A Hybrid Model of a Two-Dimensional Directional Drilling System

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A Hybrid Model of a Two-Dimensional Directional Drilling System

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Due to the exhaustion of conventional reservoirs of oil, gas and mineral resources over the recent years, the need to drill to unconventional, hard-to-reach wells has increased. Directional drilling techniques allow to drill the complex borehole geometries necessary to reach those wells. In practice, boreholes drilled with a directional drilling system often show undesired spiraling behavior. A large amount of research has been performed in the past to model the directional drilling system, in order to understand the causes of these unwanted effects. However, current models generally are based on many greatly simplifying assumptions, such as ideal contact of the stabilizers. With ideal stabilizer contact, the stabilizers of the directional drilling system always remain in contact with the borehole wall. However, the diameter of the borehole is generally larger than the diameter of the stabilizers in practice, thus resulting in non-ideal stabilizer contact.

In this research, a hybrid model of a two-dimensional directional drilling system is derived, by taking non-ideal stabilizer contact into account. The BHA dynamics are described by a two-stabilizer distributed model. Combining the BHA model with a model for the kinematic relationships and bit-rock interaction, results in a borehole propagation model. The borehole propagation model is transformed into a delay differential state-space model, subject to a complementarity condition. By analysis of this hybrid model, an explanation can be given of the effect of non-ideal stabilizer contact on the occurrence and stability of the (quasi-)stationary trajectories. Furthermore, numerical simulations provide insight in the limit cycling behavior of the directional drilling system. These limit cycle are believed to be closely related to borehole spiraling, and thus the presented model provides more insight in these oscillating effects.
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The exhaustion of conventional reservoirs of oil, gas and mineral resources has rapidly increased over the last century. The need to be able to drill boreholes enabling the exploitation of unconventional, hard-to-reach reservoirs has therefore also increased. The difficulty in reaching these unconventional reservoirs comes in part from the complex borehole geometries that need to be drilled.

Directional drilling techniques have proven to be able to drill these complex wells. In directional drilling, the bit trajectory is actively manipulated in order to produce a borehole with a desired geometry. The usage and popularity of directional drilling techniques are prevalent for several reasons.

First of all, directional drilling provides the opportunity to drill beneath obstacles (for example to reach reservoirs beneath cities). When necessary, hard-to-drill rock formations can be avoided with directional drilling. Furthermore, this technique provides the possibility of drilling away from a fault plane (a planar fracture or discontinuity in the volume of the rock), thereby avoiding damage to the well caused by fault slippage. Besides being able to avoid certain rock formations, directional drilling makes it possible to drill multiple wells from the same rig. Drainage of the well can be improved by maximizing the intersection of the well with the reservoir. Furthermore, directional drilling makes it possible to rescue an out-of-control well. In this regard, its value was shown in the Deepwater Horizon oil spill in 2010. Last but not least, directional drilling is important in the recovery of shale gas, for which near horizontal boreholes need to be drilled [33].

1-1 General description of a directional drilling system

In Figure 1-1, a schematic drawing of the directional drilling system is shown. At the surface, the rig imposes a rotary speed and a hookload (axial force) on the drillstring. The drillstring is a hollow tube with a length of up to several kilometers.

The upper part of the drillstring is in tension, due to its own weight and the imposed hookload at the rig. The lower part of the drillstring, which is called the Bottomhole assembly (BHA),
is in compression, in order to induce a desired axial force on the bit, called weight on bit. The bit is the front part of the drill that cuts away the rock. The most used bits are roller-cone bits and fixed cutter bits, also called Polycrystalline Diamond Compact (PDC) bits. In directional drilling, PDC bits are usually favored [60]. An example of a PDC bit is shown in Figure 1-2.

The BHA is generally a couple hundred meters long. It consists of heavy pipes, called drill collars, and short elements of larger diameter, called stabilizers (Figure 1-3), that center the BHA in the borehole. A BHA usually has three to five stabilizers.

Directional drilling has been accomplished using several techniques, but the most prominent one makes use of a Rotary Steerable System (RSS). An RSS is an actuator, which influences the direction of borehole propagation. The RSS actuator has the ability to steer the system during continuous rotation of the drill string. Two types of RSS actuators exist: point-the-bit and push-the-bit systems. A point-the-bit system changes the propagation direction by
pointing the drill bit in the right direction. A push-the-bit system exerts a force on the borehole and thus pushes the bit in a certain direction (Figure 1-4). Push-the-bit system are the most widely used technique for steering the bit in directional drilling [60].

![RSS force](image)

**Figure 1-4:** A push-the-bit actuator and BHA inside the borehole [32].

The RSS is located between the first stabilizer (counted from the bit upwards) and the bit (see Figure 1-1). The actuators are controlled with drilling mud, which is injected in the drillstring at the rig. After flowing out at the bit, the mud cools and lubricates the rock destruction process and transports the rock-cuttings to the surface via the annular space between borehole wall and drillpipe. The sensors at the RSS send data to the rig by the use of pulses through the drilling mud, which provides feedback on the process (typically only for monitoring purposes).

### 1-2 Challenges in directional drilling

Directional drilling with RSS is a technologically mature process, but borehole trajectories still show an undesired amount of deviation from the intended path [19], due to effects such as borehole spiraling. Borehole spiraling is the occurrence of self-induced oscillations of a drilling system, resulting in non-smooth borehole walls (Figure 1-5).

![Borehole spiraling](image)

**Figure 1-5:** Borehole spiraling, as observed from measurements [63].

Borehole spiraling is an unwanted effect in directional drilling, because the frictional resistance of the borehole is increased, which decreases rate of penetration (ROP). Furthermore, due to spirals, the accuracy of tracking a desired borehole path is decreased and inserting a casing, after the borehole has been drilled, becomes more difficult.

The causes of borehole spiraling are not completely understood, but it has been shown that stabilizer placement has an effect on the strength of the oscillations of borehole rippling, the
two-dimensional equivalent of borehole spiraling. In Figure 1-6, the occurrence of borehole rippling because of geometrical feedback of the stabilizers is shown. In the three-dimensional setting, also bit walk might be a cause of borehole spiraling. Bit walk is the natural tendency of the bit to drift in lateral direction and has a significant influence on the steady-state geometry of the borehole [60].

\[\text{Figure 1-6: Self-induced oscillations due to stabilizer placement, known as borehole rippling [40].}\]

One of the main challenges in directional drilling in the recent years is reducing the effect of borehole spiraling. Solutions are sought in BHA design [39] and active control of the system [34, 51]. In order to find solutions for these unwanted effects, the cause of them has to be understood.

For this reason, there is a demand for directional drilling models, both in industry and science. Several models for directional drilling exist (see Section 1-3). The main focus of these models is to obtain more understanding why certain phenomena, such as borehole spiraling, occur. Also, it is desired to obtain more insight in what the effect of certain parameters and BHA configurations is on these phenomena. Ultimately, controllers should be developed to avoid these unwanted effects, which leads to a desire to develop models suitable to serve as a basis for controller design.

A difficulty in the field of directional drilling is that doing experiments is difficult and expensive. Furthermore, the industry is generally not willing to publish data for validation tests. Therefore, the validity of the developed models has rarely been checked with field data. An exception to this is the PD-model [60] which has been validated against field data [41]. Controllers have primarily been tested on the models themselves, and thus have not yet proven to work in a real-life setup. Some tests were performed with hardware in the loop [42], but no detailed explanation of these tests and their results exists.

As discussed before, several models already provide insight in the causes of borehole spiraling. These models mainly predict the stability/instability of a nominal trajectory. In order to derive more realistic models, one has to relax the assumption that models were based on in the past. In [33, 40, 60] it is recommended to further extend existing models with nonlinear effects, in particular to describe steady-state borehole spiraling.

One of these nonlinear effects is bit tilt saturation. Bit tilt is the orientation difference between the bit and the borehole. In most models, the bit tilt is assumed to be proportional
to the lateral force at the bit. However, experiments show that the bit tilt saturates, due to unilateral contact between the borehole and bit gauge [23, 59].

Another nonlinearity is the non-ideal contact of the stabilizers with the borehole wall. In most models, the stabilizer is assumed to be in contact with the borehole wall on all sides, which is called ideal stabilizer contact. However, the borehole diameter is generally larger than that of the stabilizers, due to over-gauging [33]. A clearance between stabilizer and borehole thus generally exists, which leads to non-ideal stabilizer contact. Taking the non-ideal contact of the stabilizers into account could result in a model that predicts (steady-state) borehole spiraling for conditions other than assessed by current models.

In order to show what has been done on the modeling of directional drilling systems in the past, an overview of literature is presented.

1-3 Literature survey

The need to be able to describe, understand and control the physics behind the directional drilling process has led to an extensive research effort on models for directional drilling over the last decades. These models come in multiple forms (analytical or numerical, allowing for 2D or 3D deformations of the BHA, etc.). Although models differ from one to another, the statement can be made that generally directional-drilling systems are modeled as quasi-static systems. It is assumed that dynamical (i.e. inertial) effects in the BHA model do not have a strong influence on the borehole propagation when averaged over several revolutions [6]. For this reason, the independent variable used in the models is generally not time, but distance drilled. An exception to this is the work by Panchal et al. [57, 58]. In these papers, the model is strictly a kinematics-based model, with time as independent variable. The model only takes rotational transformations between different reference frames into account and has angular velocity as an input. It does not take any forces into account. Therefore, these kinematics-based models discard mechanics that are important in the directional drilling process. For this reason, the model of Panchal et al. or other kinematics-based models are not taken into account in this research.

Furthermore, it can be stated that generally in directional drilling models the influence of the drillstring is accounted for by forces acting on the upper boundary of the BHA. However, there are models which take the drillstring into account, when modeling a directional drilling system. These models are called torque and drag models, and are derived in [1, 2, 18, 26, 44].

In this section, a literature survey is presented. First of all, it is investigated what kind of directional drilling models have been derived in the literature. Second, it is researched what has been done on the modeling of nonlinearities in directional drilling. Finally, a discussion of the presented literature is given.

1-3-1 Early stages of development of directional drilling models

Already in 1953, Lubinski and Woods [36] developed an analytical model for the directional drilling process. Although earlier works exist, the work of Lubinski and Woods has been cited numerous times in literature and can be considered as a pioneering work in analytical directional drilling models. In [36], the authors describe the steady-state response of a directional
drilling system. They do this by deriving a two-dimensional, analytical BHA model. For their model, the authors assume that the bit drilling direction is the same as the direction of the force that the BHA applies on the bit. Although the paper only accounts for equilibrium solutions corresponding to straight boreholes, the authors can still derive conclusions on how to reduce crookedness of boreholes. However, the model is only designed to give qualitative insight in the physics of very specific cases. It cannot describe nor predict the propagation of complex borehole trajectories, since Lubinski and Woods did not provide a general formulation of the system kinematics.

Murphey and Cheatham [52] continued the work of Lubinski and Woods. The authors constructed a model with which equilibrium solutions for circular borehole trajectories could be derived. As for the model of Lubinski and Woods, their model did not give a general formulation of system kinematics. Also, the authors discard bit-rock interaction by assuming the bit drills in the direction of the weight on bit.

A general formulation for system kinematics was introduced by Fischer in 1974 [24]. In this paper, the author uses a numerical solver to derive solutions for borehole trajectories with piecewise constant curvature. This meant that the solution was based only on equilibrium solutions; transient behavior was not included in the model.

Computing the equilibrium solutions of a model, such as in the papers of Lubinski and Woods [36], Murphey and Cheatham [52] and Fischer [24], can give insight into the behavior of a directional drilling system. However, transient behavior is ignored in these models. The transient behavior is key to assess the stability properties, which are in turn central to the occurrence of borehole spiraling. Besides the previously mentioned papers, equilibria corresponding to straight boreholes have been researched in [10, 61]. Boreholes with constant curvature were analyzed in [7, 20, 65]. In [59], these solutions have been obtained with a numerical method and compared to the results of a laboratory setup. In [31], a circular equilibrium solution is found numerically for different system configurations, and compared to data of real-life directional drilling system configuration.

Although several authors derived analytical models for the BHA, more often numerical models were used. Generally a calculation using a finite element or finite difference model has been proposed for these models. [24] has already been mentioned; other examples are [3, 7, 8, 12, 48, 64], and, more recently, [15]. Numerical models form a powerful tool to calculate responses of different system configurations: they can be used to calculate the response of a certain specified directional drilling system configurations and to investigate what kind of system configuration gives the desired results. However, the main drawback of these models is that they give no direct insight into the physics governing the process. In an iterative way one could derive the required system configuration for a certain desired well, but it is hard to derive general laws, construct design procedures or derive controllers with these models.

1-3-2 The development of bit-rock interaction laws and 3D models

A year after Fischer’s paper was published, Bradley proposed to further develop the model of bit-rock interaction [10]. He states that if both rock and bit were isotropic, one would expect that the borehole would be drilled in the direction of the force applied to the bit by the drillstring. However, the author states that the effects of the bit combined with the failure characteristics of the rock (bit-rock interaction) result in a modification of the direction in
which the borehole progresses. Bradley argues that this effect generally is small, and therefore
the assumption, that the bit drills in the direction of the resultant force in isotropic rock,
appears to be reasonable. He however points out that this conclusion is based on a theoretical
model, and therefore a firm conclusion awaits experimental confirmation.

The experiments Bradley recommended, were done by Millheim and Warren in 1978 [50].
From their experimental data, they concluded that bit and stabilizer will cut laterally under
conditions that are similar to those experienced in a wellbore. According to the authors, it
should be possible to build a model to predict a well trajectory based on experimental data
and a finite element bottomhole assembly program. The side cutting rate of a bottomhole
assembly is shown to be a function of penetration rate, contact force, component design and
rock type.

Callas included side cutting capability in a model in 1981 [11]. In this paper, the author
allowed for a side force to work on the bit, orthogonal to the direction of the weight on bit.
He mentions that the value of the side force at the bit is not the only factor that controls the
direction in which the bit drills. The basic premise, according to Callas, is that the borehole
geometry, for a specified distance above the bit, plays a more important role. This reflects
the general idea of the 70’s and early 80’s, that the directional tendency of the directional
drilling system is mainly determined by the mechanics of the BHA. Although Callas assumes
the BHA mechanics to be more important than the bit-rock interaction, he does include
a bit-rock interaction model. In this bit-rock interaction model, the author introduces a
maximum angle between side force on bit and weight on bit. He proceeds by solving the
model numerically and comparing the simulations to real life data, finding it to be a good
match.

The idea of Bradley of combining the effects of BHA and bit-rock mechanics in a model,
formed the basis of the work of Cheatham and Ho [14], published in 1981. In this paper,
a simple model is derived, consisting of linear equations, for calculating the effects of the
bit and rock on borehole direction. An explicit distinction is made between the separate
contributions of the rock anisotropy and the bit anisotropy, the factor that contrasts the bit
axial and lateral drilling abilities. Bit constants and rock constants are introduced, which
relate drilling effectiveness to the direction of force components relative to the axis of the
bit. The authors state that experimental data are needed to determine the values of these
constants. They propose that combining the presented method with a model for interaction
of the rotating drill string can provide more insight in the phenomenon of bit walk.

Ho later refined this bit-rock interaction model [25, 27, 28, 29]. He proposed a general linear
bit-rock interaction model that also accounts for moments acting on the bit and for a turning
rate vector of the bit. Furthermore, Ho combined this bit-rock interaction model with a model
of the BHA. The model was developed using a three-dimensional geometry. Therefore, it was
capable of predicting the bit walk tendency and the build/drop tendency of a given BHA
under any drilling condition.

As discussed before, Ho derived a directional drilling model in a three-dimensional framework.
However, adding the third dimension was not straightforward and the ideas of authors on how
this should be done differed. In 1986, Bai stated that the three-dimensional problem can be
solved by decomposing it into two two-dimensional problems [4]. In this respect, he completely
ignored the effect of coupling that arises when the directional drilling system is modeled in
a three-dimensional way. However, in the same year, Chandra did acknowledge the problem
of coupling when extending a directional drilling model to three dimensions [13]. Also, he discussed the effect of bit walk in this respect.

In [62], it is stated that the bit-rock interface laws can be assumed rate-independent, due to the intrinsic symmetry of the bit-rock interaction laws averaged over one revolution. This assumption is of importance, since it allows the independent variable to be chosen as distance drilled instead of time. Furthermore, it is shown that it is possible to construct the interface laws by using the device of an equivalent blade, which embodies the properties and geometry of the bit. Besides simplifying the evaluation of the interface parameters, the introduction of the equivalent blade enables an easy separation of the contribution of the rock and of the bit in the parameters of the interface laws.

1-3-3 Borehole propagation models

Combining the bit-rock interaction, BHA mechanics and kinematic relationships in one model results in a so-called borehole propagation model. Such a model can predict the evolution of a borehole when the initial conditions and inputs are defined. Callas derived such a borehole propagation model in 1981 [11]. In this paper, the author combined a numerical solver for the BHA mechanics with a bit-rock interaction law and kinematic relationships. Other numerical borehole propagation models can be found in [9, 37, 49, 64, 65].

Compared to numerical borehole propagation models, analytical borehole propagation models provide more understanding of the physics governing the directional drilling process. Generally, analytical borehole propagation models should lead to a Delay Differential Equation (DDE) for planar trajectories. Although this makes the models complex, they can still be analyzed. Over the years, analysis techniques have become more and more sophisticated. Algorithms as in [45], have made it possible to do stability analysis for delay differential systems.

The number of publications, which derive the mathematical equations for the propagation of boreholes, is small. Contributions were first made by Neubert and Heisig. In 1996, they developed a three-dimensional, analytical borehole propagation model [55, 56]. This model is explained in much more detail in [54].

In their model, Neubert and Heisig assume that the segments, between the stabilizers and first stabilizer and bit, are linear-elastic elements modeled as Euler-Bernoulli beams. For this model, only the lower part of the BHA (from the second stabilizer to the bit) is taken into account. Furthermore, the first stabilizer is steerable, and therefore acts as a point-the-bit actuator. The authors derive equations for a kinematics model, bit-rock interaction model and BHA model in a local reference frame. By combining these equations and transforming the result to a global coordinate system, four differential equations are constructed for the translation and rotation of the BHA-axis. The unknown coefficients of the general solutions of these differential equations are found by defining constraints and initial conditions of the system. The nonlinear nature of the model is due to transformations between different reference frames and because of relating translational parameters to rotational parameters using trigonometric relations. In order to simplify the model, it is linearized. Thus, a linear delay differential equation for the states of the system is derived.

Important contributions to analytical borehole propagation models have been made by Downton. In [20], Downton derives three linear, analytical borehole propagation models. The
models are two-dimensional. In subsequent papers [22, 66], controllers have been designed for the presented models.

For these models, the author takes into account one stabilizer. At the stabilizer, the BHA is permitted to pivot about a touch-point with the borehole. The actuator is placed between the bit and the stabilizer. As only three points of borehole contact are involved (bit, actuator and stabilizer), this is called a three point steering system. In order to allow for the use of small-angle approximations, the direction of drilling is assumed to be approximately aligned to the horizon. The lateral displacement of the borehole is measured along a coordinate orthogonal to the horizon.

For the first model, the bit is assumed to be infinitely sharp and able to cut in any direction. The BHA is taken to be infinitely stiff with zero mass. In the second model, the collars and actuators are allowed to be compliant and deflect under side-loading. For these two models, the author derives a transfer function from input of the steering actuator to lateral displacement.

For the first two models, the role that force plays in the lateral borehole-propagation response was incorporated indirectly. For the third model, the forces at the bit are derived explicitly from the known geometry and input by actuators. The BHA is again taken to be infinitely stiff, but hinge flexures are added to simulate localities of compliance along the assembly. The drill bit is assumed to have a preferential drilling direction based on the applied loads. Whereas in the other two models a displacement actuator is used, now a force actuator is defined. The force actuator pushes the tool laterally using the borehole wall as a reactive surface. Also for the third model, a transfer function from actuation force to lateral displacement is derived.

The last analytical borehole propagation model that will be reviewed, is the PD-model. This three-dimensional model has been developed part by part over the years, but the most complete description is given in [60]. In [33, 51] controllers have been designed for this model.

In contrast to the models discussed previously, the borehole and BHA are represented in the PD-model by two curvilinear coordinates. Furthermore, inclination and azimuth of the borehole are chosen as states. The PD-model is built up by three different components: the BHA model, the bit-rock interface laws and the kinematic relationships. For the BHA model, the deformations of the BHA inside the borehole are presumed to be small. Hence, the BHA can be modeled statically as a series of Euler-Bernoulli beams. The stabilizers are modeled as hinges, which only apply a force on the BHA perpendicular to its axis. The stabilizers constrain the movement of the BHA. These constraints in movement are translated to mathematical constraints. Combining the solutions of the differential equations, resulting from the Euler-Bernoulli equations, and the constraints of the stabilizers, gives a model for the deflection of the BHA at every location along the BHA. In this way, expressions for the force and moment at the bit are derived. Combining all equations, results in a delay differential equation for the propagation of the borehole.

1-3-4 Modeling of nonlinearities in directional drilling

As previously mentioned, contributions to the modeling of nonlinear physical effects in directional drilling are scarce. In this section, a selection of important contributions to the modeling of bit tilt saturation and non-ideal stabilizer contact are reviewed.
Already in 1981, Callas derived a model that describes saturation of mechanics at the bit [11]. However, the author does not model saturation of the bit tilt, but saturation of the angle under which forces act on the bit. This effect has not been observed in experiments. Therefore, the modeling on this nonlinearity does not seem a logical choice. For this reason, the contribution of Callas will not be taken into account in the discussion on the modeling of nonlinearities in directional drilling.

The second reference on the modeling of bit tilt saturation is a dissertation by Marck [38]. The author models saturation of the bit tilt for the PD-model. Marck chooses to model the directional drilling system with bit tilt saturation as a hybrid switching system, switching between two modes. For the first mode, the bit tilt is not saturated; for the second mode, the tilt is saturated. By switching between these two modes, the author can calculate the response of the system. Marck researches the effect of bit tilt saturation on borehole propagation. The author states that in two dimensions, borehole rippling (the two-dimensional equivalent of spiraling) has been observed to be mainly related to fluctuations of the bit tilt. This is a consequence of the oscillating lateral force at the bit arising from the relative positions of the first two stabilizers that need to conform with the borehole geometry [40].

Research to non-ideal stabilizer contact has been done in [21]. In the paper, Downton first describes stability and response of certain system configurations. The simulations are done with a quasi-static finite element model. The non-ideal stabilizer contact is modeled as a force. This force depends on the position of the stabilizer. Downton plots different Nyquist plots of the resulting transfer function in order to perform a stability analysis of the system. However, the author does not derive an equation for the propagation of the borehole for this model.

1-3-5 Discussion of literature survey

In this literature survey, an overview has been given of the most prominent models concerning the directional drilling process. Also, the contributions of different authors in the field of modeling bit tilt saturation and non-ideal stabilizer contact have been discussed.

Most work has been done on numerical models. Numerical models are powerful tools to calculate the responses of particular directional drilling systems. The simulations can be used to investigate the BHA design needed to achieve desired results. However, the main drawback of these models is that they give no direct insight into the physics governing the process. In an iterative way one could derive the required system configuration for a certain desired well, but it is hard to derive general laws, construct design procedures or derive controllers with these models.

