Stochastic Differential Equations in Banach Spaces

Decoupling, Delay Equations, and Approximations in Space and Time

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Proefschrift

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1 Introduction

This thesis deals with various aspects of the study of stochastic partial differential equations (abbreviation: SPDEs) driven by Gaussian noise. The approach taken here is functional analytic rather than probabilistic. This means that the SPDE is interpreted as an ordinary stochastic differential equation (abbreviation: SDE) in a Banach space $(X, \|\cdot\|_X)$. Throughout this work the convention is adopted of using the term 'vector-valued' to mean 'taking values in a Banach space', which is justified to a certain extent by the fact that a Banach space is by definition a complete normed vector space.

The subtitle of this thesis indicates that the three different aspects of vector-valued SDEs that are studied here are decoupling, delay equations, and approximations in space and time. *Decoupling* is a concept that plays a role when defining the stochastic integral of a vector-valued stochastic process. *Delay equations* model processes for which the development of the current state depends on the past states. We shall study stochastic delay equations in an SDE framework. *Approximations in space and time* refers to the work presented here on convergence rates for various numerical schemes approximating the solution to an SDE. Among others, we prove optimal pathwise convergence rates for the implicit linear Euler method (a time discretization) and for the spectral Galerkin method (a space discretization).

In the next section an example of a stochastic partial differential equation is given, and it is shown how this equation may be interpreted as a vector-valued stochastic differential equation. This equation is a simple version of the stochastic partial differential equation treated in Chapter 9.

In Section 1.2 the three topics mentioned above will be explained at a level that should be accessible for a general mathematical audience. Along the way, the study of vector-valued SDEs is motivated as being mathematically interesting and perhaps even physically useful. We conclude the introduction with an outline of the thesis.

1.1 A stochastic partial differential equation

In this section we introduce a toy example that could be used to model a population of micro-organisms in a closed tube filled with motionless water. We assume the tube to be one meter long and we suppose $[0,T]$ to be the time interval over which we wish to study the population.

The micro-organisms are not equipped with any means of propulsion, so their movement through the water is determined entirely by diffusion. At a random moment a given micro-organism may divide itself ('birth'), and at a random moment it may die.

Let us assume the concentration of micro-organisms is so high that it is not necessary to keep track of each individual, instead we keep track of the concentration of live micro-organisms. We also consider the problem to be one-dimensional, i.e., we keep track of the average concentration over each cross section of the tube, thus in mg/m. Let $u(t,\xi)$ denote the concentration of live micro-organisms in mg/m at time $t \in [0,T]$ and position $\xi \in [0,1]$ in the tube.

In order to model the randomness of the births and deaths, we will use *space-time white noise*. This is an object denoted by w that assigns a Gaussian random variable to every 'rectangle' $[\xi_1,\xi_2]\times[t_1,t_2]$ in $[0,1]\times[0,T]$. This Gaussian random variable has expectation 0 and variance $(\xi_2-\xi_1)(t_2-t_1)$. If two rectangles are disjoint, then the corresponding Gaussian random variables are independent.

For $(t,\xi) \in (0,T)\times(0,1)$ and Δt, Δx sufficiently small, we assume that the change in population size in the section $[\xi,\xi+\Delta\xi]$ caused by births and deaths over the time interval $[t,t+\Delta t]$ is approximately given by

$$c_1 u(t,\xi) w([t,t+\Delta t]\times[\xi,\xi+\Delta\xi]), \tag{1.1.1}$$

where c_1 is a constant determined by the birth/death rate. The reason we multiply the white noise with the concentration $u(t,\xi)$ is that the number of births and deaths depends on the number of available live micro-organisms. The term (1.1.1) is referred to as a *multiplicative noise term*.

Let $n,m \in \{1,2,\dots\}$ and set $\Delta t := T/n$, $t_j^{(n)} = j\Delta t$, $\Delta\xi := 1/m$ and $\xi_i^{(m)} = i\Delta\xi$, $j=0,\dots,n$, $i=0,\dots,m$. The mass balance for u reads as follows:

$$\begin{aligned}
&\big[u\big(t_{j+1}^{(n)},\xi_i^{(m)}\big) - u\big(t_j^{(n)},\xi_i^{(m)}\big)\big]\Delta\xi \\
&\qquad\approx c_0\big[u\big(t_j^{(n)},\xi_{i-1}^{(m)}\big) + u\big(t_j^{(n)},\xi_{i+1}^{(m)}\big) - 2u\big(t_j^{(n)},\xi_i^{(m)}\big)\big]\frac{\Delta t}{\Delta\xi} \\
&\qquad\quad + c_1 u\big(t_j^{(n)},\xi_i^{(m)}\big) w([t,t+\Delta t]\times[\xi,\xi+\Delta\xi]),
\end{aligned} \tag{1.1.2}$$

where c_0 denotes the diffusion coefficient and $j \in \{0,\dots,n-1\}$, $i \in \{1,\dots,m-1\}$.

The mass balance above describes the behavior in the interior of the tube. At the ends, the behavior is different: as the tube is assumed to be closed, no micro-organisms diffuse in or out of tube. Therefore we impose Neumann boundary conditions at $\xi = 0$ and $\xi = 1$, in other words we assume $\frac{\partial}{\partial \xi} u(t, 0) = \frac{\partial}{\partial \xi} u(t, 1) = 0$. We also assume that the initial state is known, i.e., we assume we are given some $u_0 : [0, 1] \to \mathbb{R}_+$ that describes the concentration of micro-organisms at $t = 0$.

The problem with this model – and with stochastic partial differential equations in general – is that one would want to consider the limiting equation obtained by letting $\Delta t \downarrow 0$ and $\Delta \xi \downarrow 0$. However, a priori it is not clear whether it is possible to give a rigorous meaning to the object

$$\frac{\partial}{\partial t} w(dt, d\xi) := \lim_{\Delta\xi \downarrow 0,\, \Delta t \downarrow 0} \frac{w([\xi, \xi + \Delta\xi] \times [t, t + \Delta t])}{\Delta x \Delta t}.$$

Nevertheless, it is common practice to give the following short-hand notation of the model obtained by taking limits, with the understanding that an interpretation is yet to be given:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t, \xi) = c_0 \dfrac{\partial^2}{\partial \xi^2} u(t, \xi) + c_1 u(t, \xi) \dfrac{\partial}{\partial t} w(t, \xi), \quad (t, \xi) \in (0, T] \times (0, 1); \\ \dfrac{\partial}{\partial \xi} u(t, 0) = \dfrac{\partial}{\partial \xi} u(t, 1) = 0; \\ \quad u(0, \xi) = u_0(\xi). \end{cases} \tag{1.1.3}$$

One way to give a rigorous meaning to a solution to the stochastic partial differential equation (1.1.3) is by taking the functional-analytic approach. For that purpose we assume that for $t \in [0, T]$ the function $U(t) : [0, 1] \to \mathbb{R}$ defined by $U(t)(\xi) = u(t, \xi)$ is an element of some Banach space. In fact, for the SPDE given by (1.1.3) the reflexive Lebesgue spaces $L^p(0, 1)$, $p \in (1, \infty)$, are perfectly suitable. Recall that $L^p(0, 1)$ is the space of all Lebesgue measurable functions f for which $|f|^p$ is integrable. The norm on this space is given by $\|f\|_{L^p}^p = \int_0^1 |f(s)|^p ds$.

Fix $p \in (1, \infty)$. The weak second-order derivative $c_0 \frac{d^2}{d\xi^2}$ can be interpreted as an (unbounded) operator on $L^p(0, 1)$. We shall denote this operator with the letter A. The Neumann boundary in (1.1.3) can be incorporated in the definition of A by setting the the domain of A to be the closure in the Sobolev norm $\| \cdot \|_{H^{2,p}(0,1)}$ of all twice differentiable functions f satisfying $\frac{df}{d\xi}(0) = \frac{df}{d\xi}(1) = 0$.

Let $(h_j)_{j=0}^\infty$ be an orthonormal basis for $L^2(0, 1)$ and let $(W_j)_{j=1}^\infty$ be a sequence of independent (real-valued) Brownian motions. We define $W_{L^2} := \sum_{j=1}^\infty W_j(t) h_j$; this sum does not converge in $L^2(0, 1)$ but may be interpreted in the sense of distributions. Space-time white noise can be modeled as follows:

$$w([t_1, t_2] \times [\xi_1, \xi_2]) = \sum_{j=0}^\infty (W_j(t_2) - W_j(t_1)) \int_{\xi_1}^{\xi_2} h_j(\xi) d\xi,$$

where $0 \le t_1 < t_2 \le T$ and $0 \le \xi_1 < \xi_2 \le 1$.

Provided $u_0 \in L^p(0,1)$ we may rewrite (1.1.3) as an (ordinary) stochastic differential equation set in the Banach space $L^p(0,1)$:

$$\begin{cases} dU(t) = AU(t)\,dt + c_1 U(t) dW_{L^2}(t) & t \in [0,T]; \\ U(0) = f. \end{cases} \tag{1.1.4}$$

Note however that this is still only a formal representation of the process we are interested in. After all, a Brownian motion is not differentiable with respect to t. Moreover, even if (1.1.4) were to be interpreted as an integral equation it is not clear how to interpret $\int_0^t U(s) dW_{L^2}(s)$; recall that W_{L^2} consists of an infinite sum of Brownian motions which does not converge in $L^p(0,1)$ (not even if $p = 2$).

However, we are not far from an interpretation for a solution to (1.1.4): semigroup theory provides us with a family of operators $(e^{tA})_{t\geq 0} \subset \mathscr{L}(L^p(0,1))$ with the property (as already suggested by the notation) that for every $f \in L^p(0,1)$ one has $\frac{d}{dt} e^{tA}(t) f = Ae^{tA}(t)f$. In fact, for $h \in L^1(0,T; L^p(0,1))$ the function $u : [0,T] \to L^p(0,1)$ defined by:

$$u(t) = e^{tA} f + \int_0^t e^{(t-s)A} h(s) ds$$

satisfies $u(0) = f$ and:

$$\frac{d}{dt} u(t) = Au(t) + h(t).$$

Inspired by this we seek an interpretation of the following formula as a means to define a solution to (1.1.4):

$$U(t) = e^{tA} u_0 + c_1 \int_0^t e^{(t-s)A} U(s) dW_{L^2}(s) \quad \text{a.s. for all } t \in [0,T]. \tag{1.1.5}$$

The semigroup $(e^{tA})_{t\geq 0}$ arising from a diffusion process as considered here has smoothing properties that make the noise W_{L^2} 'well-behaved' in $L^p(0,1)$. Thus it is possible give a rigorous meaning to the vector-valued stochastic integral in (1.1.5) as an $L^p(0,1)$-valued random variable. By a fixed point argument one can then prove the existence of an $L^p(0,1)$-valued process U that indeed satisfies (1.1.5).

In short, by the functional-analytic methods of (analytic) semigroup theory, stochastic integration theory and a fixed point theorem, we have given an interpretation to a solution to (1.1.3) as a process satisfying (1.1.5).

As a minor remark concerning the accuracy of this model we note that it is not clear a priori whether $u(t,\xi) \geq 0$ for all $(t,\xi) \in [0,T] \times [0,1]$ provided $u_0 \geq 0$; which of course would be desirable. Such positivity questions for SDEs are also a field of study, albeit not one covered by this thesis.

1.2 The thesis in a nutshell

Decoupling

When one takes the approach of interpreting an SPDE as an SDE set in a Banach space X, it is essential to have a workable definition for the stochastic integral of an adapted X-valued stochastic process. The noise in the SPDEs we study is always Gaussian, and thus we consider stochastic integrals with respect to a Brownian motion.

The orthogonal decompositions that are used in the Hilbert space case to extend one-dimensional stochastic integration theory to infinite-dimensional integration theory fail for general Banach spaces. However, one can define the integral of an X-valued *function* on some interval $[0, T]$ with respect to a Brownian motion W by means of an Itô isometry in terms of the γ-radonifying norm on $L^2(0, T)$. If X is a Hilbert space, then the γ-radonifying norm is equivalent to the $L^2(0, T; X)$-norm.

The definition of the stochastic integral of an adapted X-valued *stochastic process* can be obtained subsequently by a *decoupling* argument: suppose there exist constants c_p and C_p such that if W is a one-dimensional $(\mathscr{F}_t)_{t\geq 0}$-adapted Brownian motion and Φ is a $(\mathscr{F}_t)_{t\geq 0}$-adapted, X-valued stochastic process on $[0, T]$, one has, for all $p \in (0, \infty)$:

$$c_p^{-1}\Big(\mathbb{E}\Big\| \int_0^T \Phi d\tilde{W}\Big\|_X^p\Big)^{\frac{1}{p}} \leq \Big(\mathbb{E}\sup_{0\leq t\leq T}\Big\| \int_0^t \Phi dW\Big\|_X^p\Big)^{\frac{1}{p}} \leq C_p\Big(\mathbb{E}\Big\| \int_0^T \Phi d\tilde{W}\Big\|_X^p\Big)^{\frac{1}{p}}, \tag{1.2.1}$$

where $\tilde{W}$ is a Brownian motion independent of Φ. By independence, the integral of Φ with respect to $\tilde{W}$ can be treated as if Φ were deterministic. This, in combination with the two-sided estimate (1.2.1), allows one to define the integral of Φ with respect to W as well.

Not every Banach space X allows for constants c_p and C_p such that (1.2.1) holds for all X-valued processes. It has been shown, first by Garling for processes adapted to the filtration generated by W, and later by van Neerven, Veraar, and Weis for general processes, that (1.2.1) holds for some (and then all) $p \in (1, \infty)$ if and only if X is a so-called UMD Banach space (see [47], [108]).

However, for the definition of the stochastic integral of a vector-valued stochastic process it suffices that the second estimate in (1.2.1) holds. In Chapter 3 below we study a Banach space property introduced by Kwapień and Woyczyński [87]. By the extrapolation methods of Burkholder and Gundy [20] we prove so-called p-independence of this property, and then show that it implies the second estimate in (1.2.1) for $p \in (0, \infty)$. The property we study is satisfied Banach spaces that are not UMD spaces, e.g. L^1.

Incidentally, the decoupling argument we use also allows us to obtain the two-sided estimate (1.2.1) for $p \in (0, 1]$ in the case that X is a UMD Banach space.

Stochastic delay equations

In population dynamics, the number of people born in a certain year n depends heavily on the number of people born in the period $[n-40, n-20]$ (assuming people generally have children between 20 and 40 years of age). To model such a process, i.e., a process for which past states influence the development of the current state, one makes use of *delay equations*. If in addition one wishes to account for a certain level of uncertainty in this process, then one may add a stochastic term. This leads to a *stochastic delay equation*.

The functional-analytic approach to delay equations is to take the state space to be a function space defined over what is considered to be the maximal period of influence. In that case a stochastic delay equation becomes a vector-valued SDE – precisely the object of study in this thesis.

This interpretation of a delay equation is easily demonstrated by means of an example from population dynamics. For $t \geq 0$ let $u(t)$ denote the size of a population at time t, measured in years. Arguably one may assume that the size of a human population depends on its size over the past 60 years, but not on its size before that. Thus the function $u_t : [0, 60] \to \mathbb{R}$ defined by $u_t(s) := u(t-s)$ contains all the information needed to determine $u(t)$ – except perhaps some external factors that are independent of u. The functional-analytic approach to delay equations is to study the development of the *functions* $u_t \in \mathscr{E}([0, 60]; \mathbb{R})$ instead of studying the development of $u(t) \in \mathbb{R}$. Here $\mathscr{E}([0, 60]; \mathbb{R})$ denotes some Banach function space over $[0, 60]$, and we assume that u is real-valued instead of integer-valued because this is mathematically easier to model. In population dynamics it makes sense to take $\mathscr{E}([0, 60]; \mathbb{R}) = L^1(0, 60; \mathbb{R})$.

In Chapter 4 we study SDEs arising from stochastic delay equations. We assume the delay equations to be set in a type 2 UMD Banach space X, e.g. X may be a Hilbert space, ℓ^q, or L^q, for $q \geq 2$, and we consider multiplicative noise. Following the approach Bátkai and Piazzera take in the deterministic case, we set our SDE in the function space $\mathscr{E}([0, T]; X) := L^p(0, T; X) \times X$ for $p \in [1, \infty)$ (see [5]). We prove that these delay equations allow for a unique continuous solution.

Similar results for delay equations have been obtained by Crewe [31], Liu [91], Riedle [122], and Taniguchi, Liu, and Truman [127]. The novelty of our work lies in its generality: we allow for X to be infinite-dimensional, for A to be the generator of a – not necessarily analytic – semigroup, and we allow for multiplicative noise. This combination has not been considered before.

Approximations of SPDEs

Generally speaking, functional-analytic proofs of the existence of a unique solution to a (stochastic) differential equation provide little information on the behavior of the solution. If one wishes to gain insight into the behavior, one possibility is to approximate the solution numerically. Of course, in that case it is essential to know whether the numerical approximation converges to the actual solution.

There are three aspects of developing a numerical scheme for a stochastic partial differential equation: time discretization, space discretization, and approximation of the noise. We prove optimal convergence rates for certain time and space discretization schemes for the following type of SDEs with non-linear deterministic term, and non-linear multiplicative noise:

$$\begin{cases} dU(t) = AU(t)\,dt + F(t,U(t))\,dt + G(t,U(t))\,dW_H(t); \quad t \in [0,T], \\ U(0) = x_0. \end{cases} \tag{1.2.2}$$

Here A is the generator of an analytic C_0-semigroup on a UMD Banach space X. For operators generating an analytic semigroup it is possible[1] to define fractional powers $(-A)^\theta$, $\theta \in \mathbb{R}$, and thus also $D((-A)^\theta)$. A typical example of an operator A that generates an analytic C_0-semigroup is a second-order elliptic differential operator with Dirichlet or Neumann boundary conditions on $L^p(D)$, where $D \subset \mathbb{R}^d$; more specifically, one may take A to be the Laplace operator $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$. In this case, roughly speaking the space $D((-A)^\theta)$ corresponds to the Sobolev space $H^{2\theta,p}(D)$.

The noise process W_H is a cylindrical Brownian motion in a Hilbert space H, by which we mean a process that is formally given by $W_H(t) = \sum_{j=1}^\infty W_j(t)h_j$, where $(h_j)_{j=1}^\infty$ is an orthonormal basis for H and $(W_j)_{j=1}^\infty$ is a sequence of independent real-valued Brownian motions.

The functions $F : [0,T] \times X \to D((-A)^{\theta_F})$ and $G : [0,T] \times X \to \mathscr{L}(H, D((-A)^{\theta_G}))$ satisfy appropriate global Lipschitz conditions. We assume that $\theta_F > -1 + (\frac{1}{2} - \frac{1}{\tau})$ and $\theta_G > -\frac{1}{2}$, where $\tau \in [1,2]$ is the type of the Banach space. These conditions ensure that a solution to (1.2.2) exists.

In terms of time discretizations we first prove convergence of various splitting schemes and use this to obtain convergence of the implicit-linear Euler scheme. In terms of space approximations we first prove a perturbation result for (1.2.2) and then use this to prove convergence of the Yosida approximation and – in the Hilbert space case – of certain Galerkin and finite element schemes.

From the recent review paper of Jentzen and Kloeden [70] one may conclude that stochastic numerical analysis is an active field of research at the moment. The splitting scheme has been studied in the Hilbert space setting for stochastic second order partial differential equations by Gyöngy and Krylov [56]. There are numerous articles considering the Euler scheme, e.g. [54, 62, 82, 120]. Our main contribution is that we prove pathwise convergence results in the case of multiplicative noise. This allows one to obtain convergence also for the case where F and G are merely locally Lipschitz.

Concerning perturbations of stochastic differential equations, some results may be found in the work of Brzeźniak [13] and in the recent work of Kunze

[1] To be precise, for $\lambda \geq 0$ large enough we can define $(\lambda - A)^\theta$. If A generates a bounded semigroup, one may take $\lambda = 0$.

and van Neerven [85]. However, the results of Brzeźniak cannot be used to prove convergence of numerical schemes, and the work of Kunze and van Neerven would not provide convergence rates.

There are also numerous references for convergence of space approximations such as Galerkin and finite element schemes. For pathwise convergence of the Galerkin scheme for equations with additive noise, we refer to work by Jentzen [68] and Kloeden, Lord, Neuenkirch and Shardlow [81]. For pointwise convergence of the Galerkin scheme for equations with multiplicative noise, see e.g. the work by Hausenblas [62] and Yan [133]. Again our main contribution is that we prove pathwise convergence results in the case of multiplicative noise.

The splitting scheme

The term *splitting scheme* refers to the idea of splitting (1.2.2) into two parts and solving each of them alternately over small intervals. In our case we wish to separate the linear deterministic part from the non-linear (stochastic) part. Fixing $T > 0$ and $n \in \mathbb{N}$, we define $U_0^{(n)}(0) := x_0$ and wish to successively solve, for $j = 1, \dots, n$, the problem

$$\begin{cases} dU_j^{(n)}(t) = F(t, U_j^{(n)}(t))\,dt + G(t, U_j^{(n)}(t))\,dW_H(t), \quad t \in [t_{j-1}^{(n)}, t_j^{(n)}], \\ U_j^{(n)}(t_{j-1}^{(n)}) = S(\tfrac{T}{n})U_{j-1}^{(n)}(t_{j-1}^{(n)}). \end{cases} \tag{1.2.3}$$

Here $t_j^{(n)} := \frac{jT}{n}$. For non-negative fractional indices θ_F and θ_G this scheme is well-defined. However, if either of the fractional indices θ_F and θ_G is negative, then it is not clear whether a solution to (1.2.3) exists. In order to deal with negative indices we thus consider not only the 'classical' scheme given by (1.2.3) but also a 'modified' splitting scheme, as will be explained in Chapter 6.

In Sections 6.1 and 6.2 we prove that the process obtained by either of the splitting schemes converges to the solution U of (1.2.2). To be precise, we give convergence rates for convergence in $L^\infty(0,T;L^p(\Omega,X))$ for p arbitrarily large, provided $x_0 \in L^p(\Omega; D((-A)^\eta))$ for $\eta > 0$ sufficiently large. By a Kolmogorov argument, this allows us to obtain pathwise convergence rates in a discrete Hölder norm (see Theorem 6.1). In particular, we prove that if $\delta, \eta > 0$ and $p \in [2, \infty)$ are such that:

$$\delta + \tfrac{1}{p} < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F,\ \tfrac{1}{2} + \theta_G,\ \eta,\ 1\},$$

where τ is the type of the Banach space, and $x_0 \in L^p(\Omega; D((-A)^\eta))$, then there is a constant C, independent of x_0, such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E} \sup_{1 \le j \le n} \|U(t_j^{(n)}) - U_j^{(n)}(t_j^{(n)})\|_X^p\big)^{\frac{1}{p}} \le Cn^{-\delta}(1 + \|x_0\|_{L^p(\Omega; D((-A)^\eta))}).$$

We also briefly consider a splitting scheme for linear stochastic differential equations with additive noise; i.e., equation (1.2.2) with $F \equiv 0$ and $G \equiv g \in \gamma(H, D((-A)^{\theta_G}))$. In this case the splitting scheme converges for any

Banach space, provided A is analytic. If A is not analytic, the scheme still converges provided X has type 2 (for this case we do not obtain convergence rates). We complete Chapter 6 by giving an example of a linear stochastic differential equation for which the splitting scheme does not converge.

The implicit-linear Euler scheme

In order to define the implicit-linear Euler scheme we fix $T > 0$ and $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. We set $V_0^{(n)} := x_0$ and, for $j = 1, \dots, n$, we define the random variables $V_j^{(n)}$ by

$$V_j^{(n)} = (1 - \tfrac{T}{n}A)^{-1}\big[V_{j-1}^{(n)} + \tfrac{T}{n}F\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big) + G\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big)\Delta W_j^{(n)}\big].$$

Recall that $t_j^{(n)} = \frac{jT}{n}$ and, formally, $\Delta W_j^{(n)} = W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)})$.

Chapter 7 provides convergence rates for the convergence of $(V_j^{(n)})_{j=0}^n$ to $(U(t_j^{(n)}))_{j=0}^n$, where U is the solution to (1.2.2). For this result we need the additional assumption that X is a UMD space with property (α) (this property is satisfied if X is a Hilbert space or an L^p space, $p \in [1, \infty)$). The convergence is obtained in $L^\infty(0, T; L^p(\Omega, X))$ for p arbitrarily large. Once again a Kolmogorov type argument allows us to obtain pathwise convergence rates in a discrete Hölder norm (see Theorem 7.1). In particular, we prove that if $\delta, \eta > 0$ and $p \in (2, \infty)$ are such that:

$$\delta + \tfrac{1}{p} < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + (\theta_F \wedge 0),\ \tfrac{1}{2} + (\theta_G \wedge 0),\ \eta\},$$

and $x_0 \in L^p(\Omega; D((-A)^\eta))$, then there is a constant C, independent of x_0, such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E} \sup_{1\le j\le n} \|U(t_j^{(n)}) - V_j^{(n)}\|_X^p\big)^{\frac{1}{p}} \le Cn^{-\delta}(1 + \|x_0\|_{L^p(\Omega, D((-A)^\eta))}).$$

We see that, contrary to the splitting scheme, the convergence rate does not improve as θ_F and θ_G increase above 0. However, these rates are optimal due to the way the noise discretized.

Localization

For the convergence results presented in Chapters 6 and 7 we need global Lipschitz assumptions on F and G. In Chapter 8 we demonstrate how the (pathwise) convergence results obtained in Chapters 6 and 7 can be extended to the case that F and G satisfy only *local* Lipschitz conditions, presuming they satisfy linear growth conditions. In order to do so we need an extra regularity result on the splitting scheme, which is presented in Appendix A.5.

A perturbation result

In Chapter 10 we consider the effect of perturbations of A on the solution to (1.2.2). With applications to numerical approximations in mind, we assume the perturbed equation to be set in a (possibly finite dimensional) closed subspace X_0 of X. We assume that there exists a bounded projection $P_0 : X \to X_0$ such that $P_0(X) = X_0$. Let i_{X_0} be the canonical embedding of X_0 in X and A_0 be the generator of an analytic C_0-semigroup S_0 on X_0. In the setting of numerical approximations, A_0 would be the restriction in a suitable sense of A to the finite dimensional space X_0.

The perturbed equation we consider is the following:

$$\begin{cases} dU^{(0)}(t) = A_0U^{(0)}(t)\,dt + P_0F(t,U^{(0)}(t))\,dt + P_0G(t,U^{(0)}(t))\,dW_H(t), \ t > 0; \\ U^{(0)}(0) = P_0x_0. \end{cases} \tag{SDE$_0$}$$

In Chapter 10, Theorem 10.1 we prove that if

$$D_\delta(A, A_0) := \|A^{-1} - i_{X_0}A_0^{-1}P_0\|_{\mathscr{L}(D((-A)^{\delta-1}),X)} < \infty$$

for some

$$0 \le \delta < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F,\ \tfrac{1}{2} + \theta_G\},$$

and $x_0 \in L^p(\Omega; D((-A)^\delta))$, then there exists a (unique) solution $U^{(0)}$ to (SDE$_0$) and for $p \in (2,\infty)$ satisfying $\frac{1}{p} < \frac{1}{2} + \theta_G - \delta$ we have:

$$\big(\mathbb{E}\|U - i_{X_0}U^{(0)}\|^p_{C([0,T];X)}\big)^{\frac{1}{p}} \lesssim D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega;D((-A)^\delta))}).$$

In order to prove this perturbation result we develop new methods for determining the regularity of stochastic convolutions. These results can be found in Appendix A.2.

Space approximations

Our first application of the perturbation result concerns Yosida approximations. Under the assumption that θ_F and θ_G are non-negative, we prove convergence of $U^{(n)}$ against U, where $U^{(n)}$, $n \in \mathbb{N}$, is the solution to (1.2.2) with A replaced by its n^{th} Yosida approximation $A_n := nAR(n : A)$. More precisely, for $\eta \in [0,1]$ and $p \in (2,\infty)$ such that

$$\eta < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G\}$$

we have, assuming $x_0 \in L^p(\Omega; D((-A)^\eta))$:

$$\big(\mathbb{E}\|U - U^{(n)}\|^p_{C([0,T];X)}\big)^{\frac{1}{p}} \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;D((-A)^\eta))}).$$

If (1.2.2) is set in a Hilbert space $\mathscr{H}$, i.e., $X = \mathscr{H}$, then one can derive pathwise convergence of certain Galerkin and finite element schemes from the perturbation result. In Section 11.2 we provide pathwise convergence rates for the Galerkin scheme in the case that A is a self-adjoint operator generating an eventually compact semigroup on $\mathscr{H}$. Concerning finite elements, we provide pathwise convergence rates for the case that $A : H^{2,2}(D) \to L^2(D)$ is a second-order elliptic operator.

Conclusion and ideas for future work

The work on decoupling provides new insights on the relation between the geometry of a Banach space X and the possibility to define the stochastic integral of an X-valued process. Concerning delay equations, existence and uniqueness of a quite general class of stochastic vector-valued delay equations has been established. In terms of approximations, the main achievement lies in finding pathwise convergence rates for SDEs with multiplicative noise.

There are still various interesting open problems. For example, for $\theta_G \geq 0$ and $\theta_F \geq \frac{1}{\tau} - 1$ in (1.2.2) we obtain convergence rate $n^{-\frac{1}{2}+\varepsilon}$ for the Euler scheme, where $\varepsilon > 0$ is arbitrarily small. The convergence rate $n^{-\frac{1}{2}}$ is known to be critical. Our hope is that recent results on 'stochastic maximal regularity' (see [105, 106]) can be used to obtain the critical convergence rate for the case that $X = L^q$ and A has a bounded H^∞-calculus.

From an implementation point of view, a convergence rate of $n^{-\frac{1}{2}}$ is not very satisfactory. Unfortunately, this convergence rate is critical for the type of noise discretizations we study. However, recently, Jentzen and Kloeden (see [69, 71]) have developed new techniques for noise discretization that produce better convergence rates in the Hilbert space case. It would be interesting to investigate these techniques in the Banach space setting.

There are also some interesting open problems concerning the space approximations. For example, it should be possible to prove convergence of the Galerkin scheme for the case that $X = L^q$, $q \in (1, \infty)$ (we now consider only the Hilbert space case). This would allow us to apply the Galerkin scheme to the example treated in Chapter 9.

Finally, an obvious remaining task is to combine the space and time discretizations, thereby obtaining a scheme that could in fact be implemented and tested.

1.3 Outline of the thesis

Chapter 2 contains the preliminaries on probability and stochastic analysis in Banach spaces that will be used throughout the thesis. In Chapter 3 the results on decoupling are presented, which were obtained in collaboration with Mark Veraar of the Delft University of Technology [29]. Chapter 4 contains the work on delay equations done in collaboration with Mariusz Góraski of the University of Łódź [24].

The remaining Chapters 5-11 contain results on approximation of solutions to (1.2.2). We begin with an introductory chapter which contains the standing assumptions on (1.2.2) and the relevant results concerning the existence of a solution to (1.2.2). In terms of time approximations we first consider splitting schemes, see Chapter 6. These schemes are used in Chapter 7 to study convergence of a general class of time discretizations that includes the implicit-linear Euler scheme. In both chapters we assume that the non-linear functions F and G

satisfy global Lipschitz conditions. However, as we obtain pathwise convergence estimates, it is possible to extend the results to the case that F and G are locally Lipschitz. This will be demonstrated in Chapter 8. To conclude the results on time discretizations, in Chapter 9 we demonstrate how the results apply to a parabolic partial differential equation with space-time white noise.

The results presented in Chapters 6-9 are based on joint work with Jan van Neerven of the Delft University of Technology. Most of the material is based on [27], except for Section 6.4, which is based on [28].

Concerning space approximations, we begin in Chapter 10 by studying the effect of perturbations of A on the solution to (1.2.2). In Chapter 11 we demonstrate how the results of Chapter 10 can be used to obtain pathwise estimates of space approximations schemes in the Hilbert-space case. Finally, Appendices A.1 and A.2 contain some technical lemmas that are used throughout Part III, but would disturb the flow of the text if they were to be placed elsewhere.

The results of Chapters 10 and 11 are based on joint work with Erika Hausenblas of the Montana University of Leoben, see [25] and [26].

2

Preliminaries

2.1 Some conventions

Throughout this thesis, $\mathbb{N} = \{1, 2, \ldots\}$.

We write $A \lesssim B$ if there exists a constant C, such that $A \leq CB$. Naturally $A \gtrsim B$ means $B \lesssim A$ and $A \eqsim B$ means $A \lesssim B$ *and* $B \lesssim A$. If we wish to make it explicit that the implied constant depends on some parameter p, we write $A \lesssim_p B$.

For Banach spaces X and Y we write $X \simeq Y$ to indicate that X and Y are isomorphic as Banach spaces. For more notational issues, see page 225.

2.2 Geometric Banach space properties

Most results in this thesis are proven under additional assumptions on the geometry of the Banach space involved. More specifically, we need the concept of type and cotype of a Banach space, the UMD property and property (α). In this section we give the definition of these properties and some important examples of Banach spaces satisfying them.

Definition 2.1. A *Rademacher sequence* $(r_j)_{j=1}^{\infty}$ is a sequence of independent random variables satisfying $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$.

Definition 2.2. A Banach space X is said to have *type* p, $p \in [1, 2]$, if there exists a constant C such that for all finite sequences $(x_j)_{j=1}^{n} \subset X$ one has:

$$\Big(\mathbb{E}\Big\| \sum_{j=1}^{n} r_j x_j \Big\|^p\Big)^{\frac{1}{p}} \leq C\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{1}{p}}.$$

Here $(r_j)_{j=1}^{n}$ is a Rademacher sequence. The smallest constant C for which the above holds is denoted by $\mathscr{T}_p(X)$.

A Banach space X is said to have *cotype* q, $q \in [2, \infty]$, if there exists a constant C such that for all finite sequences $(x_j)_{j=1}^n \subset X$ one has:

$$\Big(\sum_{j=1}^n \|x_j\|^q\Big)^{\frac{1}{q}} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j x_j\Big\|^q\Big)^{\frac{1}{q}},$$

with an obvious modification if $q = \infty$. The smallest constant C for which the above holds is denoted by $\mathscr{C}_q(X)$.

Every Banach space has type 1 and cotype ∞. Therefore we say that a Banach space has *non-trivial type* if it has type $p \in (1, 2]$, and *non-trivial cotype* if it has co-type $q \in [2, \infty)$. If a Banach space X has type $p' \in [1, 2]$, then it has type p for all $p \in [1, p']$. Similarly, if it has cotype $q' \in [2, \infty]$ then it has cotype q for all $q \in [q', \infty]$. For $p \in [1, \infty)$ the L^p-spaces have type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$. Hilbert spaces have type 2 and cotype 2 – in fact, any Banach space that has type 2 and cotype 2 is isomorphic to a Hilbert space. For a proof of this non-trivial fact, and for more information concerning type and cotype, we refer to [2, Section 6.2 and onwards], [41] and [117].

Definition 2.3. A Banach space X is said to be a UMD *space* (or to satisfy the UMD *property*) if for all $p \in (1, \infty)$ there exists a constant C_p such that for every finite X-valued martingale difference sequence $(d_j)_{j=1}^n \subset L^p(\Omega; X)$, and every $(\varepsilon_j)_{j=1}^n \subset \{0, 1\}^n$ one has:

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|_X^p\Big)^{\frac{1}{p}} \le C_p\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|_X^p\Big)^{\frac{1}{p}}. \tag{2.2.1}$$

Here UMD stands for *unconditional martingale difference sequences.* The least constant for which the above holds for some fixed $p \in (1, \infty)$ will be denoted by $\beta_p(X)$.

