II. VARIATIONAL DISTANCE AND ERROR PROBABILITIES

When testing two simple hypotheses (measures) $P$ and $Q$ the minimal possible sum $\inf \{a + \beta\}$ of both error probabilities satisfies a simple relation

$$\inf \{a + \beta\} = 1 - \frac{1}{2} ||P - Q|| = 1 - \frac{1}{2} \int_{\mathcal{X}} |dP - dQ|.$$  

Relation (7) and its natural generalization through the convex hull of measures for composite hypotheses was proved first by C. Kraft [4]. Much later (but independently!) it was obtained also in [3], where (see also [1]) some examples of application of a generalized version of relation (7) in testing of “very composite” hypotheses are presented.

A good collection of various estimates for $||P - Q||$ can be found in [5, Ch. 4] (where the author has learned about the reference [4] for relation (7)).

Due to relation (7) we can reformulate Corollary 1 in a more geometrical form that supplements the collection in [5].

Corollary 2: The following bounds for $||P - Q||$ are valid:

$$2 \left(1 - \exp \left(\mu(s^*) - \frac{1}{2} \sqrt{\mu''(s^*)}\right)\right) \leq ||P - Q|| \leq 2(1 - \exp \{\mu(s^*)\})$$

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where \( W \in \mathbb{C}^{(K+2L) \times (K+2L)} \) is the covariance matrix of \( \mathbf{w} \). For simplicity it will be assumed that the expectation \( E[\mathbf{w}] \) of \( \mathbf{w} \) is equal to the null vector. Next define the vector of real and complex variates \( \mathbf{v} \in \mathbb{C}^{(K+2L) \times 1} \) by
\[
\mathbf{v} = (r_1, \ldots, r_K, z_1, z_1^*, \ldots, z_L, z_L^*)^T
\]
(3)
where \( z_e = x_e + jy_e \) and \( z_e^* \) is the conjugate of \( z_e \) with \( j^2 = -1 \). Then
\[
\begin{pmatrix}
z_e \\
z_e^*
\end{pmatrix} = \mathbf{J} \begin{pmatrix} y_e \\
x_e
\end{pmatrix}
\]
(4)
where the matrix \( \mathbf{J} \in \mathbb{C}^{2 \times 2} \) is defined by
\[
\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1
\end{pmatrix}
\]
(5)
Therefore,
\[
\mathbf{v} = \mathbf{Bw}
\]
(6)
where \( \mathbf{B} \in \mathbb{C}^{(K+2L) \times (K+2L)} \) is defined as the block diagonal matrix
\[
\mathbf{B} = \text{diag} (\mathbf{I}, \mathbf{A})
\]
(7)
where \( \mathbf{I} \in \mathbb{R}^{K \times K} \) is the identity matrix of order \( K \) while \( \mathbf{A} \in \mathbb{C}^{2L \times 2L} \) is defined as the block diagonal matrix
\[
\mathbf{A} = \text{diag} (\mathbf{J}, \ldots, \mathbf{J}).
\]
(8)
From (6) it follows that the covariance matrix \( \mathbf{V} \in \mathbb{C}^{(K+2L) \times (K+2L)} \) of \( \mathbf{v} \), defined as \( E[\mathbf{v} \mathbf{v}^H] \), is equal to
\[
\mathbf{V} = \mathbf{B} E[\mathbf{w} \mathbf{w}^H] \mathbf{B}^H = \mathbf{BB}^H,
\]
and hence
\[
\mathbf{W} = \mathbf{B}^{-1} \mathbf{V} \mathbf{B}^{-H}.
\]
(10)
In these expressions, the superscript \( H \) denotes complex conjugate transposition. Since \( \mathbf{J}^{-1} = \frac{1}{2} \mathbf{J}^H \), it follows that \( \mathbf{B}^{-H} = \text{diag} (\frac{1}{2} \mathbf{J}^H) \) and \( \mathbf{B}^{-H} = \text{diag} (\frac{1}{2} \mathbf{A}^H) \). Then, by (10)
\[
\det \mathbf{W} = \left( \frac{1}{2} \right)^{2L} \det \mathbf{V} \left( -\frac{1}{2} \right)^{2L} = \det \mathbf{V}^4 / 4^L
\]
(11)
since \( \det \mathbf{J} = -2j \) and \( \det \mathbf{J}^H = 2j \). Furthermore, since \( \mathbf{w} = \mathbf{B}^{-1} \mathbf{v} \) and \( \mathbf{w}^H = \mathbf{w}^H = \mathbf{v}^H \mathbf{B}^{-H} \)
\[
\mathbf{w}^T \mathbf{W}^{-1} \mathbf{w} = \mathbf{v}^H \mathbf{B}^{-H} \mathbf{W}^{-1} \mathbf{B}^{-1} \mathbf{v} = \mathbf{v}^H (\mathbf{BB}^H)^{-1} \mathbf{v}.
\]
(12)
Therefore, by (9)
\[
\mathbf{w}^T \mathbf{W}^{-1} \mathbf{w} = \mathbf{v}^H \mathbf{V}^{-1} \mathbf{v}.
\]
(13)
Substituting (11) and (13) in (2) yields
\[
\frac{1}{2^{K+2L} \pi^{(K+2L)/2} \det \mathbf{V}^{1/2}} \exp \left( -\frac{1}{2} \mathbf{v}^H \mathbf{V}^{-1} \mathbf{v} \right)
\]
(14)
Next, rearrange the elements of \( \mathbf{v} \) as follows:
\[
\mathbf{u} = \mathbf{P} \mathbf{v}
\]
(15)
where \( \mathbf{P} \in \mathbb{R}^{(K+2L) \times (K+2L)} \) is the permutation matrix such that
\[
\mathbf{u} = (r_1, \ldots, r_K, z_1, z_1^*, \ldots, z_L, z_L^*)^T.
\]
(16)
Next, let \( \mathbf{U} \in \mathbb{C}^{(K+2L) \times (K+2L)} \) be the covariance matrix of \( \mathbf{u} \). Then by (15)
\[
\mathbf{U} = E[\mathbf{uu}^H] = \mathbf{PVP}^H.
\]
(17)
Because permutation matrices are orthogonal and since the absolute value of their determinant is equal to one [5, p. 360 and p. 25], it follows from (15) and (17) that
\[
\det \mathbf{V} = \det \mathbf{U}
\]
(18)
and
\[
\mathbf{v}^H \mathbf{V}^{-1} \mathbf{v} = \mathbf{u}^H \mathbf{U}^{-1} \mathbf{u}.
\]
(19)
Then substituting (18) and (19) in (14) yields
\[
\frac{1}{2^{K+2L} \pi^{(K+2L)/2} (\det \mathbf{U})^{1/2}} \exp \left( -\frac{1}{2} \mathbf{u}^H \mathbf{U}^{-1} \mathbf{u} \right)
\]
(20)
This is the expression for the normal probability density function of the \( K \) real variates \( r_1, \ldots, r_K \) and the \( 2L \) complex variates \( z_1, \ldots, z_L, z_1^*, \ldots, z_L^* \). It is the main result of this correspondence.

