UNIVERSITY OF CALIFORNIA
INSTITUTE OF ENGINEERING RESEARCH
BERKELEY, CALIFORNIA

ELASTIC RESPONSE OF A STIFFENED PLATE UNDER SLAMMING LOADING

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ABSTRACT

The response in the stiffened plate due to slamming loading is considered as superimposed vibration of both the whole structure and local vibrations of the panel plating. The local vibration in this case means the vibration of the plating panel having sides on the girders and stiffeners.

To obtain an approximate solution of such a problem, the energy method is used. Two theoretical analyses are developed in order to get the whole vibration due to only the effect of bending as well as the local vibration of the bottom plate due to both effects of bending and stretching. In the local vibration we discuss the problem such as anisotropic plate, in which the bottom plate is considered as one special case.

The data available to design are also given in order to decide the scantlings of a stiffened plate within the allowable amount of stress, which will be dynamically determined by experiment.

TABLE OF CONTENTS

NOMENCLATURE

α : number of side girders

β : number of stiffeners

$\beta\Delta$: time to reach F_M

Δ : duration of loading

$\alpha_i, \beta_i, \alpha_j, \beta_j$: coefficient

γ_i : coefficient

γ_{xy} : shearing strain of bottom plate

$\varepsilon_x, \varepsilon_y$: normal strain of bottom plate

τ_{xy} : shearing stress of bottom plate

τ : time

σ_x, σ_y : normal stress of bottom plate

ε : phase lag

τ_b : shearing stress on the bottom plate of stiffened plate

τ_t : shearing stress on the top plate of stiffened plate

σ_{bx} : normal stress on the bottom plate of stiffened plate in the x direction

σ_{by} : normal stress on the bottom plate of stiffened plate in the y direction

σ_{tx} : normal stress on the top plate of stiffened plate in the x direction

σ_{ty} : normal stress on the top plate of stiffened plate in the y direction

λ : effective breadth

λ_i : coefficient

μ : Poisson's ratio

ν : integer

ρ : mass per unit volume of the plate material

$\phi_i(x)$: function of x

$\phi_{ij}(t)$: function of time

θ_{ij} : coefficient concerning frequency

ω : circular frequency

a : length of stiffened plate in the x direction

b : length of stiffened plate in the y direction

a_0 : length of bottom plate in the x direction

b_0 : length of bottom plate in the y direction

A_i : coefficient concerning deflection

B_i : coefficient concerning stress

C_i : constant

a_{ij} : coefficient concerning deflection

${}_lA_p$: cross sectional area of pillar

l_p : length of pillar

$A, {}_iA_a, {}_jA_f$: cross sectional area of composite stiffener or girder with the effective breadth at maximum bending section

A_{11}, A_{ij} : coefficient concerning displacement amplitude of the stiffened plate

D : plate stiffness, $= \dfrac{Eh^3}{12(1-\mu^2)}$

E'_x, E'_y, E'' : modulus of elasticity characterizing anisotropic plate

E : Young's modulus

D_x, D_y, D_1, D_{xy} : plate stiffness

$F(t)$: slamming loading which is a function of t.

F_M : peak value of F(t)

$F_1(x), F_i(x), F_i(y)$: basic function defined by Inglis

$f_A, f_B, \varphi_A, \varphi_B$: coefficient indicating the end fixities

f, f_{ij} : frequency

$f(x,y)$: normal function

$f(x), f(y), X_1, Y_1$: beam function

F_k : function of $f(x,y)$

$f(x,y)_s$: normal function of stiffener

$f(x,y)_b$: normal function of beam

G : modulus of elasticity in shear

g : acceleration due to gravity

h : thickness of the plate

h_b, h_t: thickness respectively, of the bottom or top plate

$I_g, {}_jI_b$: moment of inertia of side girder

$I_s, {}_iI_a$: moment of inertia of stiffener

I_o : moment of inertia of central girder

$$i_x = I_g\left(\frac{\alpha+1}{b}\right)$$

$$i_y = I_s\left(\frac{\beta+1}{a}\right)$$

$$xi_y = \frac{1}{1-\mu}\left(h_b Z_b^2 + h_t Z_t^2\right)$$

I_a, I_b : geometrical moment of inertia of composite cross section in stiffener with the effective breadth of the plate about the axis through the centroid of that composite cross section

I_{pa}, I_{pb}: centroid polar moment of inertia at the cross section of stiffener

I'_a, I'_b : moment of inertia of the unit element in the lengthwise direction of the stiffener about axis through its center of gravity perpendicular to lengthwise direction.

i : integer

j : integer

K : coefficient

K_i : coefficient

$k''_a, k''_b, {}_ik''_a, {}_ik''_b, {}_jk''_a, {}_jk''_b$, : coefficient concerning shape of cross section

$k', {}_ik'_a, {}_ik'_b, {}_jk'_a, {}_jk'_b$: torsion constant

k_a, k_f : coefficient

K_b : constant concerning V_f

K_s : constant concerning V_s

L, L_i, L_{ij} : load factor

m' : mass per unit area of the plate

m, n : integer indicating the form of slamming loading

M_x, M_y : bending moment

M_{xy} : torsional moment

p_0 : circular frequency of the fundamental mode

p_k, p_{ij} : circular frequency of the kth or ij^{th} mode

p : fundamental frequency obtained by large deflection theory

q : number of side girders

r : number of stiffeners

$q_a, q_b, {}_iq_a, {}_jq_f$: weight of composite stiffener per unit length

$q_E, {}_kq_E$: uniformly distributed weight per unit area

$q_0(t)$: function of F(t)

$Q_{ij}(\tau)$: function of τ

Q_i : coefficient

R, R_{ij} : response factor

S_i, S_i' : coefficient concerning end fixities

t : time

T : total kinetic energy

T_1 : maximum kinetic energy of stiffened plate due to bending of plate and stiffeners

T_2 : maximum kinetic energy of stiffeners due to rotational inertia in their lengthwise direction and inertia force of rotation in the plane perpendicular to their lengthwise direction

T, T_{ij} : period

T_i : coefficient

$\bar{u}$: displacement in the x direction

$\bar{v}$: displacement in the y direction

v : initial velocity

V_f : strain energy due to bending

V_s : strain energy due to stretching

V_1 : maximum strain energy effected by bending of both plate and stiffeners

V_2 : maximum strain energy in stiffeners effected by both shearing force and torsion

V : total strain energy

w : deflection of either bottom plate or stiffened plate

$w_{11}(t), w_0(t), w_1(t), w_2(t), w_3(t)$: deflection function of t

$u_{11}(t), v_{11}(t)$: displacement function of t

δw : small variation of w

w_e : edge deflection of bottom plate

$\bar{w}$: amplitude of edge vibration

x : direction of ship length

y : direction of ship beam

Z_b : distance to neutral axis from the bottom plate

Z_t : distance to neutral axis from the top plate

LIST OF ILLUSTRATIONS

TABLE Page

FIGURE

I. INTRODUCTION

The structural response due to slamming loading in a plate stiffened by girders and stiffeners is here obtained. The response in the stiffened plate will be considered as superimposed vibrations of both total and local vibrations in which the local vibration means the vibration due to the plating panel having sides on the girders and stiffeners orthogonally crossing each other.

We can, therefore, analyze this problem from two points of view. One is the total vibration of the stiffened plate, and the other is the local vibration of plating panel when all edges are assumed to be clamped on the orthogonally crossed girders and stiffeners. The data available for design may easily be given, if we consider the case of an extreme condition such as each total or local vibration has the maximum value on the same side. Therefore the useful data will be obtained by the concept of adding the maximum value of the stiffened plate to that of plating panel.

Two theoretical analyses are hereby developed, applying the energy method in order to get the total vibration due to only the effect of bending as well as the local vibration due to both effects of bending and stretching. In the local vibration we discuss the problem such as anisotropic plate, in which bottom plate is considered as one special case.

Insofar as the vibration of plating panel due to only the effect of bending is concerned, some work has been done by

Greenspoon [1]*, and the comparison between theory and full-scale data has also been obtained [2].

If the deflection resulting from a large response can no longer be considered small when compared with the plate thickness, the analysis of such a problem must be carried out giving consideration to the effect of plate stretching in addition to the effect of bending, as in the case of small deflection [3]. For such a problem an anisotropic plate is chosen and solved applying the energy method, even though the application of this method requires considerable amount of computation. Useful curves and tables are also given in this paper by which scantlings of a stiffened plate may be decided within the allowable yielding point which will be dynamically determined by experiment.

The numerical calculation is given to show how to decide the plate thickness to withstand some slamming loading, applying the step-by-step procedure after using curves and tables obtained by computation.

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* Numbers in brackets refer to the bibliography at the end of the paper.

II. THEORETICAL ANALYSES

In order to get the total response of a stiffened plate we here discuss two kinds of problems applying the energy method: one of which is the vibration of a stiffened plate and the other the local vibration of bottom plate. The vibration of a stiffened plate is easily carried out by considering only the effect due to bending, but the local vibration of bottom plate is extensively discussed with the anisotropic plate considering two effects due to both bending and stretching. These two problems are discussed below.

2-1. Vibration of a Stiffened Plate

If we choose, for instance, a double bottom as shown in Fig. 1, the strain energy due to bending V_b [3] [4] [5] is

$$V_b = \frac{E}{2}\int_0^a\int_0^b \left\{ i_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + i_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{2\mu}{1-\mu^2} i_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{2}{1+\mu} i_y\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} dx\,dy$$

$$+ \int_0^a (I_0 - I_y)\left(\frac{\partial^2 w}{\partial x^2}\right)^2_{y=\frac{b}{2}} dx \quad \text{------} (1)$$

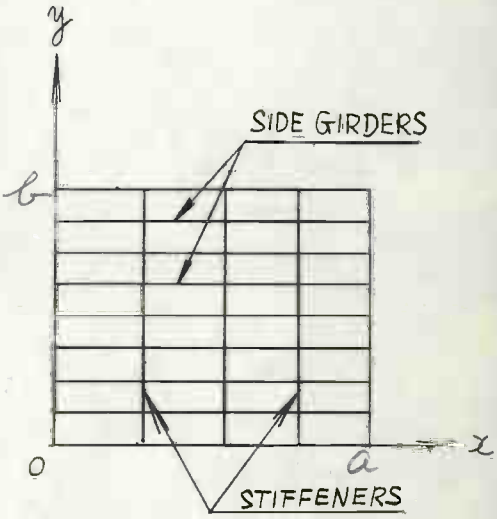


Fig. 1 Double Bottom

where i_x, i_y : distributed moment of inertia per unit length and

$$i_x = I_g\left(\frac{\alpha+1}{b}\right), \quad i_y = I_s\left(\frac{\beta+1}{a}\right), \quad x i_y = h_b Z_b^2 + h_t Z_t^2 \doteqdot i_{xy}$$

w : deflection

I_g : moment of inertia of side girder,

I_s : moment of inertia of stiffener,

I_o : moment of inertia of central girder,

h_b, h_t: thickness respectively, of the bottom or top plate

Z_b, Z_t: distance to neutral axis from the bottom or the top plate, respectively

α : number of side girders,

β : number of stiffeners.

Putting $w = w_{11}(t) f(x,y)$, and if we assume that an infinitely small variation δw of the deflection w of the plate is produced by applying the principle of virtual displacement, then the three kinds of virtual work are obtained by the effects of inertia force, elasticity force and external impulsive force $F(t)$ per unit area of the plate, respectively, as given below:

The virtual work of inertia force $= -m' \dfrac{\partial^2 w_{11}}{\partial t^2} \delta w_{11} \displaystyle\int_0^a \int_0^b [f(x,y)]^2 dx dy$

(m': the mass per unit area of the plate)

The virtual work of elasticity force $= -\dfrac{\partial V}{\partial w_{11}} \delta w_{11}$

The virtual work of impulsive force

$$\left.\begin{aligned} &= \delta w_{11}\int_0^a\int_0^b F(t)\cdot f(x,y)\,dx\,dy \\ &= F(t)\,\delta w_{11}\int_0^a\int_0^b f(x,y)\,dx\,dy \end{aligned}\right\} \quad (2)$$

Equating the entire virtual work, which is the sum of three kinds of virtual work given in Eqs. (2), to zero we obtain the equation of motion:

$$-m'\frac{\partial^2 w_{11}}{\partial t^2}\delta w_{11}\int_0^a\int_0^b \{f(x,y)\}^2\,dx\,dy - \frac{\partial V_b}{\partial w_{11}}\delta w_{11} + F(t)\,\delta w_{11}\int_0^a\int_0^b f(x,y)\,dx\,dy = 0 \quad (3)$$

Let us consider the case of all supported edges. We must thus assume suitable expression w in order to satisfy the following requirements; such as w must vanish at the boundary and the bending moments along the boundary also vanish, moreover w is an even function of x and y as concluded from symmetry. As the deflection has a rapidly converging series if we use the double trigonometric series for x and y, only the first one term of the series will be taken with sufficient accuracy. Therefore we can find w from Fig. 1

$$w = w_{11}(t)f(x,y) = w_{11}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (4)$$

In Eq. (4) $w_{11}(t)$ is a time function which will be determined later. After substituting Eq. (4) for w into Eq. (1) we have

$$V_b = K_b\left[w_{11}(t)\right]^2 \quad (5)$$

where

$$K_b = \frac{E\pi^4 ab}{8}\left[\frac{i_x}{a^4} + \frac{i_y}{b^4} + \frac{2\, i_{xy}}{a^2 b^2 (1-\mu^2)}\right] - \frac{\pi^4 (I_0 - I_g)}{2a} \tag{6}$$

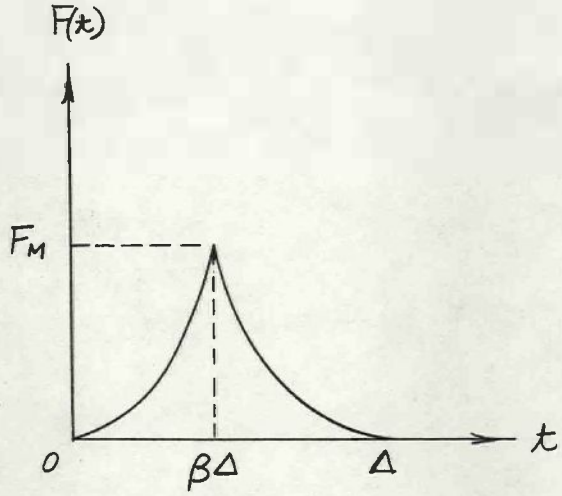
Then the equation of motion Eq. (3) that $w_{11}(t)$ must satisfy reduces to

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p_0^2 w_{11}(t) = \frac{16\, F(t)}{m' \pi^2} \tag{7}$$

in which p_0 is the circular frequency of the fundamental mode of vibration and

$$p_0^2 = \frac{8 K_b}{m' a b}$$

Let us assume that at the initial instant $t = 0$ the plate is at rest in its position of static equilibrium and the duration of impact Δ, moreover the relationship between $F(t)$ and t as shown in Fig.2:



$$\left.\begin{aligned} F(t) &= F_M\left(\frac{t}{\beta\Delta}\right)^m \text{ when } 0 < t < \beta\Delta, \\ &= F_M\left(\frac{\Delta - t}{(1-\beta)\Delta}\right)^n \text{ when } \beta\Delta < t < \Delta, \\ &= 0 \quad \text{ when } \Delta < t, \end{aligned}\right\} \tag{8}$$

where m and n are any integers and F_M is the peak value of $F(t)$

Fig. 2. Relation between $F(t)$ and t

The solution of Eq. (7), therefore, becomes

$$w=\frac{16F_M}{E\pi^6}\left[\frac{i_x}{a^4}+\frac{i_y}{b^4}+\frac{2i_{xy}}{a^2b^2(1-\mu^2)}+\frac{4}{Ea^2b^2}(I_0-I_g)\right]^{-1}\left(\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right)R, \qquad (9)$$

in which R is called the response factor and the value except R in the right side indicates the static deflection. R is the function of natural frequency P_0 which is shown below:

$$\left.\begin{aligned}
R&=\frac{1}{(\beta\Delta P_0)^m}\left[\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(P_0t)^{m-2\nu}+R_1\right] \quad \text{when } 0<t<\beta\Delta,\\
R&=\frac{1}{(\beta\Delta P_0)^m}\left[\cos P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(P_0\beta\Delta)^{m-2\nu}+\sin P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^\nu\frac{m!}{(m-2\nu-1)!}(P_0\beta\Delta)^{m-2\nu-1}+R_1\right]\\
&\quad+\frac{1}{[P_0\Delta(1-\beta)]^n}\left[\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[(\Delta-t)P_0]^{n-2\nu}-\cos P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[(1-\beta)\Delta P_0]^{n-2\nu}\right.\\
&\quad\left.+\sin P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^\nu\frac{n!}{(n-2\nu-1)!}[(1-\beta)\Delta P_0]^{n-2\nu-1}\right] \quad \text{when } \beta\Delta<t<\Delta,\\
R&=\frac{1}{(\beta\Delta P_0)^m}\left[\cos P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(P_0\beta\Delta)^{m-2\nu}+\sin P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^\nu\frac{m!}{(m-2\nu-1)!}(P_0\beta\Delta)^{m-2\nu-1}+R_1\right]\\
&\quad+\frac{1}{[P_0\Delta(1-\beta)]^n}\left[-\cos P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[P_0\Delta(1-\beta)]^{n-2\nu}+\sin P_0(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^\nu\frac{n!}{(n-2\nu-1)!}[P_0\Delta(1-\beta)]^{n-2\nu-1}\right.\\
&\quad\left.+R_3\right] \quad \text{when } t>\Delta,
\end{aligned}\right\} \qquad (10)$$

and if m and n are odd numbers, R_1 and R_3 become

$$R_1 = (-1)^{\frac{m+1}{2}} m! \sin P_0 t,$$

$$R_3 = (-1)^{\frac{n+1}{2}} n! \sin P_0 (t-\Delta),$$

if m and n are even numbers they become

$$R_1 = (-1)^{\frac{m}{2}+1} m! \cos P_0 t,$$

$$R_3 = (-1)^{\frac{n}{2}} n! \cos P_0 (t-\Delta).$$

R for m, n = 1, 2 and 3 is explicitly shown in Appendix I. Similarly, in the case of both clamped sides and both supported sides as shown in Fig. 3, we can easily obtain [6], [7]

$$w = A_{11} F_{11}(x) \left(\sin \frac{\pi y}{b}\right)(R) \tag{11}$$

in which R is given in Eq. (10), and A_{11} and $F_{11}(x)$ will be given later in numerical calculation.

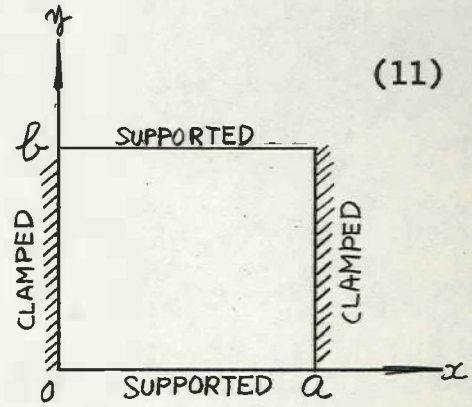


Fig. 3 Boundary Conditions

If we want more accurate value we may use the series expression containing higher modes of vibration:

$$w = \sum_{i=1,3,5\cdots}^{\infty} \sum_{j=1,3,5\cdots}^{\infty} A_{ij} F_i(x) R_{ij} \sin \frac{\pi j y}{b} \tag{12}$$

in which A_{ij}, $F_i(x)$ will be given in numerical calculation and R_{ij} is the response factor. Including even number of i or j, R_{ij} is given as follows:

$$R_{ij}=\frac{1}{(\beta\Delta P_{ij})^m}\left[\sum_{\nu=0}^{\frac{m}{2}}(-1)^{\nu}\frac{m!}{(m-2\nu)!}(P_{ij}t)^{m-2\nu}+R_1\right] \qquad \text{when } 0<t<\beta\Delta,$$

$$R_{ij}=\frac{1}{(\beta\Delta P_{ij})^m}\left[\cos P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^{\nu}\frac{m!}{(m-2\nu)!}(P_{ij}\beta\Delta)^{m-2\nu}+\sin P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^{\nu}\frac{m!}{(m-2\nu-1)!}(P_{ij}\beta\Delta)^{m-2\nu-1}+R_1\right]$$

$$+\frac{1}{[P_{ij}\Delta(1-\beta)]^n}\left[\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(\Delta-t)P_{ij}]^{n-2\nu}-\cos P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(1-\beta)\Delta P_{ij}]^{n-2\nu}\right.$$

$$\left.+\sin P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[(1-\beta)\Delta P_{ij}]^{n-2\nu-1}\right] \qquad \text{when } \beta\Delta<t<\Delta,$$

$$R_{ij}=\frac{1}{(\beta\Delta P_{ij})^m}\left[\cos P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^{\nu}\frac{m!}{(m-2\nu)!}(P_{ij}\beta\Delta)^{m-2\nu}+\sin P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^{\nu}\frac{m!}{(m-2\nu-1)!}(P_{ij}\beta\Delta)^{m-2\nu-1}+R_1\right]$$

$$+\frac{1}{[(1-\beta)\Delta P_{ij}]^n}\left[-\cos P_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[P_{ij}\Delta(1-\beta)]^{n-2\nu}\right.$$

$$\left.+\sin p_{ij}(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[P_{ij}\Delta(1-\beta)]^{n-2\nu-1}+R_3\right] \qquad \text{when } t>\Delta, \qquad (13)$$

and if m and n are odd numbers, R_1 and R_3 become

$$R_1=(-1)^{\frac{m+1}{2}}\,m!\sin P_{ij}t,$$

$$R_3=(-1)^{\frac{n+1}{2}}\,n!\sin P_{ij}(t-\Delta),$$

if m and n are even numbers they become

$$R_1 = (-1)^{\frac{m}{2}+1} m!\, \mathrm{Coo}\, P_{ij}\, t\,,$$

$$R_3 = (-1)^{\frac{n}{2}}\, n!\, \mathrm{Coo}\, P_{ij}\,(t-\Delta).$$

R_{ij} is shown in Appendix I for the k th period.

Using Eq. (12) for w, stresses are thus given by

$$\sigma_{bx} = \frac{EZ_b}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right), \qquad \sigma_{tx} = -\frac{EZ_t}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right),$$

$$\sigma_{by} = \frac{EZ_b}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right), \qquad \sigma_{ty} = -\frac{EZ_t}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right),$$

$$\tau_b = 2GZ_b \frac{\partial^2 w}{\partial y^2}, \qquad \tau_t = -2GZ_t \frac{\partial^2 w}{\partial x \partial y},$$

where $\sigma_{bx}, \sigma_{by}, \sigma_{tx}, \sigma_{ty}$ are normal stresses on the top and bottom plates in x or y direction and τ_b, τ_t are shear stresses on the top and bottom plates.

Since the stresses are space derivatives of the deflection, it may be concluded from Eqs. (12) and (14) that the dynamic response, deflection or stress, is equal to the static deflection or stress multiplied by the response factor. The maximum value of the response factor R is known as the load factor and is designated by L, hence we can determine the maximum value of deflection or stress by using L as shown in numerical calculation.

In general, only the first one term of series is enough to calculate the deflection. Therefore in order to obtain the maximum deflection quickly, it will be necessary for us just to

calculate the fundamental mode of vibration due to arbitrary end fixities at the surrounding edges of the plate, after the load factor was decided.

2-2. Frequency of the Fundamental Mode of Vibration in Stiffened Plate

In the vibration of the stiffened plate, the mode of vibration can be expressed by the Fourier's double series whatever mode it may be. If the plate supported at surrounding edges with arbitrary end fixities in only one direction, say the y direction, and simply supported in the other direction, say the x direction, the mode w of vibration is given by Inglis and Corlett [6] [7]

$$w=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} F_i(y)\sin\frac{j\pi x}{a}\phi_{ij}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(x,y)\phi_{ij}, \tag{15}$$

where ϕ_{ij} is the function of time, $F_i(y)$ is the basic function defined by Inglis, and a is the length of one side in x direction of the given rectangular plate. In Eq. (15) $F_i(y)$ is chosen to suit any end fixities and consists of a combination of hyperbolic and trigonometric functions, for which design curves for many cases must be prepared. Such curves were produced in reference [7]. Another approach to the mode of vibration w, however, may be easily done by applying the deflection of a beam due to the uniformly distributed loads with arbitrary end condition, i.e.

$$w=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(x,y)\,\phi_{ij}(t)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}\left[\left(\frac{x}{a}\right)^4+\left(\varphi_A-\varphi_B-2\right)\left(\frac{x}{a}\right)^3\right.$$

$$\left.+\left(\varphi_B-2\varphi_A+1\right)\left(\frac{x}{a}\right)^2+\varphi_A\frac{x}{a}\right]\times\left[\left(\frac{y}{b}\right)^4+\left(f_A-f_B-2\right)\left(\frac{y}{b}\right)^3\right.$$

$$\left.+\left(f_B-2f_A+1\right)\left(\frac{y}{b}\right)^2+f_A\cdot\frac{y}{b}\right]\phi_{ij}(t),$$

$$i,j=0,\ 1,\ 2,\ 3,\ \cdots\cdots \qquad (16)$$

where f_A, φ_A etc. are coefficients indicating the end fixities (for example $f_A=f_B=1$ for both ends A, B, simply supported, $f_A=f_B=0$ for both ends fixed and $f_A=\frac{1}{2}$, $f_B=0$ for one end supported at A and another end fixed at B). The other condition refers to the text book written by Vedeler [5]. The same is satisfied for φ. a is the distance along x axis and b the distance along y axis. It is easily understood that Eq. (16) contains only simple algebraic functions instead of hyperbolic functions of Eq. (15); if sufficient accuracy is obtained by application of Eq. (16), this equation can be used for the calculation of frequency. Actually, so far as the fundamental mode of vibration is concerned, the result will be quickly obtained with sufficient accuracy, as shown later. In the case of higher modes, however, the labor of calculation required seems

the same. Therefore, for the primary calculation of the fundamental frequency in any local vibration of the ship's structure or similar structures, the use of Eq. (16) will be quite convenient, and especially in case of a symmetrical boundary condition.

In calculation of frequency of the stiffened plate, the Rayleigh-Ritz method is applied, yielding a frequency somewhat larger than the correct one. We assume first the mode of vibration w using Eq. (16)

$$w = f(x, y) \cos p_0 t \tag{17}$$

and we shall start from the simplest case (A)

(A) With one pair of stiffeners and one uniformly distributed load.

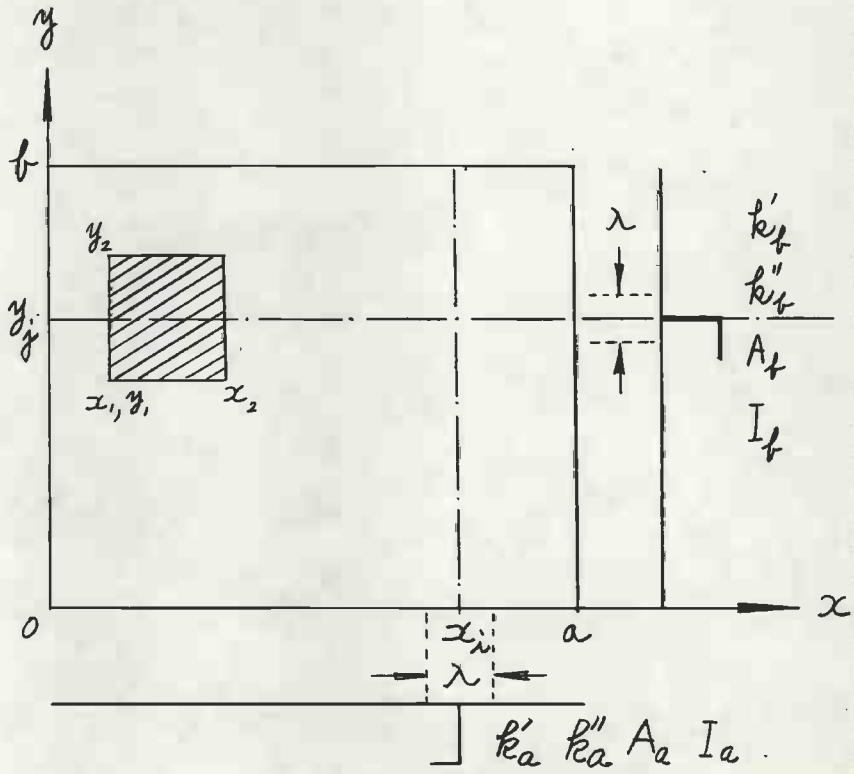


Fig. 4

Stiffened Plate

The maximum potential energy effected by bending of both the plate and two stiffeners, considering the effective breadth as defined by Schade [8] [9], is

$$V_1 = \frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right\}dx\,dy$$

$$+\frac{E}{2}\left[\int_0^b I_a\left(\frac{\partial^2 f(xy)_b}{\partial y^2}\right)^2_{x_i} dy + \int_0^a I_b\left(\frac{\partial^2 f(xy)_b}{\partial x^2}\right)^2_{y_j} dx\right] \tag{18}$$

Where I_a, I_b are the geometrical moments of inertia of the composite cross section in each stiffener with the effective breadth of the plate about the axis through the centroid of that composite cross section. λ is measured at the cross section of the maximum bending moment in the lengthwise direction. $f(xy)_b$ means the deflection due to bending only in the stiffener. μ is Poisson's ratio and $D=\frac{Eh^3}{12(1-\mu^2)}$ in which E is Yonng's modulus and h is the thickness of the plate. Neglecting the effect of warping, the maximum potential energy in stiffeners effected by both shearing force and torsion is

$$V_2 = \frac{G}{2}\left[\int_a^b k''_a A_a\left(\frac{\partial f(xy)_s}{\partial y}\right)^2_{x_i} dy + \int_0^a k''_b A_b\left(\frac{\partial f(x,y)_s}{\partial x}\right)^2_{y_j} dx\right.$$

$$\left.+\int_0^b k'_a\left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right)^2_{x_i} dy + \int_0^a k'_b\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2_{y_j} dx\right],$$

(19)

where G is the modulus of elasticity in shear. k'' is a numerical factor, depending on the shape of the cross section, and always less than 1.2 as calculated by Watanabe [10]. A is the cross-sectional area of each composite stiffener with the effective breadth at the maximum bending section. k' is the torsion constant and for the circular section k' is the polar moment of inertia generally designated as J. For other sections k' refers to the text book by Seely [11] as an example. In our case, however, the center of torsion is located at the cross point of a stiffener with the plate, hence, the modification of k' should be done by shifting the centroid of the cross-section to that cross point. Therefore, the total maximum potential energy stored during vibration by this stiffened plate is given by adding V_2 to V_1 already obtained above.

$$
\begin{aligned}
V = V_1 + V_2 = {} & \frac{E}{2}\left[\int_0^b I_a\left(\frac{\partial^2 f(x,y)_b}{\partial y^2}\right)^2_{x_i} dy + \int_0^a I_b\left(\frac{\partial^2 f(x,y)_b}{\partial x^2}\right)^2_{y_j} dx\right] \\
& + \frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right\} dx\,dy \\
& + \frac{G}{2}\left[\int_0^b k''_a A_a\left(\frac{\partial f(x,y)_s}{\partial y}\right)^2_{x_i} dy + \int_0^a k''_b A_b\left(\frac{\partial f(x,y)_s}{\partial x}\right)^2_{y_j} dx\right. \\
& \left. + \int_0^b k'_a\left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right)^2_{x_i} dy + \int_0^a k'_b\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2_{y_j} dx\right]
\end{aligned}
\tag{20}
$$

According to the textbook written by Timoshenko [12], the effect due to shearing force in the beam is only 1.3% compared to that due to bending even though the wave length of the vibration is ten times as large as the depth of the beam, and that due to torsion is always less than that due to shearing force. So far as the lowest mode of vibration is concerned, as in our present case, those effects of shearing force and torsion on the total maximum potential energy which are designated as V_2 may be neglected, and the total maximum potential energy V is approximately represented by only V_1 given by Eq. (18) after changing $f(x,y)_b$ to $f(x,y)$ which yields

$$V=\frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2+\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2+2\mu\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2\right\}dx\,dy$$
$$+\frac{E}{2}\left[\int_0^b I_a\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2_{x_i}dy+\int_0^a I_b\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2_{y_j}dx\right] \tag{21}$$

On the other hand, the maximum kinetic energy of this stiffened plate due to bending of the plate and stiffeners is

$$T_1=\frac{\rho h}{2}p_0^2\int_0^a\int_0^b\left[f(x,y)\right]^2dx\,dy+\frac{p_0^2}{2g}\left\{\int_0^b q_a\left[f(x,y)\right]^2_{x_i}dy\right.$$
$$\left.+\int_0^a q_b\left[f(x,y)\right]^2_{y_j}dx\right\}, \tag{22}$$

where ρh is the mass per unit area of the plate. g_a, g_b are the weights of composite stiffeners per unit length which means the stiffener considered with the effective breadth of the plate.

The maximum kinetic energy of stiffeners due to rotational inertia in their lengthwise direction and inertia force of rotation in the plane perpendicular to their lengthwise direction is

$$T_2 = \frac{p_0^2}{2g}\left[\int_0^b g_a I_a' \left(\frac{\partial f(x,y)}{\partial y}\right)^2_{x_i} dy + \int_0^a g_b I_b' \left(\frac{\partial f(x,y)}{\partial x}\right)^2_{y_j} dx + \int_0^b g_a I_{pa} \left(\frac{\partial f(x,y)}{\partial x}\right)^2_{x_i} dy + \int_0^a g_b I_{pb} \left(\frac{\partial f(x,y)}{\partial y}\right)^2_{y_j} dx\right], \tag{23}$$

where I' is the moment of inertia of the unit element in the lengthwise direction of the stiffener about axis through its center of gravity perpendicular to the lengthwise direction. I_p is the centroid polar moment of inertia of the cross-section, and in this case the centroid is located at the cross point of the stiffener with the plate.

The maximum kinetic energy due to one uniformly distributed weight g_E per unit area is

$$T_3 = \frac{p_0^2}{2g}\int_{y_1}^{y_2}\int_{x_1}^{x_2} g_E [f(x,y)]^2 dx\,dy \tag{24}$$

To get the total kinetic energy the amount of energy stored due to rotational inertia and inertia force of rotation which was designated as T_2 should be discussed further. The effect of rotational inertia is about 1/3.2 in comparison with that of shearing force, according to the text book by Timoshenko [12] also, when the wave length is ten times as large as the depth of the beam. The effect due to the inertia force of rotation is always less than that due to rotational inertia. Therefore, the total kinetic energy T is represented by adding T_3 to T_1:

$$T = \frac{p_0^2}{2g}\left[g\rho h\int_0^a\int_0^b [f(x,y)]^2\,dx\,dy + \int_{y_1}^{y_2}\int_{x_1}^{x_2} q_E[f(x,y)]^2\,dx\,dy + \int_0^b q_a[f(x,y)]^2_{x_i}\,dy + \int_0^a q_b[f(x,y)]^2_{y_j}\,dx\right] \tag{25}$$

Assuming no loss in energy occurs, p_0^2 is given by equating the maximum kinetic energy to the maximum potential energy designated as T of Eq. (25) or V of Eq. (20) respectively:

$$p_0^2 = \frac{2gV}{g\rho h\int_0^a\int_0^b [f(x,y)]^2\,dx\,dy + \int_0^b [f(x,y)]^2_{x_i} q_a\,dy + \int_0^a q_b[f(x,y)]^2_{y_j}\,dx + \int_{y_1}^{y_2}\int_{x_1}^{x_2} [f(x,y)]^2 q_E\,dx\,dy} \tag{26}$$

According to the Rayleigh-Ritz method [12], if Eq. (16) is taken for $f(x,y)$ then it is only necessary to determine the

coefficient $a_{ij}\,(i,j=0,1,2,3\cdots\cdots)$ in such a manner as to make the right member of Eq. (26) a minimum. In this way we arrive at a system of equations such as

$$\frac{\partial}{\partial a_{ij}}\Bigg[D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2+\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2+2\mu\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2\right\}dx\,dy$$

$$+E\left\{\int_0^b I_a\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2_{x_i}dy+\int_0^a I_b\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2_{y_j}dx\right\}-\frac{p_0^2}{g}\left\{g\rho h\int_0^a\int_0^b[f(x,y)]^2dx\,dy\right.$$

$$\left.+q_a\int_0^b[f(x,y)]^2_{x_i}dy+q_b\int_0^a[f(x,y)]^2_{y_j}dx+q_E\int_{y_1}^{y_2}\int_{x_1}^{x_2}[f(x,y)]^2dx\,dy\right\}\Bigg]=0, \tag{27}$$

which are linear with respect to the constants a_{ij}. By equating the determinant of these equations to zero the frequencies of various modes of vibration can be approximately calculated. The accuracy of the Ritz method was discussed by Tomotika [13].

The discussion explained above is further expanded to the general case (B).

(B). A plate having r stiffeners with scantlings, ${}_ik_a'$, ${}_ik_a''$, ${}_iI_a$, ${}_iA_a$ parallel to the y direction and q girders with scantlings, ${}_jI_b$, ${}_jA_b$, ${}_jk_b''$, ${}_jk_b'$ parallel to x direction together with S uniformly distributed loads, ${}_kq_E$ and, incidentally, supported by n pillars having the cross sectional area ${}_lA_p$ and length ${}_ll_p$.

Adding the effect due to pillars to the maximum strain energy Eq. (21), we have

$$V = \frac{D}{2}\int_0^a\int_0^b \left\{ \left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \right\} dx\,dy$$

$$+ \frac{E}{2}\left[\sum_{i=1}^{r} {}_iI_a \int_0^b \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2_{x_i} dy + \sum_{j=1}^{q} {}_jI_b \int_0^a \left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2_{y_j} dx + \sum_{l=1}^{n} \frac{{}_lA_p}{{}_ll_p}\left[f(x,y)\right]^2_{x_l\,y_l} \right] \tag{28}$$

And the maximum kinetic energy is

$$T = \frac{p_0^2}{2g}\left[g\rho h \int_0^a\int_0^b f^2(x,y)\,dx\,dy + \sum_{i=1}^{r} {}_ig_a \int_0^b \left[f(x,y)\right]^2_{x_i} dy \right.$$

$$\left. + \sum_{j=1}^{a} {}_jg_b \int_0^a \left[f(x,y)\right]^2_{y_j} dx + \sum_{k=1}^{s} {}_kg_E \int_{y_k'}^{y_k''}\int_{x_k'}^{x_k''} f^2(x,y)\,dx\,dy \right] \tag{29}$$

The equation corresponding to Eq. (27) therefore, in this case is

$$\frac{\partial}{\partial a_{ij}}\left[D\int_0^a\int_0^b \left\{ \left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \right\} dx\,dy \right.$$

$$+ E\left\{ \sum_{i=1}^{r} {}_iI_a \int_0^b \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2_{x_i} dy + \sum_{j=1}^{q} {}_jI_b \int_0^a \left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2_{y_j} dx \right.$$

$$\left. + \sum_{l=1}^{n} \frac{{}_lA_p}{{}_ll_p}\left[f(x,y)\right]^2_{x_l\,y_l} \right\}$$

$$- \frac{p_0^2}{g}\left\{ g\rho h \int_0^a\int_0^b f^2(x,y)\,dx\,dy + \sum_{i=1}^{r} {}_ig_a \int_0^b \left[f(x,y)\right]^2_{x_i} dy \right.$$

$$\left. \left. + \sum_{j=1}^{a} {}_jg_b \int_0^a \left[f(x,y)\right]^2_{y_j} dx + \sum_{k=1}^{s} {}_kg_E \int_{y_k'}^{y_k''}\int_{x_k'}^{x_k''} f^2(x,y)\,dx\,dy \right\} \right] = 0 \tag{30}$$

A satisfactory approximation for the frequency of the fundamental mode of vibration will be obtained by taking Eq. (16) for the mode of vibration $f(x,y)$. We may call the frequency the first approximate one only if the first term of Eq. (16) is chosen, the second approximate one if the first two terms of Eq. (16) are chosen, and so on. Determination of the first approximate frequency is the same as the Rayleigh's method which means to equate two maximum energies, potential of kinetic, to each other. Only the first term of Eq. (16) is chosen in this paper to obtain the frequency of the lowest mode of vibration, because a sufficiently accurate result will be obtained for the stiffened plate as shown in numerical calculation, i.e.

$$f(x,y) \equiv a_{00} f(x) f(y)$$

$$= a_{00}\left[\left(\frac{x}{a}\right)^4 + (\varphi_A - \varphi_B - 2)\left(\frac{x}{a}\right)^3 + (\varphi_B - 2\varphi_A + 1)\left(\frac{x}{a}\right)^2 + \varphi_A \frac{x}{a}\right]$$

$$\times \left[\left(\frac{y}{b}\right)^4 + (f_A - f_B - 2)\left(\frac{y}{b}\right)^3 + (f_B - 2f_A + 1)\left(\frac{y}{b}\right)^2 + f_A \frac{y}{b}\right] \qquad (31)$$

After substituting Eq. (31) for $f(x,y)$ into Eq. (30) p_o^2 is obtained:

$$p_0^2 = \frac{Eh^2}{\rho a^2 b^2}\left[\frac{1}{12(1-\mu^2)}\left(\frac{b^2}{a^2}\cdot\frac{S_2}{S_1} + \frac{a^2}{b^2}\cdot\frac{S_2'}{S_1'} + 2\frac{S_3}{S_1}\cdot\frac{S_3'}{S_1'}\right)\right.$$

$$\left. + \frac{1}{S_1}\cdot\frac{S_2'}{S_1'}\,\frac{a\sum_{i=1}^{r} i I_a[f(x_i)]^2}{h^3 b^2} + \frac{1}{S_1'}\frac{S_2}{S_1}\,\frac{b\sum_{j=1}^{q} j I_b[f(y_j)]^2}{h^3 a^2} + \frac{1}{S_1 S_1'}\,\frac{ab\sum_{k=1}^{n}\frac{eA_p}{el_p}[f(x,y)]_{x_k y_k}}{h^3}\right]$$

$$\times\left[1 + \frac{\sum_{i=1}^{r} i g_a[f(x_i)]^2}{g\rho h a} + \frac{1}{S_1'}\cdot\frac{\sum_{j=1}^{q} j g_b[f(y_i)]^2}{g\rho h b} + \frac{1}{S_1 S_1'}\,\frac{\sum_{k=1}^{s} k g_E F_k}{g\rho h a b}\right]^{-1} \tag{32}$$

Therefore, the frequency is

$$f = \frac{p_0}{2\pi} \quad c.p.s. \tag{33}$$

where

$$S_1 = \frac{1}{105}(\varphi_A^2+\varphi_B^2) + \frac{1}{70}\varphi_A\varphi_B + \frac{1}{140}(\varphi_A+\varphi_B) + \frac{1}{630}$$

$$S_2 = 4\left[\frac{1}{5} + \varphi_A^2 + \varphi_B^2 - \varphi_A\varphi_B\right]$$

$$S_3 = \frac{1}{15}\left[2(\varphi_A^2+\varphi_B^2) + \varphi_A\varphi_B + \varphi_A + \varphi_B + \frac{2}{7}\right]$$

$$S_1' = \frac{1}{105}\left[(f_A^2+f_B^2) + \frac{1}{70}f_A f_B + \frac{1}{140}(f_A+f_B) + \frac{1}{630}\right]$$

$$S_2' = 4\left[\frac{1}{5} + f_A^2 + f_B^2 - f_A f_B\right]$$

$$S_3' = \frac{1}{15}\left[2(f_A^2+f_B^2) + f_A f_B + f_A + f_B + \frac{2}{7}\right]$$

$$F_k = \int_{y_{k'}}^{y_{k''}} \int_{x_{k'}}^{x_{k''}} f^2(x,y)\,dx\,dy$$

$$= ab\Big[\frac{1}{9}\left(\frac{x}{a}\right)^9 + \frac{1}{4}(\varphi_A - \varphi_B - 2)\left(\frac{x}{a}\right)^8 + \frac{1}{7}\left\{(\varphi_A - \varphi_B - 2)^2 + 2(\varphi_B - 2\varphi_A + 1)\right\}\left(\frac{x}{a}\right)^7$$

$$+ \frac{1}{3}\left\{\varphi_A + (\varphi_A - \varphi_B - 2)(\varphi_B - 2\varphi_A + 1)\right\}\left(\frac{x}{a}\right)^6 + \frac{1}{5}\left\{(\varphi_B - 2\varphi_A - 2)^2 + 2\varphi_A(\varphi_A - \varphi_B - 2)\right\}\left(\frac{x}{a}\right)^5$$

$$+ \frac{1}{2}\varphi_A(\varphi_B - 2\varphi_A + 1)\left(\frac{x}{a}\right)^4 + \frac{1}{3}\varphi_A^2\left(\frac{x}{a}\right)^3\Big]_{x_{k'}}^{x_{k''}}$$

$$\times\Big[\frac{1}{9}\left(\frac{y}{b}\right)^9 + \frac{1}{4}(f_A - f_B - 2)\left(\frac{y}{b}\right)^8 + \frac{1}{7}\left\{(f_A - f_B - 2)^2 + 2(f_B - 2f_A + 1)\right\}\left(\frac{y}{b}\right)^7$$

$$+ \frac{1}{3}\left\{f_A + (f_A - f_B - 2)(f_B - 2f_A + 1)\right\}\left(\frac{y}{b}\right)^6 + \frac{1}{5}\left\{(f_B - 2f_A + 1)^2 + 2f_A(f_A - f_B - 2)\right\}\left(\frac{y}{b}\right)^5$$

$$+ \frac{1}{2}f_A(f_B - 2f_A + 1)\left(\frac{y}{b}\right)^4 + \frac{1}{3}f_A^2\cdot\left(\frac{y}{b}\right)^3\Big]_{y_{k'}}^{y_{k''}} \qquad (34)$$

Similarly another case is discussed here in (C).