The class of UMD Banach spaces has been introduced by Burkholder in [17] (see also [19] for an overview). An argument presented in [95] and attributed to Gilles Pisier implies that in order to prove that a Banach space X is a UMD space, it suffices to prove that (2.2.1) holds for *some* $p \in (1, \infty)$. An alternative proof for this *p-independence* of the UMD property was given by Burkholder [17].

The UMD property has proven to be useful when extending classical harmonic analysis [10, 45, 134] and stochastic integration [96, 108] to the vector-valued situation. More precisely, the UMD property is used as a *decoupling inequality* to define the stochastic integral of a vector-valued stochastic process. Details on this and other decoupling inequalities that allow for the definition of a stochastic integral will be presented in Chapter 3.

Examples of UMD spaces are Hilbert spaces and the spaces $L^p(\mu)$ with $1 < p < \infty$ and μ a σ-finite measure. We shall frequently use the following well-known facts:

(i) Banach spaces isomorphic to a closed subspace of a UMD space are UMD;
(ii) If X is UMD, $1 < p < \infty$ and μ is a σ-finite measure, then $L^p(\mu; X)$ is UMD;
(iii) Every UMD space is K-convex. Hence, by a theorem of Pisier [118], every UMD space has non-trivial type.
(iv) Every UMD space is (super-)reflexive. Hence a UMD space cannot contain a subspace isomorphic to c_0.

The following property was introduces by Pisier in [119]:

Definition 2.4. A Banach space X is said to satisfy *property* (α) (or *Pisier's property*) if there exists a constant C such that for all finite $(x_{j,k})_{j,k=1}^n \subset X$, and $(r''_{j,k})_{j,k=1}^n$, $(r''_j)_{j=1}^n$, and $(r'_j)_{j=1}^n$ independent Rademacher sequences one has:

$$C^{-1}\Big(\mathbb{E}\Big\|\sum_{j,k=1}^n r_{j,k}x_{j,k}\Big\|^2\Big)^{\frac{1}{2}} \le \Big(\mathbb{E}\Big\|\sum_{j,k=1}^n r'_j r''_k x_{j,k}\Big\|^2\Big)^{\frac{1}{2}} \le C\Big(\mathbb{E}\Big\|\sum_{j,k=1}^n r_{j,k}x_{j,k}\Big\|^2\Big)^{\frac{1}{2}}.$$

For an extensive discussion of this property and its use in the theory of stochastic evolution equations we refer to [77, 112]. Examples of Banach spaces with property (α) are the Hilbert spaces and the spaces $L^p(\mu)$ with $1 \le p < \infty$ and μ σ-finite. In this thesis, the relevance of property (α) lies in isomorphism (2.3.6) below.

As a final remark we mention that UMD and property (α) are independent Banach space properties: L^1 is not a UMD Banach space unless it is finite-dimensional. On the other hand, the Schatten classes $\mathscr{S}^p$ have UMD for $p \in (1, \infty)$, but fail to have property (α) unless $p = 2$.

2.3 γ-Radonifying operators

The so-called γ-radonifying norm forms the Banach space analogue of the L^2-norm in the Itô isomorphism for vector-valued stochastic integrals.

Let $(\gamma_j)_{j\ge1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathbb{P})$, let $\mathscr{H}$ be a real Hilbert space (later we shall take $\mathscr{H} = L^2(0, T; H)$, where H is another real Hilbert space) and X a real Banach space. A bounded operator R from $\mathscr{H}$ to X is called γ*-summing* if

$$\|R\|_{\gamma_\infty(\mathscr{H},X)} := \sup_h \Big(\mathbb{E}\Big\|\sum_{j=1}^k \gamma_j R h_j\Big\|_X^2\Big)^{\frac{1}{2}},$$

is finite, where the supremum is taken over all finite orthonormal systems $h = (h_j)_{j=1}^k$ in $\mathscr{H}$. It can be shown that $\|\cdot\|_{\gamma_\infty(\mathscr{H},X)}$ is indeed a norm which turns the space of γ-summing operators into a Banach space. This norm is clearly stronger than the uniform operator norm.

Every finite rank operator R from $\mathscr{H}$ to X can be represented in the form $\sum_{j=1}^k h_j \otimes x_j$, where $(h_j)_{j=1}^k$ is an orthonormal sequence in $\mathscr{H}$ and $(x_j)_{j=1}^k$ is a sequence in X. Note that we use the notation $h \otimes x$ for the rank one operator from H to X given by $(h \otimes x)(h') = [h, h']_H x$ for $h' \in H$. For such an operator we have:

$$\Big\| \sum_{j=1}^{k} h_j \otimes x_j \Big\|_{\gamma_\infty(\mathscr{H},X)} = \Big(\mathbb{E} \Big\| \sum_{j=1}^{k} \gamma_j x_j \Big\|^2 \Big)^{\frac{1}{2}}.$$

A bounded operator R from $\mathscr{H}$ to X is *γ-radonifying* if R belongs to the completion of the finite rank operators with respect to the $\gamma^\infty(\mathscr{H}, X)$-norm. We denote the space of γ-radonifying operators from $\mathscr{H}$ to X by $\gamma(H, X)$. For notational convenience we write, for $R \in \gamma(\mathscr{H}, X)$, $\|R\|_{\gamma(\mathscr{H},X)} := \|R\|_{\gamma^\infty(\mathscr{H},X)}$. It follows from a celebrated result of Kwapień and Hoffmann-Jørgensen [67, 86] that if X does not contain a closed subspace isomorphic to c_0 then $\gamma(\mathscr{H}, X) = \gamma_\infty(\mathscr{H}, X)$.

We refer to [102] for a survey on γ-summing and γ-radonifying operators. Some important observations are listed below.

Suppose $\mathscr{H}$ is separable with orthonormal basis $(h_j)_{j\geq 1}$. If $R \in \gamma(\mathscr{H}, X)$ then sum $\sum_{j\geq 1} \gamma_j R h_j$ converges in $L^2(\Omega; X)$, defining a centered X-valued Gaussian random variable. Its distribution μ is a centered Gaussian Radon measure on X whose covariance operator equals RR^*. We will refer to μ as the Gaussian measure *associated with* R. In this situation we have

$$\|R\|_{\gamma(\mathscr{H},X)} = \Big(\mathbb{E} \Big\| \sum_{j\geq 1} \gamma_j R h_j \Big\|^2 \Big)^{\frac{1}{2}}.$$

The general case may be reduced to the separable case by observing that for any $R \in \gamma(\mathscr{H}, X)$ there exists a separable closed subspace $\mathscr{H}_R$ of $\mathscr{H}$ such that R vanishes on the orthogonal complement $\mathscr{H}_R^\perp$.

In the reverse direction, if χ is a centered X-valued Gaussian random variable with reproducing kernel Hilbert space $\mathscr{H}$, then $\mathscr{H}$ is separable, the natural inclusion mapping $i : \mathscr{H} \hookrightarrow X$ is γ-radonifying, and we have

$$\|i\|^2_{\gamma(\mathscr{H},X)} = \mathbb{E}\|\chi\|^2.$$

Since convergence in $\gamma(\mathscr{H}, X)$ implies convergence in $\mathscr{L}(\mathscr{H}, X)$, every operator $R \in \gamma(\mathscr{H}, X)$, being the operator norm limit of a sequence of finite rank operators from $\mathscr{H}$ to X, is compact. Moreover, if X is a Hilbert space, then

$$\gamma(H, X) = \mathscr{L}_2(H, X), \tag{2.3.1}$$

where $\mathscr{L}_2(H, X)$ is the space of Hilbert-Schmidt operators from H to X.

The space $\gamma(\mathscr{H}, X)$ forms an operator ideal in $\mathscr{L}(\mathscr{H}, X)$: if $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces and X_1 and X_2 are Banach spaces, then for all $V \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, $R \in \gamma(\mathscr{H}_1, X_1)$, and $U \in \mathscr{L}(X_1, X_2)$ we have $URV \in \gamma(\mathscr{H}_2, X_2)$ and

$$\|URV\|_{\gamma(\mathscr{H}_2,X_2)} \leq \|U\|_{\mathscr{L}(X_1,X_2)} \, \|R\|_{\gamma(\mathscr{H}_1,X_1)} \|V\|_{\mathscr{L}(\mathscr{H}_2,\mathscr{H}_1)}. \tag{2.3.2}$$

Another useful property is the γ-Fubini isomorphism: By [108, Proposition 2.6], for any $p \in [1, \infty)$ the mapping $U : L^p(R; \gamma(\mathscr{H}, X)) \to \mathscr{L}(\mathscr{H}, L^p(R; X))$ defined by

$$((Uf)h)(r) := f(r)h, \quad r \in R, h \in \mathscr{H},$$

defines an isomorphism of Banach spaces

$$L^p(R; \gamma(\mathscr{H}, X)) \simeq \gamma(\mathscr{H}, L^p(R; X)). \tag{2.3.3}$$

Let $(S, \mathscr{S}, \mu)$ be a measure space and H a Hilbert space. In the special cases where $\mathscr{H} = L^2(S)$ or $\mathscr{H} = L^2(S; H)$ we write

$$\gamma(L^2(S); X) = \gamma(S; X), \qquad \gamma(L^2(S; H), X) = \gamma(S; H, X).$$

In particular, for $S = [0, T]$, $T > 0$, we write

$$\gamma(L^2(0, T), X) = \gamma(0, T; X), \qquad \gamma(L^2(0, T; H), X) = \gamma(0, T; H, X).$$

By covariance domination for Gaussian random variables we have, for $S' \subset S$ measurable and $g \in L^\infty(S)$ (see also [110, Corollary 4.4]):

$$\|(g\Phi)|_{S'}\|_{\gamma(S'; H, X)} \le \|g|_{S'}\|_{L^\infty(S')} \|\Phi\|_{\gamma(S; H, X)}. \tag{2.3.4}$$

If X is a type 2 Banach space, we have the following embedding (see [110]):

$$L^2(0, T; \gamma(H, X)) \hookrightarrow \gamma(0, T; H, X), \tag{2.3.5}$$

which is given by $f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x$, for $f \in L^2(S)$, $h \in H$, and $x \in X$.

If X has property (α), then for any two measure spaces $(S_1, \mathscr{S}_1, \mu_1)$ and $(S_2, \mathscr{S}_2, \mu_2)$ and any Hilbert space H we have a natural isomorphism

$$\gamma(S_1; \gamma(S_2; H, X)) \simeq \gamma(S_1 \times S_2; H, X), \tag{2.3.6}$$

which is given by the mapping $f_1 \otimes ((f_2 \otimes h) \otimes x) \mapsto ((f_1 \otimes f_2) \otimes h) \otimes x)$, where $f_1 \in L^2(S_1)$, $f_2 \in L^2(S_2)$, $h \in H$ and $x \in X$. We refer to [77, 112] for the proof and generalizations.

The following simple observation [40, Lemma 2.1] will be used frequently:

Proposition 2.5. *For all $g \in L^2(0, T)$ and $R \in \gamma(H, X)$ the function $gR : t \mapsto g(t)R$ belongs to $\gamma(0, T; H, X)$ and we have*

$$\|gR\|_{\gamma(0,T;H,X)} = \|g\|_{L^2(0,T)} \|R\|_{\gamma(H,X)}.$$

2.3.1 Besov spaces

Let $T > 0$. An important tool for estimating the $\gamma(0, T; X)$-norm is the Besov embedding given by (2.3.7) below.

Fix an interval $I = (a, b)$ with $-\infty \le a < b \le \infty$ and let X be a Banach space. For $q, r \in [1, \infty]$ and $s \in (0, 1)$ the Besov space $B^s_{q,r}(I; X)$ is defined by:

$$B^s_{q,r}(I; X) = \{f \in L^q(I; X) : \|f\|_{B^s_{q,r}(I;X)} < \infty\},$$

where

$$\|f\|_{B^s_{q,r}(I;X)} := \|f\|_{L^q(I;X)} + \Big(\int_0^1 \rho^{-sr} \sup_{|h|<\rho} \|T^I_h f - f\|^r_{L^q(I;X)} \frac{d\rho}{\rho}\Big)^{\frac{1}{r}},$$

with, for $h \in \mathbb{R}$,

$$T^I_h f(s) = \begin{cases} f(s+h); & s+h \in I, \\ 0; & s+h \notin I. \end{cases}$$

Observe that if $I' \subseteq I$ are nested intervals, then we have a natural contractive restriction mapping from $B^s_{q,r}(I;X)$ into $B^s_{q,r}(I';X)$

If (and only if) a Banach space X has type $\tau \in [1,2)$, by [108] we have a continuous embedding

$$B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(I, \gamma(H,X)) \hookrightarrow \gamma(I;H,X), \tag{2.3.7}$$

where the constant of the embedding depends on $|I|$ and the type τ constant $\mathscr{T}_\tau(X)$ of X.

2.4 Stochastic integration in Banach spaces

Throughout this section let X be a Banach space and let H be a Hilbert space. An *H-cylindrical Brownian motion* with respect to $(\mathscr{F}_t)_{t\in[0,T]}$ is a linear mapping $W_H : L^2(0,T;H) \to L^2(\Omega)$ with the following properties:

(i) for all $f \in L^2(0,T;H)$, $W_H(f)$ is Gaussian;

(ii) for all $f_1, f_2 \in L^2(0,T;H)$ we have $\mathbb{E}(W_H(f_1)W_H(f_2)) = [f_1, f_2]$;

(iii) for all $h \in H$ and $t \in [0,T]$, $W_H(1_{(0,t]} \otimes h)$ is $\mathscr{F}_t$-measurable;

(iv) for all $h \in H$ and $0 \le s \le t < \infty$, $W_H(1_{(s,t]} \otimes h)$ is independent of $\mathscr{F}_s$.

For all $f_1, \dots, f_n \in L^2(0,T;H)$ the random variables $W_H(f_1), \dots, W_H(f_n)$ are jointly Gaussian. As a consequence, these random variables are independent if and only if $f_1, \dots, f_n$ are orthogonal in $L^2(0,T;H)$. With slight abuse of notation we write $W_H(t)h := W_H(1_{[0,t]} \otimes h)$. For further details on cylindrical Brownian motions see [102, Section 3].

Formally, an H-cylindrical Brownian motion can be thought of as a 'standard Brownian motion' taking values in the Hilbert space H. Indeed, for $H = \mathbb{R}^d$, $B_t := W_{\mathbb{R}^d}([0,t])$ defines a standard Brownian motion $(B_t)_{t\in[0,T]}$ in $\mathbb{R}^d$, and every standard Brownian motion in $\mathbb{R}^d$ arises in this way.

2.4.1 Stochastic integration of functions

As announced in the previous section, the γ-radonifying norm plays an important role in the definition of the stochastic integral of an X-valued function. In fact, for X-valued functions the stochastic integral with respect to W_H can be defined simply by replacing the L^2-norm in the Itô isometry by a γ-radonifying norm, see (2.4.1) below. The definition of the stochastic integral of an X-valued *process* is slightly more complicated, as we will see in the next subsection.

A *finite rank step function* is function of the form $\sum_{n=1}^{N} 1_{(a_n,b_n]} \otimes B_n$ where each operator $B_n : H \to X$ is of finite rank. The stochastic integral with respect to W_H of such a function is defined by setting

$$\int_0^T 1_{(a,b]} \otimes (h \otimes x)\, dW_H := W_H(1_{(a,b]} \otimes h) \otimes x$$

and extending this definition by linearity. Here, for a random variable $\phi \in L^2(\Omega)$ and $x \in X$ we write $\phi \otimes x$ for the random variable $(\phi \otimes x)(\omega) = \phi(\omega)x$.

A function $\Phi : (0,T) \to \mathscr{L}(H,X)$ is said to be *stochastically integrable* with respect to W_H if there exists a sequence of finite rank step functions $\Phi_n : (0,T) \to \mathscr{L}(H,X)$ such that:

(i) for all $h \in H$ we have $\lim_{n\to\infty} \Phi_n h = \Phi h$ in measure on $(0,T)$;

(ii) the limit $\chi := \lim_{n\to\infty} \int_0^T \Phi_n \, dW_H$ exists in probability.

In this situation we write

$$\chi = \int_0^T \Phi \, dW_H$$

and call χ the *stochastic integral* of Φ with respect to W_H.

As was shown in [110], for finite rank step functions Φ one has the following analogue of the Itô isometry:

$$\Big(\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^2\Big)^{\frac{1}{2}} = \|R_\Phi\|_{\gamma(0,T;H,X)}, \tag{2.4.1}$$

where $R_\Phi : L^2(0,T;H) \to X$ is the bounded operator represented by Φ, i.e.,

$$R_\Phi f = \int_0^T \Phi(t) f(t)\, dt, \quad f \in L^2(0,T;H). \tag{2.4.2}$$

As a consequence, a function $\Phi : (0,T) \to \mathscr{L}(H,X)$ is stochastically integrable on $(0,T)$ with respect to W_H if and only if $\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in X^*$ and there exists an operator $R_\Phi \in \gamma(0,T;H,X)$ such that

$$R_\Phi^* x^* = \Phi^* x^* \quad \text{in } L^2(0,T;H) \text{ for all } x^* \in X^*.$$

The isometry (2.4.1) extends to this situation.

In this thesis we generally do not distinguish between a stochastically integrable function Φ and the corresponding operator in $\gamma(0,T,H;X)$, e.g. we simply write $\|\Phi\|_{\gamma(0,T,H;X)}$.

Note that if $\Phi \in \gamma(0,T;H,X)$, then by the Kahane-Khintchine inequalities for Gaussian random variables we have from (2.4.1) that for all $p \in (0,\infty)$ we have:

$$\Big(\mathbb{E}\Big\|\int_0^T \Phi\,dW_H\Big\|^p\Big)^{\frac{1}{p}} \eqsim_p \|\Phi\|_{\gamma(0,T;H,X)}. \tag{2.4.3}$$

2.4.2 Stochastic integration of processes

In order to define the stochastic integral of an X-valued process with respect to W_H we need to be able to 'decouple' the process from W_H. Such a decoupling is possible if X is a UMD Banach space. In fact, decoupling is possible – to some extent – for a larger class of Banach spaces. Chapter 3.4 deals with this. For the time however being we stick with stochastic integration theory for UMD Banach spaces as developed in [107, 108] and refer to these articles for more details.

We start by considering a *finite rank adapted step process* in X, i.e., a process $\Phi : (0,T) \times \Omega \to H \otimes X$ of the form

$$\Phi(t,\omega) = \sum_{n=1}^N 1_{(t_{n-1},t_n]}(t) \sum_{m=1}^M 1_{A_{nm}}(\omega) \sum_{k=1}^K h_k \otimes x_{nmk}, \tag{2.4.4}$$

where $0 \le t_0 < t_1 < ... < t_N < T$, $A_{nm} \in \mathscr{F}_{t_{n-1}}$, $x_{nmk} \in X$, and the vectors $(h_k)_{k=1}^K$ are orthonormal in H. The stochastic integral of such a process Φ with respect to W_H is defined by

$$\int_0^{t_N} \Phi\,dW_H := \sum_{n=1}^N \sum_{m=1}^M 1_{A_{nm}} \sum_{k=1}^K W_H(1_{(t_{n-1},t_n]} \otimes h_k) \otimes x_{nmk}.$$

We call a process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ *H-strongly measurable/$(\mathscr{F}_t)_{t\ge0}$-adapted* if $\Phi h : [0,T] \times \Omega \to X$ is strongly measurable/$(\mathscr{F}_t)_{t\ge0}$-adapted for all $h \in H$. In most cases it is clear what filtration Φ is adapted to – generally the same as the Brownian motion involved – and therefore a reference to the filtration is often omitted. We call Φ *scalarly in* $L^p(\Omega; L^2(0,T;X))$ if for all $x \in X^*$ we have $\Phi^* x^* \in L^p(\Omega; L^2(0,T;X))$.

Definition 2.6. Let W_H be an H-cylindrical Brownian motion adapted to $(\mathscr{F}_t)_{t\ge0}$. An H-strongly measurable, $(\mathscr{F}_t)_{t\ge0}$-adapted process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ is called *stochastically integrable with respect to* W_H if there exists a sequence of finite rank $(\mathscr{F}_t)_{t\ge0}$-adapted step processes $\Phi_n : [0,T] \times \Omega \to \mathscr{L}(H,X)$ such that:

(i) for all $h \in H$ we have $\lim_{n\to\infty} \Phi_n h = \Phi h$ in measure on $[0,T] \times \Omega$;

(ii) there exists a process $\zeta \in L^0(\Omega; C([0,T];X))$ such that

$$\lim_{n\to\infty} \int_0^{\cdot} \Phi_n\,dW_H = \zeta \qquad \text{in } L^0(\Omega; C([0,T];X)).$$

We define $\int_0^{\cdot} \Phi\,dW_H := \zeta$.

If $\zeta \in L^p(\Omega; C([0,T];X))$ for some $p \in (1,\infty)$ and

$$\lim_{n\to\infty} \int_0^{\cdot} \Phi_n \, dW_H = \zeta \qquad \text{in } L^p(\Omega; C([0,T]; X)),$$

we call Φ *L^p-stochastically integrable*.

We recall the following necessary and sufficient conditions for stochastic integrability (see [107, Theorem 2.1] and [108, Theorem 3.6 and Theorem 5.9]).

Theorem 2.7. *Let X be a* UMD *Banach space. For an H-strongly measurable adapted process $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,X)$ that is scalarly in $L^0(\Omega; L^2(0,T;H))$ the following are equivalent:*

(i) Φ is stochastically integrable with respect to W_H;

(ii) there exists a process $\xi \in L^0(\Omega; C([0,T]; X))$ such that for all $x^* \in X^*$ we have

$$\langle \xi, x^* \rangle = \int_0^{\cdot} \Phi^* x^* \, dW_H$$

in $L^0(\Omega; C([0,T]; X))$;

(iii) there exists a (necessarily unique) $R_\Phi \in L^0(\Omega; \gamma(0,T,H;X))$ such that for all $x^* \in X^*$ we have

$$R_\Phi^* x^* = \Phi^* x^*$$

in $L^0(\Omega; L^2(0,T;H))$.

In this situation one has, for all $p \in (1,\infty)$:

$$\Big(\mathbb{E} \sup_{0 \le t \le T} \Big\| \int_0^t \Phi \, dW_H \Big\|_X^p \Big)^{\frac{1}{p}} \eqsim_p \Big(\mathbb{E} \, \| R_\Phi \|^p_{\gamma(0,T,H;X)} \Big)^{\frac{1}{p}}, \tag{2.4.5}$$

whenever the right-hand side is finite; the implied constants being independent of Φ and T.

If the process ξ in (ii) *is in fact in $L^p(\Omega; C([0,T]; X))$, or if the operator R_Φ in* (iii) *is in fact in $L^p(\Omega; \gamma(0,T,H;X))$ for some $p \in (1,\infty)$, then this is equivalent to Φ being L^p-stochastically integrable.*

Remark 2.8.

(i) We refer to the estimates in (2.4.5) as *Burkholder-Davis-Gundy inequalities*. It follows from [47] that if the two-sided estimate in (2.4.5) holds for all X-valued stochastic processes, for some $p \in (1,\infty)$, then X is UMD Banach space. (I.e. the UMD condition is necessary and sufficient.)

In Chapter 3, Section 3.4 we will consider an extension of Theorem 2.7. In particular, we prove that (2.4.5) remains valid for $p \in (0,1]$ and that one obtains a one-sided estimate if one assumes that X satisfies the so-called decoupling property (which is weaker than the UMD property).

(ii) If Φ is H-strongly measurable and $R_\Phi \in \gamma(0,t;H,X)$ a.s. then by [108, Lemma 2.5, 2.7 and Remark 2.8] one automatically obtains that $R_\Phi \in L^0(\Omega;\gamma(0,t;H,X))$. In particular, in this situation one may assume without loss of generality that H and X are separable.

In this thesis we do not distinguish between an L^p-stochastically integrable process Φ and the corresponding operator in $L^p(\Omega;\gamma(0,T;H,X))$, e.g. we simply write $\|\Phi\|_{L^p(\Omega;\gamma(0,T;H,X))}$. For $p \in (1,\infty)$ we denote by

$$L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,X))$$

the completion of the space of adapted finite rank step processes with respect to the norm of $L^p(\Omega;\gamma(0,T;H,X))$ ($L^0_{\mathscr{F}}(\Omega;\gamma(0,T;H,X))$ is defined analogously). By [108] this is precisely the subspace of $L^p(\Omega;\gamma(0,T;H,X))$ containing the adapted processes.

Concerning the regularity of the stochastic integral, the following observation is obtained from Theorem 2.7 and (2.3.4). Let $\Phi \in L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,X))$, then for all $\alpha \in [0,\frac{1}{2})$ and $T_0 \in (0,T]$:

$$\begin{aligned}
&\Big\| s \mapsto \int_0^s \Phi(u)\,dW_H(u) \Big\|_{C^\alpha([0,T_0];L^p(\Omega;X))} \\
&\quad = \sup_{0\le t\le T_0} \|\Phi\|_{L^p(\Omega;\gamma(0,t;H,X))} + \sup_{0\le s<t\le T_0} (t-s)^{-\alpha} \|\Phi|_{[s,t]}\|_{L^p(\Omega;\gamma(s,t;H,X))} \\
&\quad \le \|\Phi\|_{L^p(\Omega;\gamma(0,T_0;H,X))} + T_0^\alpha \sup_{0\le t\le T_0} \|u \mapsto (t-u)^{-\alpha}\Phi(u)\|_{L^p(\Omega;\gamma(0,t;H,X))} \\
&\quad \le (T^\alpha + 1) \sup_{0\le t\le T_0} \|u \mapsto (t-u)^{-\alpha}\Phi(u)\|_{L^p(\Omega;\gamma(0,t;H,X))}
\end{aligned} \tag{2.4.6}$$

with implied constant independent of Φ and T_0.

2.4.3 Properties of the stochastic integral

The following stochastic Fubini theorem is based on [104, Theorem 3.5]. To prove it, we in fact make use of the extended version of Theorem 2.7 presented in Section 3.4, i.e., Theorem 3.29. This allows us to deal with the stochastic integral in $L^1(S,X)$, where X is a UMD space.

Lemma 2.9. *Let $(S,\mathscr{S},\mu)$ be a σ-finite measure space and let X be a* UMD *Banach space. Let $\Phi : S\times[0,t]\times\Omega \to \mathscr{L}(H,X)$ and for $s\in S$ define $\Phi_s : [0,t]\times\Omega \to \mathscr{L}(H,X)$ by $\Phi_s(u,\omega) = \Phi(s,u,\omega)$. Assume that for all $s \in S$ the section Φ_s is H-strongly measurable and adapted and that the following is satisfied:*

(i) For almost all $u\in[0,t]$ and almost all $\omega\in\Omega$ one has $\Phi(\cdot,u,\omega)h \in L^1(S;X)$ for all $h\in H$ and the operator $\int_S \Phi\,d\mu : H \to X$ defined by $\int_S \Phi\,d\mu h := \int_S \Phi h\,d\mu$ is in $\mathscr{L}(H,X)$;

(ii) The process $u \mapsto \int_S \Phi(s,u)\,d\mu(s)$ *represents an element of* $\gamma(0,t;H,X)$ *a.s.;*

(iii) The function $s \mapsto \Phi_s$ *represents an element of* $L^1(S;\gamma(0,t;H,X))$ *a.s.*

Then the function $s \mapsto \int_0^t \Phi(s,u)\,dW_H(u)$ *belongs to* $L^1(S;X)$ *a.s. and*

$$\int_S \int_0^t \Phi\,dW_H\,d\mu = \int_0^t \int_S \Phi\,d\mu\,dW_H \quad a.s. \tag{2.4.7}$$

Proof. Due to condition (iii) and the Fubini isomorphism (2.3.3) one has that Φ represents an element of $\gamma(0,t;H,L^1(S;X))$ a.s. As Φ is assumed to be H-strongly measurable we may assume H and X to be separable by Remark 2.8 (ii). This implies that Φ^*x^* is strongly measurable for all $x^* \in X^*$ by Pettis's measurability theorem, and that $\Phi_s^*x^*$ is adapted for all $x^* \in X^*$, all $s \in S$.

Moreover, because Φ represents an element of $\gamma(0,t;H,L^1(S;X))$ a.s., by Theorem 3.29 and Corollary 3.21 the process $\Psi : [0,t] \times \Omega \to \mathscr{L}(H,L^1(S;X))$ defined by

$$\Psi(u,\omega)(s) := \Phi(s,u,\omega)$$

is stochastically integrable, and by arguments similar to those in the proof of [104, Theorem 3.5] it follows that

$$\int_0^t \Phi(s,u)\,dW_H(u) = \Big(\int_0^t \Psi(u)\,dW_H(u)\Big)(s) \quad \text{a.s. for almost all } s \in S.$$

This proves that the integral with respect to μ on the left-hand side of (2.4.7) is well-defined.

Condition (i) implies that the process in condition (ii) is well-defined, and this condition in combination with Theorem 2.7 implies that the stochastic integral on the right-hand side of (2.4.7) is well-defined.

Fix $x^* \in X^*$, then $\Phi^*x^* : S \times [0,t] \times \Omega \to H$ satisfies conditions (i)-(iii) of [104, Theorem 3.5] and hence by that theorem we have:

$$\int_S \int_0^t \Phi^*x^*\,dW_H\,d\mu = \int_0^t \int_S \Phi^*x^*\,d\mu\,dW_H \quad \text{a.s.}$$

Although the null-set on which the above fails may depend on x^*, this suffices due to the fact that X^* is weak*-separable. Note that in [104, Theorem 3.5] it is assumed that Φ_s is progressive. In fact, it suffices to assume that Φ_s is adapted, see [130]. □

As in the case of the Bochner integral, a closed operator can be taken out of a stochastic integral.

Lemma 2.10. *let* X *be a* UMD *Banach space and let* $A : D(A) \subset X \to X$ *be a closed, densely defined operator. Suppose* $\Phi \in L^0_{\mathscr{F}}(\Omega,\gamma(0,T;H,X))$ *and that one has* $\Phi(s)h \in D(A)$ *for all* $s \in (0,t)$ *and all* $h \in H$ *a.s., where the null sets are independent of* h*. Suppose moreover that* $A\Phi \in L^0_{\mathscr{F}}(\Omega,\gamma(0,T;H,X))$*. Then* $\int_0^t \Phi\,dW_H \in D(A)$ *a.s. and*

$$A \int_0^t \Phi \, dW_H = \int_0^t A\Phi \, dW_H \quad a.s.$$

Proof. Define random variables $\eta := \int_0^t \Phi \, dW_H$ and $\zeta := \int_0^t A\Phi \, dW_H$ and observe that by implication (iii) $\Longrightarrow$ (ii) in Theorem 2.7, one has that for all $x^* \in X^*$:

$$\begin{aligned} \langle \eta, x^* \rangle &= \int_0^t \Phi^*(s) x^* \, dW_H(s) \quad \text{a.s.}, \\ \langle \zeta, x^* \rangle &= \int_0^t (A\Phi(s))^* x^* \, dW_H(s) \quad \text{a.s.} \end{aligned}$$

In particular for $x^* \in D(A^*)$ one has $(A\Phi(s))^* x^* = \Phi^*(s) A^* x^*$, and thus for such x^* one has:

$$\langle (\eta, \zeta), (-Ax^*, x^*) \rangle = \langle \eta, -A^* x^* \rangle + \langle \zeta, x^* \rangle = 0 \quad a.s. \tag{2.4.8}$$

Note that the null-set on which the equation above fails to hold may depend on x^*. However, as Φ and $A\Phi$ are assumed to be H-strongly measurable and in $\gamma(0,t;H,X)$ a.s. we may assume X to be separable by Remark 2.8 (ii). Hence $(X \times X)/\mathscr{G}r(A)$ is separable, where $\mathscr{G}r(A)$ is the graph of A, and thus by Hahn-Banach there exists a countable subset of $((X \times X)/\mathscr{G}r(A))^* = \mathscr{G}r(A)^{\perp}$ that separates the points of $(X \times X)/\mathscr{G}r(A)$.

Moreover, one checks that if $(x_1^*, x_2^*) \in \mathscr{G}r(A)^{\perp}$ then $x_2^* \in D(A^*)$ and $x_1^* = -A^* x_2^*$. Thus there exists a sequence $(-Ax_n^*, x_n^*)_{n \in \mathbb{N}}$ that separates points in $(X \times X)/\mathscr{G}r(A)$. As equation (2.4.8) holds for arbitrary $x^* \in D(A^*)$, it holds simultaneously for all x_n^*, on a set of measure one. Therefore $(\eta, \zeta) \in \mathscr{G}r(A)$, i.e., $\eta \in D(A)$ and $A\eta = \zeta$ a.s. $\square$

2.5 Randomized boundedness

Throughout this section let X and Y denote Banach spaces. Let $(\gamma_k)_{k \geq 1}$ denote a sequence of real-valued independent standard Gaussian random variables. A family of operators $\mathscr{R} \subseteq \mathscr{L}(X,Y)$ is called *γ-bounded* if there exists a constant $C \geq 0$ such that for all finite choices $R_1, \ldots, R_n \in \mathscr{R}$ and vectors $x_1, \ldots, x_n \in X$ we have

$$\Big(\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k R_k x_k \Big\|_Y^2\Big)^{\frac{1}{2}} \leq C\Big(\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k x_k \Big\|_X^2\Big)^{\frac{1}{2}}.$$

The least admissible constant C is called the *γ-bound* of $\mathscr{R}$, notation $\gamma(\mathscr{R})$. When we want to emphasize the domain and range spaces we write $\gamma_{[X,Y]}(\mathscr{R})$. Replacing the role of the Gaussian sequence by a Rademacher sequence we arrive at the related notion of *R-boundedness*. Every R-bounded set is γ-bounded, and the converse holds if X has non-trivial cotype. We refer to [23, 38, 84, 132] for examples and more information γ-boundedness and R-boundedness.

The following lemma is a direct consequence of the Kahane contraction principle:

Lemma 2.11. *If $\mathscr{R} \subset \mathscr{L}(X,Y)$ is γ-bounded and $M > 0$ then $M\mathscr{R} := \{aR : a \in [-M,M], R \in \mathscr{B}\}$ is γ-bounded with $\gamma_{[X,Y]}(M\mathscr{R}) \le M\gamma_{[X,Y]}(\mathscr{R})$.*

The following two results are useful for determining γ-bounded sets, they are variations on results of Kunstman and Weis, see [132, Proposition 2.5] and [84, Corollary 2.14]. Roughly speaking, the first proposition follows by writing out the definition and the second follows from the observation that $\gamma(\mathscr{R}) = \gamma(\overline{\mathrm{absco}(\mathscr{R})}^s)$, where $\overline{\mathrm{absco}(\mathscr{R})}^s$ denotes the closure in the strong operator topology of the absolute convex hull of $\mathscr{R}$.