In order to get more insight into the physics governing the directional drilling process, analytical models have been derived. Analytical models are constructed by deriving the physical laws of the process. Manipulating these equations can give insight in how the process can be controlled or used to the advantage of the driller.

The analytical models come in many different forms. Early models tended to be based on many simplifying assumptions; over the years, these assumptions have been relaxed or dropped completely. This resulted in more complicated models, but because analysis techniques improved at the same time, they were still useful for analysis of the directional drilling process.
The current state of the art in analytical models are the borehole propagation models. These models contain a set of equations, generally delay differential equations, which describe the propagation of the borehole in distance drilled. Only few authors have derived such models: the contributions of Neubert and Heisig, Downton and Perneder and Detournay were discussed in this review.

With their model, Neubert and Heisig [54, 55, 56] provide an innovative way of describing the directional drilling process. Their model contains a lot of details and a general, three-dimensional description. However, the literature on this model seems not to have reached the community that well, as no research was found that was based on the model of Neubert and Heisig. One issue with this model is that the derivations in the main source for this model [54] are not thoroughly explained. Another issue is that, because deflection is chosen as state variable, the reference frame has to be moved along with the BHA, when the borehole is propagated. This results in transformation and rotation terms, which include trigonometric functions, leading to a nonlinear model. Also, the choice for some of the variables is not made clear. All of this combined makes it difficult for the reader to get a thorough understanding of the choices that were made.

In the directional drilling models proposed by Downton [20], the BHA is treated as a three-point supported beam, whereby the transfer function from RSS input to lateral displacement is easily derived. The paper does not provide a detailed description of the bit-rock interaction law; also, due to certain assumptions (such as the borehole being approximately aligned with the horizon) the resulting model is not a general description of the process. Furthermore, if Downton’s models would be used in a description of the complete evolution of the borehole, the reference frame has to be shifted along with the bit, which complicates the procedure. The reason for this is that, just as in the model of Neubert and Heisig, the state variable is chosen to be deflection of the BHA. Although this gives the author the opportunity to do stability analysis of the borehole propagation, it results in problems when a general solution of borehole propagation needs to be formulated. Furthermore, Downton chooses the parameters of his models in a pragmatically way, but the physical meaning of the parameters is not always clear. This gives rise to problems when making interpretations about the model or when these values need to be measured for field purposes.

The model derived by Perneder and Detournay, the PD-model [60], seems to give the most complete and clearest description of the process. In contrast to the previously discussed analytical borehole propagation models, in the PD-model the state variables are chosen to be the inclination and azimuth of the borehole. This means that during borehole propagation, the reference frame, in which the state variables are measured, does not have to be moved along with the BHA. The result of this is that in the PD-model no transformation and rotation calculations between different reference frames are needed, which results in a linear model. Also, the variables in the PD-model are chosen in such a way that they represent physical quantities. Therefore, if the model is needed for field purposes, the variables can directly be measured. This increases the practical usefulness of the model. Also, it gives the model an easier physical understanding.

On the modeling of nonlinearities in directional drilling a lot of advancement is still needed. The amount of contributions on bit tilt saturation and non-ideal stabilizer contact is small, although those nonlinearities are believed to greatly affect the borehole propagation.

The main references that were reviewed on the modeling of nonlinearities are [38] and [21].
In [38], Marck models bit tilt saturation for the PD-model. In this dissertation, saturation of the bit tilt is modeled with switching modes. Through mode detection it is determined whether the dynamics of the saturated or unsaturated system are active. An improvement that could be made would be to model the hybrid system as a complementarity system. This could lead to a more compact description of the model, which would be easier to analyze.

In [21], Downton models the hybrid behavior of a directional drilling system with non-ideal stabilizer contact as a complementarity condition. This leads to a compact and easy to analyze description. However, the author does not derive an equation for the propagation of the borehole for this model. Furthermore, the derivations of the model are not stated in a clear way in this paper.

1-4 Research goals

Although a lot of work has been done on the modeling of directional drilling systems, improvement could be made on the modeling of nonlinearities in this field. In this research, the modeling of non-ideal stabilizer contact will be done for a directional drilling system.

In this thesis, the main goal is to develop a hybrid directional drilling model by incorporating non-ideal stabilizer contact. Furthermore, it should be understood what is the influence of a non-ideal stabilizer on the borehole trajectory of a directional drilling system. The focus should in particular be on the directional stability of the non-ideal system and the effect of the switching nature of the system on the (quasi-)stationary solutions.

1-5 Outline and approach of research

This thesis is structured in the following way. In Chapter 2, a two-dimensional hybrid directional drilling model is derived. As for the PD-model, the hybrid directional drilling model consists of a BHA model, bit-rock interaction and kinematic relationships. In contrast to the PD-model, the BHA is modeled as a hybrid lumped-parameter model, where the hybrid nature of the model comes from non-ideal stabilizer contact at one of the stabilizers. By combining the model for the BHA, the bit-rock interaction and the kinematic relationships, a borehole propagation model is derived. This model is transformed into a state-space description. Also the borehole propagation models and corresponding state-space descriptions of two linear subsystems of the hybrid model are derived.

Dynamics analysis of the hybrid model is done in Chapter 3. First, a relevant benchmark system is defined for the analysis. This benchmark system is then used to determine the quasi-stationary solutions of the hybrid system and the ideal subsystems. In order to research the impact of non-ideal stabilizer contact on borehole rippling, a stability analysis is performed on the ideal subsystems. Furthermore, the hybrid system is simulated and researched. In this way, the behavior of the hybrid system can be compared to that of its ideal counterparts. Thus, the effect of non-ideal stabilizer contact on the behavior of the directional drilling system can be investigated.

Finally, in Chapter 4, the conclusions of the development and analysis of the hybrid model and recommendations for further research are given.
Chapter 2

A hybrid directional drilling model

In this chapter, a hybrid directional drilling model is developed. The model is built up from a hybrid BHA model, kinematic relationships and the bit-rock interface laws. In the hybrid BHA model, a non-ideal stabilizer is taken into account. Instead of being in contact with the borehole wall at all times, this stabilizer is allowed the freedom to switch between contact and no contact. By assuming non-ideal stabilizer contact in the presented model, it is possible to analyze whether the hybrid behavior of the system results in different behavior from what is observed in models with ideal stabilizer contact.

The system is modeled as a two-dimensional lumped-parameter model. Working in two dimensions instead of three greatly reduces modeling complexity, without losing the essential effect of the nonlinearities. The choice is made for modeling the BHA model with a lumped-parameter modeling method instead of the usual distributed modeling method (as in [60]). The reason for this is that the switching behavior of the hybrid system is difficult to capture in a distributed model. When for example Euler-Bernoulli beam equations are used, one has to switch between constraints for the differential equation of the beam. In order to formulate a more concise model description, a lumped-parameter model is derived.

By combining the derived hybrid BHA model with kinematic relationships and bit-rock interaction laws, an equation for the inclination of the borehole at the bit is derived. This borehole propagation model can be transformed into a dynamic state-space model description, in order to describe the borehole propagation dynamics of the system. Both model descriptions are subject to a contact law defined for the non-ideal stabilizer.

Furthermore, together with the hybrid model, the models of two subsystems of the hybrid model are derived: Subsystem 1, for which the non-ideal stabilizer is never in contact with the borehole wall, and Subsystem 2, for which the non-ideal stabilizer is always in contact with the borehole wall. Both these subsystems are comparable to ideal directional drilling systems and will prove useful in the analysis of the hybrid model.
2-1 Description of considered directional drilling system

In order to derive a directional drilling model, a 2D directional drilling system as in Figure 2-1 is considered. In order to define positions along borehole and BHA, two curvilinear coordinates are defined. Curvilinear coordinate $s$ is measured, from the bit along the BHA. Another curvilinear coordinate, $S$, is measured from the drill rig downwards, along the borehole. The changing length of the borehole is denoted by $L$. Borehole and BHA are assumed to be approximately aligned. Therefore, any point along the BHA or the borehole (between the last stabilizer and the bit) can be described both in coordinate $s$ and $S$.

![Figure 2-1: A 2D directional drilling system with curvilinear coordinates for borehole and BHA.](image)

The BHA is chosen to be modeled as a lumped-parameter model. Such a model assumes BHA sections to be rigid. The stiffness of a physical BHA is accounted for by hinges and torsional springs. This model differs from the BHA models described in the literature that are based on Euler-Bernoulli equations (as in [60]), which lead to distributed models. However, due to the hybrid nature of the BHA presented, a lumped-parameter model is preferred in this case. Using Euler-Bernoulli equations would result in complex switching between constraints in the Euler-Bernoulli beam BHA model. The inclusion of such a distributed BHA model is left for future research.

In order to keep the derivations as simple as possible, the considered BHA consists of two stabilizers (see Figure 2-2). The first stabilizer is a non-ideal stabilizer, which has a diameter smaller than that of the borehole. This non-ideal stabilizer can switch between contact and
no contact with the borehole wall. The second stabilizer is an ideal stabilizer (which diameter is equal to that of the borehole), which is always centered in the borehole and thus acts as an ideal (bilateral) constraint. Furthermore, the BHA is modeled by three rigid links, connected by three hinges. The choice is made for three links and hinges, since in this way complexity is limited as much as possible, without eliminating key characteristics of the system. The modeling approach proposed in this chapter could easily be used to construct models with more links/stabilizers. The considered BHA configuration is shown in Figure 2-2.

The curvilinear coordinate \( s \) is shown along the BHA, where \( s \in [0, \ell_1 + \ell_2 + \ell_3] \). The BHA consists of three undeformable sections (links). The first section of the BHA is fixed to the bit, which is located at \( s = s_0 = 0 \). The length of this section is \( \ell_1 \). Attached to Section 1 is the RSS actuator. The RSS is located at a distance \( \Lambda \ell_1 \) behind the bit, where \( 0 < \Lambda < 1 \). At \( s = s_1 = \ell_1 \), Section 1 is connected to Section 2 via a frictionless hinge with a torsional spring. The torsional stiffness of this spring is \( k_1 \) [Nm]. Spring and hinge are connected to the second undeformable section of the BHA. This section has a length of \( \ell_2 \). Section 2 is connected to a second frictionless hinge and torsional spring of stiffness \( k_2 \), located at \( s = s_2 = \ell_1 + \ell_2 \). Also, a non-ideal stabilizer is attached to the hinge at \( s_2 \). This stabilizer can move freely within a certain distance from the borehole axis. When the stabilizer touches the borehole wall, the BHA model is comparable to that of a linear two-stabilizer BHA model. When the stabilizer does not touch either borehole wall, the behavior of the BHA is comparable to that of a linear one-stabilizer BHA model [60].

The frictionless hinge and torsional spring at \( s_2 \) are connected to the third BHA section. The length of this section is \( \ell_3 \). At \( s = s_3 = \ell_1 + \ell_2 + \ell_3 \), the third section is attached to a third frictionless hinge. This hinge is also attached to an ideal stabilizer. This stabilizer is always centered in the borehole.

There is no moment transmitted between a stabilizer and the borehole wall. Furthermore, only forces normal to the borehole axis can act on both stabilizers. Also the RSS force is assumed to always be applied normal to the borehole axis.

Inclinations of BHA (\( \theta(s, L) \)) and borehole (\( \Theta(S) \)) are measured anti-clockwise from the downward vertical (see Figure 2-3). The inclinations of the three BHA sections for a certain borehole length are given by \( \theta_1(L), \theta_2(L) \) and \( \theta_3(L) \). The borehole inclination at \( S \) is given by \( \Theta(S) = \Theta(L - s) \), for \( S \in [L - (\ell_1 + \ell_2 + \ell_3), L] \). Furthermore, the borehole inclination at the bit is \( \hat{\Theta} := \Theta(L) \). The BHA inclination at the bit is \( \hat{\theta} := \theta(0, L) = \theta_1(L) \). The bit tilt is defined as \( \psi := \hat{\theta} - \hat{\Theta} \).

The average inclination of the borehole axis between the bit and the non-ideal stabilizer is given by \( \langle \Theta \rangle_1 \). The average inclination of the borehole axis between the non-ideal stabilizer and the ideal stabilizer stabilizer is given by \( \langle \Theta \rangle_2 \). These average inclinations are defined as:

\[
\langle \Theta \rangle_1 = \frac{1}{S_0 - S_2} \int_{S_0}^{S_2} \Theta(\sigma) \, d\sigma, \\
\langle \Theta \rangle_2 = \frac{1}{S_2 - S_3} \int_{S_2}^{S_3} \Theta(\sigma) \, d\sigma, \tag{2-1}
\]

where \( S_i = L - s_i \), for \( i = 0, 2, 3 \).

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Figure 2-2: The considered BHA setup. Curvilinear coordinate $s$ is measured from the bit along the BHA. Located behind the bit are a hinge with torsional spring, a non-ideal stabilizer with torsional spring and an ideal stabilizer. The RSS is located a a distance $\Lambda l_1$ behind the bit, where $0 < \Lambda < 1$. All three sections of the BHA are assumed to be undeformable. The bending stiffness of a real BHA is accounted for with the torsional springs. The torsional spring at $s_1$ has a stiffness of $k_1$. The spring at $s_2$ has a stiffness of $k_2$.

It is assumed that BHA and borehole are approximately aligned. For this reason small-angle approximations hold, resulting in

$$\sin \left( \theta(s, L) - \Theta(L - s) \right) \approx \theta(s, L) - \Theta(L - s). \quad (2.3)$$

This small-angle approximation is used multiple times in the derivations of the model equations.

Figure 2-3: Inclinations of BHA sections and borehole at the bit.

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2-2 Components of directional drilling model

The model of the directional drilling system is built up from three components: the BHA model, the bit-rock interface laws and the kinematic relationships. The interaction between these three components is shown in Figure 2-4.

First, consideration of force and moment balance for the BHA leads to expressions for the force and moment at the bit that depend on the past borehole geometry. Second, the bit-rock interaction relates force and moment at the bit to the bit kinematics. Last, the relation between the bit kinematics and the borehole geometry is given by the kinematic relationships. Combining these three components results in a model for the inclination of the borehole at the bit. These three model components are described in detail in the following sections.

2-2-1 Hybrid BHA model

In order to model the relationship between the borehole geometry and the force and moment at the bit, the force and moment balance of the BHA are used. Only forces acting normal to the BHA will be taken into account in this section.

The moments and forces acting normal to the BHA are shown in Figure 2-5. The force and moment at the bit are denoted by $F_0$ and $M_0$ respectively. RSS force $F_r$ acts a distance $\Lambda \ell_1$ from the bit. On each section of the BHA, a gravity force acts. These resultants of gravity forces act on the middle of the segments. Since only forces orthogonal to borehole and BHA are taken into account in this section, the orthogonal components of the gravitational forces are used, given by

\begin{align}
F_{w,1} &= w\ell_1 \sin(\Theta)_1, \\
F_{w,2} &= w\ell_2 \sin(\Theta)_1, \\
F_{w,3} &= w\ell_3 \sin(\Theta)_2,
\end{align}

where $w$ represents the weight per unit length of the BHA. Furthermore, $F_{w,1}$, $F_{w,2}$ and $F_{w,3}$ are the resultants of the gravity forces for Section 1, 2 and 3, respectively.
Forces $F_2$ and $F_3$ represent the contact forces that act on the non-ideal and ideal stabilizer, respectively. Since the contact force exerted on the non-ideal stabilizer can only exist when the stabilizer is in contact with the borehole wall, there exists a relation between its position and the force $F_2$ exerted on it.

Unilateral contact model for non-ideal stabilizer

The deflection of the non-ideal stabilizer with respect to the borehole axis is given by $z_2$ (see Figure 2-6). The nominal clearance of this stabilizer is $R_\Delta = R_o - R_i$, where $R_o$ and $R_i$ are the radii of borehole and stabilizer, respectively. The stabilizer is in contact with the upper borehole wall when $z_2 = R_\Delta$. Contact is achieved with the lower borehole wall when $z_2 = -R_\Delta$. The relation between $F_2$ and $z_2$ can therefore be described by using Signorini’s law for unilateral contact:

\[
\begin{align*}
F_2 &\geq 0 & \text{if} \quad z_2 = -R_\Delta, \\
F_2 &= 0 & \text{if} \quad -R_\Delta < z_2 < R_\Delta, \\
F_2 &\leq 0 & \text{if} \quad z_2 = R_\Delta.
\end{align*}
\]

Equation (2-7) will be referred to as the unilateral contact law.

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Figure 2-6: The non-ideal stabilizer in the borehole. The clearance of the non-ideal stabilizer with respect to the borehole axis is denoted by $z_2$. The non-ideal stabilizer is assumed to have a radius $R_i$ and the borehole a radius $R_o$, where $R_o \geq R_i$. The maximum clearance is denoted by $R_\Delta = R_o - R_i$. Note that the displayed size difference between the non-ideal stabilizer and the borehole is increased for explanation purposes and does not represent a realistic situation.

Solving force and moment balance

From the moment and force balance of the system, the following equations are retrieved:

\[
F_3 = \frac{F_{w,3}}{2} - \frac{M_2}{\ell_3}, \quad (2-8)
\]

\[
F_2 = -\frac{M_1}{\ell_2} + \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \frac{F_{w,2} + F_{w,3}}{2}, \quad (2-9)
\]

\[
F_0 = -F_2 - F_3 - F_r + F_{w,1} + F_{w,2} + F_{w,3}, \quad (2-10)
\]

\[
\frac{M_0}{\ell_1} = -\frac{F_0}{\ell_1} - \frac{M_1}{\ell_1} - (1 - \Lambda) F_r + \frac{F_{w,1}}{2}, \quad (2-11)
\]

where $M_1 = k_1(\theta_2 - \theta_1)$ and $M_2 = k_2(\theta_3 - \theta_2)$ are the moments associated with the rotational springs. Eliminating $F_3$ from these equations results in:

\[
F_2 = \left( \frac{k_1}{\ell_2} \right) \theta_1 - \left[ \frac{k_1}{\ell_2} + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \right] \theta_2 + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \theta_3 + \frac{F_{w,2} + F_{w,3}}{2}, \quad (2-12)
\]

\[
F_0 = -\left( \frac{k_2}{\ell_3} \right) \theta_2 + \left( \frac{k_2}{\ell_3} \right) \theta_3 - F_2 - F_r + F_{w,1} + F_{w,2} + \frac{F_{w,3}}{2}, \quad (2-13)
\]

\[
\frac{M_0}{\ell_1} = \left( \frac{k_1}{\ell_1} \right) \theta_1 + \left( \frac{k_2}{\ell_3} - \frac{k_1}{\ell_1} \right) \theta_2 - \left( \frac{k_2}{\ell_3} \right) \theta_3 + F_2 + \Lambda F_r - F_{w,2} - \frac{F_{w,1} + F_{w,3}}{2}. \quad (2-14)
\]
The bit and the ideal stabilizer are assumed to be centered in the borehole at all times. This means that at, the position of the bit and the ideal stabilizer, the borehole axis and BHA axis intersect. This can mathematically be described as

\[ \ell_1 \theta_1 + \ell_2 \theta_2 + \ell_3 \theta_3 = (\ell_1 + \ell_2)\langle \Theta \rangle_1 + \ell_3 \langle \Theta \rangle_2. \]  

(2-15)

In addition to Equation (2-15), the following geometrical constraint can be defined between the BHA deflection \( z_2 \) at the non-ideal stabilizer and the inclination of borehole and BHA:

\[ \theta_3 = \frac{z_2}{\ell_3} + \langle \Theta \rangle_2. \]  

(2-16)

Using Equation (2-15) and (2-16) to eliminate \( \theta_2 \) and \( \theta_3 \) from Equations (2-12) to (2-14), results in

\[
F_2 = \left( \frac{\ell_1}{\ell_2} \right) \left( k_1 \left( \frac{1}{\ell_1} + \frac{1}{\ell_2} \right) + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \right) \theta_1 + \left( \frac{k_1}{\ell_2} + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right)^2 \right) z_2 \\
- \ell_1 \left( \frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \left( \frac{k_1}{\ell_2} + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \right) \langle \Theta \rangle_1 + k_2 \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \langle \Theta \rangle_2 + \frac{F_{w,2} + F_{w,3}}{2},
\]  

(2-17)

\[
F_0 = \frac{k_2 \ell_1}{\ell_3} \theta_1 + \left( \frac{k_2}{\ell_3} \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \right) z_2 - \left( \frac{k_2}{\ell_3} \left( 1 + \frac{\ell_1}{\ell_2} \right) \right) \langle \Theta \rangle_1 \\
+ \left( \frac{k_2}{\ell_3} \right) \langle \Theta \rangle_2 - F_2 - F_r + F_{w,1} + F_{w,2} + \frac{F_{w,3}}{2},
\]  

(2-18)

\[
M_0 = \left( k_1 \left( \frac{1}{\ell_1} + \frac{1}{\ell_2} \right) - \frac{k_2 \ell_1}{\ell_2 \ell_3} \right) \theta_1 + \left( \frac{k_1}{\ell_1 \ell_2} - \frac{k_2}{\ell_3} \left( \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \right) z_2 + \left( 1 + \frac{\ell_1}{\ell_2} \right) \left( \frac{k_2}{\ell_3} - \frac{k_1}{\ell_1} \right) \langle \Theta \rangle_1 \\
- \left( \frac{k_2}{\ell_3} \right) \langle \Theta \rangle_2 + F_2 + \Delta F_r - F_{w,2} - \frac{F_{w,1} + F_{w,3}}{2}.
\]  

(2-19)

The hybrid BHA model consists of Equation (2-17), (2-18) and (2-19), subject to Equation (2-7).

2-2-2 Kinematic relationships

In order to describe the kinematics of the bit, define two orthonormal reference frames as represented in Figure 2-7. Reference frame \( \vec{I} \) consists of \( \vec{I}_1 \) and \( \vec{I}_2 \). Vector \( \vec{I}_1 \) is a vector in the direction in which the borehole propagates. Vector \( \vec{I}_2 \) is defined orthogonal to \( \vec{I}_1 \), in anti-clockwise direction. Reference frame \( \vec{i} \) consists of \( \vec{i}_1 \) and \( \vec{i}_2 \). Vector \( \vec{i}_1 \) is in the same
direction as the bit axis. Vector $\vec{i}_2$ is orthogonal to $\vec{i}_1$, in anti-clockwise direction (see Figure 2-7).

The motion of the bit can be described by velocity $\vec{v}$ and angular velocity $\omega$. Velocity $\vec{v}$ can be decomposed into $v_1$, in the direction of $\vec{i}_1$, and $v_2$, in the direction of $\vec{i}_2$. Angular velocity $\omega$ acts around the axis perpendicular to the plane of propagation.

Motion of the bit results in penetration of the rock. We define the penetration variables

$$\vec{d} = \frac{2\pi \vec{v}}{\Omega}, \quad (2-20)$$
$$\varphi = \frac{2\pi \omega}{\Omega}, \quad (2-21)$$

where $\Omega$ is the angular velocity of the bit around $\vec{i}_1$. The penetration per revolution is described by penetration vector $\vec{d}$ in (2-20) and angular penetration $\varphi$ in (2-21). Angular penetration $\varphi$ acts around the axis perpendicular to the plane of propagation.