(C) A plate having equally spaced stiffeners with equal scantlings parallel to y axis and q girders parallel to x axis, together with i web stiffeners and j large girders parallel to y or x axis, respectively as shown in Fig. 5.

If there are many stiffeners and girders together with a few web stiffeners and large girders, the effects due to web stiffeners and large girders are only added to those of stiffeners or equally spaced girders after substituting the following relations used by Svennerud [14] into Eq. (32) we can get p_o^2

$$\sum_{i=1}^{r} i I_a \left[f(x_i) \right]^2 = I_a \frac{r+1}{a} \int_0^a \left[f(x) \right]^2 dx = I_a (r+1) S_1$$

$$\sum_{j=1}^{q} j I_b \left[f(y_j) \right]^2 = I_b \frac{q+1}{b} \int_0^b \left[f(y) \right]^2 dy = I_b (q+1) S_1'$$

$$\sum_{i=1}^{r} i q_a \left[f(x_i) \right]^2 = q_a (r+1) S_1$$

$$\sum_{j=1}^{q} j q_b \left[f(y_j) \right]^2 = q_b (q+1) S_1' \qquad (35)$$

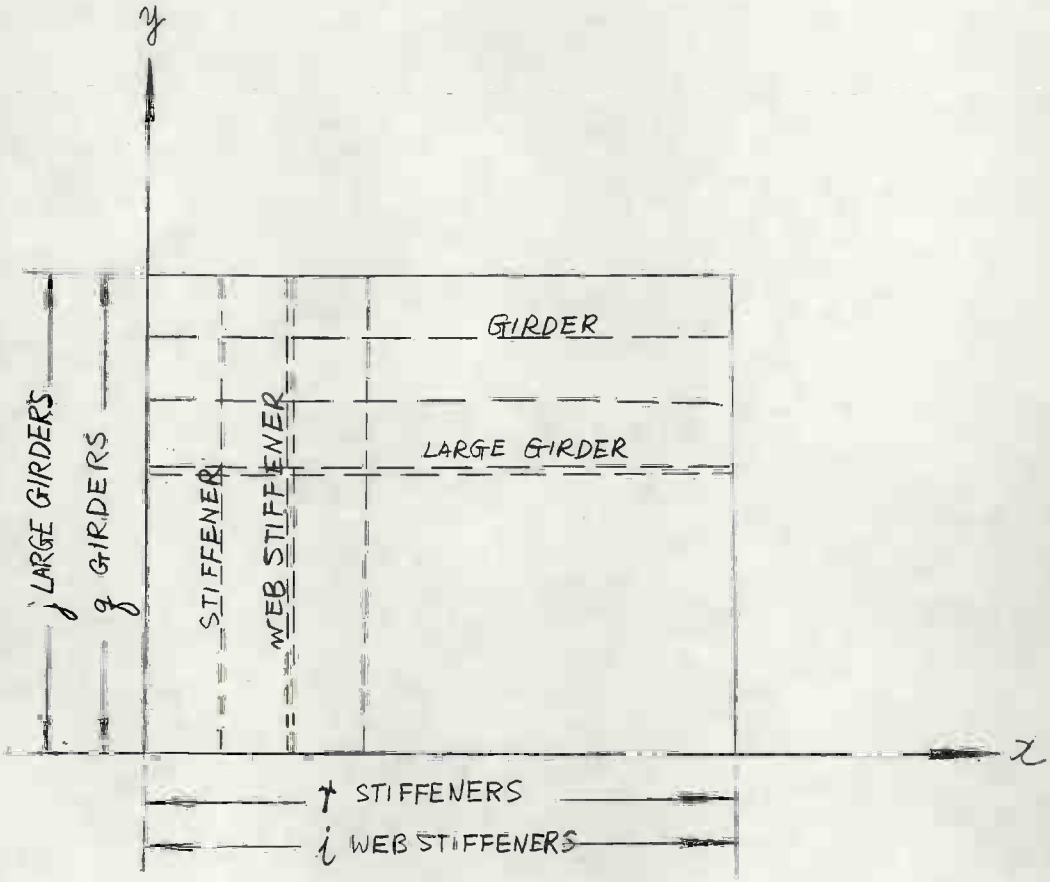


Fig. 5 Stiffened Plate

$$
\begin{aligned}
P_0^2 = \frac{Eh^2}{\rho a^2 b^2}\Bigg[\Bigg[& \frac{1}{12(1-\mu^2)}\left\{\frac{b^2}{a^2}\frac{S_2}{S_1} + \frac{a^2}{b^2}\frac{S_2'}{S_1'} + 2\frac{S_3}{S_1}\frac{S_3'}{S_1'}\right\} + \frac{S_2'}{S_1'}(r+1)\frac{aI_a}{h^3 b^2} \\
& + \frac{1}{S_1}\frac{S_2'}{S_1'}\frac{a\sum_i({}_iI_a - I_a)[f(x_i)]^2}{h^3 b^2} + \frac{S_2}{S_1}(q+1)\frac{bI_b}{h^3 a^2} \\
& + \frac{1}{S_1'}\frac{S_2'}{S_1}\frac{b\sum_j({}_jI_b - I_b)[f(y_j)]^2}{h^3 a^2} + \frac{1}{S_1 S_1'}\frac{ab}{h^3}\sum_{i=1}^{n}\frac{eA_p}{el_p}[f(x,y)]^2_{x_e y_e} \\
\Bigg]\Bigg] \times \Bigg[\Bigg[& 1 + \frac{q_a(r+1)}{g\rho h a} + \frac{1}{S_1}\frac{\sum_i({}_iq_a - q_a)[f(x_i)]^2}{g\rho h a} + \frac{q_b(q+1)}{g\rho h b} \\
& + \frac{1}{S_1'}\frac{\sum_j({}_jq_b - q_b)[f(y_j)]^2}{g\rho h b} + \frac{1}{S_1 S_1'}\frac{\sum_{k=1}^{s} {}_kq_E F_k}{g\rho h a b}\Bigg]\Bigg]^{-1}
\end{aligned}
\tag{36}
$$

Here, ${}_iI_a$ or ${}_jI_b$ is the moment of inertia of web stiffeners or large girders considered with the effective breadth of the plate at the section of the maximum bending moment, and ${}_iq_a$ or ${}_jq_b$ is the weight per unit length for each web stiffener or large girder considered with the effective breadth of the plate.

The last case is given for the double bottom of a ship as shown in (D).

(D) Double bottom of ship.

The double bottom considered as the composite structures of

both top and bottom plates as many cross stiffeners was discussed by Schade and Vedeler, and the strain energy was given as the orthotropic plate. In such a structure, a shear lag must be considered (as was discussed already by Anderson [15] in the aircraft structure), because it has the tendency to decrease the frequency considerably.

So far as the double bottom is concerned, however, it may not be necessary to consider the effect of shear lag. Before going into the discussion about the frequency of the lowest mode of vibration the following assumptions are clearly made:

(1) The double bottom has a uniformly distributed weight and load per unit area.

(2) The weights of the individual masses of machinery are uniformly distributed over their bases.

(3) The structure is regular and parallel stiffeners are identical.

The strain energy obtained by Vedeler [5] will be used and for the maximum strain energy it becomes

$$V = \frac{E}{2}\left\{\int_0^a\int_0^b\left[i_x\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2 + i_y\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)^2 + \frac{2\mu}{1-\mu}i_{xy}\frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial^2 f(x,y)}{\partial x^2}\right.\right.$$
$$\left.\left. + \frac{2}{1+\mu}\,x\,i_y\left(\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\right)^2\right]dx\,dy + \int_0^a (I_0 - I_g)\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)^2_{y=\frac{b}{2}} dx\right\} \tag{37}$$

where i_x and so on are already explained in Eq. (1).

The maximum kinetic energy stored by the double bottom, machinery and pillars is

$$T = \frac{p_0^2}{2g}\left[\rho' g \int_0^a\int_0^b f(x,y)\,dx\,dy + a_{00}^2 \sum_{k=1}^{s} {}_k g_E F_k + \sum_{l=1}^{n} P_l \left[V_0\right]_l^2\right] \tag{38}$$

where $\rho' g$ is the uniform load per unit area of double bottom, P_l is the weight of the pillar at points (x_l, y_l) and ${}_k g_E$ F_k are already defined. The energy due to pillars in this case is different from that of the case (C), because in the case of double bottom all pillars are supported by the plate which means the effects of pillars on the plate are only considered as the energy stored by the concentrated weights of pillars themselves.

After equating Eqs. (37) and (38) p_0^2 is found to be

$$p_0^2 = \frac{E\, i_x}{\rho' a^2 b^2}\, \frac{\dfrac{S_2}{S_1}\dfrac{b^2}{a^2} + \dfrac{i_y}{i_x}\dfrac{a^2}{b^2}\dfrac{S_2'}{S_1'} + \dfrac{S_3 S_3'}{S_1 S_1'}\left(\dfrac{2\mu}{1-\mu^2}\dfrac{i_{xy}}{i_x} + \dfrac{2}{1+\mu}\dfrac{i_y}{i_x}\right)}{1 + \dfrac{1}{S_1 S_1'}\dfrac{\sum_{k=1}^{s} {}_k g_E F_k}{g \rho' a b} + \dfrac{1}{S_1 S_1'}\dfrac{\sum_{l=1}^{n} P_l \left[f(x) f(y)\right]_l^2}{g \rho' a b}}, \tag{39}$$

where S_1, S_2, S_3, or S_1', S_2', S_3' are already given in Eq. (34).

In order to demonstrate the use of the formulas obtained above, the following example is given to be amenable to the practical calculation.

Example 1—Plate simply supported at all sides, with no load or pillars.

In this case, from Eq. (16) the end fixities are:

$f_A = f_B = \varphi_A = \varphi_B = 1$, from Eq. (34) we have

$$S_1 = S_1' = \frac{31}{630}$$

$$S_2 = S_2' = \frac{24}{5}$$

$$S_3 = S_3' = \frac{51}{105}$$

Therefore, after substituting these values for S into Eq. (32) p_0^2 becomes

$$p_0^2 = \frac{D}{h\rho}\,\frac{1}{a^2b^2}\left\{\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)\frac{126\times 24}{31} + 2\left(\frac{306}{31}\right)^2\right\}.$$

When $a = b$

$$p_0 = \frac{19.7476}{a^2}\sqrt{\frac{D}{\rho h}}$$

When $b = 1.5a$

$$p_0 = \frac{21.3942}{a^2}\sqrt{\frac{D}{\rho h}}$$

If however, the double series of trigonometric function is chosen as the mode of vibration as shown in the textbook by Timoshenko [12], we have

$$p_0^2 = \frac{D}{\rho h}\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2.$$

Hence, when $a=b$, $p_0 = \frac{19.7392}{a^2}\sqrt{\frac{D}{\rho h}}$ which is 0.0426% less than the first result above. When $b = 1.5\,a$, $p = \frac{21.3841}{a^2}\sqrt{\frac{D}{\rho h}}$ which is 0.0472% less than the second result above.

2-3. Local Vibration of a Bottom Plate

We discuss here the general problem such as a plating panel having three planes of symmetry with respect to its elastic properties, in which the bottom plate is considered as one special case.

Considering both effects due to bending and stretching, the total strain energy V of a plating panel $a_0 \times b_0$ (Fig. 6) is

$$V = V_b + V_s \tag{40}$$

in which V_b is the strain energy due to bending and V_s one due to stretching of the plate.

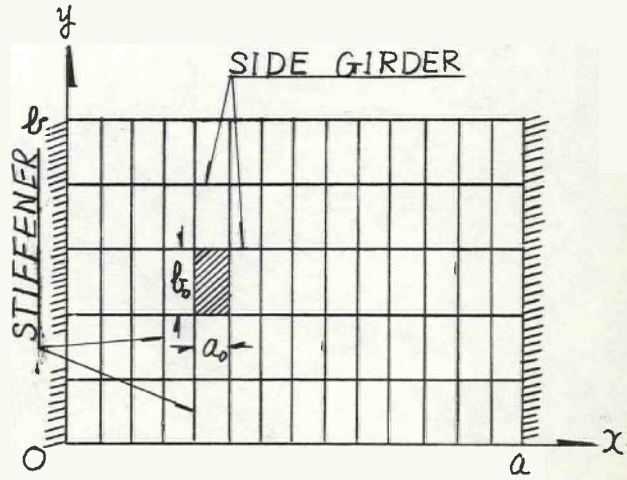


Fig. 6 Double Bottom

We now assume the following relations between stress and strain components for a case of plane stress in the xy-plane:

$$\left.\begin{aligned} \sigma_x &= E_x'\varepsilon_x + E''\varepsilon_y \\ \sigma_y &= E''\varepsilon_x + E_y'\varepsilon_y \\ \tau_{xy} &= G\gamma_{xy} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \varepsilon_x &= \frac{E''\sigma_y - E_y'\sigma_x}{E''^2 - E_x'E_y'} \\ \varepsilon_y &= \frac{E''\sigma_x - E_x'\sigma_y}{E''^2 - E_y'E_x'} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned}\right\}, \tag{41}$$

by taking these planes of symmetry as the coordinate planes, in which $\sigma_x, \sigma_y, \tau_{xy}$: stress; $\varepsilon_x, \varepsilon_y, \gamma_{xy}$: strain; E_x', E_y', E'', G: coefficient characterizing the elastic properties of a material. Then we have

$$V_b = \frac{1}{2}\int_0^{a_0}\int_0^{b_0}\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + 4D_{xy}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]dx\,dy,$$

where $D_x = \frac{E_x' h^3}{12}$, $D_y = \frac{E_y' h^3}{12}$, $D_1 = \frac{E'' h^3}{12}$, $D_{xy} = \frac{G h^3}{12}$, h: plate thickness,

and

$$V_s = \frac{h}{2}\int_0^{a_0}\int_0^{b_0}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_{xy}\gamma_{xy}\right]dx\,dy, \tag{43}$$

which reduces to

$$V_s = \frac{E_x' h}{2}\int_0^{a_0}\int_0^{b_0}\left[\varepsilon_x^2 + \frac{E_y'}{E_x'}\varepsilon_y^2 + \frac{2E''}{E_x'}\varepsilon_x\varepsilon_y + \frac{G}{E_x'}\gamma_{xy}^2\right]dx\,dy \tag{44}$$

Putting $w = w_{11}(t) f(x, y)$ as before, and using the equivalent equation of motion to Eq. (3):

$$-m \frac{\partial^2 w_{11}}{\partial t^2} \delta w_{11} \int_0^{a_0} \int_0^{b_0} f^2(x,y)\, dx\, dy - \frac{\partial V}{\partial w_{11}} \delta w_{11} + F(t) \delta w_{11} \int_0^{a_0} \int_0^{b_0} f(x,y)\, dx\, dy = 0 \qquad (45)$$

We can get the equation of motion which $w_{11}(t)$ must satisfy.

Let us solve the case of all supported edges. We must then assume suitable expressions for the displacements $\bar{u}$ and $\bar{v}$ of the directions x and y, respectively, and w, in order to satisfy the following requirements; such as the all displacements $\bar{u}$, $\bar{v}$ and w must vanish at the boundary and the bending moments along the boundary also vanish, moreover w is an even function of x and y as concluded from symmetry, whereas $\bar{u}$ and $\bar{v}$ are odd functions of x and y, respectively. From the practical purposes of initial design, only the first term of the double trigonometric series will be taken with sufficient accuracy, because the deflection has a rapidly converging series. Therefore we can find $\bar{u}$, $\bar{v}$ and w from Fig. 6 as

$$\left.\begin{aligned} w &= w_{11}(t) f(x,y) = w_{11}(t) \sin \frac{\pi x}{a_0} \sin \frac{\pi y}{b_0} \\ \bar{u} &= u_{11}(t) \sin \frac{2\pi x}{a_0} \sin \frac{\pi y}{b_0} \\ \bar{v} &= v_{11}(t) \sin \frac{2\pi y}{b_0} \sin \frac{\pi x}{a_0} \end{aligned}\right\} \qquad (46)$$

In Eqs. (46) $u_{11}(t)$, $v_{11}(t)$ and $w_{11}(t)$ are time functions which will be determined later.

By the fact that the impulsive load does not work when u_{11} or v_{11} varies, the following two relations are obtained by using the principle of virtual works:

$$\frac{\partial V}{\partial u_{11}} = 0 , \qquad \frac{\partial V}{\partial v_{11}} = 0. \tag{47}$$

Hence, after having solutions of u_{11} , v_{11} and w_{11} which satisfy both Eq. (45) and Eq. (47), and by substituting these values for u_{11}, v_{11}, and w_{11} into Eq. (46) we can obtain all displacements $\bar{u}$, $\bar{v}$ and w from which stress and strain are easily determined.

Now let us return to calculate Eq. (42) by using w of Eq. (46). Thus we have

$$V_b = K_b w_{11}^2 , \tag{48}$$

where

$$K_b = \frac{a_0 b_0 \pi^4}{8}\left[\frac{D_x}{a_0^4} + \frac{D_y}{b_0^4} + \frac{2}{a_0^2 b_0^2}(D_1 + 2D_{xy})\right], \tag{49}$$

in which if the elastic properties of the material of plate are considered as same in all directions, we have

$$E_x' = E_y' = \frac{E}{1-\mu^2}, \quad E'' = \frac{\mu E}{1-\mu^2}, \quad G = \frac{E}{2(1+\mu)}, \tag{50}$$

from which

$$D_x = D_y = \frac{Eh^3}{12(1-\mu^2)}, \quad D_1 = \frac{\mu E h^3}{12(1-\mu^2)}, \quad D_{xy} = \frac{Eh^3}{24(1+\mu)}. \tag{51}$$

In such a case K_b becomes

$$K_b = \frac{a_0 b_0 \pi^4 D}{8}\left[\frac{1}{a_0^2} + \frac{1}{b_0^2}\right]^2. \tag{52}$$

Using the following relations between strain and displacement components of the large deflection:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_y &= \frac{\partial \bar{v}}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned}\right\} \tag{53}$$

and expressions of Eq. (46) for $\bar{u}$ and $\bar{v}$, we can find from Eq. (44)

$$V_s = \frac{E_x'}{2} h \Big[w_{11}^4(t)\left(\frac{9\pi^4 b_0}{256 a_0^3} + \frac{E_y'}{E_x'}\frac{9\pi^4 a_0}{256 b_0^3} + \frac{E''}{E_x'}\frac{\pi^4}{128 a_0 b_0} + \frac{G}{E_x'}\frac{\pi^4}{64 a_0 b_0}\right)$$

$$+ w_{11}^2(t)\cdot\Big\{ u_{11}(t)\left(\frac{2 b_0 \pi^2}{3 a_0^2} - \frac{E''}{E_x'}\frac{\pi^2}{3 b_0} + \frac{G}{E_x'}\frac{\pi^2}{3 b_0}\right) + v_{11}(t)\left(\frac{2E_y'}{E_x'}\frac{\pi^2 a_0}{3 b_0^2}\right.$$

$$\left. - \frac{E''}{E_x'}\frac{\pi^2}{3 a_0} + \frac{G}{E_x'}\frac{\pi^2}{3 a_0}\right)\Big\} + u_{11}^2(t)\left(\frac{b_0 \pi^2}{a_0} + \frac{a_0 \pi^2}{4 b_0}\frac{G}{E_x'}\right)$$

$$+ v_{11}^2(t)\left(\frac{E_y'}{E_x'}\frac{\pi^2 a_0}{b_0} + \frac{b_0\pi^2}{4a_0}\frac{G}{E_x'}\right) + \frac{32}{9}u_{11}(t)\,v_{11}(t)\frac{E''+G}{E_x'}\Bigg] \tag{54}$$

Therefore we can find the total energy V from Eq. (40) by substituting Eqs. (48), and (54) for V_b , and V_s , respectively.

Substituting V thus obtained into Eqs. (47), we have

$$\left.\begin{aligned} k_1 w_{11}^2(t) + k_2 u_{11}(t) + k_3 v_{11}(t) &= 0, \\ k_4 w_{11}^2(t) + k_5 v_{11}(t) + k_3 u_{11}(t) &= 0, \end{aligned}\right\} \tag{55}$$

from which $u_{11}(t)$ and $v_{11}(t)$ are obtained:

$$\begin{aligned} u_{11}(t) &= -k_u w_{11}^2(t) \\ v_{11}(t) &= -k_v w_{11}^2(t) \end{aligned} \tag{56}$$

where k_u, k_v : positive number

$$\begin{aligned} k_1 &= \frac{1}{2}\frac{E_x' h}{a_0}\left(\frac{2b_0\pi^2}{3a_0} - \frac{E''}{E_x'}\frac{a_0\pi^2}{3b_0} + \frac{Ga_0\pi^2}{E_x' 3b_0}\right) \\ k_2 &= E_x'\left(\frac{b_0\pi^2}{a_0} + \frac{a_0\pi^2 G}{4b_0 E_x'}\right)h, \quad k_3 = \frac{16}{9}(E''+G)h \\ k_4 &= \frac{1}{2}\frac{E_y'}{b_0}\left(\frac{2a_0\pi^2}{3b_0} - \frac{E''}{E_y'}\frac{b_0\pi^2}{3a_0} + \frac{G}{E_y'}\frac{b_0\pi^2}{3a_0}\right) \\ k_5 &= E_y'\left(\frac{\pi^2 a_0}{b_0} + \frac{b_0\pi^2}{4a_0}\frac{G}{E_y'}\right)h, \quad k_u = \frac{k_3 k_4 - k_1 k_5}{k_2 k_5 - k_3^2}, \quad k_v = \frac{k_1 k_3 - k_2 k_4}{k_2 k_5 - k_3^2} \end{aligned} \tag{57}$$

From Eqs. (46) $\bar{u}$ and $\bar{v}$ thus become

$$\left.\begin{aligned} \bar{u} &= -k_u W_{\parallel}^2(t) \sin\frac{2\pi x}{a_0} \sin\frac{\pi y}{b_0} \\ \bar{v} &= -k_v W_{\parallel}^2(t) \sin\frac{2\pi y}{b_0} \sin\frac{\pi x}{a_0} \end{aligned}\right\} \tag{58}$$

Using Eqs. (54) and (56), we can also obtain

$$V_s = K_s W_{\parallel}^4(t), \tag{59}$$

where

$$\begin{aligned} K_s = \frac{E_x'}{2} h \Bigg[& \frac{9\pi^4 b_0}{256 a_0^3} + \frac{E_y'}{E_x'} \frac{9\pi^4 a_0}{256 b_0^3} + \frac{E''}{E_x'} \frac{\pi^4}{128 a_0 b_0} + \frac{G}{E_x'} \frac{\pi^4}{64 a_0 b_0} \\ & - k_u \left(\frac{2 b_0 \pi^2}{3 a_0^2} - \frac{E''}{E_x'} \frac{\pi^2}{3 b_0} + \frac{G}{E_x'} \frac{\pi^2}{3 b_0} \right) - k_v \left(\frac{2 E_y'}{E_x'} \frac{\pi^2 a_0}{3 b_0^2} \right. \\ & \left. - \frac{E''}{E_x'} \frac{\pi^2}{3 a_0} + \frac{G}{E_x'} \frac{\pi^2}{3 a_0} \right) + k_u^2 \left(\frac{b_0 \pi^2}{a_0} + \frac{a_0 \pi^2}{4 b_0} \frac{G}{E_x'} \right) \\ & + k_v^2 \left(\frac{E_y'}{E_x'} \frac{\pi^2 a_0}{b_0} + \frac{b_0 \pi^2}{4 a_0} \frac{G}{E_x'} \right) + \frac{32}{9} k_u k_v \frac{E'' + G}{E_x'} \Bigg] \end{aligned} \tag{60}$$

The strain energy V is now expressed from Eqs. (40), (48) and (59) as

$$V = K_1 w_{11}^2(t) + K_2 w_{11}^4(t) \tag{61}$$

After substituting Eq. (61) for V into Eq. (45), we have

$$-m' \frac{\partial^2 w_{11}}{\partial t^2} \frac{a_0 b_0}{4} - (2K_1 w_{11} + 4K_2 w_{11}^3) + \frac{4 a_0 b_0}{\pi^2} F(t) = 0, \tag{62}$$

from which the equation of motion that $w_{11}(t)$ must satisfy reduces to

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p_0^2 w_{11}(t) + \alpha w_{11}^3(t) = q_0(t), \tag{63}$$

in which $p_0^2 = \frac{8K_b}{m' a_0 b_0}$, $\alpha = \frac{16 K_s}{m' a_0 b_0}$, $q_0(t) = \frac{16 F(t)}{m' \pi^2}$, and $q_0(t)$ is the deflection at the center of the plate in the case of small deflection. Since from the practical point of view α may be considered as a small factor, we can first apply the method of successive approximation to the case of Eq. (63) when $q_0(t) = 0$ in order to obtain $w_{11}(t)$ and the circular frequency p of free vibration so as to satisfy $w_{11}(t) = 0$,

$\frac{\partial w_{\parallel}}{\partial t} = v$ at the instant $t = 0$.

Thus we have

$$w_{\parallel}(t) = \frac{v}{p}\sin pt + \frac{\alpha v^3}{32 p^5}(3\sin pt - \sin 3pt) + \frac{\alpha^2 v^5}{(32)^2 p^9}(31\sin pt$$

$$- 12\sin 3pt + \sin 5pt) + \frac{\alpha^3 v^7}{(32)^3 p^{13}}(429\sin pt - 174\sin 3pt$$

$$+ 20\sin 5pt - \sin 7pt), \tag{64}$$

$$p^2 = p_0^2 + \frac{3v^2\alpha}{4p_0^2} - \frac{51 v^4 \alpha^2}{128 p_0^6} - \frac{189 v^6 \alpha^3}{512 p_0^{10}},$$

as shown in appendix II.

The first equation of Eq. (64) shows that $w_{\parallel}(t)$ is the sum of terms effected due to initial velocities v, $\frac{\alpha v^3}{32 p^4}$, $\frac{\alpha^2 v^5}{(32)^2 p^8}$ and $\frac{\alpha^3 v^7}{(32)^3 p^{12}}$, respectively.

Returning to Eq. (63), in the case of impact the magnitude of the velocity increase is found from the equations:

$$\left.\begin{aligned} dv &= q_0 dt, \\ d\left[\frac{\alpha v^3}{32 p^4}\right] &= q_1 dt, \\ d\left[\frac{\alpha^2 v^5}{(32)^2 p^8}\right] &= q_2 dt, \\ d\left[\frac{\alpha^3 v^7}{(32)^3 p^{12}}\right] &= q_3 dt, \end{aligned}\right\} \tag{65}$$

in which each of q_i $(i=1\sim3)$ is some magnitude of impulsive load per unit mass of the plate produced by q_0 .

The deflection of the plate corresponding to each velocity of Eq. (65) at the instant t when each velocity is communicated at the instant τ , may be calculated by using Eq. (64). It is seen from w_{11} of Eq. (64) that, for instance, by reason of the initial velocity v the deflection at any instant t is $\frac{v}{p}\sin pt$. Hence the velocities dv and so on communicated at the instant τ to the plate produces a deflection of the plate dw_{11} at the instant t given by

$$
\begin{aligned}
dw_{11} = {} & \frac{q_0 d\tau}{p}\sin p(t-\tau) + \frac{q_1 d\tau}{p}\left[3\sin p(t-\tau) - \sin 3p(t-\tau)\right] \\
& + \frac{q_2 d\tau}{p}\left[31\sin p(t-\tau) - 12\sin 3p(t-\tau) + \sin 5p(t-\tau)\right] \qquad (66) \\
& + \frac{q_3 d\tau}{p}\left[429\sin p(t-\tau) - 174\sin 3p(t-\tau) + 20\sin 5p(t-\tau) \right. \\
& \qquad \left. - \sin 7p(t-\tau)\right]
\end{aligned}
$$

Let us assume that at the initial instant $t=0$ the plate is at rest in its position of static equilibrium and the duration of impace Δ .

Then substituting relations:

$$dv = q_0 d\tau \quad \text{and} \quad v = q_0 \tau, \tag{67}$$

into the following equation obtained from Eq. (64)

$$d\left(\frac{P}{P_0}\right)^2 = \frac{3\alpha}{2P_0^4} v\,dv - \frac{51\alpha^2}{32P_0^8} v^3 dv - \frac{567\alpha^3}{256P_0^{12}} v^5 dv, \tag{68}$$

and assuming the relationship between q_0 and slamming pulse F(t) as shown in Fig. 2:

$$\left.\begin{aligned} q_0 &= \frac{16F_M}{m'\pi^2}\left(\frac{t}{\beta\Delta}\right)^m && \text{when } 0<t<\beta\Delta, \\ &= \frac{16F_M}{m'\pi^2}\left[\frac{\Delta-t}{(1-\beta)\Delta}\right]^n && \text{when } \beta\Delta<t<\Delta, \\ &= 0 && \text{when } \Delta<t, \end{aligned}\right\} \tag{69}$$

where m and n are any integers and F_M is the peak value of F(t), we have

$$P^2 = MP_0^2 ,$$

where

$$M = 1 + \frac{3\alpha q_M^2 \Delta^2}{2P_0^4}\left[\frac{\beta^2}{2(m+1)} + (1-\beta)\left\{\frac{1}{2n+1} - \frac{1-\beta}{2(n+1)}\right\}\right]$$

$$-\frac{51\alpha^2 q_M^4}{32p^8}\left[\frac{(\beta\Delta)^4}{4(m+1)}+\Delta^4(1-\beta)\left\{\frac{1}{4n+1}-\frac{3(1-\beta)}{4n+2}+\frac{3(1-\beta)^2}{4n+3}-\frac{(1-\beta)^3}{4n+4}\right\}\right]$$

when $t>\Delta$, (70)

and $q_M=\frac{16F_M}{m'\pi^2}$

After substituting p value determined above into the first three terms of Eq. (66) and using the relations given by Eqs. (65) and (67):

$$\left.\begin{aligned} q_1 d\tau &= \frac{3\alpha q_0^3\tau^2}{32p^4}d\tau, \\ q_2 d\tau &= \frac{5\alpha^2 q_0^5\tau^4}{(32)^2p^8}d\tau, \\ q_3 d\tau &= \frac{7\alpha^3 q_0^7\tau^6}{(32)^3p^{12}}d\tau, \end{aligned}\right\} \tag{71}$$

we can find $w_{11}(t)$ in Eq. (72) when $t>\Delta$. (See Appendix III). Eq. (72) is the generalized approximate solution in the case of large deflection. Also we can find another two similar equations which correspond to both when $0<t<\beta\Delta$ and $\beta\Delta<t<\Delta$.

Two examples are given for the cases of small and large deflections of a bottom plate.

Example 2 - Small deflection of a bottom plate

Suppose, for example, that all edges of the plate will be given a simple harmonic motion having the period $2\pi/\omega$ and phase lag ε :

$$w_e = \bar{w} \sin(\omega t + \varepsilon) \tag{73}$$

in the vertical direction as the motion of ship on the sea. Then measuring deflection $w_{11}(t)$ of the plate from its equilibrium position when $w_e = 0$, the corresponding deflection to all edges will be $w_{11}(t) - w_e(t)$. Thus the equation of motion becomes from Eq. (63) neglecting α - term

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p_0^2 (w_{11} - w_e) = q_0, \tag{74}$$

from which we have

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p_0^2 w_{11}(t) = q_0(t) + \bar{w} p_0^2 \sin(\omega t + \varepsilon) \tag{75}$$

Eq. (75) is the equation of forced vibration having magnitude of $q_0(t) + \bar{w} p_0^2 \sin(\omega t + \varepsilon)$. After substituting $q_0(t) + \bar{w} p_0^2 \sin(\omega t + \varepsilon)$ instead of $q_0(t)$ into Eq. (71), and using the maximum static deflection

$\frac{q_M}{p_0^2} = \frac{16 F_M}{m' \pi^2 p_0^2} = \frac{16 F_M}{\pi^6 D \left(\frac{1}{a_0^2} + \frac{1}{b_0^2}\right)^2}$, we have $w(xyt)$ from Eqs. (46) and (66):

$$w = \frac{16 F_M}{m' \pi^2 p_0^2} \left(\sin \frac{\pi x}{a_0} \sin \frac{\pi y}{b_0} \right) \cdot R, \tag{76}$$

where R is the response factor as shown below

$$R=\frac{1}{(\beta\Delta p_o)^m}\left[\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(p_o t)^{m-2\nu}+R_1\right]+R_2 \quad \underline{\text{when } 0<t<\beta\Delta}$$

$$R=\frac{1}{(\beta\Delta p_o)^m}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(p_o\beta\Delta)^{m-2\nu}+\sin p_o(t-\beta\Delta)$$

$$\times\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^\nu\frac{m!}{(m-2\nu-1)!}(p\beta\Delta)^{m-2\nu-1}+R_1\Big]+\frac{1}{[p_o\Delta(1-\beta)]^n}\Big[\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[(\Delta-t)p_o]^{n-2\nu}$$

$$-\cos p_o(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[(1-\beta)\Delta p_o]^{n-2\nu}+\sin p_o(t-\beta\Delta)$$

$$\times\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^\nu\frac{n!}{(n-2\nu-1)!}[(1-\beta)\Delta p_o]^{n-2\nu-1}\Big]+R_2 \quad \underline{\text{when } \beta\Delta<t<\Delta} \qquad (77)$$

$$R=\frac{1}{(\beta\Delta p_o)^m}\Big[\cos p_o(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(p_o\beta\Delta)^{m-2\nu}+\sin p_o(t-\beta\Delta)$$

$$\times\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^\nu\frac{m!}{(m-2\nu-1)!}(p_o\beta\Delta)^{m-2\nu-1}+R_1\Big]+\frac{1}{[(1-\beta)\Delta p_o]^n}\Big[-\cos p_o(t-\beta\Delta)$$

$$\times\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[p_o\Delta(1-\beta)]^{n-2\nu}+\sin p_o(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^\nu\frac{n!}{(n-2\nu-1)}[p_o\Delta(1-\beta)]^{n-2\nu-1}$$

$$+R_3\Big]+R_2 \qquad \underline{\text{when } t>\Delta}$$

and

$$R_2=\frac{p_o^2\bar{\omega}\pi^2}{16F_M}\left[\frac{\sin\frac{t}{2}(p_o+\omega)\cos\left[\frac{t}{2}(\omega-p_o)+\varepsilon\right]}{\frac{\omega}{p_o}+1}+\frac{\sin\frac{t}{2}(\omega-p_o)\cos\left[\frac{t}{2}(p_o+\omega)+\varepsilon\right]}{\frac{\omega}{p_o}-1}\right],$$

if m and n are odd numbers, R_1 and R_3 become

$$R_1=(-1)^{\frac{m+1}{2}}m!\sin p_o t$$

$$R_3=(-1)^{\frac{n+1}{2}}n!\sin p_o(t-\Delta)$$

if m and n are even numbers they become

$$R_1=(-1)^{\frac{m}{2}+1}m!\cos p_o t$$

$$R_3=(-1)^{\frac{n}{2}}n!\cos p_o(t-\Delta)$$

Since the stresses are space derivatives of the deflection, it may be concluded from Eq. (76) that the dynamic response, deflection or stress, is equal to the static deflection or stress multiplied by the response factor. The maximum value of the response factor R is known as the load factor and is designated by L. If we want more accurate value we may use the series expression containing higher modes of vibration:

$$[w]_{max} = \frac{16 F_M}{\pi^6 D} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{\frac{i+j}{2}-1}}{ij\left(\frac{i^2}{a_0^2} + \frac{j^2}{b_0^2}\right)^2} L_{ij}, \tag{78}$$

in which L_{ij} will be obtained from Eqs. (77) for each natural frequency P_{ij}, which is expressed by $P_{ij} = \frac{\pi^4 D}{m'}\left(\frac{i^2}{a_0^2} + \frac{j^2}{b_0^2}\right)^2$, instead of P_0

Also if we use from Eq. (46)

$$w = \frac{16 F_M}{\pi^6 D} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\sin\frac{i\pi x}{a_0} \sin\frac{j\pi y}{b_0}}{ij\left(\frac{i^2}{a_0^2} + \frac{j^2}{b_0^2}\right)^2} R_{ij} \tag{79}$$

the moments and stresses are easily given by

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

(80)

$$M_{xy} = D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

where h is the plate thickness (81)

$$\sigma_x = \frac{6M_x}{h^2}$$

$$\sigma_y = \frac{6M_y}{h^2}$$

Example 3 - Large deflection of a bottom plate neglecting α^2-term of Eq. (72), we have in the case of $m = n = 1$

$$w = w_{11}(t)\sin\frac{\pi x}{a_0}\sin\frac{\pi y}{b_0}$$

where

$$w_{11}(t) = \frac{q_M}{\beta\Delta p_1^3}(p_1 t - \sin p_1 t)$$

$$+\frac{3\alpha}{32 p_1^5}\frac{q_M^3}{(\beta\Delta)^3}\left[3\left\{\frac{t^5}{p_1} - \frac{20t^3}{p_1^3} + \frac{5!\,t}{p_1^5} - \frac{5!}{p_1^6}\sin 3p_1 t\right\}\right. \tag{82}$$

$$\left. -\left\{\frac{t^5}{3p_1} - \frac{20t^3}{(3p_1)^3} + \frac{5!\,t}{(3p_1)^5} - \frac{5!}{(3p_1)^6}\sin 3p_1 t\right\}\right]$$

when $0 < t < \beta\Delta$,

$$
\begin{aligned}
w_{11} = {} & \frac{q_M}{\beta(1-\beta)p_1^3\Delta}\Big[\beta p_1(\Delta - t) - (1-\beta)\sin p_1 t + \sin p_1(t-\beta\Delta)\Big] \\
& + \frac{3\alpha q_M^3}{32 p_1^5 (1-\beta)^3 (\beta\Delta)^3} \\
& \times \Bigg\{ \frac{3(1-\beta)^3}{p_1^6}\Big[Q_{51}(\beta\Delta p_1)\cos p_1(t-\beta\Delta) + Q_{52}(\beta\Delta p_1)\sin p_1(t-\beta\Delta) - 5!\sin p_1 t\Big] \\
& - \frac{(1-\beta)^3}{(3p_1)^6}\Big[Q_{51}(3p_1\beta\Delta)\cos 3p_1(t-\beta\Delta) + Q_{52}(3p_1\beta\Delta)\sin 3p_1(t-\beta\Delta) - 5!\sin 3p_1 t\Big] \\
& - \frac{3\beta^3\Delta^2}{p_1^4}\Big[-Q_{31}p_1(\Delta-t) + \cos p_1(t-\beta\Delta)Q_{31}(p_1\Delta(1-\beta)) - \sin p_1(t-\beta\Delta)Q_{32}(p_1\Delta(1-\beta))\Big] \qquad (82) \\
& + \frac{6\Delta\beta^3}{p_1^5}\Big[-Q_{41}p_1(\Delta-t) + \cos p_1(t-\beta\Delta)Q_{41}p_1\Delta(1-\beta) - \sin p_1(t-\beta\Delta)Q_{42}p_1\Delta(1-\beta)\Big] \\
& - \frac{3\beta^3}{p_1^6}\Big[-Q_{51}p_1(\Delta-t) + \cos p_1(t-\beta\Delta)Q_{51}p_1\Delta(1-\beta) - \sin p_1(t-\beta\Delta)Q_{52}p_1\Delta(1-\beta)\Big] \\
& + \frac{\Delta^2}{(3p_1)^4}\Big[-Q_{31}3p_1(\Delta-t) + \cos 3p_1(1-\beta\Delta)Q_{31}3p_1\Delta(1-\beta) - \sin 3p_1(t-\beta\Delta)Q_{32}3p_1\Delta(1-\beta)\Big] \\
& - \frac{2\Delta\beta^3}{(3p_1)^5}\Big[-Q_{41}3p_1(\Delta-t) + \cos 3p_1(t-\beta\Delta)Q_{41}3p_1\Delta(1-\beta) - \sin 3p_1(t-\beta\Delta)Q_{42}3p_1\Delta(1-\beta)\Big]
\end{aligned}
$$

$$+\frac{1}{(3p_1)^6}\left[-Q_{5-1}3p_1(\Delta-t)+\cos 3p_1(t-\beta\Delta)Q_{5-1}3p_1\Delta(1-\beta)-\sin 3p_1(t-\beta\Delta)Q_{5-2}3p_1(1-\beta)\right]\Bigg\}$$

when $\beta\Delta < t < \Delta$,

$$W_{11}=\frac{q_M}{\beta(1-\beta)p_1^3\Delta}\left[\sin p_1(t-\beta\Delta)-(1-\beta)\sin p_1 t-\sin p_1(t-\Delta)\right]$$

$$+\frac{3\alpha q_M^3}{32p_1^5(1-\beta)^3(\beta\Delta)^3}$$

$$\times\Bigg\{\frac{3(1-\beta)^3}{p_1^6}\left[Q_{5-1}(\beta\Delta p_1)\cos p_1(t-\beta\Delta)+Q_{5-2}(\beta\Delta p_1)\sin p_1(t-\beta\Delta)-5!\sin p_1 t\right] \tag{82}$$

$$-\frac{(1-\beta)^3}{(3p_1)^6}\left[Q_{5-1}(3p_1\Delta\beta)\cos 3p_1(t-\beta\Delta)+Q_{5-2}(3p_1\beta\Delta)\sin 3p_1(t-\beta\Delta)-5!\sin 3p_1 t\right]$$

$$-\frac{3\beta^3\Delta^2}{p_1^4}\left[-3!\sin p_1(t-\Delta)+\cos p_1(t-\beta\Delta)Q_{3-1}p_1\Delta(1-\beta)-\sin p_1(t-\beta\Delta)Q_{3-2}p_1\Delta(1-\beta)\right]$$

$$+\frac{6\beta^3\Delta}{p_1^5}\left[-4!\cos p_1(t-\Delta)+\cos p_1(t-\beta\Delta)Q_{4-1}p_1\Delta(1-\beta)-\sin p_1(t-\beta\Delta)Q_{4-2}p\Delta(1-\beta)\right]$$

$$-\frac{3\beta^3}{p_1^6}\left[5!\sin p_1(t-\Delta)+\cos p_1(t-\beta\Delta)Q_{5-1}p_1\Delta(1-\beta)-\sin p_1(t-\Delta)Q_{5-2}p_1\Delta(1-\beta)\right]$$

$$+\frac{\beta^3\Delta^2}{(3p_1)^4}\left[-3!\sin 3p_1(t-\Delta)+\cos 3p_1(t-\beta\Delta)Q_{31}3p_1\Delta(1-\beta)-\sin 3p_1(t-\beta\Delta)Q_{32}3p_1\Delta(1-\beta)\right]$$

$$-\frac{2\beta^3\Delta}{(3p_1)^5}\left[-4!\cos 3p_1(t-\Delta)+\cos 3p_1(t-\beta\Delta)\,Q_{41}3p_1\Delta(1-\beta)-\sin 3p_1(t-\beta\Delta)Q_{42}3p_1\Delta(1-\beta)\right]$$

$$+\frac{\beta^3}{(3p_1)^6}\left[5!\sin 3p_1(t-\Delta)+\cos 3p_1(t-\beta\Delta)Q_{51}3p_1\Delta(1-\beta)-\sin 3p_1(t-\Delta)Q_{52}3p_1\Delta(1-\beta)\right]\Bigg\}.$$

when $t>\Delta$,

and

$$Q_{51}(\tau)=\sum_{\nu=0}^{\frac{5}{2}}(-1)^{\nu}\frac{5!}{(5-2\nu)!}\tau^{5-2\nu}=\tau^5-\frac{5!}{3!}\tau^3+5!\tau \tag{82}$$

$$Q_{52}(\tau)=\sum_{\nu=0}^{2}(-1)^{\nu}\frac{5!}{(4-2\nu)!}\tau^{4-2\nu}=\frac{5!}{4!}\tau^4-\frac{5!}{2!}\tau^2+5!$$

$$Q_{31}(\tau)=\sum_{\nu=0}^{\frac{3}{2}}(-1)^{\nu}\frac{3!}{(3-2\nu)!}\tau^{3-2\nu}=\tau^3-3!\tau$$

$$Q_{32}(\tau)=\sum_{\nu=0}^{1}(-1)^{\nu}\frac{3!}{(2-2\nu)!}\tau^{2-2\nu}=3\tau^2-3!$$

$$Q_{41}(\tau)=\sum_{\nu=0}^{2}(-1)^{\nu}\frac{4!}{(4-2\nu)!}\tau^{4-2\nu}=\tau^4-\frac{4!}{2!}\tau^2+4!$$

$$Q_{42}(\tau)=\sum_{\nu=0}^{\frac{3}{2}}(-1)^{\nu}\frac{4!}{(3-2\nu)!}\tau^{3-2\nu}=4\tau^3-4!\tau$$

Similarly, we may obtain the deflections in any case of edge conditions by employing the beam functions already applied. For instance, in the case of all clamped edges [16], assuming the deflection:

$$w = w_{11}(t) X_1 Y_1 \tag{83}$$

where
$$\left.\begin{aligned} X_1 &= \cosh\frac{\beta_1 x}{a_0} - \cos\frac{\beta_1 x}{a_0} - \alpha_1\left(\sinh\frac{\beta_1 x}{a_0} - \sin\frac{\beta_1 x}{a_0}\right) \\ Y_1 &= \cosh\frac{\beta_1 y}{b_0} - \cos\frac{\beta_1 y}{b_0} - \alpha_1\left(\sinh\frac{\beta_1 y}{b_0} - \sin\frac{\beta_1 y}{b_0}\right) \end{aligned}\right\} \tag{84}$$

and α_1, β_1 are given in numerical calculation, we can obtain the equation of motion that $w_{11}(t)$ must satisfy

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p^2 w_{11}(t) + \alpha w_{11}^3(t) = q_0(t) \tag{85}$$

in which α is given by $\frac{4K_s}{m' a_0 b_0}$ assuming K_s approximately equal to that of all supported edges; $q_0(t) = \frac{0.69039}{m'} F(t)$ and

$$p^2 = \frac{D}{m'}(500.54665)\left[\frac{1}{a_0^4} + \frac{1}{b_0^4} + \frac{0.60475}{a_0^2 b_0^2}\right]$$

Therefore, if we use $q_M = \frac{0.69039}{m'} F_M$, we can get the same expressions as Eq. (72) for the solution of Eq. (85). The maximum deflections are graphically represented later for one special case in order to show the effect due to the plate stretching under increase of slamming load. The same tendency also may be given for the maximum stresses which will happen at the middle

of the long sides.