Proposition 2.12. *Let $\Phi : (0,T) \to \mathscr{L}(X,Y)$ be such that for all $x \in X$ the function $t \mapsto \Phi(t)x$ is continuously differentiable. Suppose there exists a $g \in L^1(0,T)$ such that for almost all $t \in [0,T]$ and all $x \in X$ we have:*

$$\|\Phi'(t)x\|_Y \le g(t)\|x\|_X.$$

Then the set $\mathscr{F} := \{\Phi(t) : \ t \in (0,T)\}$ is γ-bounded in $\mathscr{L}(X,Y)$ and

$$\gamma_{[X,Y]}(\mathscr{F}) \le \|\Phi(0+)\|_{\mathscr{L}(X,Y)} + \|g\|_{L^1},$$

where part of the assertion is that $\Phi(0+) = \lim_{t\downarrow 0}\Phi(t)$ exists in the strong operator topology.

Proposition 2.13. *Let $(S, \mathscr{S}, \mu)$ be a σ-finite measure space and let $\mathscr{R} \subset \mathscr{L}(X,Y)$ be γ-bounded. Suppose $\Phi : S \to \mathscr{L}(X,Y)$ is such that Φx is strongly measurable for all $x \in X$ and $\Phi(s) \in \mathscr{R}$ for almost all $s \in S$. For $f \in L^1(S)$ define $T_f^\Phi \in \mathscr{L}(X,Y)$ by*

$$T_f^\Phi x = \int_S f\Phi d\mu, \qquad x \in X.$$

Then $\gamma_{[X,Y]}(\{T_f^\Phi \ : \ f \in L^1(S)\}) \le \gamma_{[X,Y]}(\mathscr{R})$.

The following γ-multiplier result, due to Kalton and Weis [77] (see also [102]), establishes a relation between stochastic integrability and γ-boundedness.

Theorem 2.14 (γ-Multiplier theorem). *Suppose $M : (0,T) \to \mathscr{L}(X,Y)$ is a strongly measurable function (in the sense that $t \mapsto M(t)x$ is strongly measurable for every $x \in X$) with γ-bounded range $\mathscr{M} = \{M(t) : \ t \in (0,T)\}$. Then for every finite rank simple function $\Phi : (0,T) \to \gamma(H,X)$ we have that $M\Phi$ represents an element of $\gamma_\infty(0,T;H,Y)$ and*

$$\|M\Phi\|_{\gamma_\infty(0,T;H,Y)} \le \gamma(\mathscr{M})\, \|\Phi\|_{\gamma(0,T;H,X)}.$$

As a result, the map $\widetilde{M} : \Phi \mapsto M\Phi$ has a unique extension to a bounded operator

$$\widetilde{M} : \gamma(0,T;H,X) \to \gamma_\infty(0,T;H,Y)$$

of norm $\|\widetilde{M}\| \le \gamma(\mathscr{M})$.

In view of Theorem 2.7, Theorem 2.14 implies that if X and Y are UMD spaces, then for all $\Phi \in L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,X))$, $p \in (1,\infty)$, the function $M\Phi : (0,T)\times\Omega \to \mathscr{L}(H,Y)$ is L^p-stochastically integrable and

$$\Big\| \int_0^T M\Phi \, dW_H \Big\|_{L^p(\Omega;Y)} \lesssim \gamma_{[X,Y]}(\mathscr{M}) \Big\| \int_0^T \Phi \, dW_H \Big\|_{L^p(\Omega;X)}.$$

Here we use that if Y is a UMD Banach space, then Y does not contain a subspace isomorphic to c_0 and hence $\gamma_\infty(0,T;H,Y) = \gamma(0,T;H,Y)$. If we wish to apply the multiplier theorem to stochastic integrals in general Banach spaces, then we have to check that $M\Phi \in \gamma(0,T;H,Y)$.

The γ-multiplier theorem will frequently by applied in conjunction with the following basic result due to Kaiser and Weis [73, Corollary 3.6]:

Theorem 2.15. *Let X be a Banach space with non-trivial cotype. Define, for every $h \in H$, the operator $U_h : X \to \gamma(H,X)$ by*

$$U_h x := h \otimes x, \quad x \in X.$$

Then the family $\{U_h : \|h\| \le 1\}$ is γ-bounded.

2.6 Analytic semigroups

Throughout this section X denotes a Banach space. Recall that a C_0-*semigroup* on X is a family of operators $(S_t)_{t\ge0}$ such that $S(0) = I$, $S(t+s) = S(t)S(s)$ for $t,s \ge 0$, and $t \mapsto S(t)$ is strongly continuous.

For $\delta \in [0,\pi]$ we define $\Sigma_\delta := \{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \delta\}$. We recall the definition of an analytic C_0-semigroup [114, Chapter 2.5]:

Definition 2.16. Let $\delta \in (0,\frac{\pi}{2})$. A C_0-semigroup $(S(t))_{t\ge0}$ on X is called *analytic in Σ_δ* if

(i) S extends to an analytic function $S : \Sigma_\delta \to \mathscr{L}(X)$;

(ii) $S(z_1+z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Sigma_\delta$;

(iii) $\lim_{z\to0; z\in\Sigma_\delta} S(z)x = x$ for all $x \in X$.

Typical examples of operators generating analytic C_0-semigroups are second-order elliptic operators. The theorem below is obtained from [114, Theorem 2.5.2] by straightforward adaptations and gives some useful characterizations of analytic C_0-semigroups.

Theorem 2.17. *Let $(S(t))_{t\ge0}$ be a C_0-semigroup on X. Let $\omega \in \mathbb{R}$ be such that $(e^{-\omega t}S(t))_{t\ge0}$ is exponentially stable. Let A be the generator of S. The following statements are equivalent:*

(i) S is an analytic C_0-semigroup on Σ_δ for some $\delta \in (0,\frac{\pi}{2})$ and for every $\delta' < \delta$ there exists a constant $C_{1,\delta'}$ such that $\|e^{-\omega z}S(z)\| \le C_{1,\delta'}$ for all $z \in \Sigma_{\delta'}$.

(ii) There exists a $\theta \in (0, \frac{\pi}{2})$ *such that* $\omega + \Sigma_{\frac{\pi}{2}+\theta} \subset \varrho(A)$*, and for every* $\theta' \in (0, \theta)$ *there exists a constant* $C_{2,\theta'} > 0$ *such that:*

$$|\lambda - \omega| \|R(\lambda : A)\| \leq C_{2,\theta'}, \quad \textit{for all } \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta'}.$$

(iii) S *is differentiable (in the uniform operator topology) for* $t > 0$*,* $\frac{d}{dt}S = AS$*, and there exists a constant* C_3 *such that:*

$$t\|AS(t)\| \leq C_3 e^{\omega t}, \quad \textit{for all } t > 0.$$

This theorem justifies the following definition:

Definition 2.18. Let A be the generator of an analytic C_0-semigroup on X. We say that A is *of type* (ω, θ, K), where $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $K > 0$, if $\omega + \Sigma_{\frac{\pi}{2}+\theta} \subseteq \varrho(A)$, $(e^{-\omega t}S(t))_{t\geq 0}$ is exponentially stable, and

$$|\lambda - \omega| \|R(\lambda : A)\|_{\mathscr{L}(X)} \leq K \quad \text{for all } \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}.$$

Remark 2.19. It follows from the aforementioned proof in [114] that the constants $\delta, C_{1,\delta'}$; $\delta' \in (0, \delta)$, $C_{2,\theta'}$; $\theta' \in (0, \theta)$, and C_3 in Theorem 2.17 can be expressed explicitly in terms of ω, θ, and K; for example we may take $C_3 = \frac{K}{\pi \cos\theta}$.

Note that if A is the generator of an analytic C_0 semigroup of type (ω, θ, K) then for all $\lambda \in \omega(1 + 2(\cos\theta)^{-1}) + \Sigma_{\frac{\pi}{2}+\theta}$ one has (noting that the choice of λ implies $|\lambda| > 2|\omega|$ and hence $|\lambda - \omega| > \big||\lambda| - |\omega|\big| \geq \frac{1}{2}|\lambda|$):

$$\|AR(\lambda : A)\|_{\mathscr{L}(X)} = \|\lambda R(\lambda : A) - I\| \leq 1 + 2K. \tag{2.6.1}$$

If A is the generator of an analytic semigroup of type (ω, θ, K) on X, then for $\lambda \in \mathbb{C}$ such that $\Re e(\lambda) > \omega$ it is possible to define negative fractional powers of $\lambda I - A$ (see [114, Chapter 2.6]): for $\alpha \in (0, 1)$ and $x \in X$ we set:

$$(\lambda I - A)^{-\alpha} x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\lambda t} S(t)x \, dt. \tag{2.6.2}$$

Setting $(\lambda I - A)^0 := I$ this allows us to define $(\lambda I - A)^{-\alpha}$ for all $\alpha \geq 0$. For $\alpha \geq 0$ the *extrapolation space* $X^A_{-\alpha}$ denotes the closure of X under the norm $\|x\|_{X^A_{-\alpha}} := \|(\lambda I - A)^{-\alpha} x\|_X$. One may check that regardless of the choice of λ the extrapolation spaces are uniquely determined up to isomorphisms.

The negative fractional powers of $\lambda I - A$ are injective. Therefore, it is possible to define their (unbounded) inverse: for $\alpha \geq 0$ we set $(\lambda I - A)^{\alpha} := [(\lambda I - A)^{-\alpha}]^{-1}$. By X^A_α we denote the fractional domain space: $X^A_\alpha := D((\lambda I - A)^\alpha)$ for $\alpha \geq 0$.

We tend to write X_α instead of X^A_α if it is clear from the context that X_α is defined in terms of A.

For α, $\beta \in \mathbb{R}$ one has $(\lambda I - A)^\alpha (\lambda I - A)^\beta = (\lambda I - A)^{\alpha+\beta}$ on X_γ, where $\gamma = \max\{\beta, \alpha + \beta\}$ (see [114, Theorem 2.6.8]). Moreover, for $\alpha > 0$ and $\lambda, \mu \in \mathbb{C}$, $\Re e(\lambda), \Re e(\mu) > \omega$, one has $(\lambda I - A)^\alpha (\mu I - A)^{-\alpha} \in \mathscr{L}(X)$ and:

$$\|(\lambda I - A)^{\alpha}(\mu I - A)^{-\alpha}\|_{\mathscr{L}(X)} \leq C(\omega, \theta, K, \lambda, \mu),$$

where $C(\omega, \theta, K, \lambda, \mu)$ denotes a constant depending only on $\omega, \theta, K, \lambda$, and μ.

Statement (iii) in Theorem 2.17 can be extended; from the proof of [114, Theorem 2.6.13] we obtain that for an analytic C_0-semigroup S of type (ω, θ, K) generated by A one has, for $\alpha > 0$:

$$\|S(t)\|_{\mathscr{L}(X,X_\alpha)} \leq 2[\tfrac{K}{\pi \cos \theta}]^{\lceil \alpha \rceil} t^{-\alpha} e^{\omega t}. \tag{2.6.3}$$

Another important property of analytic semigroups (see again [114, Theorem 2.6.13]) is that for every $T > 0$ there exists a constant C depending only on (ω, θ, K) such that for all $\alpha > 0$ one has:

$$\|S(t) - I\|_{\mathscr{L}(X_\alpha,X)} \leq C t^{\alpha \wedge 1}; \quad \text{for all } t \in (0, T]. \tag{2.6.4}$$

The following interpolation result holds for the fractional domain spaces (see [114, Theorem 2.6.10]):

Theorem 2.20. *Let A be the generator of an analytic C_0-semigroup on X of type (ω, θ, K). Let $\alpha \in (0, 1)$ and $\lambda \in \mathbb{C}$ such that $\mathscr{R}e(\lambda) > \omega$. Then for every $x \in D(A)$ we have:*

$$\|(\lambda I - A)^{\alpha} x\| \leq 2(1 + K)\|x\|^{1-\alpha}\|(\lambda I - A)x\|^{\alpha}.$$

For more properties of X_α, $\alpha \in \mathbb{R}$, we refer to [114, Section 2.6].

2.6.1 A γ-boundedness result for analytic semigroups

The following γ-boundedness result for analytic semigroups is essential for proving the existence of a solution to an X-valued stochastic differential equation, as well as for proving convergence of time discretizations.

Lemma 2.21. *Let A generate an analytic C_0-semigroup S and let $T > 0$.*

(1) For all $0 \leq \alpha < \beta$ and $t \in (0, T)$ the set $\mathscr{S}_{\beta,t} = \{s^{\beta} S(s) : s \in [0, t]\}$ is γ-bounded in $\mathscr{L}(X, X_\alpha)$ and we have

$$\gamma_{[X,X_\alpha]}(\mathscr{S}_{\beta,t}) \lesssim t^{\beta-\alpha}, \quad t \in (0, T),$$

with implied constant independent of $t \in (0, T)$.

(2) For all $0 < \alpha \leq 1$ and $t \in (0, T)$ the set $\mathscr{S}_t = \mathscr{S}_{0,t} = \{S(s) : s \in [0, t]\}$ is γ-bounded in $\mathscr{L}(X_\alpha, X)$ and we have

$$\gamma_{[X_\alpha,X]}(\mathscr{S}_t) \lesssim t^{\alpha}, \quad t \in (0, T),$$

with implied constant independent of $t \in (0, T)$.

(3) For all $0 < \alpha \leq 1$ *and* $t \in (0,T)$ *the set* $\mathscr{T}_t = \{S(s) - I : \ s \in [0,t]\}$ *is* γ*-bounded in* $\mathscr{L}(X_\alpha, X)$ *and we have*

$$\gamma_{[X_\alpha,X]}(\mathscr{T}_t) \lesssim t^\alpha, \quad t \in (0,T),$$

with implied constant independent of $t \in [0,T]$.

Let us emphasize that the constants in Lemma 2.21 may depend on the final time T: all we are asserting is that, given T, the constants are independent of $t \in [0,T]$. Also, as the semigroup S commutes with the fractional powers of A, for any $\theta \in \mathbb{R}$ and $0 \leq \alpha < \beta$ we have, from Lemma 2.21 (1):

$$\gamma_{[X_\theta,X_{\theta+\alpha}]}(\mathscr{S}_{\beta,t}) = \gamma_{[X,X_\alpha]}(\mathscr{S}_{\beta,t}).$$

Analogous statements hold for parts (2) and (3) of Lemma 2.21.

Proof (of Lemma 2.21). For the proof of (1) we refer to [40] or [101, Lemma 10.17]. The proof of (2) is obtained similarly, both proofs are based on Proposition 2.12.

To prove (3) it will be shown that for any fixed and large enough $w \in \mathbb{R}$ the set

$$\mathscr{T}^w_{\alpha,t} := \{e^{-ws}S(s) - I : \ s \in [0,t]\}$$

is γ-bounded in $\mathscr{L}(X_\alpha, X)$ with γ-bound $\lesssim t^\alpha$. From this we deduce that $\{S(s) : s \in [0,t]\}$ is γ-bounded in $\mathscr{L}(X_\alpha, X)$ with γ-bound $\lesssim 1$. In view of the identity

$$S(s) - I = (e^{-ws}S(s) - I) + (1 - e^{-ws})S(s)$$

and noting that $1 - e^{-ws} \lesssim s$, this will prove the assertion of the lemma.

For all $x \in X$ and $0 \leq s \leq t$,

$$e^{-ws}S(s)x - x = \int_0^s e^{-wr}(A - w)S(r)x \, dr.$$

By (2.6.3) and Proposition 2.12 the set $\mathscr{T}^w_{\alpha,t}$ is γ-bounded in $\mathscr{L}(X_\alpha, X)$ and $\gamma(\mathscr{T}^w_{\alpha,t}) \lesssim \int_0^t s^{\alpha-1}\, ds \lesssim t^\alpha$. □

Part I

Decoupling

3

Decoupling

In this chapter, which is based on [29], we study a decoupling inequality for X-valued tangent sequences, where X is a (quasi-)Banach space (see Section 3.1). Our motivation lies in the role this inequality plays in the development of theory for stochastic integration in Banach spaces [96], [108]; we shall elaborate on this below. However, the decoupling inequality has attracted attention in its own right, see [36], [88] and references therein. Let us begin with a formal definition of the decoupling inequality.

Let X be a (quasi-)Banach space. Let $(\Omega, (\mathscr{F}_n)_{n\geq 1}, \mathscr{A}, \mathbb{P})$ be a complete probability space and let $(d_n)_{n\geq 1}$ be an $(\mathscr{F}_n)_{n\geq 1}$-adapted sequence of X-valued random variables. We adopt the convention that $\mathscr{F}_0 = \{\Omega, \varnothing\}$. Set $\mathscr{F}_\infty := \sigma(\mathscr{F}_n : n \geq 1)$. A *$\mathscr{F}_\infty$-decoupled tangent sequence of* $(d_n)_{n\geq 1}$ is a sequence $(e_n)_{n\geq 1}$ of X-valued random variables on $(\Omega, \mathscr{A}, \mathbb{P})$ satisfying two properties. Firstly, we assume that for any $B \in \mathscr{B}(X)$ (the Borel-measurable sets of X) we have:

$$\mathbb{P}(d_n \in B \,|\, \mathscr{F}_{n-1}) = \mathbb{P}(e_n \in B \,|\, \mathscr{F}_\infty),$$

and secondly we assume that $(e_n)_{n\geq 1}$ is $\mathscr{F}_\infty$-conditionally independent, i.e., for every $n \geq 1$ and every $B_1, \ldots, B_n \in \mathscr{B}(X)$ we have:

$$\mathbb{P}(e_1 \in B_1, \ldots, e_n \in B_n \,|\, \mathscr{F}_\infty) = \mathbb{P}(e_1 \in B_1 \,|\, \mathscr{F}_\infty) \cdot \ldots \cdot \mathbb{P}(e_n \in B_n \,|\, \mathscr{F}_\infty).$$

Here $\mathbb{P}(C \,|\, \mathscr{F}_{n-1}) = \mathbb{E}(\mathbf{1}_C \,|\, \mathscr{F}_{n-1})$ if $C \in \mathscr{A}$. We wish to emphasize that the definition of a decoupled tangent sequence depends on a filtration and on a sequence adapted to that filtration. However, in what follows we shall omit the reference to the σ-algebra if it is clear from the context.

Kwapień and Woyczyński introduced the concept of decoupled tangent sequences in [87]. For details on the subject we refer to the monographs [36, 88] and the references therein. It is shown there that given a sequence $(d_n)_{n\geq 1}$ of $(\mathscr{F}_n)_{n\geq 1}$-adapted random variables on $(\Omega, \mathscr{A}, \mathbb{P})$ one can, by an extension of the probability space, construct a decoupled tangent sequence of $(d_n)_{n\geq 1}$. One easily checks that any two $\mathscr{F}_\infty$-decoupled tangent sequences of a $(\mathscr{F}_n)_{n\geq 1}$-adapted sequence $(d_n)_{n\geq 1}$ share the same law.

We recall the following basic example (see also Lemma 3.10 below; and [88, Section 4.3] and [36, Chapter 6] where many more examples can be found).

Example 3.1. Let $d_n = \xi_n v_n$, where $(\xi_n)_{n\geq 1}$ is a sequence of independent random variables with values in $\mathbb{R}$ and $(v_n)_{n\geq 1}$ is an X-valued $(\mathscr{F}_n)_{n\geq 0}$-predictable sequence; $\mathscr{F}_n := \sigma(\xi_1, \ldots, \xi_n)$ for $n \geq 1$ and $\mathscr{F}_0 := \{\Omega, \varnothing\}$. Let $(\tilde{\xi}_n)_{n\geq 1}$ be an independent copy of $(\xi_n)_{n\geq 1}$, then a decoupled tangent sequence of d_n is given by $e_n = \tilde{\xi}_n v_n$ for $n \geq 1$.

Let $(\xi_n)_{n\geq 1}$ be a sequence (of X-valued random variables). The *difference sequence of* $(\xi_n)_{n\geq 1}$ is the sequence $(\xi_n - \xi_{n-1})_{n\geq 1}$, with the understanding that $\xi_0 \equiv 0$.

Definition 3.2. Let X be a (quasi-)Banach space and let $p \in (0,\infty)$. We say that *the decoupling inequality holds in X for p* if there exists a constant D_p such that for all complete probability spaces $(\Omega, \mathscr{A}, (\mathscr{F}_n)_{n\geq 1}, \mathbb{P})$ and every X-valued $(\mathscr{F}_n)_{n\geq 1}$-adapted L^p-sequence f (i.e., $f = (f_n)_{n\geq 1} \subseteq L^p(\Omega, X)$):

$$\|f_n\|_p \leq D_p \|g_n\|_p, \tag{3.0.1}$$

for every $n \geq 1$, where g is a sequence whose difference sequence is an $\mathscr{F}_\infty$-decoupled tangent sequence of the difference sequence of f. The least constant D_p for which (3.0.1) holds is denoted by $D_p(X)$.

We will refer to a sequence $g = (g_n)_{n\geq 1}$ whose difference sequence is a $\mathscr{F}_\infty$-decoupled tangent sequence of the difference sequence of f, where f is adapted to $(\mathscr{F}_n)_{n\geq 1}$, as *a $\mathscr{F}_\infty$-decoupled sum sequence of f*. As before, we omit the reference to the σ-algebra if it is obvious. Note that inequality (3.0.1) holds for all $\mathscr{F}_\infty$-decoupled sum sequences of f if it holds for some $\mathscr{F}_\infty$-decoupled sum sequence of f as they are identical in law.

A natural question to ask is whether a (quasi-)Banach space X that satisfies the decoupling inequality for some $p \in (0,\infty)$, automatically satisfies it for all $q \in (0,\infty)$. In [30] it was shown that if the decoupling inequality is satisfied in a Banach space X for some $p \in [1,\infty)$, then it is satisfied for all $q \in (p,\infty)$. This also follows from results presented in [49], see Remark 3.13. One of the main results of this chapter is that the decoupling inequality is in fact satisfied for all $q \in (0,\infty)$ if it is satisfied for some $p \in (0,\infty)$, see Theorem 3.16 in Section 3.3. As a result of that Theorem 3.16 we may speak of a (quasi-)Banach space X for which *the decoupling inequality holds*, meaning a space for which it holds for some, and hence all, $p \in (0,\infty)$.

A necessary condition for a Banach space to satisfy the decoupling inequality is that X has finite cotype. This has been proven in [48, Theorem 2], see also [30, Example 3], by proving that c_0 does not satisfy the decoupling inequality and then appealing to the Maurey-Pisier theorem. In fact, by Lemma 3.20 below the decoupling property is local: if a Banach space X satisfies the decoupling inequality for some $p \in (0,\infty)$ and Banach space Y is finitely representable in X then Y satisfies the decoupling inequality for p, and $D_p(Y) \leq D_p(X)$. Moreover, it was demonstrated in [30] that $D_p(L^p(S;Y)) = D_p(Y)$ whenever Y is a Banach space satisfying the decoupling inequality for some $p \in [1,\infty)$. In Section 3.3 we show that this extends to $p \in (0,\infty)$, thus the L^p-spaces with $p \in (0,\infty)$ satisfy the decoupling inequality. This indicates that quasi-Banach spaces are a natural setting in which to study the decoupling inequality.

Although we refer to inequality (3.0.1) as *the* decoupling inequality, various other types of decoupling inequalities have been studied. Below we shall elaborate on some related inequalities and results, in particular we will shall consider the (randomized) UMD inequality.

Remark 3.3. In situations where only the laws of $(d_n)_{n\geq 1}$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$ are relevant, as is the case in Definition 3.2, it suffices to consider the probability space $([0,1]^{\mathbb{N}}, \mathscr{B}([0,1]^{\mathbb{N}}), \lambda_{\mathbb{N}})$ where $\lambda_{\mathbb{N}}$ is the Lebesgue product measure. This has been demonstrated in [99], where it also has been shown that one may assume the sequences $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ to have a certain structure on that probability space, which is useful when trying to gain insight in the properties of decoupled sequences. However, the details are rather technical, so we will stick to the definition involving arbitrary probability spaces.

Other decoupling inequalities

Hitczenko [63] and McConnell [96] have independently proven that X is a UMD Banach space if and only if for all (for some) $1 < p < \infty$ there exist constants C_p and D_p such that one has:

$$C_p^{-1}\|g_n\|_p \leq \|f_n\|_p \leq D_p\|g_n\|_p, \tag{3.0.2}$$

for all $n \geq 1$ and all X-valued L^p-martingales f adapted to some filtration $(\mathscr{F}_n)_{n\geq 1}$, and any g that is a decoupled sum sequence of f. The least constants for which (3.0.2) holds are denoted by $C_p(X)$ and $D_p(X)$. From the proofs in [63,96] it follows that $\max\{C_p(X), D_p(X)\} \leq \beta_p(X)$ where $\beta_p(X)$ is the UMD constant of X.

The second inequality in (3.0.2) corresponds to the decoupling inequality (3.0.1) for $p \in (1,\infty)$, the only difference being that f in (3.0.2) is assumed to be a $(\mathscr{F}_n)_{n\geq 1}$-martingale. It follows from Lemma 3.15 below that this difference is artificial. For this we use that every $(\mathscr{F}_n)_{n\geq 1}$-adapted sequence $(f_n)_{n\geq 1}$ in $L^p(\Omega, X)$ with $p \in [1,\infty)$ such that $f_n - f_{n-1}$ is $\mathscr{F}_{n-1}$-conditionally symmetric is a martingale difference sequence. The reason we choose not to work with martingale difference sequences is that they are not well-defined for $p < 1$.

The inequalities (3.0.2) allow for a way to 'split' the UMD property into two weaker properties. The aforementioned randomized UMD spaces, which were introduced in [48], are obtained by 'splitting' the UMD property in a different way, leading to the following inequalities:

$$[\beta_p^+]^{-1}\Big\|\sum_{k=1}^n r_k d_k\Big\|_p \leq \Big\|\sum_{k=1}^n d_k\Big\|_p \leq \beta_p^-\Big\|\sum_{k=1}^n r_k d_k\Big\|_p; \quad n \geq 1, \tag{3.0.3}$$

where $(d_k)_{k\geq 1}$ is an X-valued martingale difference sequence, $(r_k)_{k\geq 1}$ is a Rademacher sequence independent of $(d_k)_{k\geq 1}$, and β_p^-, β_p^+ are constants independent of $(d_k)_{k\geq 1}$ and n. It is an open question whether there exists a Banach space X that fails to be a UMD space but for which the first inequality in (3.0.3) holds for all X-valued martingale difference sequence. However, it was demonstrated in [50] that for p fixed there fails to exist a constant c such that $\beta_p(X) \leq cC_p(X)$ for all Banach spaces X.

Note that the inequalities (3.0.3) coincide with the inequalities (3.0.2) if one considers only those f which are adapted to the dyadic filtration (Paley-Walsh martingales). However, in general the inequalities are different. For $X = \mathbb{R}$ the constant in (3.0.1) is bounded as $p \to \infty$ (see (3.0.4) below). However, the optimal constant for the second inequality in (3.0.3) is $\mathscr{O}(\sqrt{p})$ as $p \to \infty$. Indeed, by the Khintchine inequalities there is a constant C such that

$$\Big\| \sum_{k=1}^n d_k \Big\|_p \leq \beta_p^-(\mathbb{R}) \Big\| \sum_{k=1}^n r_k d_k \Big\|_p \leq C\sqrt{p}\beta_p^-(\mathbb{R}) \Big\| \Big(\sum_{k=1}^n |d_k|^2 \Big)^{\frac{1}{2}} \Big\|_p.$$

Since the best constant in the above square function inequality is $p-1$ if $p \geq 2$ (see [18, Theorem 3.3]) we deduce that $C\sqrt{p}\beta_p^-(\mathbb{R}) \geq p-1$ and therefore the above claim follows. Furthermore, there is an example in [48] showing that there exist Banach lattices with finite cotype that do *not* satisfy (3.0.3). However, the martingales constructed to prove this are *not* Paley-Walsh martingales, which means there is hope that all Banach lattices with finite cotype satisfy the decoupling inequality (3.0.1).

The monographs [36] and [117] and the references therein provide a good overview of the various decoupling inequalities that have been studied. For the case that $X = \mathbb{R}$ the decoupling inequality (3.0.1) has been studied among others by De la Peña, Giné, Hitczenko and Montgomery-Smith, see [36, Chapter 6 and 7], [65] and [66]. Another important decoupling inequality that is studied in [36, Chapters 3-5] has been proven to hold in all Banach spaces [37], whereas it has been demonstrated by Kalton [76] that it fails in some quasi-Banach spaces.

Decoupling and vector-valued stochastic integrals

Our main motivation for studying the decoupling inequality is its role in the definition of vector-valued stochastic integrals. To be precise, decoupling is used to derive Burkholder-Davis-Gundy type inequalities, i.e., the inequalities of equation (2.4.5) in Theorem 2.7 on page 21, see also equations (3.4.2) and (3.4.3) below. This not a novel idea: in [47] Burkholder-Davis-Gundy type inequalities are obtained from the randomized UMD inequalities (3.0.3). However, the approach in [47] requires the stochastic process in the integrand to be adapted to the filtration generated by the Brownian motion, which we do not wish to assume. A decoupling argument based on the UMD inequality is given in [108], which is the source of Theorem 2.7. As mentioned in Remark 2.8, the approach in [108] does not give Burkholder-Davis-Gundy type inequalities for p^{th} moments when

$p \in (0, 1]$. In Section 3.4 we present yet another decoupling argument which *does* give Burkholder-Davis-Gundy type inequalities for p^{th} moments when $p \in (0, 1]$ in UMD spaces. This argument is based on the equivalence of the UMD property and the two-sided decoupling inequality of equation (3.0.2).

The argument by which the two-sided Burkholder-Davis-Gundy inequality is obtained from (3.0.2) can be used to obtain the second estimate of the Burkholder-Davis-Gundy inequality for Banach spaces satisfying the one-sided decoupling inequality (3.0.1). This is also demonstrated in Section 3.4. This one-sided Burkholder-Davis-Gundy inequality suffices in order to define the stochastic integral of vector-valued processes.

The role of decoupling inequalities in the definition of vector-valued stochastic integrals is not limited to integrals with respect to the Brownian motion: in recent work by Dirksen, Maas and van Neerven decoupling inequalities are used to define stochastic integrals with respect to a Poisson random measure, see [42].

Best constants in the decoupling inequality

The behavior of the decoupling constant $D_p(X)$ in (3.0.1) is of interest as it can be used to obtain the right (optimal) behavior of the constants in inequalities such as the Burkholder-Davis-Gundy inequality and the Rosenthal inequality for martingale difference sequences as p tends to ∞ (see [36, Theorem 7.3.2]). In [65] Hitczenko proves the remarkable result that if $X = \mathbb{R}$ then there is a universal constant $D_{\mathbb{R}}$ such that (3.0.1) holds for all $p \in [1, \infty]$ with $D_p = D_{\mathbb{R}}$, i.e.

$$D_{\mathbb{R}} := \sup_{p \in [1,\infty]} D_p(\mathbb{R}) < \infty, \tag{3.0.4}$$

see also [36, Chapter 7]. In [66] the existence of a universal constant in (3.0.1) has been proven with the L^p-norms replaced by a large class of Orlicz norms and rearrangement invariant norms on Ω. The traditional approach to proving such extrapolation results is by methods as introduced in [20]. In [49], such extrapolation results have been stated in a BMO-framework with which one obtains estimates in a more general setting (see also Remark 3.13).

We will show that if H is a Hilbert space, then $\sup_{p \in [1,\infty)} D_p(H) \le D_{\mathbb{R}}$, where $D_{\mathbb{R}}$ is defined as in equation (3.0.4) (see Corollary 3.24). It remains an open problem whether the constant D in (3.0.1) can be taken independently of p in the general case that X is a (quasi-)Banach space.

3.1 Random sequences in quasi-Banach spaces

As explained above we consider decoupling inequalities in the setting of quasi-Banach spaces. The definition of a quasi-Banach space is identical to that of a Banach space, except that the triangle inequality is replaced by

$$\|x + y\| \le C(\|x\| + \|y\|),$$

for all $x, y \in X$, where C is some constant independent of x and y. We shall only need some basic results on such spaces, and we refer the reader to [75] and references therein for general theory and more advanced results.

Definition 3.4. Let X be a quasi-Banach space. We say that X is an r-normable quasi-Banach space for some $0 < r \leq 1$ if there exists a constant $C > 0$ such that

$$\Big\| \sum_{j=1}^{n} x_j \Big\|^r \leq C \sum_{j=1}^{n} \|x_j\|^r$$

for any sequence $(x_j)_{j=1}^n \subseteq X$.

The space $L^p(0,1)$ with $p \in (0,1)$ is an example of a p-normable quasi-Banach space. In fact, by the Aoki-Rolewicz Theorem [4], [124], any quasi-Banach space X may be equivalently re-normed so it is r-normable for some $r \in (0,1]$, with $C = 1$. It easily follows that every quasi-Banach space is a (not necessarily locally convex) F-space. Whenever we speak of an r-normable quasi-Banach space X in this chapter, we implicitly assume $C = 1$. Observe that if X is a r-normable quasi-Banach space and $x, y \in X$ then

$$\big| \|x\|^r - \|y\|^r \big| \leq \|x - y\|^r, \tag{3.1.1}$$

and hence the map $x \to \|x\|$ is continuous and therefore Borel measurable.

Recall that an X-valued random variable is a Borel measurable mapping from Ω into X with separable range (see [128, Section I.1.4]). We say that an X-valued random variable ξ on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with $\mathscr{G} \subseteq \mathscr{A}$ a σ-algebra is *$\mathscr{G}$-conditionally symmetric* if for all $B \in \mathscr{B}(X)$ one has $\mathbb{P}(\xi \in B \,|\, \mathscr{G}) = \mathbb{P}(-\xi \in B \,|\, \mathscr{G})$. We sometimes omit the σ-algebra if it is obvious from the context.

Also recall the following notation: if $(\zeta_i)_{i \in I}$ is a set of X-valued random variables indexed by an ordered set I, then $\zeta_i^* = \sup_{j \leq i} \|\zeta_j\|$ and $\zeta^* = \sup_{j \in I} \|\zeta_j\|$.

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space. We recall some probabilistic lemmas to be used later on. Since we need them in the quasi-Banach setting, we provide the short proofs which might be well-known to experts. The following version of Lévy's inequality holds in quasi-Banach spaces:

Lemma 3.5. *Let X be an r-normable quasi-Banach space. Let $\mathscr{G} \subseteq \mathscr{A}$ be a sub-σ-algebra. Let $(\xi_k)_{k=1}^n$ be a sequence of $\mathscr{G}$-conditionally independent and $\mathscr{G}$-conditionally symmetric X-valued random variables. Then for all $t > 0$ one has:*

$$\mathbb{P}\Big(\max_{k=1,\ldots,n} \Big\| \sum_{j=1}^{k} \xi_j \Big\| > t \,\Big|\, \mathscr{G} \Big) \leq 2\mathbb{P}\Big(\Big\| \sum_{j=1}^{n} \xi_j \Big\| > 2^{1-\frac{1}{r}} t \,\Big|\, \mathscr{G} \Big)$$

and

$$\mathbb{P}\Big(\max_{k=1,\ldots,n} \|\xi_j\| > t \,\Big|\, \mathscr{G} \Big) \leq 2\mathbb{P}\Big(\Big\| \sum_{j=1}^{n} \xi_j \Big\| > 2^{1-\frac{1}{r}} t \,\Big|\, \mathscr{G} \Big).$$

Let $\xi = (\xi_1, \ldots, \xi_n)$. For the proof note that there is a regular version $\mu : \Omega \times \mathscr{B}(X^n) \to [0,1]$ of $\mathbb{P}(\xi \in \cdot \,|\, \mathscr{G})$, with the following properties: for all $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a probability measure on $(X^n, \mathscr{B}(X^n))$ and for all $B \in \mathscr{B}(X^n)$ one has $\mu(\cdot, B) = \mathbb{P}(\xi \in B \,|\, \mathscr{G})$ a.s. (see [74, Theorems 6.3 and 6.4]). Moreover, for any Borel function $\phi : X^n \to \mathbb{R}_+$ one has:

$$\int_{X^n} \phi(x)\, \mu(\omega, dx) = \mathbb{E}(\phi(\xi) \,|\, \mathscr{G})(\omega), \text{ for almost all } \omega \in \Omega,$$

whenever the latter exists. For the existence of the regular version note that X^n is a separable complete metric space; ξ_j is separably valued for each $1 \leq j \leq n$ and the metric is given by $d(x,y) = \|x-y\|^r$. The regular version μ of the conditional probability can be used to reduce the proof of the lemma to the case without conditional probabilities. Indeed, let $\tilde{\xi} = (\tilde{\xi}_j)_{j=1}^n : X^n \to X^n$ be given by $\tilde{\xi}(x) = x$. Then one can argue with the random variable $\tilde{\xi}$ and probability measure $\mu(\omega, \cdot)$ on X^n with $\omega \in \Omega$ fixed. We use this method below.