For the description of special cases of this probability density, the covariance matrix \( \mathbf{U} \) is partitioned as follows:
\[
\mathbf{U} = \begin{pmatrix} \mathbf{R} & \mathbf{Q} & \mathbf{Q}' \\
\mathbf{Q}^H & \mathbf{Z} & \mathbf{S} \\
\mathbf{Q}' \mathbf{Z} & \mathbf{S}' & \mathbf{Z}'
\end{pmatrix}
\]
(21)
with
\[
\mathbf{N} = \begin{pmatrix} \mathbf{Z} \\
\mathbf{S}' \mathbf{Z} \\
\mathbf{Z}'
\end{pmatrix}
\]
(23)
and the probability density function becomes
\[
\frac{1}{\sqrt{2^{K+2L} \pi^{(K+2L)/2} (\det \mathbf{R})^{1/2} (\det \mathbf{N})^{1/2}}} \exp \left( -\frac{1}{2} \left( \mathbf{r}^H \mathbf{R}^{-1} \mathbf{r} + \mathbf{z}^H \mathbf{N}^{-1} \mathbf{z} \right) \right)
\]
(24)
and, therefore, the probability density becomes
\[
\frac{1}{\sqrt{2^{K+2L} \pi^{(K+2L)/2} (\det \mathbf{R})^{1/2} (\det \mathbf{Z})^{1/2}}} \exp \left( -\frac{1}{2} \mathbf{r}^H \mathbf{R}^{-1} \mathbf{r} - \mathbf{z}^H \mathbf{Z}^{-1} \mathbf{z} \right)
\]
(26)
For \( K = L = 1 \), \( \mathbf{r} \) and \( \mathbf{z} \) become scalars \( r \) and \( z \) with probability density function
\[
\frac{1}{\sqrt{2 \pi} \sigma_r \sigma_z} \exp \left( -\frac{1}{2} \frac{r^2}{\sigma_r^2} - \frac{z^2}{\sigma_z^2} \right)
\]
(27)
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Zero-Crossing Rates of Mixtures 
and Products of Gaussian Processes

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Abstract—Formulas for the expected zero-crossing rate of non-Gaussian 
mixtures and products of Gaussian processes are obtained. The approach 
we take is to first derive the expected zero-crossing rate in discrete time 
and then obtain the rate in continuous time by an appropriate limiting 
argument. The processes considered, which are non-Gaussian but derived 
from Gaussian processes, serve to illustrate the variability of the zero-
crossing rate in terms of the normalized autocorrelation function \( \rho(t) \) 
of the process. For Gaussian processes, Rice’s formula gives the expected 
zero-crossing rate in continuous time as \( \frac{\pi}{\sqrt{1-\rho^2(0)}} \). We show 
processes exist with expected zero-crossing rates given by \( \frac{\pi}{\sqrt{1-\rho^2(0)}} \) 
either \( \kappa \geq 1 \) or \( \kappa \leq 1 \). Consequently, such processes can have an arbitrarily 
large or small zero-crossing rate as compared to a Gaussian process with 
the same autocorrelation function.