Since stresses due to stretching are easily given from Eqs. (41), (53) and (58) using W of Eq. (82), total stresses are obtained by adding stresses or stretching to those in Eq. (81) of bending.

In the case of all supported edges, if all edges of the plate have a simple harmonic motion of Eq. (73), we can find from Eq. (72)

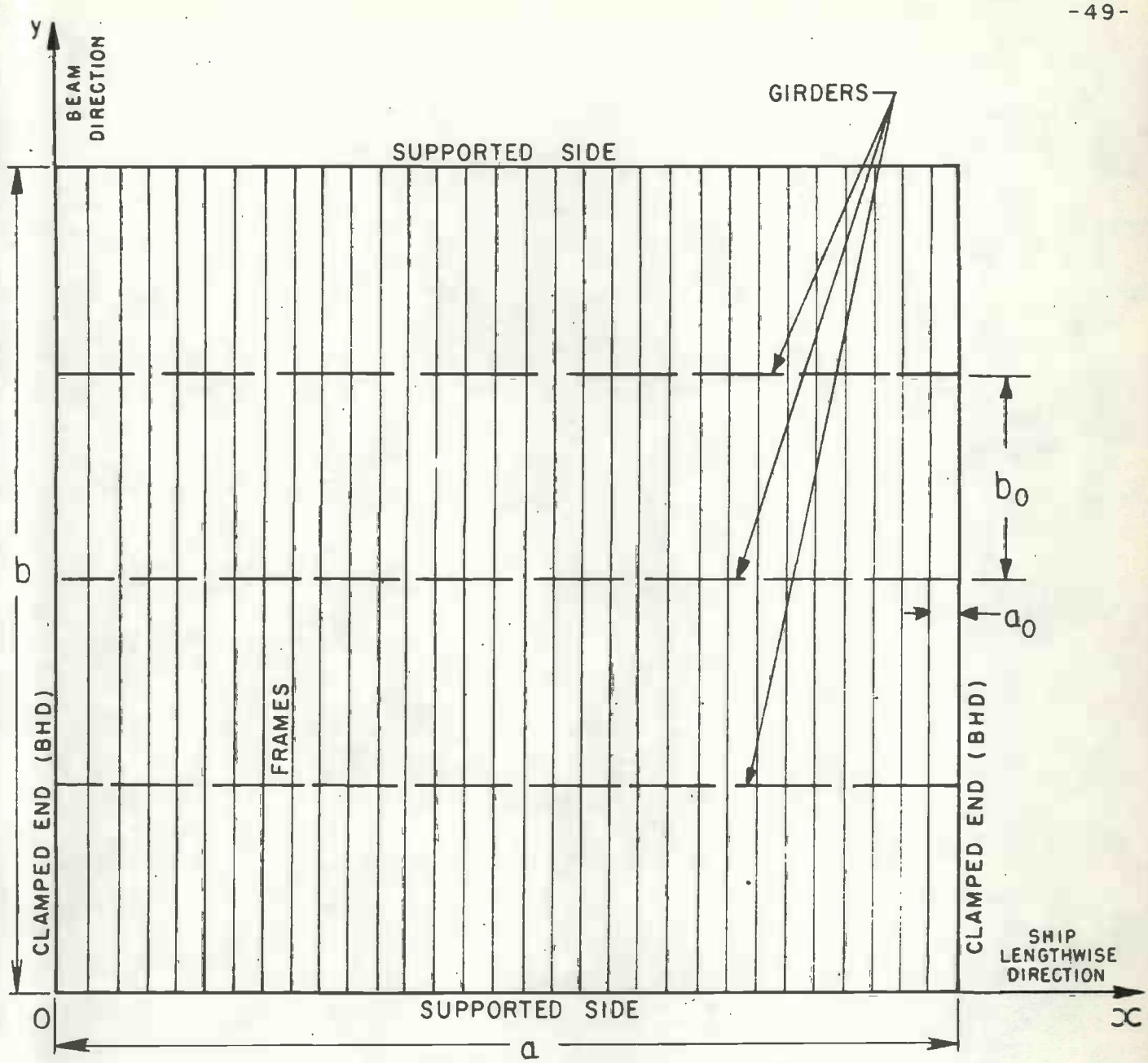
$$w = w \text{ given in Eq (82)} + \left[\bar{w}\left(\frac{\sin\frac{t}{2}(p+\omega)\cos\left[\frac{t}{2}(\omega-p)+\varepsilon\right]}{\frac{\omega}{p}+1} + \frac{\sin\frac{t}{2}(\omega-p)\cos\left[\frac{t}{2}(p+\omega)+\varepsilon\right]}{\frac{\omega}{p}-1}\right) + \frac{3\alpha}{32p^5}F(\bar{w})\right]\sin\frac{\pi x}{a_0}\sin\frac{\pi y}{b_0}, \tag{86}$$

by neglecting the effect of the edge motion to the third term when compared with the second term in the left side of Eq. (63). $F(\bar{w})$ of Eq. (86) is given in appendix III. And next, in order to apply the results obtained above explicitly, we would like to discuss in the following section the response of a double bottom having two sides supported and the other two sides clamped.

III. Numerical Calculation

As the stiffened plate, we use the double bottom, as shown in Fig. 7., of the same size as that given by Corlett [7], so that comparison can easily be carried out. We assume the double bottom to be supported on both sides and clamped on both ends, moreover, all the sides of bottom plate are clamped along the cross-stiffened boundaries. As one special case the loading condition is assumed as follows:

Total area of the double bottom is subjected to slamming loading such as $F_M = 15$ psi, $\beta = \frac{1}{2}$ and $\Delta = 0.02$ sec, and on some bottom plate around the central girder the slamming loading such as $F_M = 60$ psi, $\beta = \frac{1}{2}$ and $\Delta = 0.02$ sec is acting.



α	(NUMBER OF LONGITUDINALS)	= 3	$E = 30 \times 10^6$ PSI
β	(NUMBER OF FRAMES)	= 30	$g = 386$ IN/SEC^2
I_g	(MOMENT OF INERTIA OF GIRDER)	= 50,000 IN^4	$\rho g = 0.5$ PSI
I_s	(MOMENT OF INERTIA OF FRAME)	= 14,000 IN^4	$i_x = I_g(\alpha+1/b) = 273.224$ IN^3
a	(LENGTH BETWEEN TWO BULKHEADS)	= 72'-4" (868")	$i_y = I_s(\beta+1/a) = 500.000$ IN^3
a_0	(LENGTH BETWEEN TWO FRAMES)	= 28"	$i_{xy} = 404$ IN^3
b	(LENGTH BETWEEN TWO SIDES)	= 61'-0" (732")	$a/b = 1.18579$
b_0	(LENGTH BETWEEN TWO LONGITUDINALS)	= 183"	$b_0/a_0 = 6.53571$
μ	(POISSON'S RATIO)	= 0.3	

FIG. 7 DOUBLE BOTTOM

Before starting discussion with this example, we would like to prepare useful curves by Eq. (39), in order to get first fundamental frequencies easily by using these curves.

For the sake of convenience, curves are plotted for two different boundary conditions such as:

1. All clamped sides, and
2. Both ends clamped and both sides simply supported, or both sides clamped and both ends simply supported.

The ratios of width and length $\frac{b}{a}$ are 3, 2, 1, 2/3 and 1/2. For each ratio, different combinations of the value $\frac{i_y}{i_x}$ or $\frac{i_{xy}}{i_x}$ are chosen to produce separate curves on 30 sheets from Fig. 8 to Fig. 37 by which first fundamental frequencies may be easily obtained in the practical uses. In each figure the curves are plotted with $fab\sqrt{\frac{\rho g}{i_x}}$ as ordinate and $\frac{k^q_E F_k}{\rho g a b}$ as abscissa.

The curves in Fig. 8-37 are plotted only for the case having machinery and no pillars, but they can be used to a case having machinery and pillars and also one having pillars and no machinery. For the case having machinery and pillars, we may simply add the term which governs the effect of pillars $\frac{1}{\rho g a b}\sum P_l[f(x)f(y)]^2_l$ to the term of machinery $\frac{k^q_E F_k}{\rho g a b}$, and then use this sum instead of $\frac{k^q_E F_k}{\rho g a b}$. For the case having pillars and no machinery we may simply use the value of $\frac{1}{\rho g a b}\sum P_l[f(x)f(y)]^2_l$ instead of $\frac{k^q_E F_k}{\rho g a b}$. And the case of $\frac{k^q_E F_k}{\rho g a b} = 0$ corresponds to that having neither machinery nor pillar.

In Table 1 the values of $\frac{k^q_E F_k}{\rho g a b}$ are shown, corresponding to the ratios of $fab\sqrt{\frac{\rho g}{i_x}}$ given in column ⑦. In the case of

$\frac{b\delta_E^g F_R}{\rho g a b} > 1.6 \times 10^{-3}$, corresponding value of $fab\sqrt{\frac{\rho g}{i_x}}$ can be obtained by extrapolation or by extending the curves to the point corresponding to $\frac{b\delta_E^g F_R}{\rho g a b} = 2.2782 \times 10^{-3}$ on the abscissa, where the value of $fab\sqrt{\frac{\rho g}{i_x}}$ is also equal to 70% of that on $\frac{b\delta_E^g F_R}{\rho g a b} = 10^{-3}$ as shown in Column 7 of Table 1.

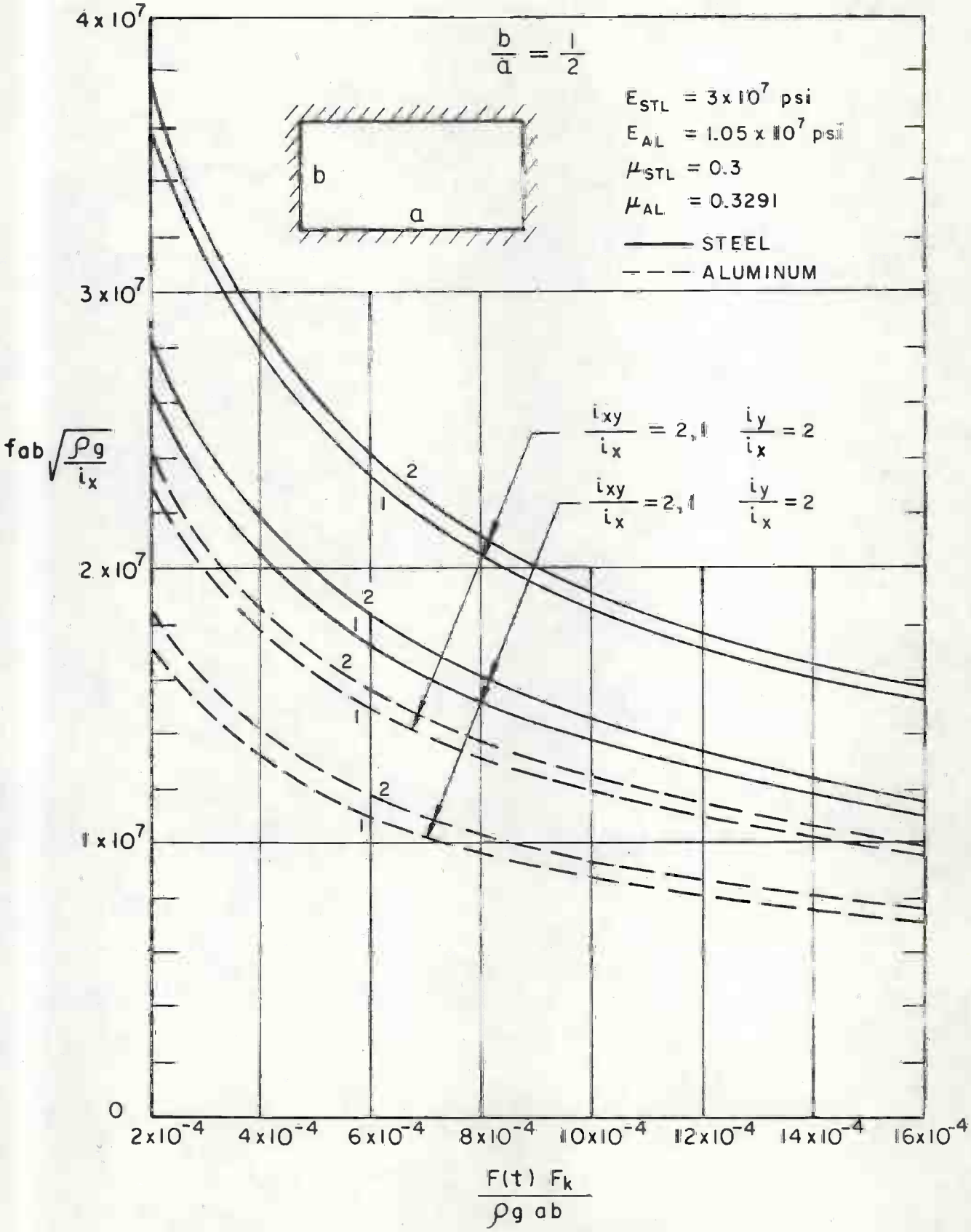

$\frac{b}{a} = \frac{1}{2}$
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
b
a
$\frac{i_{xy}}{i_x} = 2, 1$ $\frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 2, 1$ $\frac{i_y}{i_x} = 2$
$fab\sqrt{\frac{\rho g}{i_x}}$
4×10^7
3×10^7
2×10^7
1×10^7
0
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\ F_k}{\rho g\ ab}$

FIG. 8

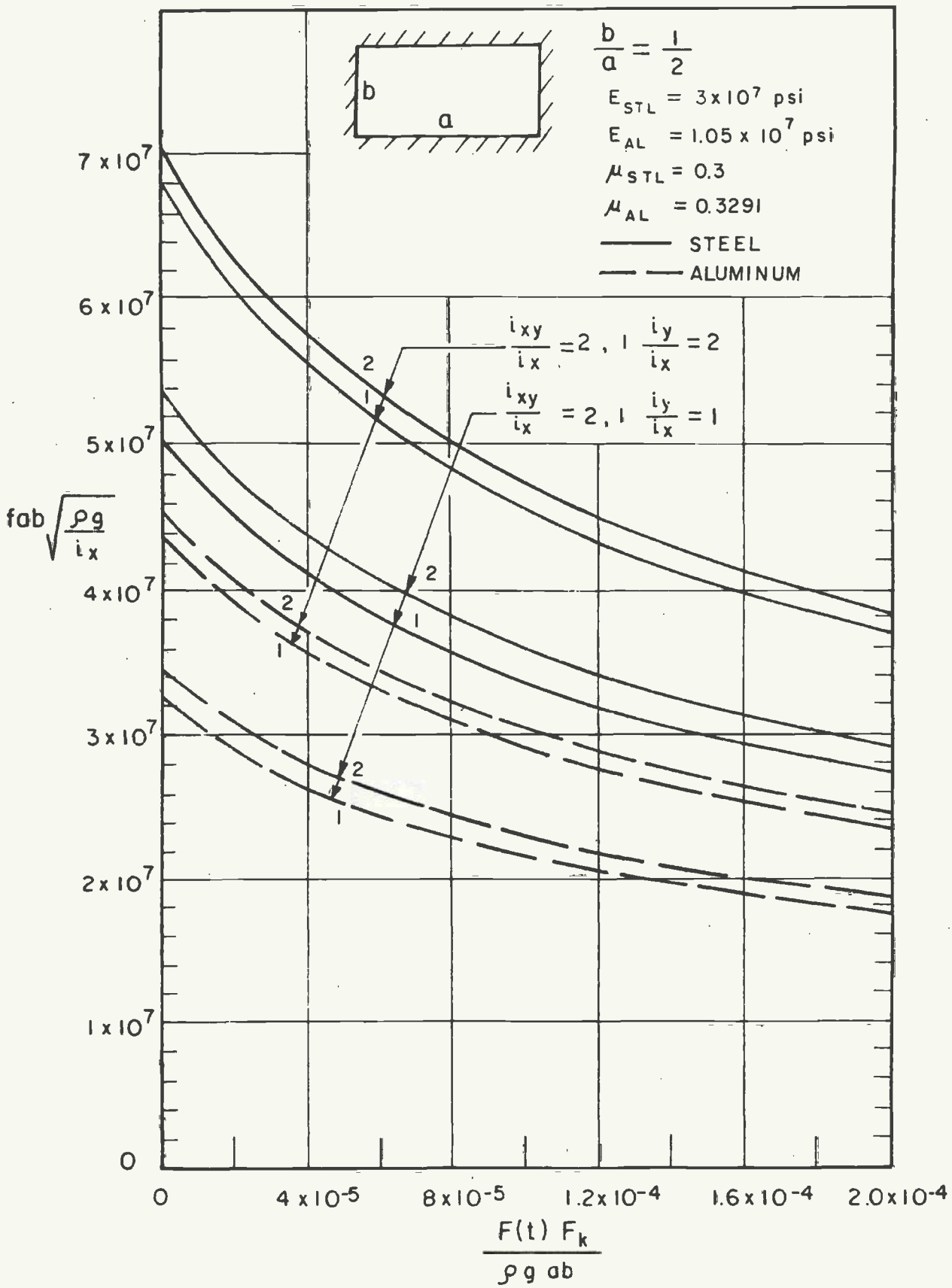

b/a = 1/2
E_STL = 3x10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
i_xy/i_x = 2, 1 i_y/i_x = 2
i_xy/i_x = 2, 1 i_y/i_x = 1
7 x 10^7
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/i_x)
0
4 x 10^-5
8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) F_k / ρg ab

FIG. 9

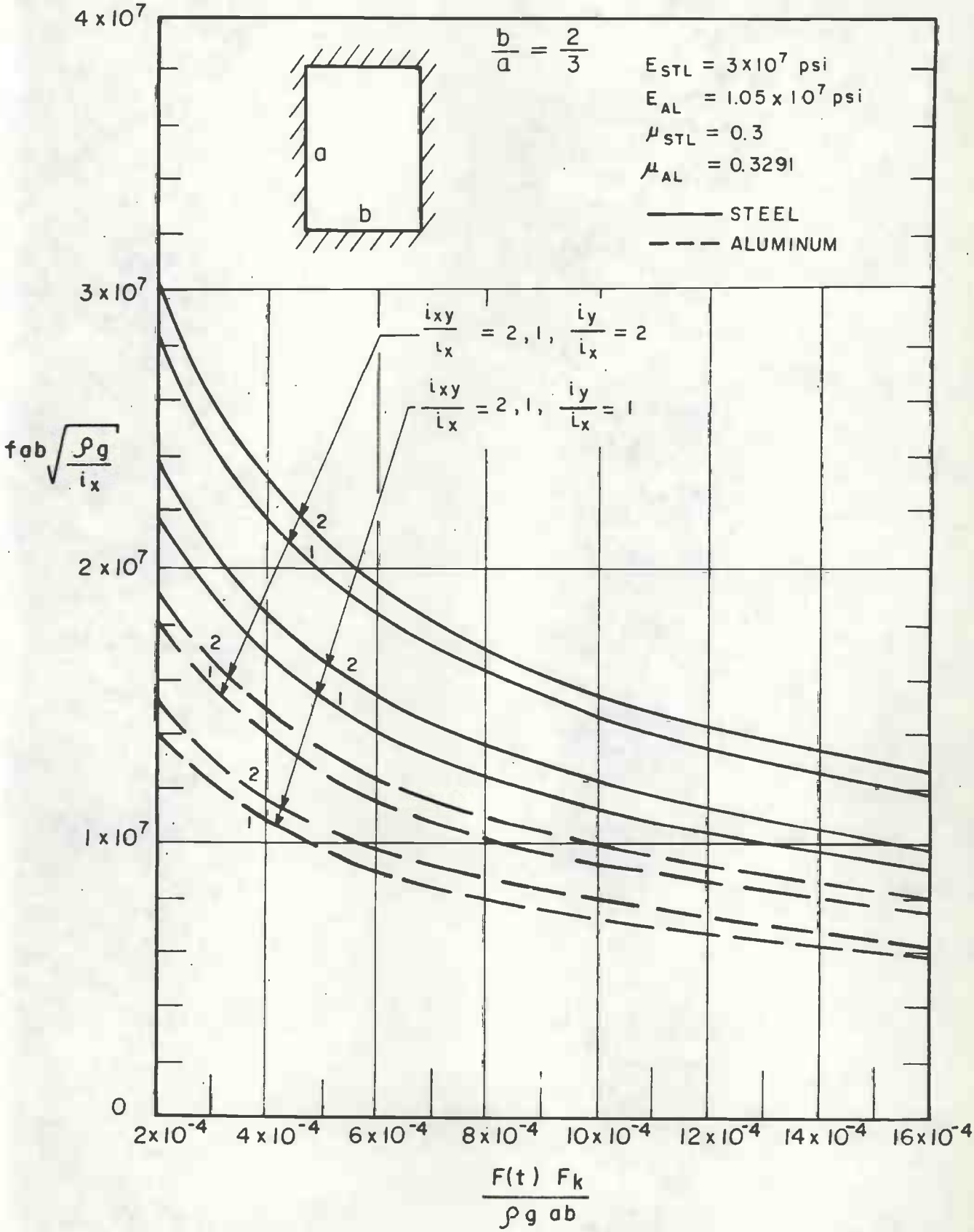

$\frac{b}{a} = \frac{2}{3}$
a
b
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
$\frac{i_{xy}}{i_x} = 2, 1, \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 2, 1, \frac{i_y}{i_x} = 1$
$f ab \sqrt{\frac{\rho g}{i_x}}$
4×10^7
3×10^7
2×10^7
1×10^7
0
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\ F_k}{\rho g\ ab}$

FIG. 10

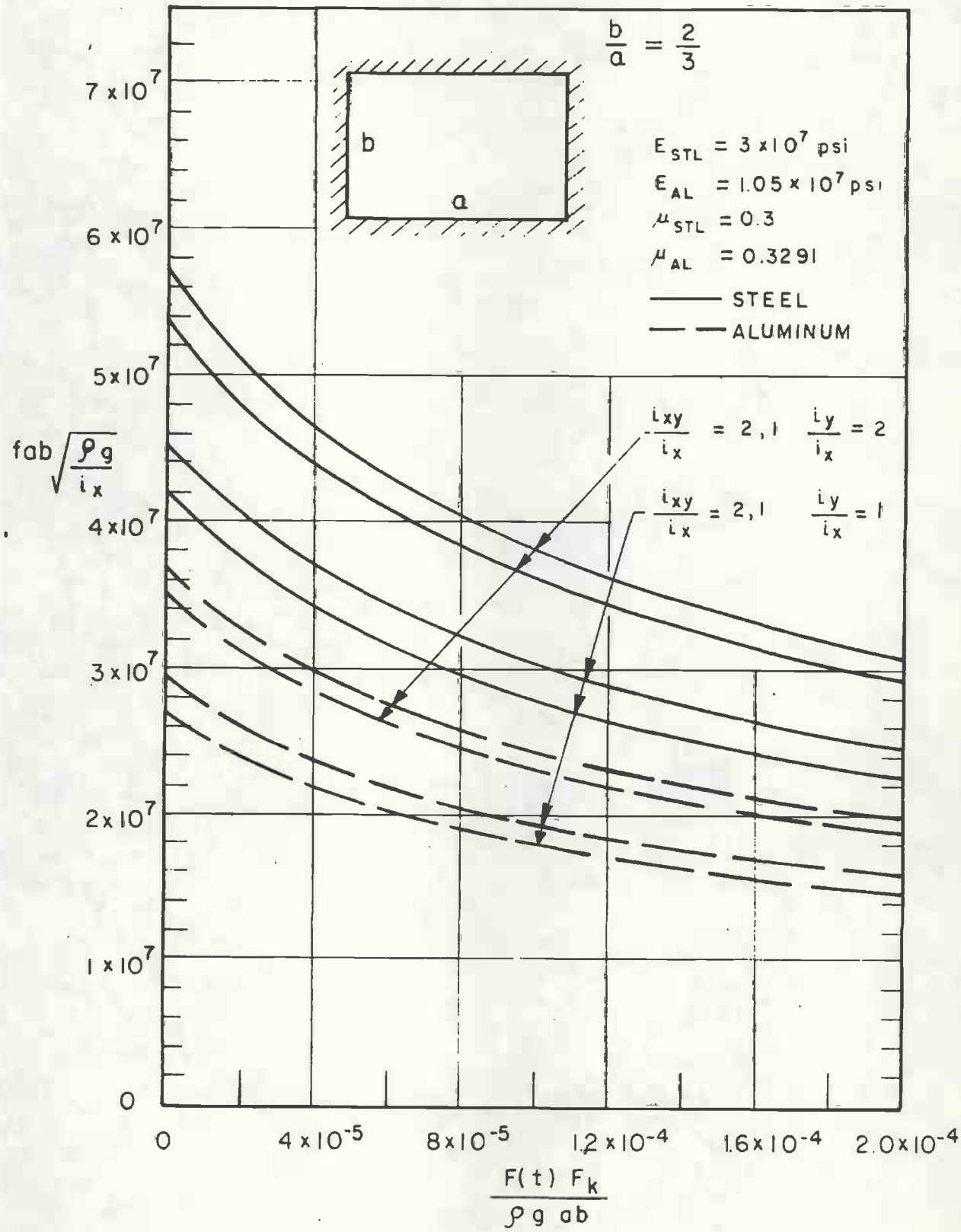

b/a = 2/3
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
i_xy/i_x = 2,1 i_y/i_x = 2
i_xy/i_x = 2,1 i_y/i_x = 1
fab √(ρg/i_x)
7 x 10^7
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
0
4 x 10^-5
8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) F_k / ρ g ab

FIG. 11

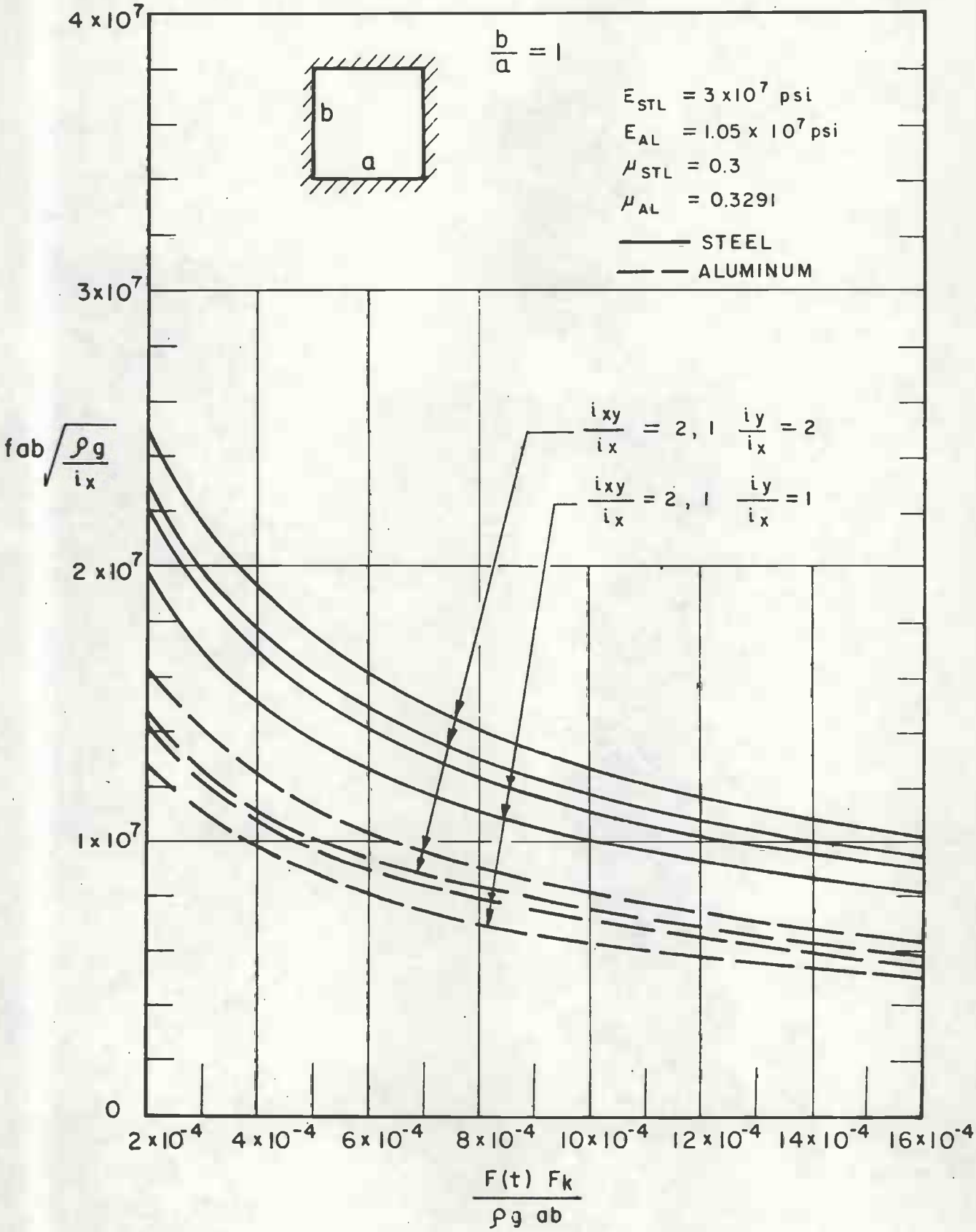

$\frac{b}{a} = 1$
b
a
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
$\frac{i_{xy}}{i_x} = 2, 1 \quad \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 2, 1 \quad \frac{i_y}{i_x} = 1$
4×10^7
3×10^7
2×10^7
1×10^7
0
$fab\sqrt{\frac{\rho g}{i_x}}$
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\ F_k}{\rho g\ ab}$

FIG. 12

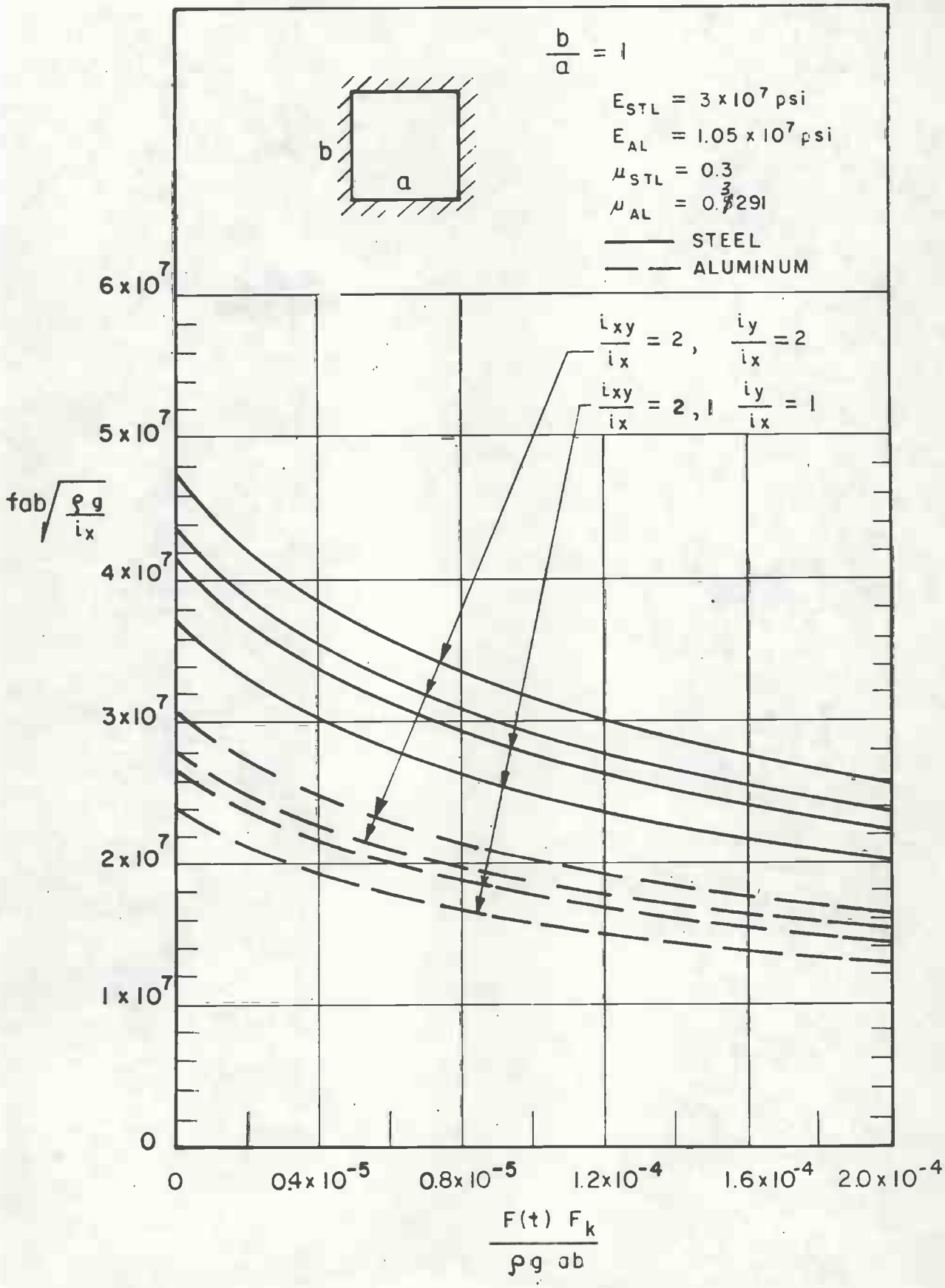

b/a = 1
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.291
STEEL
ALUMINUM
b
a
i_xy/i_x = 2 , i_y/i_x = 2
i_xy/i_x = 2 , 1 i_y/i_x = 1
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/i_x)
0
0.4 x 10^-5
0.8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) F_k / ρg ab

FIG. 13

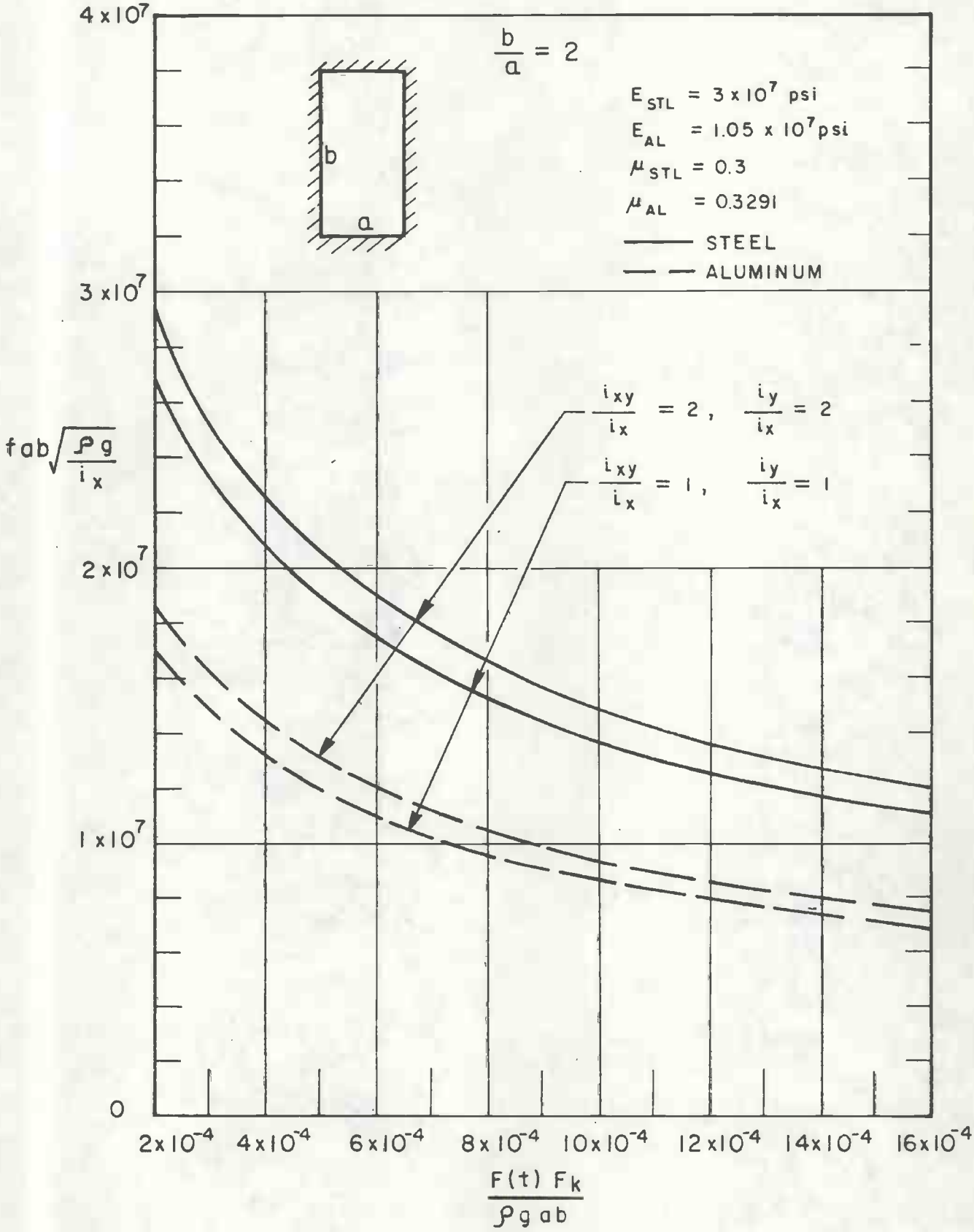

$\frac{b}{a} = 2$
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
b
a
$\frac{i_{xy}}{i_x} = 2, \quad \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 1, \quad \frac{i_y}{i_x} = 1$
4×10^7
3×10^7
2×10^7
1×10^7
0
$fab\sqrt{\frac{\rho g}{i_x}}$
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\,F_k}{\rho g\, ab}$