Proof. We only give a proof for the first statement, which is a modification of the proof given both in [36, Theorem 1.1.1] and in [88, Proposition 1.1.1]. These monographs also provide a proof of the second statement which is very similar to that of the first.

As explained before the lemma we can leave out the conditional probabilities. For $k = 1, \ldots, n$ define $S_k = \sum_{j=1}^k \xi_j$ and

$$A_k = \{\|S_j\| \leq t \text{ for all } j = 1, \ldots, k-1; \|S_k\| > t\}.$$

Note that the sets A_k, $k = 1, \ldots, n$, are mutually disjoint. Define $S_n^{(k)} := S_k - \xi_{k+1} - \ldots - \xi_n$. Observe that by symmetry and independence of the random variables $(\xi_k)_{k=1}^n$ the random variables S_n and $S_n^{(k)}$ have the same conditional distribution with respect to $\sigma\big(\bigcup_{j=1}^k \xi_j\big)$. Hence

$$\mathbb{P}(A_k \cap \{\|S_n\| > 2^{1-\frac{1}{r}} t\}) = \mathbb{P}(A_k \cap \{\|S_n^{(k)}\| > 2^{1-\frac{1}{r}} t\}).$$

On the other hand, because for any $x, y \in X$ one has

$$\|x\| \leq 2^{\frac{1}{r}-1} \max\{\|x+y\|, \|x-y\|\},$$

on the set A_k one has $t < \|S_k\| \leq 2^{\frac{1}{r}-1} \max\{\|S_n\|, \|S_n^{(k)}\|\}$ and thus

$$A_k = (A_k \cap \{\|S_n\| > 2^{1-\frac{1}{r}} t\}) \cup (A_k \cap \{\|S_n^{(k)}\| > 2^{1-\frac{1}{r}} t\}).$$

Therefore

$$\begin{aligned} \mathbb{P}(S_n^* > t) &= \sum_{k=1}^n \mathbb{P}(A_k) \leq 2 \sum_{k=1}^n \mathbb{P}(A_k \cap \{\|S_n\| > 2^{1-\frac{1}{r}} t\}) \\ &= 2\mathbb{P}\Big(\bigcup_{k=1}^n A_k \cap \{\|S_n\| > 2^{1-\frac{1}{r}} t\}\Big) \leq 2\mathbb{P}(\|S_n\| > 2^{1-\frac{1}{r}} t). \end{aligned}$$

□

As a consequence we obtain the following peculiar result which we need twice below. It is a "toy"-version of the Kahane contraction principle.

Corollary 3.6. *Assume the conditions of Lemma 3.5 hold. Let $(v_j)_{j=1}^n$ be a $\{0,1\}$-valued sequence of random variables such that $(v_j\xi_j)_{j=1}^n$ is again $\mathscr{G}$-conditionally independent and $\mathscr{G}$-conditionally symmetric. Then for all $t \geq 0$ one has:*

$$\mathbb{P}\Big(\Big\|\sum_{j=1}^n v_j\xi_j\Big\| > t \,\Big|\, \mathscr{G}\Big) \leq 2\mathbb{P}\Big(\Big\|\sum_{j=1}^n \xi_j\Big\| > 2^{1-\frac{1}{r}} t \,\Big|\, \mathscr{G}\Big).$$

Nigel Kalton kindly showed us how to obtain a Kahane contraction principle for tail probabilities for r-normable quasi-Banach space. However, the standard convexity proof for $r = 1$ (cf. [88, Corollary 1.2.]) does not extend to the case $r < 1$, and the constants are more complicated. Since we do not need the more general version we only consider the situation of Corollary 3.6.

Proof. As in Lemma 3.5, using a regular conditional probability for the X^{2n}-valued random variable $\big((v_j\xi_j)_{j=1}^n, (\xi_j)_{j=1}^n\big)$, one can reduce to the case without conditional probabilities.

Let $(r_k)_{k\geq 1}$ be a Rademacher sequence on an independent complete probability space, where $\mathbb{E}_r$ and $\mathbb{P}_r$ denote the the expectation and probability measure with respect to the Rademacher sequence. We obtain:

$$\begin{aligned}
\mathbb{P}\Big(\Big\|\sum_{j=1}^n v_j\xi_j\Big\| > t\Big) &\overset{(i)}{=} \mathbb{E}_r\mathbb{E}\mathbf{1}_{\{\|\sum_{j=1}^n r_j v_j \xi_j\| > t\}} \\
&= \mathbb{E}\mathbb{P}_r\Big(\Big\|\sum_{j=1}^n r_j v_j\xi_j\Big\| > t\Big) \\
&\overset{(ii)}{\leq} 2\mathbb{E}\mathbb{P}_r\Big(\Big\|\sum_{j=1}^n r_j\xi_j\Big\| > 2^{1-\frac{1}{r}} t\Big) \\
&\overset{(iii)}{=} 2\mathbb{P}\Big(\Big\|\sum_{j=1}^n \xi_j\Big\| > 2^{1-\frac{1}{r}} t\Big),
\end{aligned}$$

where $\mathbf{1}$ denotes the indicator function. In (i) and (iii) we used the independence and symmetry of $(v_j\xi_j)_{j=1}^n$ and of $(\xi_j)_{j=1}^n$. In (ii) we applied Lemma 3.6 to the random variables $(r_j v_j(\omega)\xi_j(\omega))_{j=1}^n$ where $\omega \in \Omega$ is fixed and we used that $v_j \in \{0,1\}$. $\square$

Recall that for $a, b \geq 0$ and $p \geq 1$ one has:

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p),$$

the latter inequality following by convexity. For $0 < p \leq 1$ the reversed inequalities hold, hence by defining

$$l_p := 2^{1-p} \vee 1 \quad \text{and} \quad u_p := 2^{p-1} \vee 1, \quad p \in (0,\infty), \tag{3.1.2}$$

we obtain the following general statement for $p \in (0,\infty)$ and $a, b \geq 0$:

$$l_p^{-1}(a^p + b^p) \leq (a+b)^p \leq u_p(a^p + b^p). \tag{3.1.3}$$

Note that $2^{1-p}u_p = l_p$. A tiny yet useful Lemma:

Lemma 3.7. *Let X be an r-normable quasi-Banach space and let $\mathscr{G} \subseteq \mathscr{A}$ be a sub-σ-algebra. Let ξ and ζ be $\mathscr{G}$-conditionally independent X-valued random variables. If ζ is $\mathscr{G}$-conditionally symmetric, then for all $p \in (0,\infty)$ one has:*

$$\mathbb{E}[\|\xi\|^p \,|\, \mathscr{G}] \leq 2^{1-p}u_{p/r}\mathbb{E}[\|\xi + \zeta\|^p \,|\, \mathscr{G}],$$

where $u_{p/r}$ is as defined in (3.1.2).

Proof. As in Lemma 3.5 it suffices to prove the estimate without conditional expectations.

Because ξ and ζ are independent and ζ is symmetric, $\xi + \zeta$ and $\xi - \zeta$ are identically distributed. By (3.1.3) one has:

$$\begin{aligned}
\mathbb{E}\|\xi\|^p &\leq 2^{-p}\mathbb{E}(\|\xi+\zeta\|^r + \|\xi-\zeta\|^r)^{\frac{p}{r}} \\
&\leq 2^{-p}u_{p/r}\mathbb{E}(\|\xi+\zeta\|^p + \|\xi-\zeta\|^p) = 2^{1-p}u_{p/r}\mathbb{E}\|\xi+\zeta\|^p.
\end{aligned}$$

□

From [89, p. 161] we adapt to the quasi-Banach space setting a reverse Kolmogorov inequality:

Lemma 3.8. *Let X be an r-normable quasi-Banach space and let $p \in (0,\infty)$. Let $(\xi_k)_{k=1}^n$ be a sequence of $\mathscr{G}$-conditionally independent and $\mathscr{G}$-conditionally symmetric X-valued random variables. Then for all $t > 0$ one has:*

$$\mathbb{P}\Big(\max_{k=1,\ldots,n}\Big\|\sum_{j=1}^k \xi_j\Big\| > t \,\Big|\, \mathscr{G}\Big) \geq 2^{p-1}\left[u_{p/r}^{-2} - \frac{t^p + \mathbb{E}(|\xi^*|^p \,|\, \mathscr{G})}{\mathbb{E}(\|\sum_{j=1}^n \xi_j\|^p \,|\, \mathscr{G})}\right].$$

In particular, if $r = 1$ this corresponds to the result as stated in [89, p. 161].

Proof. As in the last two lemmas it suffices to consider the situation without conditioning. Set $S_k = \sum_{j=1}^k \xi_j$ $(k = 1,\ldots,n)$, $S_0 = 0$, and define the stopping time

$$\tau := \inf\{k : \|S_k\| > t\}.$$

On the set $\{\tau = k\}$ one has, by applying (3.1.3) twice:

$$\begin{aligned}
\|S_n\|^p &\leq u_{p/r}(u_{p/r}[\|S_{k-1}\|^p + \|\xi_k\|^p] + \|S_n - S_k\|^p) \\
&\leq u_{p/r}(u_{p/r}[t^p + (\xi_n^*)^p] + \|S_n - S_k\|^p).
\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\|S_n\|^p &\le t^p\mathbb{P}(S_n^* \le t) + \sum_{k=1}^n \int_{\{\tau=k\}} \|S_n\|^p d\mathbb{P} \\ &\le t^p\mathbb{P}(S_n^* \le t) + u_{p/r}\sum_{k=1}^n \int_{\{\tau=k\}} (u_{p/r}[t^p + (\xi_n^*)^p] + \|S_n - S_k\|^p) d\mathbb{P}.\end{aligned}$$

Because $S_n - S_k$ is independent of $\{\tau = k\}$ and $\sum_{k=1}^n \mathbb{P}(\tau = k) = \mathbb{P}(S_n^* > t)$ the above can be estimated by:

$$\begin{aligned}\mathbb{E}\|S_n\|^p \le t^p\mathbb{P}(S_n^* \le t) &+ u_{p/r}^2\big[t^p\mathbb{P}(S_n^* > t) + \mathbb{E}(\xi_n^*)^p\big] \\ &+ u_{p/r} \sup_{1\le k\le n} \mathbb{E}\|S_n - S_k\|^p\mathbb{P}(S_n^* > t).\end{aligned}$$

By Lemma 3.7 we have $\mathbb{E}\|S_n - S_k\|^p \le 2^{1-p}u_{p/r}\mathbb{E}\|S_n\|^p$ and thus, observing that $u_{p/r} \ge 1$,

$$\mathbb{E}\|S_n\|^p \le u_{p/r}^2\big[t^p + \mathbb{E}(\xi_n^*)^p + 2^{1-p}\mathbb{E}\|S_n\|^p\mathbb{P}(S_n^* > t)\big],$$

from which the desired estimate follows. $\square$

The next lemma relates the distribution of e^* and d^* if $(e_n)_{n\ge1}$ is a decoupled tangent sequence of $(d_n)_{n\ge1}$ (see [64, Lemma 1] or [88, Theorem 5.2.1]):

Lemma 3.9. *Let X be an r-normable quasi-Banach space and let $(e_n)_{n\ge1}$ be a decoupled tangent sequence of $(d_n)_{n\ge1}$. Then for each $t > 0$ one has:*

$$\mathbb{P}(e^* > t) \le 2\mathbb{P}(d^* > t) \qquad \textit{and} \qquad \mathbb{P}(d^* > t) \le 2\mathbb{P}(e^* > t).$$

(The proof requires no adaptation; if $(e_n)_{n\ge1}$ is a decoupled tangent sequence of $(d_n)_{n\ge1}$ then the sequence $(\|e_n\|)_{n\ge1}$ is a decoupled tangent sequence of $(\|d_n\|)_{n\ge1}$.)

The following lemma is well-known to experts, but we could not find a reference.

Lemma 3.10. *Let X be a complete separable metric space, and let (S, Σ) be a measurable space. Suppose $(d_n)_{n\ge1}$ is an $(\mathscr{F}_n)_{n\ge1}$-adapted X-valued sequence and let $(v_n)_{n\ge1}$ be an $(\mathscr{F}_n)_{n\ge0}$-predictable S-valued sequence. For $n \ge 1$ let $h_n : X \times S \to X$ be a $\mathscr{B}(X) \otimes \Sigma$-measurable function. Then $(h_n(e_n, v_n))_{n\ge1}$ is a decoupled tangent sequence of $(h_n(d_n, v_n))_{n\ge1}$ whenever $(e_n)_{n\ge1}$ is a decoupled tangent sequence of $(d_n)_{n\ge1}$.*

Moreover, if the function h_n satisfies $-h_n(x, s) = h_n(-x, s)$ for all $x \in X, s \in S$ for some $n \ge 1$, then $h_n(d_n, v_n)$ is $\mathscr{F}_{n-1}$-conditionally symmetric and $h_n(e_n, v_n)$ is $\mathscr{F}_\infty$-conditionally symmetric whenever d_n is.

Proof (of Lemma 3.10). Fix $k \ge 1$. Let $\mu_1, \mu_2 : \Omega \times \mathscr{B}(X) \to [0, 1]$ be regular conditional probabilities for $\mathbb{P}(d_k \in \cdot \,|\, \mathscr{F}_{k-1})$ and $\mathbb{P}(e_k \in \cdot \,|\, \mathscr{F}_\infty)$. Note that

$\mathbb{P}(e_k \in \cdot \,|\, \mathscr{F}_\infty) = \mathbb{P}(e_k \in \cdot \,|\, \mathscr{F}_{k-1})$. Then by the fact that $(e_n)_{n\geq 1}$ is a decoupled tangent sequence of d_n we have $\mu_1(\omega,\cdot) = \mu_2(\omega,\cdot)$ for almost all $\omega \in \Omega$. Let $\tilde{d}_k(x) = x$ and $\tilde{e}_k(x) = x$. Let $B \subseteq X$ be a Borel set. Then by disintegration (also see [74, Theorems 6.3 and 6.4]) for almost all $\omega \in \Omega$ one has:

$$\begin{aligned}\mathbb{P}(h_k(d_k, v_k) \in B \,|\, \mathscr{F}_{k-1})(\omega) &= \int_X \mathbf{1}_{h_k(\tilde{d}_k(x), v_k(\omega)) \in B}\, \mu_1(\omega, dx)\\ &= \int_X \mathbf{1}_{h_k(\tilde{e}_k(x), v_k(\omega)) \in B}\, \mu_2(\omega, dx)\\ &= \mathbb{P}(h_k(e_k, v_k) \in B \,|\, \mathscr{F}_{k-1})(\omega).\end{aligned}$$

The claim concerning the conditional symmetry of $h_n(d_n, v_n)$ and $h_n(e_n, v_n)$ can be proven in a similar fashion.

Therefore, it remains to prove the conditional independence. Fix $n \geq 1$. Let $\mu : \Omega \times \mathscr{B}(X^n) \to [0,1]$ be a regular conditional probability for $(e_k)_{k=1}^n$. Let $\tilde{e} : X^n \to X^n$ be given by $\tilde{e}(x) = x$. Then for each $\omega \in \Omega$, $(\tilde{e}_k)_{k=1}^n$ are independent random variables with respect to the probability measure with respect to $\mu(\omega,\cdot)$. In this part of the argument we only require that v_n is $\mathscr{F}_\infty$-measurable. By disintegration one obtains that for all Borel sets $B_1, \ldots, B_n \subseteq X$ and almost all $\omega \in \Omega$ one has:

$$\begin{aligned}&\mathbb{P}(h_1(e_1, v_1) \in B_1, \ldots, h_n(e_n, v_n) \in B_n \,|\, \mathscr{F}_\infty)(\omega)\\ &\quad= \int_{X^n} \prod_{k=1}^n \mathbf{1}_{h_k(\tilde{e}_k(x), v_n(\omega))) \in B_k}\, \mu(\omega, dx)\\ &\quad= \prod_{k=1}^n \int_{X^n} \mathbf{1}_{h_k(\tilde{e}_k(x), v_n(\omega))) \in B_k}\, \mu(\omega, dx) \quad \text{(by independence)}\\ &\quad= \prod_{k=1}^n \mathbb{P}(h_k(e_k, v_n) \in B_k \,|\, \mathscr{F}_\infty)(\omega).\end{aligned}$$

□

3.2 Extrapolation lemmas

Throughout this section let X be a fixed r-normable quasi-Banach space, and let $(\Omega, (\mathscr{F}_n)_{n\geq 1}, \mathscr{A}, \mathbb{P})$ be a fixed complete probability space. As usual we define $\mathscr{F}_\infty = \sigma(\mathscr{F}_n : n \geq 1)$. Moreover, in this section and the next $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ always denote the respective difference sequences of the sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$.

Let $\mathscr{M}_\infty$ be the set of all $(\mathscr{F}_n)_{n\geq 1}$-adapted uniformly bounded X-valued sequences $f = (f_n)_{n\geq 1}$ such that d_n is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$ and for which there exists an $N \in \mathbb{N}$ such that $d_n = 0$ for all $n \geq N$. We define

$$\mathscr{D}_\infty := \{f \in \mathscr{M}_\infty \,:\, \text{there exists a } \mathscr{F}_\infty\text{-decoupled sum sequence } g \text{ of } f \\ \text{on the space } (\Omega, \mathscr{A}, \mathbb{P})\},$$

(It would be more precise to refer to $\mathscr{D}_\infty$ as $\mathscr{D}_\infty(\Omega, (\mathscr{F}_n)_{n\geq 1}, \mathscr{A}, \mathbb{P}; X)$ but we have assumed the space X and the probability space to be fixed throughout this section.)

The operator $T_p : \mathscr{D}_\infty \to L^0(\Omega, \mathscr{A}, \mathbb{R}_+)$ is defined as follows:

$$T_p(f) = \Big(\mathbb{E}\Big[\Big\|\sum_{k\geq 1} e_k\Big\|^p \,\Big|\, \mathscr{F}_\infty\Big]\Big)^{\frac{1}{p}},$$

where $(e_k)_{n\geq 1}$ is a $\mathscr{F}_\infty$-decoupled tangent sequence of $(d_n)_{n\geq 1}$ on $(\Omega, \mathscr{A}, \mathbb{P})$. In the next remark it is shown that T_p is well-defined.

Remark 3.11. Observe that although the sequence $(e_k)_{n\geq 1}$ is not uniquely defined on $(\Omega, \mathscr{A}, \mathbb{P})$, its conditional distribution given $\mathscr{F}_\infty$ is unique. Indeed, if $(\tilde{e}_k)_{k\geq 1}$ is another $\mathscr{F}_\infty$-decoupled tangent sequence for $(d_k)_{k\geq 1}$ on $(\Omega, \mathscr{A}, \mathbb{P})$, then by definition we have:

$$\mathbb{P}(\tilde{e}_1 \in B_1, \ldots, \tilde{e}_n \in B_n \,|\, \mathscr{F}_\infty) = \mathbb{P}(d_1 \in B_1 \,|\, \mathscr{F}_0) \cdot \ldots \cdot \mathbb{P}(d_n \in B_n \,|\, \mathscr{F}_{n-1}), \tag{3.2.1}$$

for all $n \geq 1$ and all Borel sets $B_1, \ldots, B_n$ and the same holds with $(\tilde{e}_k)_{k=1}^n$ replaced by $(e_k)_{k=1}^n$. A monotone class argument implies that for all Borel functions $\phi : X^n \to \mathbb{R}_+$ one has $\mathbb{E}[\phi(e_1, \ldots, e_n) \,|\, \mathscr{F}_\infty] = \mathbb{E}[\phi(\tilde{e}_1, \ldots, \tilde{e}_n) \,|\, \mathscr{F}_\infty]$. In particular, taking $\phi(x_1, \ldots, x_n) = \big\|\sum_{k=1}^n x_k\big\|^p$ it follows that $T_p(f)$ is unique. Moreover, from (3.2.1) with $\tilde{e}_k$ replaced by e_k, $k = 1, \ldots, n$, one also sees that $\mathbb{E}[\phi(e_1, \ldots, e_n) \,|\, \mathscr{F}_\infty]$ is $\mathscr{F}_{n-1}$-measurable.

The following properties of T_p are well-known and easy to prove:

(i) T_p is local, i.e., $T_p f = 0$ on the set $\bigcap_{n\geq 1}\{\mathbb{E}[\|d_n\| \,|\, \mathscr{F}_{n-1}] = 0\}$.

(ii) T_p is monotone when $r = 1$, i.e., $T_p(f^n) \leq T_p(f^{n+1})$ (see Lemma 3.7).

(iii) T_p is predictable, i.e., $T_p(f^n)$ is $\mathscr{F}_{n-1}$-measurable (see Remark 3.11).

(iv) T_p is quasilinear for all $p \in (0, \infty)$ and $r \in (0, 1]$, and sub-linear if $p \in [1, \infty)$ and $r = 1$.

For $f \in \mathscr{D}_\infty$ let $T_p^*(f) := \sup_{n\geq 1} T_p(f^n)$ and $\|f\| := \lim_{n\to\infty} \|f_n\|$, both of which are well-defined by definition of $\mathscr{D}_\infty$. Observe that if g is a decoupled sum sequence of f then $\|T_p(f)\|_p = \|g\|_p$.

The first lemma we prove employs the well-known Burkholder stopping-time technique (see for example [20], [17]). The assumption given by (3.2.2) below can be interpreted as a BMO-condition, this approach has been introduced in [49].

Let τ be an $(\mathscr{F}_n)_{n\geq 1}$-stopping time and f an $(\mathscr{F}_n)_{n\geq 1}$-adapted sequence. The stopped sequence f^τ is defined by $f^\tau := (\mathbf{1}_{\{\tau\geq n\}} d_n)_{n\geq 1}$ and the started sequence by ${}^\tau\! f := (\mathbf{1}_{\{\tau<n\}} d_n)_{n\geq 1}$. If ν is another stopping time then ${}^\tau\! f^\nu := f^\nu - f^\tau$. It follows from Lemma 3.10 that ${}^\tau\! f^\nu \in \mathscr{D}_\infty$ whenever $f \in \mathscr{D}_\infty$. (Thus in particular $T_p({}^\tau\! f^\nu)$ is well-defined if $f \in \mathscr{D}_\infty$.)

Lemma 3.12. *Let $p \in (0,\infty)$ and let $\mathscr{D}_\infty$ be as defined above. Suppose that for some $b \in (0,1)$ and $A > 0$ we have:*

$$\sup_{f\in\mathscr{D}_\infty} \sup_{0\le k\le l} \sup_{B\in\mathscr{F}_k, B\neq\varnothing} \mathbb{P}(\|{}^k\!f^l\| > A\|T_p({}^k\!f^l)\|_\infty \,|\, B) < b. \tag{3.2.2}$$

Then there exist $\delta \in (0,1)$, $\beta \in (1,2)$ depending directly on A and on $\min\{p,r\}$ such that for all $f \in \mathscr{D}_\infty$ one has:

$$\mathbb{P}(f^* \ge \beta\lambda, T_p^*(f) \vee d^* < \delta\lambda) \le b\mathbb{P}(f^* > \lambda), \quad \lambda > 0. \tag{3.2.3}$$

(For example setting $\bar{r} := \min\{p,r\}$ one can pick $\beta = \frac{3}{2}$ and $\delta = \big[\frac{(3/2)^{\bar{r}}-1}{2A^{\bar{r}}+1}\big]^{\frac{1}{\bar{r}}}$.)

The proof is quite standard. For convenience of the reader we give the details.

Proof. Because any r-normable quasi-Banach space is also $\bar{r}$-normable for any $\bar{r} \le r$ we may assume $r \le p$ (noting that any dependence on r is actually a dependence on $\min\{r,p\}$). Let $\beta,\delta > 0$ be such that $\beta^r > 1 + \delta^r$. We will pick more specific values for δ and β later on. Let $f \in \mathscr{D}_\infty$ and let $\lambda > 0$ be arbitrary. Define the following stopping times:

$$\begin{aligned}
\mu &= \inf\{n \ge 1 : \|f_n\| > \lambda\};\\
\nu &= \inf\{n \ge 1 : \|f_n\| > \beta\lambda\};\\
\sigma &= \inf\{n \ge 1 : T_p(f^{n+1}) \vee \|d_n\| > \delta\lambda\}.
\end{aligned}$$

On the set $\{\nu < \infty, \sigma = \infty\}$ one has by (3.1.1) that:

$$\begin{aligned}
\|{}^\mu\!f^{\nu\wedge\sigma}\|^r &\ge \|f^{\nu\wedge\sigma}\|^r - \|f^{\mu-1}\|^r - \|d_\mu\|^r\\
&> (\beta\lambda)^r - \lambda^r - (\delta\lambda)^r = (\beta^r - 1 - \delta^r)\lambda^r.
\end{aligned}$$

We show that

$$\|T_p({}^\mu\!f^{\nu\wedge\sigma})\|_\infty \le 2^{\frac{1}{r}}\delta\lambda. \tag{3.2.4}$$

On the set $\{\mu \ge \sigma\}$ one has $T_p({}^\mu\!f^{\nu\wedge\sigma}) = 0$. On the set $\{\mu < \sigma\}$ one has:

$$\begin{aligned}
[T_p({}^\mu\!f^{\nu\wedge\sigma})]^r &= [T_p({}^{\mu\wedge\sigma}\!f^{\nu\wedge\sigma})]^r = [T_p(f^{\nu\wedge\sigma} - f^{\mu\wedge\sigma})]^r\\
&\le [T_p(f^{\nu\wedge\sigma})]^r + [T_p(f^{\mu\wedge\sigma})]^r,
\end{aligned}$$

using that if $p \ge r$ and X is r-normable, then $L^p(\Omega, X)$ is r-normable. By definition of σ one has $T_p(f^{\nu\wedge\sigma}) \le \delta\lambda$ and $T_p(f^{\mu\wedge\sigma}) \le \delta\lambda$, from which (3.2.4) follows.

We obtain:

$$\begin{aligned}
\mathbb{P}(f^* > \beta\lambda, T_p^*(f) \vee d^* \le \delta\lambda) &= \mathbb{P}(\nu < \infty, \sigma = \infty)\\
&\le \mathbb{P}(\|{}^\mu\!f^{\nu\wedge\sigma}\| > (\beta^r - 1 - \delta^r)^{\frac{1}{r}}\lambda)
\end{aligned}$$

$$\leq \mathbb{P}(\|{}^{\mu}\!f^{\nu\wedge\sigma}\| > 2^{-\frac{1}{r}}\delta^{-1}(\beta^r - 1 - \delta^r)^{\frac{1}{r}}\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty). \tag{3.2.5}$$

Pick $\beta > 1$ and pick $\delta \in (0,1)$ such that $\beta^r > 1+\delta^r$ and $2^{-\frac{1}{r}}\delta^{-1}(\beta^r-1-\delta^r)^{\frac{1}{r}} \geq A$ (e.g. take $\beta = \frac{3}{2}$ and $\delta = \left[\frac{(3/2)^r-1}{2A^r+1}\right]^{\frac{1}{r}}$). Then by assumption (3.2.2) we have:

$$\begin{aligned}
&\mathbb{P}(\|{}^{\mu}\!f^{\nu\wedge\sigma}\| > 2^{-\frac{1}{r}}\delta^{-1}(\beta^r - 1 - \delta^r)^{\frac{1}{r}}\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty \,|\, \mu < \infty) \\
&\qquad\leq \mathbb{P}({}^{\mu}\!f^{\nu\wedge\sigma} > A\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty \,|\, \mu < \infty) \\
&\qquad= \mathbb{P}(\mu < \infty)^{-1}\sum_{k=1}^{\infty}\mathbb{P}({}^{\mu}\!f^{\nu\wedge\sigma} > A\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty \,|\, \mu = k)\mathbb{P}(\mu = k) \\
&\qquad\leq b\mathbb{P}(\mu < \infty)^{-1}\sum_{k=1}^{\infty}\mathbb{P}(\mu = k) = b.
\end{aligned}$$

As ${}^{\mu}\!f^{\nu\wedge\sigma} = 0$ on $\{\mu = \infty\}$ we have:

$$\|{}^{\mu}\!f^{\nu\wedge\sigma}\| \leq 2^{-\frac{1}{r}}\delta^{-1}(\beta^r - 1 - \delta^r)^{\frac{1}{r}}\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty$$

on that set. Combining the above we obtain:

$$\mathbb{P}(\|{}^{\mu}\!f^{\nu\wedge\sigma}\| > 2^{-\frac{1}{r}}\delta^{-1}(\beta^r - 1 - \delta^r)^{\frac{1}{r}}\|T_p({}^{\mu}\!f^{\nu\wedge\sigma})\|_\infty) \leq b\mathbb{P}(\mu < \infty) = bP(f^* > \lambda),$$

which, when inserted in equation (3.2.5), gives (3.2.3). □

Remark 3.13. Suppose X is a Banach space, i.e., $r = 1$. In [49] it has been demonstrated how extrapolation results can be obtained from BMO-type assumptions like (3.2.2) in Lemma 3.12. In particular, from Corollary 6.3 and Proposition 7.3 in [49] one can deduce that if assumption (3.2.2) is satisfied, then there exists a constant $c_{X,b,p}$ such that for all $1 \leq q < \infty$ and all $f \in \mathscr{D}_\infty$ one has:

$$\|f^*\|_q \leq c_{X,b,p}\, q\|T_p(f)\|_q.$$

Observe that for $q \geq p$ we have $\|T_p(f)\|_q \leq \|g\|_q$ by the conditional Hölder's inequality, where g is a decoupled sum sequence of f. However, it seems that this approach fails when $q < p$ as well as in the more general setting that we consider in Theorem 3.16. Thus we proceed in a different manner.

Let $q \in (0,\infty)$. Following the notation in [66] we shall use F_q to denote the set of all non-decreasing, continuous functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\Phi(0) = 0$ and:

$$\Phi(st) \leq s^q\Phi(t), \quad \text{for all } s,t \in \mathbb{R}_+. \tag{3.2.6}$$

Proposition 3.14. *Let $p \in (0,\infty)$. Let $\mathscr{D}_\infty$, and $\Phi \in F_q$ for some $q \in (0,\infty)$. Suppose that* (3.2.2) *holds for $b = 2^{p-\frac{2p}{r}-q-1}$ and some $A > 0$. Then for all $f \in \mathscr{D}_\infty$ we have:*

$$\mathbb{E}\Phi(f^*) \le C_{X,r,p,q}\mathbb{E}\Phi(\|g\|),$$

where g is a decoupled sum sequence of f and $C_{X,r,p,q}$ as in (3.2.19) *below. In particular, for $r = 1$ and $p \ge 1$ one can take:*

$$C_{X,1,p,q} = 2^{2q+2}\big[2^{p+4q+2}(2A+1)^q\big]. \tag{3.2.7}$$

For a positive random variable we can write $\mathbb{E}\Phi(\xi) = \int_0^\infty \mathbb{P}(\xi > \lambda)\, d\Phi(\lambda)$, where the integral is of Lebesgue-Stieltjes type.

Proof. Without loss of generality we assume $p \ge r$. (Noting that any dependence on r is actually a dependence on $\min\{r,p\}$.)

Let $f \in \mathscr{D}_\infty$ be given. The Davis decomposition of $(d_n)_{n\ge1}$ is given by $d_n = d'_n + d''_n$ where $d'_1 := 0$, $d''_1 := d_1$ and for $n \ge 2$:

$$d'_n = d_n \mathbf{1}_{\{\|d_n\| \le 2d^*_{n-1}\}} \quad\text{and}\quad d''_n = d_n \mathbf{1}_{\{\|d_n\| > 2d^*_{n-1}\}},$$

and $f'_n = \sum_{k=1}^n d'_k$ and $f''_n = \sum_{k=1}^n d''_k$. It follows from Lemma 3.10 that $f', f'' \in \mathscr{D}_\infty$ and that $\mathscr{F}_\infty$-decoupled tangent sequences of $(d'_n)_{n\ge1}$ and $(d''_n)_{n\ge1}$ are given by $e'_1 := 0, e''_1 := e_1$ and for $n \ge 2$:

$$e'_n = e_n \mathbf{1}_{\{\|e_n\| \le 2d^*_{n-1}\}}, \quad\text{and}\quad e''_n = e_n \mathbf{1}_{\{\|e_n\| > 2d^*_{n-1}\}}. \tag{3.2.8}$$

Moreover, the random variable d'_n is bounded by the $\mathscr{F}_{n-1}$-measurable random variable $2d^*_{n-1}$. On the other hand, for the sequence $(d''_n)_{n\ge1}$ we have on the set $\{\|d_n\| > 2d^*_{n-1}\}$, $n \ge 2$,

$$(2^r - 1)\|d''_n\|^r + (2d^*_{n-1})^r \le (2^r - 1 + 1)\|d''_n\|^r \le 2^r (d^*_n)^r,$$

whence $\|d''_n\|^r \le (1 - 2^{-r})^{-1}[(d^*_n)^r - (d^*_{n-1})^r]$ and thus

$$\|f''\|^r \le \sum_{n=1}^\infty \|d''_n\|^r \le (1 - 2^{-r})^{-1}(d^*)^r \tag{3.2.9}$$

(for $r = 1$ see [35] or [16, inequality (4.5)]).

By (3.2.9) and due to the fact that Φ is non-decreasing we have:

$$\begin{aligned}
\mathbb{E}\Phi(f^*) &\le \mathbb{E}\Phi(2^{\frac{1}{r}-1}[f'^* + f''^*]) \le \mathbb{E}\Phi(2^{\frac{1}{r}-1}[f'^* + (1-2^{-r})^{-\frac{1}{r}} d^*]) \\
&= \int_0^\infty \mathbb{P}(2^{\frac{1}{r}-1}[f'^* + (1-2^{-r})^{-\frac{1}{r}} d^*] > \lambda)\, d\Phi(\lambda) \\
&\le \mathbb{E}\Phi(2^{\frac{1}{r}} f'^*) + \int_0^\infty \mathbb{P}(2^{\frac{1}{r}} d^* > (1-2^{-r})^{\frac{1}{r}} \lambda)\, d\Phi(\lambda).
\end{aligned}$$

Using Lemma 3.9 and the Lévy inequality applied conditionally (Lemma 3.5) we can estimate the right-most term in the above:

$$
\begin{aligned}
\int_0^\infty \mathbb{P}(d^* > 2^{-\frac{1}{r}}(1-2^{-r})^{\frac{1}{r}}\lambda)\, d\Phi(\lambda) &\le 2\int_0^\infty \mathbb{P}(e^* > 2^{-\frac{1}{r}}(1-2^{-r})^{\frac{1}{r}}\lambda)\, d\Phi(\lambda) \\
&\le 4\int_0^\infty \mathbb{P}(\|g\| > 2^{1-\frac{2}{r}}(1-2^{-r})^{\frac{1}{r}}\lambda)\, d\Phi(\lambda) \\
&= 4\mathbb{E}\Phi(2^{\frac{2}{r}-1}(1-2^{-r})^{-\frac{1}{r}}\|g\|).
\end{aligned}
\tag{3.2.10}
$$

Therefore, we conclude that

$$
\mathbb{E}\Phi(f^*) \le 2^{\frac{q}{r}}\mathbb{E}\Phi(f'^*) + 2^{\frac{2q}{r}-q+2}(1-2^{-r})^{-\frac{q}{r}}\mathbb{E}\Phi(\|g\|), \tag{3.2.11}
$$

with q as in (3.2.6). It remains to estimate $\mathbb{E}\Phi(f'^*)$, for which we use Lemma 3.12.