Penetration vector $\vec{d}$ can be partitioned into $d_1$ and $d_2$, where $\vec{d} = d_1 \vec{i}_1 + d_2 \vec{i}_2$. By definition, the direction in which penetration vector $\vec{d}$ points is the direction in which the borehole will propagate. Therefore, $\vec{d}$ is inclined by $\hat{\Theta}$ on the vertical axis. Vector $d_1 \vec{i}_1$ is defined as the component of $\vec{d}$ in the direction in which the bit is pointed. Therefore, the angle $\beta$ between $d_1 \vec{i}_1$ and $\vec{d}$ can be expressed as $\beta = \hat{\Theta} - \hat{\theta}$. Recall that the bit tilt was defined as $\psi = \hat{\theta} - \hat{\Theta}$. Therefore, it can be stated that

$$\psi = -\beta = -\arctan \left( \frac{d_2}{d_1} \right). \quad (2-22)$$

Furthermore, since bits are designed to drill axially, the assumption can be made that $d_1 \gg d_2$, leading to the following approximation for the bit tilt:

$$\psi \approx -\frac{d_2}{d_1}. \quad (2-23)$$
The change of inclination of the bit with respect to distance drilled can be written as

\[ \frac{d\hat{\theta}}{dL} = \frac{dt}{dL} \frac{\hat{\theta}}{dt} = \frac{\omega}{\|v\|} = \frac{\varphi}{d} \approx \frac{\varphi}{d_1}. \]  

(2-24)

**2-2-3 Bit-rock interface laws**

The forces and moment acting on the bit are related to the penetrations by the bit-rock interface laws. In this respect, define \( F_{ax} \) as the axial force acting on the bit. It is defined as

\[ F_{ax} = F_{ax}|_{S_3} + w_1 \cos(\Theta)_1 + w_2 \cos(\Theta)_1 + w_3 \cos(\Theta)_2, \]  

(2-25)

where \( F_{ax}|_{S_3} \) is the axial force transmitted by the upper drill-string to the BHA (i.e. at the ideal stabilizer). The bit-rock interface laws can be expressed as [62]:

\[
\begin{bmatrix}
F_{ax} \\
F_0 \\
M_0
\end{bmatrix} = - \begin{bmatrix}
G_1 \\
H_1 & 0 & 0 \\
0 & H_2 & 0 \\
0 & 0 & H_3
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2 \\
\varphi
\end{bmatrix},
\]

(2-26)

where \( G_1 \) is a measure of bit bluntness. Coefficients \( H_i \), for \( i = 1, 2, 3 \), relate penetration variables to the forces required for these penetrations. Generally, \( H_1 \ll H_2 \), since the bit is designed to drill in the axial direction. With these definitions, the active weight on bit can be defined by \( W = H_1 d_1 = -(F_{ax} + G_1) \). The active weight on bit is the part of the weight on bit that is directly associated to the bit advancement into the rock.

**2-3 Scaling**

In order to identify characteristic parameters of the directional drilling system and derive a general model description, the considered system is expressed in dimensionless form. Therefore, the parameters and variables which define the model, are scaled.

For the scaling of lengths of the BHA segments, a characteristic length of \( \ell_1 \) is chosen. Thus scaled lengths of the BHA sections become \( \lambda_i = \ell_i / \ell_1 \), for \( i = 1, 2, 3 \). The length of the borehole after scaling is \( \xi = \ell_2 / \ell_1 \). Using these scaled variables, define

\[ \xi_i = \xi - \sum_{j=1}^{i} \lambda_j \quad \text{for} \quad i = 2, 3, \]

(2-27)

and \( \xi_0 = \xi \). Quantities with dimension of length that are orthogonal to the BHA are scaled with the maximum clearance \( R_\Delta \) at the non-ideal stabilizer. The position of the non-ideal stabilizer with respect to the borehole axis becomes \( \tilde{z}_2 = \tilde{z}_2 / R_\Delta \) after scaling. The scaled radii are \( \tilde{R}_i = R_i / R_\Delta \) and \( \tilde{R}_o = R_o / R_\Delta \). After scaling, the maximum clearance becomes \( \tilde{R}_\Delta = R_\Delta / R_\Delta = 1 \).
Since axial and lateral lengths are scaled differently, parameter $\varrho = \frac{R\Delta}{l_1}$ is defined to account for this difference.

Spring constants are scaled with $k_1$. Therefore, the scaled spring constants of the spring at $\xi_1$ and $\xi_2$ become $\kappa_1 = \frac{k_1}{k_1} = 1$ and $\kappa_2 = \frac{k_2}{k_1}$, respectively.

Before scaling, the forces and moments acting on the BHA are denoted by $F$ and $M$. In order to scale the forces and moments, $F^* = \frac{k_1}{k_1}$ is chosen as a characteristic force. Scaling of the RSS force leads to $\tilde{F}_r = \frac{F_r}{F^*}$. Scaling of the force at the bit and the contact forces results in $\tilde{F}_i = \frac{F_i}{F^*}$, for $i = 0, 2, 3$. The scaled orthogonal components of the gravity forces are determined with

\begin{align}
\tilde{F}_{w,1} &= \tilde{w}\lambda_1 \sin(\Theta)_1, \\
\tilde{F}_{w,2} &= \tilde{w}\lambda_2 \sin(\Theta)_1, \\
\tilde{F}_{w,3} &= \tilde{w}\lambda_3 \sin(\Theta)_2,
\end{align}

where $\tilde{w} = \frac{w_{\xi_1}}{F^*}$ for $i = 1, 2, 3$.

Scaled active weight on bit is given by $\Pi = \frac{W}{F^*}$. The moment at the bit is scaled with $M^* = k_1$, resulting in $\tilde{M}_0 = \frac{M_0}{F^*}$.

### 2-4 Borehole propagation model

Combining the hybrid BHA model, kinematic relationships and the bit-rock interaction, results in a borehole propagation model. Borehole propagation models are derived for the hybrid system and the ideal subsystems: Subsystem 1, for which the non-ideal stabilizer is
never in contact with the borehole wall, and Subsystem 2, for which the non-ideal stabilizer is always in contact with the borehole wall.

### 2-4-1 Borehole propagation model of hybrid system

In order to derive the borehole propagation model of the hybrid system, define the following parameters

\[
\eta = \frac{H_2}{H_1}, \quad (2-31)
\]
\[
\chi = \frac{H_3}{H_1 \ell_1}, \quad (2-32)
\]
\[
\epsilon = \frac{\chi}{\eta}, \quad (2-33)
\]

where \(\eta\) and \(\chi\) are the lateral and angular steering resistance, respectively. The ratio between the angular and lateral steering resistance is given by \(\epsilon\). Combining these definitions with the kinematic relationships (Equation (2-23) and (2-24)) and the bit-rock interaction (Equation (2-26)) and applying scaling, results in

\[
\tilde{F}_0 = -\eta \Pi (\hat{\Theta} - \hat{\theta}) = -\eta \Pi (\hat{\Theta} - \theta_1), \quad (2-34)
\]
\[
\tilde{M}_0 = -\epsilon \eta \Pi \frac{d\hat{\theta}}{d\xi} = -\epsilon \eta \Pi \frac{d\theta_1}{d\xi}. \quad (2-35)
\]

Generally, \(\eta \gg \chi\), which yields \(\epsilon \approx 0\), see [38]. Furthermore, \(\eta \Pi\) is at most of order \(O(1)\) and \(\frac{d\theta_1}{d\xi}\) of order \(O(0.1)\). Therefore, the assumption can be made that

\[
\tilde{M}_0 = 0. \quad (2-36)
\]

By combining the BHA model with Equation (2-34) and (2-36), the following model is derived:

\[
\hat{\Theta} = A_1(\Theta) + A_2(\Theta)_2 + A_3 \tilde{F}_2 + A_4 \tilde{F}_r + A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3}, \quad (2-37)
\]
\[
\tilde{z}_2 = B_1(\Theta) + B_2(\Theta)_2 + B_3 \tilde{F}_2 + B_4 \tilde{F}_r + B_5 \tilde{F}_{w,1} + B_6 \tilde{F}_{w,2} + B_7 \tilde{F}_{w,3}, \quad (2-38)
\]

where the coefficients \(A_i\) and \(B_i\), for \(i = 1, \ldots, 7\), can be found in Appendix A. Equation (2-38) is subject to the scaled unilateral contact law:

\[
\tilde{F}_2 \geq 0 \quad \text{if} \quad \tilde{z}_2 = -1,
\]
\[
\tilde{F}_2 = 0 \quad \text{if} \quad -1 < \tilde{z}_2 < 1,
\]
\[
\tilde{F}_2 \leq 0 \quad \text{if} \quad \tilde{z}_2 = 1. \quad (2-39)
\]
Now, Equation (2-37) to (2-39) constitute the hybrid borehole evolution model. Note that it reflects a dynamic delay system model, due to the distributed delay terms induced by the terms \( \langle \Theta \rangle_i \), for \( i = 1, 2 \). It is a hybrid model due to the complementarity condition in (2-39). Also note that Equation (2-38), subject to (2-39), has one unique solution if \( B_3 > 0 \).

2-4-2 Borehole propagation models of ideal subsystems

In order to compare the hybrid system to its ideal subsystems, the borehole propagation models of those systems are derived as well.

Borehole propagation model of Subsystem 1

In the case where the non-ideal stabilizer is not in contact with the borehole wall, contact force \( \tilde{F}_2 \) is zero. For this situation, Equation (2-37) becomes

\[
\dot{\Theta} = A_1(\Theta)_1 + A_2(\Theta)_2 + A_4 \tilde{F}_r + A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3},
\]

which forms the borehole propagation model in case of no contact at the non-ideal stabilizer.

Borehole propagation model of Subsystem 2

In order to derive the equation for the inclination of the borehole for the situation where the non-ideal stabilizer is in contact with the borehole wall, Equation (2-38) is first rewritten as

\[
\tilde{F}_2 = C_1(\Theta)_1 + C_2(\Theta)_2 + C_3 \tilde{z}_2 + C_4 \tilde{F}_r + C_5 \tilde{F}_{w,1} + C_6 \tilde{F}_{w,2} + C_7 \tilde{F}_{w,3},
\]

subject to Equation (2-39). Coefficients \( C_i \), for \( i = 1, ..., 7 \), are given in Appendix A. For Subsystem 2, the stabilizer touches the upper or lower borehole wall borehole wall. This means that \( \tilde{z}_2 \) is either 1 or -1 for upper or lower wall respectively. For \( \tilde{z}_2 = \pm 1 \), Equation (2-41) becomes

\[
F_2 = C_1(\Theta)_1 + C_2(\Theta)_2 \pm C_3 + C_4 F_r + C_5 F_{w,1} + C_6 F_{w,2} + C_7 F_{w,3}.
\]

Combining this with Equation (2-37) results in the borehole propagation model for Subsystem 2:

\[
\dot{\Theta} = (A_1 + A_3 C_1)(\Theta)_1 + (A_2 + A_3 C_2)(\Theta)_2 + (A_4 + A_3 C_4) \tilde{F}_r \\
+ (A_5 + A_3 C_5) \tilde{F}_{w,1} + (A_6 + A_3 C_6) \tilde{F}_{w,2} + (A_7 + A_3 C_7) \tilde{F}_{w,3} \pm A_3 C_3,
\]

\[
= D_1(\Theta)_1 + D_2(\Theta)_2 \pm D_3 + D_4 \tilde{F}_r + D_5 \tilde{F}_{w,1} + D_6 \tilde{F}_{w,2},
\]

where the coefficients \( D_i \), for \( i = 1, ..., 6 \), are given in Appendix A.
2-5 State-space model description

The borehole propagation models give the direction in which the borehole will propagate for a certain static configuration. In order to describe the propagation dynamics, the state-space models for the hybrid system, Subsystem 1 and Subsystem 2 are derived.

2-5-1 State-space model description of hybrid system

In order to derive a state space model description, the state is chosen to be

\[ x = \begin{bmatrix} \langle \Theta \rangle_1 \\ \langle \Theta \rangle_2 \end{bmatrix} \]  

(2-45)

Taking the derivative of the state \( x \) with respect to \( \xi \) (the spatial independent variable in this problem) results in

\[ \frac{dx(\xi)}{d\xi} = x'(\xi) = \begin{bmatrix} \langle \Theta \rangle'_1(\xi) \\ \langle \Theta \rangle'_2(\xi) \end{bmatrix} = \begin{bmatrix} \frac{\Theta(\xi_2) - \Theta(\xi_3)}{\xi_2 - \xi_3} \\ \frac{\Theta(\xi_2) - \Theta(\xi_3)}{\xi_2 - \xi_3} \end{bmatrix} = \begin{bmatrix} \frac{\Theta(\xi_0) - \Theta(\xi_2)}{1 + \lambda_2} \\ \frac{\Theta(\xi_2) - \Theta(\xi_3)}{\lambda_3} \end{bmatrix} \]  

(2-46)

Using Equation (2-37), it can be stated that

\[ \Theta(\xi_i) = A_1(\Theta)_1(\xi_i) + A_2(\Theta)_2(\xi_i) + A_3 \hat{F}_2(\xi_i) + A_4 \hat{F}_r(\xi_i) + A_5 \hat{F}_{w,1}(\xi_i) + A_6 \hat{F}_{w,2}(\xi_i) + A_7 \hat{F}_{w,3}(\xi_i), \]  

(2-47)

for \( i = 0, 2, 3 \). Equation (2-47) and (2-46) are used to define the state-space system dynamics. The first output of the model, \( y_1 \), is the inclination of the borehole at the bit \( \hat{\Theta} \), given by Equation (2-37). The second output, \( y_2 \), is the scaled position of the non-ideal stabilizer, \( \tilde{z}_2 \), given by Equation (2-38), which is subject to the complementarity condition in (2-39). The initial condition of the state-space model is defined as a straight borehole under an inclination \( \Theta_0 \), with a constant initial RSS force, defined such that it is a solution of Equation (2-37) and (2-38), subject to (2-39). Thus, the state-space formulation can be expressed as
\[ x'(\xi) = \begin{bmatrix} \frac{A_1}{1+\lambda_2} & \frac{A_2}{1+\lambda_2} \\ 0 & 0 \end{bmatrix} x(\xi) + \begin{bmatrix} \frac{-A_1}{1+\lambda_2} & \frac{-A_2}{1+\lambda_2} \\ \frac{A_1}{\lambda_3} & \frac{A_2}{\lambda_3} \end{bmatrix} x(\xi_2) + \begin{bmatrix} 0 & 0 \\ \frac{-A_1}{\lambda_3} & \frac{-A_2}{\lambda_3} \end{bmatrix} x(\xi_3) + \\
\begin{bmatrix} \frac{A_4}{1+\lambda_2} \\ 0 \end{bmatrix} \tilde{F}_2(\xi) + \begin{bmatrix} 0 \\ -\frac{A_1}{\lambda_3} \end{bmatrix} \tilde{F}_2(\xi_2) + \begin{bmatrix} 0 \\ -\frac{A_1}{\lambda_3} \end{bmatrix} \tilde{F}_2(\xi_3) + \begin{bmatrix} \frac{A_4}{1+\lambda_2} \\ 0 \end{bmatrix} \tilde{F}_r(\xi) + \begin{bmatrix} \frac{-A_1}{\lambda_3} \end{bmatrix} \tilde{F}_r(\xi_2) + \begin{bmatrix} 0 \\ \frac{-A_1}{\lambda_3} \end{bmatrix} \tilde{F}_r(\xi_3) + \\
\begin{bmatrix} \frac{-A_5}{1+\lambda_2} & \frac{-A_6}{1+\lambda_2} & \frac{-A_7}{1+\lambda_2} \\ \frac{A_5}{\lambda_3} & \frac{A_6}{\lambda_3} & \frac{A_7}{\lambda_3} \end{bmatrix} \tilde{F}_{w,1}(\xi_2) + \begin{bmatrix} \frac{-A_5}{\lambda_3} & \frac{-A_6}{\lambda_3} & \frac{-A_7}{\lambda_3} \end{bmatrix} \tilde{F}_{w,2}(\xi_2) + \begin{bmatrix} \frac{A_4}{1+\lambda_2} \\ 0 \end{bmatrix} \tilde{F}_{w,1}(\xi_3) + \begin{bmatrix} \frac{A_4}{1+\lambda_2} \\ 0 \end{bmatrix} \tilde{F}_{w,2}(\xi_3) + \begin{bmatrix} \frac{-A_1}{\lambda_3} \end{bmatrix} \tilde{F}_{w,3}(\xi_3) + \begin{bmatrix} \frac{-A_1}{\lambda_3} \end{bmatrix} \tilde{F}_{w,3}(\xi_3) \right), \tag{2-48} \]

\[ y_1(\xi) = \begin{bmatrix} A_1 & A_2 \end{bmatrix} x(\xi) + A_3 \tilde{F}_2(\xi) + A_4 \tilde{F}_r(\xi) + A_5 \tilde{F}_{w,1}(\xi) + A_6 \tilde{F}_{w,2}(\xi) + A_7 \tilde{F}_{w,3}(\xi), \tag{2-49} \]

\[ y_2(\xi) = \begin{bmatrix} B_1 & B_2 \end{bmatrix} x(\xi) + B_3 \tilde{F}_2(\xi) + B_4 \tilde{F}_r(\xi) + B_5 \tilde{F}_{w,1}(\xi) + B_6 \tilde{F}_{w,2}(\xi) + B_7 \tilde{F}_{w,3}(\xi), \tag{2-50} \]

subject to

\[ \tilde{F}_2(\xi) \geq 0 \quad \text{if} \quad y_2(\xi) = -1, \]
\[ \tilde{F}_2(\xi) = 0 \quad \text{if} \quad -1 < y_2(\xi) < 1, \tag{2-51} \]
\[ \tilde{F}_2(\xi) \leq 0 \quad \text{if} \quad y_2(\xi) = 1, \]

with initial conditions

\[ x(\xi) = \Theta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for} \quad -\sum_{j=1}^{3} \lambda_j \leq \xi \leq 0, \tag{2-52} \]
\[ \tilde{F}_r(\xi) = \tilde{F}_{r_0} \quad \text{for} \quad -\sum_{j=1}^{3} \lambda_j \leq \xi \leq 0, \tag{2-53} \]

where \( \Theta_0 \in \mathbb{R}^{1 \times 1} \) and \( \tilde{F}_{r_0} \in \mathbb{R}^{1 \times 1} \) should be taken such that they comply with Equation (2-37) and (2-38), subject to (2-39).

### 2-5-2 State-space model description of ideal subsystems

With the borehole propagation models of the ideal subsystems (Equation (2-40) and (2-44)), state-space descriptions of these system can be derived.
State-space model description of Subsystem 1

Deriving a state-space description for Equation (2-40) is equivalent to taking $\dot{F}_2 = 0$ for the state-space description of Equation (2-48) and (2-49). This results in the following state-space description of Subsystem 1:

\begin{align}
    x'(\xi) &= \begin{bmatrix} A_1 & \frac{A_2}{1 + \lambda_2} \\ 0 & \frac{A_3}{1 + \lambda_3} \end{bmatrix} x(\xi) + \begin{bmatrix} A_4 \\ 0 \end{bmatrix} \tilde{F}_r(\xi) + \begin{bmatrix} A_5 & 0 & A_7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \\ \tilde{F}_{w,3}(\xi) \end{bmatrix} + \\
    &+ \begin{bmatrix} -\frac{A_1}{1 + \lambda_2} & -\frac{A_2}{1 + \lambda_2} \\ -\frac{A_4}{1 + \lambda_2} \end{bmatrix} x(\xi) + \begin{bmatrix} -\frac{A_1}{1 + \lambda_2} & \frac{A_4}{1 + \lambda_2} \\ -\frac{A_3}{1 + \lambda_3} \end{bmatrix} \dot{F}_r(\xi) + \begin{bmatrix} -\frac{A_5}{1 + \lambda_2} & -\frac{A_6}{1 + \lambda_2} & -\frac{A_7}{1 + \lambda_2} \\ -\frac{A_3}{1 + \lambda_3} & -\frac{A_6}{1 + \lambda_3} & -\frac{A_7}{1 + \lambda_3} \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \\ \tilde{F}_{w,3}(\xi) \end{bmatrix} + \\
    &+ \begin{bmatrix} 0 & 0 & 0 \\ -\frac{A_1}{1 + \lambda_2} & -\frac{A_2}{1 + \lambda_2} \end{bmatrix} x(\xi) + \begin{bmatrix} 0 \\ -\frac{A_3}{1 + \lambda_3} \end{bmatrix} \dot{F}_r(\xi) + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{A_3}{1 + \lambda_3} & -\frac{A_6}{1 + \lambda_3} & -\frac{A_7}{1 + \lambda_3} \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \\ \tilde{F}_{w,3}(\xi) \end{bmatrix},
\end{align}

\begin{align}
    y_1(\xi) &= \begin{bmatrix} A_1 & A_2 \end{bmatrix} x(\xi) + A_4 \dot{F}_r(\xi) + A_5 \tilde{F}_{w,1}(\xi) + A_6 \tilde{F}_{w,2}(\xi) + A_7 \tilde{F}_{w,3}(\xi),
\end{align}

with initial conditions of Equation (2-52) and (2-53).

State-space model description of Subsystem 2

The borehole propagation of Subsystem 2 is governed by Equation (2-44). Deriving the state-space model for this case results in:

\begin{align}
    x'(\xi) &= \begin{bmatrix} D_1 & \frac{D_2}{1 + \lambda_2} \\ 0 & \frac{D_3}{1 + \lambda_3} \end{bmatrix} x(\xi) + \begin{bmatrix} D_4 \\ 0 \end{bmatrix} \tilde{F}_r(\xi) + \begin{bmatrix} D_5 & 0 & D_7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \end{bmatrix} + \\
    &+ \begin{bmatrix} -\frac{D_1}{1 + \lambda_2} & -\frac{D_2}{1 + \lambda_2} \\ -\frac{D_4}{1 + \lambda_2} \end{bmatrix} x(\xi) + \begin{bmatrix} -\frac{D_1}{1 + \lambda_2} & \frac{D_4}{1 + \lambda_2} \\ -\frac{D_3}{1 + \lambda_3} \end{bmatrix} \dot{F}_r(\xi) + \begin{bmatrix} -\frac{D_5}{1 + \lambda_2} & -\frac{D_6}{1 + \lambda_2} & -\frac{D_7}{1 + \lambda_2} \\ -\frac{D_3}{1 + \lambda_3} & -\frac{D_6}{1 + \lambda_3} & -\frac{D_7}{1 + \lambda_3} \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \end{bmatrix} + \\
    &+ \begin{bmatrix} 0 & 0 & 0 \\ -\frac{D_1}{1 + \lambda_2} & -\frac{D_2}{1 + \lambda_2} \end{bmatrix} x(\xi) + \begin{bmatrix} 0 \\ -\frac{D_3}{1 + \lambda_3} \end{bmatrix} \dot{F}_r(\xi) + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{D_3}{1 + \lambda_3} & -\frac{D_6}{1 + \lambda_3} & -\frac{D_7}{1 + \lambda_3} \end{bmatrix} \begin{bmatrix} \tilde{F}_{w,1}(\xi) \\ \tilde{F}_{w,2}(\xi) \end{bmatrix},
\end{align}

\begin{align}
    y_1(\xi) &= \begin{bmatrix} D_1 & D_2 \end{bmatrix} x(\xi) \pm D_3 + D_4 \dot{F}_r(\xi) + D_5 \tilde{F}_{w,1}(\xi) + D_6 \tilde{F}_{w,2}(\xi),
\end{align}

with initial conditions of Equation (2-52) and (2-53).