FIG. 14

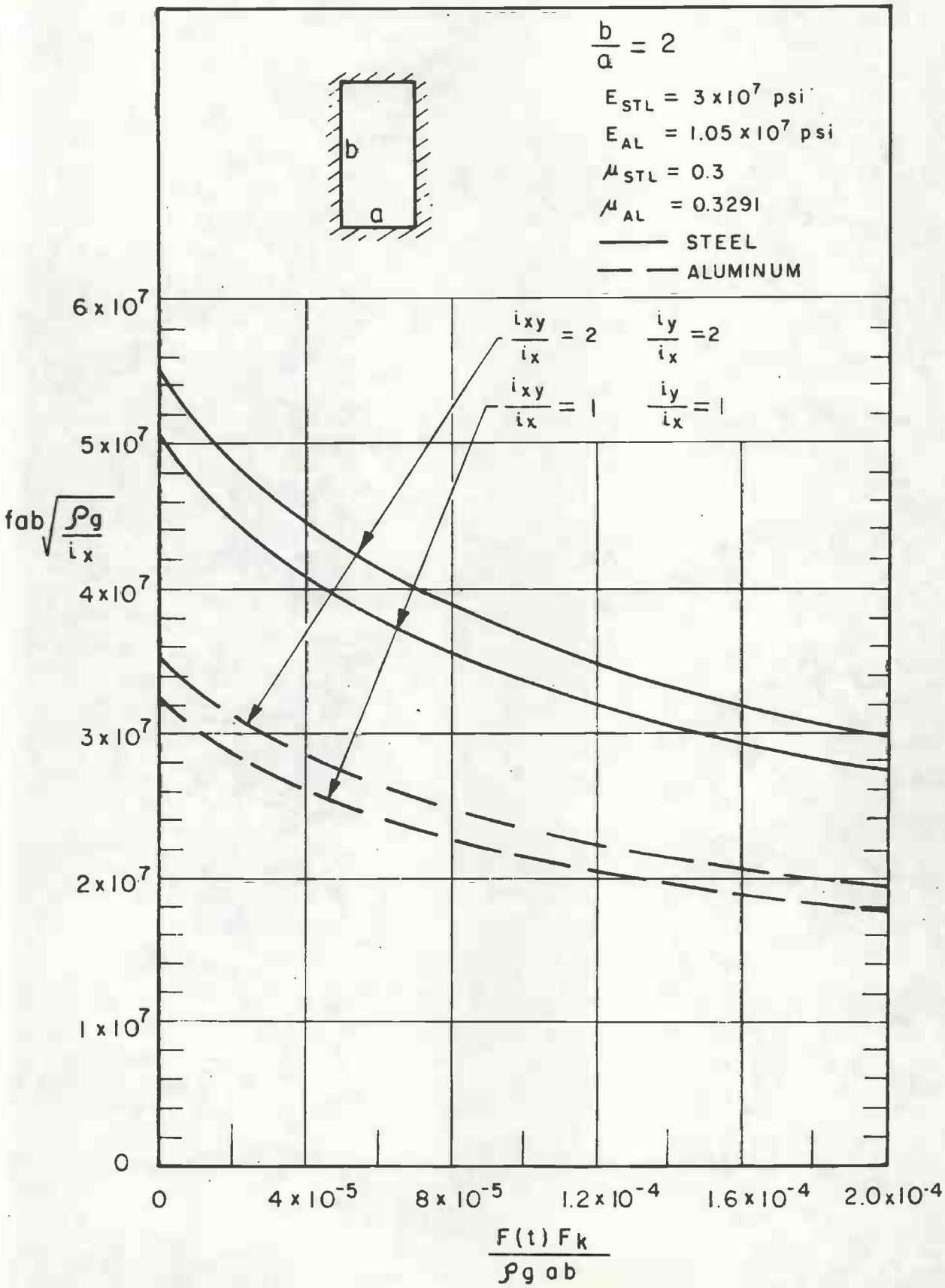

b/a = 2
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2
iy/ix = 2
ixy/ix = 1
iy/ix = 1
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/ix)
0
4 x 10^-5
8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) Fk / ρg ab

FIG. 15

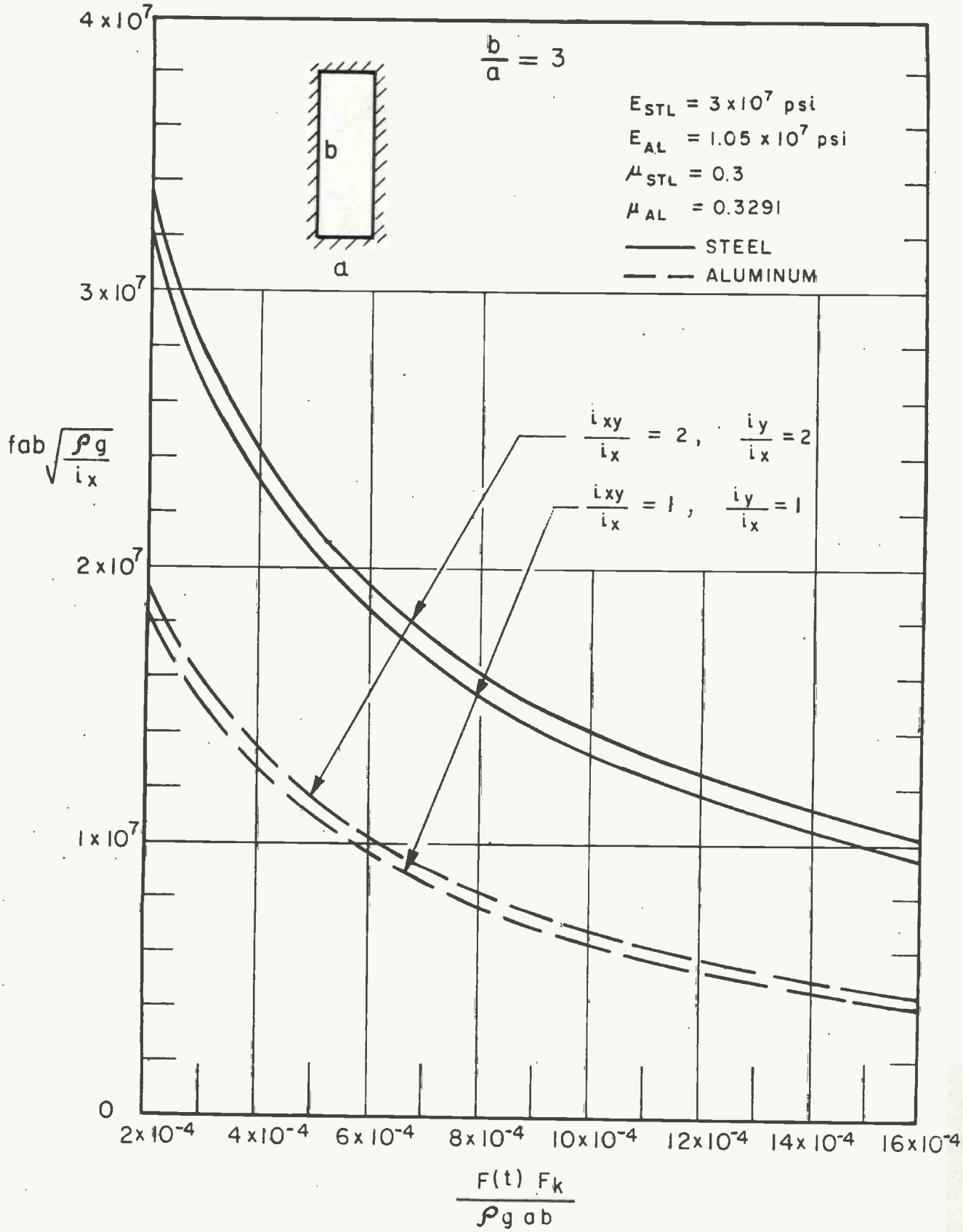

$\frac{b}{a} = 3$
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
b
a
$\frac{i_{xy}}{i_x} = 2, \quad \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 1, \quad \frac{i_y}{i_x} = 1$
$fab\sqrt{\frac{\rho g}{i_x}}$
4×10^7
3×10^7
2×10^7
1×10^7
0
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\ F_k}{\rho g\ ab}$

FIG. 16

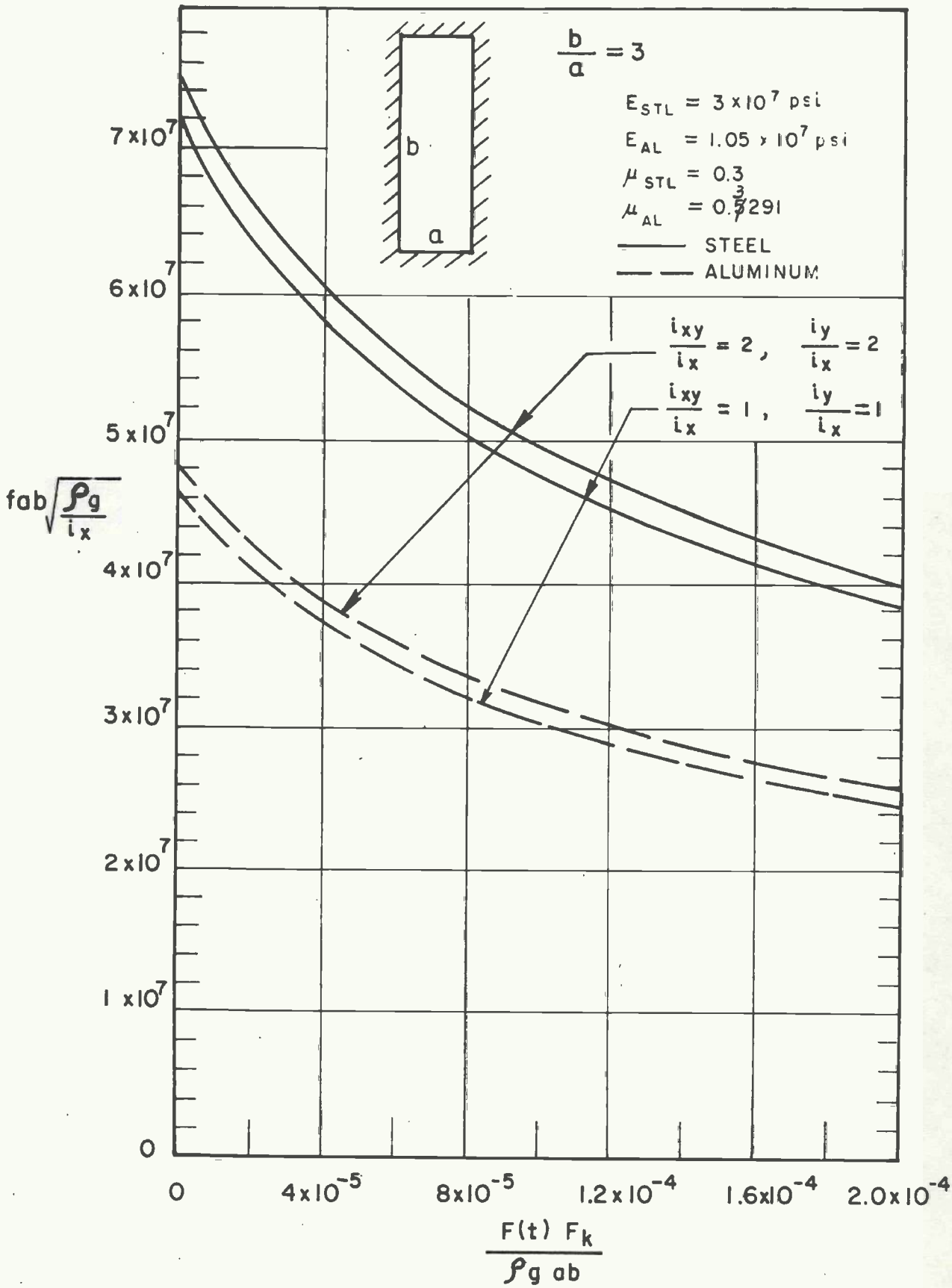

b/a = 3
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2, iy/ix = 2
ixy/ix = 1, iy/ix = 1
fab √(ρg/ix)
7x10^7
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1 x10^7
0
0
4x10^-5
8x10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t) Fk / ρg ab

FIG. 17

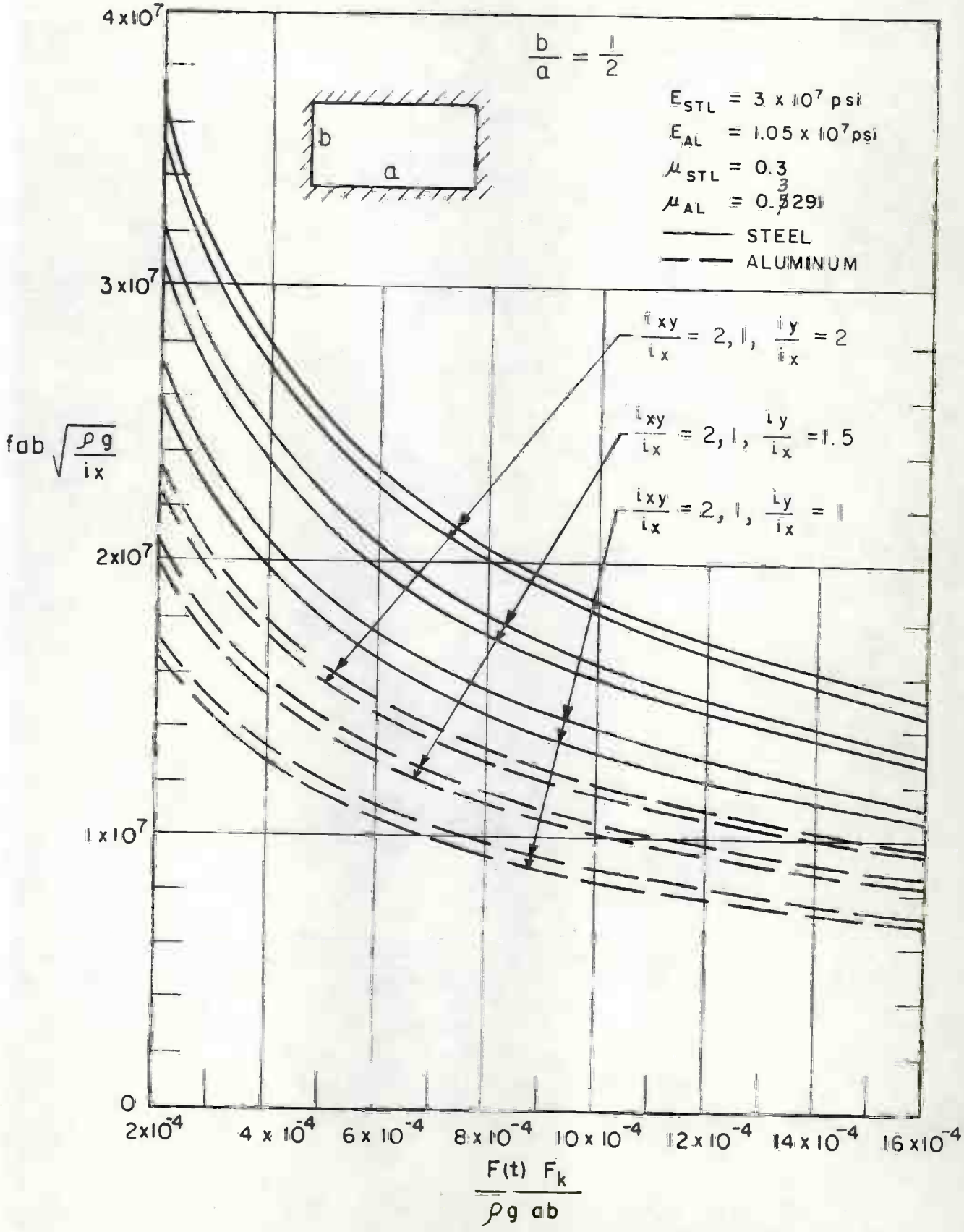

4x10^7
3x10^7
2x10^7
1x10^7
0
fab √(ρg/ix)
b/a = 1/2
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
ixy/ix = 2, 1, iy/ix = 2
ixy/ix = 2, 1, iy/ix = 1.5
ixy/ix = 2, 1, iy/ix = 1
2x10^-4
4 x 10^-4
6 x 10^-4
8 x 10^-4
10 x 10^-4
12 x 10^-4
14 x 10^-4
16 x 10^-4
F(t) F_k / ρg ab

FIG. 18

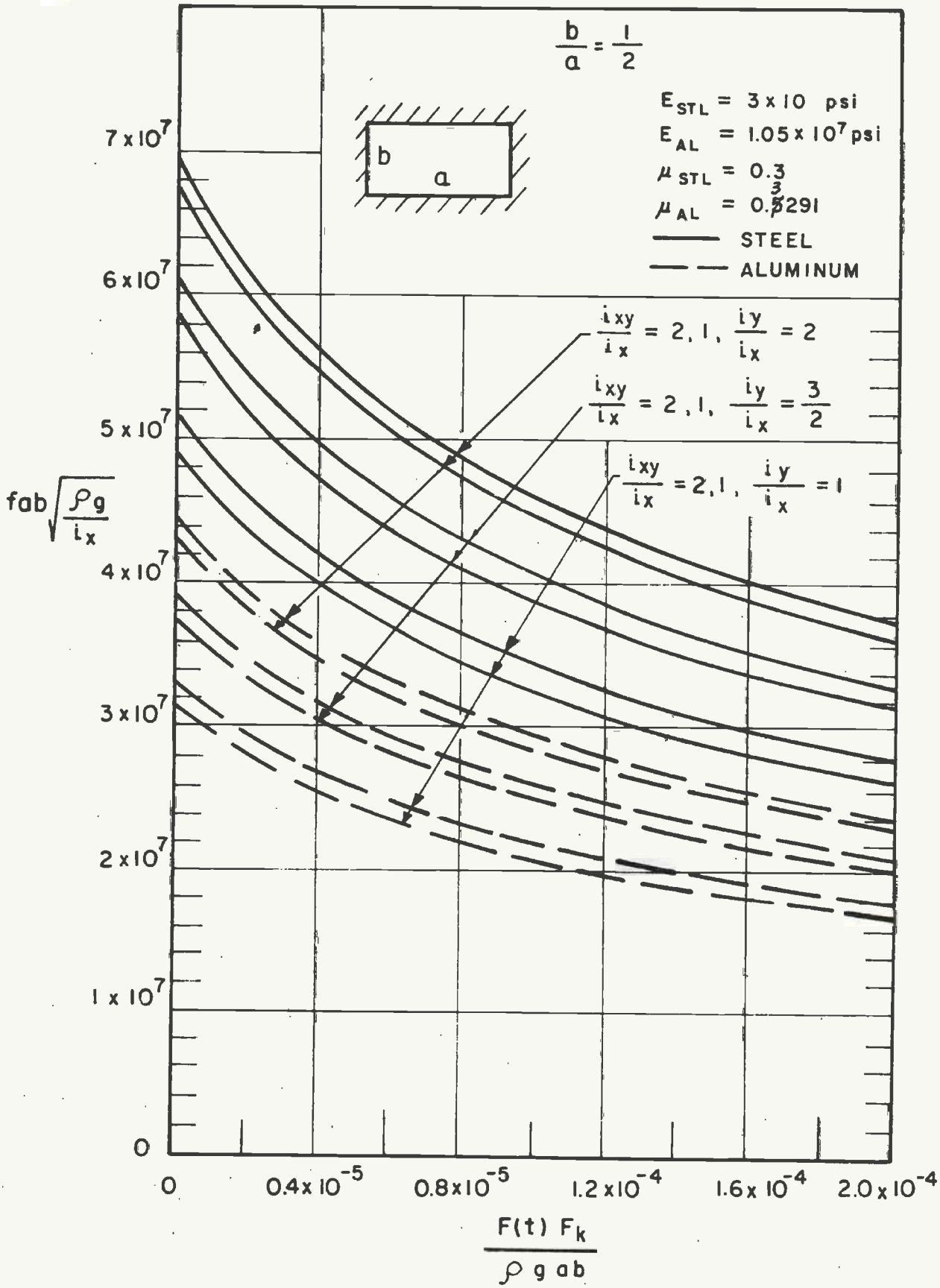

b/a = 1/2
E_STL = 3 x 10 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2, 1, iy/ix = 2
ixy/ix = 2, 1, iy/ix = 3/2
ixy/ix = 2, 1, iy/ix = 1
7 x 10^7
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/ix)
0
0.4 x 10^-5
0.8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) Fk / ρ g ab

FIG. 19

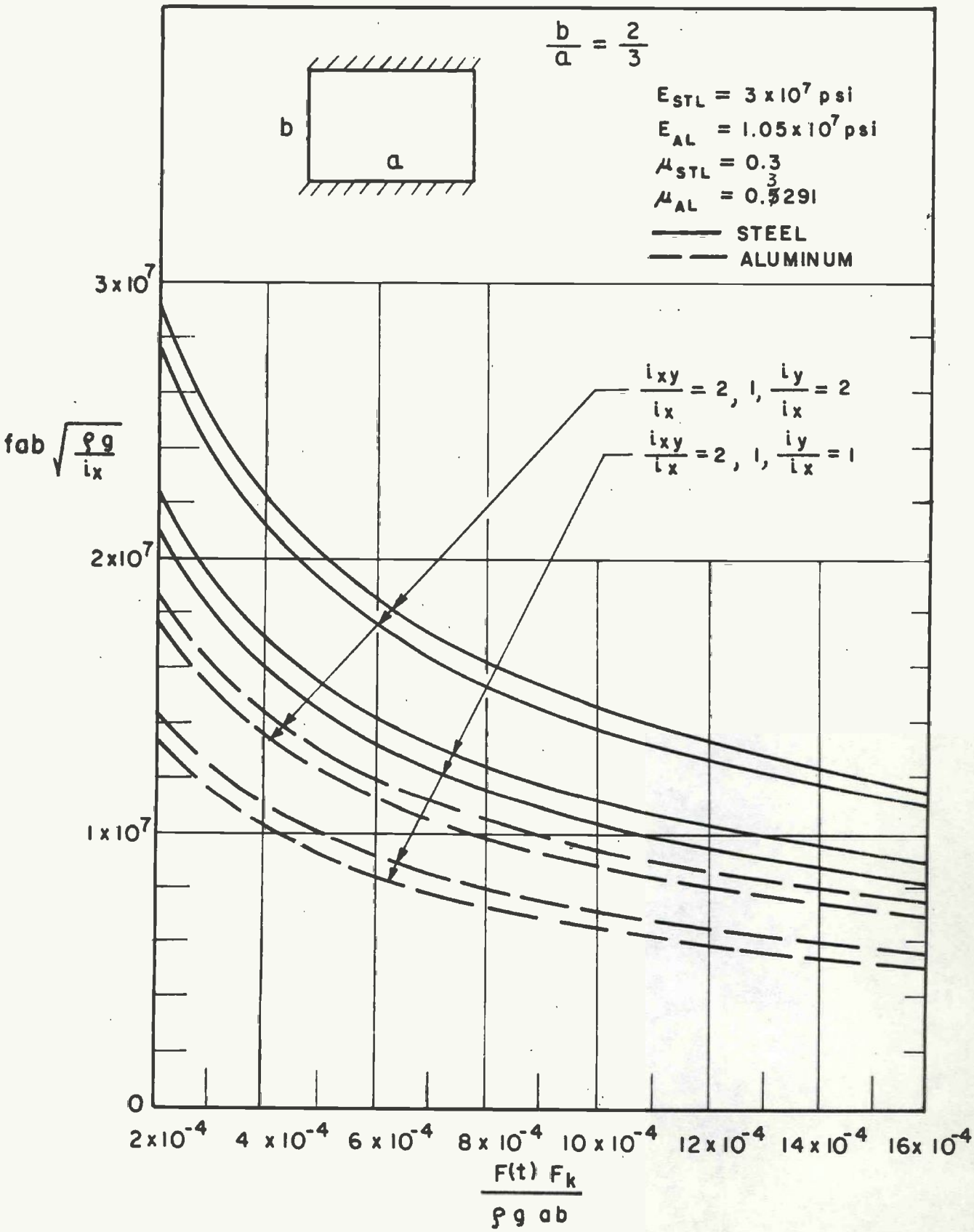

b/a = 2/3
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
ixy/ix = 2, 1, iy/ix = 2
ixy/ix = 2, 1, iy/ix = 1
fab √(ρg/ix)
3 x 10^7
2 x 10^7
1 x 10^7
0
2 x 10^-4
4 x 10^-4
6 x 10^-4
8 x 10^-4
10 x 10^-4
12 x 10^-4
14 x 10^-4
16 x 10^-4
F(t) Fk / ρg ab

FIG. 20

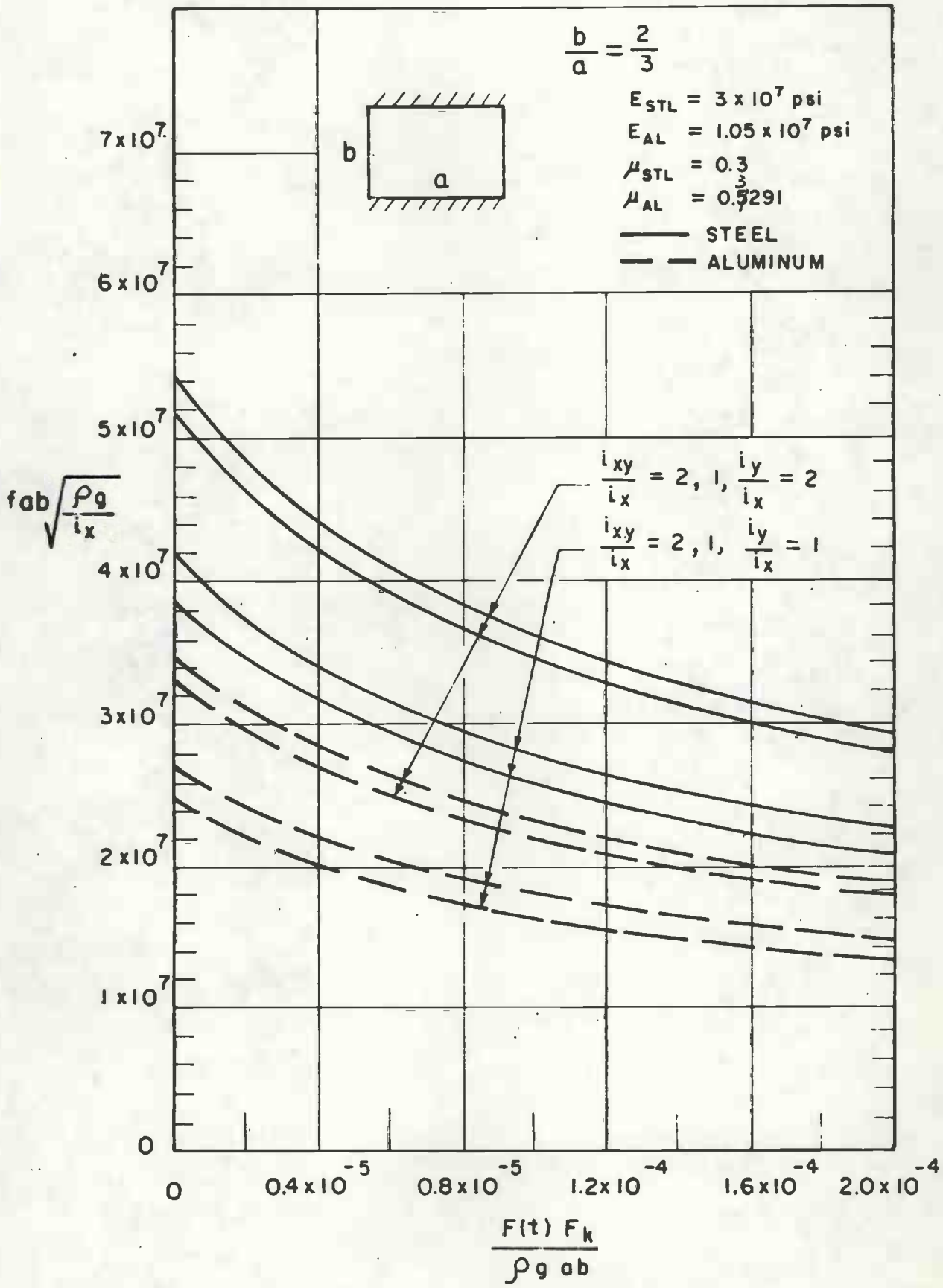

$\frac{b}{a} = \frac{2}{3}$
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
b
a
$\frac{i_{xy}}{i_x} = 2, 1, \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 2, 1, \frac{i_y}{i_x} = 1$
$fab\sqrt{\frac{\rho g}{i_x}}$
7×10^7
6×10^7
5×10^7
4×10^7
3×10^7
2×10^7
1×10^7
0
0
0.4×10^{-5}
0.8×10^{-5}
1.2×10^{-4}
1.6×10^{-4}
2.0×10^{-4}
$\frac{F(t)\, F_k}{\rho g\, ab}$

FIG. 21

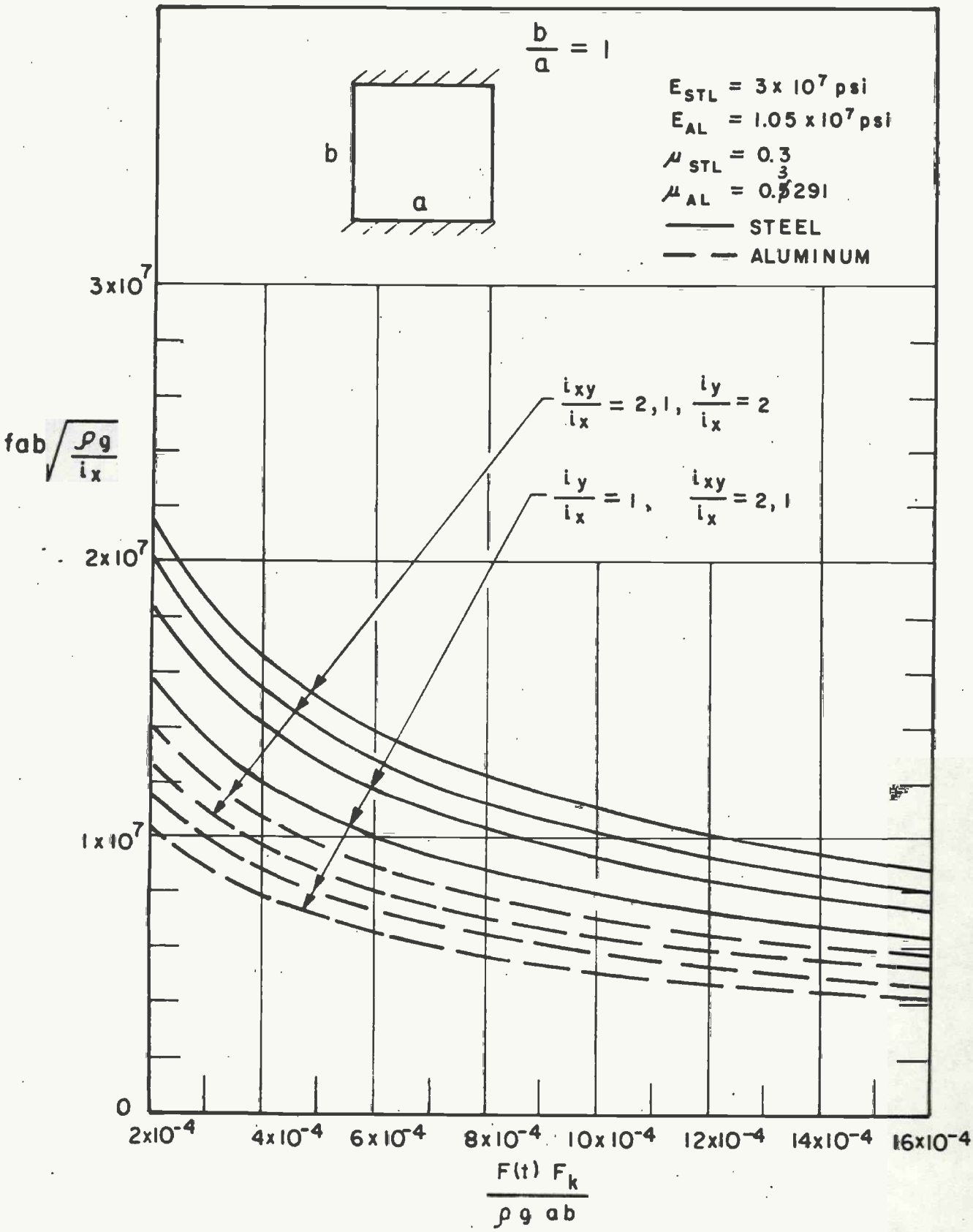

b/a = 1
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
3x10^7
2x10^7
1x10^7
0
fab √(ρg/i_x)
i_xy/i_x = 2,1, i_y/i_x = 2
i_y/i_x = 1, i_xy/i_x = 2,1
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) F_k / ρ g ab

FIG. 22

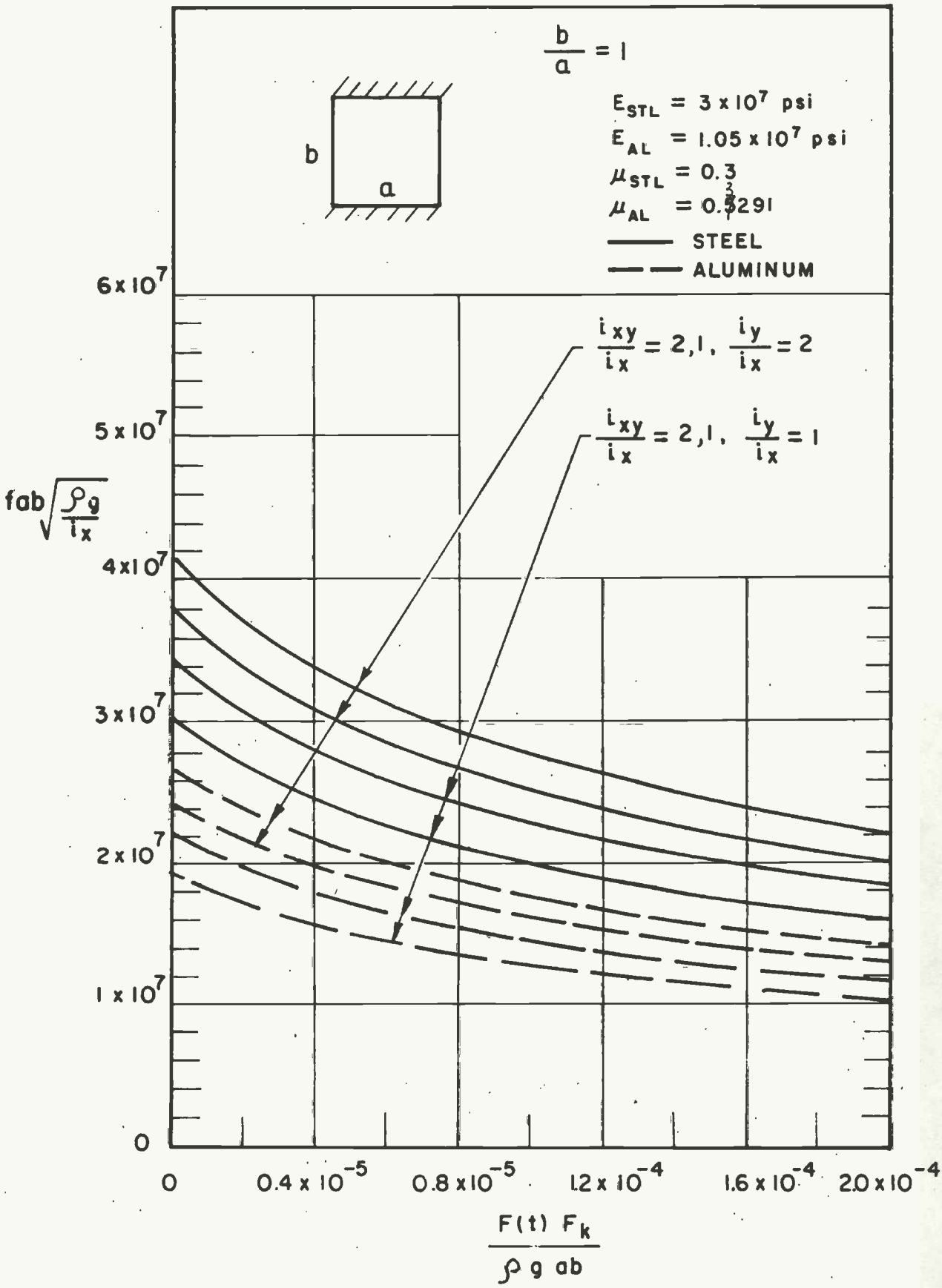

b/a = 1
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2,1, iy/ix = 2
ixy/ix = 2,1, iy/ix = 1
fab √(ρg/ix)
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1x10^7
0
0
0.4 x 10^-5
0.8 x10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x10^-4
F(t) Fk / ρ g ab

FIG. 23

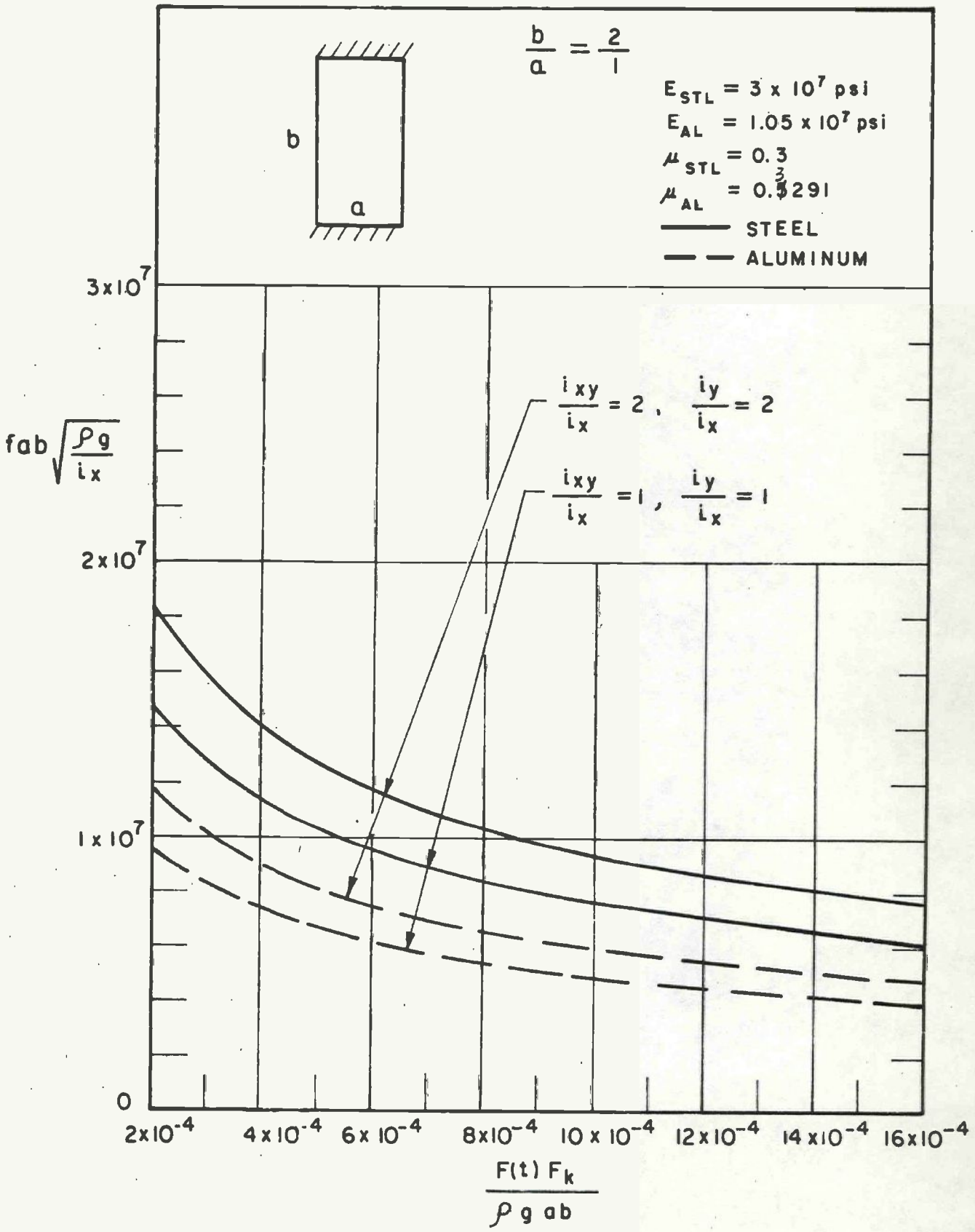

b/a = 2/1
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
ixy/ix = 2 , iy/ix = 2
ixy/ix = 1 , iy/ix = 1
fab √(ρg/ix)
3x10^7
2x10^7
1x10^7
0
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) Fk / ρ g ab

FIG. 24

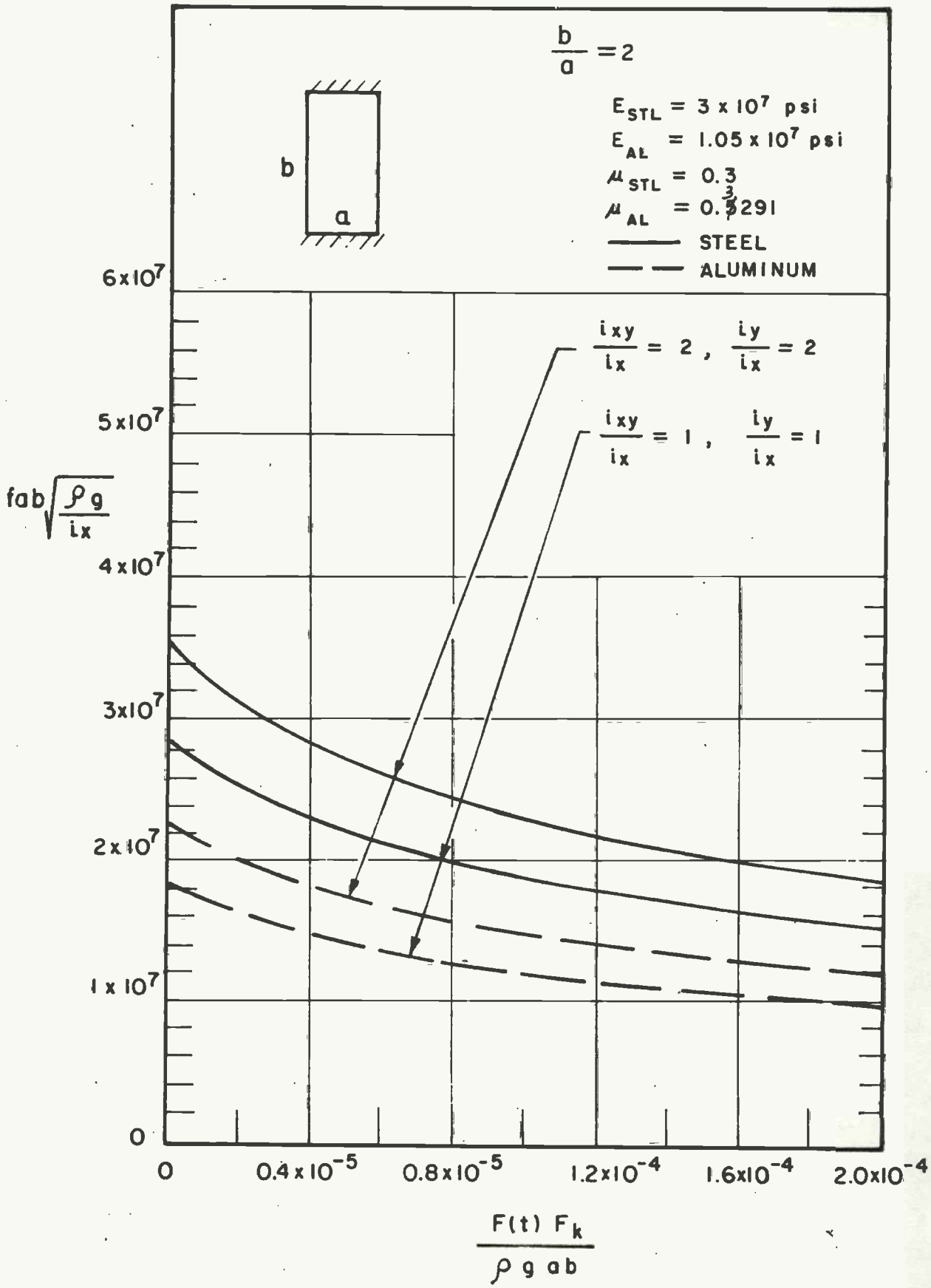

b/a = 2
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2 , iy/ix = 2
ixy/ix = 1 , iy/ix = 1
fab√(ρg/ix)
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1 x 10^7
0
0
0.4 x 10^-5
0.8 x 10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t) F_k / ρ g ab