We follow the proof of [66, Lemma 2.2] to show that for $\delta \in (0,1)$ and $\beta \in (1,2)$ as in Lemma 3.12 one has:

$$
\begin{aligned}
&\mathbb{P}(f'^* \ge \beta\lambda, g'^* < \delta_2\lambda) \\
&\qquad \le b\mathbb{P}(f'^* \ge \lambda) + \mathbb{P}(2d^* \ge \delta_2\lambda) + (1-2^{p-\frac{2p}{r}})\mathbb{P}(f'^* \ge \beta\lambda),
\end{aligned}
\tag{3.2.12}
$$

where $\delta_2 = 4^{-\frac{1}{r}}\delta$ and $g' = \sum_{n\ge 1} e'_n$. Indeed,

$$
\begin{aligned}
\mathbb{P}(f'^* \ge \beta\lambda, g'^* < \delta_2\lambda) \le \mathbb{P}(f'^* \ge \beta\lambda, T_p^*(f') < \delta\lambda, 2d^* < \delta_2\lambda) + \mathbb{P}(2d^* \ge \delta_2\lambda) \\
+ \mathbb{P}(f'^* \ge \beta\lambda, T_p^*(f') \ge \delta\lambda, 2d^* < \delta_2\lambda, g'^* < \delta_2\lambda).
\end{aligned}
\tag{3.2.13}
$$

As $\delta_2 \le \delta$ it follows from the definition of $(d'_n)_{n\ge 1}$ and from Lemma 3.12 that for the first probability on the right-hand side of (3.2.13) one has:

$$
\begin{aligned}
\mathbb{P}(f'^* \ge \beta\lambda, T_p^*(f') < \delta\lambda, 2d^* < \delta_2\lambda) &\le \mathbb{P}(f'^* \ge \beta\lambda, T_p^*(f') < \delta\lambda, d'^* < \delta\lambda) \\
&\le b\mathbb{P}(f'^* \ge \lambda).
\end{aligned}
\tag{3.2.14}
$$

It remains to estimate the last probability in (3.2.13). Since f'^*, d^* and $T_p^*(f')$ are all $\mathscr{F}_\infty$-measurable, by conditioning on $\mathscr{F}_\infty$ we see that this probability is equal to

$$
\mathbb{E}[\mathbf{1}_{\{f'^* \ge \beta\lambda, T_p^*(f') \ge \delta\lambda, 2d^* < \delta_2\lambda\}}\mathbb{P}(g'^* < \delta_2\lambda \,|\, \mathscr{F}_\infty)]. \tag{3.2.15}
$$

By Lemma 3.8 we have:

$$
\mathbb{P}(g'^* < \delta_2\lambda \,|\, \mathscr{F}_\infty) \le 1 - 2^{p-1}\left[2^{-\frac{2p}{r}+2} - \frac{(\delta_2\lambda)^p + \mathbb{E}[(e'^*)^p \,|\, \mathscr{F}_\infty]}{\mathbb{E}(\|g'\|^p \,|\, \mathscr{F}_\infty)}\right],
$$

observing that $u_{p/r} = 2^{\frac{p}{r}-1}$ as $p \ge r$. Note that $\mathbb{E}(\|g'\|^p \,|\, \mathscr{F}_\infty) = T_p(f')$ and by (3.2.8) we have $e'^* \le 2d^*$, and thus on the set

$$
S := \{f'^* \ge \beta\lambda, T_p^*(f') \ge \delta\lambda, 2d^* < \delta_2\lambda\}
$$

one has:

$$\mathbb{P}(g'^* < \delta_2\lambda \,|\, \mathscr{F}_\infty) \le 1 - 2^{p-1}\left[2^{-2\frac{p}{r}+2} - \frac{2(\delta_2\lambda)^p}{(\delta\lambda)^p}\right] = 1 - 2^{p-\frac{2p}{r}}.$$

Therefore we find:

$$\mathbb{E}[\mathbf{1}_S\mathbb{P}(g'^* < \delta_2\lambda \,|\, \mathscr{F}_\infty)] \le \mathbb{E}[\mathbf{1}_S(1 - 2^{p-\frac{2p}{r}})] \le (1 - 2^{p-\frac{2p}{r}})\mathbb{P}(f^* \ge \beta\lambda). \quad (3.2.16)$$

Combining equations (3.2.13), (3.2.14), (3.2.15) and (3.2.16) gives (3.2.12).

It follows from (3.2.12) that

$$\begin{aligned}&\mathbb{P}(f'^* \ge \beta\lambda)\\ &\quad\le b\mathbb{P}(f'^* \ge \lambda) + \mathbb{P}(2d^* \ge \delta_2\lambda) + (1 - 2^{p-\frac{2p}{r}})\mathbb{P}(f'^* \ge \beta\lambda) + \mathbb{P}(g'^* \ge \delta_2\lambda).\end{aligned}$$

Collecting terms and integrating with respect to $d\Phi(\lambda)$ gives that

$$\mathbb{E}\Phi(f'^*/\beta) \le 2^{\frac{2p}{r}-p}[b\mathbb{E}\Phi(f'^*) + \mathbb{E}\Phi(2d^*/\delta_2) + \mathbb{E}\Phi(g'^*/\delta_2)].$$

From this we see that (because $\beta < 2$ and Φ is non-decreasing)

$$\begin{aligned}\mathbb{E}\Phi(f'^*) &= \mathbb{E}\Phi(\beta f'^*/\beta) \le 2^q\mathbb{E}\Phi(f'^*/\beta)\\ &\le 2^{\frac{2p}{r}-p+q}[b\mathbb{E}\Phi(f'^*) + \mathbb{E}\Phi(2d^*/\delta_2) + \mathbb{E}\Phi(g'^*/\delta_2)].\end{aligned}$$

Since $b = 2^{p-\frac{2p}{r}-q-1}$ and $\delta_2 = 4^{-\frac{1}{r}}\delta = 2^{-\frac{2}{r}}\left(\frac{(3/2)^r-1}{2A^r+1}\right)^{\frac{1}{r}}$ we have:

$$\begin{aligned}\mathbb{E}\Phi(f'^*) &\le 2^{\frac{2p}{r}-p+q+1}[\mathbb{E}\Phi(2d^*/\delta_2) + \mathbb{E}\Phi(g'^*/\delta_2)]\\ &\le 2^{\frac{2p}{r}-p+\frac{2q}{r}+q+1}\left(\frac{2A^r+1}{(3/2)^r-1}\right)^{\frac{q}{r}}[2^q\mathbb{E}\Phi(d^*) + \mathbb{E}\Phi(g'^*)].\end{aligned} \quad (3.2.17)$$

As before in (3.2.10) one can prove that $\mathbb{E}\Phi(d^*) \le 2^{\frac{q}{r}-q+2}\mathbb{E}\Phi(\|g\|)$. By the Lévy inequality we obtain $\mathbb{E}\Phi(g'^*) \le 2^{\frac{q}{r}-q+1}\mathbb{E}\Phi(\|g'\|)$. By Corollary 3.6 and the definition of $(e'_n)_{n\ge1}$ we have:

$$\begin{aligned}\mathbb{E}\Phi(\|g'\|) &= \mathbb{E}\int_0^\infty \mathbb{P}\Big(\Big\|\sum_{k=1}^n \mathbf{1}_{\{\|e_k\|\le d^*_{k-1}\}}e_k\Big\| > \lambda \,\Big|\, \mathscr{F}_\infty\Big)d\Phi(\lambda)\\ &\le 2\int_0^\infty \mathbb{P}\Big(\Big\|\sum_{k=1}^n e_k\Big\| > 2^{1-\frac{1}{r}}\lambda \,\Big|\, \mathscr{F}_\infty\Big)d\Phi(\lambda) \le 2^{\frac{q}{r}-q+1}\mathbb{E}\Phi(\|g\|).\end{aligned} \quad (3.2.18)$$

Combining equations (3.2.11) and (3.2.17) with the estimates above gives:

$$\mathbb{E}\Phi(f^*) \le C_{X,r,p,q}\mathbb{E}\Phi(\|g\|),$$

for all $f \in \mathscr{D}_\infty$, where

$$C_{X,r,p,q} = 2^{\frac{2q}{r}-q+2}\Big[2^{\frac{2p}{r}-p+\frac{2q}{r}+1}(2^{2q}+2^{\frac{q}{r}})\Big(\frac{2A^r+1}{(3/2)^r-1}\Big)^{\frac{q}{r}}+(1-2^{-r})^{-\frac{q}{r}}\Big]. \tag{3.2.19}$$

However, recall that we assumed $r \leq p$ at the beginning of the proof, thus in the above one should read $\min\{r,p\}$ at every occasion of r. □

Finally, we recall the following lemma, which can be proven like [36, Corollary 6.4.3]. The inequalities in this lemma are to be interpreted in the sense that the left-hand side is finite whenever the right-hand side is so.

Lemma 3.15. *Let X be an r-normable quasi-Banach space and let $\Phi \in F_q$ for some $q \in (0,\infty)$. Suppose that exists a $C \geq 0$ such that for every complete probability space $(\Omega,(\mathscr{F}_n)_{n\geq1},\mathscr{A},\mathbb{P})$ and every $(\mathscr{F}_n)_{n\geq1}$-adapted X-valued sequence $(f_n)_{n\geq1}$, where f_n-f_{n-1} is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$ ($f_0 \equiv 0$), and every decoupled sum sequence g of f we have:*

$$\mathbb{E}\Phi(f_n^*) \leq C\mathbb{E}\Phi(g_n^*), \quad n \geq 1.$$

Then for every complete probability space $(\Omega,(\mathscr{F}_n)_{n\geq1},\mathscr{A},\mathbb{P})$ and every X-valued sequence $(f_n)_{n\geq1}$ adapted to $(\mathscr{F}_n)_{n\geq1}$ we have:

$$\mathbb{E}\Phi(f_n^*) \leq 2^{\frac{q}{r}}(2^{1+\frac{q}{r}}C+1)\mathbb{E}\Phi(g_n^*), \quad n \geq 1.$$

The same result holds with f_n^* and g_n^* replaced by f_n and g_n in both the assumption and the assertions.

3.3 p-Independence and the decoupling constant

The p-independence of the decoupling inequality follows from taking $\Phi(s) = s^q$ in Theorem 3.16 below.

Theorem 3.16. *Let X be an r-normable quasi-Banach space in which the decoupling inequality* (3.0.1) *holds for some $p \in (0,\infty)$, then for $\Phi \in F_q$ for some $q \in (0,\infty)$ there exists a constant $K = K_{X,r,p,q}$ such that for all complete probability spaces $(\Omega,\mathscr{A},(\mathscr{F}_n)_{n\geq1},\mathbb{P})$ and $(\mathscr{F}_n)_{n\geq1}$-adapted sequences $(f_n)_{n\geq1}$ one has:*

$$\mathbb{E}\Phi(\|f_n\|) \leq K\mathbb{E}\Phi(\|g_n\|) \quad \text{and} \quad \mathbb{E}\Phi(f_n^*) \leq K\mathbb{E}\Phi(g_n^*), \quad n \geq 1, \tag{3.3.1}$$

where g is a $\mathscr{F}_\infty$-decoupled sum sequence of f.

Now assume X is a Banach space and $p \geq 1$. Then the constant K can be estimated by:

$$K \leq 2^{6+p+11q+\frac{q}{p}+\frac{q^2}{p}} D_p^q(X), \tag{3.3.2}$$

and in particular,

$$D_q(X) \leq 2^{11+\frac{1}{p}+\frac{6}{q}+\frac{p}{q}+\frac{q}{p}} D_p(X),$$

for all $q \in (0,\infty)$. *Moreover, for* $q \in (p,\infty)$ *there exists a constant* $k_{X,p}$ *such that*

$$D_q(X) \leq k_{X,p} q. \tag{3.3.3}$$

We interpret (3.3.1) in the sense that the left-hand side is finite whenever the right-hand side is so. Note that (3.3.3) is an improvement of (3.3.2) as it implies that $D_q(X)$ grows at most linearly in q whereas (3.3.2) gives an exponential bound.

Proof. By assumption the decoupling inequality holds in the r-normable quasi-Banach space X for some $p \in (0,\infty)$. Lemma 3.15 states the following: If there exists a constant $C_{X,r,p,q}$ such that for every complete probability space $(\Omega, (\mathscr{F}_n)_{n\geq 1}, \mathscr{A}, \mathbb{P})$ and every $(\mathscr{F}_n)_{n\geq 1}$-adapted $(f_n)_{n\geq 1}$, for which d_n is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$, one has:

$$\mathbb{E}\Phi(f^*) \leq C_{X,r,p,q} \mathbb{E}\Phi(\|g\|), \tag{3.3.4}$$

where g is a decoupled sum sequence of f, then (3.3.1) holds with

$$K_{X,p,q} = K_{X,r,p,q} \leq 2^{\frac{q}{r}}(2^{1+\frac{q}{r}} C_{X,r,p,q} + 1). \tag{3.3.5}$$

Fix a complete probability space $(\Omega, (\mathscr{F}_n)_{n\geq 1}, \mathscr{A}, \mathbb{P})$. We wish to apply Proposition 3.14; i.e., we wish to prove that assumption 3.2.2 is satisfied for $b = 2^{p-\frac{2p}{r}-q-1}$ and some $A > 0$ (independent of the probability space). Let $(f_n)_{n\geq 1} \in \mathscr{D}_\infty$ where $\mathscr{D}_\infty$ is as defined on page 44, and let g be a decoupled sum sequence of f on $(\Omega, \mathscr{A}, \mathbb{P})$. Pick $0 \leq k \leq l$ and let $B \in \mathscr{F}_k$. Observe that $T_p({}^k\!f^l \mathbf{1}_B) = T_p({}^k\!f^l)\mathbf{1}_B$. By applying Chebyshev's inequality in the final line we obtain:

$$\begin{aligned}
\mathbb{P}(\{\|{}^k\!f^l\| > A\|T_p({}^k\!f^l)\|_\infty\} \cap B) &= \mathbb{P}(\|{}^k\!f^l \mathbf{1}_B\| > A\|T_p({}^k\!f^l)\|_\infty \mathbf{1}_B) \\
&\leq \mathbb{P}(\|{}^k\!f^l \mathbf{1}_B\| > A\|T_p({}^k\!f^l)\mathbf{1}_B\|_\infty) \\
&\leq A^{-p}\|T_p({}^k\!f^l)\mathbf{1}_B\|_\infty^{-p} \|{}^k\!f^l \mathbf{1}_B\|_p^p.
\end{aligned} \tag{3.3.6}$$

By Lemma 3.10 we have that $({}^k\!g_n^l \mathbf{1}_B)_{n\geq 1}$ is a decoupled sum sequence of $({}^k\!f_n^l \mathbf{1}_B)_{n\geq 1}$. Thus, because the decoupling inequality holds in X for p, we have:

$$\begin{aligned}
\|{}^k\!f^l \mathbf{1}_B\|_p^p &\leq D_{p,X}^p \|{}^k\!g_n^l \mathbf{1}_B\|_p^p = D_{p,X}^p \|T_p({}^k\!f_n^l \mathbf{1}_B)\|_p^p \\
&= D_{p,X}^p \|T_p({}^k\!f_n^l \mathbf{1}_B)\mathbf{1}_B\|_p^p \leq D_{p,X}^p \|T_p({}^k\!f_n^l \mathbf{1}_B)\|_\infty^p \mathbb{P}(B).
\end{aligned} \tag{3.3.7}$$

Thus setting $A = b^{-\frac{1}{p}} D_p(X)$ one obtains:

$$\mathbb{P}(\{\|{}^k\!f^l\| > A\|T_p({}^k\!f^l)\|_\infty\} \cap B) \leq b\mathbb{P}(B).$$

Thus condition (3.2.2) in Proposition 3.14 is satisfied, and therefore (3.3.4) holds for all $f \in \mathscr{D}_\infty$ with a constant $C_{X,p,q,r}$ as given in that proposition. For general $(\mathscr{F}_n)_{n\geq 1}$-adapted sequences $(f_n)_{n\geq 1}$ with decoupled sum sequence g defined on $(\Omega, \mathscr{A}, \mathbb{P})$ such that d_n is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$, we can reduce to the former case as follows:

$$\begin{aligned}\mathbb{E}\Phi(f^*) &\overset{(i)}{\leq} \liminf_{j\to\infty} \mathbb{E}\Phi\Big(\sup_{n\geq 1}\Big\| \sum_{k=1}^{n\vee j} d_k \mathbf{1}_{\|d_k\|\leq j}\Big\|\Big) \\ &\overset{(ii)}{\leq} C_{X,p,q,r} \liminf_{j\to\infty} \mathbb{E}\Phi\Big(\Big\| \sum_{k=1}^{j} e_k \mathbf{1}_{\|e_k\|\leq j}\Big\|\Big) \overset{(iii)}{\leq} 2^{\frac{q}{r}-q+1} C_{X,p,q,r} \mathbb{E}\Phi\|g\|.\end{aligned}$$

In (i) we used Fatou's lemma. We applied (3.3.4) in (ii), where we use that by Lemma 3.10 $(e_k\mathbf{1}_{\|e_k\|\leq j})_{k=1}^n$ is a $\mathscr{F}_\infty$-conditionally symmetric decoupled tangent sequence of $(d_k\mathbf{1}_{\|d_k\|\leq j})_{k=1}^n$. In (iii) we used Corollary 3.6 as in (3.2.18).

We have thus proven that (3.3.4) holds for an arbitrary yet fixed complete probability space, with a constant $C_{X,r,p,q}$ independent of the probability space. This completes the proof of inequality (3.3.1).

If $r = 1$ and $p \geq 1$ then one can pick $b = 2^{-p-q-1}$ in the proof of Proposition 3.14, and then $A = b^{-\frac{1}{p}} D_p(X) = 2^{1+\frac{1}{p}+\frac{q}{p}} D_p(X)$. Entering this in equation (3.2.7) in Proposition 3.14 leads to the following estimate for the constant in (3.3.4):

$$C_{X,1,p,q} \leq 2^{2q+2}\Big[2^{p+4q+2}(2^{2+\frac{1}{p}+\frac{q}{p}} D_p(X) + 1)^q\Big], \tag{3.3.8}$$

which, in combination with (3.3.5) and some rough estimates, leads to equation (3.3.2).

Now let X be a Banach space. Inequality (3.3.3) follows from Remark 3.13 in the following manner: by the calculations in (3.3.6) and (3.3.7) it follows that the conditions of Proposition 3.14 hold by fixing some $b \in (0, 1)$ and taking $A := b^{-\frac{1}{p}} D_p(X)$ in (3.2.2). Thus it follows from Remark 3.13 there exists a constant $c_{X,p}$ such that:

$$\|f^*\|_q \leq c_{X,p} q \|T_p(f)\|_q \leq c_{X,p} q \|g\|_q, \tag{3.3.9}$$

for all $q \in (p, \infty)$ and all $f \in D_\infty$ with decoupled sum sequence g. (Note that $c_{X,p}$ also depends on our choice of b.) By applying the same approximation and symmetrization arguments as before we obtain inequality (3.3.3). $\square$

From the proof above we obtain a somewhat stronger result, i.e., a maximal inequality for conditionally symmetric adapted sequences:

Corollary 3.17. *Let X be a Banach space in which the decoupling inequality* (3.0.1) *holds for some $p \in (0, \infty)$, then for every $\Phi \in F_q$, $q \in (0, \infty)$, and every $(\mathscr{F}_n)_{n\geq 1}$-adapted X-valued sequence $(f_n)_{n\geq 1}$ such that d_n is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$ one has:*

$$\mathbb{E}\Phi(f_n^*) \leq C_{X,r,p,q}\mathbb{E}\Phi(g_n), \quad n \geq 1,$$

where g is a decoupled sum sequence of f and $C_{X,r,p,q}$ is as given in (3.3.8)*. Moreover, if X is a Banach space we have $C_{X,1,p,q} \leq C_{X,p}q$ for $q \in (p,\infty)$, for some constant $C_{X,p}$.*

Remark 3.18. From the proof of Theorem 3.16 it follows that in order to check whether a (quasi-)Banach space satisfies the decoupling inequality it suffices to check whether the following weak estimate holds: for some $p \in (0,\infty)$ and some $b \in (0,1)$ there exists an $A = A(b,X,r,p)$ such that

$$\sup_{f\in\mathscr{D}_\infty} \sup_{0\leq k\leq l} \sup_{B\in\mathscr{F}_k, B\neq\varnothing} \mathbb{P}(\|{}^k\!f^l\| \geq A\|T_p({}^k\!f^l)\|_\infty \,|\, B) \leq b.$$

After all, if this holds for some $b \in (0,1)$, there will be a $p \in (0,\infty)$ such that $b \leq 2^{-\frac{2p}{r}-1}$. We then take $\Phi = x^p$ in Proposition 3.14 (i.e., $q = p$) and obtain that (3.3.4) holds for $f \in \mathscr{D}_\infty$ on a arbitrary yet fixed complete probability space $(\Omega, (\mathscr{F}_n)_{n\geq1}, \mathscr{A}, \mathbb{P})$. By the same arguments as in the proof of Theorem 3.16 above we find that the decoupling inequality holds in p for X, and thus, by Theorem 3.16, X is a Banach space for which the decoupling inequality holds.

Corollary 3.19. *If X is a* UMD *space, then the decoupling inequality holds.*

Proof. As explained on page 35, if X is a UMD space then (3.0.2) holds for all martingale difference sequences and for all $p \in (1,\infty)$. Therefore, by Lemma 3.15 and Theorem 3.16 every UMD space satisfies the decoupling inequality. □

The lemma below implies that the decoupling property is a super-property: if X is a quasi-Banach space satisfying the decoupling inequality and Y is a quasi-Banach space that is finitely representable in X, then Y satisfies the decoupling inequality and $D_p(Y) \leq D_p(X)$, $p \in (0,\infty)$. For the definition of finite representability we refer to [2].

Lemma 3.20. *A quasi-Banach space X satisfies the decoupling inequality in $p \in (0,\infty)$ with constant $D_p(X)$ if and only if* (3.0.1) *in Definition 3.2 holds with constant $D_p(X)$ for every finitely-valued X-valued $(\mathscr{F}_n)_{n\geq1}$-adapted finite sequence $f = (f_k)_{k=1}^n$, for any probability space $(\Omega, \mathscr{A}, (\mathscr{F}_n)_{n\geq1}, \mathbb{P})$.*

Proof. Fix $p \in (0,\infty)$. It is clear from the definition that it suffices to consider finite sequences. Let $(\Omega, \mathscr{A}, (\mathscr{F}_n)_{n\geq1}, \mathbb{P})$ be a probability space and let $(f_k)_{k=1}^n$ be a X-valued, $(\mathscr{F}_k)_{k=1}^n$-adapted L^p-sequence, and let $(g_k)_{k=1}^n$ be the decoupled sum sequence of $(f_k)_{k=1}^n$. By strong measurability we may assume that $(f_k)_{k=1}^n$ and $(g_k)_{k=1}^n$ take values in a separable subspace $X_0 \subseteq X$. Let $(x_n)_{n\geq1}$ be a dense subset of X_0 such that $x_1 = 0$. For $m \in \mathbb{N}$ we define $\phi_m : X \to \mathbb{R}$ by

$$\phi_m(x) = \min_{1\leq n\leq m} \{\|x - x_n\| : \|x_n\| \leq \|x\|\}.$$

For $n, m \in \mathbb{N}$, $n \leq m$ define $E_{n,m} := \{x \in X : \|x - x_n\| = \phi_m(x)\}$. Define $\psi_m : X \to \{x_1, \ldots, x_m\}$ by

$$\psi_m(x) = x_n; \qquad x \in E_{n,m} \setminus \bigcup_{j=1}^{n-1} E_{j,m}.$$

Clearly, ψ_m is $\mathscr{B}(X)$-measurable. Moreover, for all $x \in X$ one has $\|\psi_m(x)-x\| \to 0$ as $m \to \infty$, and $\psi_m(x) \le \|x\|$. Thus by the dominated convergence theorem we have $\psi_m(f_k) \to f_k$ and $\psi_m(g_k) \to g_k$ in $L^p(X)$, for all $k = 1, \ldots, n$. By Lemma 3.10, $\psi_m(g_k)$ is the decoupled sum sequence for $\psi_m(f_k)$ for all $m \in \mathbb{N}$, so if (3.0.1) holds for the pairs $\psi_m(f_k)$ and $\psi_m(g_k)$ for all m with some constant D_p, then it also holds for $(f_k)_{k=1}^n$ and $(g_k)_{k=1}^n$ with the same constant. □

Corollary 3.21. *Let Y be a space for which the decoupling inequality* (3.0.1) *holds. Let (S, Σ, μ) be a nonzero measure space and let $q \in (0, \infty)$. Then $X = L^q(S; Y)$ satisfies the decoupling inequality. Moreover, $D_p(L^p(S;Y)) = D_p(Y)$.*

Proof (of Corollary 3.21). By Lemma 3.20 it suffices to consider finite sequences taking values in a finite subset of $L^p(S;Y)$. Thus without loss of generality we may assume that (S, Σ, μ) is σ-finite. Then the proof follows from Theorem 3.16 by the same method as in [30, Theorem 14], where $q = 1$ has been considered.□

In particular, we have the following examples.

Example 3.22. Let (S_i, Σ_i, μ_i) be a measure space and let $q_i \in (0, \infty)$ for $1 \le i \le n$. Let $X = L^{q_1}(S_1; L^{q_2}(S_2; \ldots L^{q_n}(S_n)))$, then the decoupling inequality holds for X. Note these spaces are not UMD spaces if $q_i \le 1$ for some i.

Example 3.23. Let (S, Σ) be a measurable space. Let X be the space of bounded σ-additive measures on (S, Σ) equipped by the variation norm. Then X is a Banach lattice where $\mu_1 \le \mu_2$ if $\mu_1(A) \le \mu_2(A)$ for all $A \in \Sigma$. Moreover, X is an abstract L^1-space and hence by [3, Theorem 4.27], the decoupling property holds for X.

A consequence of Corollary 3.21 is the following result for Hilbert spaces X.

Corollary 3.24. *Let X be a Hilbert space. Then for every $p \in [1, \infty]$ and every adapted X-valued f in $L^p(\Omega; X)$ one has:*

$$\|f_n\|_p \le D_{\mathbb{R}} \|g_n\|_p,$$

for all $n \in \mathbb{N}$, where g is a decoupled sum sequence of f and $D_{\mathbb{R}}$ as in (3.0.4).

Using this we prove a similar statement for estimates of type (3.3.1), see inequality (3.3.10). Note that it has been proven that a Hilbert space X satisfies the decoupling inequality in [36, Corollary 6.4.3], but it has not been proven that the constants $D_p(X)$ are uniformly bounded. It seems that the arguments of [65], [36, Chapter 7] do *not* extend to the vector-valued situation and a different argument is used.

Proof. Let X be a Hilbert space. Let $p \in [1, \infty)$ be given and let f be an adapted X-valued L^p-sequence. Because f is strongly measurable we may assume that X is separable. As every separable Hilbert space is isometrically isomorphic to a closed subspace of ℓ^2 we may assume f to be an adapted ℓ^2-valued L^p-sequence. It is known that ℓ^2 embeds isometrically in $L^p(0,1)$ for all $1 \leq p < \infty$ (see [2, Proposition 6.4.13]), let $J_p : \ell^2 \to L^p(0,1)$ denote this isometric embedding. Let g be a decoupled sum sequence of f. Observe that $J_p f$ is an adapted $L^p(0,1)$-valued L^p-sequence with decoupled sum sequence $J_p g$. By equation (3.0.4) on page 37 and Corollary 3.21 it follows that, for all $n \geq 1$,

$$\begin{aligned}\|f_n\|_{L^p(\Omega;\ell^2)} = \|J_p f_n\|_{L^p(\Omega;L^p(0,1))} &\leq D_{\mathbb{R}} \|J_p g_n\|_{L^p(\Omega;L^p(0,1))} \\ &= D_{\mathbb{R}} \|g_n\|_{L^p(\Omega;\ell^2)}.\end{aligned}$$

□

Remark 3.25. We mention some direct consequences of Corollary (3.24). Let X be a Hilbert space.

(i) Let $\Phi \in F_q$ for some $q \in (0, \infty)$, with F_q as defined on page 46. By Corollary 3.24 we have $D_q(X) \leq D_{\mathbb{R}}$; using this and substituting $p = q$ in (3.3.2) we obtain:

$$\mathbb{E}\Phi(\|f\|) \leq 2^{7+13q} [D_{\mathbb{R}}]^q \mathbb{E}\Phi(\|g\|), \tag{3.3.10}$$

for all X-valued sequences f. This improves [36, Corollary 6.4.3] where this estimate has been proven without giving a bound on the constant.

(ii) In [65, Section 6] it has been observed that if $D_p(X)$ is uniformly bounded in p then using Taylor expansions one obtains estimates for $\mathbb{E}\Phi(\|f_n\|)$ even if Φ does not satisfy (3.2.6). This applies for example to the exponential function. I.e. by Corollary 3.24 and Taylor expansions one has:

$$\mathbb{E}\exp(\|f_n\|) \leq \mathbb{E}\exp(D_{\mathbb{R}}\|g_n\|),$$

for all X-valued adapted sequences f. For the real case this estimate also follows for mean-zero sequences from a result in [36, Section 6.2] (with constant 2 instead of $D_{\mathbb{R}}$).

We conclude this section with some observations. In [48, p. 105] it has been proven that c_0 does not have the decoupling property by proving that for any dimension d one has $D_p(\ell^\infty_{(d)}) \geq 4^{-1} K_{p,2}^{-1} [\frac{\log d}{\log 2}]^{\frac{1}{2}}$ where $K_{p,2}$ is the optimal constant in the Kahane-Khintchine inequality. We have the following upper estimate for $D_p(\ell^\infty_{(d)})$ for p large:

Corollary 3.26. *Let $d \in \mathbb{N}$ and $p \geq \frac{\log d}{\log 2}$, then $D_p(\ell^\infty_{(d)}) \leq 2D_{\mathbb{R}}$.*

Proof. Recall from Corollary 3.21 that $D_p(\ell_p) = D_{\mathbb{R}}$ and hence for any $\ell^\infty_{(d)}$ valued L^p-sequence f with decoupled sum sequence g and any $n \in \mathbb{N}$, one has by Hölder's inequality:

$$(\mathbb{E}\|f_n\|^p_{\ell^\infty_{(d)}})^{\frac{1}{p}} \leq (\mathbb{E}\|f_n\|^p_{\ell^p_{(d)}})^{\frac{1}{p}} \leq D_{\mathbb{R}}(\mathbb{E}\|g_n\|^p_{\ell^p_{(d)}})^{\frac{1}{p}}$$
$$\leq D_{\mathbb{R}} d^{\frac{1}{p}}(\mathbb{E}\|g_n\|^p_{\ell^\infty_{(d)}})^{\frac{1}{p}} \leq 2D_{\mathbb{R}}(\mathbb{E}\|g_n\|^p_{\ell^\infty_{(d)}})^{\frac{1}{p}}.$$

□

Remark 3.27. As in [66] the L^p-norms in the decoupling inequality (3.0.1) can be replaced by certain rearrangement invariant quasi-norms: Let X be a quasi-Banach space satisfying the decoupling inequality and let Y be a (p,q)-K-interpolation space for some $0 < p,q < \infty$ on some complete probability space $(\Omega, \Sigma, \mathbb{P})$. Then there exists a constant D such that for all sequences $(f_n)_{n\geq 1}$ for which $\|f_n\|_X \in Y$ for all $n \geq 1$, with decoupled sum sequence $(g_n)_{n\geq 1}$ one has:

$$\big\|\|f_n\|_X\big\|_Y \leq D\big\|\|g_n\|_X\big\|_Y, \quad \text{for all } n \geq 1.$$

The proof of this statement is entirely analogous to [66, Corollary 1.4]. Examples of (p,q)-K-interpolation spaces include all (p,q)-interpolation spaces with $1 \leq p,q \leq \infty$ and the Lorentz spaces $L_{p,q}$ for $0 < p,q < \infty$. Recall that a rearrangement invariant space Y is an (p,q)-interpolation space if the Boyd indices p_0, q_0 satisfy $p < p_0$, $q > q_0$ [11].

Remark 3.28. Let X be a UMD space and let $\mathscr{H}$ be the Hilbert transform on $L^p(\mathbb{R};X)$ (or equivalently the periodic Hilbert transform on $L^p(0,2\pi;X)$). The estimate $\|\mathscr{H}\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq \beta_p(X)^2$, where $\beta_p(X)$ is the UMD constant of X, is the usual estimate in the literature (see [19, 47]). As the proofs in [47] and [19] involve only Paley-Walsh martingales, it follows that one actually has:

$$\|\mathscr{H}\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq C_p(X) D_p(X),$$

where $C_p(X)$ and $D_p(X)$ are as in (3.0.2). Recall that $\max\{C_p(X), D_p(X)\} \leq \beta_p(X)$. Moreover, the behavior of $D_p(X)$ as $p \downarrow 1$ is better than $\beta_p(X)$. Indeed, according to Theorem 3.16 one has $\sup_{p\in[1,2]} D_p(X) < \infty$, but $\beta_p(X) \to \infty$ as $p \downarrow 1$. Although we do not know whether $\sup_{p\in[2,\infty)} D_p(X) < \infty$, still a similar behavior occurs for the norm of $\mathscr{H}$ as $p \to \infty$. This follows from a duality argument. Indeed, recall that X^* is a UMD space again, and if $p \in [2,\infty)$, then with $1/p + 1/p' = 1$, we find:

$$\|\mathscr{H}\|_{\mathscr{L}(L^p(\mathbb{R};X))} = \|\mathscr{H}^*\|_{\mathscr{L}(L^{p'}(\mathbb{R};X^*))} \leq C_{p'}(X^*) D_{p'}(X^*).$$

Now $\sup_{p\in[2,\infty)} D_{p'}(X^*) < \infty$. Moreover, $C_{p'}(X^*) \leq \beta_{p'}(X^*) \leq \beta_p(X)$ by a duality argument.