2-6 Discussion

In this chapter, a hybrid model for a directional drilling system with clearance between a stabilizer and the borehole wall has been derived. The system is modeled as a two-dimensional lumped-parameter model.
The hybrid nature of the model originates from the fact that one of the stabilizers is considered to show non-ideal contact behavior. Instead of being in contact with the borehole wall at all times, it is allowed the freedom to switch between contact and no contact.

For the hybrid model, a borehole propagation model and corresponding state space description are derived. Both model descriptions are subject to a contact law, defined for the non-ideal stabilizer. The state-space model is in the form of a delay complementarity system.

The hybrid model could be viewed as consisting of two linear models: Subsystem 1, for which the non-ideal stabilizer is never in contact with the borehole wall, and Subsystem 2, for which the non-ideal stabilizer is always in contact with the borehole wall. For both these ideal subsystems, a borehole propagation model and state-space model description are derived.

In the next chapter, these lumped-parameter models will be used for the analysis of the effect of the hybrid nature, induced by the clearance between a stabilizer and the borehole wall, on the behavior of directional drilling systems.
In this chapter, the two-dimensional hybrid directional drilling model, constructed in Chapter 2, is analyzed.

First of all, a realistic value or value range is chosen for the parameters in Section 3-1. Values for the parameters of distributed models are found in [33, 51]. These values are transformed to lumped-parameter model parameters using a method for comparative analysis, discussed in Appendix B. The comparative analysis ensures that the lumped-parameter models are representative of physical directional drilling systems.

Second, the quasi-stationary solutions are derived in Section 3-2. The quasi-stationary solutions are defined as the solutions for which the curvature is uniform at the scale of the BHA. First the quasi-stationary solutions of the hybrid model are derived. The quasi-stationary solutions of the hybrid system are then compared to the quasi-stationary solutions of the two ideal subsystems. In order to determine whether the system will converge to these quasi-stationary trajectories, the stability of the perturbation dynamics of the ideal subsystems is examined in Section 3-3. This is done by computing the right-most poles for both the contact and non-contact case. In this way, it will be found out for which situations the ideal subsystems will converge to the quasi-stationary solutions. In order to make a statement about whether the hybrid system will converge to the quasi-stationary solutions, numerical simulations of the hybrid system are performed in Section 3-4. These simulations provide insight in the stability and behavior of the hybrid system. The quasi-stationary solutions and stability analysis of the ideal subsystems are taken into account when analyzing the simulation results. Finally, statements are made about the effect of the hybrid nature of the model on the stability and behavior of the system in Section 3-5.
3-1  Case study description

In order to conduct simulations, numerical values are needed for the parameters of the model. A set of parameters is listed in Table 3-1, based on information given in [33, 51].

<table>
<thead>
<tr>
<th>ℓ₁ [m]</th>
<th>ℓ₂ [m]</th>
<th>ℓ₃ [m]</th>
<th>Δℓ₁ [m]</th>
<th>Lᵣ [m]</th>
<th>Oᵣ [m]</th>
<th>E [N/m²]</th>
<th>ρ [kg/m³]</th>
<th>η [−]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.83</td>
<td>1.83</td>
<td>3.66</td>
<td>0.6096</td>
<td>0.0533</td>
<td>2 · 10¹¹</td>
<td>7800</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Parameters ℓ₁, ℓ₂, ℓ₃ and Δℓ₁ are defined as in Figure 2-2. E and ρ are respectively the Young’s modulus and density of steel. Lᵣ and Oᵣ give the values for the inner radius and outer radius of the cross section of the collars of the BHA. The moment of inertia of the cross section of the collars of the BHA is given by

\[ I = \frac{\pi}{4}(Oᵣ^4 - Lᵣ^4) = 3.6 \cdot 10^{-5} \text{ m}^4. \] (3-1)

Furthermore, the weight per unit length is

\[ w = \pi(Oᵣ^2 - Lᵣ^2) \cdot \rho \cdot 9.8 = 1.08 \cdot 10^3 \frac{N}{m}. \] (3-2)

The active weight on bit of a BHA with push-the-bit RSS is in the range of 20 to 45 kN [41]. Furthermore, a realistic value for \( R_Δ \) is from \( 10^{-3} \) m to \( 10^{-2} \) m [16]. In this analysis, a value of \( R_Δ = ℓ₁ \cdot 10^{-3} = 1.83 \cdot 10^{-3} \text{ m} \) is taken.

Parameters \( E \) and \( I \) are parameters of a distributed model. A method is presented in Appendix B to establish an approximate equivalence between the lumped-parameter model and the corresponding distributed model. With this method, values for spring constants \( k₁ \) and \( k₂ \) of the lumped-parameter model are obtained by solving an optimization problem, namely the minimization of cost function \( J \). This cost function is given by adding up the squares of the errors of certain variables, such as the inclination of the bit and the position of the non-ideal stabilizer, between the lumped-parameter model and an equivalent distributed model (see Appendix B). By minimizing \( J \), the errors between these variables, which determine the propagation dynamics, is reduced as much as possible. For the parameter values given in previously in this section, \( J \) is given by

\[
J(k₁, k₂) = \frac{1}{k₁² k₂² (k₁ + 0.25k₂)²} \left( k₁(Q₁ - Q₂k₂)k₂² + Q₃k₂⁴ + k₁²k₂²(Q₄ + (Q₅ + Q₆k₂)k₂) + k₁²k₂(Q₇ + (Q₈ + Q₇k₂)k₂) + k₁²(Q₁₀ + (Q₁₁ + Q₁₂k₂)k₂) \right),
\] (3-3)

where \( Qᵢ \), for \( i = 1, 2, ..., 12 \), are given in Appendix A. A contour plot of function of \( J(k₁, k₂) \) is shown in Figure 3-1.
Equation (B-48) can be solved using the \textit{fminsearch} algorithm in Matlab [43], which is based on the simplex search method of Lagarias et al. [35]. With this method, the minimal cost was found to be $J = 1.5738 \cdot 10^{-8}$ for values of spring constants $k_1 = 2.9402 \cdot 10^6 \text{ Nm}$ and $k_2 = 3.4415 \cdot 10^6 \text{ Nm}$. These values yield an error vector of

$$
e = \begin{bmatrix} 1.6553 \cdot 10^{-5} \\ 7.2782 \cdot 10^{-8} \\ 6.1086 \cdot 10^{-5} \\ 1.0832 \cdot 10^{-4} \end{bmatrix},$$

where the first two entries of the error vector give the error between the lumped-parameter and distributed model in inclination of the bit and position of the non-ideal stabilizer respectively, in case the non-ideal stabilizer is not in contact with the borehole wall. The third and fourth entry of the error vector are the errors in inclination of the bit between the lumped-parameter and distributed model, in case the non-ideal stabilizer is in contact with the upper and lower borehole wall respectively.

\textbf{Remark 1.} The values of the individual entries of $e$ are observed to be negligible for the previously mentioned spring constants. In order to calculate these values, specific values were assigned to $R_\Delta$ and the applied distributed load $\bar{q}$. However, the individual errors should be small for all realistic values of $R_\Delta$ and $\bar{q}$. The effect of these two parameters on the error vector can be discerned from Equation (B-46) in Appendix B. Taking values for $R_\Delta$ in the range $[0, 0.1]$ m and the distributed load in $[0, 10^5]$ Nm, results in individual errors of 0.01 or less. Therefore, the error values will remain small for all realistic values of $R_\Delta$ and loading of the BHA.

Finally, as discussed in Section 2-3, the parameters have to be scaled. The values for the weight on bit given before, result in a scaled active weight on bit $\Pi$ of approximately $1.25 \cdot 10^{-2}$ to $2.81 \cdot 10^{-2}$, given that $F^* = 1.61 \cdot 10^6$ N. Consequently, the interval of $\eta\Pi$ should be taken as
\( \eta \Pi \in [0.38, 0.84] \). In order to keep the analysis as general as possible, \( \eta \Pi \in [0.1, 1] \) is taken.

The scaled weight per unit length is \( \tilde{w} = w \cdot \ell_1 / F^* = 1.23 \cdot 10^{-3} \). The force imposed by RSS actuators can go up to 16 kN \([16]\). This corresponds to a maximum value of \( \tilde{F}_r = 10^{-2} \).

Thus taking the values as shown in Table 3-2 for the lumped-parameter model, will result in a system with behavior comparable to that of the distributed system with parameters as in Table 3-1.

### Table 3-2: Scaled lumped-parameter model parameter values.

| \( \lambda_2 \) | \( \lambda_3 \) | \( \Lambda \) | \( 
\tilde{F}^* \) | \( 
\tilde{M}^* \) | \( \kappa_2 \) | \( \eta \Pi \) |
<table>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.33</td>
<td>( 1.61 \cdot 10^6 ) N</td>
<td>( 2.94 \cdot 10^6 ) Nm</td>
<td>1.17</td>
<td>([0.1, 1])</td>
</tr>
</tbody>
</table>

#### 3-2 Quasi-stationary solutions

The quasi-stationary solutions are defined as the solutions for which the curvature is uniform at the scale of the BHA. For these solutions, the bit tilt and all forces acting on the BHA are constant, resulting in a circular borehole trajectory with constant radius. In order to derive these solutions, it is assumed that the average inclinations of the borehole, \( \langle \Theta \rangle_1 \) and \( \langle \Theta \rangle_2 \), change slowly on an intermediate length scale (see Remark 2). Therefore, the gravity forces (Equation (2-28) to (2-30)) are assumed to be quasi-constant for the analysis of the quasi-stationary solutions. This approximation is motivated by the fact that we wish to evaluate the steady-state behavior and transient dynamics of the directional drilling system on this intermediate length scale of order \( \mathcal{O}(10^1) \).

**Remark 2.** Three length scales can be identified in the response of the borehole to a sudden change in weight \( \Pi \) or RSS force \( \tilde{F}_r \): a short length scale of order \( \mathcal{O}(10^0) \), an intermediate length scale of order \( \mathcal{O}(10^1) \), and a large length scale of order \( \mathcal{O}(10^2 \sim 10^3) \) \([60]\). The short scale evolution is associated with fast dynamics, caused by a sudden change in direction at the start. When the effect of the initial discontinuity diminishes, the borehole inclination is expected to vary slowly. This corresponds to the intermediate scale evolution. Because of the slowly changing borehole inclination, the gravity terms can be assumed to be quasi-constant. The intermediate length scale is at least an order of magnitude smaller than the radius of curvature of the borehole. If the system is directionally stable, the transient behavior evolves in such a way that the system reaches quasi-stationarity and the borehole converges toward solutions corresponding to boreholes with quasi-constant curvatures. On the large length scale, the effect of the gravity forces becomes more significant. The result is that the system will converge to a stationary solution (constant inclination), if a balance between the RSS force and the influence of the weight is achieved. For a large RSS force, the borehole inclination might keep increasing on the large length scale. In Section 3-4-4, simulations are performed to investigate whether the orders and assumptions of these length scales are valid.

**Remark 3.** Note that these quasi-stationary solutions correspond to equilibrium solutions from the reference frame of a (weightless) BHA. However, the state-space models presented, are formulated from the perspective of an earth-bound reference frame. By deriving the perturbation dynamics, a state-space system is derived for which the equilibria correspond to these quasi-stationary trajectories. This analysis is performed in Section 3-3.
For a weightless BHA, the quasi-stationary solutions correspond to the large \( \xi \) limit. When gravity is taken into account, the curvature of the quasi-stationary solution is slowly evolving. Eventually, the system will evolve toward a constant inclination, if a balance between RSS force and gravity forces is achievable. The solutions the system will converge to on the large length scale, is referred to as the long term solutions and are discussed in Section 3-2-4.

In order to define quasi-stationary borehole trajectories, curvature \( \tilde{K} \) is introduced. The curvature is defined as

\[
\tilde{K} = \frac{d\hat{\Theta}}{d\xi}.
\] (3-5)

For quasi-stationary borehole trajectories, \( \tilde{K} = \tilde{K}_c \) is constant. It is related to the radius of the quasi-stationary borehole trajectory, \( \tilde{R}_c \), by \( \tilde{K}_c = 1/\tilde{R}_c \).

For a constant curvature, the following equations follow from geometrical considerations:

\[
\langle \Theta \rangle_1 - \langle \Theta \rangle_2 = \tilde{K}_c \left( \frac{1 + \lambda_2 + \lambda_3}{2} \right),
\] (3-6)

\[
\langle \Theta \rangle_1 - \Theta(\xi_2) = \Theta(\xi_0) - \langle \Theta \rangle_1,
\] (3-7)

\[
\langle \Theta \rangle_2 - \Theta(\xi_3) = \Theta(\xi_2) - \langle \Theta \rangle_2.
\] (3-8)

For boreholes with constant curvature, it can also be stated that

\[
\langle \Theta \rangle'_1 = \langle \Theta \rangle'_2 = \tilde{K}_c.
\] (3-9)

Using Equation (2-46), (3-7) and (3-8), the derivatives of the states become

\[
\langle \Theta \rangle'_1 = \frac{\Theta(\xi_0) - \Theta(\xi_2)}{1 + \lambda_2} = \frac{2\hat{\Theta} - 2\langle \Theta \rangle_1}{1 + \lambda_2},
\] (3-10)

\[
\langle \Theta \rangle'_2 = \frac{\Theta(\xi_2) - \Theta(\xi_3)}{\lambda_3} = \frac{4\langle \Theta \rangle_1 - 2\langle \Theta \rangle_2 - 2\hat{\Theta}}{\lambda_3}.
\] (3-11)

With the definition in Equation (3-9), the right-hand sides of Equation (3-10) and (3-11) can be equated, resulting in

\[
\hat{\Theta} = \left( \frac{2(1 + \lambda_2) + \lambda_3}{1 + \lambda_2 + \lambda_3} \right) \langle \Theta \rangle_1 - \left( \frac{1 + \lambda_2}{1 + \lambda_2 + \lambda_3} \right) \langle \Theta \rangle_2.
\] (3-12)

By substitution of Equation (2-37) into (3-12), the following is obtained

\[
\langle \Theta \rangle_1 - \langle \Theta \rangle_2 = A_3 \tilde{F}_2 + A_4 \tilde{F}_r + A_5 \tilde{F}_w,1 + A_6 \tilde{F}_w,2 + A_7 \tilde{F}_w,3.
\] (3-13)

With Equation (3-6), the constant curvature is given by

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\[
\tilde{K}_c = \left(\frac{2}{1 + \lambda_2 + \lambda_3}\right) \left(A_3 \tilde{F}_2 + A_4 \tilde{F}_r + A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3}\right).
\] (3-14)

Thus the curvature of the quasi-stationary solutions of the hybrid system is determined by Equation (3-14). Contact force \( \tilde{F}_2 \) for this case can be determined by combining Equation (2-38) with Equation (3-13), resulting in

\[
\tilde{z}_2 = B_1 (\langle \Theta \rangle_1 - \langle \Theta \rangle_2) + B_3 \tilde{F}_2 + B_4 \tilde{F}_r + B_5 \tilde{F}_{w,1} + B_6 \tilde{F}_{w,2} + B_7 \tilde{F}_{w,3},
\] (3-15)

\[
= (B_1 A_3 + B_3) \tilde{F}_2 + (B_1 A_4 + B_4) \tilde{F}_r + (B_1 A_5 + B_5) \tilde{F}_{w,1} + (B_1 A_6 + B_6) \tilde{F}_{w,2} + (B_1 A_7 + B_7) \tilde{F}_{w,3},
\] (3-16)

\[
= E_3 \tilde{F}_2 + E_4 \tilde{F}_r + E_5 \tilde{F}_{w,1} + E_6 \tilde{F}_{w,2} + E_7 \tilde{F}_{w,3},
\] (3-17)

subject to complementarity condition (2-39). Coefficients \( E_i \), for \( i = 3, ..., 7 \), can be found in Appendix A. Note that Equation (3-17), subject to (2-39), has one unique solution if \( E_3 > 0 \).

### 3-2-1 Quasi-stationary solutions for Subsystem 1

In case the non-ideal stabilizer is not in contact with the borehole wall, contact force \( \tilde{F}_0 = 0 \). Taking \( \tilde{F}_2 = 0 \) for Equation (3-14) thus results in the quasi-stationary solutions for Subsystem 1:

\[
\tilde{K}_c = \left(\frac{2}{1 + \lambda_2 + \lambda_3}\right) \left(A_4 \tilde{F}_r + A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3}\right).
\] (3-18)

### 3-2-2 Quasi-stationary solutions for Subsystem 2

When the non-ideal stabilizer is in contact with the borehole wall, the position of the non-ideal stabilizer is either 1 or \(-1\). Substituting \( \tilde{z}_2 = \pm 1 \) in Equation (3-17) and writing the equation in terms of \( \tilde{F}_2 \) results in:

\[
\tilde{F}_2 = \left(\frac{\pm 1 - E_4 \tilde{F}_r - E_5 \tilde{F}_{w,1} - E_6 \tilde{F}_{w,2} - E_7 \tilde{F}_{w,3}}{E_3}\right).
\] (3-19)

By substituting this equation for the contact force into Equation (3-14), the equation for the quasi-stationary solution of Subsystem 2 is derived:

\[
\tilde{K}_c = \left(\frac{2}{1 + \lambda_2 + \lambda_3}\right) \left[\pm \left(\frac{A_3}{E_3}\right) + \left(A_4 - A_3 \frac{E_4}{E_3}\right) \tilde{F}_r + \left(A_5 - A_3 \frac{E_5}{E_3}\right) \tilde{F}_{w,1} + \left(A_6 - A_3 \frac{E_6}{E_3}\right) \tilde{F}_{w,2} + \left(A_7 - A_3 \frac{E_7}{E_3}\right) \tilde{F}_{w,3}\right].
\] (3-20)
3-2-3 Analysis of quasi-stationary solutions

The quasi-stationary solutions will be analyzed by first looking at the case of a weightless BHA. Subsequently, the effect of the quasi-constant gravity forces will be investigated. Finally, the long term solutions of the hybrid system are derived from the quasi-stationary solutions and analyzed.

Quasi-stationary solutions for a weightless BHA

In Figure 3-2, the curvature of the quasi-stationary solution of the hybrid system (Equation (3-14) and (3-17) subject to Equation (2-39)) is plotted with respect to RSS force $\tilde{F}_r$, for different $\eta \Pi$. In the same plots the quasi-stationary solutions of Subsystem 1 are shown (Equation (3-18)). Also the quasi-stationary solutions of Subsystem 2 (Equation (3-20)) are shown, for both $\tilde{z}_2 = -1$ and $\tilde{z}_2 = 1$. For these plots, the parameter values of Table 3-2 were used. Furthermore, the components of the gravity forces are taken as zero ($\tilde{F}_{w,i} = 0$ for $i = 1, 2, 3$) and $g$ is set to be $10^{-3}$.

First of all, it can be noted that when $\eta \Pi$ is increased, a larger RSS force $\tilde{F}_r$ is needed to generate the same curvature $\tilde{K}_c$. This means that a larger weight on bit increases the tendency of the borehole to go in a straight line. Therefore, a larger RSS force is needed to steer the borehole in a certain direction.

Secondly, it is observed that the switching between contact and no contact at the non-ideal stabilizer is clearly visible in Figure 3-2. The behavior of the hybrid system for high positive values and high negative values of $\tilde{F}_r$ is seen to match exactly that of Subsystem 2 for $\tilde{z}_2 = 1$ and $\tilde{z}_2 = -1$, respectively. When the absolute value of the RSS force is lower than a certain critical RSS force $\tilde{F}_{r\text{crit}}$ ($|\tilde{F}_r| \leq \tilde{F}_{r\text{crit}}$), the behavior of the hybrid system is same as for Subsystem 1. This critical RSS force is defined to always be larger that 0, i.e. $\tilde{F}_{r\text{crit}} > 0$.

This means that the effect of the RSS force on the curvature of the quasi-stationary solution of the hybrid system can be divided into three parts. At the first part, corresponding to $\tilde{F}_r \leq -\tilde{F}_{r\text{crit}}$, the non-ideal stabilizer is pushed onto the lower borehole wall ($\tilde{z}_2 = -1$) by the RSS actuator. When $\tilde{F}_r = -\tilde{F}_{r\text{crit}}$, the switching takes place. At the second part, corresponding to $-\tilde{F}_{r\text{crit}} < \tilde{F}_r < \tilde{F}_{r\text{crit}}$, the absolute value of the RSS force is too low to push the non-ideal stabilizer against either one of the borehole walls. When $\tilde{F}_r = \tilde{F}_{r\text{crit}}$, the switching from no contact to contact with the upper borehole wall ($\tilde{z}_2 = 1$) occurs. At the third part, corresponding to $\tilde{F}_{r\text{crit}} \leq \tilde{F}_r$, the non-ideal stabilizer is pushed onto the upper borehole wall by the RSS actuator.

When $\eta \Pi$ is increased, critical force $\tilde{F}_{r\text{crit}}$ is observed to increase. The effect of $\eta \Pi$ on $\tilde{F}_{r\text{crit}}$ can be calculated with the previously derived quasi-stationary solutions. At the point where the system switches from non-touching to touching mode ($|\tilde{F}_r| = \tilde{F}_{r\text{crit}}$), the quasi-stationary curvature of Subsystem 1 and 2 is the same. Thus equating the right-hand sides of Equation (3-18) and (3-20) yields

$$\tilde{F}_{r\text{crit}} = \pm \left( \frac{1}{E_4} \right) - \left( \frac{E_6}{E_4} \right) \tilde{F}_{w,1} - \left( \frac{E_6}{E_4} \right) \tilde{F}_{w,2} - \left( \frac{E_7}{E_4} \right) \tilde{F}_{w,3}. \quad (3-21)$$

Taking the values of Table 3-2 and $\tilde{F}_{w,i} = 0$ for $i = 1, 2, 3$, the following expression is derived:
(a) The quasi-stationary solutions for \( \eta \Pi = 0.10 \). The non-ideal stabilizer achieves contact when \( |\tilde{F}_r| \geq \tilde{F}_{crit} = 1.09 \times 10^{-4} \).

(b) The quasi-stationary solutions for \( \eta \Pi = 0.30 \). The non-ideal stabilizer achieves contact when \( |\tilde{F}_r| \geq \tilde{F}_{crit} = 3.26 \times 10^{-4} \).

(c) The quasi-stationary solutions for \( \eta \Pi = 0.62 \). The non-ideal stabilizer achieves contact when \( |\tilde{F}_r| \geq \tilde{F}_{crit} = 6.72 \times 10^{-4} \).