FIG. 25

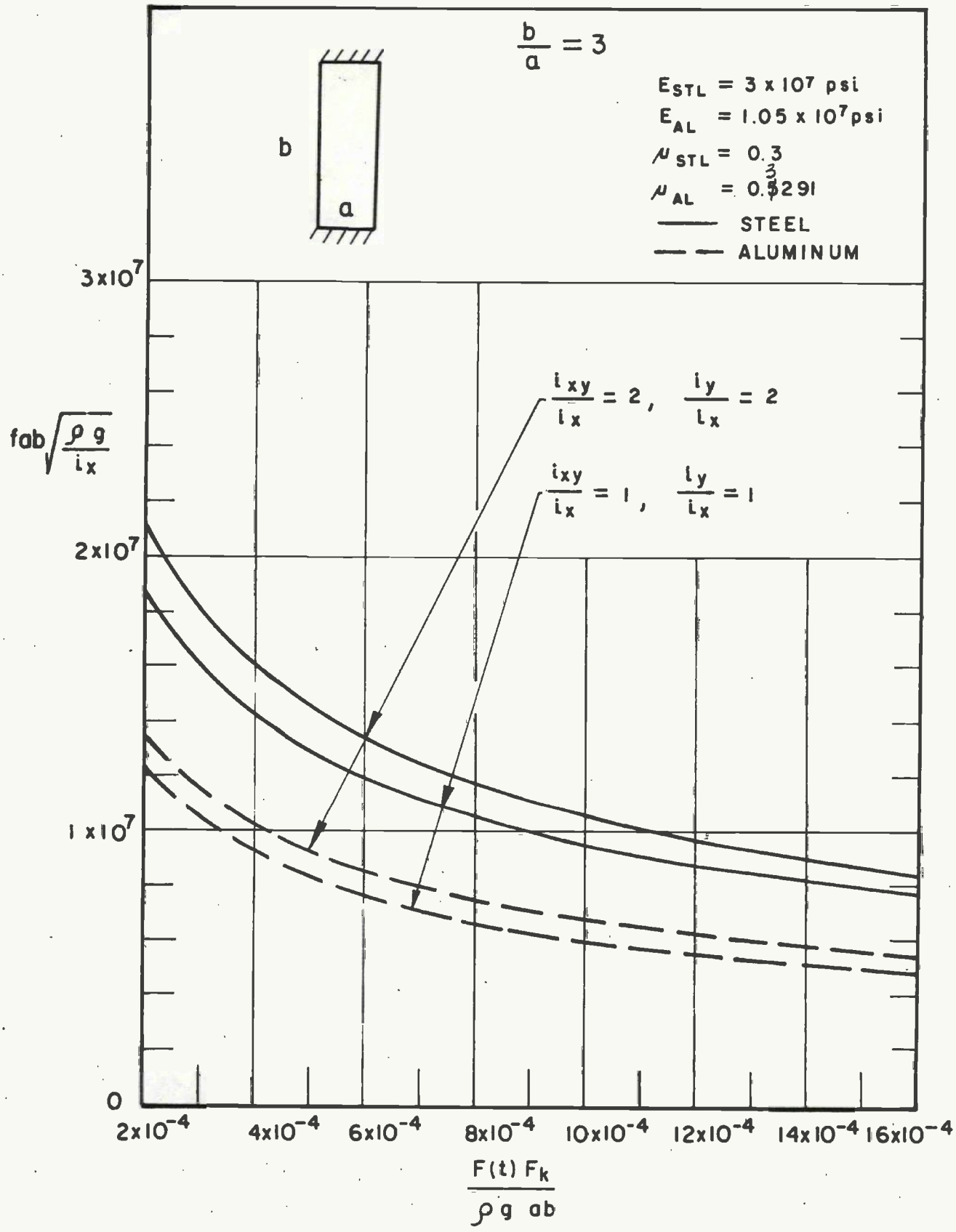

b/a = 3
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
i_xy/i_x = 2, i_y/i_x = 2
i_xy/i_x = 1, i_y/i_x = 1
fab √(ρ g / i_x)
3x10^7
2x10^7
1 x10^7
0
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) F_k / ρ g ab

FIG. 26

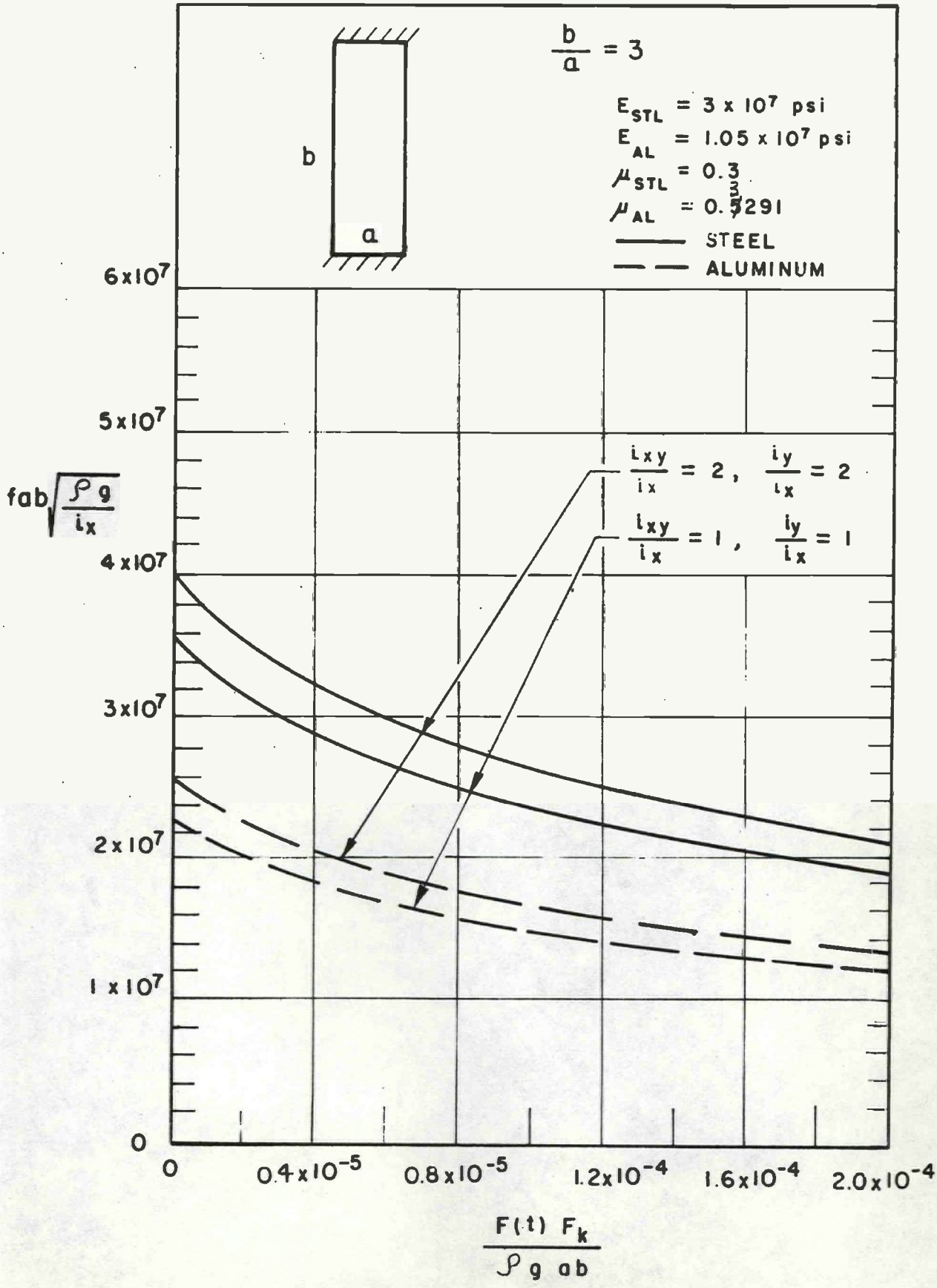

b/a = 3
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2, iy/ix = 2
ixy/ix = 1, iy/ix = 1
fab √(ρg/ix)
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1x10^7
0
0.4x10^-5
0.8x10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t) Fk / ρg ab

FIG. 27

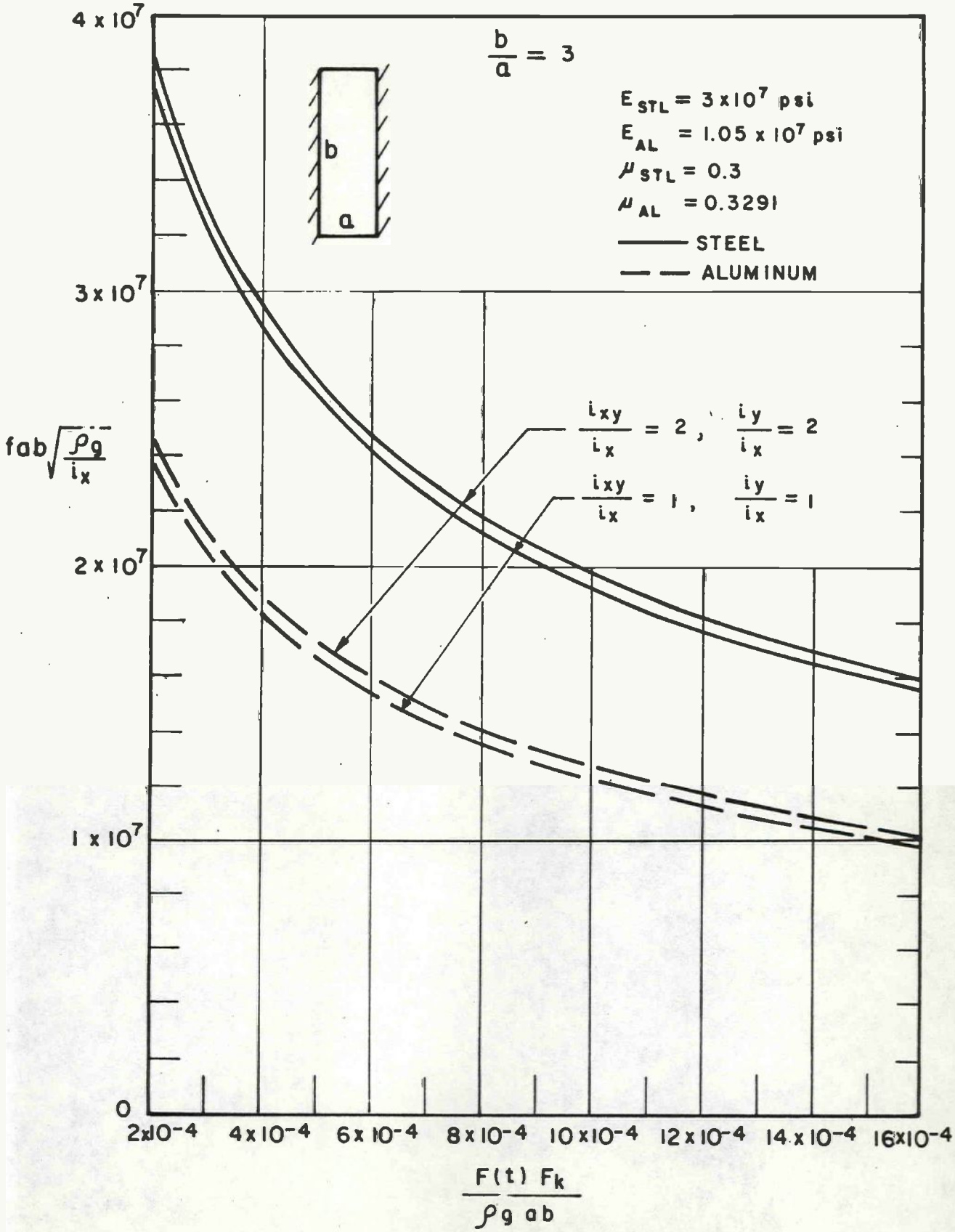

$\frac{b}{a} = 3$
$E_{STL} = 3 \times 10^7$ psi
$E_{AL} = 1.05 \times 10^7$ psi
$\mu_{STL} = 0.3$
$\mu_{AL} = 0.3291$
STEEL
ALUMINUM
b
a
$\frac{i_{xy}}{i_x} = 2, \quad \frac{i_y}{i_x} = 2$
$\frac{i_{xy}}{i_x} = 1, \quad \frac{i_y}{i_x} = 1$
$fab\sqrt{\frac{\rho g}{i_x}}$
4×10^7
3×10^7
2×10^7
1×10^7
0
2×10^{-4}
4×10^{-4}
6×10^{-4}
8×10^{-4}
10×10^{-4}
12×10^{-4}
14×10^{-4}
16×10^{-4}
$\frac{F(t)\,F_k}{\rho g\, ab}$

FIG. 28

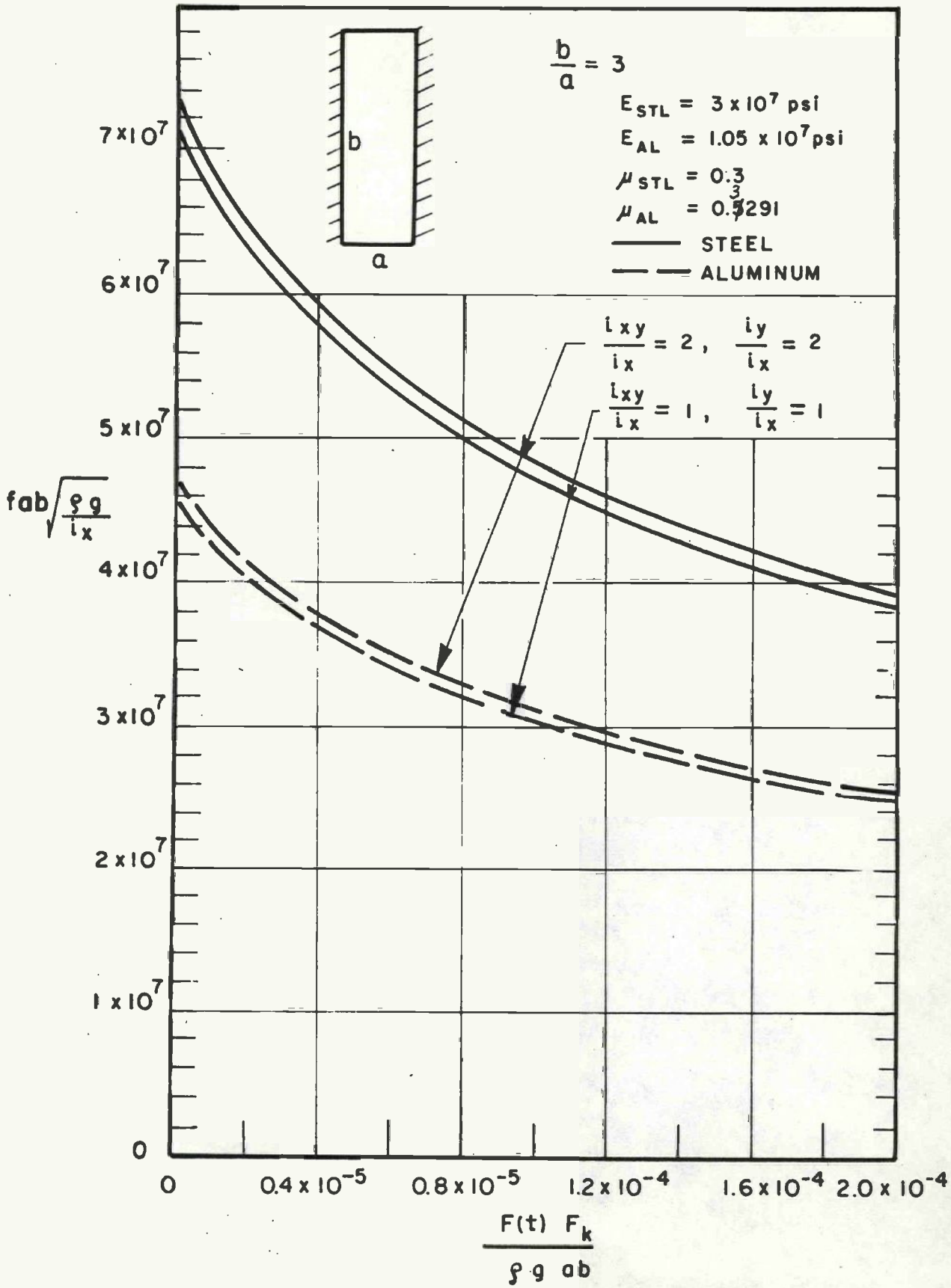

b/a = 3
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
i_xy/i_x = 2, i_y/i_x = 2
i_xy/i_x = 1, i_y/i_x = 1
7x10^7
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1x10^7
0
fab √(ρg/i_x)
0
0.4 x 10^-5
0.8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) F_k / ρ g ab

FIG. 29

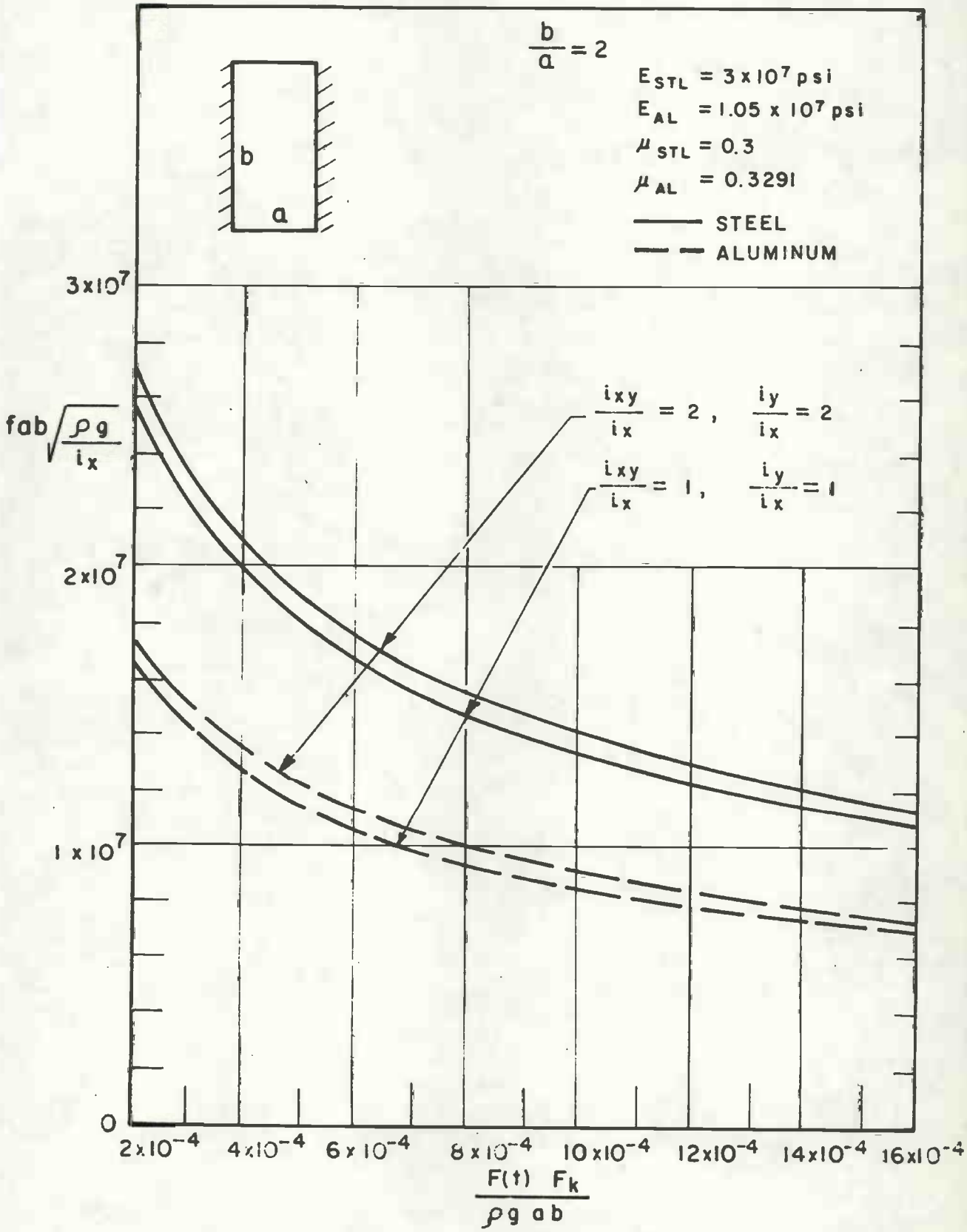

b/a = 2
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
3x10^7
2x10^7
1 x 10^7
0
fab √(ρg/ix)
ixy/ix = 2 , iy/ix = 2
ixy/ix = 1 , iy/ix = 1
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) Fk / ρg ab

FIG. 30

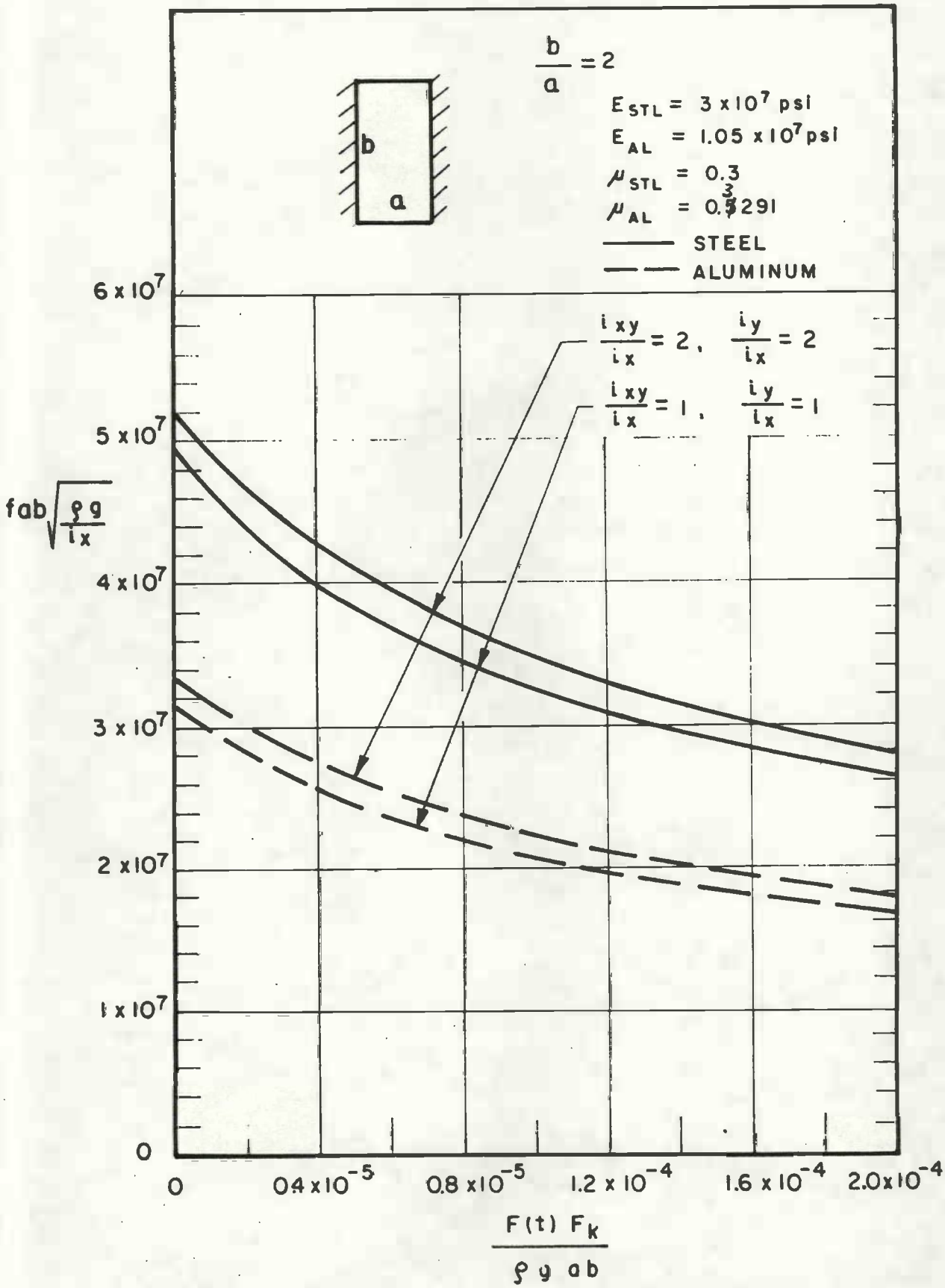

b/a = 2
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
i_xy/i_x = 2, i_y/i_x = 2
i_xy/i_x = 1, i_y/i_x = 1
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/i_x)
0
0.4 x 10^-5
0.8 x 10^-5
1.2 x 10^-4
1.6 x 10^-4
2.0 x 10^-4
F(t) F_K / ρ g ab

FIG. 31

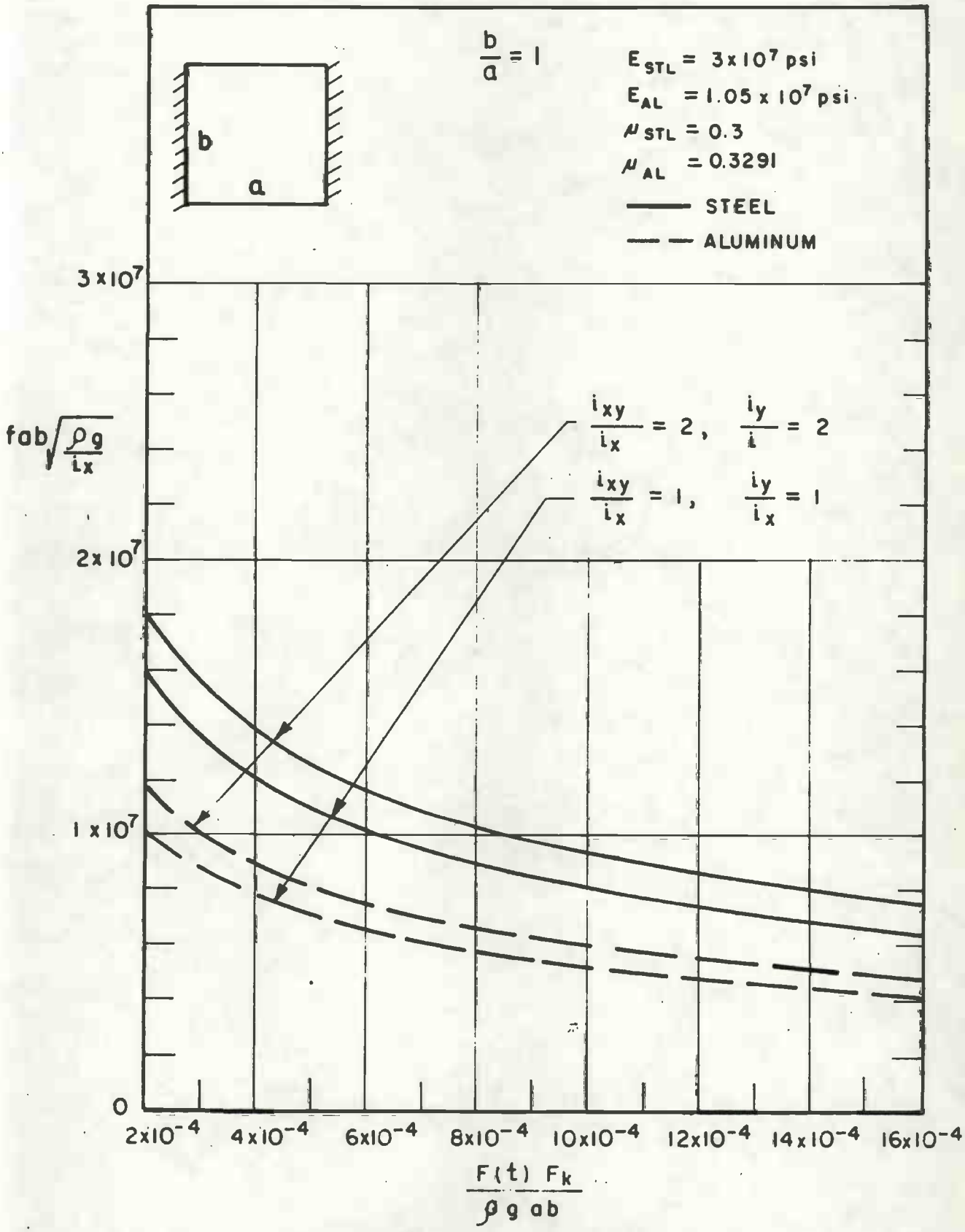

b/a = 1
E_STL = 3x10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
3x10^7
2x10^7
1x10^7
0
fab√(ρg/ix)
ixy/ix = 2, iy/i = 2
ixy/ix = 1, iy/ix = 1
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) Fk / ρ g ab

FIG. 32

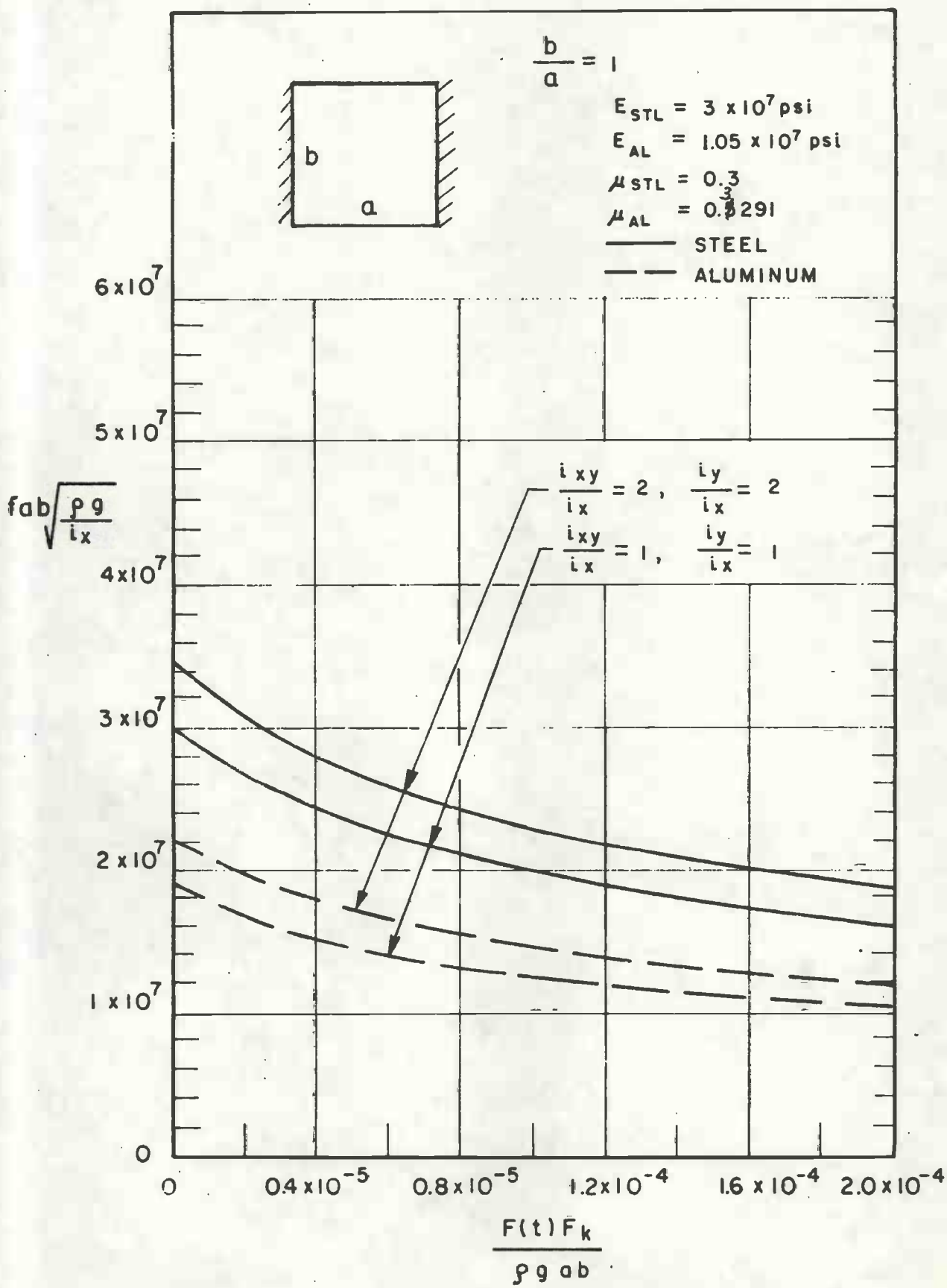

b/a = 1
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2, iy/ix = 2
ixy/ix = 1, iy/ix = 1
fab√(ρg/ix)
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1x10^7
0
0.4x10^-5
0.8x10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t)Fk / ρgab

FIG. 33

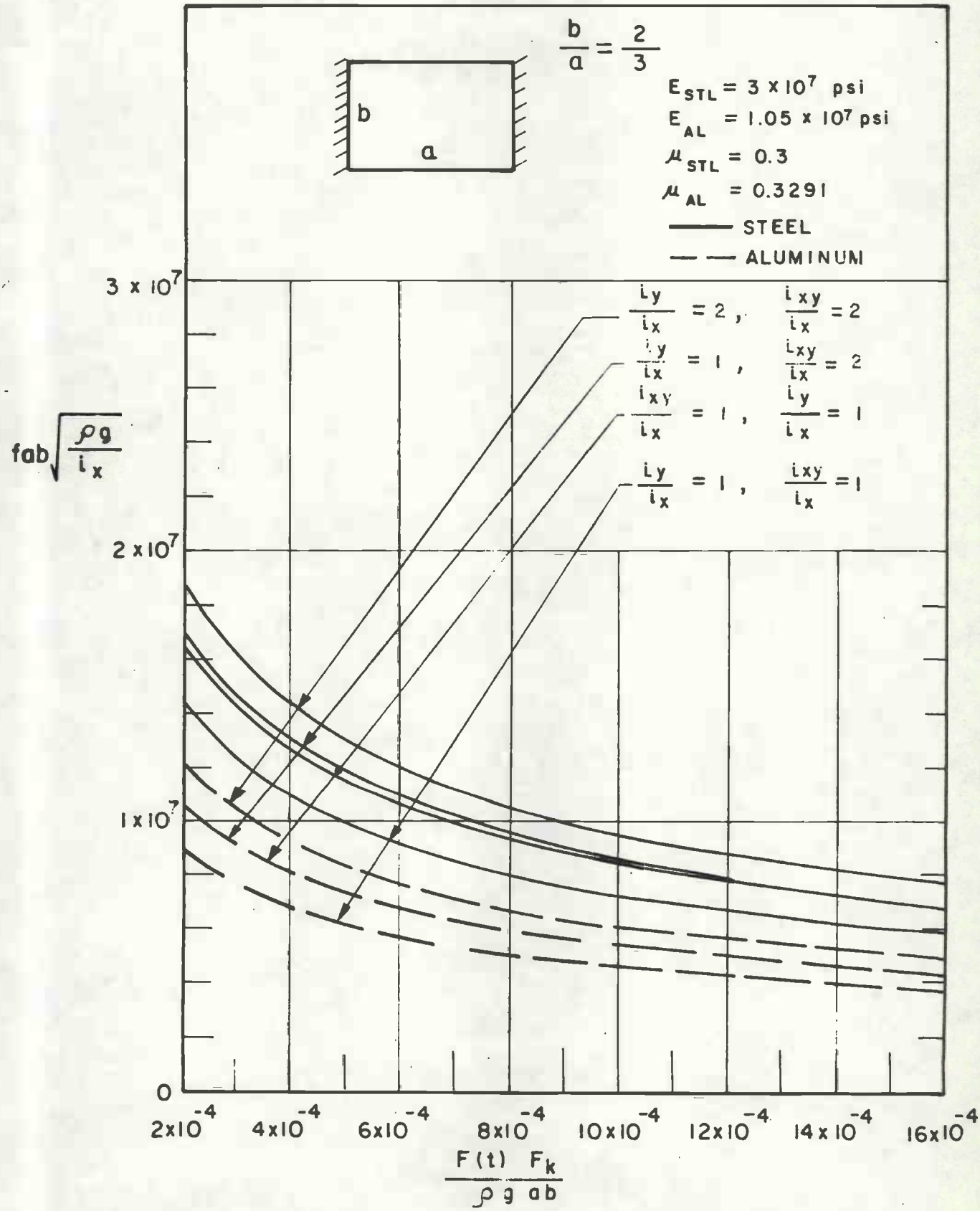

b/a = 2/3
b
a
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
iy/ix = 2, ixy/ix = 2
iy/ix = 1, ixy/ix = 2
ixy/ix = 1, iy/ix = 1
iy/ix = 1, ixy/ix = 1
fab √(ρg/ix)
3 x 10^7
2 x 10^7
1 x 10^7
0
2x10^-4
4x10^-4
6x10^-4
8x10^-4
10x10^-4
12x10^-4
14x10^-4
16x10^-4
F(t) Fk / ρ g ab

FIG. 34

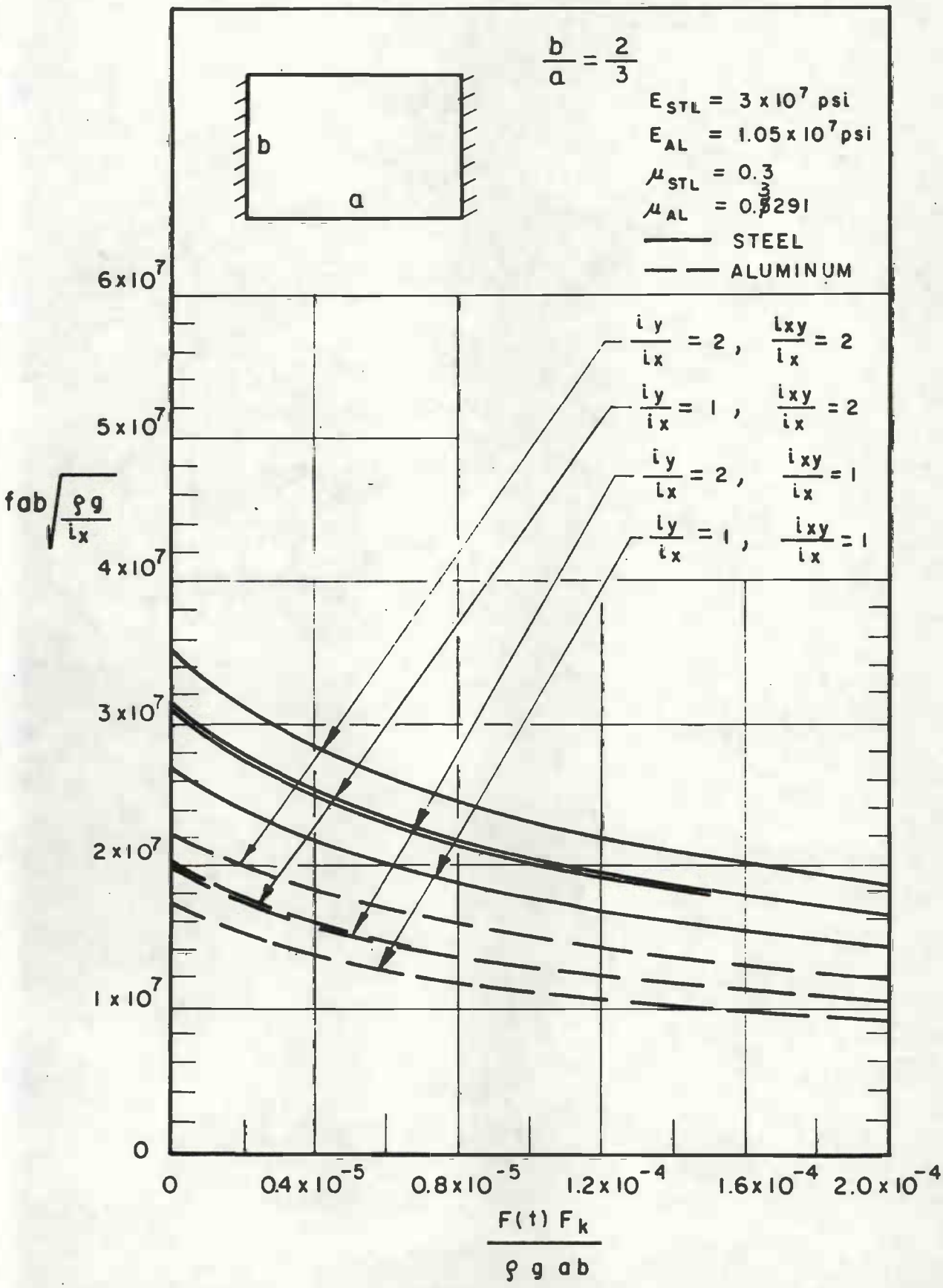

b/a = 2/3
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
iy/ix = 2, ixy/ix = 2
iy/ix = 1, ixy/ix = 2
iy/ix = 2, ixy/ix = 1
iy/ix = 1, ixy/ix = 1
fab √(ρg/ix)
6x10^7
5x10^7
4x10^7
3x10^7
2x10^7
1x10^7
0
0
0.4x10^-5
0.8x10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t) Fk / ρ g ab

