the one-dimensional diffusive part $\frac{1}{h^2}\begin{bmatrix}-1 & 2-1\end{bmatrix}_h$ one obtains

$$\ell_{-2}^{2h} = 0, \quad \ell_{-1}^{2h} = \ell_1^{2h} = -\frac{1}{h^2}\frac{4}{16} = -\frac{1}{(2h)^2}, \quad \ell_0^{2h} = \frac{1}{h^2}\frac{8}{16} = \frac{1}{(2h)^2}2,$$

i.e., $L_{2h}^{\mathrm{FCA}} = L_{2h}^{\mathrm{DCA}}$. This means that FCA with $J_{2h} = J^4$ (or with the straightforwardly modified pattern J^7 in two dimensions) leads to appropriate coarse grid operators to be applied within a multigrid solver for convection-diffusion-type problems with dominant convection.

The discussion of the convection-diffusion operator demonstrates another interesting feature of the FCA approach. That is, it can be used to identify proper coarse grid stencil patterns for a given fine grid discretization or to judge between coarsening strategies. Note that the character of the subspace F_{2h} (5) is governed by the prescribed stencil pattern and the coarsening strategy. Hence, if it is not possible to find a good approximation for $\widetilde{L}_h$ in this subspace it should be usually due to the fact that the stencil pattern J_{2h} or the coarsening strategy is not appropriate.

3.3 Further applications

Similar analytical expressions for sequences of coarse grid stencil entries can be obtained for other fine and coarse grid stencil patterns. Apart from the examples presented above the method has been successfully applied to anisotropic diffusion equations, operators with mixed derivatives, and problems with jumping coefficients. For jumping coefficients it is sometimes not sufficient to use the fine grid stencil entries of the coarse (and fine) grid point under consideration to construct the coarse grid stencil entries. Instead, one has to use averages of fine grid stencil entries of neighboring fine grid points. Then the FCA approach yields similar coarse grid stencils to GCA with operator-dependent transfers.

4 CONCLUSIONS

It has been shown that the FCA approach is an interesting alternative to construct coarse grid approximations to be used within a multigrid solution method. The sequence of coarse grid stencils can be computed by simple analytical formulas composed of the fine grid stencil entries. The size and pattern of the coarse grid stencils can be varied which is a very useful feature. In this way it is possible to control the accuracy of the coarse grid approximation and the computational work. The FCA strategy automatically adapts to the fine grid operator under consideration. That means, the suggested coarse grid operators are similar or equal to those operators obtained by the DCA approach whenever they are an adequate choice. If more accurate or sophisticated coarse grid operators are necessary to obtain an efficient multigrid method, FCA resembles Galerkin-type coarse grid approximations. In this way, FCA yields proper coarse grid operators in a black-box fashion for various applications like (anisotropic) diffusion equations, convection-diffusion with dominant convection, and problems involving mixed derivatives or jumping coefficients. Moreover, the underlying minimization procedure for the design of coarse grid operators can even be used to judge between coarsening strategies and different patterns of coarse grid stencils.

Finally, we would like to mention that the presented approach can be easily modified for other regular coarsening strategies like semi or red-black coarsening and to three-dimensional applications.

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