(d) The quasi-stationary solutions for \( \eta \Pi = 1.00 \). The non-ideal stabilizer achieves contact when \( |\tilde{F}_r| \geq \tilde{F}_{crit} = 1.08 \times 10^{-3} \).

**Figure 3-2:** The effect of changing RSS force \( \tilde{F}_r \) on the curvature \( \tilde{K}_c \) of the quasi-stationary solutions, for different \( \eta \Pi \). As \( \eta \Pi \) is increased, a larger RSS force \( \tilde{F}_r \) is required to reach a certain curvature \( \tilde{K}_c \). At the same time, when \( \eta \Pi \) is increased, it takes more RSS force \( \tilde{F}_r \) to make the non-ideal stabilizer touch the borehole wall.
\[ \tilde{F}_{r \text{crit}} = \frac{0.114612 \eta \Pi}{105.061 + \eta \Pi}, \]  

which, for the specified domain of \( \eta \Pi \) in Table 3-2, can be approximated by

\[ \tilde{F}_{r \text{crit}} \approx 1.08062 \cdot 10^{-3} \eta \Pi \quad \text{for} \quad \eta \Pi \in [0.1, 1]. \]  

This means that the RSS force necessary to make the non-ideal stabilizer touch the borehole wall increases approximately linearly with \( \eta \Pi \).

At low values of \( \eta \Pi \), the section corresponding to Subsystem 1 shows a steeper incline than the sections corresponding to Subsystem 2 (Figure 3-2a). At high values of \( \eta \Pi \), this behavior is opposite (Figure 3-2d). For \( \eta \Pi = 0.62 \), the three segments fall exactly on the same line.

This behavior can be described in a more detailed way by looking at the equations of the corresponding lines. Equation (3-18) can be expressed as

\[ \tilde{K}_c = C_1 \tilde{F}_r + 2 \left( \frac{A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3}}{1 + \lambda_2 + \lambda_3} \right), \]  

where

\[ C_1 = \left( \frac{2A_4}{1 + \lambda_2 + \lambda_3} \right). \]  

Coefficient \( C_1 \) represents the effect that a change in RSS force has on the curvature of the quasi-stationary solution of Subsystem 1. In the same way, Equation (3-20) can be expressed as

\[ \tilde{K}_c = C_2 \tilde{F}_r + 2 \left( \pm \left( \frac{A_4}{E_3} \right) + \left( A_5 - A_3 \frac{E_3}{E_4} \right) \tilde{F}_{w,1} + \left( A_6 - A_3 \frac{E_3}{E_5} \right) \tilde{F}_{w,2} + \left( A_7 - A_3 \frac{E_3}{E_7} \right) \tilde{F}_{w,3} \right), \]  

where

\[ C_2 = 2 \left( \frac{A_4 - A_3 \frac{E_3}{E_4}}{1 + \lambda_2 + \lambda_3} \right). \]  

Thus coefficient \( C_2 \) represents the effect that a change in RSS force has on the curvature of the quasi-stationary solution of Subsystem 2. Using the parameter values of Table 3-2, these coefficients can be expressed as
Figure 3-3: The directional coefficients $C_i$, for $i = 1, 2$, with respect to $\eta \Pi$. $C_i$ is a measure of how large the effect of a change in RSS force is on the change in curvature.

$$C_1 = -0.129363 + \frac{0.458333}{\eta \Pi},$$

(3-28)

$$C_2 = -0.0504164 + \frac{0.291667}{\eta \Pi} + \frac{1.18036}{5.63855 + \eta \Pi}.$$  

(3-29)

Equation (3-28) and (3-29) are shown in Figure 3-3. It can be observed that for low $\eta \Pi$, coefficient $C_1$ is significantly larger than $C_2$. The two coefficients are equal at $\eta \Pi = 0.62$. For $\eta \Pi > 0.62$, $C_1$ is smaller than $C_2$ by a small amount. The reason for this can be found in the fact that $C_1$ and $C_2$ represent the directional tendencies of a one-stabilizer and two-stabilizer system, respectively. The presence of an extra stabilizer constrains the movement of Subsystem 2, resulting in a limited directional tendency. The BHA of Subsystem 1, on the other hand, is allowed more freedom to move in the lateral direction. Therefore, both $\tilde{F}_r$ and $\eta \Pi$ have a large influence on the directional tendency of this system.

Remark 4. It should be noted that when $\eta \Pi$ is increased to 4 or higher (for this particular system), the system starts showing a non-minimum phase response, corresponding to negative values of $C_1$ and/or $C_2$. This critical value of $\eta \Pi$ was referred to as $\eta \Pi|_{\phi}$ in [60]. However, this behavior is outside of the interval of $\eta \Pi$ defined in Table 3-2. Therefore, it will not be taken into account in analysis.

Quasi-stationary solutions for quasi-constant gravity forces

In Figure 3-4, the effect of the quasi-constant gravity forces on the curvature of the quasi-stationary solutions is shown, for different $\eta \Pi$. For each case, two functions are shown. For the dashed line, $F_{w,i} = 0$, for $i = 1, 2, 3$, is used, corresponding to a vertical borehole. For...
3-2 Quasi-stationary solutions

Figure 3-4: The effect of the gravitational forces on the quasi-stationary solutions. The dashed line shows the quasi-stationary solutions for quasi-constant gravity forces of $\hat{F}_{w,i} = 0$ for $i = 1, 2, 3$, so when the BHA is positioned vertically. The solid line gives the quasi-stationary solutions when the quasi-constant gravity forces are $\hat{F}_{w,i} = \hat{w}_\lambda$ for $i = 1, 2, 3$, so when the BHA is positioned horizontally.

the solid line, gravity forces are taken to be $\hat{F}_{w,i} = \hat{w}_\lambda$ for $i = 1, 2, 3$, where $\hat{w} = 1.23 \cdot 10^{-3}$, corresponding to a horizontal borehole.

As can be seen in Figure 3-4, a larger RSS force $\hat{F}_r$ is needed to reach a given positive curvature for a larger weight. The reason for this is that the RSS force and gravity forces act in opposite direction. Therefore, the net amount of force acting on the BHA is lower, for a larger weight. For the same reason, negative RSS forces lead to a larger (negative) curvature when weight is increased.

Another effect that is observed in Figure 3-4, is that increasing the weight results in a shift of the RSS force for which the system switches from contact to no contact. This means that a larger RSS force is needed to lift the non-ideal stabilizer from the lower borehole wall. Again, this can be explained by the fact that the gravity forces act in the opposite direction of the RSS force. For this reason, a larger RSS force is required to impose a certain net force on the non-ideal stabilizer, when the gravity forces increase.

For the values of $\hat{F}_r$ around which the switching behavior takes place, the effect of adding weight is significantly different for different values of $\eta \Pi$. When $\eta \Pi$ is lower than 0.62, the resulting decrease in $\hat{K}_c$ is significant. For $\eta \Pi$ higher than 0.62, the decrease in $\hat{K}_c$ is very small. This means that for low $\eta \Pi$, the gravity forces have a significant effect on the shape of the (quasi-stationary) borehole. For high $\eta \Pi$, this effect is much less prominent.

Remark 5. Note that the analysis of the effect of the (quasi-constant) gravity forces on the quasi-stationary solutions is limited to the case where $\langle \Theta \rangle_1$ and $\langle \Theta \rangle_2$ are in the domain $[0, \pi]$. The conclusions can trivially be extended to the case where these inclinations are in the domain $[\pi, 2\pi]$. 

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3-2-4 Long term solutions

As discussed before, the system will, on a large length scale, converge to a straight borehole when a balance between RSS force and gravity forces can be achieved. When this balance is not achieved, the system will not reach steady-state on the long-term.

For the case in which balance is achieved between the RSS force and the gravity forces, the system system is in equilibrium. For any RSS force, chosen such that this balance exists, there are two equilibria: one corresponding to an upward borehole and one to a downward borehole.

The equilibrium corresponding to the upward borehole (($\Theta_i \in [\pi/2, 3\pi/2]$, for $i = 1, 2$) is unstable: if the inclination decreases by a small amount, the gravity forces exceed the RSS force and the borehole diverges from this equilibrium. For a small increase in inclination, the unstable: if the inclination decreases by a small amount, the gravity forces exceed the RSS force and the borehole diverges from this equilibrium. For a small increase in inclination, the gravity forces decrease, resulting in a divergence of the equilibrium as well. On the other hand, the gravity forces have a stabilizing effect on the equilibrium corresponding to the downward borehole ($\Theta_i \in [-\pi/2, \pi/2]$, for $i = 1, 2$). However, whether these long-term solutions are stable, depends also on the internal dynamics of the system. The stability of these solutions will be further analyzed in Section 3-3 and 3-4.

The inclination of the long-term steady-state borehole, $\Theta_\infty$, can easily be extracted from the quasi-stationary solutions. By taking $K_r = 0$ and $F_{w,i} = \tilde{w}\lambda_i \sin \Theta_\infty$, for $i = 1, 2, 3$, for Equation (3-14), the inclination of the long term solution borehole can be calculated with

$$\Theta_\infty = \sin^{-1}\left(-\frac{A_3\tilde{F}_2 + A_4\tilde{F}_r}{\tilde{w}(A_5 + A_6\lambda_2 + A_7\lambda_3)}\right), \quad (3-30)$$

where contact force $\tilde{F}_2$ can be determined with Equation (3-17), which reduces to

$$\tilde{z}_2 = \left[B_3 - \left(\frac{B_5 + B_6\lambda_2 + B_7\lambda_3}{A_5 + A_6\lambda_2 + A_7\lambda_3}\right)A_3\right] \tilde{F}_2 + \left[B_4 - \left(\frac{B_5 + B_6\lambda_2 + B_7\lambda_3}{A_5 + A_6\lambda_2 + A_7\lambda_3}\right)A_4\right] \tilde{F}_r, \quad (3-31)$$

subject to the unilateral contact law of Equation (2-39).

It should be noted that the solution of an arcsine, as in Equation (3-30), is only defined for an argument in the domain $[-1, 1]$. This corresponds to an output $\Theta_\infty$ that is defined on the (principal value) interval $\Theta_\infty \in [-\pi/2, \pi/2]$. This interval conveniently excludes the upward equilibrium boreholes ($\Theta_i \in [\pi/2, 3\pi/2]$, for $i = 1, 2$), which were already shown to be unstable.

In Figure 3-5, the long-term inclination $\Theta_\infty$ of Equation (3-30) is shown for different $\eta II$. Inclination $\Theta_\infty$ is seen to increase with $\tilde{F}_r$, until the value $\pm \pi/2$. The RSS force for which $\Theta_\infty = \pm \pi/2$, is denoted by $\tilde{F}_{r\text{limit}}$. If $\tilde{F}_r > \tilde{F}_{r\text{limit}}$, the system will not reach steady-state on the long-term, since no balance can be achieved between the RSS force and the gravity forces.

An expression for $\tilde{F}_{r\text{limit}}$, the RSS force for a horizontal borehole ($\Theta_\infty = \pi/2$), can be given by taking $\Theta_\infty = \pi/2$ for Equation (3-30). $\tilde{F}_{r\text{limit}}$ is then determined with

$$\tilde{F}_{r\text{limit}} = -\frac{1}{A_4} \left(A_3\tilde{F}_2 + \tilde{w}(A_5 + A_6\lambda_2 + A_7\lambda_3)\right). \quad (3-32)$$

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Figure 3-5: The long-term inclination $\Theta_\infty$ resulting from a constant RSS force $\tilde{F}_r$, for different $\eta\Pi$. The solution is only defined on the interval $\Theta_\infty \in [-\pi/2, \pi/2]$. The borders of this interval are shown in the plot with two dashed lines.

Contact force $\tilde{F}_2$ can be determined by substitution of Equation (3-32) in (3-31), resulting in

$$\ddot{z}_2 = \left( B_3 - B_4 \frac{A_3}{A_4} \right) \tilde{F}_2 + \ddot{w} \left[ \left( B_5 - B_4 \frac{A_5}{A_4} \right) + \left( B_6 - B_4 \frac{A_6}{A_4} \right) \lambda_2 + \left( B_7 - B_4 \frac{A_7}{A_4} \right) \lambda_3 \right].$$ (3-33)

subject to the unilateral contact law of Equation (2-39).

With the values of Table 3-2, the value of $\tilde{F}_{r\text{limit}}$ is shown in Figure 3-6, for varying $\eta\Pi$. It is observed that the values of $\tilde{F}_{r\text{limit}}$ are for all $\eta\Pi$ significantly lower than the maximum applicable RSS force ($10^{-2}$).

3-3 Stability analysis of ideal subsystems

In this section, the stability properties of Subsystem 1 and Subsystem 2 around the quasi-stationary solutions of Section 3-2 are examined. In order to do so, first the perturbation dynamics around the quasi-stationary solutions are determined. Since the long-term solutions are a subset of the quasi-stationary solutions, the stability analysis presented in this section is also applicable to the trajectory on the large length scale.

3-3-1 Perturbation dynamics of ideal subsystems

A perturbation around the quasi-stationary solution can be described as
Analysis of hybrid directional drilling model

Figure 3-6: The value of $\tilde{F}_{\text{fr limit}}$ for varying $\eta \Pi$. For $\tilde{F}_{r} \leq \tilde{F}_{\text{fr limit}}$, the system will converge to a steady-state solution on the long term.

\[ x(\xi) = x_{c}(\xi) + x_{\Delta}(\xi), \quad (3-34) \]
\[ \Theta(\xi) = \Theta_{c}(\xi) + \Theta_{\Delta}(\xi), \quad (3-35) \]

where $x_{c}(\xi)$ and $\Theta_{c}(\xi) = \tilde{K}_{c} \xi + \Theta_{c}(0)$ are the state and inclination of the borehole at $\xi$ for the quasi-stationary solution; $x_{\Delta}(\xi)$ and $\Theta_{\Delta}(\xi)$ are the perturbed state and inclination of the borehole around the quasi-stationary solution at $\xi$.

By substitution of Equation (3-6) into (3-12), it is retrieved that

\[ \langle \Theta \rangle_{2c} = \tilde{K}_{c} \left( \frac{2(1 + \lambda_{2}) + \ell_{3}}{2} \right). \quad (3-36) \]

Thus, with the help of Equation (3-6), state $x_{c}$ at $\xi$ can be described as

\[ x_{c}(\xi) = \begin{bmatrix} \Theta_{c}(\xi) - \tilde{K}_{c} \left( \frac{1 + \lambda_{2}}{2} \right) \\ \Theta_{c}(\xi) - \tilde{K}_{c} \left( \frac{2(1 + \lambda_{2}) + \lambda_{3}}{2} \right) \end{bmatrix}. \quad (3-37) \]

Recall that the borehole inclinations of Subsystem 1 and Subsystem 2 are determined from Equation (2-40) and (2-44), respectively. Substitution of Equation (3-34), (3-35) and (3-37) into those equations results in
Equation (3-38) and (3-39), gives

\[ \Theta^1_\Delta(\xi) = \left[ A_1 A_2 \right] \Delta^1_{\Delta}(\xi) - \frac{\tilde{K}^1_\Delta}{2} \left[ A_1 A_2 \right] \left[ 1 + \lambda_2 \right] \frac{1}{2(1 + \lambda_2) + \lambda_3} + A_4 \tilde{F}_r \]
\[ + (A_1 + A_2 - 1) \Theta_c(\xi) + A_5 \tilde{F}_{w,1} + A_6 \tilde{F}_{w,2} + A_7 \tilde{F}_{w,3}, \]  \tag{3-38}

\[ \Theta^II_\Delta(\xi) = \left[ D_1 D_2 \right] \Delta^II_{\Delta}(\xi) - \frac{\tilde{K}^{II}_\Delta}{2} \left[ D_1 D_2 \right] \left[ 1 + \lambda_2 \right] \frac{1}{2(1 + \lambda_2) + \lambda_3} + D_3 + D_4 \tilde{F}_r \]
\[ + (D_1 + D_2 - 1) \Theta_c(\xi) + D_5 \tilde{F}_{w,1} + D_6 \tilde{F}_{w,2}, \]  \tag{3-39}

where the I and II superscripts denote variables and parameters of Subsystem 1 and Subsystem 2 respectively. Thus, Equation (3-38) describes the perturbation dynamics of Subsystem 1 and Equation (3-39) describes the perturbation dynamics of Subsystem 2.

Substitution of the curvature equations of both subsystems (Equation (3-18) and (3-20)) into Equation (3-38) and (3-39), gives

\[ \Theta^1_\Delta(\xi) = \left[ A_1 A_2 \right] \Delta^1_{\Delta}(\xi), \]  \tag{3-40}
\[ \Theta^II_\Delta(\xi) = \left[ D_1 D_2 \right] \Delta^II_{\Delta}(\xi). \]  \tag{3-41}

These equations can be transformed into state-space formulations, resulting in

\[ \{ \Delta^1_{\Delta}(\xi) \}' = \left[ \begin{array}{cc} A_1^{-1} + \lambda_2 & A_2^{-1} + \lambda_2 \\ 0 & 0 \end{array} \right] \Delta^1_{\Delta}(\xi) + \left[ \begin{array}{cc} A_1^{-1} & \frac{-A_2}{\lambda_3} \\ \frac{-A_1}{\lambda_3} & \frac{A_2}{\lambda_3} \end{array} \right] \Delta^1_{\Delta}(\xi) + \left[ \begin{array}{cc} 0 & 0 \\ \frac{-D_1}{\lambda_3} & \frac{-D_2}{\lambda_3} \end{array} \right] \Delta^1_{\Delta}(\xi). \]  \tag{3-42}
\[ \{ \Delta^II_{\Delta}(\xi) \}' = \left[ \begin{array}{cc} D_1^{-1} + \lambda_2 & D_2^{-1} + \lambda_2 \\ 0 & 0 \end{array} \right] \Delta^II_{\Delta}(\xi) + \left[ \begin{array}{cc} D_1^{-1} & \frac{-D_2}{\lambda_3} \\ \frac{-D_1}{\lambda_3} & \frac{D_2}{\lambda_3} \end{array} \right] \Delta^II_{\Delta}(\xi) + \left[ \begin{array}{cc} 0 & 0 \\ \frac{-D_1}{\lambda_3} & \frac{-D_2}{\lambda_3} \end{array} \right] \Delta^II_{\Delta}(\xi). \]  \tag{3-43}

The state-space models of the perturbation dynamics of Subsystem 1 (Equation (3-42)) and Subsystem 2 (Equation (3-43)) are of the general form

\[ \Delta^\prime_{\Delta}(\xi) = \mathcal{A}_0 \Delta_{\Delta}(\xi_0) + \mathcal{A}_2 \Delta_{\Delta}(\xi_2) + \mathcal{A}_3 \Delta_{\Delta}(\xi_3). \]  \tag{3-44}

Delayed state-space systems of the form of Equation (3-44) are known to have infinitely many poles [45]. The characteristic equation of Equation (3-44) can be expressed as

\[ \det \left( -\mu I + \mathcal{A}_0 + \mathcal{A}_2 e^{-\nu(1+\lambda_2)\mu} + \mathcal{A}_3 e^{-\nu(1+\lambda_2+\lambda_3)\mu} \right) = 0, \]  \tag{3-45}

where \( \mu \) denotes the poles of the system. The poles of a delay differential state-space system such as Equation (3-45) can be found with a Matlab package [46] developed by Wu and Michiels, based on a spectral discretization method [47].

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### 3-3-2 Stability analysis of Subsystem 1

Using Equation (3-45), the characteristic equation of the perturbation dynamics of Subsystem 1, described by Equation (3-42), is given by

\[
\mu \left( e^{-(1 + \lambda_2 + \lambda_3)\mu} + (1 + \lambda_2 + \lambda_3)\mu - 1 \right) = 0. \tag{3-46}
\]

Taking \( \lambda_2 = 1 \) and \( \lambda_3 = 2 \) results in

\[
\frac{1}{4} \mu (e^{-4\mu} + 4\mu - 1) = 0. \tag{3-47}
\]

It is noted that Equation (3-47) has two poles in 0. Average inclinations \( \langle \Theta \rangle_1 \) and \( \langle \Theta \rangle_2 \) have been introduced as states in order to formulate the DDE’s of the state-space models (Equation (2-48), (2-54) and (2-56)). Due to defining average inclinations (being integral versions of the momentary borehole inclination), two poles were added to the state-space models at the origin. However, these roots do not affect the stability of the underlying system [33]. If all the poles of Equation (3-47), apart from the two additional poles at zero, are in the open left-half complex plane, then the quasi-stationary solutions for Subsystem 1 are globally asymptotically stable. Therefore, these two poles at zero will not be taken into account during stability analysis.

The right-most poles of the perturbation dynamics of Subsystem 1 are shown in Figure 3-7.

![Figure 3-7](image)

**Figure 3-7:** The right-most poles of the perturbation dynamics of Subsystem 1 \( (\tilde{F}_2 = 0) \) for \( \lambda_2 = 1 \) and \( \lambda_3 = 2 \). The highest real part of the poles of this system is -0.52. As a result, the perturbation state-space system is asymptotically stable. Therefore, the response of Subsystem 1 will converge to a quasi-stationary solution.

The real part of all poles of the perturbation dynamics of Subsystem 1 is equal to or smaller than -0.5222. Therefore, the perturbation state-space system of Subsystem 1 is asymptotically
stable. This means that the response of Subsystem 1 will converge to a quasi-stationary solution. This statement is always true for the chosen benchmark system, since the stability only depends on the values of $\lambda_2$ and $\lambda_3$.

**Remark 6.** Note that Subsystem 1 corresponds to a one-stabilizer system. These systems are always asymptotically stable [38].

### 3-3-3 Stability analysis of Subsystem 2

Using Equation (3-45), the characteristic equation of the state-space model of the perturbation dynamics of Subsystem 2, described by Equation (3-43), becomes

$$e^{-(2+2\lambda_2+\lambda_3)\mu} \frac{\mu}{\eta \Pi (\kappa_2 + (1 + \lambda_2)^2) \lambda_3} \left( e^{(2+2\lambda_2+\lambda_3)\mu} \lambda_3 (\kappa_2 - \eta \Pi (1 + \kappa_2 + \lambda_2) + \eta \Pi (\kappa_2 + (1 + \lambda_2)^2)) \mu \right. \\
+ \kappa_2 (-1 + (1 + \eta \Pi) \lambda_2 + (-1 + \eta \Pi) \lambda_3) + e^{(1+\lambda_2)\mu} \kappa_2 (1 + \lambda_2 - \eta \Pi \lambda_2) \\
+ e^{(1+\lambda_2+\lambda_3)\mu} (\eta \Pi (1 + \lambda_2) \lambda_3) = 0.$$  

(3-48)

After substitution of $\lambda_2 = 1$, $\lambda_3 = 2$ and $\kappa_2 = 1.17$, the location of the poles can be determined with the characteristic equation

$$\left( -0.613 - 0.113 e^{-4\mu} + 0.726 e^{-2\mu} + \frac{0.226 + 0.226 e^{-4\mu} - 0.453 e^{-2\mu}}{\eta \Pi} + \mu \right) \mu = 0.$$  

(3-49)

Also for the perturbation dynamics of Subsystem 2 it is noted that the characteristic equation contains two poles in zero (for the same reason as for Subsystem 1). For the same reasons addressed in the stability analysis of the perturbation dynamics of Subsystem 1, these poles will not be taken into account in the stability analysis of Subsystem 2.