3.4 Applications to stochastic integration

In this section let X be a Banach space, $(\Omega, (\mathscr{F}_t)_{t\geq 0}, P)$ a complete probability space and H a real separable Hilbert space. Our aim is to prove an extension of Theorem 2.7, see Theorem 3.29 below. The extension concerns proving that if X is a UMD space, then the Burkholder-Davis-Gundy inequalities (2.4.5) hold for p^{th} moments, for all $p \in (0,\infty)$ (i.e., not only for $p \in (1,\infty)$). We also prove that the one-sided Burkholder-Davis-Gundy inequality, equation (3.4.3) below, holds in spaces that satisfy the decoupling inequality. The idea of the proof of Theorem 3.29 is taken from [108, Lemma 3.5], an alternative approach would be to use the extrapolation results in [90].

In the UMD setting, we use a version of the decoupling inequalities in equation (3.0.2). To be precise, if X is a UMD space, then by [63, Theorem 3'] (see also [30, Proposition 2]) one has, for all $p \in (0,\infty)$:

$$\|f^*\|_p \eqsim_{p,X} \|g\|_p, \tag{3.4.1}$$

for all $(\mathscr{F}_n)_{n\geq 1}$-adapted X-valued L^p-sequences $(f_n)_{n\geq 1}$ on some complete probability space such that $f_n - f_{n-1}$ is $\mathscr{F}_{n-1}$-conditionally symmetric for all $n \geq 1$, and g a decoupled sum sequence of f.

Theorem 3.29. *Let X be a Banach space and H be a separable Hilbert space. Let W_H be an H-cylindrical $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion. Let $\Psi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ be an H-strongly measurable and $(\mathscr{F}_t)_{t\geq 0}$-adapted process which is scalarly in $L^0(\Omega; L^2(0,T;H))$.*

(1) If X is a UMD *space, then the following assertions are equivalent:*

(i) Ψ is stochastically integrable with respect to W_H;

(ii) $\Psi \in \gamma(0,T;H,X)$ a.s.

Moreover, for $p \in (0,\infty)$ the following continuous time Burkholder-Davis-Gundy inequalities hold:

$$\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t \Psi \, dW_H \Big\|^p \eqsim_{p,X} \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)}. \tag{3.4.2}$$

(2) If X satisfies the decoupling inequality then (ii) $\Rightarrow$ (i) above still holds and for $p \in (0,\infty)$ there exists a constant $\kappa_{p,X}$ such that:

$$\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t \Psi \, dW_H \Big\|^p \leq \kappa_{p,X}^p \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)}, \tag{3.4.3}$$

whenever the right-hand side is finite. Moreover, one can take $\kappa_{p,X}$ such that $\sup_{p\geq 1} \kappa_{p,X}/p < \infty$.

Remark 3.30. The constants in (3.4.2) and (3.4.3) are independent of T, and it is not difficult to see that one can also take $T = \infty$. Since every UMD space satisfies the decoupling inequality (see Corollary 3.19), the estimate (3.4.3) holds for UMD spaces X with the same behavior of the constant $\kappa_{p,X}$. Already for $X = L^q$ with $q \neq 2$, it is an open problem whether the optimal constant $\kappa_{p,X}$ satisfies $\sup_{p\geq 1} \kappa_{p,X}/\sqrt{p} < \infty$. For Hilbert spaces this is indeed the case.

Remark 3.31.

(i) Let $(\Omega, \mathscr{F}, (\mathscr{F}_i)_{i=1}^n, \mathbb{P})$ be a probability space endowed with a filtration. Let $n \in \mathbb{N}$ and let $g_1, \ldots, g_n$ be independent standard Gaussian random variables on $(\Omega, \mathscr{F}, (\mathscr{F}_i)_{i=1}^n, \mathbb{P})$ such that g_i is independent of $\mathscr{F}_i$. Let $(\tilde{g}_1, \ldots, \tilde{g}_n)$ be a copy of $(g_1, \ldots, g_n)$ independent of $(\Omega, \mathscr{F}, (\mathscr{F}_i)_{i=1}^n, \mathbb{P})$. From the proof of Theorem 3.29 it follows that in order to prove that the Burkholder-Davis-Gundy inequality (5.3) is satisfied for processes in a Banach space X, for some $p \in (0, \infty)$, it suffices to prove that there exists a constant c_p such that

$$\Big\| \sup_{1\leq j\leq n} \Big\| \sum_{i=1}^{j} g_i v_{i-1} \Big\| \Big\|_{L^p(\Omega)} \leq c_p \Big\| \sum_{i=1}^{n} \tilde{g}_i v_{i-1} \Big\|_{L^p(\Omega\times\tilde{\Omega},X)} \tag{3.4.4}$$

for all $(v_i)_{i=1}^n$ an $(\mathscr{F}_i)_{i=1}^n$-adapted sequence of X-valued simple random variables and all $n \in \mathbb{N}$. For this it is sufficient that the decoupling inequality holds for p, but we do not know whether it is necessary.

It *is* known that if instead of a one-sided estimate, one has a two-sided estimate in (3.4.4) for some $p \in (1, \infty)$, then X is a UMD Banach space, see [47] and Remark 2.8.

(ii) By studying the proof of [63, Theorem 3'] one may conclude that if (3.4.1) holds for some $p \in (1, \infty)$, it holds for all $p \in (1, \infty)$. As a result, and by considering Paley-Walsh martingales, one can also prove that if (3.4.1) holds in a Banach space X for some $p \in (1, \infty)$, for all X-valued L^p-sequences f with conditionally symmetric increments, then X is a UMD space.

(iii) Suppose the filtration $(\mathscr{F}_t)_{t\geq 0}$ in Theorem 3.29 has the form

$$\mathscr{F}_t = \sigma(W_H(s)h : s \leq t, h \in H)$$

for each $t \in [0, \infty)$. In this case (3.4.2) for some $p \in (0, \infty)$ can be derived from (3.4.1) for Paley–Walsh martingales for that p. Similarly, (3.4.3) for some $p \in (0, \infty)$ can be derived from the corresponding one-sided estimate in (3.4.1) for Paley–Walsh martingales for that p. This follows from a central limit theorem argument as in [51, Theorem 3.1]. Conversely, (3.4.3) implies the corresponding one-sided estimate in (3.4.1) for Paley–Walsh martingales (see [131]).

Proof. (1): Let $p \in (0, \infty)$ be fixed and let Ψ be a finite-rank step process of the form (2.4.4) on page 20 with $\xi_{nm} \in L^\infty(\mathscr{F}_{t_{n-1}}, X)$, W_H an H-cylindrical $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion and let $\widetilde{W}_H$ be a copy of W_H that is independent of $\mathscr{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathscr{F}_t)$. Then

$$\left(\sum_{m=1}^{M} (W_H(t_n)h_m - W_H(t_{n-1})h_m)\xi_{nm} \right)_{n=1}^{N}$$

is a $\mathscr{F}_\infty$-conditionally symmetric sequence and a $\mathscr{F}_\infty$-decoupled version is given by $\Big(\sum_{m=1}^M (\widetilde{W}_H(t_n)h_m - \widetilde{W}_H(t_{n-1})h_m)\xi_{nm}\Big)_{n=1}^N$. One has:

$$\begin{aligned}
&\mathbb{E}\sup_{1\le j\le N}\Big\|\int_0^{t_j}\Psi\,dW_H\Big\|^p\\
&\qquad= \mathbb{E}\sup_{1\le j\le N}\Big\|\sum_{n=1}^{j}\sum_{m=1}^{M}(W_H(t_n)h_m - W_H(t_{n-1})h_m)\xi_{nm}\Big\|^p\\
&\qquad\overset{(i)}{\eqsim}_{p,X} \mathbb{E}\Big\|\sum_{n=1}^{N}\sum_{m=1}^{M}(\widetilde{W}_H(t_n)h_m - \widetilde{W}_H(t_{n-1})h_m)\xi_{nm}\Big\|^p\\
&\qquad\overset{(ii)}{\eqsim}_{p,X} \Big(\mathbb{E}\Big\|\sum_{n=1}^{N}\sum_{m=1}^{M}(\widetilde{W}_H(t_n)h_m - \widetilde{W}_H(t_{n-1})h_m)\xi_{nm}\Big\|^2\Big)^{\frac{p}{2}}\\
&\qquad= \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)},
\end{aligned}$$

where equation (i) follows from equation (3.4.1) and equation (ii) follows by the Kahane-Khintchine inequality (see [36, Section 1.3]).

For $n\in\mathbb{N}$ let D_n be the n^{th} dyadic partition of $[0,T]$, i.e., $D_n := \{\frac{k}{2^n} : k = 0,1,2,\ldots\}\cap[0,T]$ and define $\widetilde{D}_n := D_n\cup\{t_1,\ldots,t_N\}$. Then by the above one has:

$$\mathbb{E}\sup_{t\in\widetilde{D}_n}\Big\|\int_0^t\Psi\,dW_H\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)},\quad n\in\mathbb{N}.$$

By the monotone convergence theorem and path continuity of the integral process one has:

$$\mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t\Psi\,dW_H\Big\|^p = \lim_{n\to\infty}\mathbb{E}\sup_{t\in\widetilde{D}_n}\Big\|\int_0^t\Psi\,dW_H\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)}.$$

Hence equation (3.4.2) holds for finite-rank step processes.

Now let Ψ be any stochastically integrable process. Suppose

$$\mathbb{E}\sup_{0\le t\le T}\Big\|\int_0^t\Psi\,dW_H\Big\|^p < \infty,$$

then by an approximation argument as in the proof of [108, Theorem 5.12]) one has:

$$\mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)} \lesssim_{p,X} \mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t\Psi\,dW_H\Big\|^p.$$

Hence it suffices to prove (3.4.2) under the assumption that $\mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)} < \infty$.

By a straightforward adaptation of the proof of [108, Proposition 2.12] one can prove that there exists a sequence of finite-rank $(\mathscr{F}_t)_{t\ge0}$-adapted step processes

$(\Psi_n)_{n\geq 1}$ that converges to Ψ in $L^p(\Omega, \gamma(0,T;H,X))$. Hence in particular for all $x^* \in X^*$, $\Psi_n^* x^* \to \Psi^* x^*$ in $L^0(\Omega; L^2(0,T;H))$. Because (3.4.2) holds for finite-rank $(\mathscr{F}_t)_{t\geq 0}$-adapted step processes the sequence $\big(\int_0^\cdot \Psi_n \, dW_H\big)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega, C([0,T];X))$. In particular $(\Psi_n)_{n\geq 1}$ approximates the stochastic integral of Ψ in the sense of Definition 2.6 and

$$\begin{aligned}\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t \Psi \, dW_H(t) \Big\|^p &= \lim_{n\to\infty} \mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t \Psi_n \, dW_H(t) \Big\|^p \\ &\eqsim_{p,X} \lim_{n\to\infty} \mathbb{E}\|\Psi_n\|^p_{\gamma(0,T;H,X)} = \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)}.\end{aligned}$$

(2): Suppose X satisfies the decoupling inequality. In this case the proof for finite-rank $(\mathscr{F}_t)_{t\geq 0}$-adapted step processes given above can be repeated using the inequality in Corollary 3.17 instead of equation 3.4.1. To prove (3.4.3) for arbitrary processes we repeat the argument in (1) concerning the case that $\mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)} < \infty$.

However, in order to obtain the estimate $\sup_{p\geq 1} \kappa_{X,p}/p < \infty$ we use inequality (3.3.9) in the proof of Theorem 3.16 in the following manner: when $p \in [1,\infty)$ we have

$$\begin{aligned}&\mathbb{E} \sup_{1\leq j\leq N} \Big\| \int_0^{t_j} \Psi \, dW_H \Big\|^p \\ &= \mathbb{E} \sup_{1\leq j\leq N} \Big\| \sum_{n=1}^{j} \sum_{m=1}^{M} (W_H(t_n)h_m - W_H(t_{n-1})h_m)\xi_{nm} \Big\|^p \\ &\leq c_{X,2}\, p \,\Big\| \mathbb{E}\Big(\Big\| \sum_{n=1}^{j} \sum_{m=1}^{M} (\widetilde{W}_H(t_n)h_m - \widetilde{W}_H(t_{n-1})h_m)\xi_{nm} \Big\|^2 \,\Big|\, \mathscr{F}_\infty \Big)^{1/2} \Big\|_p^p \\ &= c_{X,2}\, p\, \mathbb{E}\|\Psi\|^p_{\gamma(0,T;H,X)}.\end{aligned}$$

□

If X has type 2, then by embedding (2.3.5) one obtains the following Corollary from Theorem 3.29 (2) (as was already observed in [108]). This inequality has been proven for $p \in (1,\infty)$ in [14], [15] using different techniques.

Corollary 3.32. *If X is a Banach space satisfying the decoupling inequality (e.g. a* UMD *space) and X has type 2 then for each $p \in (0,\infty)$ there is a constant $\mathscr{C}_{p,X}$ such that one has:*

$$\mathbb{E} \sup_{0\leq t\leq T} \Big\| \int_0^t \Psi \, dW_H \Big\|^p \leq \mathscr{C}_{p,X} \mathbb{E}\|\Psi\|^p_{L^2(0,T;\gamma(H,X))}, \tag{3.4.5}$$

whenever the right-hand side is finite.

As in Theorem 3.29 one again has $\sup_{p\geq 1} \mathscr{C}_{p,X}/p < \infty$ if $\mathscr{C}_{p,X}$ is the optimal constant in (3.4.5). However, in [126] it has been recently proved that one has $\sup_{p\geq 1} \mathscr{C}_{p,X}/\sqrt{p} < \infty$ in (3.4.5).

Part II

Stochastic Delay Equations

4

Delay equations in type 2 UMD spaces

Let X be a type 2 UMD Banach space and let H be a Hilbert space. In this chapter, which is based on [24], we study the following stochastic delay equation in X:

$$\begin{cases} dU(t) = AU(t)\,dt + BU_t\,dt + G(U(t), U_t)\,dW_H(t),\ t > 0; \\ \ U(0) = x_0; \\ \quad U_0 = f_0, \end{cases} \tag{4.0.1}$$

where for a strongly measurable function $x : [-1, \infty) \to X$ and $t \geq 0$ we define $x_t : [-1, 0] \to X$ by

$$x_t(s) := x(t+s), \quad s \in [-1, 0].$$

We assume that $A : D(A) \subset X \to X$ is closed, densely defined and linear, and generates a C_0-semigroup. We assume that $B \in \mathscr{L}(H^{1,p}(-1, 0; X), X)$ for some $p \in (1, \infty)$, and that $G : \mathscr{E}^p(X) \to \gamma(H, X)$ is a Lipschitz function.

We follow the semigroup approach to the delay equation as given in the monograph of Bátkai and Piazzera [5]. This requires additional assumptions on B, as stated in [5, Theorem 3.26]. A typical example of an operator B that satisfies these assumptions is an operator defined in terms of a Riemann-Stieltjes integral

$$Bf := \int_{-1}^{0} f\,d\eta,$$

where $\eta : [-1, 0] \to \mathscr{L}(X)$ is of bounded variation. Note that this defines an element of $\mathscr{L}(H^{1,p}(-1, 0; X), X)$ by the Sobolev embedding.

For $p \in (1, \infty)$ define $\mathscr{E}^p(X) := X \times L^p(-1, 0; X)$. One can define a closed operator $\mathscr{A}$ on $\mathscr{E}^p(X)$ by

$$\begin{aligned} D(\mathscr{A}) &= \{[x, f] \in D(A) \times H^{1,p}(-1, 0; X) \,:\, f(0) = x\}; \\ \mathscr{A} &= \begin{bmatrix} A & B \\ 0 & \frac{d}{dt} \end{bmatrix}. \end{aligned} \tag{4.0.2}$$

This operator generates a C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$ on $\mathscr{E}^p(X)$ (see [5, Theorem 3.29]) and the stochastic delay equation can be rewritten as a stochastic Cauchy problem in $\mathscr{E}^p(X)$ given by

$$\begin{cases} dV(t) = \mathscr{A}V(t)dt + \mathscr{G}(V(t))\,dW_H(t),\ t\geq 0; \\ V(0) = \begin{bmatrix} x_0 \\ f_0 \end{bmatrix}, \end{cases} \tag{4.0.3}$$

where $\mathscr{G}(V(t)) := [G(V(t)), 0]^T$.

The approach we take is to prove existence, uniqueness and continuity of a solution to the stochastic Cauchy problem (4.0.3) and then translate these results to corresponding results for the stochastic delay equation (4.0.1). The monograph by Da Prato and Zabczyk [33] gives an extensive treatment of the stochastic Cauchy problem in Hilbert spaces. The stochastic Cauchy problem in Banach spaces has been considered in the work by Brzeźniak [13] and van Neerven, Veraar and Weis [109], however, they both consider the case that $\mathscr{A}$ generates an analytic semigroup. Nevertheless their approach is a valuable starting point for studying (4.0.3).

Following the approach of the above mentioned authors we consider the following variation of constants formula:

$$V(t) = \mathscr{T}(t)V(0) + \int_0^t \mathscr{T}(t-s)\mathscr{G}(V(s))\,dW_H(s). \tag{4.0.4}$$

A process satisfying (4.0.4) is usually referred to as a *mild solution*, see Definition 4.2 below. In Section 4.1 we give general conditions under which a mild solution is equivalent to what we call a *generalized strong solution* of the stochastic Cauchy problem. This terminology was suggested by Mark Veraar, who independently gave a proof of the equivalence of mild and generalized strong solutions.

The existence of a mild solution to the stochastic Cauchy problem arising from the delay equation, i.e., equation (4.0.3), is proven by a fixed-point argument in Section 4.2, see Theorem 4.11. Using the factorization method we prove the continuity of a mild solution to (4.0.3), see Theorem 4.12.

Finally, in Subsection 4.2.4 we show how solutions to (4.0.3) relate to solutions to the corresponding delay equation (4.0.1). We do so by means of the generalized strong solution. Combining all these results we obtain existence, uniqueness and continuity of a solution to (4.0.1), see Corollaries 4.17 and 4.18.

Apart from the applications described above, the equivalence of solutions to (4.0.1) and (4.0.3) may be used to translate other results of the stochastic abstract Cauchy problem to delay equations. For example, one may obtain more information concerning invariant measures, see Remark 4.21. We also have that the solution to (4.0.3) is a Markov process, whereas the solution to (4.0.1) is not.

For the theory of stochastic delay equations in the case that X is finite-dimensional we refer to the monographs by Mohammed [98] and Mao [94] and references therein. In particular we wish to mention [21], where equivalence of solutions to the stochastic delay equation and the corresponding abstract Cauchy problem has been shown by Chojnowska-Michalik for the Hilbert space case, i.e., the case that $p = 2$ and X is finite-dimensional. Similar results concerning the abstract Cauchy problem arising from delay equations with state space $C([0,1])$ and with additive noise are given by van Neerven and Riedle [103]. For a general class of spaces including the $\mathscr{E}^p$-spaces the variation of constants formula for finite-dimensional delay equations with additive noise and a bounded delay operator is discussed in Riedle [122]. The latter articles both consider the stochastic convolution as a stochastic integral in a locally convex space. So far there is no suitable interpretation for the stochastic integral of a stochastic process in a locally convex space, hence this approach fails for equations with multiplicative noise.

Stochastic delay equations where X is a Hilbert space and $p = 2$ have been considered by Taniguchi, Liu, and Truman [127], Liu [91] and Bierkens, van Gaans and Verduyn-Lunel [9]. Both [127] and [91] prove existence and uniqueness of solutions to (4.0.1); in [127] it is assumed that A generates an analytic semigroup, whereas in [91] the noise is assumed to be additive. In [9] the existence of an invariant measure has been studied. Very recently, Crewe [31] has taken it upon himself to prove existence, uniqueness and regularity properties of (4.0.1) in UMD Banach spaces under the assumption that A generates an analytic semigroup.

Remark 4.1. For delay equations arising from population dynamics the L^1-spaces are natural as a state space, i.e., one assumes $B \in \mathscr{L}(H^{1,1}(-1,0;X),X)$ in (4.0.1) and considers (4.0.3) with state space $\mathscr{E}^1(X) = X \times L^1(-1,0;X)$ (see [5, Example 3.16]). Note however that $\mathscr{E}^1(X)$ is not a UMD space. However, by Example 3.21, the space $\mathscr{E}^1(X)$ satisfies the decoupling inequality studied in Chapter 3. In particular, the stochastic integration theory of Section 3.4 is available. This suffices in order to obtain all the above-mentioned results concerning delay equations for this specific case. For details we refer to [24].

4.1 The stochastic Cauchy problem

In this section we consider the stochastic Cauchy problem (4.1.1) in general. In the next section, the results obtained here will be applied to the stochastic Cauchy problem arising from a delay equation.

Let Y be a UMD Banach space and H a Hilbert space, and let $A : D(A) \subset Y \to Y$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Y. Let W_H be an H-cylindrical Brownian motion and let $G : Y \to \mathscr{L}(H,Y)$ be continuous (where $\mathscr{L}(H,Y)$ is endowed with the strong operator topology). We consider the following problem:

$$\begin{cases} dV(t) = AV(t)dt + G(V(t))\,dW_H(t),\ t \geq 0; \\ V(0) = V_0. \end{cases} \tag{4.1.1}$$

We shall consider the following two solution concepts for (4.1.1):

Definition 4.2. An H-strongly measurable adapted process V is called a *mild solution* to (4.1.1) if for all $t \in [0,T]$ the process $s \mapsto T(t-s)G(V(s))1_{s\in[0,t]}$ is stochastically integrable and for all $t \in [0,T]$ one has:

$$V(t) = T(t)Y_0 + \int_0^t T(t-s)G(V(s))dW_H(s) \quad a.s. \tag{4.1.2}$$

Definition 4.3. An H-strongly measurable adapted process V is called a *generalized strong solution* to (4.1.1) if V is a.s. locally Bochner integrable and for all $t > 0$:

(i) $\int_0^t V(s)\,ds \in D(A)$ a.s.,

(ii) $G(V)$ is stochastically integrable on $[0,t]$,

and

$$V(t) - V_0 = A\int_0^t V(s)\,ds + \int_0^t G(V(s))\,dW_H(s) \quad a.s.$$

We use the term 'generalized strong solution' to distinguish this solution concept from the conventional definition of a 'strong solution', which concerns a process satisfying $V(t) \in D(A)$ a.s. for all $t \geq 0$ (see [33]). This assumption is not suitable for our situation, see Remark 4.19 below.

Theorem 4.4. *Let V be an Y-valued H-strongly measurable adapted process. For $t \geq 0$ define $\int_0^t T(s)G(V(u))\,ds \in \mathscr{L}(H,Y)$ by*

$$\Big(\int_0^t T(s)G(V(u))\,ds\Big)h := \int_0^t T(s)G(V(u))h\,ds.$$

Assume that for all $t > 0$ the following processes are in $\gamma(0,t;H,Y)$ a.s.:

(a) $G(V)$;

(b) $u \mapsto T(t-u)G(V(u))$;

(c) $u \mapsto \int_0^{t-u} T(s)G(V(u))\,ds$;

and that for all $t > 0$

$$\int_0^t \|T(s-\cdot)G(V(\cdot))\|_{\gamma(0,s,H;Y)}\,ds < \infty. \tag{4.1.3}$$

Then V is a generalized strong solution to (4.1.1) if and only if it is a mild solution to (4.1.1).

Remark 4.5.

(i) If V is strongly measurable and adapted then the processes in (a), (b) and (c) are H-strongly measurable and adapted.

(ii) If $G : Y \to \gamma(H, Y)$ then for all $u \in [0, t]$ almost all paths $s \mapsto T(s)G(V(u))$ are locally Bochner integrable in $\gamma(H, Y)$ because $G(V(u))$ is the limit of finite-rank operators in $\gamma(H, Y)$.

(iii) If A has a bounded H^∞-calculus on a sector Σ_θ, $\theta \in (0, \pi)$, then by [78] the set $(T(s))_{0\le s\le T}$ is γ-bounded, and thus by the multiplier Theorem 2.14 assumptions (b) and (c) follow from (a). More generally, if T is analytic, then by Lemma 2.21 the set $(s^\alpha T(s))_{0\le s\le T}$ is γ-bounded for $\alpha > 0$, and thus in order to check whether (a), (b) and (c) hold, it suffices to check whether $s \mapsto (t-s)^{-\alpha} G(V(s)) 1_{s\in[0,t)}$ is stochastically integrable for all $t \in [0, T]$, for some $\alpha > 0$.

Proof (of Theorem 4.4).

Step 1. We apply Lemmas 2.9 and 2.10 to obtain the key equations for the proof of Theorem 4.4, equations (4.1.6) and (4.1.7) below. Consider the following process:

$$\Phi : [0, t] \times [0, t] \times \Omega \to \mathscr{L}(H, Y); \quad \Phi(s, u, \omega) := 1_{u\le s\le t} T(t - s)G(V(u)).$$

Because V is strongly measurable and adapted, and because $G : Y \to \mathscr{L}(H, Y)$ is continuous with respect to the strong operator topology and the semigroup $T(s)$ is strongly continuous it follows that Φ is H-strongly measurable and adapted. Thus conditions measurability and adaptedness conditions of Lemma 2.9 are satisfied. One easily checks that condition (i) of Lemma 2.9 is satisfied by Φ. Condition (ii) in Lemma 2.9 follows from assumption (c). Condition (iii) in Lemma 2.9 follows from the definition of $\gamma(0, t; H, Y)$, assumption (a) and the exponential boundedness of the semigroup: let $(h_k)_{k=1}^n$ be an arbitrary orthonormal sequence in $L^2(0, t; H)$, then

$$\begin{aligned}
&\int_0^t \Big(\mathbb{E}\, \Big\| \sum_{k=1}^n \gamma_k \int_0^t T(t-s)G(V(u))h_k(u)1_{[0,s]}(u)\, du \Big\|_Y^2\Big)^{\frac12} ds \\
&\le \int_0^t \|T(t-s)\|_{\mathscr{L}(Y)} \Big(\mathbb{E}\, \Big\| \sum_{k=1}^n \gamma_k \int_0^t G(V(u))h_k(u)1_{[0,s]}(u)\, du \Big\|_Y^2\Big)^{\frac12} ds \\
&\le M_t \int_0^t \|G(V)1_{[0,s])}\|_{\gamma(0,t;H,Y)}\, ds \le tM_t \|G(V)\|_{\gamma(0,t;H,Y)} < \infty,
\end{aligned}$$

where $M_t := \sup_{0\le s\le t} \|T(s)\|_{\mathscr{L}(Y)}$. Note that by (2.3.4) one has

$$\|G(V)1_{[0,s]}\|_{\gamma(0,t;H,Y)} \le \|G(V)\|_{\gamma(0,t;H,Y)}.$$

Thus Φ satisfies all the conditions of the stochastic Fubini theorem (Lemma 2.9) and we obtain:

$$\int_0^t T(t-s) \int_0^s G(V(u))\, dW_H(u)\, ds = \int_0^t \int_u^t T(t-s)G(V(u))\, ds\, dW_H(u) \text{ a.s.} \tag{4.1.4}$$

Observe that for all $h \in H$ one has $\int_u^t T(t-s)G(V(u))h\,ds \in D(A)$. Hence by assumptions (a) and (b) we can apply Lemma 2.10 to obtain that the stochastic integral on the right-hand side of equation (4.1.4) above is in $D(A)$ a.s., and we have:

$$\begin{aligned} A\int_0^t \int_u^t T(t-s)G(V(u))\,ds\,dW_H(u) \\ &= \int_0^t A\int_0^{t-u} T(s)G(V(u))\,ds\,dW_H(u) \\ &= \int_0^t (T(t-u)-I)G(V(u))\,dW_H(u) \quad \text{a.s.} \end{aligned} \tag{4.1.5}$$

Combining equations (4.1.4) and (4.1.5) we obtain that almost surely:

$$A\int_0^t T(t-s)\int_0^s G(V(u))\,dW_H(u)\,ds = \int_0^t (T(t-u)-I)G(V(u))\,dW_H(u). \tag{4.1.6}$$

Similarly, using assumption (4.1.3) one can prove that for $0 \le s \le t$ the stochastic integrals in the equation below are well-defined and one has the following identity:

$$A\int_0^t \int_0^s T(s-u)G(V(u))\,dW_H(u)\,ds = \int_0^t (T(t-u)-I)G(V(u))\,dW_H(u). \tag{4.1.7}$$

Step 2. Assume V is a generalized strong solution to (4.1.1), we prove that (4.1.2) holds. By (4.1.6) and by the definition of a generalized strong solution we have:

$$\begin{aligned} V(t) - V_0 - A\int_0^t V(s)\,ds &= \int_0^t G(V(s))\,dW_H(s) \\ &= \int_0^t T(t-s)G(V(s))\,dW_H(s) - A\int_0^t T(t-s)\int_0^s G(V(u))\,dW_H(u)\,ds. \end{aligned}$$

Let us consider the final term above. By assumption and by Fubini's theorem one has:

$$\begin{aligned} &\int_0^t T(t-s)\int_0^s G(V(u))\,dW_H(u)\,ds \\ &\quad= \int_0^t T(t-s)\left[V(s) - V_0 - A\int_0^s V(u)\,du\right]\,ds \\ &\quad= \int_0^t T(t-s)V(s)\,ds - \int_0^t T(t-s)V_0\,ds - A\int_0^t\int_u^t T(t-s)V(u)\,ds\,du \\ &\quad= -\int_0^t T(t-s)V_0\,ds + \int_0^t V(s)\,ds, \end{aligned}$$

which, when substituted to the earlier equation, gives:

$$\begin{aligned}V(t) - V_0 - A\int_0^t V(s)\,ds \\ &= \int_0^t T(t-s)G(V(s))\,dW_H(s) + T(t)V_0 - V_0 - A\int_0^t V(s)\,ds.\end{aligned}$$

This proves that $V(t)$ is a mild solution.

On the other hand, if V is a mild solution, then $\int_0^t V(s)\,ds$ exists and takes values in $D(A)$ a.s. by (4.1.7), and therefore using this equation we obtain:

$$\begin{aligned}A\int_0^t V(s)\,ds &= A\int_0^t T(s)V_0\,ds + A\int_0^t\int_0^s T(s-u)G(V(u))\,dW_H(u)\,ds \\ &= T(t)V_0 - V_0 + \int_0^t [T(t-u) - 1]\,G(V(u))\,dW_H(u) \\ &= V(t) - V_0 - \int_0^t G(V(u))\,dW_H(u),\end{aligned}$$

whence V is a generalized strong solution. □

Continuity of a process satisfying (4.1.2) can be proved by means of the factorization method as introduced in [32, Section 2]. We give the proof below; it is a straightforward adaptation of the proof of [129, Theorem 3.3].

Theorem 4.6. *Let $(T(t))_{t\geq 0}$ be a semigroup on a* UMD *Banach space Y. Let $\Phi : [0,t]\times\Omega \to \mathscr{L}(H,Y)$ be an H-strongly measurable adapted process. Suppose that there exists $\alpha, p > 0$, $\frac{1}{p} < \alpha < \frac{1}{2}$ and $M > 0$ such that*

$$\sup_{0\leq s\leq t} \|u \mapsto (s-u)^{-\alpha} T(s-u)\Phi(u)\|_{L^p(\Omega;\gamma(0,s,H;Y))} \leq M. \tag{4.1.8}$$

Then the process

$$s \mapsto \int_0^s T(s-u)\Phi(u)\,dW_H(u)$$

is well-defined and has a version with continuous paths. Moreover we have

$$\mathbb{E}\sup_{0\leq s\leq t}\Big\|\int_0^s T(s-u)\Phi(u)\,dW_H(u)\Big\|_Y^p < \infty.$$

Before giving the proof of this theorem we mention the following corollary:

Corollary 4.7. *Consider the stochastic Cauchy problem* (4.1.1) *set in a* UMD *Banach space Y. The process $V : [0,t]\times\Omega \to Y$ satisfying the variation of constants formula* (4.1.2) *belongs to $L^p(\Omega; C([0,t];Y))$ if there exists $\alpha, p > 0$, $\frac{1}{p} < \alpha < \frac{1}{2}$ such that*

$$\sup_{0\leq s\leq t} \|u \mapsto (s-u)^{-\alpha} T(s-u)V(u)\|_{L^p(\Omega;\gamma(0,s,H;Y))} < \infty.$$

Proof (of Theorem 4.6). By assumption (4.1.8) and Theorem 2.7 it follows that for all $s \in [0,t]$ we can define

$$\Psi_1(s) := \int_0^s (s-u)^{-\alpha} T(s-u)\Phi(u)\, dW_H(u).$$

By [109, Proposition A.1] the process Φ_1 has a version which is adapted and strongly measurable. Moreover, by assumption and the Burkholder-Davis-Gundy inequalities (2.4.5) one has, for all $s \in [0,t]$,

$$\mathbb{E}\, \|\Psi_1(s)\|_Y^p \leq M, \tag{4.1.9}$$

whence $\Psi_1 \in L^p(0,t;L^p(\Omega;Y))$. Thus, by Fubini, $\Psi_1 \in L^p(\Omega;L^p(0,t;Y))$. Let $\Omega_0 \subset \Omega$ denote the set on which $\Psi_1 \in L^p(0,t;Y)$; we have $\mathbb{P}(\Omega_0) = 1$.

By the domination principle for Gaussian random variables, i.e., equation (2.3.4), it follows that for all $s \in [0,t]$ one has, almost surely,

$$\|u \mapsto T(s-u)\Phi(u,\omega)\|_{\gamma(0,s,H;Y)} \leq t^\alpha \|u \mapsto (s-u)^{-\alpha} T(s-u)\Phi(u,\omega)\|_{\gamma(0,s,H;Y)}.$$

Thus by assumption we can define, for all $s \in [0,t]$,

$$\Psi_2(s) := \int_0^s T(s-u)\Phi(u)\, dW_H(u),$$

which again has a version that is adapted and strongly measurable.

It is proved in [32] that one may define a bounded operator $R_\alpha : L^p(0,t;Y) \to C([0,t];Y)$ by setting

$$(R_\alpha f)(s) := \int_0^s (s-u)^{\alpha-1} T(s-u) f(u)\, du.$$

Thus it remains to show that for almost all $\omega \in \Omega_0$ one has:

$$\Psi_2(s) = \frac{\sin \pi\alpha}{\pi} (R_\alpha \Psi_1)(s), \tag{4.1.10}$$

for all $s \in [0,T]$, i.e., that for all $x^* \in Y^*$ one has

$$\langle \Psi_2(s), x^* \rangle = \frac{\sin \pi\alpha}{\pi} \int_0^s (s-u)^{\alpha-1} \langle T(s-u)\Psi_1(u), x^* \rangle\, du \quad \text{a.s.}$$

This follows from a Fubini argument, see [104, Theorem 3.5] and [32]. The conditions necessary to apply the Fubini theorem follow from the assumption (4.1.8).

By (4.1.10) and (4.1.9) one has

$$\mathbb{E} \sup_{0 \leq s \leq t} \|\Psi_2(s)\|_Y^p \leq C\mathbb{E} \int_0^t \|\Psi_1(s)\|_Y^p\, ds \leq tCM,$$

where C is independent of Φ. Thus the final estimate follows. $\square$

4.2 The stochastic delay equation

4.2.1 The variation of constants formula

Now we wish to apply the results of the previous section to stochastic Cauchy problem (4.0.3) on page 64 that arose from the stochastic delay equation (4.0.1) as presented on page 63. Recall that we assumed that (4.0.1) is set in a type 2 UMD Banach space X and that the related Cauchy problem is set in $\mathscr{E}^p(X) = X \times L^p(-1,0;X)$ for some $p \in (1,\infty)$.