FIG. 35

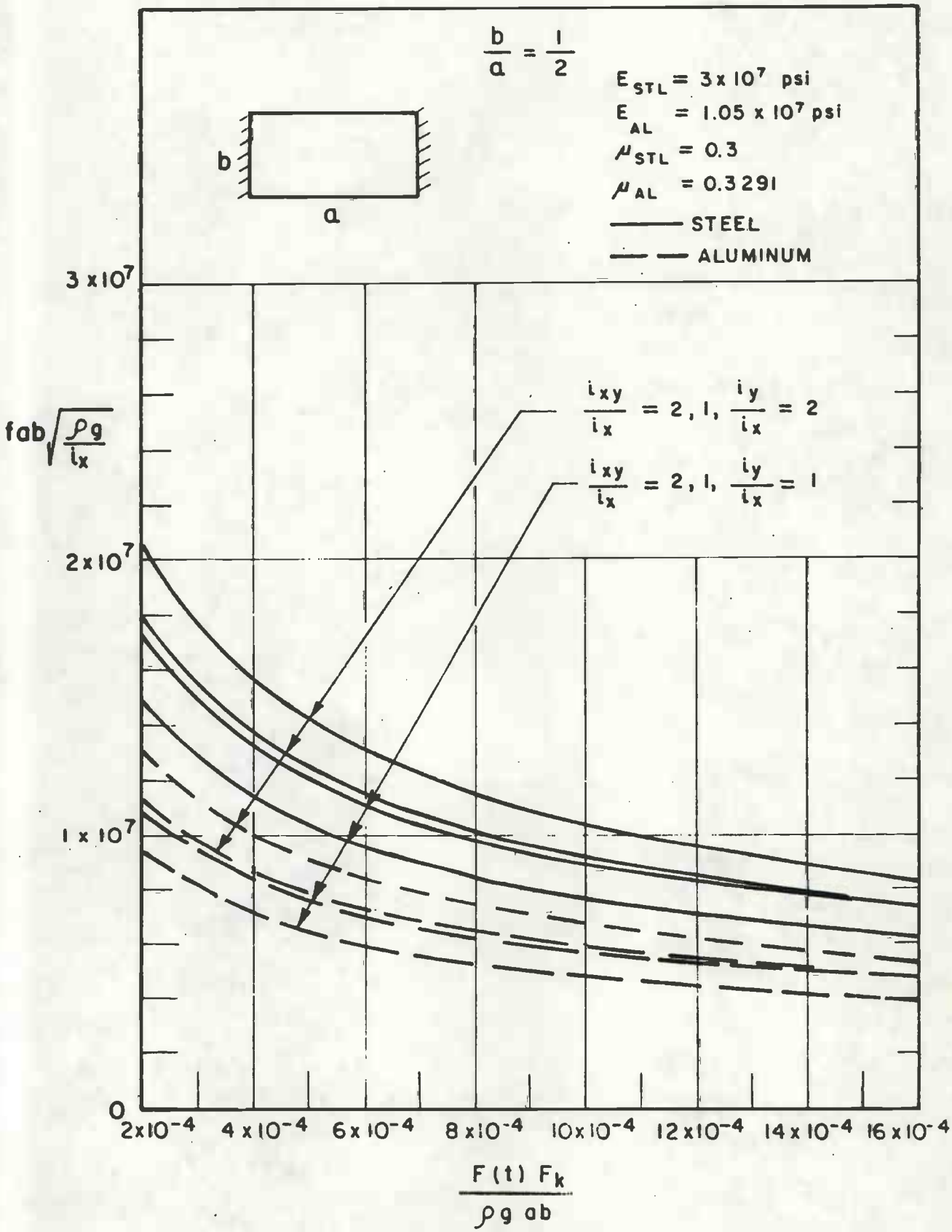

b/a = 1/2
E_STL = 3 x 10^7 psi
E_AL = 1.05 x 10^7 psi
μ_STL = 0.3
μ_AL = 0.3291
STEEL
ALUMINUM
b
a
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/i_x)
i_xy/i_x = 2, 1, i_y/i_x = 2
i_xy/i_x = 2, 1, i_y/i_x = 1
2x10^-4
4 x 10^-4
6 x 10^-4
8 x 10^-4
10x10^-4
12x10^-4
14 x 10^-4
16 x 10^-4
F(t) F_k / ρg ab

FIG. 36

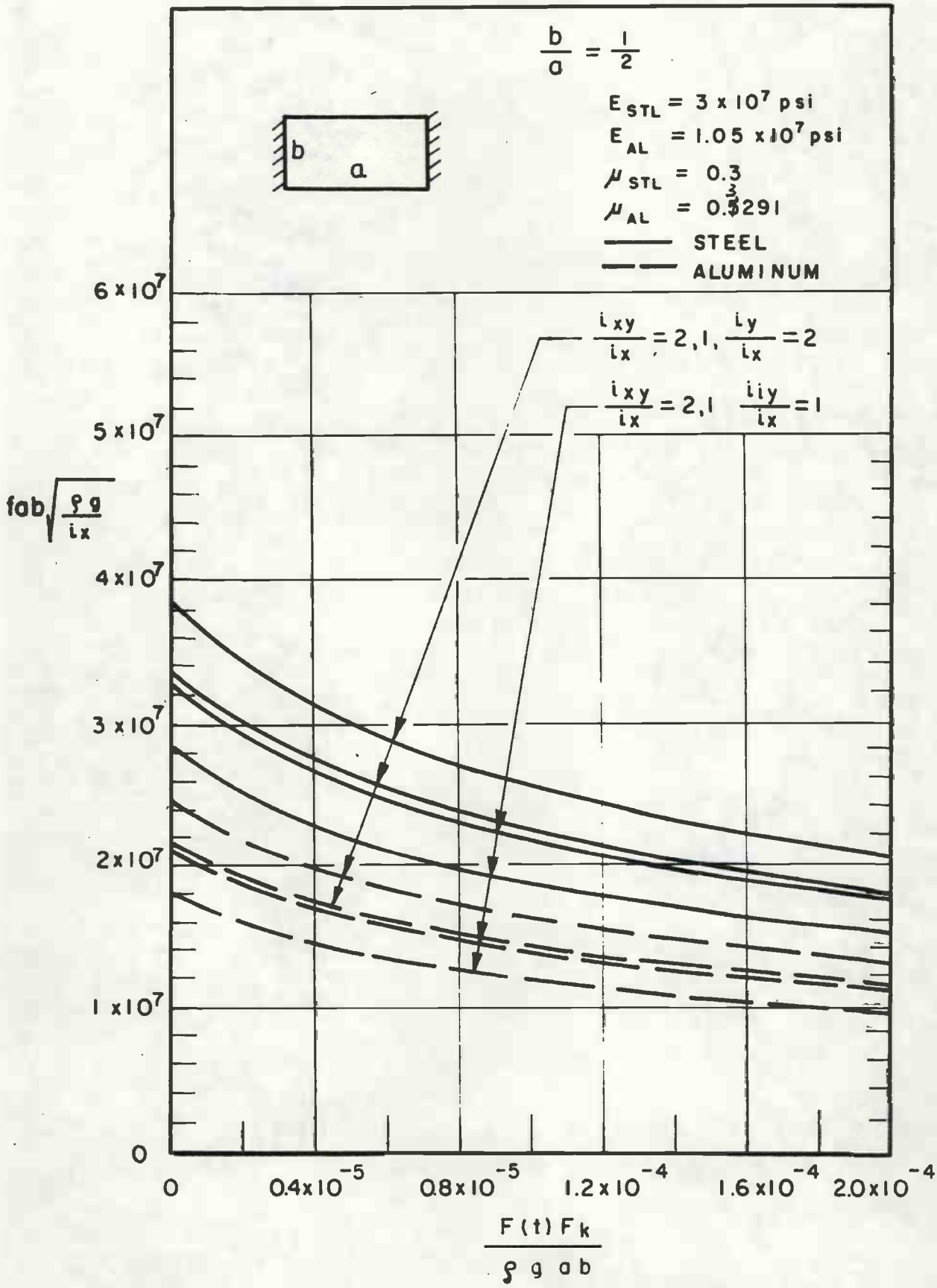

b/a = 1/2
E STL = 3 x 10^7 psi
E AL = 1.05 x 10^7 psi
μ STL = 0.3
μ AL = 0.3291
STEEL
ALUMINUM
b
a
ixy/ix = 2,1, iy/ix = 2
ixy/ix = 2,1 iiy/ix = 1
6 x 10^7
5 x 10^7
4 x 10^7
3 x 10^7
2 x 10^7
1 x 10^7
0
fab √(ρg/ix)
0
0.4x10^-5
0.8x10^-5
1.2x10^-4
1.6x10^-4
2.0x10^-4
F(t) Fk / ρ g a b

FIG. 37

TABLE 1 VALUES OF $\frac{g f_E F_k}{\rho g a b}$ CORRESPONDING TO THE RATIO OF $f a b \sqrt{\frac{\rho g}{i x}}$

	①	②	③	④	⑤		⑥		⑦
n	$\frac{g f_E F_k}{\rho g a b}$	$\frac{1}{S_1 S_1'}$	1+②①	$\sqrt{③}$	$\frac{60}{2\pi}\sqrt{gE}\left[\frac{lb^{\frac{1}{2}}}{in\ sec}\right]$		⑤/④		$\frac{⑥_n}{⑥_9}$
					Al	Steel	Al	Steel	
1	0	12,803	1	1	650,248	1,028,140	650,245	1,028,137	3.7152
2	0.4167×10^{-4}		1.53360	1.2384			525,069	830,214	3.0000
3	0.9439×10^{-4}		2.20850	1.4861			437,557	691,845	2.4999
4	1.914×10^{-4}		3.45067	1.8576			350,046	553,476	2.0000
5	2.546×10^{-4}		4.26009	2.0640			315,041	498,128	1.8000
6	3.430×10^{-4}		5.39168	2.3220			280,037	442,781	1.6000
7	4.719×10^{-4}		7.04212	2.6537			245,033	387,433	1.4000
8	6.676×10^{-4}		9.54810	3.0960			210,028	332,086	1.2000
9	10.000×10^{-4}		13.8032	3.7152			175,023	276,738	1.0000
10	12.529×10^{-4}		17.0404	4.1280			157,521	249,064	0.9000
11	16.067×10^{-4}		21.5714	4.6440			140,020	221,390	0.8000
12	22.7820×10^{-4}		29.1684	5.3074			122,516	193.716	0.7000

Values of $fab\sqrt{\frac{\rho g}{i_x}}$ for $\frac{k\gamma_E F_k}{\rho g a b} = 10^{-3}$ in the case of both sides fixed and both ends fixed are given in Table 2, which corresponds to $S_1 = S_1' = \frac{1}{630}$, $S_2 = S_2' = \frac{4}{5}$ and $S_3 = S_3' = \frac{2}{105}$

In Table 3 we show the case of both sides fixed but both ends supported, which corresponds to

$$S_1 = \frac{31}{630}, \quad S_2 = \frac{24}{5}, \quad S_3 = \frac{51}{105}$$

$$S_1' = \frac{1}{630} \quad S_2' = \frac{4}{5}, \quad S_3' = \frac{2}{105}$$

Also in Table 4 we show the case of both sides supported and both ends clamped, which corresponds to

$$S_1 = \frac{1}{630}, \quad S_2 = \frac{4}{5}, \quad S_3 = \frac{2}{105}$$

$$S_1' = \frac{31}{630}, \quad S_2' = \frac{24}{5}, \quad S_3' = \frac{51}{105}$$

Let us now explain how to use the curves to get the value of $fab\sqrt{\frac{\rho g}{i_x}}$ for the case of $\frac{i_y}{i_x} = 2$, $\frac{i_{xy}}{i_x} = 1$, $\frac{k\gamma_E F_k}{\rho g a b} = 1.8\text{x}10^{-3}$ and with all clamped sides.

From Table 2 we find

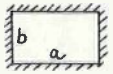
$$fab\sqrt{\frac{\rho g}{i_x}} = 18.5 \times 10^7 \quad \text{for} \quad \frac{k\gamma_E F_k}{\rho g a b} = 10^{-3}$$

Then from Table 1 we can get

$$fab\sqrt{\frac{\rho g}{i_x}} = 18.51 \times 10^7 \times 0.7 = 12.957 \times 10^7$$

$$\text{for} \quad \frac{k\gamma_E F_k}{\rho g a b} = 22.782 \times 10^{-4}$$

TABLE 2 VALUES OF $fab\sqrt{\frac{\rho g}{i_x}}$ WHEN $\frac{g t_E F_k}{\rho g a b} = 0.001$ FOR A DOUBLE BOTTOM WITH ALL SIDES CLAMPED



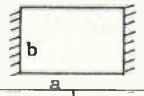
$S_1 = \frac{1}{630}$, $S_1' = \frac{1}{630}$, $S_2 = \frac{4}{5}$, $S_2' = \frac{4}{5}$, $S_3 = \frac{2}{105}$, $S_3' = \frac{2}{105}$

(1)	(2)	(3)	(4)	(5)		(6)	(7)		(8)	(9)	(10)		(11)*	
$\frac{S_2}{S_1}$	$\left(\frac{b}{a}\right)^2$	(1)x(2)	$\frac{S_3}{S_1} \times \frac{S_3'}{S_1'}$	$\frac{2}{1-\mu^2}$		$\frac{i_{xg}}{i_x}$	(4)x(5)x(6)		$\frac{i_y}{i_x}$	(3) / (2) x $\frac{S_2'}{S_1'}$	(3)+(7)+(9)		$fab\sqrt{\frac{\rho g}{i_x}} = \lambda\sqrt{(10)}$	
				Al	Stl		Al	Stl			Al	Stl	Al	Stl
504.00	3^2	4536.0	118.451	2.2429	2.1978	1	322.977	316.483	1	56.00	4915.00	4908.50	12.432×10^7	19.387×10^7
						2	645.955	632.966	2	112.00	5293.9	5281.0	12.901	20.110
	2^2	2016.0				1	322.977	316.483	1	126.00	2465.0	2458.5	8.804	13.720
						1			2	252.00	2591.0	2584.5	9.026	14.067
						2	645.955	632.966	1	126.00	2788.0	2775.0	9.363	14.578
						2			2	252.00	2914.0	2901.0	9.572	14.905
	1^2	504.00				1	322.977	316.483	1	504.00	1331.0	1324.5	6.469	10.069
						1			2	1008.00	1835.0	1828.5	7.596	11.832
						2	645.955	632.966	1	504.00	1653.9	1640.9	7.209	11.210
						2			2	1008.00	2157.9	2144.9	8.237	12.817
	$(2/3)^2$	223.77				1	322.977	316.483	1	1135.13	1681.9	1675.4	7.237	11.325
						1			1.5	1702.69	2249.4	2242.9	8.412	13.106
						1			2	2270.26	2817.0	2810.5	9.411	14.669
						2	645.955	632.966	1	1135.13	2004.8	1998.3	7.938	12.369
						2			1.5	1702.69	2572.4	2559.4	8.993	13.999
						2			2	2270.26	3140.0	3127.0	9.940	15.475
	$\left(\frac{1}{2}\right)^2$	126.00				1	322.977	316.483	1	2016.00	2465.0	2458.5	8.804	13.720
						1			1.5	3024.00	3473.0	3466.5	10.450	16.292
						1			2	4032.00	4481.0	4474.5	11.870	18.510
						2	645.955	632.966	1	2016.00	2787.9	2874.9	9.361	14.575
						2			1.5	3024.00	3795.9	3782.9	10.924	17.025
						2			2	4032.00	4804.0	4791.0	12.290	19.155

* $\lambda = \frac{60}{2\pi}\left[\frac{gE}{1+\frac{1}{S_1 S_1'}\frac{g t_E F_k}{\rho g a b}}\right]^{\frac{1}{2}}$ = 175,023 for Aluminum

276,738 for Mild Steel

TABLE 3 VALUES OF $fab\sqrt{\frac{\rho g}{i_x}}$ WHEN $\frac{\frac{g}{k}E_k}{\rho g a b} = 0.001$ FOR A DOUBLE BOTTOM WITH BOTH ENDS CLAMPED AND BOTH SIDES SIMPLY SUPPORTED

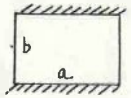


$S_1 = \frac{1}{630}, S_2 = \frac{4}{5}, S_3 = \frac{2}{105}, S_1' = \frac{31}{630}, S_2' = \frac{24}{5}, S_3' = \frac{51}{105}$

(1)	(2)	(3)	(4)	(5)		(6)	(7)		(8)	(9)	(10)		(11)*	
$\frac{S_2}{S_1}$	$\left(\frac{b}{a}\right)^2$	(1)x(2)	$\frac{S_2}{S_1} \times \frac{S_3'}{S_1'}$	$\frac{2}{1-\mu^2}$		$\frac{i_{xg}}{i_x}$	(4)x(5)x(6)		$\frac{i_g}{i_x}$	$\frac{(8)}{(2)} \times \frac{S_2'}{S_1'}$	(3)+(7)+(9)		$fab\sqrt{\frac{\rho g}{i_x}} = \lambda\sqrt{(10)}$	
				Al	Stl		Al	Stl			Al	Stl	Al	Stl
504.00	3^2	4536.00	118.451	2.2429	2.1978	1	265.674	260.331	1	10.838	4812.5	4807.2	12.301	19.186×10^7
						2	531.348	520.662	2	21.676	5089.1	5078.5	12.650	19.731
	2^2	2016.00				1	265.674	260.331	1	24.387	2306.1	2300.8	8.515	13.283
						1			2	48.774	—	—	—	—
						2	531.348	520.662	1	24.387	—	—	—	—
						2			2	48.774	2596.20	2585.6	9.035	14.086
	1^2	504.00				1	265.674	260.331	1	97.548	867.2	862.0	5.222	8.136
						1			2	195.096	964.8	959.8	5.508	8.579
						2	531.348	520.662	1	97.548	1132.9	1122.9	5.969	9.298
						2			2	195.096	1230.5	1220.5	6.042	9.409
	$\left(\frac{2}{3}\right)^2$	223.77				1	265.674	260.331	1	219.483	708.9	703.9	4.719	7.350
						1			2	438.966	928.4	923.4	5.434	8.468
						2	531.348	520.662	1	219.483	974.6	964.6	5.534	8.606
						2			2	438.966	1194.1	1184.1	6.127	9.547
	$\left(\frac{1}{2}\right)^2$	126.00				1	265.674	260.331	1	390.192	781.9	776.9	4.959	7.721
						1			1.5	585.288	977.0	972.0	5.543	8.634
						1			2	780.192	1172.0	1167.0	5.893	9.186
						2	531.348	520.662	1	390.192	1047.6	1037.6	5.733	8.936
						2			1.5	585.288	1242.7	1232.7	6.072	9.463
						2			2	390.192	1437.8	1427.8	6.722	10.473

* $\lambda = \frac{60}{2\pi}\left[\frac{gE}{1+\frac{1}{S_1 S_1'}\frac{\frac{g}{k}E_k}{\rho g a b}}\right]^{\frac{1}{2}}$ = 175,023 for Aluminum
276,738 for Mild Steel

TABLE 4 VALUES OF $fab\sqrt{\frac{\rho g}{i_x}}$ WHEN $\frac{gEF_k}{\rho gab} = 0.001$ FOR A DOUBLE BOTTOM WITH BOTH SIDES CLAMPED AND BOTH ENDS SIMPLY SUPPORTED



$S_1 = \frac{1}{630}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{2}{105}$, $S_1' = \frac{31}{630}$, $S_2' = \frac{24}{5}$, $S_3' = \frac{51}{105}$

(1)	(2)	(3)	(4)	(5)		(6)	(7)		(8)	(9)	(10)		(11)*	
$\frac{S_2}{S_1}$	$\left(\frac{b}{a}\right)^2$	(1)x(2)	$\frac{S_3}{S_1} \times \frac{S_3'}{S_1'}$	$\frac{2}{1-\mu^2}$		$\frac{i_{xg}}{i_x}$	(4)x(5)x(6)		$\frac{i_y}{i_x}$	$\frac{(8)}{(2)} \times \frac{S_2'}{S_1'}$	(3)+(7)+(9)		$fab\sqrt{\rho g/i_x} = \lambda\sqrt{(10)}$	
				Al	Stl		Al	Stl			Al	Stl	Al	Stl
97.548	3^2	877.93	118.451	2.2429	2.1978	1	265.674	260.331	1	56.00	1199.6	1194.3	6.1402×10^7	9.5924×10^7
						2	531.348	520.662	2	112.00	1521.3	1510.7	6.9158	10.7535
	2^2	390.19				1	265.674	260.331	1	126.00	781.9	776.6	4.9571	7.7088
						1			2	252.00	907.9	902.6	5.3429	8.3132
						2	531.348	520.662	1	126.00	1047.5	1036.9	5.7379	8.9070
						2			2	252.00	1173.5	1162.9	6.0707	9.4334
	1^2	97.55				1	265.674	260.331	1	504.00	867.2	861.9	5.2205	8.1247
						1			2	1008.00	1371.2	1365.9	6.5660	10.2279
						2	531.348	520.662	1	504.00	1132.9	1122.3	5.9662	9.2696
						2			2	1008.00	1636.9	1626.3	7.1724	11.1589
	$(\frac{2}{3})^2$	43.31				1	265.674	260.331	1	1135.13	1444.1	1438.8	6.7385	10.4939
						1			1.5	1702.69	2011.6	2006.3	7.9521	12.3945
						1			2	2270.26	2579.2	2573.9	9.0055	14.0372
						2	531.348	520.662	1	1135.13	1700.8	1690.2	7.3113	11.3764
						2			1.5	1707.69	2277.3	2266.7	8.4620	13.1791
						2			2	2270.26	2835.9	2825.3	9.4435	14.7089
	$(\frac{1}{2})^2$	24.39				1	265.674	260.331	1	2016.00	2306.0	2300.7	8.5154	13.2718
						1			1.5	3024.00	3314.0	3308.7	10.2083	15.9165
						1			2	4032.00	4322.0	4316.7	11.6578	18.1952
						2	531.348	520.662	1	2016.00	2571.7	2561.1	8.9915	14.0054
						2			1.5	3024.00	3579.7	3569.1	10.6085	16.5326
						2			2	4032.00	4587.8	4577.2	12.0100	18.7426

* $\lambda = \frac{60}{2\pi}\left[\frac{gE}{1+\frac{1}{S_1S_1'}\frac{gEF_k}{\rho gab}}\right]^{\frac{1}{2}} =$ 175,023 for Aluminum
276,738 for Mild Steel

From the curve

$$fab\sqrt{\frac{\rho g}{i_x}} = 15.2 \times 10^7 \quad \text{for} \quad \frac{k\,\overset{g}{b}_E \cdot F_k}{\rho g a b} = 16.00 \times 10^{-4}$$

We, therefore, can obtain by interpolation

$$fab\sqrt{\frac{\rho g}{i_x}} = \left(12.957 + 2.243 \times \frac{4.782}{6.782}\right) \times 10^7 = 14.538 \times 10^7 \quad \text{for} \quad \frac{k\,\overset{g}{b}_E F_k}{\rho g a b} = 18.00 \times 10^{-4}$$

On the other hand we may easily obtain, by connecting both points of 18.51×10^7 and 12.957×10^7 with reasonable curve corresponding to 10^{-3} and 2.2782×10^{-3} on the abscissa

$$fab\sqrt{\frac{\rho g}{i_x}} = 14.4 \times 10^7 \qquad \frac{k\,\overset{g}{b}_E F_k}{\rho g a b} = 18.00 \times 10^{-4}$$

Now let us return to the problem of the double bottom having the scantlings and material characteristics as already given in Fig. 7.

In this case there is neither machinery nor pillar, hence

$$\frac{k\,\overset{g}{b}_E F_k}{\rho g a b} = 0$$

From the curves of Fig. (32) and Fig. (34) we have

$fab\sqrt{\frac{pg}{i_x}}$	$\frac{i_y}{i_x}$	$\frac{i_{xy}}{i_x}$	$\frac{b}{a}$
3.475×10^7	2	2	1
3.025×10^7	1	1	1
3.530×10^7	2	2	2/3
3.18×10^7	1	2	2/3
3.13×10^7	2	1	2/3
2.725×10^7	1	1	2/3
	1.830	1.478	0.8433

Applying interpolation to above data we can obtain

$$(3.025 + \left(\frac{0.830+0.478}{2}\right) \times 0.45) \times 10^7 = 3.317 \times 10^7$$

$$(3.18 + 0.35 \times 0.83) \times 10^7 = 3.470 \times 10^7$$

$$(2.725 + 0.405 \times 0.83) \times 10^7 = 3.061 \times 10^7$$

$$(3.061 + 0.409 \times 0.478) \times 10^7 = 3.256 \times 10^7$$

which reduces

$$f = 3.256 \times 10^7 \times \sqrt{\frac{273.2}{0.5}} \times \frac{1}{635,376} = 1198.11 \text{ CPM}$$

or = 19.968 cps

From Eqs (10) and Appendix I we may obtain the relation between load factor and time ratio $\frac{\Delta}{T}$, as shown in Fig. 39 with Fig. 38, after substituting $m = n = 1$, $\beta = \frac{1}{2}$.

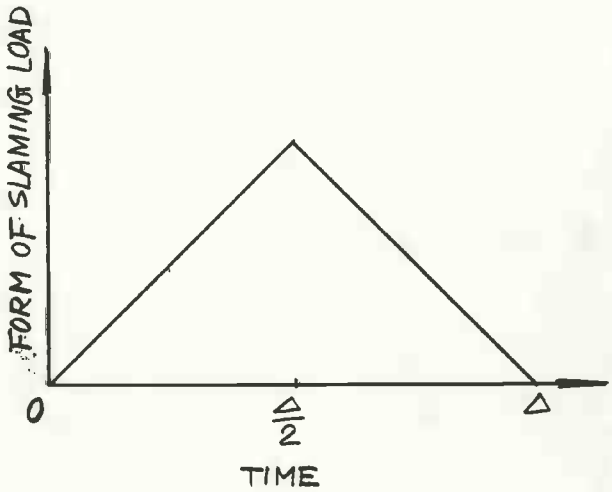


Fig. 38 Relation between Slamming Loading and Time

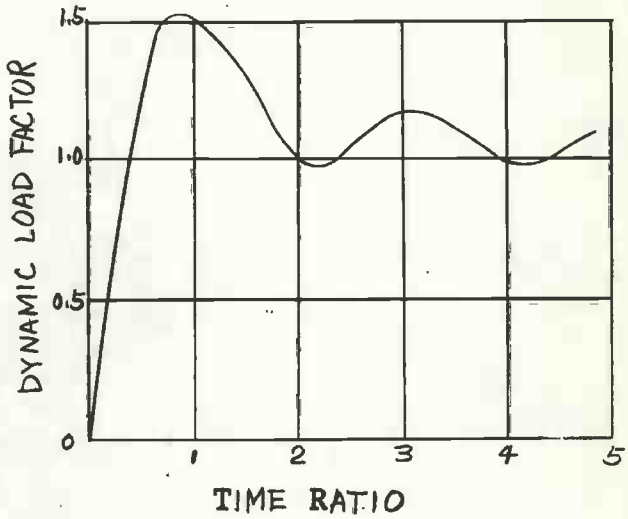


Fig. 39 Relation between Load Factor and Time Ratio

Therefore, the load factor L_{11} becomes 1.00 corresponding to $\frac{\Delta}{T}$ = 0.02 x 19.968 = 0.39936, because of Δ = 0.02 sec in this case.

Then the maximum value of deflection w becomes from Eq. (11)

$$w_{max} = A_{11} L_{11} \left[F_1(x) \right]_{x=\frac{a}{2}}, \tag{87}$$

where if i, j = 1,2,3 [6] [7]

$$A_{ij} = \frac{4a^4 F_M \lambda_i b}{E \pi j} \left[\gamma_i I_q(\alpha+1) + \left(\frac{a}{b}\right)^3 \pi^4 j^4 T_i^2 I_s(\beta+1) + \pi^2 j^2 K_i^2 \frac{i_{zy}}{1-\mu^2} \frac{a^2}{b} \right]^{-1} * \tag{88}$$

$\lambda_i = \frac{1}{a}\int_0^a F_i(x)dx$ in which $F_i(x)$ is the basic function given by Inglis and

---

* The third term in bracket is revised from the paper already published. [7]

$$F_i(x) = T_i\left(\cosh\beta_i\frac{x}{a} - \cos\beta_i\frac{x}{a}\right) - \left(\sinh\beta_i\frac{x}{a} - \sin\beta_i\frac{x}{a}\right),$$

$$T_i = \frac{\sinh\beta_i - \sin\beta_i}{\cosh\beta_i - \cos\beta_i}$$

β_i is the solution of $\cosh\beta_i \cos\beta_i = 1$ (89)

$$K_i^2 = -a\int_0^a \left(\frac{dF_i(x)}{dx}\right)^2 dx = \int_0^a F_i(x)\left(\frac{d^2F_i(x)}{dx^2}\right)dx$$

$$\gamma_i = (\beta_i)^4 T_i^2$$

λ_i and so on will appear in Table 5 for i = 1~3. After substituting all data into A_{11}, then we have

$$A_{11} = 0.94758,$$

which reduces

$$w_{max} = 0.94758 \times 1.00 \times 1.6165 = 1.53174 \text{ in}$$

TABLE 5 VALUES OF THE COEFFICIENTS AVAILABLE FOR A_{ij}

	$i=1$	$i=2$	$i=3$	
γ_i	518.53	3,799	14,620	
λ_i	0.845	0	0.365	
T_i	1.0178	0.99922	1.000034	
β_i	4.73004	7.8540	10.9956	
K_i^2	13.12	47	99	
$F_i(x)_{x=\frac{a}{2}}$	1.6165	0	1.4059	
$\phi_i(x)_{x=\frac{a}{2}}$	1.2374	0	1.4226	
$\frac{dF_i(x)}{dx}$	+4.9222 -4.9222	—	+1.2150 -1.2150	at $x=0.2a$ at $x=0.8a$
$\left.\frac{d^2F_i(x)}{dx^2}\right\|_{x=\frac{a}{2}}$	$\frac{27.6846}{a^2}$	0	$\frac{171.9969}{a^2}$	$\frac{d^2F_i}{dx^2}=\beta_i^2\phi_i(x)$ see Fig. 40
$\phi_i(x)_{x=0,a}$	2.0356	—	2.0001	
$\left.\frac{d^2F_i(x)}{dx^2}\right\|_{x=0.2a,\,0.8a}$	$\frac{45.542}{a^2}$	—	$\frac{241.818}{a^2}$	

Using Eqs. (11), (87) and (14) we can obtain stresses approximately together with the values $\left[\frac{d^2F_i(x)}{dx^2}\right]$ given in Table 5 and Fig. 40, assuming $Z_b = Z_t = 1.5$.

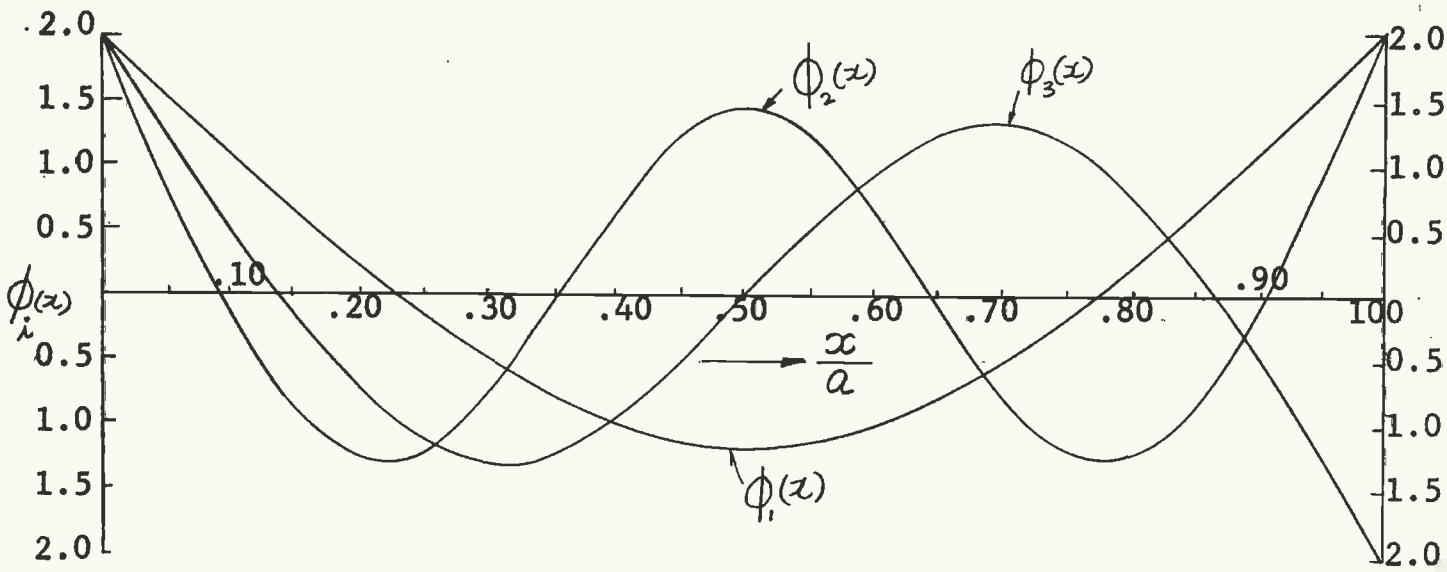


Fig. 40 $\phi_i(x)$ FUNCTION

We have maximum normal stress at $x=0, a$ and $y=\frac{b}{2}$

$$\sigma_{bx} = 11{,}329.5 \text{ PSI}, \qquad \sigma_{tx} = -11{,}329.5 \text{ PSI}$$

and at the center

$$\sigma_x = \pm 8{,}771.2 \text{ PSI}$$

$$\sigma_y = \pm 8{,}346.17 \text{ PSI}$$ on bottom and top plates.

Other stresses are shown in Table 6.

The maximum deflection in the case of all supported sides is also given in Table 6. The method of calculation is now explained

TABLE 6 COMPARISON BETWEEN TWO METHODS

	By author's method	By another method given in [7]*	Difference %
Dynamic Deflection	1.53174 in	1.49121 in (4 terms)	2.65%
		1.50109 in (1st terms)	2.00%
Dynamic Stress at Center of Plate	σ_x = 25,750.4 psi	σ_x = 25,041.0 psi	2.75%
	σ_y = 23,161.8 psi	σ_y = 21,209.0 psi	8.43%
	τ = 0	τ = 0	
Max. Dynamic Stress at $x=0$, a, $y=\frac{b}{2}$	σ_x = 33,988.5 psi	σ_x = 33,643.5 psi	1.01%
	σ_y = 10,196.4 psi	σ_y = 10,093.5 psi	1.01%
Max. Shear Stress about $x=\frac{a}{4}, \frac{3a}{4}$, $y=0, b$	τ = 9,579.5 psi	τ = 9,303.6 psi	2.88%
Dynamic deflection of the double bottom, but simply supported all sides	2.2635 in		

* To obtain these figures, the results given in reference [7] were used after revised.

briefly. Using Eq. (39) we have p_0 = 314.76 rad/sec which reduces f = 50.09 cps and T = 0.01996 sec. Hence L_{11} becomes 1.5 using $\frac{\Delta}{T} = \frac{0.02}{0.01996}$ = 1.002, we have then the maximum value of w at the center of the plate from Eq. (9) by inserting L_{11} instead of R: w = 2.2635 in.

If we use Eq. (12) in the above case, higher accuracy will be expected than that by Eq. (11).

Eq. (12) now reduces

$$w_{max} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij} L_{ij} \left[F_i(x)\right]_{x=\frac{a}{2}} \sin\frac{j\pi}{2}, \tag{90}$$

taking until $i = j$ = 3 because of rapid convergency.

Using Table 5 we have

A_{11} = 0.94758, A_{31} = 0.003244, A_{13} = 0.008280,

A_{33} = 0.000966

In order to get the load factors L_{ij}, we have to calculate frequencies by using following formulas [7].

$$f_{ij} = \frac{\pi}{2}\sqrt{\frac{E g \theta_{ij}}{1.036 \rho g + \sum_{1}^{r} \frac{g}{k_E} k_a k_b}} \tag{91}$$

and

$$\theta_{ij} = \frac{1}{\pi^4 b a^4}\left\{\gamma_i I_g(\alpha+1) + \frac{\pi^4 j^4 a^3}{b^3} T_i^2 I_s(1+\beta) + \frac{\pi^2 j^2 K_i^2 i_{xg} a^2}{b(1-\mu^2)}\right\}^* \tag{92}$$

---

* The third term in bracket is revised from the paper already published [7].

where

$$k_a = \frac{1}{a}\int_0^a F_1^2(x)\,dx = 1.03, \quad k_b = \frac{1}{b}\int_0^b \sin^2\frac{\pi y}{b}\,dy = 0.5$$

After substituting all data and values given in Table 5 into the equation of θ_{ij}, frequencies f_{ij} are determined by putting $\sum_{1}^{r}\sum_{1}^{q} f_E\, k_a\, k_b = 0$:

f_{11} = 17.939 cps, f_{13} = 94.5223 cps, f_{31} = 68.513 cps and f_{33} = 131.38 cps,

which reduce

$$\frac{t_1}{T_{11}} = 0.02 \times 17.939 = 0.35878$$

$$\frac{t_1}{T_{13}} = 0.02 \times 94.522 = 1.8904$$

$$\frac{t_1}{T_{31}} = 0.02 \times 68.513 = 1.3702$$

$$\frac{t_1}{T_{33}} = 0.02 \times 131.38 = 2.6276$$

Therefore load factors L_{ij} are now obtained from Fig. 39 corresponding to each period such as

L_{11} = 0.98, L_{13} = 1.1, L_{31} = 1.4 and L_{33} = 1.1

Then w_{max} becomes, using again Table 5 for $[F_i(x)]_{x=\frac{a}{2}}$

$$w_{max} = A_{11}L_{11}[F_1(x)]_{x=\frac{a}{2}} - A_{13}L_{13}[F_1(x)]_{x=\frac{a}{2}} + A_{31}L_{31}[F_3(x)]_{x=\frac{a}{2}} - A_{33}L_{33}[F_3(x)]_{x=\frac{a}{2}}$$

= 0.94758 x 0.98 x 1.6165-0.00828 x 1.1 x 1.6165 + 0.003246 x 1.4 x 1.4059 - 0.000966 x 1.1 x 1.4059 = 1.49121 inches.

We can obtain stresses by using Eqs. (12) and (14) together with the values $\frac{dF_i(x)}{dx}$ and $\frac{d^2F_i(x)}{dx^2}$ given in Table 5 and Fig. 40. We have maximum normal stresses at $x=0,\ a$ and $y=\frac{t}{2}$

$$\sigma_{bx} = 33{,}643.5 \text{ psi}, \qquad \sigma_{tx} = -33{,}643.5 \text{ psi}$$

and at the center

$$\sigma_x = 25{,}041.0 \qquad \sigma_y = 21{,}209.0 \text{ psi on bottom and top}$$

plates. Other stresses are shown in Table 6.

Comparing both methods, the results given by the author's method show good accuracy as practical purpose, and moreover the response of a stiffened plate having any kind of boundary conditions is rapidly determined. The central normal stress thus obtained is superimposed to that of the bottom plate.

Now, let us obtain the deflection and stress of the bottom plate $a_0 \times b_0$ with all sides clamped. [See Fig. 7] As in the case of all supported sides, we can easily obtain the deflection equivalent to Eq. (79):

$$w = \sum_{i=1,3\cdots}^{\infty}\sum_{j=1,3,5\cdots}^{\infty} X_i Y_j \frac{F_M \int_0^{a_0}\int_0^{b_0} X_i Y_j \, dx\,dy}{m' t p_{ij}^2 \int_0^{a_0}\int_0^{b_0} (X_i Y_j)^2 \, dx\,dy} R_{ij} \tag{93}$$

in which X_i has the same meaning as $F_i(x)$ and

also

$$\left.\begin{aligned} X_i &= \cosh\frac{\beta_i x}{a_0} - \cos\frac{\beta_i x}{a_0} - \alpha_i\left(\sinh\frac{\beta_i x}{a_0} - \sin\frac{\beta_i x}{a_0}\right), \\ Y_j &= \cosh\frac{\beta_j y}{b_0} - \cos\frac{\beta_j y}{b_0} - \alpha_j\left(\sinh\frac{\beta_j y}{b_0} - \sin\frac{\beta_j y}{b_0}\right) \end{aligned}\right\} \tag{94}$$

X_i, Y_j values are given in Table 7 [1].

Therefore maximum normal stresses are obtained by Eq. (80):

$$\sigma_x = \frac{6M_x}{t^2}, \qquad \sigma_y = \frac{6M_y}{t^2} \tag{95}$$

in which

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = -D\sum_{i=1,3\cdots}^{\infty}\sum_{j=1,3,5\cdots}^{\infty}(X_i''Y_j + \mu X_i Y_j'')\frac{F_M\int_0^{a_0}\int_0^{b_0} X_i Y_j\,dx\,dy}{m' + p_{ij}^2\int_0^{a_0}\int_0^{b_0}(X_i Y_j)^2\,dx\,dy}R_{ij}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = -D\sum_{i=1,3\cdots}^{\infty}\sum_{j=1,3,5\cdots}^{\infty}(X_i Y_j'' + \mu X_i'' Y_j)\frac{F_M\int_0^{a_0}\int_0^{b_0} X_i Y_j\,dx\,dy \cdot R_{ij}}{m' + p_{ij}^2\int_0^{a_0}\int_0^{b_0}(X_i Y_j)^2\,dx\,dy} \tag{96}$$

and $X_i'' = \dfrac{d^2 X_i}{dx^2}$, $\qquad Y_j'' = \dfrac{d^2 Y_j}{dy^2}$

The natural frequency of any bottom plate $a_0 \times b_0$ shown in Fig. 6 is given by the general formula [1].

$$f = \frac{K}{2\pi a_0^2}\sqrt{\frac{D}{\rho h}},$$

which reduces for mild steel

$$f = 9730\,\frac{Kh}{a_0^2} \quad \text{cps, } (h,\ a_0 \text{ should be measured in inches}) \tag{97}$$

after substituting $E = 30 \times 10^6$ psi, $\mu = 0.3$ and $\rho = \frac{0.284}{386} \frac{lb\text{-}sec^2}{in^4}$. In Eq. (97), K is given in Table 7 for the case of all clamped sides. Then from Eq. (93) the maximum deflection w_{max} which appears at the center of the plate is approximately given by

$$w_{max} = \frac{F_M a_0^4}{E h^3}\left[A_1 L_1 + A_2 L_2 + A_3 L_3 + A_4 L_4\right], \tag{98}$$

where A_1, A_2, A_3 and A_4 are tabulated in Table 7 and L_1, L_2, L_3 and L_4 are load factors for modes corresponding to A_1, A_2, A_3 and A_4 to be read off in Fig. 39.