The right-most poles of the perturbation state-space model of Subsystem 2 are shown in Figure 3-8 for different values of $\eta \Pi$. For $\eta \Pi \geq 0.18$, the real parts of the poles of the perturbation state-space system of Subsystem 2 are negative. This results in asymptotically stable perturbation dynamics. When $\eta \Pi < 0.18$, the system has poles with a positive real part, which yields unstable perturbation dynamics.

Therefore, it can be stated that Subsystem 1, for the chosen benchmark system, will always converge to the quasi-stationary solution. Subsystem 2 is shown to asymptotically converge to the quasi-stationary solution when $\eta \Pi \geq \eta \Pi_{\text{crit}} = 0.18$. When $\eta \Pi < 0.18$, Subsystem 2 will not converge to the quasi-stationary solution.

However, it should be noted that choosing $\eta \Pi \geq 0.18$ does not guarantee stability of the quasi-stationary solutions of the hybrid system, nor does taking $\eta \Pi < 0.18$ guarantee unstable quasi-stationary solutions. Stability or instability of the subsystems does not guarantee stability or instability of the hybrid system [17]. However, claims about stability of the hybrid system can still be made in the subsequent section with the use of simulations of the propagation of
The right-most poles of Subsystem 2 for $\eta \Pi = 0.30$. The real part of all poles is equal to or smaller than $-0.17$.

The right-most poles of Subsystem 2 for $\eta \Pi = 0.18$. The real part of all poles is equal to or smaller than $-0.0015$.

The right-most poles of Subsystem 2 for $\eta \Pi = 0.17$. The real part of all poles is equal to or smaller than $0.014$.

The right-most poles of Subsystem 2 for $\eta \Pi = 0.10$. The real part of all poles is equal to or smaller than $0.12$.

Figure 3-8: The right-most poles of the perturbation dynamics of Subsystem 2 ($\tilde{\varepsilon}_2 = \pm 1$) for different values of $\eta \Pi$. The real parts of the poles of Subsystem 2 are negative for $\eta \Pi \geq 0.18$. When $\eta \Pi < 0.18$, the system has poles with a positive real part, which yields an unstable system.
the hybrid system. The stability analysis of the ideal subsystems performed, proves to be a helpful asset in the analysis of the simulations.

Moreover, the stability analysis of the subsystems can still provide information about the local asymptotic stability of the constant curvature solutions (if the latter solution is a feasible solution for one of the subsystem dynamics). We note that the stability analysis of delay complementarity systems, such as the hybrid directional drilling model presented here, is an open issue in the literature in general.

### 3-4 Numerical simulations of the hybrid system

In order to analyze the dynamics of the hybrid system, simulations of the borehole propagation are performed with the hybrid model.

#### 3-4-1 Discretized state-space description

In order to be able to perform numerical simulations, the state-space description of the hybrid system needs to be transformed into a discretized state-space formulation. Using the forward Euler discretization method, the following can be defined:

\[
x'(\xi) \approx \frac{x(\xi + \Delta\xi) - x(\xi)}{\Delta\xi},
\]  

(3-50)

for \( \Delta\xi \) small and where \( \Delta\xi \) is a spatial discretization step size. Equation (3-50) can be used to give an expression for the states a distance \( \Delta\xi \) ahead:

\[
x(\xi + \Delta\xi) = x(\xi) + \Delta\xi x'(\xi).
\]  

(3-51)

Define \( k = \frac{\xi}{\Delta\xi} \) and \( k_i = \frac{\xi_i}{\Delta\xi} \), for \( i = 0, 2, 3, \) where \( \xi \in \{0, \Delta\xi, 2\Delta\xi, 3\Delta\xi, \ldots\} \). Let us define \( x(k) = x(\xi) \) (with some abuse of notation). Then, Equation (3-51) results in

\[
x(k + 1) = x(k) + \Delta\xi x'(k).
\]  

(3-52)

Using the continuous state-space description of the hybrid system (defined by Equation (2-48) to (2-53)), the following discretized state-space description is derived:
\[ x(k + 1) = \begin{bmatrix} 1 + \frac{\Delta \xi_A}{1 + \lambda_2^2} & \frac{\Delta \xi_A}{1 + \lambda_2^2} \\ \frac{\Delta \xi_A}{1 + \lambda_2^2} & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{\Delta \xi_A}{1 + \lambda_3^2} & \frac{\Delta \xi_A}{1 + \lambda_3^2} \\ \frac{\Delta \xi_A}{1 + \lambda_3^2} & -\frac{\Delta \xi_A}{1 + \lambda_3^2} \end{bmatrix} x(k_2) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(k_3) + \]

\[
\begin{bmatrix}
\frac{\Delta \xi_A}{1 + \lambda_2^2} \\
\frac{\Delta \xi_A}{1 + \lambda_3^2}
\end{bmatrix} \tilde{F}_2(k) + \begin{bmatrix}
-\frac{\Delta \xi_A}{1 + \lambda_2^2} \\
-\frac{\Delta \xi_A}{1 + \lambda_3^2}
\end{bmatrix} \tilde{F}_2(k_2) + \begin{bmatrix}
0 \\
0
\end{bmatrix} \tilde{F}_2(k_3) + \begin{bmatrix}
0 \\
0
\end{bmatrix} \tilde{F}_r(k) + \]

\[
\begin{bmatrix}
-\frac{\Delta \xi_A}{1 + \lambda_2^2} \\
\frac{\Delta \xi_A}{1 + \lambda_3^2}
\end{bmatrix} \tilde{F}_r(k_2) + \begin{bmatrix}
0 \\
0
\end{bmatrix} \tilde{F}_r(k_3) + \begin{bmatrix}
\frac{\Delta \xi_A}{1 + \lambda_2^2} & \frac{\Delta \xi_A}{1 + \lambda_2^2} \\
\frac{\Delta \xi_A}{1 + \lambda_3^2} & \frac{\Delta \xi_A}{1 + \lambda_3^2}
\end{bmatrix} \tilde{F}_{w,1}(k) + \]

\[
\begin{bmatrix}
\frac{\Delta \xi_A}{1 + \lambda_2^2} & \frac{\Delta \xi_A}{1 + \lambda_2^2} \\
\frac{\Delta \xi_A}{1 + \lambda_3^2} & \frac{\Delta \xi_A}{1 + \lambda_3^2}
\end{bmatrix} \tilde{F}_{w,2}(k) + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\tilde{F}_{w,1}(k_3) \\
\tilde{F}_{w,2}(k_3) \\
\tilde{F}_{w,3}(k_3)
\end{bmatrix},
\]

\[ (3-53) \]

\[
y_1(k) = \begin{bmatrix} A_1 & A_2 \end{bmatrix} x(k) + A_3 \tilde{F}_2(k) + A_4 \tilde{F}_r(k) + A_5 \tilde{F}_{w,1}(k) + A_6 \tilde{F}_{w,2}(k) + A_7 \tilde{F}_{w,3}(k),
\]

\[ (3-54) \]

\[
y_2(k) = \begin{bmatrix} B_1 & B_2 \end{bmatrix} x(k) + B_3 \tilde{F}_2(k) + B_4 \tilde{F}_r(k) + B_5 \tilde{F}_{w,1}(k) + B_6 \tilde{F}_{w,2}(k) + B_7 \tilde{F}_{w,3}(k),
\]

\[ (3-55) \]

subject to

\[
\begin{align*}
\tilde{F}_2(k) & \geq 0 & \text{if} & y_2(k) = -1, \\
\tilde{F}_2(k) & = 0 & \text{if} & -1 < y_2(k) < 1, \\
\tilde{F}_2(k) & \leq 0 & \text{if} & y_2(k) = 1,
\end{align*}
\]

\[ (3-56) \]

with initial conditions

\[
\begin{align*}
x(k) & = \Theta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{for} & k \in \left[-\sum_{j=1}^{3} \frac{\lambda_j}{\lambda_2}, 1 - \sum_{j=1}^{3} \frac{\lambda_j}{\lambda_2}, \ldots, 0\right], \\
\tilde{F}_r(k) & = \tilde{F}_{r_0} & \text{for} & k \in \left[-\sum_{j=1}^{3} \frac{\lambda_j}{\lambda_2}, 1 - \sum_{j=1}^{3} \frac{\lambda_j}{\lambda_2}, \ldots, 0\right],
\end{align*}
\]

\[ (3-57) \]

\[ (3-58) \]

where \( \Theta_0 \in \mathbb{R}^{1 \times 1} \) and \( \tilde{F}_{r_0} \in \mathbb{R}^{1 \times 1} \) should be taken such that they comply with Equation (2-37) and (2-38), subject to the unilateral contact law of (2-39). Furthermore, all variables, such as e.g. \( y_i(k) \), for \( i = 1, 2 \), are defined analogous to \( x(k) \).

\[ 3-4-2 \quad \text{Transforming the contact problem into a Linear Complementarity Problem} \]

Equation (3-55), subject to the unilateral contact law of (3-56), can be expressed as a Linear Complementarity Problem (LCP). Linear Complementarity Problems are a widely used...
framework in which contact problems can be solved [17]. In order to express the contact problem as an LCP, rewrite Equation (3-55) as

\[ \tilde{F}_2(k) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(k) + C_3 g_2(k) + C_4 \tilde{F}_r(k) + C_5 \tilde{F}_{w,1}(k) + C_6 \tilde{F}_{w,2}(k) + C_7 \tilde{F}_{w,3}(k), \]  

(3-59)

where coefficients \( C_i \), for \( i = 1, ..., 7 \), are defined in Appendix A.

Transforming Equation (3-59), subject to Equation (3-56), into a linear complementarity problem, results in:

\[
\begin{bmatrix} \Gamma_1(k) \\ \Gamma_2(k) \end{bmatrix} := \begin{bmatrix} C_4 & 0 \\ 0 & C_3 \end{bmatrix} \begin{bmatrix} \gamma_1(k) \\ \gamma_2(k) \end{bmatrix} + \begin{bmatrix} -C_3 + \left[ C_1 \\ C_2 \right] x(k) + C_4 \tilde{F}_r(k) + C_5 \tilde{F}_{w,1}(k) + C_6 \tilde{F}_{w,2}(k) + C_7 \tilde{F}_{w,3}(k) \\ -C_3 - \left[ C_1 \\ C_2 \right] x(k) + C_4 \tilde{F}_r(k) + C_5 \tilde{F}_{w,1}(k) + C_6 \tilde{F}_{w,2}(k) + C_7 \tilde{F}_{w,3}(k) \end{bmatrix},
\]  

(3-60)

subject to

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} \Gamma_1(k) \\ \Gamma_2(k) \end{bmatrix} \perp \begin{bmatrix} \gamma_1(k) \\ \gamma_2(k) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]  

(3-61)

from where the contact force and position of the non-ideal stabilizer are retrieved with

\[
\text{if } |\gamma_2(k) - 1| > |\gamma_1(k) - 1| : \quad \tilde{F}_2(k) = \Gamma_1(k) \quad \land \quad \tilde{z}_2(k) = \gamma_1(k) - 1,
\]

\[
\text{else } \quad \tilde{F}_2(k) = -\Gamma_2(k) \quad \land \quad \tilde{z}_2(k) = -(\gamma_2(k) - 1).
\]  

(3-62)

The \( \perp \) symbol denotes that the product of the vectors is zero. Here, the variables \( \gamma_i \) and \( \Gamma_i \), for \( i = 1, 2 \), are introduced, which transform the contact problem of both borehole walls to two separate unilateral contact problems. Equation (3-60), subject to (3-61), can be solved with the \textit{lemke} algorithm [30] in Matlab [43], which uses the complementarity pivot algorithm [53], generally referred to as Lemke’s algorithm, to solve the LCP. For \( C_3 > 0 \), this returns two solutions, \( (\gamma_1(k), \Gamma_1(k)) \) and \( (\gamma_2(k), \Gamma_2(k)) \). At least one of those solutions meets the constraint of the unilateral contact law of Equation (3-56). This solution is selected with Equation (3-62), from which \( \tilde{F}_2(k) \) and \( \tilde{z}_2(k) \) are retrieved.

### 3-4-3 Numerical algorithm

The parameters for the simulation are listed in Table 3-2. A step size of \( \Delta \xi = 10^{-3} \) is chosen. The scaled distributed weight is taken as \( \bar{w} = 1.23 \cdot 10^{-3} \). The maximum scaled clearance is set at \( \varrho = 10^{-3} \).

Define the iteration counter for the initial conditions as
Equation (3-57) is substituted. In this way, the LCP of the initial conditions is described as

\[ k_0 \in \left\{ -3 \sum_{j=1}^{3} \frac{\lambda_j}{D_j}, 1 - 3 \sum_{j=1}^{3} \frac{\lambda_j}{D_j}, 2 - 3 \sum_{j=1}^{3} \frac{\lambda_j}{D_j}, \ldots, 0 \right\} . \tag{3-63} \]

As discussed before, the initial borehole is defined to be straight, under an inclination \( \Theta_0 \). Thus, the states for \( k = k_0 \) are given by Equation (3-57). The initial gravity forces are determined with

\[ \tilde{F}_{w,i}(k_0) = \{ \tilde{F}_{w,i} \}_0 = \tilde{w}_i \lambda_i \sin(\Theta_0) , \tag{3-64} \]

for \( i = 1, 2, 3 \).

The initial RSS force is chosen such that the solution of Equation (2-37) for the initial conditions is the same as the inclination of the initial borehole. In order to determine this RSS force, first the contact force needs to be determined. Equation (3-14) is taken for \( \tilde{K}_c = 0 \) and written in terms of \( \tilde{F}_r \). This equation is then substituted into the LCP (Equation (3-60)). In the first row \( \tilde{F}_2 \) is replaced by \( \Gamma_1 \); in the second row, \( \tilde{F}_2 \) is replaced by \(-\Gamma_2 \). Furthermore, Equation (3-57) is substituted. In this way, the LCP of the initial conditions is described as

\[
\begin{bmatrix}
\Gamma_1(k_0) \\
\Gamma_2(k_0)
\end{bmatrix} = 
\begin{bmatrix}
\frac{A_1}{B_1 D_1} & 0 \\
0 & \frac{A_2}{B_2 D_2}
\end{bmatrix}
\begin{bmatrix}
\gamma_1(k_0) \\
\gamma_2(k_0)
\end{bmatrix} + 
\begin{bmatrix}
\frac{A_1 C_5 - A_5 C_6}{D_4} \tilde{F}_{w,1}(k_0) + \frac{A_4 C_6 - A_6 C_4}{D_4} \tilde{F}_{w,2}(k_0) + \frac{1}{2} \tilde{F}_{w,3}(k_0) \\
\frac{A_1 C_5 - A_5 C_6}{D_4} \tilde{F}_{w,1}(k_0) - \frac{A_4 C_6 - A_6 C_4}{D_4} \tilde{F}_{w,2}(k_0) - \frac{1}{2} \tilde{F}_{w,3}(k_0)
\end{bmatrix},
\]

subject to

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} \Gamma_1(k_0) \\ \Gamma_2(k_0) \end{bmatrix} - \begin{bmatrix} \gamma_1(k_0) \\ \gamma_2(k_0) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{3-66} \]

where the initial RSS force and position of the non-ideal stabilizer are determined with

\[
\begin{aligned}
\text{if } |\gamma_2(k_0) - 1| > |\gamma_1(k_0) - 1| & : \quad \tilde{F}_2(k_0) = \Gamma_1(k_0) \quad \text{and} \quad \tilde{z}_2(k_0) = 1 + \gamma_1(k_0), \\
\text{else} & : \quad \tilde{F}_2(k_0) = -\Gamma_2(k_0) \quad \text{and} \quad \tilde{z}_2(k_0) = 1 - \gamma_1(k_0). \tag{3-67}
\end{aligned}
\]

The required RSS force for a straight initial borehole can be calculated using Equation (3-14) for \( \tilde{K}_c = 0 \), resulting in:

\[ \tilde{F}_r(k_0) = \tilde{F}_{r,0} = -\frac{A_3 \tilde{F}_2(k_0) + A_5 \tilde{F}_{w,1}(k_0) + A_6 \tilde{F}_{w,2}(k_0) + A_7 \tilde{F}_{w,3}(k_0)}{A_4} . \tag{3-68} \]
After having determined the parameter values and the initial conditions, the borehole is propagated in a step-wise fashion. At every step, a value for the RSS force at that step \((\tilde{F}_r(k))\) is given. With current state \(x(k)\), determined in the previous step, the resultants of the gravity forces can be determined at this step, using Equation (2-28), (2-29) and (2-30).

The numerical algorithm proceeds by solving the Linear Complementarity Problem at step \(k\), described by Equation (3-60) subject to (3-61). The values of \(\tilde{F}_2(k)\) and \(y_2(k)\) are then determined with Equation (3-62).

Finally, the inclination of the borehole at the bit \((\hat{\theta}(k) = y_1(k))\) can be determined with Equation (3-54). Furthermore, the state at the next step, \(x(k+1)\), is determined with Equation (3-53).

### 3-4-4 Simulation results

In Section 3-3, it was concluded that the perturbation dynamics of Subsystem 1 are asymptotically stable for all \(\eta\Pi\). Furthermore, for the values of Table 3-2, the perturbation dynamics of Subsystem 2 are asymptotically stable if \(\eta\Pi \geq \eta\Pi_{\text{crit}} = 0.18\).

If \(\varrho = \frac{R}{\Delta} = 0\), the non-ideal stabilizer always touches the borehole wall. Therefore, the hybrid system displays the dynamics of Subsystem 2 in such a case. For this case, the conclusions on the stability of the perturbation dynamics of Subsystem 2 of Section 3-3 apply directly to the hybrid system for this case.

In order to derive conclusions on the stability properties of the hybrid system for the case where \(\varrho > 0\), simulations of the hybrid system are performed. The hybrid system is simulated as described in Section 3-4-3.

#### Simulations for stationary gravity forces

In Figure 3-9, the results of a simulation of the hybrid system for \(\eta\Pi = 0.30\) are depicted. In order to make the effect of the clearance on the quasi-stationary solutions as clear as possible, the gravity forces are set to be zero. The initial borehole is taken as a straight, vertical borehole. From Section 3-4-3 it then follows that \(\tilde{F}_{w,1}(k_0) = \tilde{F}_{w,2}(k_0) = \tilde{F}_{w,3}(k_0) = \tilde{F}_2(k_0) = \tilde{F}_r(k_0) = 0\). The RSS force is constant and taken as \(\tilde{F}_r = 1 \cdot 10^{-3}\). From Equation (3-14) and (3-17), subject to (2-39), it is derived that an RSS force of \(\tilde{F}_r = 1 \cdot 10^{-3}\) will yield a quasi-stationary solution with a curvature of \(\tilde{K}_c = 9.45 \cdot 10^{-4}\). This can also be observed in Figure 3-2b. Furthermore, for this quasi-stationary solution, the non-ideal stabilizer will be in contact with the borehole wall. The simulation for this case is shown in Figure 3-9a. In the first plot, bit inclination \(\hat{\theta}\) and borehole inclination at the bit \(\hat{\Theta}\) are shown. In the second and third plot, position of the non-ideal stabilizer \(\tilde{z}_2\) and contact force acting on the non-ideal stabilizer \(\tilde{F}_2\) are shown, respectively. As can be observed in Figure 3-9a, the RSS force brings the non-ideal stabilizer into contact with the upper borehole wall \((\tilde{z}_2 = 1)\). After contact is achieved, the non-ideal stabilizer remains in contact with the borehole wall. The contact force \(\tilde{F}_2\) at the non-ideal stabilizer remains seen to oscillate at the start, but converges to a constant value. In Figure 3-9b, the evolution of the derivatives of the states are shown. In the same plot, the quasi-stationary curvature, \(\tilde{K}_c = 9.45 \cdot 10^{-4}\), is plotted. It is observed that the state derivatives converge to a curvature corresponding to the curvature of the quasi-stationary...
solution. Increasing the RSS force for this case will still result in a response which converges to the relevant quasi-stationary solution. The quasi-stationary solutions, for a weightless BHA, are always locally asymptotically stable if $\eta \Pi \geq 0.18$. Although the latter fact is no guarantee for the global asymptotic stability of the quasi-stationary solutions of the hybrid system, the simulations performed seem to suggest that the hybrid system asymptotically converges to the quasi-stationary solutions on the intermediate length scale, for $\eta \Pi \geq 0.18$.

From Section 3-2 and 3-3, it is concluded that, for $\eta \Pi < 0.18$, the quasi-stationary solutions of the ideal subsystems are locally asymptotically stable for $\tilde{F}_r < \tilde{F}_{r, \text{crit}} = 1.08062 \cdot 10^{-3} \eta \Pi$. In order to investigate whether the quasi-stationary solutions of the hybrid system are globally asymptotically stable for $\eta \Pi < 0.18$, a simulation is performed for $\eta \Pi = 0.10$, where a constant RSS force of $\tilde{F}_r(k) = 4 \cdot 10^{-4}$ is applied. From Equation (3-14) and (3-17), subject to (2-39), it is derived that this RSS force will yield a quasi-stationary solution with $\tilde{K}_c = 7.25 \cdot 10^{-4}$, which can also be observed in Figure 3-2a. Furthermore, in this quasi-stationary solution the non-ideal stabilizer will be in contact with the borehole wall. Since $\eta \Pi < \eta \Pi|_{\text{crit}}$ and $\tilde{F}_r > \tilde{F}_{r, \text{crit}}$, this quasi-stationary solution is known to be locally unstable. The simulation for this system is shown in Figure 3-10. As can be seen in Figure 3-10a, the hybrid system shows limit cycling behavior. At the start, the RSS force brings the non-ideal stabilizer upwards, until it touches the upper borehole wall ($\tilde{z}_2 = 1$). After this, switching between contact and no contact is observed to continue for the entire simulation. It is also noted that, after some transient, the stabilizer position reaches the same minimum value every non-contact phase. At that moment, the hybrid system has entered a limit cycle. Each cycle, the system converges to the quasi-stationary solution, which results in contact at the non-ideal stabilizer. Due to the instability of the quasi-stationary solution for Subsystem 2, the system is then driven out of contact. Because contact is lost, the system again converges to the quasi-stationary solution, because the stable dynamics of Subsystem 1 are activated. The limit cycle occurs, because a balance is achieved in the switching between contact and no contact. The limit cycle that the hybrid system enters is also visible in Figure 3-10b. In this figure, the derivatives of the states are shown. Their trajectory is seen to grow at the start of the simulation, but eventually reaches a limit cycle. As stated before, the equilibrium solution of the situation presented would be a borehole with curvature $\tilde{K}_c = 1.6 \cdot 10^{-3}$, which is also shown in Figure 3-10b. The limit cycle is seen to encircle the quasi-stationary solution.