Index Terms—Autocorrelation, cosine formula, expected zero-crossing 
rate, non-Gaussian processes, Rice’s formula.

I. INTRODUCTION

Consider a zero-mean, strictly stationary Gaussian process \( \{Z(t)\} \), 
\( -\infty < t < \infty \), with autocovariance \( R(t) \) and autocorrelation func-
tion \( \rho(t) \). We assume throughout that the variance of the underlying 
Gaussian process \( \{Z(t)\} \) is one so that \( R(0) = \rho(0) = 1 \). If \( \{Z(t)\} \) 
is mean-square-differentiable, that is, if \( \rho''(0) \) exists and is finite, 
then the expected number of zero crossings per unit time is given by 
Rice’s formula ([17], [19])

\[
E[D_c] = \frac{1}{\pi} \sqrt{\rho''(0)}
\]  

(1)

where \( D_c \) (for continuous) is the number of zero crossings of 
\( \{Z(t)\} \) for \( t \) in the unit interval \([0,1]\), and \( \rho''(0) \) is the second 
derivative of the autocorrelation function of \( \{Z(t)\} \) at \( 0 \). In the sequel 
we shall continue to use \( D_c \) to denote the zero-crossing rate in 
continuous time regardless of the process.

The analogous formula for a discrete-time, zero-mean, unit vari-
ance, stationary Gaussian sequence \( \{Z(k)\} \), \( k = 0, \pm 1, \pm 2 \cdots \) 
is given by ([14], [19], [9])

\[
\rho_1 = \cos \frac{\pi E[D_1]}{N-1}
\]  

(2)

where \( D_1 \) is the number of sign changes or zero crossings in 
\( \{Z(1), \cdots, Z(N)\} \), \( \rho_k = E[Z(k+j)Z(j)] \) is the correlation 
sequence of \( \{Z(k)\} \), and \( E[D_1]/(N-1) \) is the expected zero-
crossing rate in discrete time. We refer to (2) as the “cosine formula.”

In this correspondence we present extensions of Rice’s formula 
of the form \( \frac{\pi}{\sqrt{1-\rho^2(0)}} \) where \( \kappa \leq 1 \) or \( \kappa \geq 1 \), and \( \rho(t) \) is the 
autocorrelation function of the process in question.

Our approach is to first derive the expected zero-crossing rate in 
discrete time (to obtain a cosine formula) and by an appropriate 
limiting argument arrive at the zero-crossing rate in continuous 
time. In particular, we derive analogs of the “cosine formula” and 
“Rice’s formula” for a scaled-time mixture of a Gaussian process, 
for general mixtures of Gaussian processes, and for products of Gaussian 
processes.

Mixtures and products of Gaussian processes are used, in both 
engineering and physics, as models in such diverse areas as: rainfall, 
body weights, crushing processes, diffusive transport in random 
media, and multifractal processes (see [10], [7], and [16]). Hence, 
knowing the zero-crossing rates for such processes is of practical 
value.

To motivate our investigation, we first discuss a formal “orthant 
probability formula” for random processes satisfying mild stationarity 
requirements. Using a formal “cosine formula,” a formal “orthant 
probability formula” is obtained from which we argue that, in general,

\[
E[D_c] = \frac{\kappa}{\pi} \sqrt{\rho''(0)}
\]  

(3)

for sufficiently smooth processes. Moreover, the fact that \( \kappa \) may be 
quite different than one in (3) serves as a warning that Rice’s formula, 
(1), may not be indiscriminately applied in the non-Gaussian case 
(e.g. [3, p. 149], [8, p. 236], and [15, p. 1398]).

A. A Formal Orthant Probability Formula

Let \( \{Z(t)\} \), \( -\infty < t < \infty \), be a stochastic process consisting of 
continuous random variables with mean zero and satisfying the “stationarity” 
requirement

\[
\Pr [Z(t) \geq 0] = \frac{1}{2}
\]

\[
\Pr [Z(t) \geq 0, Z(t+s) \geq 0] = g(|t-s|)
\]

for some function \( g(\cdot) \). For \( t \in [0, 1] \) and for a positive integer \( N > 2 \) 
we define the discrete time process 
\( Z_k \equiv Z((k-1)\Delta) \), \( k = 1, 2, \cdots, N \)

such that

\[
(N-1)\Delta = 1.
\]  

(4)

The interval \([0, 1]\) is now partitioned into \( N-1 \) subintervals each of 
length \( \Delta \) so that \( \{Z_k\} \) is simply \( \{Z(t)\} \) evaluated at the endpoints 
of the subintervals. Define the indicator

\[
d_k = I_{\text{sign change in } Z_k, Z_{k+1}} = I_{Z_k > 0, Z_{k+1} < 0 \lor Z_{k+1} > 0, Z_k < 0}.
\]