The maximum tensile or compression stress occurs at the middle of the long side in the direction of the short side; and there is a tensile stress on the surface at the edge where the slamming loading is applied, and on the contrary, a compressive stress on the inside surface. In both the amount of maximum stress is approximately given from Eq. (95)

$$\sigma_{max} = \frac{6}{h^2} F_M a_0^2 \left[B_1 L_1 + B_2 L_2 + B_3 L_3 + B_4 L_4\right], \tag{99}$$

where B_1, B_2, B_3 and B_4 are tabulated in Table 7 and L_1, L_2, L_3 and L_4 are load factors to be read off in Fig. 39. As shown later, the maximum local response of bottom plate happens immediately after passing the peak value of slamming load. On the other hand, the maximum response of double bottom is induced

TABLE 7 VALUES OF THE COEFFICIENTS AVAILABLE FOR f, W_{MAX} AND σ_{MAX}

$\frac{b_o}{a_o}$	FIRST SYMMETRICAL MODE $W = X_1Y_1$			SECOND SYMMETRICAL MODE $W = X_1Y_3$			THIRD SYMMETRICAL MODE $W = X_1Y_5$			FOURTH SYMMETRICAL MODE $W = X_3Y_1$			FIRST ANTI-SYM. MODE $W=X_1Y_2$	SECOND ANTI-SYM. MODE $W=X_2Y_2$	STATIC VALUES DEFLECTION $w=\gamma\frac{p_0a^4}{Eh^3}$ $\gamma=A_1-A_2+A_3-A_4$	STATIC VALUES STRESS $\sigma=\delta\frac{6p_0a^2}{h^2}$ $\delta=B_1-B_2+B_3\pm B_4$
	K	A_1	B_1	K	A_2	B_2	K	A_3	B_3	K	A_4	B_4	K	K	γ	δ
1.0	36.00	0.0146	0.0376	132.5	0.0004	0.0011	310.0	0.00005	0.00012	132.5	0.00042	0.0066	73.8	109.0	0.0138	0.0432
1.2	30.00	.0199	.0515	96.1	.0008	.0021	219.0	.00010	.00026	128.7	.00045	.0070	56.0	92.5	.0188	.0567
1.4	28.00	.0242	.0624	74.5	.0013	.0034	164.0	.00017	.00045	126.5	.00046	.0073	45.5	83.3	.0226	.0668
1.6	26.35	.0273	.0705	60.6	.0020	.0052	129.0	.00028	.00073	125.1	.00047	.0074	39.0	77.6	.0251	.0734
1.8	25.35	.0296	.0764	51.5	.0028	.0072	105.0	.00043	.00111	124.2	.00048	.0075	34.8	73.8	.0268	.0778
2.0	24.60	.0313	.0807	45.0	.0036	.0094	87.5	.00061	.00158	123.5	.00048	.0076	32.8	71.5	.0278	.0805
2.2	24.15	.0325	.0839	40.3	.0045	.0117	75.0	.00084	.00215	123.0	.00049	.0076	30.0	69.5	.0284	.0820
2.4	23.80	.0334	.0863	36.9	.0054	.0139	65.6	.00109	.00282	122.7	.00049	.0077	28.5	67.5	.0286	.0829
2.6	23.85	.0341	.0881	34.4	.0062	.0161	58.4	.00138	.00357	122.4	.00049	.0077	27.4	67.1	.0288	.0833
2.8	23.35	.0347	.0896	32.4	.0070	.0181	52.6	.00169	.00432	122.2	.00049	.0078	26.6	66.4	.0289	.0837
3.0	23.20	.0352	.0907	30.8	.0078	.0200	48.0	.00203	.00525	122.0	.00049	.0078	25.9	65.6	.0289	.0838
4.0	22.80	.0364	.0940	26.6	.0104	.0268	35.3	.00376	.00971	121.5	.00050	.0079	24.2	63.8	.0293	.0848
5.0	22.60	.0370	.0955	24.9	.0119	.0306	30.0	.00523	.01348	121.3	.00050	.0079	23.5	63.0	.0298	.0863
6.0	22.50	.0373	.0963	24.0	.0127	.0328	27.4	.00627	.01618	121.2	.00050	.0079	23.1	62.5	.0304	.0876
7.0	22.46	.0375	.0968	23.6	.0133	.0343	25.8	.00706	.01821	121.1	.00050	.0079	22.9	62.3	.0308	.0886
8.0	22.42	.0376	.0971	23.2	.0136	.0352	24.8	.00761	.01964	121.1	.00050	.0079	22.7	62.1	.0311	.0894
9.0	22.40	.0377	.0973	23.0	.0139	.0358	24.2	.00794	.02050	121.0	.00050	.0079	22.6	62.0	.0312	.0899
10.0	22.40	.0378	0.0974	22.8	.0141	0.0364	23.8	.00822	0.02120	121.0	0.00050	0.0079	22.6	61.9	0.0314	0.0901
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

at much more delayed time, so that we need not consider the superimposed condition in the case of slamming phenomena of the double bottom. Hence the scantlings of the bottom plate may be determined separately without consideration of the behavior of double bottom.

The problem is now to decide the thickness h when the plate 28" x 183" is subjected to a triangular slamming load having maximum F_M = 60 psi of 0.02 sec duration as shown in Fig. 38. We here assume the maximum dynamic yielding point of $\sigma_{max} = 60 \times 10^3$ psi. Then by choosing two kinds of thickness 3/4" and 1" we repeat the following procedure:

1. Compute the natural frequency f (cps) by Eq. (97)
2. Insert this to obtain the period T for that mode
3. For a given slamming loading in Fig. 38 calculate $\frac{\Delta}{T}$ etc. and use the curve in Fig. 39 to determine the load factor
4. Compute σ_{max} by Eq. (99) using Table 7.

Applying interpolation to above two values or using Fig. 41, 42, and 43, we can approximately determine

$$h = 0.913''$$

Finally W_{max} is obtained from Eq. (98) by inserting $h = 0.913''$ as

$$W_{max} = 0.097''$$

From these values, the deflection becomes 10.62% when compared with the thickness, and it will be therefore expected that the

effect due to stretching is negligible.

In order to show this phenomena, the ratio of the maximum deflection due to both effects with the maximum deflection due to only bending effect is now computed as the function of F_M from Eq. (82) by choosing h = 1/4", 1/2" and 1" under the above slamming loading condition: $\beta = \frac{1}{2}$, $m = n = 1$ and Δ = 0.02 sec. with arbitrary F_M . That relation is given in Fig. 44 and it shows the effect due to stretching is very small when h = 0.913". Incidentally the maximum local response happens at t = 0.012 sec. on the bottom plate.

In order to design the double bottom, therefore, it is only necessary to consider the maximum stress of bottom plate. However, the deflection of a double bottom becomes totally larger after impulse due to slamming loading; for instance, in the above case Table 6 shows that the maximum deflection reaches about 1.532" when the plate thickness h = 0.913", after passing the local peak value 0.097" on bottom plate.

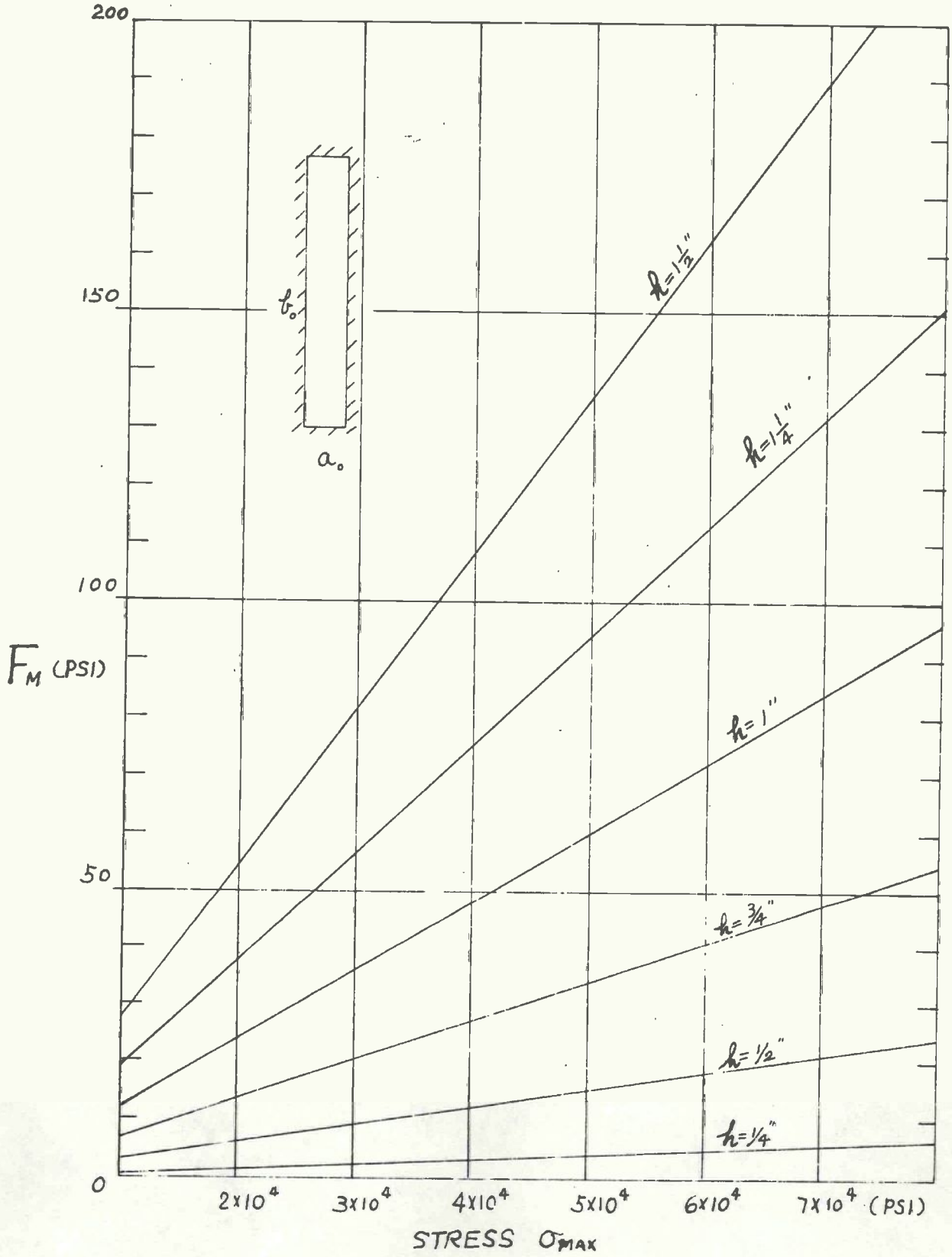


FIG 41 RELATION BETWEEN F_M AND STRESS σ_{MAX}

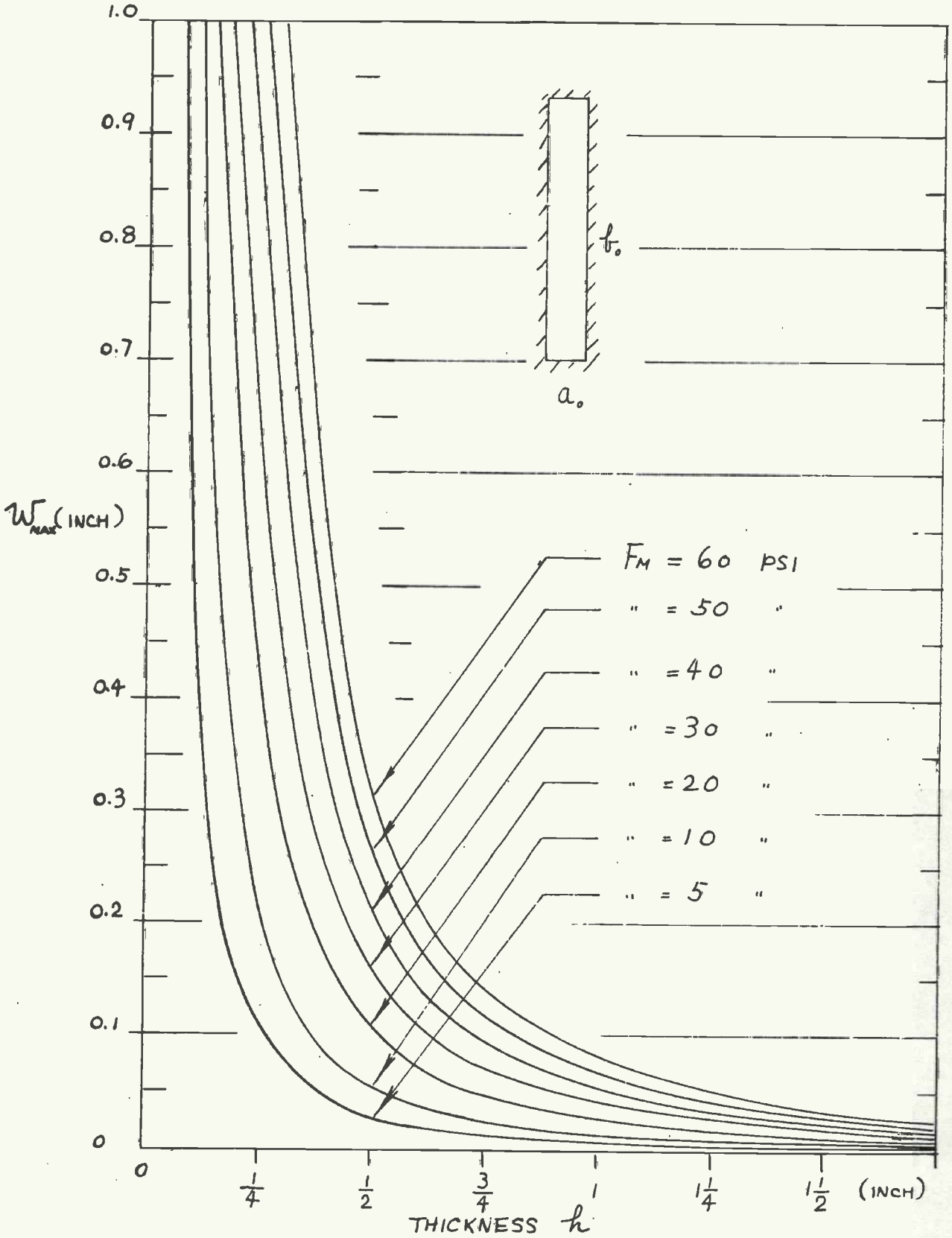


FIG 42 RELATION BETWEEN DEFLECTION AND THICKNESS

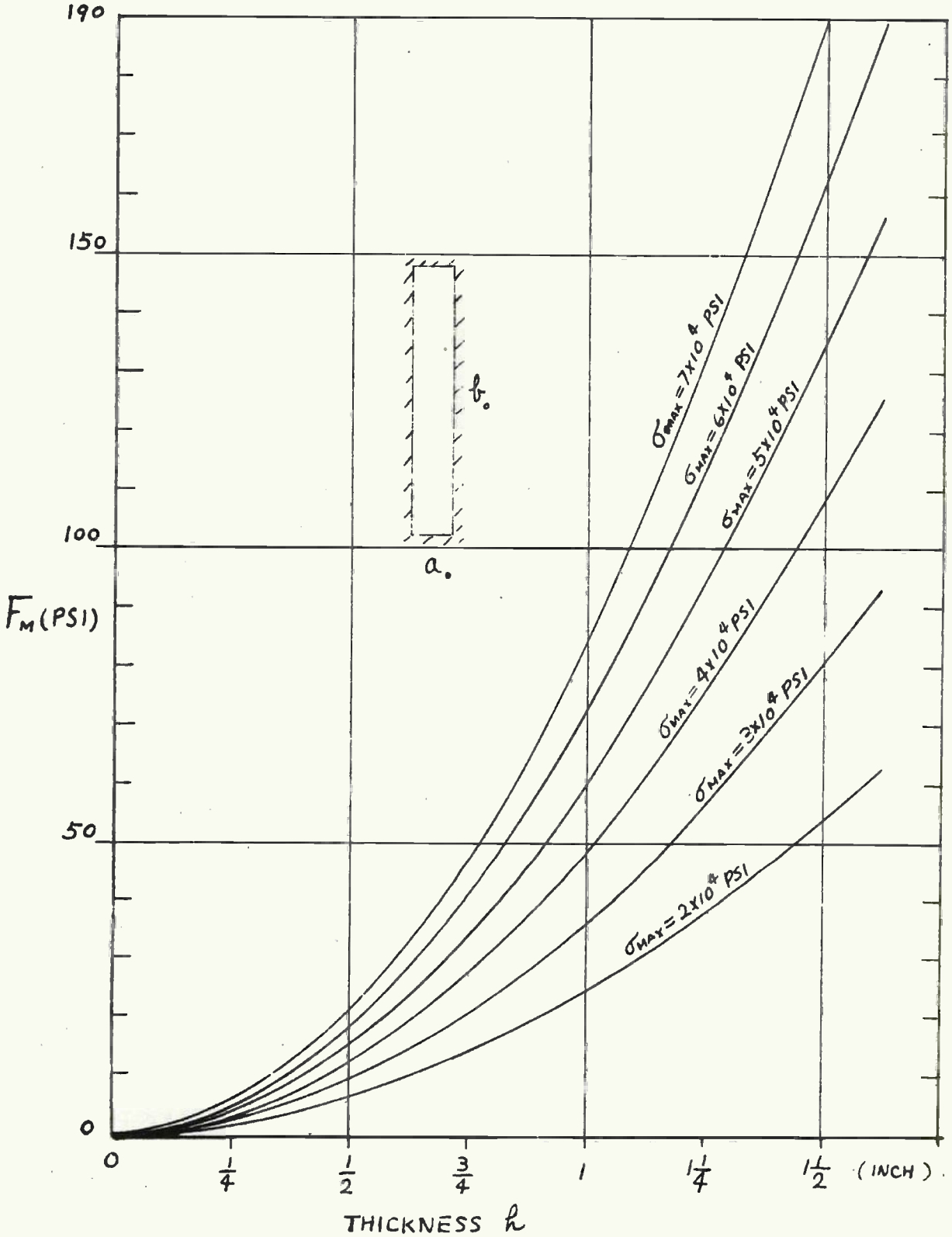


FIG 43 RELATION BETWEEN F_M AND THICKNESS h

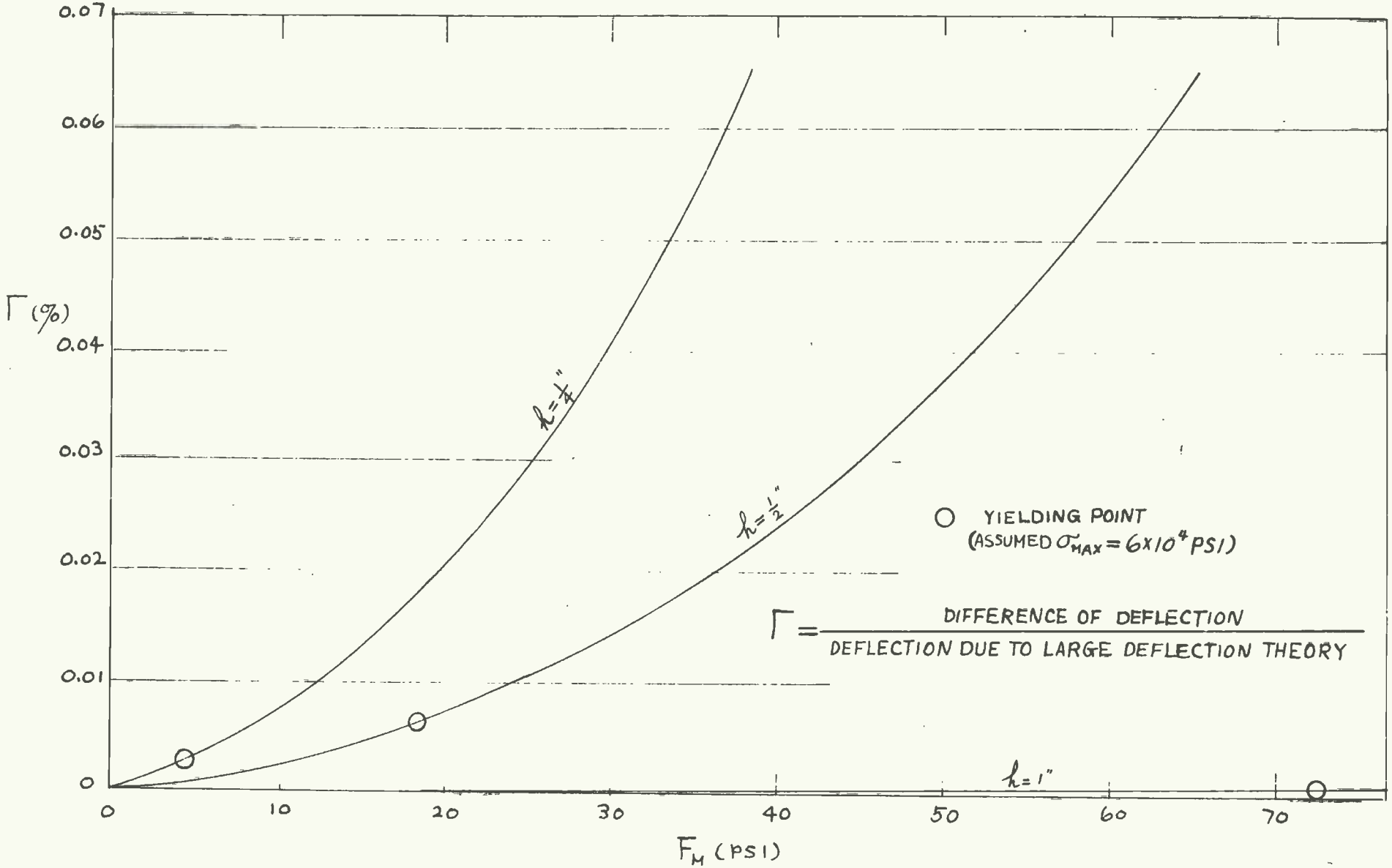


FIG 44 COMPARISON BETWEEN SMALL AND LARGE DEFLECTION THEORIES

IV CONCLUSIONS

Results obtained in this paper are summerized as follows:

(1) Applying the energy method, the structural response to slamming loading in a stiffened plate is obtained.

(2) Two theoretical analyses are obtained both for the stiffened plate and the anisotropic plate, in which only the effect due to bending is taken into consideration for the former, but on the contrary both effects due to bending and stretching are considered for the latter. However, the small deflection theory can be approximately applied to both cases of the double bottom and the bottom plate, with good accuracy.

(3) Taking the double bottom as an example, the application of theoretical results to practical purpose is numerically carried out.

(4) Judging from the results obtained by computation we can generally conclude for design of a double bottom that insofar as the dynamic response due to slamming loading is concerned it is only necessary to consider the maximum stress of bottom plate due to bending.

(5) In order to get maximum deflection and stress on the bottom plate under any kind of loading, Eqs. (98) and (99) are used together with Table 7 after computing load factor by the response factor equation in Appendix I.

(6) Also, on the double bottom, in order to get the maximum deflection at the center and the maximum stress along boundaries under any kind of loading, Eqs. (11), (87) and (14) are used

together with Table 5 and Fig. 40 after computing load factor by the response factor equation in Appendix I.

(7) If we can decide the allowable yielding point dynamically by experiment, curves and tables obtained in this paper are all available to design purpose of the double bottom.

(8) In order to apply our procedure to the other parts constructed by much thinner plates than the bottom plate, the curves such as Fig. 41 must be prepared previously under reasonable combinations of m and n following the form of impulsive loading, since the effect due to stretching becomes not negligible when compared with that of bending.

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APPENDIX I

Response Factor

9 cases

General Case

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta P_k)^m}\left[\sum_{\nu=0}^{\frac{m}{2}} (-1)^\nu \frac{m!}{(m-2\nu)!}(P_k t)^{m-2\nu} - (-1)^{\frac{m-1}{2}} m! \sin P_k t\right]$$

if m is odd number

$$= \frac{1}{(\beta\Delta P_k)^m}\left[\sum_{\nu=0}^{\frac{m}{2}} (-1)^\nu \frac{m!}{(m-2\nu)!}(P_k t)^{m-2\nu} + (-1)^{\frac{m}{2}+1} m! \cos P_k t\right]$$

if m is even number.

(2) When $\beta\Delta \le t \le \Delta$

$$R_k = \frac{1}{(P_k\beta\Delta)^m}\left[Q_1(P_k\beta\Delta)^{m-2\nu} + Q_2(P_k\beta\Delta)^{m-2\nu-1} - (-1)^{\frac{m-1}{2}} \cdot m! \cdot \sin P_k t\right]$$

$$+ \frac{1}{[P_k\Delta(1-\beta)]^n}\left\{\sum_{\nu=0}^{\frac{n}{2}} (-1)^\nu \frac{n!}{(n-2\nu)!}[(\Delta-t)P_k]^{n-2\nu} - Q_1[(1-\beta)\Delta P_k]^{n-2\nu}\right.$$

$$\left. + Q_2[(1-\beta)\Delta P_k]^{n-2\nu-1}\right\}$$

if m and n are both odd numbers.

And

$$R_k = \frac{1}{(P_k\beta\Delta)^m}\left[Q_1(P_k\beta\Delta)^{m-2\nu} + Q_2(P_k\beta\Delta)^{m-2\nu-1} + (-1)^{\frac{m}{2}+1} \cdot m! \cos P_k t\right]$$

$$+ \frac{1}{[P_k\Delta(1-\beta)]^n}\left\{\sum_{\nu=0}^{\frac{n}{2}} (-1)^\nu \frac{n!}{(n-2\nu)!}[(\Delta-t)P_k]^{n-2\nu} - Q_1[(1-\beta)\Delta P_k]^{n-2\nu}\right.$$

$$+ Q_2[(1-\beta)\Delta P_k]^{n-2\gamma-1} \Big\}$$

if m and n are both even numbers

(3) When $t \geq \Delta$

$$R_k = \frac{1}{(P_k\beta\Delta)^m}\Big[Q_1(P_k\beta\Delta)^{m-2\gamma} + Q_2(P_k\beta\Delta)^{m-2\gamma-1} - (-1)^{\frac{m-1}{2}}\cdot m!\sin P_k t\Big]$$

$$+ \frac{1}{[(1-\beta)\Delta P_k]^n}\Big[-(-1)^{\frac{n-1}{2}}\cdot n!\sin P_k(t-\Delta) - Q_1[(1-\beta)\Delta P_k]^{n-2\gamma}$$

$$+ Q_2[(1-\beta)\Delta P_k]^{n-2\gamma-1}\Big]$$

if m and n are both odd numbers.

And

$$R_k = \frac{1}{(P_k\beta\Delta)^m}\Big[Q_1(P_k\beta\Delta)^{m-2\gamma} + Q_2(P_k\beta\Delta)^{m-2\gamma-1} + (-1)^{\frac{m}{2}+1}\cdot m!\cdot\cos P_k t\Big]$$

$$+ \frac{1}{[(1-\beta)\Delta P_k]^n}\Big[-(-1)^{\frac{n}{2}+1}\cdot n!\cos P_k(t-\Delta) - Q_1[(1-\beta)\Delta P_k]^{n-2\gamma}$$

$$+ Q_2[(1-\beta)\Delta P_k]^{n-2\gamma-1}\Big]$$

if m and n are both even numbers.

Where

$$Q_1 = \cos P_k(t-\Delta\beta)\sum_{\gamma=0}^{\frac{n}{2}}(-1)^{\gamma}\frac{n!}{(n-2\gamma)!}$$

$$Q_2 = \sin P_k(t-\Delta\beta)\sum_{\gamma=0}^{\frac{n-1}{2}}(-1)^{\gamma}\frac{n!}{(n-2\gamma-1)!}$$

(I) Case of $m = 1, n = 1$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)}(p_k t - \sin p_k t)$$

(2) When $\Delta\beta \le t \le \Delta$

$$R_k = \frac{1}{\beta(1-\beta)p_k\Delta}\left[\beta(p_k\Delta - p_k t) - (1-\beta)\sin p_k t + \sin(p_k t - \beta p_k \Delta)\right]$$

(3) When $t \ge \Delta$

$$R_k = \frac{1}{\beta(1-\beta)p_k\Delta}\left[\sin(p_k t - \beta p_k\Delta) - (1-\beta)\sin p_k t - \sin(p_k t - \beta p_k\Delta)\right]$$

(II) Case of $m = 1$, $n = 2$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)}(p_k t - \sin p_k t)$$

(2) When $\Delta\beta \le t \le \Delta$

$$R_k = \frac{1}{[p_k\beta\Delta(1-\beta)]^2}\Big\{2\beta^2\cos p_k(t-\beta\Delta) + 2\beta^2[(1-\beta)\Delta p_k]\sin p_k(t-\Delta\beta)$$
$$+\beta^2[(\Delta - t)p_k]^2 - 2\Big\}$$

(3) When $t \ge 0$

$$R_k = \frac{2}{[p_k\Delta(1-\beta)]^2}\Big\{[(1-\beta)\Delta p_k]\sin p_k(t-\Delta\beta) + \cos p_k(t-\Delta\beta)$$
$$-\cos p_k(t-\Delta)\Big\}$$

(III) Case of $m = 2$, $n = 1$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)^2}\left[(p_k t)^2 + 2\cos p_k t - 2\right]$$

(2) When $\beta\Delta \le t \le \Delta$

$$R_k = \frac{1}{(1-\beta)(p_k\beta\Delta)^2}\Big[p_k\beta\Delta(2-\beta)\sin p_k(t-\Delta\beta) - p_k\Delta\beta^2\sin p_k(t-\Delta)$$
$$- 2(1-\beta)\cos p_k(t-\Delta\beta) + 2(1-\beta)\cos p_k t\Big]$$

(3) When $t \ge \Delta$

$$R_k = \frac{1}{(1-\beta)(p_k\beta\Delta)^2}\Big\{-2(1-\beta)\cos p_k(t-\Delta\beta) + (2-\beta)(p_k\beta\Delta)\sin p_k(t-\Delta\beta)$$
$$+ 2(1-\beta)\cos p_k t - \beta^2\Delta p_k \sin p_k(t-\Delta)\Big\}$$

(IV) Case of $m = 2$, $n = 3$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)^2}\left[(p_k t)^2 + 2\cos p_k t - 2\right]$$

(2) When $\beta\Delta \le t \le \Delta$

$$R_k = \frac{1}{[p_k\Delta\beta(1-\beta)]^3}\Big\{\Big\{\{-2(1-\beta)^3 p_k\Delta\beta + 3!\beta^3[(1-\beta)\Delta p_k]\}\cos p_k(t-\Delta\beta)$$
$$+\{3\beta^3[(1-\beta)\Delta p_k]^2 - 3!\beta^3\}\sin p_k(t-\Delta\beta) + 2(1-\beta)^3 p_k\Delta\beta\cos p_k t$$
$$+ \beta^3[(\Delta-t)p_k]^3 - 3!\beta^3[(\Delta-t)p_k]\Big\}$$

(3) When $t \geq \Delta$

$$R_k = \frac{1}{[p_k \Delta \beta (1-\beta)]^3} \Big\{ \Big\{ -2(1-\beta)^3 p_k \Delta\beta + 3!\beta^3 [(1-\beta)\Delta p_k] \Big\} \cos p_k (t - \Delta\beta)$$

$$+ \Big\{ 3\beta^3 [(1-\beta)\Delta p_k]^2 - 3!\beta^3 \Big\} \sin p_k (t - \Delta\beta) + 2(1-\beta)^3 p_k \Delta\beta \cos p_k t$$

$$+ 3!\beta^3 \sin p_k (t - \Delta) \Big\}$$

(V) Case of $m = 3$, $n = 2$

(1) When $0 \leq t \leq \beta\Delta$

$$R_k = \frac{1}{(\beta \Delta p_k)^3} \Big[(p_k t)^3 - 3!(p_k t) + 3! \sin p_k t \Big]$$

(2) When $\beta\Delta \leq t \leq \Delta$

$$R_k = \frac{1}{(1-\beta)^2 (p_k \beta \Delta)^3} \Big\{ \Big[-3!(1-\beta)^2 (p_k \beta \Delta) + 2\beta^3 p_k \Delta \Big] \cos p_k (t - \Delta\beta)$$

$$+ \Big[3(1-\beta)^2 (p_k \beta \Delta)^2 - 3!(1-\beta)^2 \Big] \sin p_k (t - \Delta\beta) + 3!(1-\beta)^2 \sin p_k t$$

$$+ \beta^3 p_k \Delta [(\Delta - t) p_k]^2 - 2\beta^3 p_k \Delta \Big\}$$

(3) When $t \geq \Delta$

$$R_k = \frac{1}{(1-\beta)^2 (p_k \beta \Delta)^3} \Big\{ -3!(p_k \beta \Delta)(1-\beta)^2 \cos p_k (t - \Delta\beta) + \Big[3(1-\beta)^2 (p_k \beta \Delta)^2$$

$$- 3!(1-\beta)^2 + 2(\Delta p_k \beta)^2 \beta (1-\beta) \Big] \sin p_k (t - \Delta\beta) + 3!(1-\beta)^2 \sin p_k t \Big\}$$

(VI) Case of $m = 1$, $n = 3$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)}(p_k t - \sin p_k t)$$

(2) When $\beta\Delta \le t \le \Delta$

$$R_k = \frac{1}{[p_k\beta\Delta(1-\beta)]^3}\Big\{\Big\{[\Delta p_k\beta(1-\beta)]^2(1+2\beta) - 3!\beta^3\Big\}\sin p_k(t-\Delta\beta)$$

$$+ 3!\beta^3[(1-\beta)\Delta p_k]\cos p_k(t-\Delta\beta) - (p_k\beta\Delta)^2(1-\beta)^3\sin p_k t$$

$$+\beta^3[(\Delta-t)p_k]^3 - 3!\beta^3[(\Delta-t)p_k]\Big\}$$

(3) When $t \ge \Delta$

$$R_k = \frac{1}{[p_k\Delta\beta(1-\beta)]^3}\Big\{\Big\{[\Delta p_k\beta(1-\beta)]^2(1+2\beta) - 3!\beta^3\Big\}\sin p_k(t-\Delta\beta)$$

$$+3!\beta^3[(1-\beta)\Delta p_k]\cos p_k(t-\Delta\beta) + 3!\beta^3[(1-\beta)\Delta p_k]\cos p_k(t-\Delta\beta)$$

$$-(p_k\beta\Delta)^2(1-\beta)^3\sin p_k t\Big\}$$

(VII) Case of $m = 3$, $n = 1$

(1) When $0 \le t \le \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta p_k)^3}\left[(p_k t)^3 - 3!(p_k t) + 3!\sin p_k t\right]$$

(2) When $\beta\Delta \leq t \leq \Delta$

$$R_k = \frac{1}{(1-\beta)(\Delta P_k \beta)^3} \Big\{ \Big[3(1-\beta)(P_k\beta\Delta)^2 + \beta^3 P_k^2 \Delta^2 - 3!(1-\beta) \Big]$$

$$\times \sin P_k(t-\Delta\beta) + 3!(1-\beta)\sin P_k t - \beta^3 P_k^2 \Delta^2 \sin P_k(t-\Delta)$$

$$- 3!(1-\beta)(P_k\beta\Delta)\cos P_k(t-\Delta\beta) \Big\}$$

(3) When $t \geq \Delta$

$$R_k = \frac{1}{(1-\beta)(P_k\beta\Delta)^3} \Big\{ -3! P_k\beta\Delta(1-\beta)\cos P_k(t-\Delta\beta)$$

$$+ \Big[(3-2\beta)(P_k\beta\Delta)^2 - 3!(1-\beta) \Big] \sin P_k(t-\Delta\beta) + 3!(1-\beta)\sin P_k t$$

$$- \beta^3 \Delta^2 P_k^2 \sin P_k(t-\Delta) \Big\}$$

(VIII) Case of $m = n = 2$

(1) When $0 \leq t \leq \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta P_k)^2} \Big[(P_k t)^2 - 2 + 2\cos P_k t \Big]$$

(2) When $\beta\Delta \leq t \leq \Delta$

$$R_k = \frac{1}{[P_k\beta\Delta(1-\beta)]^2} \Big\{ 2(2\beta-1)\cdot\cos P_k(t-\beta\Delta) + 2\beta(1-\beta)\Delta P_k$$

$$\times \sin P_k(t-\beta\Delta) + 2(1-\beta)^2\cos P_k t + \beta^2\Big[(\Delta-t)^2 P_k^2 - 2\Big] \Big\}$$

(3) When $t \geqq 0$

$$R_k = \frac{1}{\{P_k \beta \Delta (1-\beta)\}^2}\Big[2(2\beta-1)\cos P_k(t-\beta\Delta) + 2\beta(1-\beta)\Delta P_k \sin P_k(t-\beta\Delta)$$

$$+2(1-\beta)^2 \cos P_k t - 2\beta^2 \cos P_k(\Delta - t)\Big]$$

(IX) Case of $m = n = 3$

(1) When $0 \leqq t \leqq \beta\Delta$

$$R_k = \frac{1}{(\beta\Delta P_k)^3}\Big[(P_k t)^3 - 3!(P_k t) + 3!\sin P_k t\Big]$$

(2) When $\beta\Delta \leqq t \leqq \Delta$

$$R_k = \frac{1}{\{\beta P_k \Delta(1-\beta)\}^3}\Big[3!\,\beta\Delta P_k(-2\beta^2+3\beta-1)\cos P_k(t-\Delta\beta)$$

$$+\Big\{3(P_k\beta\Delta)^2(\beta-1)^2 - 3!(1-3\beta+3\beta^2)\Big\}\sin P_k(t-\Delta\beta)$$

$$+3!(1-\beta)^3\sin P_k t + 3!\beta^3 \sin P_k(t-\Delta)\Big]$$

(3) When $t \geqq \Delta$

$$R_k = \frac{1}{\{\beta P_k \Delta(1-\beta)\}^3}\Big[3!\,\beta\Delta P_k(-2\beta^2+3\beta-1)\cos P_k(t-\Delta\beta)$$

$$+\Big\{3(P_k\beta\Delta)^2(\beta-1)^2 - 3!(1-3\beta+3\beta^2)\Big\}\sin P_k(t-\Delta\beta)$$

$$+3!(1-\beta)^3\sin P_k t + 3!\beta^3 \sin P_k(t-\Delta)\Big]$$

APPENDIX II

Solution of

$$\frac{\partial^2 w_{11}(t)}{\partial t^2} + p_0^2 w_{11}(t) + \alpha w_{11}^3(t) = 0$$

Method of successive approximation [12] is applied to solve the equation

$$\frac{\partial^2 w_{11}}{\partial t^2} + p_0^2 w_{11} + \alpha w_{11}^3 = 0 \tag{100}$$

Take the series:

$$\left.\begin{aligned} w_{11} &= w_0 + \alpha w_1 + \alpha^2 w_2 + \alpha^3 w_3 + \text{---------} \\ p_0^2 &= p^2 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + \text{----------} \end{aligned}\right\} \tag{101}$$

which contain higher powers of the small quantity α . In these series W_0, W_1, W_2, - - - are unknown functions of time t, p is the frequency, which will be determined later, and C_1, C_2 - - are constants which will be chosen so as to eliminate condition of resonance. By increasing the number of terms in Eq. (101) we can calculate as many successive approximations as we desire. In the following discussion we limit our calculations by omitting all the terms containing α in a power higher than the third. Substituting Eq. (101) into Eq. (100) we obtain

$$\begin{aligned} &\ddot{w}_0 + \alpha\ddot{w}_1 + \alpha^2\ddot{w}_2 + \alpha^3\ddot{w}_3 + (p^2 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3)(w_0 + \alpha w_1 + \alpha^2 w_2 + \alpha^3 w_3) \\ &\quad + \alpha(w_0 + \alpha w_1 + \alpha^2 w_2 + \alpha^3 w_3)^3 = 0. \end{aligned} \tag{102}$$

After making the indicated algebraic operations and neglecting all the terms containing α in a power higher than the third, we can have the following system of equations since each factor for each of the four powers of α must be zero:

$$\left.\begin{aligned} \ddot{w}_0 + p^2 w_0 &= 0 \\ \ddot{w}_1 + p^2 w_1 &= -c_1 w_0 - w_0^3 \\ \ddot{w}_2 + p^2 w_2 &= -c_2 w_0 - c_1 w_1 - 3 w_0^2 w_1 \\ \ddot{w}_3 + p^2 w_3 &= -c_3 w_0 - c_2 w_1 - c_1 w_2 - 3 w_0^2 w_2 - 3 w_0 w_1^2 \end{aligned}\right\} \tag{103}$$

Assuming that at the initial instant, t=0, we have $w_{11} = 0$, $\dot{w}_{11} = v$, and substituting for w_{11} of Eq. (101), we obtain

$$\left.\begin{aligned} w_0(0)+\alpha w_1(0)+\alpha^2 w_2(0)+\alpha^3 w_3(0) &= 0 \\ \dot{w}_0(0)+\alpha \dot{w}_1(0)+\alpha^2 \dot{w}_2(0)+\alpha^3 \dot{w}_3(0) &= v \end{aligned}\right\} \tag{104}$$

Again, since these equations must hold for any magnitude of α, we have

$$\left.\begin{aligned} w_0(0) &= 0, & \dot{w}_0(0) &= v, \\ w_1(0) &= 0, & \dot{w}_1(0) &= 0, \\ w_2(0) &= 0, & \dot{w}_2(0) &= 0, \\ w_3(0) &= 0, & \dot{w}_3(0) &= 0. \end{aligned}\right\} \tag{105}$$

Considering the first of Eq. (103) and the corresponding initial conditions represented by the first row of Eq. (105) we find

$$w_0 = \frac{v}{p} \sin pt \tag{106}$$

Substituting this first approximation into the right side of the second of Eq. (103) we obtain

$$\ddot{w}_1 + p^2 w_1 = -\left(\frac{C_1 v}{p} + \frac{3v^3}{4p^3}\right)\sin pt + \frac{v^3}{4p^3}\sin 3pt. \tag{107}$$

To eliminate the condition of resonance we will choose the constant C, so as to make the first term on the right side of the equation equal to zero. Then

$$\frac{C_1 v}{p} + \frac{3v^3}{4p^3} = 0,$$

and we find

$$C_1 = -\frac{3v^2}{4p^2} \tag{108}$$

The general solution for W_1 then becomes

$$W_1 = C_1 \cos pt + C_2 \sin pt - \frac{v^3}{32p^5} \sin 3pt$$

To satisfy the initial conditions given by the second row of Eq. (105), we put

$$C_1 = 0 \ , \quad C_2 = \frac{3v^3}{32p^5} ,$$

Thus

$$W_1 = \frac{v^3}{32p^5}\left[3 \sin pt - \sin 3pt\right] \tag{109}$$

To obtain the third approximation we substitute the expressions Eqs. (106), (108) and (109) into the right side of the third of Eq. (103) and obtain

$$\ddot{W}_2 + p^2 W_2 = -\left(C_2 \frac{v}{p} + \frac{21v^5}{128p^7}\right) \sin pt - \frac{3v^5}{32p^7} \sin 3pt - \frac{3v^5}{128p^7} \sin 5pt \tag{110}$$

Again, to eliminate the condition of resonance, we put

$$C_2 = -\frac{21v^4}{128p^6} \tag{111}$$

Then the general solution for W_2 becomes

$$W_2 = \frac{v^5}{1024p^9}\left[31 \sin pt - 12 \sin 3pt + \sin 5pt\right] \tag{112}$$

by using the constants of integration determined so as to satisfy the third row of Eq. (105).