If, for the case of $\eta \Pi = 0.10$, the RSS force would be slightly increased, the amplitude of $\tilde{z}_2$ in the non-contact phase of the limit cycle becomes larger. When the RSS force equals $\tilde{F}_r = 4.5 \cdot 10^{-4}$ or more, the amplitude of $\tilde{z}_2$ in the non-contact phase becomes so large, that the non-ideal stabilizer starts to touch the lower borehole wall as well. In Figure 3-11, a simulation of the hybrid system for a RSS force of $\tilde{F}_r = 4.5 \cdot 10^{-4}$ is shown. It can be observed that for this RSS force, the movement of the non-ideal stabilizer has increased to the point where it switches between contact with the upper borehole wall ($\tilde{z}_2 = 1$) and the lower borehole wall ($\tilde{z}_2 = -1$) in every cycle. This switching of the non-ideal stabilizer converges to a constant contact duration on either side. The (spatial) duration of the non-contact phase is observed to decrease every cycle and converges to 0. Therefore, the system effectively starts to display only the dynamics of Subsystem 2. Since Subsystem 2 is unstable, this may explain the observed unstable behavior. The contact force, $\tilde{F}_2$, is seen to increase with every cycle. Also the derivatives of the states of the system are seen to grow unbounded (Figure 3-11b).

It should be noted that dynamics (inertial effects) are not taken into account in this model. In this simulation, the non-contact phase was observed to converge to a (spatial) duration of
0. Thus, the assumption that dynamical effects do not have to be taken into account, might be invalid for this simulation.

(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The derivatives of the states of the system converge to a constant value.

Figure 3-9: Simulation of the weightless hybrid system for $\eta_{II} = 0.30$. A constant RSS force $F_r = 1 \cdot 10^{-3}$ is applied. After a transient, the non-ideal stabilizer stays in contact with the upper borehole wall ($\tilde{z}_2 = 1$).
(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The system enters a limit cycle.

**Figure 3-10:** Simulation of the weightless hybrid system for $\eta\Pi = 0.10$. A constant RSS force $F_r = 4 \times 10^{-4}$ is applied. Due to this RSS force, the non-ideal stabilizer is brought into contact with the upper borehole wall ($z_2 = 1$). However, the system is unstable when in contact. Therefore, the system starts to switch between contact and no contact. It enters a limit cycle.
(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The derivatives of the states of the system grow unbounded.

**Figure 3-11:** Simulation of the weightless hybrid system for $\eta_{II} = 0.10$, with an increased constant RSS of force $\tilde{F}_r = 4.5 \cdot 10^{-4}$. This RSS force results in switching between contact at the lower borehole wall ($\tilde{z}_2 = -1$) and upper borehole wall ($\tilde{z}_2 = 1$). This yields an unstable hybrid system.
Simulation for evolving gravity forces

As discussed in Section 3-2-4, the system will converge to a straight borehole trajectory on the long term, if \( \tilde{F}_r \leq \tilde{F}_{r, \text{limit}} \). This can also be observed in Figure 3-12, where the results of a simulation of the hybrid system are shown, for \( \eta I = 0.10 \) and \( \tilde{F}_r = 4.5 \cdot 10^{-4} \). In this simulation, the gravity forces are described by Equation (2-28), (2-29) and (2-30), for \( \tilde{w} = 1.23 \cdot 10^{-3} \). The simulation is performed for \( 8 \cdot 10^5 \) iterations, which corresponds to a borehole length of \( \xi = 800 \). For this simulation, the same parameter values were taken as for the simulation of Figure 3-11. It can be observed that, due to the fact that gravity forces are now taken into account, the system has become stable. The reason for this is that the net force acting on the BHA is reduced by the gravity forces. For this reason, the non-ideal stabilizer does not reach the lower wall anymore during the cycle and the system does not become unstable. Furthermore, it is observed that the system shows transient oscillating behavior, but no limit cycle. Due to the gravity forces present, the response observed in the plot of the evolution of the derivatives of the states (Figure 3-12b) reduces in size as \( \xi \) increases. In the transient, the non-ideal stabilizer intermittently loses contact with the upper borehole wall. In contrast to the case where the gravity forces were zero, the amplitude in the position of the stabilizer during the non-contact phase, decreases as the borehole progresses. Eventually, the non-ideal stabilizer loses contact and does not return to contact anymore. The derivatives of the states converge to zero, resulting in a straight borehole. The inclination of the borehole at the end of the simulation is \( 0.182 \text{ rad} \), which corresponds to the value of \( \Theta_\infty \) that is found with Equation (3-30).

In Section 3-2-4, it was found that the borehole inclination keeps increasing on a large length scale, if \( \tilde{F}_r > \tilde{F}_{r, \text{limit}} \). A simulation displaying this behavior, is shown in Figure 3-13. The hybrid system is simulated for \( \eta I = 0.3 \) and the maximum RSS force, \( \tilde{F}_r(k) = 1 \cdot 10^{-2} \). The system is simulated for \( 1.4 \cdot 10^6 \) iterations, resulting in a borehole of \( \xi = 1400 \). In Figure 3-13a and 3-14, it can be observed that the borehole inclination keeps increasing. Figure 3-13b shows that trajectory of the state derivatives. After the transient response, the trajectory of the state derivatives remains on a small cycle, resulting in a borehole with an approximately constant curvature. This can also be observed in Figure 3-14, where the resulting borehole trajectory is shown.

In Figure 3-16, a simulation is shown for \( \eta I = 0.10 \). For the initial borehole, an inclination of \( \Theta_0 = \pi/2 \) is chosen. The other initial parameters are derived as in Section 3-4-3. Due to the weight forces present, the non-ideal stabilizer is initially in contact with the lower borehole wall (\( \tilde{z}_2 = -1 \)). The RSS force is chosen as \( \tilde{F}_r = 2.5 \cdot 10^{-3} \). In Figure 3-4a it can be observed that, by choosing this RSS force, the non-ideal stabilizer is not in contact with the borehole wall for the quasi-stationary solution of this system. This means the quasi-stationary solution is locally asymptotically stable. However, the simulation shows that a limit cycle occurs. The reason for this is that, due to the initial conditions, the system starts in the domain of (unstable) Subsystem 2. The initial instability results in a system that does not asymptotically converge to the quasi-stationary solution. If the RSS force is increased to \( \tilde{F}_r = 2.28 \cdot 10^{-3} \), the system shows an asymptotically stable response. The reason for this is that for this RSS force, the unstable transient behavior at the start is suppressed and the quasi-stationary solution is locally asymptotically stable. However, the domain of the RSS force that yields a stable system, is very small. For any RSS force outside of the domain \( [2.4, 2.9] \cdot 10^{-3} \), the response of the system is unstable.
**3-5 Discussion**

In this chapter, the hybrid directional drilling model, derived in Chapter 2, was analyzed. First, parameter values for the case study were selected. Second, expressions for the quasi-stationary solutions of the hybrid model were derived. Due to the switching between no contact and contact at the non-ideal stabilizer, the quasi-stationary solutions of the hybrid system are a combination of the quasi-stationary solutions of Subsystem 1 and 2. A stability analysis was performed on the perturbation dynamics of these quasi-stationary solutions. It was found that quasi-stationary solutions for which the non-stabilizer is not in contact, corresponding to Subsystem 1, are always (locally) asymptotically stable. Quasi-stationary solutions for which the non-ideal stabilizer is in contact with the borehole wall, corresponding to Subsystem 2, are (locally) asymptotically stable for $\eta \Pi > \eta \Pi_{\text{crit}}$.

The stability of the hybrid system was investigated by numerically simulating the dynamics of the system. These simulations showed that if $\eta \Pi \geq \eta \Pi_{\text{crit}}$, the hybrid system converged asymptotically to the quasi-stationary solutions.

For $\eta \Pi < \eta \Pi_{\text{crit}}$, stability or instability of the quasi-stationary solutions of the hybrid system is not guaranteed. For an initial condition for which the non-ideal stabilizer is not in contact, the system converges to the quasi-stationary solution at the start of the simulation. If the loading on the BHA is such that the quasi-stationary solution lays in the domain of Subsystem 1, the system converges to the quasi-stationary solution. For a quasi-stationary solution in the domain of Subsystem 2, the system will either enter a limit cycle or become unstable, depending on the load that is applied. When the non-ideal stabilizer is in contact with the borehole wall for the initial conditions, the system shows unstable behavior from the start. For a specific load, the system will converge to a quasi-stationary solution of Subsystem 1, or enter a limit cycle around this solution. For any other load, the system is unstable.

For the quasi-stationary solutions, the gravity forces were assumed to be quasi-constant. Because of the slow evolution of the inclination of the borehole, this assumption is valid on...
an intermediate length scale. On a large length scale, the borehole inclination converges to a straight, downward trajectory, if the system is stable and the RSS force is smaller than $\tilde{F}_{\text{limit}}$. For $\tilde{F}_r > \tilde{F}_{\text{limit}}$, the inclination of stable systems keeps increasing.

It should be pointed out that for some simulations, the switching of the non-ideal stabilizer seems to happen on a high frequency. In the case where the hybrid system was unstable, the duration of the switching between contact with the two borehole wall converged to zero. This means the assumption that the BHA is quasi-static might not be valid for modeling of a non-ideal stabilizer.
(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The derivatives of the states of the system eventually converge to zero.

**Figure 3-12:** Simulation of the hybrid system for $\eta = 0.10$ and non-zero gravity forces. A constant RSS force $\tilde{F}_r = 4.5 \cdot 10^{-4}$ is applied.
(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The derivatives of the states of the system eventually converge to zero.

**Figure 3-13:** Simulation of the hybrid system for $\eta_{II} = 0.30$ and non-zero gravity forces. A constant RSS force $\tilde{F}_r = 1 \cdot 10^{-2}$ is applied.
**Figure 3-14:** The borehole resulting from a simulation of the hybrid system for \( \eta \Pi = 0.30 \) and non-zero gravity forces. A constant RSS force \( \tilde{F}_r = 1 \cdot 10^{-2} \) is applied. It can be seen that this large RSS force produces a curved borehole on a large length scale.

**Figure 3-15:** The length scales shown with the trajectory of \( \hat{\Theta}' \). On the short length scale of \( O(1) \), discontinuities can be seen. A step on the RSS force resulted in a sudden directional change of the borehole inclination. Furthermore, the non-ideal stabilizer switches almost immediately from non-contact to contact, resulting in a discontinuity. This discontinuities keep prevalent in the transient response, due to the geometrical feedback of the stabilizers. After approximately \( \xi = 10 \), their effect has become insignificant and the system converges to a quasi-constant curvature.
(a) The borehole and BHA inclination at the bit are displayed in the first plot. In the second and third plot, the position of the non-ideal stabilizer and the contact force acting on the non-ideal stabilizer are shown respectively.

(b) Trajectory of the state derivative. The system enters a limit cycle.

**Figure 3-16:** Simulation of the hybrid system for $\eta\Pi = 0.10$ and non-zero gravity forces. A constant RSS force $\tilde{F}_r = 2.5 \cdot 10^{-3}$ is applied. Furthermore, the initial borehole is under an inclination of $\Theta_0 = \pi/2$. 

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4-1 Conclusions

The main goal of this research was to develop a hybrid directional drilling model by incorporating non-ideal stabilizer contact. Furthermore, we aimed to understand what the influence of a non-ideal stabilizer on the borehole trajectory of a directional drilling system is, with the main focus on the directional stability of the non-ideal system and the effect of the switching nature of the system on the (quasi-)stationary solutions.

In order to achieve the main goal, a hybrid model of a two-dimensional directional drilling system was derived, by taking non-ideal stabilizer contact into account. The BHA dynamics were described by a two-stabilizer distributed model. Combining the BHA model with a model for the kinematic relationships and bit-rock interaction, resulted in a borehole propagation model. The borehole propagation model was transformed into a delay differential state-space model, subject to a complementarity condition. In order to provide an asset for the analysis of the hybrid model, the system equations of two ideal subsystems of this model were also derived.

In order to determine what the influence of a non-ideal stabilizer on the borehole trajectory of a directional drilling system is, analysis was performed on a benchmark BHA. On an intermediate length scale, the asymptotes of the system were described by the quasi-stationary solutions, for which the gravity forces were assumed to be quasi-constant. It was found that the quasi-stationary solutions of the hybrid system correspond directly to the quasi-stationary solutions of the ideal subsystems, for the relevant intervals. On a large length scale, the asymptotes of the system correspond to a subset of those quasi-stationary solutions, due to the evolving gravity forces. The long term solutions result in a straight, downward borehole, if the RSS force is taken smaller than $\tilde{F}_{\text{r,limit}}$ and the system is directionally stable. For $\tilde{F}_r > \tilde{F}_{\text{r,limit}}$, the inclination of the system keeps increasing, resulting in a rotating borehole. The hybrid system was found to converge to these quasi-stationary solutions, for $\eta \Pi \geq \eta \Pi|_{\text{crit}}$. For $\eta \Pi < \eta \Pi|_{\text{crit}}$, the switching behavior of the hybrid system determines the stability of the quasi-stationary trajectories on an intermediate length scale. Depending on initial condition
Conclusions and recommendations

and loading, the response will either be unstable or enter a limit cycle. These limit cycles are believed to be closely related to borehole spiraling, and thus the presented model provides more insight in these oscillating effects.

4-2 Recommendations

Several recommendations could be made for future research:

- In this research, the choice was made to model the BHA with two stabilizers, of which one a non-ideal stabilizer. In [60] it is shown that the influence of considering more stabilizers on the forces and moment on the bit diminishes rapidly after three stabilizers. Therefore, it is recommended for future research to take into account three stabilizers, of which two non-ideal. The modeling methods presented can be extended for this effort.

- In order to model the hybrid behavior of the system, the choice was made in this research to model the system as a lumped-parameter model. A comparative analysis was performed between this lumped-parameter model and the generally used distributed model. It would be interesting to investigate more thoroughly whether the lumped-parameter modeling method serves as a proper replacement of the standard distributed modeling method. Also a validation, using measurement data, is recommended. Furthermore, the choice to model the hybrid system with a distributed modeling method might be an interesting approach to the problem.

- Depending on the rate of penetration, the switching of the non-ideal stabilizer could occur on a frequency where the dynamics of the system start playing a role. In this research, the BHA system was modeled as a quasi-static system, and dynamics were therefore disregarded. It might be interesting to check whether this assumption really holds, when modeling non-ideal stabilizer contact.

- Besides non-ideal stabilizer contact, bit tilt saturation is shown to affect the behavior of directional drilling systems. Including both non-ideal stabilizer contact and bit tilt saturation in a model, could lead to results which explain the behavior of directional drilling systems in much more detail than current models do.

- The hybrid directional drilling model presented was modeled in two dimensions. In practice, boreholes are drilled in three dimensions, and therefore it would be recommended for future research to model non-ideal stabilizer contact in three dimensions. However, modeling in three dimensions makes model derivations significantly more complicated, because effects such as bit walk should be incorporated in the model.

- In the research presented, the moment at the bit was assumed to be zero. Relaxing this assumption leads to more difficult derivations, but should also lead to a more realistic representation of the process.

- It would be interesting to develop controllers for a hybrid directional drilling model, which can deal with the switching of the hybrid system.
Appendix A

Model coefficients

For simplility, the equations and derivations of this thesis are described with coefficients $A_i$, $B_i$, and $C_i$, for $i = 1,\ldots,7$, $D_i$, for $i = 1,\ldots,6$, $E_i$, for $i = 3,\ldots,7$ and $Q_i$, for $i = 1,\ldots,12$.

Coefficients $A_i$, for $i = 1,\ldots,7$, are given by

\begin{align*}
A_1 &= \frac{1 + \lambda_2}{1 + \lambda_2 + \lambda_3}, \\
A_2 &= \frac{\lambda_3}{1 + \lambda_2 + \lambda_3}, \\
A_3 &= \frac{\lambda_3(-\eta\Pi(1 + \lambda_2)\lambda_3 + \kappa_2(1 + \lambda_2 - \eta\Pi\lambda_3 + \lambda_3 - \eta\Pi\lambda_3))}{\kappa_2\eta\Pi(1 + \lambda_2 + \lambda_3)^2}, \\
A_4 &= \frac{1}{\eta\Pi} - \frac{(\eta\Pi\lambda_3^2 + \kappa_2(1 + \lambda_2 + \eta\Pi\lambda_3^2 + \lambda_3 + 2\eta\Pi\lambda_2\lambda_3 + \eta\Pi\lambda_3^2))\Lambda}{\eta\Pi\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \\
A_5 &= \frac{\eta\Pi\lambda_3^2 + \kappa_2(-1 + (-2 + \eta\Pi)\lambda_3^2 + \lambda_3(-3 + (-2 + \eta\Pi)\lambda_3) + \lambda_2(-3 + 2(-2 + \eta\Pi)\lambda_3))}{2\kappa_2\eta\Pi(1 + \lambda_2 + \lambda_3)^2}, \\
A_6 &= \frac{\eta\Pi(2 + \lambda_2)\lambda_3^2 + \kappa_2(\lambda_2 + 2\lambda_3)(-1 + (-1 + \eta\Pi)\lambda_2 + (-1 + \eta\Pi)\lambda_3)}{2\kappa_2\eta\Pi(1 + \lambda_2 + \lambda_3)^2}, \\
A_7 &= \frac{\lambda_3(\eta\Pi(1 + \lambda_2)\lambda_3 + \kappa_2(-1 + (-1 + \eta\Pi)\lambda_2 + (-1 + \eta\Pi)\lambda_3))}{2\kappa_2\eta\Pi(1 + \lambda_2 + \lambda_3)^2}.
\end{align*}
Coefficients $B_i$, for $i = 1, \ldots, 7$, are given by

\begin{align*}
B_1 &= \frac{(1 + \lambda_2)\lambda_3}{\varrho(1 + \lambda_2 + \lambda_3)}, \quad \text{(A-8)} \\
B_2 &= -\frac{(1 + \lambda_2)\lambda_3}{\varrho(1 + \lambda_2 + \lambda_3)}, \quad \text{(A-9)} \\
B_3 &= \frac{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3^2}{\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \quad \text{(A-10)} \\
B_4 &= \frac{\lambda_3(\kappa_2\lambda_2 + (1 + \kappa_2 + \lambda_2)\lambda_3)\lambda}{\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \quad \text{(A-11)} \\
B_5 &= -\frac{\lambda_3(\kappa_2\lambda_2 + (1 + \kappa_2 + \lambda_2)\lambda_3)}{2\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \quad \text{(A-12)} \\
B_6 &= -\frac{\lambda_3((1 + \lambda_2)(2 + \lambda_2)\lambda_3 + \kappa_2(\lambda_2 + 2\lambda_3))}{2\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \quad \text{(A-13)} \\
B_7 &= -\frac{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3^2}{2\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}, \quad \text{(A-14)}
\end{align*}

Coefficients $C_i$, for $i = 1, \ldots, 7$, are given by

\begin{align*}
C_1 &= -\frac{\kappa_2(1 + \lambda_2)(1 + \lambda_2 + \lambda_3)}{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3}, \quad \text{(A-15)} \\
C_2 &= \frac{\kappa_2(1 + \lambda_2)(1 + \lambda_2 + \lambda_3)}{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3}, \quad \text{(A-16)} \\
C_3 &= \frac{\varrho\kappa_2(1 + \lambda_2 + \lambda_3)^2}{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3^2}, \quad \text{(A-17)} \\
C_4 &= -\frac{(\kappa_2\lambda_2 + (1 + \kappa_2 + \lambda_2)\lambda_3)\lambda}{(\kappa_2 + (1 + \lambda_2)^2)\lambda_3}, \quad \text{(A-18)} \\
C_5 &= \frac{\kappa_2\lambda_2 + (1 + \kappa_2 + \lambda_2)\lambda_3}{2(\kappa_2 + (1 + \lambda_2)^2)\lambda_3}, \quad \text{(A-19)} \\
C_6 &= \frac{(1 + \lambda_2)(2 + \lambda_2)\lambda_3 + \kappa_2(\lambda_2 + 2\lambda_3)}{2(\kappa_2 + (1 + \lambda_2)^2)\lambda_3}, \quad \text{(A-20)} \\
C_7 &= \frac{1}{2}, \quad \text{(A-21)}
\end{align*}
Coefficients \(D_i\), for \(i = 1, \ldots, 6\), are given by

\[
D_1 = \frac{(1 + \lambda_2)(-\kappa_2 + \eta \Pi (1 + \kappa_2 + \lambda_2))}{\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2}, \tag{A-22}
\]

\[
D_2 = \frac{\kappa_2(1 + \lambda_2 - \eta \Pi \lambda_2)}{\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2}, \tag{A-23}
\]

\[
D_3 = \frac{\sigma(-\eta \Pi (1 + \lambda_2)\lambda_3 + \kappa_2(1 + \lambda_2 - \eta \Pi \lambda_2 + \lambda_3 - \eta \Pi \lambda_3))}{\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2} \lambda_3 \tag{A-24}
\]

\[
D_4 = \frac{1}{\eta \Pi} - \frac{(1 + \kappa_2 + \lambda_2 + \eta \Pi \lambda_2^2) \Lambda}{\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2}, \tag{A-25}
\]

\[
D_5 = -\frac{1 + \kappa_2 + \lambda_2(3 - (-2 + \eta \Pi)\lambda_2)}{2\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2}, \tag{A-26}
\]

\[
D_6 = \frac{\lambda_2(-1 + (-1 + \eta \Pi)\lambda_2)}{2\eta \Pi \kappa_2^2 + (1 + \lambda_2)^2}. \tag{A-27}
\]

Coefficients \(E_i\), for \(i = 3, \ldots, 7\), are given by

\[
E_3 = \frac{\lambda_3^2 (\kappa_2(1 + \lambda_2)(1 + \lambda_2 + \lambda_3) + \eta \Pi (1 + \lambda_2)^2 - \kappa_2(-1 + \lambda_2(2 + \lambda_2 + \lambda_3))))}{\eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3}, \tag{A-28}
\]

\[
E_4 = \frac{\lambda_3}{\eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3} \left( \eta \Pi \lambda_2^2(1 + \lambda_2)^2 - \kappa_2 \Delta \lambda_2 \left(2 + \lambda_3 + \eta \Pi (-1 + \lambda_3^2)\right) \right. \tag{A-29}
\]

\[
- \kappa_2 \Delta \left(1 + \eta \Pi \lambda_2^3 + \lambda_3 - \eta \Pi \lambda_3 + \lambda_3^2(1 + 2\eta \Pi \lambda_3)\right) + \kappa_2(1 + \lambda_2)(1 + \lambda_2 + \lambda_3)^2, \right.
\]

\[
E_5 = \frac{-\frac{\kappa_2 \lambda_3}{2\eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3} \left( \lambda_2 \left(-2 + \eta \Pi \right) \lambda_3 - 4 - \eta \Pi - 7\lambda_3 \right) + (\eta \Pi - 2)\lambda_3^3 \right. \tag{A-30}
\]

\[
- \lambda_3(3 + \eta \Pi + 2\lambda_3) - 1 + \lambda_3^2(2(\eta \Pi - 2)\lambda_3 - 5) \right) - \frac{\eta \Pi (1 + \lambda_2)^2 \lambda_3^3}{2 \eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3},
\]

\[
E_6 = \frac{-\frac{\lambda_3}{2\eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3} \left( -\eta \Pi (1 + \lambda_2)^2(2 + \lambda_2 + \lambda_3) \right. \tag{A-31}
\]

\[
+ \kappa_2(\lambda_2 + 2\lambda_3)\left(\eta \Pi (\lambda_2 + \lambda_3 - 1) - (1 + \lambda_2)(1 + \lambda_2 + \lambda_3)\right) \right),
\]

\[
E_7 = \frac{\lambda_3}{2\eta \Pi \kappa_2^2 (1 + \lambda_2 + \lambda_3)^3} \left( -\kappa_2(1 + \lambda_2)(1 + \lambda_2 + \lambda_3) + \eta \Pi (-1 + \lambda_2)^3 + \kappa_2(-1 + \lambda_2(2 + \lambda_2 + \lambda_3))) \right). \tag{A-32}
\]
Coefficients $Q_i$, for $i = 1, \ldots, 12$, are

\[
\begin{align*}
Q_1 &= 3.06526 \cdot 10^7, \quad \text{(A-33)} \\
Q_2 &= 4.72636, \quad \text{(A-34)} \\
Q_3 &= 2.57488 \cdot 10^6, \quad \text{(A-35)} \\
Q_4 &= 1.35031 \cdot 10^8, \quad \text{(A-36)} \\
Q_5 &= -48.3365, \quad \text{(A-37)} \\
Q_6 &= 2.36481 \cdot 10^{-6}, \quad \text{(A-38)} \\
Q_7 &= 2.55014 \cdot 10^8, \quad \text{(A-39)} \\
Q_8 &= -159.272, \quad \text{(A-40)} \\
Q_9 &= 0.0000186945, \quad \text{(A-41)} \\
Q_{10} &= 1.88314 \cdot 10^8, \quad \text{(A-42)} \\
Q_{11} &= -166.193, \quad \text{(A-43)} \\
Q_{12} &= 0.0000375384. \quad \text{(A-44)}
\end{align*}
\]
Comparative analysis of lumped-parameter model and distributed model

In order to compare the lumped parameter model to distributed models, a simple distributed model is derived for both the case in which the non-ideal stabilizer is not in contact and the case for which it is in contact with the borehole wall. These non-contact and contact case correspond to a one-stabilizer and two-stabilizer distributed model respectively.