Let $(\mathscr{T}(t))_{t\geq 0}$ denote the semigroup generated by $\mathscr{A}$, where $\mathscr{A}$ is the operator in (4.0.3) defined by (4.0.2) on page 63. We define the projections $\pi_1 : \mathscr{E}^p(X) \to X$ and $\pi_2 : \mathscr{E}^p(X) \to L^p(-1,0;X)$ as follows:

$$\pi_1 \begin{bmatrix} x \\ f \end{bmatrix} = x; \qquad\qquad \pi_2 \begin{bmatrix} x \\ f \end{bmatrix} = f.$$

The following property of $(\mathscr{T}(t))_{t\geq 0}$ is intuitively obvious and useful in the following:

$$\left(\pi_2 \mathscr{T}(t) \begin{bmatrix} x \\ f \end{bmatrix}\right)(u) = \pi_1 \mathscr{T}(t+u) \begin{bmatrix} x \\ f \end{bmatrix}. \tag{4.2.1}$$

for $f \in \mathscr{E}^p(X), u \in [-1,0], t > -u$ (for a proof see [5, Proposition 3.11]).

The proof of the following lemma is straightforward and thus left to the reader:

Lemma 4.8. *Let $t > 0$, $p \in (1,\infty)$ and $x \in L^p(-1,t;X)$. Then the function $y : [0,t] \to L^p(-1,0;X)$, $y(s) := x_s$ is (Bochner) integrable and*

$$\int_0^t y(s)\, ds \in H^{1,p}(-1,0;X); \qquad \left(\int_0^t y(s)\, ds\right)(u) = \int_0^t x(s+u)\, ds \quad a.s.$$

Generalized strong solutions to (4.0.3) are equivalent to mild solutions:

Theorem 4.9. *Let X be a type 2 UMD Banach space and let $p \in (1,\infty)$. Consider (4.0.3); i.e., let $\mathscr{A}$ defined by (4.0.2) be the generator of the C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$ on $\mathscr{E}^p(X) = X \times L^p(-1,0;X)$. Let $\mathscr{G} : \mathscr{E}^p(X) \to \gamma(H, \mathscr{E}^p(X))$ be given by $\mathscr{G}([x,f]^T) = [G([x,f]),0]^T$, where $G : \mathscr{E}^p(X) \to \gamma(H,X)$ is Lipschitz continuous. Finally, let W_H be an H-cylindrical Brownian motion adapted to $(\mathscr{F}_s)_{s\geq 0}$.*

Let $V : [0,\infty) \times \Omega \to \mathscr{E}^p(X)$ be a strongly measurable, adapted process satisfying

$$\int_0^t \|V(s)\|^2_{\mathscr{E}^p(X)}\, ds < \infty \quad \text{a.s. for all } t > 0;$$

Then V is a generalized strong solution to (4.0.3) if and only if V is a mild solution to (4.0.3).

Proof. We apply Theorem 4.4 to obtain the above assertion, for which we need to check condition (4.1.3) and that the processes given by (a), (b) and (c) in that theorem are elements of $\gamma(0,t;H,\mathscr{E}^p(X))$ a.s. for all $t>0$. Let $t>0$ be fixed.

Process (a) **in Theorem 4.4**. By the embedding (2.3.5) and the Lipschitz-continuity of $\mathscr{G}$ we have:

$$\begin{aligned}\|s\mapsto \mathscr{G}(V(s))\|_{\gamma(0,t;H,\mathscr{E}^p(X))} &= \|s\mapsto G(V(s))\|_{\gamma(0,t;H,X)}\\ &\lesssim \|s\mapsto G(V(s))\|_{L^2(0,t;\gamma(H,X))}\\ &\lesssim t^{\frac12}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))},\end{aligned}$$

where K is the Lipschitz-constant of G.

Process (b) **in Theorem 4.4**. By the γ-Fubini isomorphism (2.3.3) and embedding (2.3.5) we have:

$$\begin{aligned}&\|u\mapsto \mathscr{T}(t-u)\mathscr{G}(V(u))\|_{\gamma(0,t;H,\mathscr{E}^p(X))}\\ &\qquad\lesssim_p \|u\mapsto \pi_1\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{\gamma(0,t;H,X)}\\ &\qquad\quad + \|u\mapsto \pi_2\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{L^p(-1,0;\gamma(0,t;H,X))}\\ &\qquad\le \|u\mapsto \pi_1\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{L^2(0,t;\gamma(H,X))}\\ &\qquad\quad + \|u\mapsto \pi_2\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{L^p(-1,0;L^2(0,t;\gamma(H,X))}.\end{aligned}$$

Set $M_t := \sup_{u\in[0,t]}\|\mathscr{T}(u)\|_{\mathscr{L}(\mathscr{E}^p(X))}$. By the ideal property of the γ-radonifying operators and the Lipschitz-continuity of $\mathscr{G}$ we have:

$$\begin{aligned}&\|u\mapsto \pi_1\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{L^2(0,t;\gamma(H,X))}\\ &\qquad\le M_t\Big[t^{\frac12}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))}\Big],\end{aligned}$$

where K is the Lipschitz-constant of G, and, by equality (4.2.1),

$$\begin{aligned}&\|u\mapsto \pi_2\mathscr{T}(t-u)\mathscr{G}(V(u))\|_{L^p(-1,0;L^2(0,t;\gamma(H,X))}\\ &\qquad= \Big(\int_{-1}^0 \|\pi_1\mathscr{T}(t-u+s)\mathscr{G}(V(u))\|^p_{L^2(0,t+s;\gamma(H,X)}\,ds\Big)^{\frac1p}\\ &\qquad\le M_t\Big[t^{\frac12}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))}\Big].\end{aligned}$$

Process (c) **in Theorem 4.4**. Note that by Remark 4.5 we may interpret

$$\int_0^{t-u}\mathscr{T}(s)\mathscr{G}(V(u))\,ds$$

as a $\gamma(H,\mathscr{E}^p(X))$-valued Bochner integral. To prove that the process

$$u\mapsto \int_0^{t-u}\mathscr{T}(s)\mathscr{G}(V(u))\,ds \in \gamma(0,t;H,\mathscr{E}^p(X)) \quad \text{a.s.},$$

observe that by γ-Fubini isomorphism (2.3.3) and embedding (2.3.5) we have:

$$\begin{aligned}
&\Big\| u \mapsto \int_0^{t-u} \mathscr{T}(s)\mathscr{G}(V(u))\,ds \Big\|_{\gamma(0,t;H,\mathscr{E}^p(X))} \\
&\qquad \lesssim_p \Big\| u \mapsto \pi_1 \int_0^{t-u} \mathscr{T}(s)\mathscr{G}(V(u))\,ds \Big\|_{L^2(0,t;\gamma(H,X))} \\
&\qquad\quad + \Big\| u \mapsto \pi_2 \int_0^{t-u} \mathscr{T}(s)\mathscr{G}(V(u))\,ds \Big\|_{L^p(-1,0;L^2(0,t;\gamma(H,X))}.
\end{aligned}$$

By Minkowski's integral inequality, the ideal property for γ-radonifying operators and the Lipschitz-continuity of $\mathscr{G}$ we have:

$$\begin{aligned}
&\Big\| u \mapsto \pi_1 \int_0^{t-u} \mathscr{T}(s)\mathscr{G}(V(u))\,ds \Big\|_{L^2(0,t;\gamma(H,X))} \\
&\qquad \le tM_t\Big[t^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))}\Big],
\end{aligned}$$

and by equation (4.2.1) and Lemma 4.8 we have:

$$\begin{aligned}
&\Big\| u \mapsto \pi_2 \int_0^{t-u} \mathscr{T}(s)\mathscr{G}(V(u))\,ds \Big\|_{L^p(-1,0;L^2(0,t;\gamma(H,X))} \\
&\qquad = \Big(\int_{-1}^0 \Big\| u \mapsto \pi_1 \int_0^{t-u+r} \mathscr{T}(s+r)\mathscr{G}(V(u))\,ds \Big\|^p_{L^2(0,t;\gamma(H,X))} dr \Big)^{\frac{1}{p}} \\
&\qquad \le tM_t\Big[t^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))}\Big].
\end{aligned}$$

Condition (4.1.3) **in Theorem 4.4.** From the estimates for process (c) above we obtain:

$$\begin{aligned}
&\int_0^t \|u \mapsto \mathscr{T}(s-u)\mathscr{G}(V(u))\|_{\gamma(0,s;H,\mathscr{E}^p(X))}\,ds \\
&\qquad \lesssim_p 2tM_t\Big[t^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + K\|V\|_{L^2(0,t;\mathscr{E}^p(X))}\Big].
\end{aligned}$$

Having checked condition (4.1.3) and that all processes are in $\gamma(0,t;H,X)$ a.s. we may apply Theorem 4.4 to obtain the desired result. □

Remark 4.10. Let p' be such that $\frac{1}{p}+\frac{1}{p'}=1$. By testing the stochastic convolution in the variation of constants formula (4.1.2) for V against elements of $X^* \times L^{p'}(-1,0;X^*)$, which is norming for $\mathscr{E}^p(X)$, and applying equality (4.2.1) one shows that almost surely:

$$\int_0^t \pi_2\mathscr{T}(t-s)\mathscr{G}(V(s))\,dW_H(s) = u \mapsto \int_0^{t+u} \pi_1\mathscr{T}(t-s+u)\mathscr{G}(V(s))\,dW_H(s).$$

It thus follows from the variation of constants formula (4.1.2) that if V is a generalized strong solution to (4.0.3) then $\pi_2V(t)(u) = \pi_1V(t+u)$; in particular it follows that $\pi_1V \in L^p_{loc}(0,\infty;X)$ a.s.

4.2.2 Existence and uniqueness of a solution to (SDCP)

Recall that $(\mathscr{F}_s)_{s\geq 0}$ denotes the filtration to which W_H is adapted. For $t>0$, $q\in[1,\infty)$ and $r\in[1,\infty]$ let $L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$ be the space of $(\mathscr{F}_s)_{s\geq 0}$ adapted processes in $L^r(0,t;L^q(\Omega;\mathscr{E}^p(X)))$. In particular, $L^\infty_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$ is the Banach space of $(\mathscr{F}_s)_{s\geq 0}$ adapted processes V such that

$$\|V\|_{L^\infty_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))} = \sup_{0\leq s\leq t}\left(\mathbb{E}\,\|V(s)\|^q_{\mathscr{E}^p(X)}\right)^{\frac{1}{q}} < \infty.$$

Theorem 4.11. *Let the assumptions of Theorem 4.9 hold and assume that $V_0 := [x_0,f_0]^T \in L^q(\mathscr{F}_0,\mathscr{E}^p(X))$ for some $q\in[2,\infty)$. Then for every $t>0$ and every $r\in[2,\infty]$ there exists a unique mild solution $V\in L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$ to (4.0.3). In particular, this process is in $L^\infty_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$.*

Proof. The final remark of the theorem is a direct consequence of there being a solution in $L^\infty_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$ and of the uniqueness of the solution in $L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$.

Fix $r\in(2,\infty]$ and let $t>0$. Define

$$L : L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X))) \to L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$$

as follows:

$$L(\Phi)(s) := \mathscr{T}(s)V_0 + \int_0^s \mathscr{T}(s-u)\mathscr{G}(\Phi(u))\,dW_H(u),$$

where $s\in[0,t]$. Set $M_t := \sup_{0\leq u\leq t}\|\mathscr{T}(u)\|_{\mathscr{L}(\mathscr{E}^p(X))}$. To prove that $L(\Phi)$ is indeed in $L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$, first observe that by inequality (2.4.5) and the proof of Theorem 4.9 we have:

$$\begin{aligned}
&(\mathbb{E}\,\|L(\Phi)(s)\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} \\
&\quad\lesssim_q (\mathbb{E}\,\|\mathscr{T}(s)V_0\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} + \|u\mapsto \mathscr{T}(s-u)\mathscr{G}(\Phi(s))\|_{L^q(\Omega,\gamma(0,s;H,\mathscr{E}^p(X)))} \\
&\quad\leq M_t\Big[(\mathbb{E}\,\|V_0\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} \\
&\qquad\qquad + \Big(\mathbb{E}\,\Big[s^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + K\Big(\int_0^s \|\Phi(u)\|^2_{\mathscr{E}^p(X)}\,du\Big)^{\frac{1}{2}}\Big]^q\Big)^{\frac{1}{q}}\Big],
\end{aligned}$$

and thus from Minkowski's integral inequality, the Hölder inequality and the fact that $r\geq q\geq 2$ we obtain:

$$\begin{aligned}
&(\mathbb{E}\,\|L(\Phi)(s)\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} \\
&\leq M_t\Big[(\mathbb{E}\,\|V_0\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} + t^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + K\Big(\int_0^s \big[\mathbb{E}\,\|\Phi(u)\|^q_{\mathscr{E}^p(X)}\big]^{\frac{2}{q}}\,du\Big)^{\frac{1}{2}}\Big] \\
&\leq M_t\Big[(\mathbb{E}\,\|V_0\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}} + t^{\frac{1}{2}}\|G(0)\|_{\gamma(H,X)} + Ks^{\frac{1}{2}-\frac{1}{r}}\|\Phi\|_{L^r(0,t;L^q(\Omega;\mathscr{E}^p(X)))}\Big],
\end{aligned}$$

for every $s \in [0,t]$, where K is the Lipschitz constant of G. (In the case $r = \infty$ we interpret $\frac{1}{r} = 0$.) Taking r^{th} powers in the above and integrating with respect to s gives that $L(\Phi) \in L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$.

By similar arguments one has, for $\Phi_1, \Phi_2 \in L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$;

$$\|L(\Phi_1) - L(\Phi_2)\|_{L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))} \lesssim_q Kt^{\frac{1}{2}} M_t \|\Phi_1 - \Phi_2\|_{L^r_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))},$$

so this is a strict contraction for $t = t_0$, where t_0 is chosen to be sufficiently small. Hence by the Banach fixed-point theorem there exists a unique $V \in L^r_{\mathscr{F}}(0,t_0;L^q(\Omega;\mathscr{E}^p(X)))$ that satisfies the variation of constants formula (4.1.2) for (4.0.3). By repeating this argument, but now on the interval $[t_0, 2t_0]$ with initial value

$$\tilde{Y}_0 := \mathscr{T}(t_0)Y_0 + \int_0^{t_0} \mathscr{T}(t-u)\mathscr{G}(V(u))dW_H(u),$$

and continuing in this manner until the desired end time is reached, one obtains a solution for arbitrary $t > 0$. □

4.2.3 Continuity of the solution to (SDCP)

Theorem 4.12. *Let the assumptions of Theorem 4.9 hold. In addition, assume that $V_0 := [x_0, f_0]^T \in L^q(\mathscr{F}_0, \mathscr{E}^p(X))$ for some $q \in (2,\infty)$. Let $t > 0$. Then the solution $V \in L^\infty_{\mathscr{F}}(0,t;L^q(\Omega;\mathscr{E}^p(X)))$ to* (4.0.3) *as given by Theorem 4.11 satisfies $V \in L^q(\Omega;C([0,t];\mathscr{E}^p(X)))$.*

Proof. The statement follows from Corollary 4.7 once we have established that for some $\alpha \in (\frac{1}{q}, \frac{1}{2})$ we have:

$$\sup_{0\le s\le t} \|u \mapsto (s-u)^{-\alpha}\mathscr{T}(s-u)\mathscr{G}(V(u))\|_{L^q(\Omega,\gamma(0,s;\mathscr{E}^p(X)))} < \infty. \tag{4.2.2}$$

Fix $\alpha \in (\frac{1}{q}, \frac{1}{2})$ and $s \in [0,t]$. By γ-Fubini isomorphism (2.3.3) and embedding (2.3.5) we have:

$$\begin{aligned}
&\|u \mapsto (s-u)^{-\alpha}\mathscr{T}(s-u)\mathscr{G}(V(u))\|_{\gamma(0,s;\mathscr{E}^p(X))}\\
&\qquad \lesssim_p \|u \mapsto \pi_1(s-u)^{-\alpha}\mathscr{T}(s-u)\mathscr{G}(V(u))\|_{L^2(0,s;\gamma(H,X))}\\
&\qquad\quad + \|u \mapsto (s-u)^{-\alpha}\pi_2\mathscr{T}(s-u)\mathscr{G}(V(u))\|_{L^p(-1,0;L^2(0,s;\gamma(H,X))},
\end{aligned} \tag{4.2.3}$$

where $M_t := \sup_{u\in[0,t]} \|\mathscr{T}(u)\|_{\mathscr{L}(\mathscr{E}^p(X))}$. Concerning the final term in (4.2.3); by (4.2.1) and by the ideal property of the γ-radonifying operators we have:

$$\begin{aligned}
&\|u \mapsto (s-u)^{-\alpha}\pi_2\mathscr{T}(s-u)\mathscr{G}(V(u))\|_{L^p(-1,0;L^2(0,s;\gamma(H,X))}\\
&\quad = \Big[\int_{-1}^0 \Big(\int_0^{s+r} (s-u)^{-2\alpha}\|\pi_1\mathscr{T}(s-u+r)\mathscr{G}(V(u))\|^2_{\gamma(H,X)}\,du\Big)^{\frac{p}{2}} dr\Big]^{\frac{1}{p}}
\end{aligned}$$

$$\leq M_t \Big[\int_{-1}^{0} \Big(\int_0^s (s-u)^{-2\alpha} \|G(V(u))\|^2_{\gamma(H,X)}\, du \Big)^{\frac{p}{2}} dr \Big]^{\frac{1}{p}}$$
$$= M_t \Big(\int_0^s (s-u)^{-2\alpha} \|G(V(u))\|^2_{\gamma(H,X)}\, du \Big)^{\frac{1}{2}}.$$

As $q > 2$, and using in addition the Lipschitz-continuity of G, it follows that:

$$\|u \mapsto (s-u)^{-\alpha} \pi_2 \mathscr{T}(s-u) \mathscr{G}(V(u))\|_{L^q(\Omega; L^p(-1,0; L^2(0,s;\gamma(H,X))))}$$
$$\leq M_t \Big(\int_0^s (s-u)^{-2\alpha} \big[\mathbb{E}\|G(V(u))\|^q_{\gamma(H,X)}\big]^{\frac{2}{q}}\, du \Big)^{\frac{1}{2}}$$
$$\leq (1-2\alpha)^{-\frac{1}{2}} M_t s^{\frac{1}{2}-\alpha} \big[\|G(0)\|_{\gamma(H,X)} + K \sup_{u\in[0,s]} (\mathbb{E}\, \|V(u)\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}}\big] < \infty,$$

where K is the Lipschitz constant of G. The estimate for the first term on the right-hand side of (4.2.3) is similar, but slightly simpler; one obtains:

$$\|u \mapsto (s-u)^{-\alpha} \pi_1 \mathscr{T}(s-u) \mathscr{G}(V(u))\|_{L^q(\Omega; L^2(0,s;\gamma(H,X)))}$$
$$\leq (1-2\alpha)^{-\frac{1}{2}} M_t s^{\frac{1}{2}-\alpha} \big[\|G(0)\|_{\gamma(H,X)} + K \sup_{u\in[0,s]} (\mathbb{E}\, \|V(u)\|^q_{\mathscr{E}^p(X)})^{\frac{1}{q}}\big] < \infty.$$

From the above estimates and the fact that $s^{\frac{1}{2}-\alpha} \leq t^{\frac{1}{2}-\alpha}$ because $\alpha < \frac{1}{2}$, we conclude that (4.2.2) holds. □

4.2.4 Equivalence of solutions to (SDE) and (SDCP)

Before translating the results on the stochastic Cauchy problem to the stochastic delay equation, let us define what we mean by a solution to (4.0.1). Let $p \in (1,\infty)$ be such that $B \in \mathscr{L}(H^{1,p}(-1,0;X), X)$.

Definition 4.13. A process $U : [-1,\infty) \times \Omega \to X$ is called a *strong solution* to (4.0.1) if it is measurable and adapted to $(\mathscr{F}_t)_{t\geq 0}$ and for all $t \geq 0$ one has:

(i) $\int_0^t |U(s)|^{2\vee p}\, ds < \infty$ a.s.;

(ii) $U|_{[-1,0)} = f_0$,

(iii) $\int_0^t U(s)\, ds \in D(A)$ for all $t > 0$ a.s.;

and

$$U(t) - x_0 = A \int_0^t U(s)\, ds + B \int_0^t U_s\, ds + \int_0^t G(U(s), U_s)\, dW_H(s) \quad \text{a.s.} \quad (4.2.4)$$

Remark 4.14. Note that by condition (i) and Lemma 4.8 one has $\int_0^t U_s\, ds \in H^{1,p}(-1,0;X)$ a.s. Moreover, for any $t > 0$; by Minkowski's integral inequality one has:

$$\Big(\int_0^t \|U_s\|^2_{L^p(X)}\, ds \Big)^{\frac{1}{2}} = \Big(\int_0^t \Big[\int_{s-1}^{s} \|U(u)\|^p_X\, du \Big]^{\frac{2}{p}} ds \Big)^{\frac{1}{2}}$$

$$\leq \|f_0\|_{L^p} + \Big(\int_0^t \Big[\int_0^t \|U(u)\|_X^{p\vee 2}\,du\Big]^{\frac{p\wedge 2}{p\vee 2}}\,ds\Big)^{\frac{1}{p\wedge 2}}$$
$$= \|f_0\|_{L^p} + t^{\frac{1}{p\wedge 2}}\Big[\int_0^t \|U(u)\|_X^{p\vee 2}\,du\Big]^{\frac{1}{p\vee 2}} < \infty \quad \text{a.s.}$$

Hence by condition (i) the stochastic integral on right hand side of (4.2.4) is well defined.

Theorem 4.15. *(i) Let U be a strong solution to* (4.0.1)*, then the process V defined by $V(t) := [U(t), U_t]^T$ is a generalized strong solution to* (4.0.3)*.*

(ii) On the other hand, if V is a generalized strong solution to (4.0.3) *then the process defined by $U|_{[-1,0)} = f_0$, $U(t) := \pi_1(V(t))$ for $t \geq 0$ is a strong solution to* (4.0.1)*.*

Proof. Part (i). In the proof of Theorem 4.9 we saw that $s \mapsto \mathscr{G}(V(s))$ is stochastically integrable if $V \in L^2(0,t;\mathscr{E}^p(X))$ a.s. By Remark 4.14 it follows from the definition of a strong solution to (4.0.1) that this is indeed the case. From Lemma 4.8 above it follows that V is integrable a.s.:

$$\int_0^t V(s)\,ds = \begin{bmatrix} \int_0^t U(s)\,ds \\ \int_0^t U_s\,ds \end{bmatrix} \quad \text{a.s.}$$

and that $\int_0^t U_s\,ds \in H^{1,p}(-1,0;X)$ a.s. and $\int_0^t U_s\,ds(0) = \int_0^t U(s)\,ds \in D(A)$. Hence $\int_0^t V(s)\,ds \in D(\mathscr{A})$ a.s. and again by Lemma 4.8 and by assumption we have, a.s.:

$$\mathscr{A}\int_0^t V(s)\,ds = \begin{bmatrix} A\int_0^t U(s)\,ds + B\int_0^t U_s\,ds \\ U_t - f_0 \end{bmatrix}$$
$$= \begin{bmatrix} U(t) - x_0 - \int_0^t G(U(s))\,dW_H(s) \\ U_t - f_0 \end{bmatrix}.$$

Combining this equality with the following:

$$\int_0^t \mathscr{G}(V(s))\,dW_H(s) = \int_0^t \begin{bmatrix} G(V(s), V_s) \\ 0 \end{bmatrix} dW_H(s)$$
$$= \begin{bmatrix} \int_0^t G(U(s), U_s)\,dW_H(s) \\ 0 \end{bmatrix},$$

we see V is a generalized strong solution.

Part (ii). Let V be a generalized strong solution to (4.0.3) and define $U|_{[-1,0)} = f_0$, $U(t) := \pi_1(V(t))$ for $t \geq 0$. Recall from Remark 4.10 that $\pi_2 V(t) = u \mapsto \pi_1 V(s+u) = U_s$. Thus from the definitions of a generalized strong solution and from the generator $\mathscr{A}$ we obtain

$$U(s) - x_0 = A\int_0^t U(s)\,ds + B\int_0^t U_s\,ds + \int_0^t G(U(s), U_s)\,dW_H(s) \quad \textit{a.s.}$$

□

Corollary 4.16. *U is a strong solution to* (4.0.1) *if and only if U satisfies*

$$U(t) = \pi_1 \mathscr{T}(t) \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} + \int_0^t \pi_1 \mathscr{T}(t-s) G(U(s))\, dW_H(s) \quad a.s.$$

From Theorem 4.11 and Theorem 4.15 we obtain:

Corollary 4.17. *Consider* (4.0.1) *with $x_0 \in L^q(\mathscr{F}_0; X)$ and $f_0 \in L^q(\mathscr{F}_0; L^p)$ for some $p \in (1, \infty)$, $q \in [2, \infty)$. Then* (4.0.1) *has a unique strong solution in $L^r(0, t; L^q(\Omega; X))$ for every $r \in [2, \infty]$ and every $t > 0$.*

Combining Theorem 4.12 and Theorem 4.15 we obtain:

Corollary 4.18. *Consider* (4.0.1) *with $x_0 \in L^q(\mathscr{F}_0; X)$ and $f_0 \in L^q(\mathscr{F}_0; L^p)$ for some $p \in (1, \infty)$, $q \in (2, \infty)$. The strong solution $U \in L^\infty(0, t; L^q(\Omega; X))$ to* (4.0.1) *given by Corollary 4.17 satisfies $U \in L^q(\Omega; C([0, t]; X))$.*

Remark 4.19. One cannot hope to obtain a strong solution to (4.0.3) as defined in the monograph of Da Prato and Zabczyk [33], i.e., a process V such that $V(t) \in D(\mathscr{A})$ a.s. for all $t \geq 0$ and

$$V(t) - \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} = \int_0^t \mathscr{A} V(s)\, ds + \int_0^t \mathscr{G}(V(s))\, dW_H(s) \quad \text{a.s. for all } t \geq 0,$$

unless the problem is deterministic, because of the following:

Proposition 4.20. *Let $X = \mathbb{R}$. If a generalized strong solution V to* (4.0.3) *satisfies $V(s) \in D(\mathscr{A})$ a.s. for all $s \in [0, t]$ then $\mathscr{T}(s)[x_0, f_0]^T \in Null(\mathscr{G})$ and $V(s) = \mathscr{T}(s)[x_0, f_0]^T$ a.s. for almost all $s \in [0, t]$, i.e.* (4.0.3) *is deterministic.*

Proof. Define $U := \pi_1(V)$, then U is a generalized strong solution to (4.0.1) by Theorem 4.15. If $V(s) \in D(\mathscr{A})$ for all $s \in [0, t]$ a.s. then $U \in H^{1,p}(0, t)$ a.s., i.e., by Lemma 4.8 the process $I(\mathscr{G}(V)) : [0, t] \times \Omega \to \mathbb{R}$ defined by $I(\mathscr{G}(V))(s) = \int_0^s \mathscr{G}(V(u))\, dW_H(u)$ is in $H^{1,p}(0, t)$ a.s. Recall that the quadratic variation of $I(\mathscr{G}(V))$ is given by

$$V_t^2(I(\mathscr{G}(V)) = \int_0^t \mathscr{G}^2(V(s))\, ds,$$

and hence by [79, Problem 1.5.11] the process $I(\mathscr{G}(V))$ can only be of bounded variation (and hence only possibly in $H^{1,p}(0, t)$) on the set

$$\left\{ \omega \in \Omega : \int_0^t \mathscr{G}^2(V(s, \omega))\, ds = 0 \right\}$$
$$= \{ \omega \in \Omega : V(s, \omega) \in Null(\mathscr{G}) \text{ for almost all } s \in [0, t] \}.$$

Thus if $I(\mathscr{G}^2(V(s)))$ is to be in $H^{1,p}(0, t)$ a.s. then one has

$$V(s) - \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} = \mathscr{A} \int_0^s V(u)\, du \quad \text{a.s. for all } s \in [0, t],$$

which implies that $V(s) = \mathscr{T}(s)[x_0, f_0]^T$ and $\mathscr{T}(s)[x_0, f_0]^T \in Null(\mathscr{G})$ a.s. for all $s \in [0, t]$. $\square$

Remark 4.21. If the noise in (4.0.1) is additive, i.e., if $G \equiv g \in \gamma(H, X)$, then Theorem 4.15 can be applied to prove existence of a stationary solution to (4.0.1). After all, in this case it follows from [111, Proposition 4.4] that (4.0.3) admits invariant measure if and only if the function

$$t \mapsto \mathscr{T}(t)[g, 0]^T$$

represents an element of $\gamma(0, \infty; H, \mathscr{E}^p(X))$. By γ-Fubini isomorphism (2.3.3), embedding (2.3.5) and equality (4.2.1) this is the case if $\pi_1 \mathscr{T}(t)[g, 0]^T \in L^2(0, \infty; \gamma(H, X))$, i.e., in particular if $(\mathscr{T}(t))_{t \geq 0}$ is exponentially stable.

Part III

Approximating Stochastic Differential Equations

5

Introduction: the SDE to be approximated

Let X be a UMD Banach space and let $T > 0$. The upcoming Chapters 6-11 concern approximations in space and time of the following stochastic differential equation in X:

$$\begin{cases} dU(t) = AU(t)\,dt + F(t, U(t))\,dt + G(t, U(t))\,dW_H(t); \quad t \in [0, T], \\ U(0) = x_0. \end{cases} \tag{SDE}$$

Here A is the generator of an analytic C_0-semigroup on X, W_H is a cylindrical Brownian motion in a Hilbert space H, and the functions $F : [0, T] \times X \to X_{\theta_F}$ and $G : [0, T] \times X \to \mathscr{L}(H, X_{\theta_G})$ satisfy appropriate Lipschitz conditions. Recall that X_δ denotes the fractional domain or extrapolation space of A, see Section 2.6. In Section 5.1 we state the standing assumptions on A, F, and G. Section 5.2 contains an existence and uniqueness result for solutions to (SDE) under these assumptions.

5.1 Assumptions on A, F and G

When considering approximations to (SDE), we make the following standing assumptions on the Banach space X, the operator A, and the functions F and G. (Except in Chapter 8, where **(F)** and **(G)** are replaced by the local Lipschitz assumptions **($\mathbf{F_{loc}}$)** and **($\mathbf{F_{loc}}$)**.)

(A) A generates an analytic C_0-semigroup on the UMD Banach space X.

(F) For some $\theta_F > -1 + (\frac{1}{\tau} - \frac{1}{2})$, where τ is the type of X, the function $F : [0, T] \times X \to X_{\theta_F}$ is measurable in the sense that for all $x \in X$ the mapping $F(\cdot, x) : [0, T] \to X_{\theta_F}$ is strongly measurable. Moreover, F is uniformly Lipschitz continuous and uniformly of linear growth in its second variable.

That is to say, there exist constants C_0 and C_1 such that for all $t \in [0, T]$ and all $x, x_1, x_2 \in X$:

$$\|F(t, x_1) - F(t, x_2)\|_{X_{\theta_F}} \le C_0 \|x_1 - x_2\|_X,$$

$$\|F(t,x)\|_{X_{\theta_F}} \le C_1(1 + \|x\|_X).$$

The least constant C_0 such that the above holds is denoted by $\mathrm{Lip}(F)$, and the least constant C_1 such that the above holds is denoted by $M(F)$.

(G) For some $\theta_G > -\frac{1}{2}$, the function $G : [0,T]\times X \to \mathscr{L}(H, X_{\theta_G})$ is measurable in the sense that for all $h \in H$ and $x \in X$ the mapping $G(\cdot, x)h : [0,T] \to X_{\theta_G}$ is strongly measurable. Moreover, G is uniformly L^2_γ-Lipschitz continuous and uniformly of linear growth in its second variable.

That is to say, there exist constants C_0 and C_1 such that for all $\alpha \in [0, \frac{1}{2})$, all $t \in [0,T]$, and all simple functions ϕ_1, ϕ_2, $\phi : [0,T] \to X$ one has:

$$\begin{aligned}
&\|s \mapsto (t-s)^{-\alpha}[G(s,\phi_1(s)) - G(s,\phi_2(s))]\|_{\gamma(0,t;H,X_{\theta_G})} \\
&\qquad \le C_0\|s \mapsto (t-s)^{-\alpha}[\phi_1 - \phi_2]\|_{L^2(0,t;X)\cap\gamma(0,t;X)}; \\
&\|s \mapsto (t-s)^{-\alpha}G(s,\phi(s))\|_{\gamma(0,t;H,X_{\theta_G})} \\
&\qquad \le C_1\big(1 + \|s \mapsto (t-s)^{-\alpha}\phi(s)\|_{L^2(0,t;X)\cap\gamma(0,t;X)}\big).
\end{aligned}$$

The least constant C_0 such that the above holds is denoted by $\mathrm{Lip}_\gamma(G)$, and the least constant C_1 such that the above holds is denoted by $M_\gamma(G)$.

Our definition of L^2_γ-Lipschitz continuity is a slight adaptation of the definition given in [109]. Examples of L^2_γ-Lipschitz continuous operators can be found in that article. In particular:

(i) If G is defined by an Nemytskii map on $[0,T] \times L^p(R)$, where $p \in [1,\infty)$ and $(R, \mathscr{R}, \mu)$ a σ-finite measure space, then G is L^2_γ-Lipschitz continuous (see [109, Example 5.5]).

(ii) if $G : [0,T] \times X \to \gamma(H, X_{\theta_G})$ is Lipschitz continuous, uniformly in $[0,T]$, and X is a type 2 space, then G is L^2_γ-Lipschitz continuous (see [109, Lemma 5.2]).

5.2 Existence and uniqueness of a solution

The solution concept we shall be working with is the following (see also Definition 4.2):

Definition 5.1. An adapted, strongly measurable process $U : [0,T] \times \Omega \to X$ is called a *mild solution* to (SDE) if, for all $t \in [0,T]$,

(i) $s \mapsto S(t-s)F(S,U(s)) \in L^0(\Omega, L^1(0,T;X))$,

(ii) $s \mapsto S(t-s)G(s,U(s))$ is H-strongly measurable, adapted and almost surely in $\gamma(0,t;H,X)$,

and moreover U satisfies:

$$U(t) = S(t)x_0 + \int_0^t S(t-s)F(s,U(s))\,ds + \int_0^t S(t-s)G(s,U(s))\,dW_H(s) \quad (5.2.1)$$

almost surely for all $t \in [0,T]$.

Recall from Remark 2.8 that condition (ii) above is equivalent to saying $s \mapsto S(t-s)G(s, U(s)) \in L^0(\Omega, \gamma(0,t;H,X))$.

In [109] existence and uniqueness of a mild solution to (SDE) is proven under the conditions **(A)**, **(F)**, **(G)**. Theorem 5.3 below is a variation on that theorem that is more suitable for our needs when considering time discretizations. Before presenting the theorem, however, we define the spaces in which we shall work.