Substituting the expressions for W_0, W_1, W_2, C_1 and C_2 in the last of Eq. (103), and proceeding as before, we obtain

the fourth approximation

$$w_3 = \frac{v^7}{(32)^3 p^{13}}\left[429 \sin pt - 174 \sin 3pt + 20 \sin 5pt - \sin 7pt\right], \quad (113)$$

together with

$$C_3 = -\frac{66 v^6}{1024 p^{10}} \quad (114)$$

Thus we finally obtain from Eq. (101)

$$w_{\parallel} = \frac{v}{p}\Big(\sin pt + \frac{\alpha v^3}{32 p^5}(3 \sin pt - \sin 3pt) + \frac{\alpha^2 v^5}{(32)^2 p^9}(31 \sin pt - 12 \sin 3pt + \sin 5pt) + \frac{\alpha^3 v^7}{(32)^3 p^3}(429 \sin pt - 174 \sin 3pt + 20 \sin 5pt - \sin 7pt)\Big), \quad (115)$$

and

$$p^2 = p_0^2 + \frac{3 v^2 \alpha}{4 p_0^2} - \frac{51 v^4 \alpha^2}{128 p_0^6} - \frac{189 v^6 \alpha^3}{512 p_0^{10}}$$

APPENDIX III

Eq. (72)

and

$F(\overline{W})$ of Eq. (86)

$$w_{11}(t)=\frac{q_M}{p^2}\Bigg[\frac{1}{(\beta\Delta p)^m}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m}{2}}(-1)^\nu\frac{m!}{(m-2\nu)!}(p\beta\Delta)^{m-2\nu}$$

$$+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m-1}{2}}(-1)^\nu\frac{m!}{(m-2\nu-1)!}(p\beta\Delta)^{m-2\nu-1}+Q_1\Big\}+\frac{1}{[(1-\beta)\Delta p]^n}\Big\{\sin p(t-\beta\Delta)\cdot$$

$$\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^\nu\frac{n!}{(n-2\nu-1)!}[(1-\beta)\Delta p]^{n-2\nu-1}-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^\nu\frac{n!}{(n-2\nu)!}[(1-\beta)\Delta p]^{n-2\nu}+Q_2\Big\}\Bigg]$$

$$+\frac{3\alpha}{32p^5}\Bigg[\frac{3q_M^3}{(\beta\Delta p)^{3m}p^3}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3m+2}{2}}(-1)^\nu\frac{(3m+2)!}{(3m+2-2\nu)!}\cdot$$

$$(p\beta\Delta)^{3m+2-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3m+1}{2}}(-1)^\nu\frac{(3m+2)!}{(3m+1-2\nu)!}(\beta\Delta p)^{3m+1-2\nu}+Q_3\Big\}-\frac{q_M^3}{(3p\beta\Delta)^{3m}(3p)^3}\cdot$$

$$\Big\{\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3m+2}{2}}(-1)^\nu\frac{(3m+2)!}{(3m+2-2\nu)!}(3p\beta\Delta)^{3m+2-2\nu}+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3m+1}{2}}(-1)^\nu\frac{(3m+2)!}{(3m+1-2\nu)!}\cdot$$

$$(3p\beta\Delta)^{3m+1-2\nu}+Q_4\Big\}$$

$$+\frac{3q_M^3}{(1-\beta)^{3n}\Delta^{3n}}\Bigg[\frac{\Delta^2}{p^{3n+1}}\Big\{-\cos[p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n}{2}}(-1)^\nu\frac{3n!}{(3n-2\nu)!}[p\Delta(1-\beta)]^{3n-2\nu}+\sin[p(t-\beta\Delta)]\cdot$$

$$\sum_{\nu=0}^{\frac{3n-1}{2}}(-1)^\nu\frac{3n!}{(3n-2\nu-1)!}[p\Delta(1-\beta)]^{3n-2\nu-1}+Q_5\Big\}+\frac{2\Delta}{p^{3n+2}}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3n+1}{2}}(-1)^\nu\frac{(3n+1)!}{(3n+1-2\nu)!}\cdot$$

$$[p\Delta(1-\beta)]^{3n+1-2\nu}-\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3n}{2}}(-1)^\nu\frac{(3n+1)!}{(3n-2\nu)!}[p\Delta(1-\beta)]^{3n-2\nu}+Q_6\Big\}$$

$$+\frac{1}{p^{3n+3}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3n+2}{2}}(-1)^\nu\frac{(3n+2)!}{(3n+2-2\nu)!}[p\Delta(1-\beta)]^{3n+2-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3n+1}{2}}(-1)^\nu\cdot$$

$$\frac{(3n+2)!}{(3n+1-2\nu)!}[p\Delta(1-\beta)]^{3n+1-2\nu}+Q_7\Big\}\Bigg]$$

$$-\frac{q_M^3}{(1-\beta)^{3n}\Delta^{3n}}\Bigg[\frac{\Delta^2}{(3p)^{3n+1}}\Big\{-\cos[3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n}{2}}(-1)^\nu\frac{3n!}{(3n-2\nu)!}[3p\Delta(1-\beta)]^{3n-2\nu}$$

$$+\sin[3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n-1}{2}}(-1)^\nu\frac{3n!}{(3n-2\nu-1)!}[3p\Delta(1-\beta)]^{3n-2\nu-1}+Q_8\Big\}$$

$$+\frac{2\Delta}{(3p)^{3n+2}}\Big\{\cos[3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n+1}{2}}(-1)^{\nu}\frac{(3n+1)!}{(3n+1-2\nu)!}[3p\Delta(1-\beta)]^{3n+1-2\nu}$$

$$-\sin[3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n}{2}}(-1)^{\nu}\frac{(3n+1)!}{(3n-2\nu)!}[3p\Delta(1-\beta)]^{3n-2\nu}+Q_9\Big\}+\frac{1}{(3p)^{3n+3}}\cdot$$

$$\Big\{-\cos[3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{3n+2}{2}}(-1)^{\nu}\frac{(3n+2)!}{(3n+2-2\nu)!}[3p\Delta(1-\beta)]^{3n+2-2\nu}+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{3n+1}{2}}(-1)^{\nu}\cdot$$

$$\frac{(3n+2)!}{(3n+1-2\nu)!}[3p\Delta(1-\beta)]^{3n+1-2\nu}+Q_{10}\Big\}\Bigg]\Bigg\}$$

$$+\frac{5\alpha^2}{(32)^2p^9}\Bigg\{\frac{q_M^5}{(\beta\Delta)^{5m}}\Bigg[\frac{31}{p^{5(m+1)}}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+4}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+4-2\nu)!}\cdot$$

$$(p\beta\Delta)^{5m+4-2\nu}+\sin(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+3}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+3-2\nu)!}[p\beta\Delta]^{5m+3-2\nu}+Q_{11}\Big\}$$

$$-\frac{12}{(3p)^{5(m+1)}}\Big\{\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+4}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+4-2\nu)!}[3p\beta\Delta]^{5m+4-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+3}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+3-2\nu)!}[3p\beta\Delta]^{5m+3-2\nu}+Q_{12}\Big\}$$

$$+\frac{1}{(5p)^{5(m+1)}}\Big\{\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+4}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+4-2\nu)!}[5p\beta\Delta]^{5m+4-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5m+3}{2}}(-1)^{\nu}\frac{(5m+4)!}{(5m+3-2\nu)!}[5p\beta\Delta]^{5m+3-2\nu}+Q_{13}\Big\}\Bigg]$$

$$+\frac{31q_M^5}{(1-\beta)^{5n}\Delta^{5n}}\Bigg[\frac{\Delta^4}{p^{5n+1}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu)!}[p\Delta(1-\beta)]^{5n-2\nu}$$

$$+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n-1}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu-1)!}[p\Delta(1-\beta)]^{5n-2\nu-1}+Q_{14}\Big\}$$

$$-\frac{4\Delta^3}{p^{5n+2}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+1)!}{(5n+1-2\nu)!}[p\Delta(1-\beta)]^{5n+1-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\cdot$$

$$\frac{(5n+1)!}{(5n-2\nu)!}[p\Delta(1-\beta)]^{5n-2\nu}+Q_{15}\Big\}$$

$$+\frac{6\Delta^2}{p^{5n+3}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+2-2\nu)!}[p\Delta(1-\beta)]^{5n+2-2\nu}$$

$$+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+1-2\nu)!}[p\Delta(1-\beta)]^{5n-2\nu+1}+Q_{16}\Big\}$$

$$-\frac{4\Delta}{p^{5n+4}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+3-2\nu)!}[p\Delta(1-\beta)]^{5n+3-2\nu}$$

$$+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+2-2\nu)!}[p\Delta(1-\beta)]^{5n+2-2\nu}+Q_{17}\Big\}$$

$$+\frac{1}{p^{5(n+1)}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+4}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+4-2\nu)!}[p\Delta(1-\beta)]^{5n+4-2\nu}$$

$$+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+3-2\nu)!}[p\Delta(1-\beta)]^{5n+3-2\nu}+Q_{18}\Big\}\Bigg]$$

$$-\frac{12\,q_M^5}{(1-\beta)^{5n}\,\Delta^{5n}}\Bigg[\frac{\Delta^4}{(3p)^{5n+1}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu)!}[3p\Delta(1-\beta)]^{5n-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n-1}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu-1)!}[3p\Delta(1-\beta)]^{5n-2\nu-1}+Q_{19}\Big\}$$

$$-\frac{4\Delta^3}{(3p)^{5n+2}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+1)!}{(5n+1-2\nu)!}[3p\Delta(1-\beta)]^{5n+1-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\frac{(5n+1)!}{(5n-2\nu)!}[3p\Delta(1-\beta)]^{5n-2\nu}+Q_{20}\Big\}$$

$$+\frac{6\Delta^2}{(3p)^{5n+3}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+2-2\nu)!}[3p\Delta(1-\beta)]^{5n+2-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+1-2\nu)!}[3p\Delta(1-\beta)]^{5n-2\nu+1}+Q_{21}\Big\}$$

$$-\frac{4\Delta}{(3p)^{5n+4}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+3-2\nu)!}[3p\Delta(1-\beta)]^{5n+3-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+2-2\nu)!}[3p\Delta(1-\beta)]^{5n+2-2\nu}+Q_{22}\Big\}$$

$$+\frac{1}{(3p)^{5(n+1)}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+4}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+4-2\nu)!}[3p\Delta(1-\beta)]^{5n+4-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+3-2\nu)!}[3p\Delta(1-\beta)]^{5n+3-2\nu}+Q_{23}\Big\}\Bigg]$$

$$+\frac{q_M^5}{(1-\beta)^{5n}\Delta^{5n}}\Bigg[\frac{\Delta^4}{(5p)^{5n+1}}\Big\{-\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu)!}[5p\Delta(1-\beta)]^{5n-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n-1}{2}}(-1)^{\nu}\frac{5n!}{(5n-2\nu-1)!}[5p\Delta(1-\beta)]^{5n-2\nu-1}+Q_{24}\Big\}$$

$$-\frac{4\Delta^3}{(5p)^{5n+2}}\Big\{-\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+1)!}{(5n+1-2\nu)!}[5p\Delta(1-\beta)]^{5n+1-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n}{2}}(-1)^{\nu}\frac{(5n+1)!}{(5n-2\nu)!}[5p\Delta(1-\beta)]^{5n-2\nu}+Q_{25}\Big\}$$

$$+\frac{6\Delta^2}{(5p)^{5n+3}}\Big\{-\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+2-2\nu)!}[5p(1-\beta)\Delta]^{5n+2-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+1}{2}}(-1)^{\nu}\frac{(5n+2)!}{(5n+1-2\nu)!}[5p\Delta(1-\beta)]^{5n-2\nu+1}+Q_{26}\Big\}$$

$$-\frac{4\Delta}{(5p)^{5n+4}}\Big\{-\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+3-2\nu)!}[5p(1-\beta)\Delta]^{5n+3-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+2}{2}}(-1)^{\nu}\frac{(5n+3)!}{(5n+2-2\nu)!}[5p\Delta(1-\beta)]^{5n+2-2\nu}+Q_{27}\Big\}$$

$$+\frac{1}{(5p)^{5(n+1)}}\Big\{-\cos 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+4}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+4-2\nu)!}[5p\Delta(1-\beta)]^{5n+4-2\nu}$$

$$+\sin 5p(t-\beta\Delta)\sum_{\nu=0}^{\frac{5n+3}{2}}(-1)^{\nu}\frac{(5n+4)!}{(5n+3-2\nu)!}[5p\Delta(1-\beta)]^{5n+3-2\nu}+Q_{28}\Big\}\Bigg]\Bigg\},$$

where if m and n are odd numbers Q_i's are as follows:

$$Q_1 = (-1)^{\frac{m+1}{2}} m! \sin pt,$$

$$Q_2 = (-1)^{\frac{n+1}{2}} n! \sin p(t-\Delta),$$

$$Q_3 = (-1)^{\frac{3m+3}{2}} (3m+2)! \sin pt,$$

$$Q_4 = (-1)^{\frac{3m+3}{2}} (3m+2)! \sin 3pt,$$

$$Q_5 = (-1)^{\frac{3n+1}{2}} (3n)! \sin p(t-\Delta),$$

$$Q_6 = (-1)^{\frac{3n+3}{2}} (3n+1)! \cos p(t-\Delta),$$

$$Q_7 = (-1)^{\frac{3n+3}{2}} (3n+2)! \sin p(t-\Delta),$$

$$Q_8 = (-1)^{\frac{3n+1}{2}} (3n)! \sin 3p(t-\Delta),$$

$$Q_9 = (-1)^{\frac{3n+3}{2}} (3n+1)! \cos 3p(t-\Delta),$$

$$Q_{10} = (-1)^{\frac{3n+3}{2}} (3n+2)! \sin 3p(t-\Delta),$$

$$Q_{11} = (-1)^{\frac{5m+5}{2}} (5m+4)! \sin pt,$$

$$Q_{12} = (-1)^{\frac{5m+5}{2}} (5m+4)! \sin 3pt,$$

$$Q_{13} = (-1)^{\frac{5m+5}{2}} (5m+4)! \sin 5pt,$$

$$Q_{14} = (-1)^{\frac{5n+1}{2}} (5n)! \sin p(t-\Delta),$$

$$Q_{15} = (-1)^{\frac{5n+1}{2}} (5n+1)! \cos p(t-\Delta),$$

$$Q_{16} = (-1)^{\frac{5n+3}{2}} (5n+2)! \sin p(t-\Delta),$$

$$Q_{17} = (-1)^{\frac{5n+3}{2}} (5n+3)! \cos p(t-\Delta),$$

$$Q_{18} = (-1)^{\frac{5n+5}{2}} (5n+4)! \sin p(t-\Delta),$$

$$Q_{19} = (-1)^{\frac{5n+1}{2}} (5n)! \sin 3p(t-\Delta),$$

$$Q_{20} = (-1)^{\frac{5n+3}{2}} (5n+1)! \cos 3p(t-\Delta),$$

$$Q_{21} = (-1)^{\frac{5n+3}{2}} (5n+2)! \sin 3p(t-\Delta),$$

$$Q_{22} = (-1)^{\frac{5n+3}{2}} (5n+3)! \cos 3p(t-\Delta),$$

$$Q_{23} = (-1)^{\frac{5n+5}{2}} (5n+4)! \sin 3p(t-\Delta),$$

$$Q_{24} = (-1)^{\frac{5n+1}{2}} (5n)! \sin 5p(t-\Delta),$$

$$Q_{25} = (-1)^{\frac{5n+3}{2}} (5n+1)! \cos 5p(t-\Delta),$$

$$Q_{26} = (-1)^{\frac{5n+3}{2}} (5n+2)! \sin 5p(t-\Delta),$$

$$Q_{27} = (-1)^{\frac{5n+3}{2}} (5n+3)! \cos 5p(t-\Delta),$$

$$Q_{28} = (-1)^{\frac{5n+5}{2}} (5n+4)! \sin 5p(t-\Delta),$$

and if m and n are even numbers they become

$$Q_1=(-1)^{\frac{m}{2}+1}\, m!\cos pt,$$

$$Q_2=(-1)^{\frac{n}{2}}\, n!\cos p(t-\Delta),$$

$$Q_3=(-1)^{\frac{3m}{2}}(3m+2)!\cos pt,$$

$$Q_4=(-1)^{\frac{3m}{2}}(3m+2)!\cos 3pt,$$

$$Q_5=(-1)^{\frac{3n}{2}}(3n)!\cos p(t-\Delta),$$

$$Q_6=(-1)^{\frac{3n}{2}}(3n+1)!\sin p(t-\Delta),$$

$$Q_7=(-1)^{\frac{3n}{2}+1}(3n+2)!\cos p(t-\Delta),$$

$$Q_8=(-1)^{\frac{3n}{2}}(3n)!\cos 3p(t-\Delta),$$

$$Q_9=(-1)^{\frac{3n}{2}}(3n+1)!\sin 3p(t-\Delta),$$

$$Q_{10}=(-1)^{\frac{3n}{2}+1}(3n+2)!\cos 3p(t-\Delta),$$

$$Q_{11}=(-1)^{\frac{5m}{2}+1}(5m+4)!\cos pt,$$

$$Q_{12}=(-1)^{\frac{5m}{2}+1}(5m+4)!\cos 3pt,$$

$$Q_{13}=(-1)^{\frac{5m}{2}+1}(5m+4)!\cos 5pt,$$

$$Q_{14}=(-1)^{\frac{5n}{2}}(5n)!\cos p(t-\Delta),$$

$$Q_{15}=(-1)^{\frac{5n}{2}+1}(5n+1)!\sin p(t-\Delta),$$

$$Q_{16}=(-1)^{\frac{5n}{2}+1}(5n+2)!\cos p(t-\Delta),$$

$$Q_{17}=(-1)^{\frac{5n}{2}}(5n+3)!\sin p(t-\Delta),$$

$$Q_{18}=(-1)^{\frac{5n}{2}}(5n+4)!\cos p(t-\Delta),$$

$$Q_{19}=(-1)^{\frac{5n}{2}}(5n)!\cos 3p(t-\Delta),$$

$$Q_{20}=(-1)^{\frac{5n}{2}+1}(5n+1)!\sin 3p(t-\Delta),$$

$$Q_{21}=(-1)^{\frac{5n}{2}+1}(5n+2)!\cos 3p(t-\Delta),$$

$$Q_{22}=(-1)^{\frac{5n}{2}}(5n+3)!\sin 3p(t-\Delta),$$

$$Q_{23}=(-1)^{\frac{5n}{2}}(5n+4)!\cos 3p(t-\Delta),$$

$$Q_{24}=(-1)^{\frac{5n}{2}}(5n)!\cos 5p(t-\Delta),$$

$$Q_{25}=(-1)^{\frac{5n}{2}+1}(5n+1)!\sin 5p(t-\Delta),$$

$$Q_{26}=(-1)^{\frac{5n}{2}+1}(5n+2)!\cos 5p(t-\Delta),$$

$$Q_{27}=(-1)^{\frac{5n}{2}}(5n+3)!\sin 5p(t-\Delta),$$

$$Q_{28}=(-1)^{\frac{5n}{2}}(5n+4)!\cos 5p(t-\Delta).$$

$$U = \bar{w} p_0^2$$

$$F(\bar{w}) = \frac{9U^3}{8}\Bigg[\frac{1}{(p+\omega)^3}\Big\{2t(p+\omega)\cos(\omega t+\varepsilon) - 2\sin(\omega t+\varepsilon)$$

$$+ t^2(p+\omega)^2\sin(\omega t+\varepsilon) - 2\sin(pt-\varepsilon)\Big\} - \frac{1}{(\omega-p)^3}\Big\{2t(\omega-p)\cos(\omega t+\varepsilon)$$

$$-2\sin(\omega t+\varepsilon) + t^2(\omega-p)^2\sin(\omega t+\varepsilon) + 2\sin(pt+\varepsilon)\Big\}\Bigg]$$

$$-\frac{3U^3}{8}\Bigg[\frac{1}{(p+3\omega)^3}\Big\{2t(p+3\omega)\cos 3(\omega t+\varepsilon) - 2\sin 3(\omega t+\varepsilon) + t^2(p+3\omega)^2\cdot$$

$$\sin 3(\omega t+\varepsilon) - 2\sin(pt-3\varepsilon)\Big\} - \frac{1}{(3\omega-p)^3}\Big\{2t(3\omega-p)\cos 3(\omega t+\varepsilon)$$

$$-2\sin 3(\omega t+\varepsilon) + t^2(3\omega-p)^2\sin 3(\omega t+\varepsilon) + 2\sin(pt+3\varepsilon)\Big\}\Bigg]$$

$$-\frac{3U^3}{2}\Bigg[\frac{1}{(3p+\omega)^3}\Big\{2t(3p+\omega)\cos(\omega t+\varepsilon) - 2\sin(\varepsilon+\omega t) + (3p+\omega)^2t^2\sin(\omega t+\varepsilon)$$

$$-2\sin(3pt-\varepsilon)\Big\} - \frac{1}{(3p-\omega)^3}\Big\{2(3p-\omega)t\cos(\omega t+\varepsilon) + 2\sin(\omega t+\varepsilon)$$

$$-t^2(3p-\omega)^2\sin(\omega t+\varepsilon) - 2\sin(3pt+\varepsilon)\Big\}\Bigg]$$

$$+\frac{U^3}{2}\Bigg[\frac{1}{(3p+3\omega)^3}\Big\{6t(p+\omega)\cos 3(\omega t+\varepsilon) - 2\sin 3(\omega t+\varepsilon) + t^2(3p+3\omega)^2\cdot$$

$$\sin 3(\omega t+\varepsilon) - 2\sin 3(pt-\varepsilon)\Big\} - \frac{1}{(3p-3\omega)^3}\Big\{6(p-\omega)t\cos 3(\omega t+\varepsilon)$$

$$+2\sin 3(\omega t+\varepsilon) - (3p-3\omega)^2t^2\sin 3(\omega t+\varepsilon) - 2\sin 3(pt+\varepsilon)\Big\}\Bigg]$$

$$+\frac{3\mathcal{G}_m U^2}{2(\beta\Delta)^m}\Bigg[\frac{3}{p^{m+3}}\Big\{\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^\nu\frac{(m+2)!}{(m+2-2\nu)!}\cdot$$

$$(p\beta\Delta)^{m+2-2\nu} + \sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^\nu\frac{(m+2)!}{(m+1-2\nu)!}(p\beta\Delta)^{m+1-2\nu} + Q_{29}\Big\}$$

$$-\frac{1}{(3p)^{m+3}}\Big\{\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+2-2\nu)!}(3p\beta\Delta)^{m+2-2\nu}+\sin 3p(t-\beta\Delta)\cdot$$

$$\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+1-2\nu)!}(3p\beta\Delta)^{m+1-2\nu}+Q_{30}\Big\}$$

$$+\frac{3}{2(2\omega-p)^{m+3}}\Big\{\cos[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+2-2\nu)!}[(2\omega-p)\beta\Delta]^{m+2-2\nu}$$

$$-\sin[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+1-2\nu)!}[(2\omega-p)\beta\Delta]^{m+1-2\nu}+Q_{31}\Big\}$$

$$-\frac{1}{2(2\omega-3p)^{m+3}}\Big\{\cos[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+2-2\nu)!}[(2\omega-3p)\beta\Delta]^{m+2-2\nu}$$

$$-\sin[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+1-2\nu)!}[(2\omega-3p)\beta\Delta]^{m+1-2\nu}+Q_{32}\Big\}$$

$$+\frac{3}{2(2\omega+p)^{m+3}}\Big\{-\cos[2(\varepsilon+\omega\beta\Delta)-p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+2-2\nu)!}[(p+2\omega)\beta\Delta]^{m+2-2\nu}$$

$$+\sin[2(\varepsilon+\omega\beta\Delta)-p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+1-2\nu)!}[(2\omega+p)\beta\Delta]^{m+1-2\nu}+Q_{33}\Big\}$$

$$-\frac{1}{2(2\omega+3p)^{m+3}}\Big\{-\cos[2(\varepsilon+\omega\beta\Delta)-3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{m+2}{2}}(-1)^{\nu}\frac{(m+2)!}{(m+2-2\nu)!}[(2\omega+3p)\beta\Delta]^{m+2-2\nu}$$

$$+\sin[2(\varepsilon+\omega\beta\Delta)-3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{m+1}{2}}(-1)^{2}\frac{(m+2)!}{(m+2-2\nu)!}[(2\omega+3p)\beta\Delta]^{m+1-2\nu}+Q_{34}\Big\}\Bigg]$$

$$+\frac{3\,q_M U^2}{2(1-\beta)^n\Delta^n}\Bigg[\frac{3\Delta^2}{p^{n+1}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}\cdot$$

$$[p(1-\beta)\Delta]^{n-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[p\Delta(1-\beta)]^{n-2\nu-1}+Q_{35}\Big\}$$

$$+\frac{\Delta^2}{(3p)^{n+1}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[3p\Delta(1-\beta)]^{n-2\nu}$$

$$+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[3p(1-\beta)\Delta]^{n-2\nu-1}+Q_{36}\Big\}$$

$$-\frac{6\Delta}{p^{n+2}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+1)!}{(n+1-2\nu)!}[p\Delta(1-\beta)]^{n+1-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\cdot$$

$$\frac{(n+1)!}{(n-2\nu)!}[p\Delta(1-\beta)]^{n-2\nu}+Q_{37}\Big\}$$

$$-\frac{2\Delta}{(3p)^{n+2}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+1)!}{(n+1-2\nu)!}[3p\Delta(1-\beta)]^{n+1-2\nu}+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\cdot$$

$$\frac{(n+1)!}{(n-2\nu)!}[3p\Delta(1-\beta)]^{n-2\nu}+Q_{38}\Big\}$$

$$+\frac{3}{p^{n+3}}\Big\{-\cos p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[p\Delta(1-\beta)]^{n+2-2\nu}+\sin p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\cdot$$

$$\frac{(n+2)!}{(n+1-2\nu)!}[p\Delta(1-\beta)]^{n+1-2\nu}+Q_{39}\Big\}$$

$$+\frac{1}{(3p)^{n+3}}\Big\{-\cos 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[3p\Delta(1-\beta)]^{n+2-2\nu}+\sin 3p(t-\beta\Delta)\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\cdot$$

$$\frac{(n+2)!}{(n+1-2\nu)!}[3p\Delta(1-\beta)]^{n+1-2\nu}+Q_{40}\Big\}$$

$$-\frac{3\Delta^2}{2(2\omega-p)^{n+1}}\Big\{-\cos[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(p-2\omega)(1-\beta)\Delta]^{n-2\nu}$$

$$+\sin[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[(p-2\omega)(1-\beta)\Delta]^{n-2\nu-1}+Q_{41}\Big\}$$

$$-\frac{\Delta^2}{2(3p-2\omega)^{n+1}}\Big\{-\cos[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(3p-2\omega)(1-\beta)\Delta]^{n-2\nu}$$

$$+\sin[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[(3p-2\omega)(1-\beta)\Delta]^{n-2\nu-1}+Q_{42}\Big\}$$

$$+\frac{3\Delta}{(2\omega-p)^{n+2}}\Big\{-\cos[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+1)!}{(n+1-2\nu)!}[(p-2\omega)(1-\beta)\Delta]^{n+1-2\nu}$$

$$+\sin[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{(n+1)!}{(n-2\nu)!}[(p-2\omega)(1-\beta)\Delta]^{n-2\nu}+Q_{43}\Big\}$$

$$+\frac{\Delta}{(3p-2\omega)^{n+2}}\Big\{-\cos[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+1)!}{(n+1-2\nu)!}[(3p-2\omega)(1-\beta)\Delta]^{n+1-2\nu}$$

$$+\sin[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(3p-2\omega)(1-\beta)\Delta]^{n-2\nu}+Q_{44}\Big\}$$

$$-\frac{3}{2(2\omega-p)^{n+3}}\Big\{-\cos[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[(p-2\omega)(1-\beta)\Delta]^{n+1-2\nu}$$

$$+\sin[p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+1-2\nu)!}[(p-2\omega)(1-\beta)\Delta]^{n+1-2\nu}+Q_{45}\Big\}$$

$$-\frac{1}{2(3p-2\omega)^{n+3}}\Big\{-\cos[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[(3p-2\omega)(1-\beta)\Delta]^{n+2-2\nu}$$

$$+\sin[3p(t-\beta\Delta)+2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+1-2\nu)!}[(3p-2\omega)(1-\beta)\Delta]^{n+1-2\nu}+Q_{46}\Big\}$$

$$-\frac{3\Delta^2}{2(p+2\omega)^{n+1}}\Big\{-\cos[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(p+2\omega)(1-\beta)\Delta]^{n-2\nu}$$

$$+\sin[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[(p+2\omega)(1-\beta)\Delta]^{n-2\nu-1}+Q_{47}\Big\}$$

$$-\frac{\Delta^2}{2(3p+2\omega)^{n+1}}\Big\{-\cos[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu)!}[(3p+2\omega)(1-\beta)\Delta]^{n-2\nu}$$

$$+\sin[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n-1}{2}}(-1)^{\nu}\frac{n!}{(n-2\nu-1)!}[(3p+2\omega)(1-\beta)\Delta]^{n-2\nu-1}+Q_{48}\Big\}$$

$$+\frac{3\Delta}{(p+2\omega)^{n+1}}\Big\{-\cos[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+1)!}{(n+1-2\nu)!}[(p+2\omega)(1-\beta)\Delta]^{n-2\nu+1}$$

$$+\sin[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{(n+1)!}{(n-2\nu)!}[(p+2\omega)(1-\beta)\Delta]^{n-2\nu}+Q_{49}\Big\}$$

$$+\frac{\Delta}{(3p+2\omega)^{n+2}}\Big\{-\cos[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}\frac{(n+1)!}{(n+1-2\nu)!}[(3p+2\omega)(1-\beta)\Delta]^{n+1-2\nu}$$

$$+\sin[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n}{2}}(-1)^{\nu}\frac{(n+1)!}{(n-2\nu)!}[(3p+2\omega)(1-\beta)\Delta]^{n-2\nu}+Q_{50}\Big\}$$

$$-\frac{3}{2(p+2\omega)^{n+1}}\Big\{-\cos[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[(p+2\omega)(1-\beta)\Delta]^{n+2-2\nu}$$

$$+\sin[p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+1-2\nu)!}[(p+2\omega)(1-\beta)\Delta]^{n+1-2\nu}+Q_{51}\Big\}$$

$$-\frac{1}{(3p+2\omega)^{n+3}}\Big\{-\cos[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+2}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+2-2\nu)!}[(3p+2\omega)(1-\beta)\Delta]^{n+2-2\nu}$$

$$+\sin[3p(t-\beta\Delta)-2(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{n+1}{2}}(-1)^{\nu}\frac{(n+2)!}{(n+1-2\nu)!}[(3p+2\omega)(1-\beta)\Delta]^{n+1-2\nu}+Q_{52}\Big\}\Big]$$

$$+\frac{3q_M^2 U}{2(\beta\Delta)^{2m}}\Bigg[\frac{3}{(p+\omega)^{2m+3}}\Big\{\sin[\varepsilon+\beta\Delta\omega-p(t-\beta\Delta)]\sum_{\nu=0}^{m+1}(-1)^{\nu}\cdot$$

$$\frac{(2m+2)!}{(2m+2-2\nu)!}[\Delta\beta(p+\omega)]^{2m+2-2\nu}+\cos[\varepsilon+\beta\Delta\omega-p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{2m+1}{2}}(-1)^{\nu}\frac{(2m+2)!}{(2m+1-2\nu)!}[\beta\Delta(p+\omega)]^{2m+1-2\nu}+Q_{53}\Big\}$$

$$-\frac{1}{(3p+\omega)^{2m+3}}\Big\{\sin[\omega\beta\Delta+\varepsilon-3p(t-\beta\Delta)]\sum_{\nu=0}^{m+1}(-1)^{\nu}\frac{(2m+2)!}{(2m+2-2\nu)!}[\beta\Delta(3p+\omega)]^{2m+2-2\nu}$$

$$+\cos[\omega\beta\Delta+\varepsilon-3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{2m+1}{2}}(-1)^{\nu}\frac{(2m+2)!}{(2m+1-2\nu)!}[\beta\Delta(3p+\omega)]^{2m+1-2\nu}+Q_{54}\Big\}$$

$$-\frac{3}{(\omega-p)^{2m+3}}\Big\{\sin[\varepsilon+\beta\Delta\omega+p(t-\beta\Delta)]\sum_{\nu=0}^{m+1}(-1)^{\nu}\frac{(2m+2)!}{(2m+2-2\nu)!}[\beta\Delta(\omega-p)]^{2m+2-2\nu}$$

$$+\cos[\varepsilon+\beta\Delta\omega+p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{2m+1}{2}}(-1)^{\nu}\frac{(2m+2)!}{(2m+1-2\nu)!}[\beta\Delta(\omega-p)]^{2m+1-2\nu}+Q_{55}\Big\}$$

$$+\frac{1}{(\omega-3p)^{2m+3}}\Big\{\sin[\omega\beta\Delta+\varepsilon+3p(t-\beta\Delta)]\sum_{\nu=0}^{m+1}(-1)^{\nu}\frac{(2m+2)!}{(2m+2-2\nu)!}[\beta\Delta(\omega-3p)]^{2m+2-2\nu}$$

$$+\cos[\omega\beta\Delta+\varepsilon+3p(t-\beta\Delta)]\sum_{\nu=0}^{\frac{2m+1}{2}}(-1)^{\nu}\frac{(2m+2)!}{(2m+1-2\nu)!}[\beta\Delta(\omega-3p)]^{2m+1-2\nu}+Q_{56}\Big\}\Bigg]$$

$$+\frac{3q_M^2 U}{2(1-\beta)^{2n}\Delta^{2n}}\Bigg[\frac{3\Delta^2}{(p+\omega)^{2n+1}}\Big\{\sin[p(t-\beta\Delta)-(\omega\beta\Delta+\varepsilon)]\sum_{\nu=0}^{n}(-1)^{\nu}\cdot$$

$$\frac{2n!}{(2n-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n-2\nu}+\cos[p(t-\beta\Delta)-(\omega\beta\Delta+\varepsilon)]\sum_{\nu=0}^{\frac{2n-1}{2}}\frac{2n!}{(2n-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n-2\nu-1}+Q_{57}\Big\}$$

$$+\frac{\Delta^2}{(3p+\omega)^{2n+1}}\Big\{\sin[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{2n!}{(2n-2\nu)!}[(3p+\omega)(1-\beta)\Delta]^{2n-2\nu}$$

$$+\cos[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{2n-1}{2}}(-1)^{\nu}\frac{2n!}{(2n-2\nu-1)!}[(3p+\omega)(1-\beta)\Delta]^{2n-2\nu-1}+Q_{58}\Big\}$$

$$-\frac{6\Delta}{(p+\omega)^{2n+2}}\Big\{\sin[p(t-\beta\Delta)-(\omega\beta\Delta+\varepsilon)]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+1)!}{(2n+1-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n+1-2\nu}$$

$$+\cos[p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{(2n+1)!}{(2n-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n-2\nu}+Q_{59}\Big\}$$

$$-\frac{2\Delta}{(3p+\omega)^{2n+2}}\Big\{\sin[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+1)!}{(2n+1-2\nu)!}[(3p+\omega)(1-\beta)\Delta]^{2n+1-2\nu}$$

$$+\cos[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{(2n+1)!}{(2n-2\nu)!}[(3p+\omega)(1-\beta)\Delta]^{2n-2\nu}+Q_{60}\Big\}$$

$$+\frac{3}{(p+\omega)^{2n+3}}\Big\{\sin[p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{n+1}(-1)^{\nu}\frac{(2n+2)!}{(2n+2-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n+2-2\nu}$$

$$+\cos[p(t-\beta\Delta)-(\omega\beta\Delta+\varepsilon)]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+2)!}{(2n+1-2\nu)!}[(p+\omega)(1-\beta)\Delta]^{2n+1-2\nu}+Q_{61}\Big\}$$

$$+\frac{1}{(3p+\omega)^{2n+3}}\Big\{\sin[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{n+1}(-1)^{\nu}\frac{(2n+2)!}{(2n+2-2\nu)!}[(3p+\omega)(1-\beta)\Delta]^{2n+2-2\nu}$$

$$+\cos[3p(t-\beta\Delta)-(\varepsilon+\omega\beta\Delta)]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+2)!}{(2n+1-2\nu)!}[(3p+\omega)(1-\beta)\Delta]^{2n+1-2\nu}+Q_{62}\Big\}$$

$$-\frac{3\Delta^2}{(p-\omega)^{2n+1}}\Big\{\sin[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{2n!}{(2n-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n-2\nu}$$

$$+\cos[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{\frac{2n-1}{2}}(-1)^{\nu}\frac{2n!}{(2n-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n-2\nu-1}+Q_{63}\Big\}$$

$$-\frac{\Delta^2}{(3p-\omega)^{2n+1}}\Big\{\sin[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{2n!}{(2n-2\nu)!}[(3p-\omega)(1-\beta)\Delta]^{2n-2\nu}$$

$$+\cos[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{\frac{2n-1}{2}}(-1)^{\nu}\frac{2n!}{(2n-2\nu-1)!}[(3p-\omega)(1-\beta)\Delta]^{2n-2\nu-1}+Q_{64}\Big\}$$

$$+\frac{6\Delta}{(p-\omega)^{2n+2}}\Big\{\sin[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+1)!}{(2n+1-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n+1-2\nu}$$

$$+\cos[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{(2n+1)!}{(2n-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n-2\nu}+Q_{65}\Big\}$$

$$+\frac{2\Delta}{(3p-\omega)^{2n+2}}\Big\{\sin[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+1)!}{(2n+1-2\nu)!}[(3p-\omega)(1-\beta)\Delta]^{2n+1-2\nu}$$

$$+\cos[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{n}(-1)^{\nu}\frac{(2n+1)!}{(2n-2\nu)!}[(3p-\omega)(1-\beta)\Delta]^{2n-2\nu}+Q_{66}\Big\}$$

$$-\frac{3}{(p-\omega)^{2n+3}}\Big\{\sin[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{n+1}(-1)^{\nu}\frac{(2n+2)!}{(2n+2-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n+2-2\nu}$$

$$+\cos[p(t-\beta\Delta)+\omega\beta\Delta+\varepsilon]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+2)!}{(2n+1-2\nu)!}[(p-\omega)(1-\beta)\Delta]^{2n-1-2\nu}+Q_{67}\Big\}$$

$$-\frac{1}{(3p-\omega)^{2n+3}}\Big\{\sin[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{n+1}(-1)^{\nu}\frac{(2n+2)!}{(2n+2-2\nu)!}[(3p-\omega)(1-\beta)\Delta]^{2n+2-2\nu}$$

$$+\cos[3p(t-\beta\Delta)+\varepsilon+\omega\beta\Delta]\sum_{\nu=0}^{\frac{2n+1}{2}}(-1)^{\nu}\frac{(2n+2)!}{(2n+1-2\nu)!}[(3p-\omega)(1-\beta)\Delta]^{2n+1-2\nu}+Q_{68}\Big\}\Bigg],$$

where if m and n are odd numbers Q_i's are as follows:

$$Q_{29}=(-1)^{\frac{m+1}{2}+1}(m+2)!\sin pt,$$

$$Q_{30}=(-1)^{\frac{m+1}{2}+1}(m+2)!\sin 3pt,$$

$$Q_{31}=(-1)^{\frac{m+1}{2}}(m+2)!\sin(pt+2\varepsilon),$$

$$Q_{32}=(-1)^{\frac{m+1}{2}}(m+2)!\sin(3pt+2\varepsilon),$$

$$Q_{33}=(-1)^{\frac{m+1}{2}}(m+2)!\sin(pt-2\varepsilon),$$

$$Q_{34}=(-1)^{\frac{m+1}{2}}(m+2)!\sin(3pt-2\varepsilon),$$

$$Q_{35}=(-1)^{\frac{n+1}{2}}n!\sin p(t-\Delta),$$

$$Q_{36}=(-1)^{\frac{n+1}{2}}n!\sin 3p(t-\Delta),$$

$$Q_{37}=(-1)^{\frac{n+1}{2}}(n+1)!\cos p(t-\Delta),$$

$$Q_{38}=(-1)^{\frac{n+1}{2}}(n+1)!\cos 3p(t-\Delta),$$

$$Q_{39}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin p(t-\Delta),$$

$$Q_{40}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin 3p(t-\Delta),$$

$$Q_{41}=(-1)^{\frac{n+1}{2}}n!\sin[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{42}=(-1)^{\frac{n+1}{2}}n!\sin[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{43}=(-1)^{\frac{n+1}{2}}(n+1)!\cos[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{44}=(-1)^{\frac{n+1}{2}}(n+1)!\cos[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{45}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{46}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{47}=(-1)^{\frac{n+1}{2}}n!\sin[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{48}=(-1)^{\frac{n+1}{2}}n!\sin[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

$$Q_{49}=(-1)^{\frac{n+1}{2}}(n+1)!\cos[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{50}=(-1)^{\frac{n+1}{2}}(n+1)!\cos[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

$$Q_{51}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{52}=(-1)^{\frac{n+1}{2}+1}(n+2)!\sin[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

and if m and n are even numbers they become

$$Q_{29}=(-1)^{\frac{m}{2}}(m+2)!\cos pt,$$

$$Q_{30}=(-1)^{\frac{m}{2}}(m+2)!\cos 3pt,$$

$$Q_{31}=(-1)^{\frac{m}{2}}(m+2)!\cos(pt+2\varepsilon),$$

$$Q_{32}=(-1)^{\frac{m}{2}}(m+2)!\cos(3pt+2\varepsilon),$$

$$Q_{33}=(-1)^{\frac{m}{2}+1}(m+2)!\cos(pt-2\varepsilon),$$

$$Q_{34}=(-1)^{\frac{m}{2}+1}(m+2)!\cos(3pt-2\varepsilon),$$

$$Q_{35}=(-1)^{\frac{n}{2}}n!\cos p(t-\Delta),$$

$$Q_{36}=(-1)^{\frac{n}{2}}n!\cos 3p(t-\Delta),$$

$$Q_{37}=(-1)^{\frac{n}{2}+1}(n+1)!\sin p(t-\Delta),$$

$$Q_{38}=(-1)^{\frac{n}{2}+1}(n+1)!\sin 3p(t-\Delta),$$

$$Q_{39}=(-1)^{\frac{n}{2}+1}(n+2)!\cos p(t-\Delta),$$

$$Q_{40}=(-1)^{\frac{n}{2}+1}(n+2)!\cos 3p(t-\Delta),$$

$$Q_{41}=(-1)^{\frac{n}{2}}n!\cos[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{42}=(-1)^{\frac{n}{2}}n!\cos[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{43}=(-1)^{\frac{n}{2}+1}(n+1)!\sin[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{44}=(-1)^{\frac{n}{2}+1}(n+1)!\sin[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{45}=(-1)^{\frac{n}{2}+1}(n+2)!\cos[pt+2\varepsilon+(2\omega-p)\Delta],$$

$$Q_{46}=(-1)^{\frac{n}{2}+1}(n+2)!\cos[3pt+2\varepsilon+(2\omega-3p)\Delta],$$

$$Q_{47}=(-1)^{\frac{n}{2}}n!\cos[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{48}=(-1)^{\frac{n}{2}}n!\cos[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

$$Q_{49}=(-1)^{\frac{n}{2}+1}(n+1)!\sin[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{50}=(-1)^{\frac{n}{2}+1}(n+1)!\sin[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

$$Q_{51}=(-1)^{\frac{n}{2}+1}(n+2)!\cos[pt-2\varepsilon-(p+2\omega)\Delta],$$

$$Q_{52}=(-1)^{\frac{n}{2}+1}(n+2)!\cos[3pt-2\varepsilon-(3p+2\omega)\Delta],$$

and if m and n are odd or even numbers they become

$$Q_{53}=(-1)^{m+3}(2m+2)!\sin(pt-\varepsilon),$$

$$Q_{54}=(-1)^{m+3}(2m+2)!\sin(3pt-\varepsilon),$$

$$Q_{55}=(-1)^{m+2}(2m+2)!\sin(pt+\varepsilon),$$

$$Q_{56}=(-1)^{m+2}(2m+2)!\sin(3pt+\varepsilon),$$

$$Q_{57}=(-1)^{n+1}(2n)!\sin[p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{58}=(-1)^{n+1}(2n)!\sin[3p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{59}=(-1)^{n+1}(2n+1)!\cos[p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{60}=(-1)^{n+1}(2n+1)!\cos[3p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{61}=(-1)^{n+2}(2n+2)!\sin[p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{62}=(-1)^{n+2}(2n+2)!\sin[3p(t-\Delta)-(\varepsilon+\omega\Delta)],$$

$$Q_{63}=(-1)^{n+1}(2n)!\sin[p(t-\Delta)+\varepsilon+\omega\Delta],$$

$$Q_{64}=(-1)^{n+1}(2n)!\sin[3p(t-\Delta)+\varepsilon+\omega\Delta],$$

$$Q_{65}=(-1)^{n+1}(2n+1)!\cos[p(t-\Delta)+\varepsilon+\omega\Delta],$$

$$Q_{66}=(-1)^{n+1}(2n+1)!\cos[3p(t-\Delta)+\varepsilon+\omega\Delta],$$

$$Q_{67}=(-1)^{n+2}(2n+2)!\sin[p(t-\Delta)+\varepsilon+\omega\Delta],$$

$$Q_{68}=(-1)^{n+2}(2n+2)!\sin[3p(t-\Delta)+\varepsilon+\omega\Delta].$$