For the non-contact case, the propagation of the borehole is governed by the inclination of the bit and the dynamics of the non-ideal stabilizer. Therefore, the lumped-parameter model should have an inclination of the bit and deflection of the non-ideal stabilizer, comparable to those values of the distributed model.

For the contact case, the propagation of the borehole is only governed by the inclination of the bit. Therefore, there should be similarity between the angle at the bit for both the lumped-parameter and distributed model for this case.

For this comparative analysis, for simplicity the borehole is assumed to be straight and vertical. The gravity forces in Equation (2-28), (2-29) and (2-30) are replaced by the constant loads $\tilde{F}_{w,i} = -\tilde{q}\lambda_i$ for $i = 1, 2, 3$. The effect of the RSS force will not be taken into account ($\tilde{F}_r = 0$).

The distributed models will be modeled using Euler-Bernoulli beam theory. Define

$$\frac{d^4\tilde{z}(\xi)}{d\xi^4} = \frac{\tilde{q}}{\rho EI}, \quad (B-1)$$

where $\tilde{z}(\xi)$ is the scaled deflection of the BHA with respect to the borehole axis of the distributed model. Deflection $\tilde{z}(\xi)$ is positive in the same direction as $\tilde{z}_2$ for the lumped-parameter model. For this section, $\xi$ is defined to be 0 at the ideal stabilizer. Furthermore,
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Comparative analysis of lumped-parameter model and distributed model

$q$ is a scaled distributed load, defined positive in the direction of $\bar{z}$. Rigidity of the beam is defined by dimensionless parameter $\tilde{EI} = \frac{EI}{k\ell_1}$, where $E \left[ \frac{N}{m^2} \right]$ is the Young’s modulus of the BHA and $I \left[ m^4 \right]$ the moment of Inertia of the intersection of the BHA.

Distributed model of Subsystem 1

For the non-contact case, Equation (B-1) results in

$$\ddot{\bar{z}}_1(\xi) = P_{0,4}\xi^4 + P_{0,3}\xi^3 + P_{0,2}\xi^2 + P_{0,1}\xi + P_{0,0}, \tag{B-2}$$

where $\ddot{\bar{z}}_1(\xi)$ denotes the deflection of the BHA at $\xi$ for the non-contact case of the distributed model and $P_{0,i}$ for $i = 0, \ldots, 4$ are constant coefficients. The values of these coefficients can be determined using the following constraints

$$\ddot{\bar{z}}_1 = 0 \quad \text{for} \quad \xi = 0, \tag{B-3}$$

$$\frac{d^2\ddot{\bar{z}}_1}{d\xi^2} = 0 \quad \text{for} \quad \xi = 0, \tag{B-4}$$

$$\ddot{\bar{z}}_1 = 0 \quad \text{for} \quad \xi = 1 + \lambda_2 + \lambda_3, \tag{B-5}$$

$$\frac{d^2\ddot{\bar{z}}_1}{d\xi^2} = 0 \quad \text{for} \quad \xi = 1 + \lambda_2 + \lambda_3. \tag{B-6}$$

This results in the coefficients

$$P_{0,4} = \left( \frac{\hat{q}}{24qEI} \right), \tag{B-7}$$

$$P_{0,3} = -\left( \frac{\hat{q}}{12qEI} \right)(1 + \lambda_2 + \lambda_3), \tag{B-8}$$

$$P_{0,2} = 0, \tag{B-9}$$

$$P_{0,1} = \left( \frac{\hat{q}}{24qEI} \right)(1 + \lambda_2 + \lambda_3)^3, \tag{B-10}$$

$$P_{0,0} = 0. \tag{B-11}$$

Comparative analysis of inclination of the bit for Subsystem 1

Equation (B-2) results in the following equation for the inclination of the bit for the distributed model of Subsystem 1

$$\ddot{\bar{\theta}}_1 = \frac{q}{12qEI}(1 + \lambda_2 + \lambda_3) = -\left( \frac{\hat{q}}{24qEI} \right)(1 + \lambda_2 + \lambda_3)^3. \tag{B-12}$$

Taking Equation (2-17), (2-18) and (2-19) for $\tilde{F}_2 = 0$, $\tilde{M}_0 = 0$, $\langle \Theta \rangle_1 = \langle \Theta \rangle_2 = 0$, $\tilde{F}_r = 0$ and $\tilde{F}_{\nu,i} = -\hat{q}\lambda_i$ results in an expression for the inclination of the bit for the lumped-parameter model of Subsystem 1.

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\[ \hat{\theta}^I = -\frac{(1 + \lambda_2)\lambda_2^2 + \kappa_2(\lambda_2 + \lambda_3)^2}{2\kappa_2(1 + \lambda_2 + \lambda_3)} \hat{q}. \]  

(B-13)

Parameters need to be chosen such that Equation (B-12) and (B-13) give as close values as possible. Therefore

\[ e_1 = \left\| \hat{\theta}^I - \hat{\theta}^I \right\| \]

(B-14)

\[ = \left\| -\left( \frac{\hat{q}}{24EI} \right) (1 + \lambda_2 + \lambda_3)^3 + \frac{(1 + \lambda_2)\lambda_2^2 + \kappa_2(\lambda_2 + \lambda_3)^2}{2\kappa_2(1 + \lambda_2 + \lambda_3)} \hat{q} \right\|. \]

(B-15)

needs to be minimized.

**Comparative analysis of position of the non-ideal stabilizer for Subsystem 1**

Equation (B-2) results in the following equation for the position of the non-ideal stabilizer for the distributed model of Subsystem 1

\[ \bar{z}_{I2} = \bar{z}^I(\lambda_3) = \frac{(1 + \lambda_2)\lambda_3((1 + \lambda_2)^2 + 3(1 + \lambda_2)\lambda_3 + \lambda_3^2)}{24\varrho E I} \hat{q}. \]

(B-16)

Taking Equation (2-17), (2-18) and (2-19) for \( \bar{F}_2 = 0 \), \( \bar{M}_0 = 0 \), \( \langle \Theta \rangle_1 = \langle \Theta \rangle_2 = 0 \), \( \bar{F}_r = 0 \) and \( \bar{F}_{w,i} = -\hat{q}\lambda_2 \) results in an expression for the position of the non-ideal stabilizer for the lumped-parameter model of Subsystem 1

\[ \tilde{z}_{I2} = \frac{\lambda_2((1 + \lambda_2)^2\lambda_3 + \kappa_2(\lambda_2 + \lambda_3))}{2\varrho \kappa_2(1 + \lambda_2 + \lambda_3)} \hat{q}. \]

(B-17)

The absolute difference in position needs to be minimized. Therefore, parameters need to be chosen such that the unscaled versions of Equation (B-16) and (B-17) give as close values as possible. Therefore

\[ e_2 = \left\| R_\Delta \left( \bar{z}_{I2} - \tilde{z}_{I2} \right) / \ell_1 \right\| \]

(B-18)

\[ = \left\| \left( \frac{(1 + \lambda_2)\lambda_3((1 + \lambda_2)^2 + 3(1 + \lambda_2)\lambda_3 + \lambda_3^2)}{24\varrho E I} \right) \hat{q} \right\| \]

(B-19)

\[ - \left( \frac{\lambda_2((1 + \lambda_2)^2\lambda_3 + \kappa_2(\lambda_2 + \lambda_3))}{2\varrho \kappa_2(1 + \lambda_2 + \lambda_3)} \right) \hat{q} \]

needs to be minimized.

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The second entry of the error vector, the error in the position of the second stabilizer, is defined accordingly to yield a dimensionless error. The choice to scale the error with \( \ell \) instead of with \( R_\Delta \), is made in order to keep the error in the position of the second stabilizer independent on the value of \( R_\Delta \). If one would choose to scale the second entry of the error vector with \( R_\Delta \), the error value would yield high values for low \( R_\Delta \). This would mean that the error of the position with respect to \( R_\Delta \) would be high. However, since at these low values of \( R_\Delta \) the error in position does not matter anymore, this effect does not have to be taken into account.

**Distributed model of Subsystem 2**

When the non-ideal stabilizer is touching, Equation (B-1) leads to

\[
\bar{z}^{II,i}(\xi) = P_{i,4}\xi^4 + P_{i,3}\xi^3 + P_{i,2}\xi^2 + P_{i,1}\xi + P_{i,0},
\]

(B-20)

where \( \bar{z}^{II,i}(\xi) \) denotes the deflection of the BHA at \( \xi \) for the touching case of the distributed model. Furthermore, \( i = 1 \) for \( \xi \in [0, \lambda_3] \) and \( i = 2 \) for \( \xi \in [\lambda_3, 1 + \lambda_2 + \lambda_3] \). Coefficients \( P_{i,j} \) for \( i = 1, 2 \) and \( j = 0,\ldots,4 \) are constant coefficients. The values of these coefficients can be determined with a set of constraints. For \( \bar{z}^{II,1} \) and \( \bar{z}^{II,2} \), these constraints are the following.

\[
\begin{align*}
\bar{z}^{II,1} &= 0, & \text{for } \xi = 0, \\
\frac{d^2\bar{z}^{II,1}}{d\xi^2} &= 0, & \text{for } \xi = 0, \\
\bar{z}^{II,1} &= \bar{z}^{II,2} = \pm 1, & \text{for } \xi = \lambda_3, \\
\frac{d\bar{z}^{II,1}}{d\xi} &= \frac{d\bar{z}^{II,2}}{d\xi}, & \text{for } \xi = \lambda_3, \\
\frac{d^2\bar{z}^{II,1}}{d\xi^2} &= \frac{d^2\bar{z}^{II,2}}{d\xi^2}, & \text{for } \xi = \lambda_3, \\
\bar{z}^{II,2} &= 0, & \text{for } \xi = 1 + \lambda_2 + \lambda_3, \\
\frac{d^2\bar{z}^{II,2}}{d\xi^2} &= 0, & \text{for } \xi = 1 + \lambda_2 + \lambda_3.
\end{align*}
\]

(B-21) \hspace{1cm} (B-22) \hspace{1cm} (B-23) \hspace{1cm} (B-24) \hspace{1cm} (B-25) \hspace{1cm} (B-26) \hspace{1cm} (B-27)

This results in the coefficients

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\[ P_{1,4} = \frac{-\tilde{q}}{24\varrho EI}, \]  
\[ P_{1,3} = \frac{\pm \frac{24}{1+2\lambda_2} + \lambda_3(-1+\lambda_2)^2+(1+\lambda_2)\lambda_3+3\lambda_3^2)\tilde{q}}{48\lambda_3^2} \]  
\[ P_{1,2} = 0, \]  
\[ P_{1,1} = \frac{\pm\frac{1}{2+2\lambda_2} + \frac{1}{\lambda_3}}{48\varrho EI} \]  
\[ P_{1,0} = 0, \]  
\[ P_{2,4} = \frac{-\tilde{q}}{24\varrho EI}, \]  
\[ P_{2,3} = \frac{(\pm24\varrho EI - (1+\lambda_2)\lambda_3(5(1+\lambda_2)^2+7(1+\lambda_2)\lambda_3+\lambda_3^3))\tilde{q}}{48\varrho EI(1+\lambda_2)^2\lambda_3} \]  
\[ P_{2,2} = \frac{(1+\lambda_2+\lambda_3)(\pm24\varrho EI + (1+\lambda_2)\lambda_3((1+\lambda_2)^2+3(1+\lambda_2)\lambda_3+\lambda_3^3)\tilde{q})}{16\varrho EI(1+\lambda_2)^2\lambda_3}, \]  
\[ P_{2,1} = \frac{1}{48\varrho EI(1+\lambda_2)^2\lambda_3} \left( \pm24\varrho EI \left( 2(1+\lambda_2)^2+4(1+\lambda_2)\lambda_3+3\lambda_3^2 \right) \right. \]  
\[ \left. - (1+\lambda_2)^2 \left( 4(1+\lambda_2)^3+11(1+\lambda_2)^2+11(1+\lambda_2)\lambda_3+3\lambda_3^3 \right) \tilde{q} \right), \]  
\[ P_{2,0} = \frac{\lambda_3(1+\lambda_2+\lambda_3)(\pm24\varrho EI + (1+\lambda_2)\lambda_3((1+\lambda_2)^2+3(1+\lambda_2)\lambda_3+\lambda_3^3)\tilde{q})}{48\varrho EI(1+\lambda_2)^2}. \]  

This leads to an inclination at the bit for the distributed model of Subsystem 2 given by

\[ \tilde{\theta}^{II} = \frac{d\tilde{\theta}^{II}}{d\xi} = \frac{\varrho(1+\lambda_2+2\lambda_3)}{2(1+\lambda_2)\lambda_3} - \frac{(1+\lambda_2)(1+\lambda_2^2+\lambda_3-\lambda_3^2+\lambda_2(2+\lambda_3))\tilde{q}}{48EI}, \]  

where subscript ± denotes a solution for contact with the upper borehole wall (+) or contact with the lower borehole wall (−).

Combining Equation (2-17), (2-18) and (2-19) for \( \tilde{\varphi}_2 = \pm 1, \tilde{M}_0 = 0, \langle \Theta \rangle_1 = \langle \Theta \rangle_2 = 0, \tilde{F}_r = 0 \) and \( \tilde{F}_{w,i} = -\tilde{q}\lambda_i \) gives the inclination of the bit for the lumped-parameter model of Subsystem 2

\[ \tilde{\theta}^{I} = \frac{-\pm2\varrho(\kappa_2\lambda_2 + (1+\kappa_2+\lambda_2)\lambda_3) + \lambda_3^2(1+\lambda_2)\lambda_3\tilde{q}}{2(\kappa_2+(1+\lambda_2)^2)\lambda_3} \]  

Equation (B-39) and (B-40) need to be close in both the case when the non-ideal stabilizer touches the upper and the lower borehole wall.
In case the non-ideal stabilizer is in contact with the upper borehole wall ($\tilde{z}_2 = 1$), define

$$
e_3 = \left\| \tilde{\hat{\theta}}_{+}^I - \hat{\theta}_{+}^I \right\| = \left\| \left( \frac{\varrho(1 + \lambda_2 + 2\lambda_3)}{2(1 + \lambda_2)\lambda_3} - \frac{(1 + \lambda_2)(1 + \lambda_2^2 + \lambda_3 - \lambda_3^2 + \lambda_2(2 + \lambda_3))}{48EI} \right) \tilde{q} \right\|.
$$

(B-41)

When the non-ideal stabilizer is in contact with the lower borehole wall ($\tilde{z}_2 = -1$), define

$$
e_4 = \left\| \tilde{\hat{\theta}}_{-}^I - \hat{\theta}_{-}^I \right\| = \left\| \left( \frac{-\varrho(1 + \lambda_2 + 2\lambda_3)}{2(1 + \lambda_2)\lambda_3} - \frac{(1 + \lambda_2)(1 + \lambda_2^2 + \lambda_3 - \lambda_3^2 + \lambda_2(2 + \lambda_3))}{48EI} \right) \tilde{q} \right\|.
$$

(B-42)

Combining lumped-parameter and distributed models

In order to assure that the lumped-parameter model is representative of the distributed model, the following error vector is defined

$$
e = \begin{bmatrix} 
e_1 \\ 
e_2 \\ 
e_3 \\ 
e_4 
\end{bmatrix},
$$

(B-45)

where $\ne_1$, $\ne_2$, $\ne_3$ and $\ne_4$ are defined by Equation (B-15), (B-19), (B-42) and (B-44) respectively. After substitution of

$$\kappa_2 = \frac{k_2}{k_1},$$

$$\bar{E}I = \frac{EI}{k_1\ell_1},$$

$$\tilde{q} = \frac{w\ell_1^2}{k_1},$$

$$\varrho = \frac{R_\Delta}{\ell_1},$$

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the errors become

\[
\begin{bmatrix}
\frac{\ell_1^2 w}{24(1 + \lambda_2 + \lambda_3)} \left( \frac{\ell_1(1 + \lambda_2 + \lambda_3)^4}{EI} - \frac{12 (1 + \lambda_2) \lambda_3^2 + \frac{k_2(\lambda_2 + \lambda_3)^2}{k_1}}{k_2} \right)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\ell_2^2 \lambda_3 w}{24k_1} \left( \frac{k_1 \ell_1 (1 + \lambda_2) \left( (1 + \lambda_2)^2 + 3(1 + \lambda_2) \lambda_3 + \lambda_3^2 \right)}{EI}
+ \frac{12 (k_1 (1 + \lambda_2)^2 \lambda_3 + k_2 (\lambda_2 + \lambda_3))}{k_2 (1 + \lambda_2 + \lambda_3)} \right)
\end{bmatrix}
\]

\[
e = \begin{bmatrix}
\frac{1}{48 \ell_1} \left( \frac{\ell_1^4 (1 + \lambda_2) (1 + \lambda_2^2 + \lambda_3 - \lambda_3^2 + \lambda_2 (2 + \lambda_3)) w}{EI}
+ \frac{24 (1 + \lambda_2 + 2 \lambda_3) R_\Delta}{(1 + \lambda_2) \lambda_3}
- \frac{24 (2k_2 (\lambda_2 + \lambda_3) R_\Delta + (1 + \lambda_2) \lambda_3 (2k_1 R_\Delta + \ell_1^2 \lambda_3^2 w))}{(k_2 + k_1 (1 + \lambda_2)^2) \lambda_3}
\right)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{48 \ell_1} \left( \frac{\ell_1^4 (1 + \lambda_2) (1 + \lambda_2^2 + \lambda_3 - \lambda_3^2 + \lambda_2 (2 + \lambda_3)) w}{EI}
+ \frac{24 (1 + \lambda_2 + 2 \lambda_3) R_\Delta}{(1 + \lambda_2) \lambda_3}
- \frac{24 (2k_2 (\lambda_2 + \lambda_3) R_\Delta + (1 + \lambda_2) \lambda_3 (2k_1 R_\Delta - \ell_1^2 \lambda_3^2 w))}{(k_2 + k_1 (1 + \lambda_2)^2) \lambda_3}
\right)
\end{bmatrix}
\]

Define the following function

\[
J = e^T e. \tag{B-47}
\]

Finally, values for \(k_1\) and \(k_2\) are retrieved by solving the following minimization problem:

\[
\min_{k_1, k_2} J(k_1, k_2). \tag{B-48}
\]
Bibliography


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Glossary

List of Acronyms

PDC  Polycrystalline Diamond Compact
DDE  Delay Differential Equation
LCP  Linear Complementarity Problem
RSS  Rotary Steerable System
BHA  Bottomhole assembly

List of Symbols

Diacritic characters

′  Denotes derivative to $\xi$
¯  Denotes variable of distributed model (comparative analysis)
^  Denotes variable at the bit
˜  Denotes scaled variables

Greek symbols

$\beta$  Angle between penetration vector and bit axis
$\chi$  Angular steering resistance
$\kappa$  Scaled spring constant
$\epsilon$  Ratio angular and lateral steering resistance
$q$  Compensation different scaling
$\eta$  Lateral steering resistance
$\Lambda$  Dimensionless RSS position
$\langle \Theta \rangle$  Average inclination
$\xi$  Scaled length of borehole
$\mu$  Poles of state-space systems
Ω  Angular velocity of bit around own axis
ω  Angular velocity of bit around axis perpendicular to the plane of propagation
μ  Pole
ψ  Bit tilt angle
ρ  Material density
Θ  Inclination borehole
θ  Inclination BHA
φ  Angular velocity of bit around axis perpendicular to the plane of propagation

**Latin symbols**

\[ A, B, C, D, E \]  Modeling coefficients
\[ k \]  Spring constant
\[ C \]  Directional coefficient constant curvature solutions
\[ e \]  Error comparative analysis
\[ F \]  Force
\[ P \]  Coefficients distributed model (comparative analysis)
\[ k \]  Iteration counter
\[ \hat{K} \]  Scaled curvature
\[ \lambda \]  Scaled length
\[ L \]  Length of borehole
\[ \ell \]  Length of BHA section
\[ M \]  Moment
\[ \hat{R}_c \]  Radius of steady-state solution
\[ R_\Delta \]  Maximum clearance
\[ R_i \]  Radius non-ideal stabilizer
\[ R_o \]  radius borehole
\[ s \]  Curvilinear coordinate of BHA
\[ S \]  Curvilinear coordinate of borehole
\[ x \]  State of state-space system
\[ \vec{d} \]  Penetration vector
\[ \vec{v} \]  Velocity vector
\[ z_2 \]  Position of non-ideal stabilizer with respect to borehole axis
\[ E \]  Young's modulus
\[ G_1 \]  Measure of bit bluntness
\[ H_1, H_2, H_3 \]  Coefficients of bit/rock interaction
\[ I \]  Area moment of inertia of the BHA
\[ I_r \]  Inner radius of the BHA pipes
\[ J \]  Cost function comparative analysis
\[ O_r \]  Outer radius of the BHA pipes
\[ W \]  Active weight on bit
\[ w \]  Weight per unit length of BHA
\( y \)  \hspace{1em} \text{Output of state-space system}

\textbf{Units}

\begin{tabular}{ll}
  kg & Kilogram \\
  m & Meter \\
  N & Newton \\
\end{tabular}