Definition 5.2. For $\alpha \geq 0$ and $0 \leq a < b < \infty$ we define $\mathscr{V}^{\alpha,p}_{\infty}([a,b]\times\Omega;X)$ to be the space consisting of those $\Phi \in L^p_{\mathscr{F}}(\Omega;\gamma(a,b;X))$ for which the following norm is finite:

$$\begin{aligned}&\|\Phi\|_{\mathscr{V}^{\alpha,p}_{\infty}([a,b]\times\Omega;X)}\\&\qquad = \|\Phi\|_{L^\infty(a,b;L^p(\Omega;X))} + \sup_{a\leq t\leq b}\|s\mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(a,t;X))}.\end{aligned}$$

Similarly, we define $V^{\alpha,p}_{\infty}([a,b]\times\Omega;X)$ to be the space consisting of those $\Phi \in L^p_{\mathscr{F}}(\Omega;\gamma(a,b;X))$ for which the following norm is finite:

$$\begin{aligned}&\|\Phi\|_{V^{\alpha,p}_{\infty}([a,b]\times\Omega;X)}\\&\qquad = \|\Phi\|_{L^p(\Omega;L^\infty(a,b;X))} + \sup_{a\leq t\leq b}\|s\mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(a,t;X))}.\end{aligned}$$

Finally, we define $V^{\alpha,p}_{\mathrm{c}}([a,b]\times\Omega;X)$ to be the space consisting of those $\Phi \in L^p_{\mathscr{F}}(\Omega;\gamma(a,b;X))$ that are continuous and for which the following norm is finite:

$$\begin{aligned}&\|\Phi\|_{V^{\alpha,p}_{\mathrm{c}}([a,b]\times\Omega;X)}\\&\qquad = \|\Phi\|_{L^p(\Omega;C([a,b];X))} + \sup_{a\leq t\leq b}\|s\mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(a,t;X))}.\end{aligned}$$

Note that the finite rank adapted step processes are not contained in $V^{\alpha,p}_{\mathrm{c}}$. Moreover, although they are contained in $V^{\alpha,p}_{\infty}$ and $\mathscr{V}^{\alpha,p}_{\infty}$, they are not dense in these spaces, see Appendix A.4.

For $0 \leq \beta \leq \alpha < \frac{1}{2}$ covariance domination implies (see (2.3.4)):

$$\|\Phi\|_{\mathscr{V}^{\beta,p}_{\infty}([a,b]\times\Omega;X)} \leq (b-a)^{\alpha-\beta}\|\Phi\|_{\mathscr{V}^{\alpha,p}_{\infty}([a,b]\times\Omega;X)}. \tag{5.2.2}$$

Moreover, for $a \leq c < d \leq b$,

$$\|\Phi|_{[c,d]}\|_{\mathscr{V}^{\alpha,p}_{\infty}([a,b]\times\Omega;X)} = \|\Phi|_{[c,d]}\|_{\mathscr{V}^{\alpha,p}_{\infty}([c,d]\times\Omega;X)}. \tag{5.2.3}$$

Finally, if $G : [0,T]\times X \to \mathscr{L}(H, X_{\theta_G})$ satisfies **(G)** and $\Phi_1, \Phi_2 \in \mathscr{V}^{\alpha,p}_{\infty}([0,T]\times\Omega;X)$ for some $p \geq 2$, then:

$$\begin{aligned}&\sup_{0\leq t\leq T}\|s\mapsto (t-s)^{-\alpha}[G(s,\Phi_1(s)) - G(s,\Phi_2(s))]\|_{L^p(\Omega;\gamma(0,t;X_{\theta_G}))}\\&\quad\leq \mathrm{Lip}_\gamma(G)\sup_{0\leq t\leq T}\|s\mapsto (t-s)^{-\alpha}[\Phi_1(s)-\Phi_2(s)]\|_{L^p(\Omega;L^2(0,t;X))\cap L^p(\Omega;\gamma(0,t;X))}\\&\quad\leq \mathrm{Lip}_\gamma(G)\sup_{0\leq t\leq T}\|s\mapsto (t-s)^{-\alpha}[\Phi_1(s)-\Phi_2(s)]\|_{L^2(0,t;L^p(\Omega;X))\cap L^p(\Omega;\gamma(0,t;X))}\\&\quad\leq (1+T^{\frac{1}{2}-\alpha})\mathrm{Lip}_\gamma(G)\|\Phi_1-\Phi_2\|_{\mathscr{V}^{\alpha,p}_{\infty}([0,T]\times\Omega;X)},\end{aligned} \tag{5.2.4}$$

and, by a similar argument one has, for $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$:

$$\begin{aligned}\sup_{0\le t\le T} &\|s \mapsto (t-s)^{-\alpha} G(s,\Phi(s))\|_{L^p(\Omega;\gamma(0,t;X_{\theta_G}))} \\ &\le (1+T^{\frac{1}{2}-\alpha}) M_\gamma(G)\big(1+\|\Phi\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)}\big).\end{aligned} \tag{5.2.5}$$

One may check that equations (5.2.2)-(5.2.5) remain valid if the $\mathscr{V}_\infty^{\alpha,p}$-norms are replaced by $V_\infty^{\alpha,p}$- or $V_{\mathrm{c}}^{\alpha,p}$-norms.

Set

$$\eta_{\max} := \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F,\ \tfrac{1}{2} + \theta_G\}. \tag{5.2.6}$$

The proof of the following theorem, which is entirely analogous to the proof of [109, Theorem 6.2], is presented in Appendix A.3.

Theorem 5.3. *Let $0 \le \eta < \eta_{\max}$ and $p \in [2,\infty)$. For all initial values $x_0 \in L^p(\Omega;\mathscr{F}_0;X_\eta)$ and all $\alpha \in [0,\frac{1}{2})$, the problem* (SDE) *has a unique mild solution U in $\mathscr{V}_\infty^{\alpha,p}([0,t]\times\Omega;X_\eta)$. It satisfies*

$$\|U\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X_\eta)} \lesssim 1 + \|x_0\|_{L^p(\Omega;X_\eta)}. \tag{5.2.7}$$

Moreover, if $\frac{1}{p} < \frac{1}{2} + \theta_G$ and $0 \le \eta < \min\{1 - (\frac{1}{\tau} - \frac{1}{2}) + \theta_F,\ \frac{1}{2} + \theta_G - \frac{1}{p}\}$, then $U \in V_{\mathrm{c}}^{\alpha,p}([0,T]\times\Omega;X_\eta)$ for all $\alpha \in [0,\frac{1}{2})$.

Remark 5.4. The extra factor $\frac{1}{p}$ in the second statement above arises from a Kolmogorov-type estimate on the stochastic convolution in (5.2.1).

The reason we need this adapted version of Theorem [109, Theorem 6.2] is rather technical. Note that the approximations obtained by the splitting scheme and the Euler scheme are not continuous. Accordingly we first prove convergence of the various schemes in $\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)$ for arbitrarily large $p \in (2,\infty)$. Pathwise convergence results (in the grid points) can then be obtained by a Kolmogorov argument. However, if we were to use the existence and uniqueness results of [109], we would lose a factor $\frac{1}{p}$ twice.

6

The splitting scheme

This chapter concerns the convergence of splitting schemes for the stochastic differential equation (SDE) under the assumptions **(A)**, **(F)**, **(G)** of Section 5.1. For reasons to be explained shortly, we shall consider two different schemes that we shall refer to as the *modified splitting scheme* and the *classical splitting scheme*. The modified scheme is defined as follows: fix $T > 0$ and an integer $n \in \mathbb{N}$. For $j = 1, \dots, n$ we define the process $U_j^{(n)} : [t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega \to X$ as the mild solution to the problem

$$\begin{cases} dU_j^{(n)}(t) = S(\tfrac{T}{n})\big[F(t, U_j^{(n)}(t))\, dt + G(t, U_j^{(n)}(t))\, dW_H(t)\big], \quad t \in [t_{j-1}^{(n)}, t_j^{(n)}]; \\ U_j^{(n)}(t_{j-1}^{(n)}) = S(\tfrac{T}{n}) U_{j-1}^{(n)}(t_{j-1}^{(n)}), \end{cases} \tag{6.0.1}$$

where we set $U_0^{(n)}(0) := x_0$, and recall that $t_j^{(n)} = \frac{jT}{n}$.

The existence of a unique mild solution to (6.0.1) in $\mathscr{V}_\infty^{\alpha,p}([t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega; X)$, for $\alpha \in [0, \frac{1}{2})$ and $p \in [2, \infty)$, is guaranteed by Theorem 5.3. Here we use that $S(\frac{T}{n})F : [0,T] \times X \to X$ satisfies **(F)** and that $S(\frac{T}{n})G : [0,T] \times X \to \mathscr{L}(H, X)$ satisfies **(G)**.

For $j = 1, \dots, n$ we define $I_j^{(n)} := [t_{j-1}^{(n)}, t_j^{(n)})$. Observe that the adapted process $U^{(n)} : [0,T) \times \Omega \to X$ defined by

$$U^{(n)} := \sum_{j=1}^{n} 1_{I_j^{(n)}}(t)\, U_j^{(n)}(t), \quad t \in [0,T), \tag{6.0.2}$$

defines an element of $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$. In the next section we prove convergence of $U^{(n)}$ against U, where U is the mild solution to (SDE), in $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$, for $p \in [2, \infty)$ (provided $x_0 \in L^p(\Omega, X_\eta)$ for $\eta > 0$).

As we obtain convergence in $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$ for p arbitrarily large, by a Kolmogorov argument we obtain convergence in the discrete Hölder norm: For a sequence $x = (x_j)_{j=0}^n$ in X (we think of the x_j as the values $f(t_j^{(n)})$ of some function $f : [0,T] \to X$) and $0 \le \gamma \le 1$ we define the discrete Hölder norm

$\|\cdot\|_{c_\gamma^{(n)}}$ as follows:

$$\|x\|_{c_\gamma^{(n)}([0,T];X)} := \sup_{0\le j\le n} \|x_j\|_X + \sup_{0\le i<j\le n} \frac{\|x_j - x_i\|_X}{|t_j^{(n)} - t_i^{(n)}|^\gamma}.$$

For $n \in \mathbb{N}$ fixed set $u := (U(t_j^{(n)}))_{j=0}^n$ and $u^{(n)} := (U^{(n)}(t_j^{(n)}))_{j=0}^n$. The proof of the following theorem is contained in Section 6.3:

Theorem 6.1 (Hölder convergence of the splitting scheme). *Let X be a* UMD *Banach space and let $\tau \in (1,2]$ be the type of X. Suppose $p \in [2,\infty)$ and $\gamma, \delta \in [0,1)$ and $\eta > 0$ satisfy*

$$\gamma + \delta + \tfrac{1}{p} < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F,\ \tfrac{1}{2} + \theta_G,\ \eta,\ 1\},$$

and suppose that $x_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$. There is a constant C, independent of x_0, such that for all $n \in \mathbb{N}$,

$$\big(\mathbb{E}\|u - u^{(n)}\|^p_{c_\gamma^{(n)}([0,T];X)}\big)^{\frac{1}{p}} \le C n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}). \tag{6.0.3}$$

By a Borel-Cantelli argument, (6.0.3) implies that for almost all $\omega \in \Omega$ there exists a constant C depending on ω but independent of n, such that:

$$\sup_{j\in\{0,\dots,n\}} \|u(\omega) - u^{(n)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} \le C n^{-\delta}.$$

There is a subtle difference between the modified splitting scheme and the *classical splitting scheme*, which is defined by solving, for $j \in \{1,\dots,n\}$;

$$\begin{cases} d\tilde{U}_j^{(n)}(t) = F(t, \tilde{U}_j^{(n)}(t))\,dt + G(t, \tilde{U}_j^{(n)}(t))\,dW_H(t), \quad t \in [t_{j-1}^{(n)}, t_j^{(n)}]; \\ \tilde{U}_j^{(n)}(t_{j-1}^{(n)}) = S(\tfrac{T}{n})\tilde{U}_{j-1}^{(n)}(t_{j-1}^{(n)}), \end{cases} \tag{6.0.4}$$

with $\tilde{U}_0^{(n)}(0) := x_0$. The existence of a unique mild solution $\tilde{U}_j^{(n)}$ to (6.0.4) in $\mathscr{V}_\infty^{\alpha,p}([t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega; X)$, for every $\alpha \in [0, \frac{1}{2})$ and $p \in [2,\infty)$, is again guaranteed by Theorem 5.3 *provided that* $\theta_F \ge 0$ and $\theta_G \ge 0$. However, if $\theta_F < 0$ or $\theta_G < 0$, then we have no means to define a solution to (6.0.4) in X, since we cannot guarantee that the integrals corresponding to F and G in the definition of a mild solution take values in X. In the modified splitting scheme, this problem is overcome by the extra operator $S(\frac{T}{n})$ with provides the required additional smoothing.

Once convergence of the modified splitting scheme has been established, convergence of the classical splitting scheme is derived from it under the additional assumptions $\theta_F \ge 0$ and $\theta_G \ge 0$ (Theorem 6.4).

In the final section of this chapter we consider the case that (SDE) is linear with additive noise, i.e., $F \equiv 0$ and $G = g \in \gamma(H, X_{\theta_G})$. In that case Theorem 6.1 holds in arbitrary Banach spaces (i.e., we need not assume that the space satisfies the UMD property). Moreover, we also have convergence of the splitting scheme if S is not analytic, *provided* X is a type 2 space. The section is concluded with an example for which the splitting scheme does *not* converge.

At the cost of some extra work it is possible to obtain pathwise convergence of the splitting schemes over $[0,T]$ instead of over the grid points, i.e., convergence in $L^p(\Omega; C([0,T];X))$ and in $C([0,T];X)$ almost surely. See Appendix A.5.

For semi-linear (Stratonovich type) SPDEs governed by second order elliptic operators on $\mathbb{R}^d$ and driven by multiplicative noise, convergence in $X = L^2(\mathbb{R}^d)$ of splitting schemes like the one considered here has been proved by various authors [6, 7, 46, 56, 100]. In particular, pathwise uniform convergence of the splitting scheme, with rate n^{-1} (which is the rate corresponding to exponents $\theta_F \geq \frac{1}{\tau} - \frac{1}{2}$ and $\theta_G \geq \frac{1}{2}$ in our framework, provided we take $\eta \geq 1$), has been obtained by Gyöngy and Krylov [56] for again a different splitting scheme, namely one for which the solution to (6.0.1) is guaranteed by adding (roughly speaking) a term $\varepsilon AU(t)\,dt$ to the right-hand side of (6.0.1).

Our results concerning convergence in the case that S is not analytic (see Section 6.4) extend and simplify previous work by Kühnemund and van Neerven [83, Theorems 4.3 and 5.2].

6.1 Convergence of the modified splitting scheme

As always we assume that **(A)**, **(F)**, **(G)** hold, and we denote by U the mild solution to the problem (SDE) with initial value x_0. For technical reasons we shall consider the modified splitting scheme with initial value $U_0^{(n)}(0) := y_0$, where $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$, $p \in [2,\infty)$, may be taken different from x_0. Of course, in applications it is natural to assume $y_0 = x_0$ or y_0 is a close approximation of x_0. Thus we define $U^{(n)}$ as in (6.0.2), with $(U_j^{(n)})_{j=1}^n$ defined by (6.0.1) but with $U_0^{(n)}(0) := y_0$.

Theorem 6.2. *Let $0 \leq \eta \leq 1$ satisfy $\eta < \eta_{\max}$, where $\eta_{\max}$ is as on page 86, and assume that $x_0 \in L^p(\mathscr{F}_0, X_\eta)$ and $y_0 \in L^p(\mathscr{F}_0, X)$ for some $p \in [2,\infty)$. Then for all $\alpha \in [0, \frac{1}{2})$ one has:*

$$\|U - U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}\big(1 + \|x_0\|_{L^p(\Omega;X_\eta)}\big), \tag{6.1.1}$$

with implied constants independent of n, x_0 and y_0.

For $t \in I_j^{(n)}$ we define

$$\underline{t} := t_{j-1}^{(n)}, \quad \overline{t} := t_j^{(n)}.$$

In particular, $\overline{t}_{j-1}^{(n)} = t_j^{(n)}$. It should be kept in mind that in the notation $\underline{x}$ and $\overline{x}$ we suppress the dependence on n and T.

The proof of Theorem 6.2 uses the strong resemblance between identity (6.1.2) for $U^{(n)}$ below and the identity (5.2.1) satisfied by U.

Claim. *Let $U^{(n)}$ be defined by* (6.0.2) *but with $U_0^{(n)}(0) = y_0$. Almost surely, for all $t \in [0, T)$ we have:*

$$U^{(n)}(t) = S(\overline{t})y_0 + \int_0^t S(\overline{t} - \underline{s})F\big(s, U^{(n)}(s)\big)\, ds \\ + \int_0^t S(\overline{t} - \underline{s})G\big(s, U^{(n)}(s)\big)\, dW_H(s). \tag{6.1.2}$$

Proof. It suffices to prove that for any $j \in \{1, \ldots, n\}$, almost surely the following identity holds for all $t \in I_j^{(n)}$:

$$\begin{aligned} U_j^{(n)}(t) = S(t_j^{(n)})y_0 &+ \int_{t_{j-1}^{(n)}}^t S(\tfrac{T}{n})F(s, U_j^{(n)}(s))\, ds \\ &+ \int_{t_{j-1}^{(n)}}^t S(\tfrac{T}{n})G(s, U_j^{(n)}(s))\, dW_H(s) \\ &+ \sum_{k=1}^{j-1} \int_{I_k^{(n)}} S(t_{j-k+1}^{(n)})F(s, U_k^{(n)}(s))\, ds \\ &+ \sum_{k=1}^{j-1} \int_{I_k^{(n)}} S(t_{j-k+1}^{(n)})G(s, U_k^{(n)}(s))\, dW_H(s). \end{aligned} \tag{6.1.3}$$

By definition, the process $U_j^{(n)}$, being a mild solution to (6.0.1), satisfies:

$$\begin{aligned} U_j^{(n)}(t) = S(\tfrac{T}{n})U_{j-1}^{(n)}(t_{j-1}^{(n)}) &+ \int_{t_{j-1}^{(n)}}^t S(\tfrac{T}{n})F(s, U_j^{(n)}(s))\, ds \\ &+ \int_{t_{j-1}^{(n)}}^t S(\tfrac{T}{n})G\big(s, U_j^{(n)}(s)\big)\, dW_H(s) \end{aligned} \tag{6.1.4}$$

almost surely for all $t \in I_j^{(n)}$. For $j = 1$ (6.1.3) follows directly from (6.1.4), and for $j \in \{2, \ldots, n\}$ it follows by induction. $\square$

Proof (of Theorem 6.2). Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\{\tfrac{1}{2}, 1 - 2\alpha, \eta_{\max} - \eta\}.$$

In particular we have $\varepsilon < \frac{1}{2} + \theta_G$ and thus, by replacing $\alpha \in [0, \frac{1}{2})$ by some larger value if necessary, we may assume that

$$\max\{1 - \tfrac{4}{3}\varepsilon, \varepsilon - 2\theta_G\} < 2\alpha < 1 - \varepsilon.$$

We split the proof of (6.1.1) into several parts. In each part, constants will be allowed to depend on the final time T. Thus, the statement '$A(t) \lesssim B$ with a constant independent of $t \in [0,T]$' is to be interpreted as short-hand for 'there is a constant C, possibly depending on T, such that $\sup_{t\in[0,T]} A(t) \le CB$'.

Part 1. Fix $n \in \mathbb{N}$. Let $x, y \in L^p(\Omega; X_\eta)$. By the identities (5.2.1) and (6.1.2), for all $s \in [0,T]$ we have:

$$\begin{aligned} U(s) - U^{(n)}(s) &= (S(s) - S(\overline{s}))x_0 \\ &\quad + S(\overline{s})(x_0 - y_0) \\ &\quad + \int_0^s [S(s-u) - S(\overline{s} - \underline{u})]F(u, U(u))\,du \\ &\quad + \int_0^s S(\overline{s} - \underline{u})[F(u,U(u)) - F(u, U^{(n)}(u))]\,du \\ &\quad + \int_0^s [S(s-u) - S(\overline{s} - \underline{u})]G(u, U(u))\,dW_H(u) \\ &\quad + \int_0^s S(\overline{s} - \underline{u})[G(u,U(u)) - G(u, U^{(n)}(u))]\,dW_H(u). \end{aligned} \tag{6.1.5}$$

We shall estimate the $\mathscr{V}_\infty^{\alpha,p}([0,T_0] \times \Omega; X)$-norm of each of the six terms separately for arbitrary $T_0 \in [0,T]$. In the fourth and sixth term (part 1d and 1f below) it will be necessary to keep track of the dependence on T_0.

Part 1a. We start with the first term in (6.1.5). Writing $S(s) - S(\overline{s}) = (I - S(\overline{s} - s))S(s)$ and $S(s) = s^{-\frac{1}{2}\varepsilon} s^{-\frac{1}{2}\varepsilon} S(s)$, from Lemma 2.21 (1) and (3) and Theorem 2.14 we obtain, almost surely for all $t \in [0,T]$:

$$\begin{aligned} \|s \mapsto (t-s)^{-\alpha}(S(s) - S(\overline{s}))x_0\|_{\gamma(0,t;X)} &\lesssim n^{-\eta} \|s \mapsto (t-s)^{-\alpha} S(s)x_0\|_{\gamma(0,t;X)} \\ &\lesssim n^{-\eta} \|s \mapsto (t-s)^{-\alpha} s^{-\frac{1}{2}\varepsilon} x_0\|_{\gamma(0,t;X)} \\ &= n^{-\eta} \|s \mapsto (t-s)^{-\alpha} s^{-\frac{1}{2}\varepsilon}\|_{L^2(0,t)} \|x_0\|_X \\ &\lesssim n^{-\eta} \|x_0\|_X, \end{aligned}$$

with implied constants independent of n, t, and x_0. Also, by (2.6.4),

$$\begin{aligned} &\|s \mapsto (S(s) - S(\overline{s}))x_0\|_{L^\infty(0,T_0;X)} \\ &\quad \le \sup_{s\in[0,T_0]} \|S(s)\|_{\mathscr{L}(X)} \|I - S(\overline{s} - s)\|_{\mathscr{L}(X_\eta, X)} \|x_0\|_{X_\eta} \lesssim n^{-\eta} \|x_0\|_{X_\eta}. \end{aligned}$$

By taking p^{th} moments it follows that that for every $T_0 \in [0,T]$ we have:

$$\|s \mapsto (S(s) - S(\underline{s}))x_0\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim n^{-\eta} \|x_0\|_{L^p(\Omega;X_\eta)}, \tag{6.1.6}$$

with implied constant independent of n, T_0 and x_0.

Part 1b. Concerning the second term on the right-hand side in (6.1.5) we note that, almost surely:

$$\|s \mapsto S(\overline{s})(x_0 - y_0)\|_{L^\infty(0,T;X)} \lesssim \|x_0 - y_0\|_X,$$

with implied constant independent of n, x_0 and y_0. Also, by Lemma 2.21 (1) and Theorem 2.14, almost surely we have, for all $t \in [0,T]$:

$$\begin{aligned}\|s \mapsto (t-s)^{-\alpha} S(\overline{s})(x_0 - y_0)\|_{\gamma(0,t;X)} &\lesssim \|s \mapsto (t-s)^{-\alpha} (\overline{s})^{-\frac{1}{2}\varepsilon}(x_0 - y_0)\|_{\gamma(0,t;X)} \\ &= \|s \mapsto (t-s)^{-\alpha}(\overline{s})^{-\frac{1}{2}\varepsilon}\|_{L^2(0,t)}\|x_0 - y_0\|_X \\ &\lesssim \|x_0 - y_0\|_X\end{aligned}$$

with implied constants are independent of n, t, x_0 and y_0.

Combining these estimates we obtain, for all $T_0 \in [0,T]$:

$$\|s \mapsto S(\underline{s})(x_0 - y_0)\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} \tag{6.1.7}$$

with implied constants independent of n, T_0, x_0 and y_0.

Part 1c. Concerning the third term on the right-hand side in (6.1.5) we observe that:

$$S(s-u)-S(\overline{s}-\underline{u}) = (I-S(\overline{s}-s))S(s-u)+S(\overline{s}-s)S(s-u)(I-S(u-\underline{u})), \tag{6.1.8}$$

and hence

$$\begin{aligned}&\int_0^s [S(s-u) - S(\overline{s}-\underline{u})]F(u,U(u))\,du \\ &\qquad = (I - S(\overline{s}-s))\int_0^s S(s-u)F(u,U(u))\,du \\ &\qquad\quad + S(\overline{s}-s)\int_0^s S(s-u)(I - S(u-\underline{u}))F(u,U(u))\,du.\end{aligned}$$

Let $T_0 \in [0,T]$. It follows from Lemma 2.21, part (2) (with exponent $\frac{1}{2}\varepsilon$) and part (3) (with exponent $\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon$) and and the γ-multiplier theorem (Theorem 2.14) that:

$$\begin{aligned}&\Big\|s \mapsto \int_0^s [S(s-u) - S(\overline{s}-\underline{u})]F(u,U(u))\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\ &\quad\lesssim n^{-\min\{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon,1\}} \\ &\qquad\qquad \times \Big\|s \mapsto \int_0^s S(s-u)F(u,U(u))\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon})} \\ &\qquad + \Big\|s \mapsto \int_0^s S(s-u)(I-S(u-\underline{u}))F(u,U(u))\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{2}\varepsilon})},\end{aligned} \tag{6.1.9}$$

with implied constants independent of n and T_0. We shall estimate the two terms on the right-hand side of (6.1.9) separately.

We begin with the first term. Recall that $U \in \mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)$ and therefore, by **(F)**, we have $F(\cdot,U(\cdot)) \in L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))$. By Lemma A.7

(applied with $Y = X_{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon}$, $\Phi(u) = F(u, U(u))$, and $\delta = -\frac{3}{2} + \frac{1}{\tau} + \varepsilon$) we obtain, for all $t \in [0, T_0]$:

$$\begin{aligned}\Big\| s \mapsto \int_0^s S(s-u)F(u,U(u))\,du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon})} \\ \lesssim \|u \mapsto F(u,U(u))\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))} \\ \lesssim (1 + \|U\|_{L^\infty(0,T_0;L^p(\Omega;X))}),\end{aligned}$$

with implied constants independent of n, x_0 and T_0.

For the second term in the right-hand side of (6.1.9) we apply Lemma A.7 (with $Y = X_{\frac{1}{2}\varepsilon}$, $\delta = -\frac{3}{2} + \frac{1}{\tau} + \frac{1}{2}\varepsilon$ and $\Phi(u) = (I - S(u-\underline{u}))F(u, U(u))$). Note that $\Phi \in L^\infty(0,T; L^p(\Omega; X_{-\frac{3}{2}+\frac{1}{\tau}+\varepsilon}))$ by the boundedness of $u \mapsto (I - S(u - \underline{u}))$ in $\mathscr{L}(X_{\theta_F}, X_{-\frac{3}{2}+\frac{1}{\tau}+\varepsilon})$, the linear growth condition in **(F)** and the fact that $U \in \mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)$. We obtain:

$$\begin{aligned}\Big\| s \mapsto \int_0^s S(s-u)(I-S(u-\underline{u}))F(u,U(u))\,du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{2}\varepsilon})} \\ \lesssim \|u \mapsto (I-S(u-\underline{u}))F(u,U(u))\|_{L^\infty(0,T_0;L^p(\Omega;X_{-\frac{3}{2}+\frac{1}{\tau}+\varepsilon}))} \\ \lesssim n^{-\min\{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon,1\}}\|u \mapsto F(u,U(u))\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))} \\ \lesssim n^{-\min\{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon,1\}}(1 + \|U\|_{L^\infty(0,T_0;L^p(\Omega;X))}),\end{aligned}$$

with implied constants independent of n, x_0 and T_0. For the penultimate estimate we used (2.6.4).

Combining these estimates, applying (5.2.7), and recalling the assumptions $\eta \le 1$ and $\eta < \frac{3}{2} - \frac{1}{\tau} - \varepsilon + \theta_F$, we obtain:

$$\begin{aligned}\Big\| s \mapsto \int_0^s [S(s-u) - S(\overline{s}-\underline{u})]F(u,U(u))\,du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\ \lesssim n^{-\min\{\frac{3}{2}+\theta_F-\frac{1}{\tau}-\varepsilon,1\}}(1 + \|U\|_{L^\infty(0,T_0;L^p(\Omega;X))}) \\ \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}),\end{aligned} \tag{6.1.10}$$

with implied constants independent of n, x_0 and T_0.

Part 1d. Concerning the fourth term on the right-hand side in (6.1.5) we first apply Theorem 2.14 and Lemma 2.21 (2) (with exponent $\frac{1}{2}\varepsilon$) and then apply Lemma A.7 (with $Y = X_{\frac{1}{2}\varepsilon}$, $\delta = \theta_F - \varepsilon$ and $\Phi(u) = S(u-\underline{u})[F(u,U(u)) - F(u, U^{(n)}(u))]$). Observe that $\Phi \in L^\infty(0,T; L^p(\Omega;X))$ by the fact that both U and $U^{(n)}$ belong to $\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)$, **(F)**, and the uniform boundedness of $u \mapsto S(u-\underline{u})$ in $\mathscr{L}(X_{\theta_F}, X_{\theta_F - \frac{1}{2}\varepsilon})$. We obtain:

$$\Big\| s \mapsto S(\overline{s}-s)\int_0^s S(s-\underline{u})[F(u,U(u)) - F(u,U^{(n)}(u))]\,du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}$$

$$
\begin{aligned}
&\lesssim \Big\| s \mapsto \int_0^s S(s-\underline{u})[F(u,U(u)) - F(u,U^{(n)}(u))]\,du \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_{\frac12\varepsilon})} \\
&\lesssim (T_0^{1-(\theta_F-\varepsilon)^-} + T_0^{\frac12-\alpha}) \qquad (6.1.11)\\
&\qquad \times \|u \mapsto S(u-\underline{u})[F(u,U(u)) - F(u,U^{(n)}(u))]\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F-\frac12\varepsilon}))} \\
&\lesssim (T_0^{1-(\theta_F-\varepsilon)^-} + T_0^{\frac12-\alpha})\|U - U^{(n)}\|_{L^\infty(0,T_0;L^p(\Omega;X))}, \qquad (6.1.12)
\end{aligned}
$$

with implied constants independent of n and T_0.

Part 1e. For the fifth term on the right-hand side in (6.1.5) we proceed as in part 1c. Using (6.1.8), Lemma 2.21, part (2) (with exponent $\frac13\varepsilon$) and part (3) (with exponent $\frac12 + \theta_G - \frac23\varepsilon$), and Theorem 2.14, we obtain:

$$
\begin{aligned}
&\Big\| s \mapsto \int_0^s [S(s-u) - S(\overline{s}-\underline{u})]G(u,U(u))\,dW_H(u) \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X)} \\
&\quad \lesssim n^{-\min\{\frac12+\theta_G-\frac23\varepsilon,1\}} \\
&\qquad\qquad \times \Big\| s \mapsto \int_0^s S(s-u)G(u,U(u))\,dW_H(u) \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_{\frac12+\theta_G-\frac23\varepsilon})} \\
&\qquad + \Big\| s \mapsto \int_0^s S(s-u)(I - S(u-\underline{u}))G(u,U(u))\,dW_H(u) \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_{\frac13\varepsilon})}.
\end{aligned}
\qquad (6.1.13)
$$

Now we apply Lemma A.8 to the two terms on the right-hand side of (6.1.13). For the first term we apply Lemma A.8 with $Y = X_{\frac12+\theta_G-\frac23\varepsilon}$, $\delta = -\frac12 + \frac23\varepsilon$ and $\Phi(u) = G(u,U(u))$, noting that $\alpha > \frac12 - \frac23\varepsilon = -\delta$. Assumption (A.2.1) is satisfied due to (5.2.5) and the fact that $U \in \mathscr{V}^{\alpha,p}_\infty([0,T]\times\Omega;X)$. By Lemma A.8 and (5.2.5) we obtain:

$$
\begin{aligned}
\Big\| s \mapsto \int_0^s S(s-u)G(u,U(u))\,dW_H(u) \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_{\frac12+\theta_G-\frac23\varepsilon})}
&\lesssim \sup_{s\in[0,T_0]} \|u \mapsto (s-u)^{-\alpha}G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;X_{\theta_G}))} \\
&\lesssim 1 + \|U\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X)},
\end{aligned}
$$

with implied constants independent of n, x_0 and T_0.

For the second term we take $Y = X_{\frac13\varepsilon}$, $\delta = -\frac12 + \frac23\varepsilon$ and $\Phi(u) = (I - S(u-\underline{u}))G(u,U(u))$ in Lemma A.8, noting that $\alpha > \frac12 - \frac23\varepsilon = -\delta$; assumption (A.2.1) is satisfied because of $U \in \mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X)$, (5.2.5), and the fact that the operators $I - S(u-\underline{u})$ are γ-bounded from X_{θ_G} to $X_{-\frac12+\varepsilon}$ by Lemma 2.21 (3).

By Lemma A.8, Lemma 2.21 (3) (applied with exponent $\frac12 + \theta_G - \varepsilon$), and (5.2.5) we obtain:

$$
\Big\| s \mapsto \int_0^s S(s-u)(I - S(u-\underline{u}))G(u,U(u))\,dW_H(u) \Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_{\frac13\varepsilon})}
$$

$$
\begin{aligned}
&\lesssim \sup_{s\in[0,T_0]} \|u\mapsto (s-u)^{-\alpha}(I-S(u-\underline{u}))G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;X_{-\frac12+\varepsilon}))}\\
&\lesssim n^{-\min\{\frac12+\theta_G-\varepsilon,1\}} \sup_{s\in[0,T_0]} \|u\mapsto (s-u)^{-\alpha}G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;X_{\theta_G}))}\\
&\lesssim n^{-\eta}(1+\|U\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}),
\end{aligned}
$$

with implied constants independent of n, x_0 and T_0.

Combining these estimates and applying (5.2.7) we obtain:

$$
\begin{aligned}
\Big\|s\mapsto \int_0^s [S(s-u)-S(\overline{s}-\underline{u})]G(u,U(u))\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}&\\
\lesssim n^{-\eta}(1+\|U\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)})&\\
\lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}),&
\end{aligned}
\tag{6.1.14}
$$

with implied constants independent of n, x_0 and T_0.

Part 1f. For the final term in (6.1.5) we proceed as in part 1d. First we apply Theorem 2.14 in combination with Lemma 2.21 (2) (with exponent $\frac14\varepsilon$) to get rid of the term $S(\overline{s}-s)$. Then we apply Lemma A.8 (with $Y=X_{\frac14\varepsilon}$, $\delta=\theta_G-\frac12\varepsilon$, and $\Phi=S(u-\underline{u})G(u,U(u))-G(u,U^{(n)}(u))$). Note that $\alpha>\frac12\varepsilon-\theta_G=-\delta$. Assumption (A.2.1) is satisfied because U and $U^{(n)}$ are in $\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)$, condition **(G)** holds, and the operators $S(u-\underline{u})$ are γ-bounded from X_{θ_G} to $X_{\theta_G-\frac14\varepsilon}$. Finally, we apply Theorem 2.14 again in combination with Lemma 2.21 (2) (with exponent $\frac14\varepsilon$) to get rid of the term $S(u-\underline{u})$. We obtain that there exists an $\epsilon>0$, independent of $T_0\in[0,T]$, such that:

$$
\begin{aligned}
&\Big\|s\mapsto \int_0^s S(\overline{s}-\underline{u})[G(u,U(u))-G(u,U^{(n)}(u))]\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}\\
&\quad\lesssim \Big\|s\mapsto \int_0^s S(s-\underline{u})[G(u,U(u))-G(u,U^{(n)}(u))]\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac14\varepsilon})}\\
&\quad\lesssim T_0^{\epsilon}\sup_{0\le s\le T_0}\|u\mapsto (s-u)^{-\alpha}S(u-\underline{u})\\
&\qquad\qquad\times[G(u,U(u))-G(u,U^{(n)}(u))]\|_{L^p(\Omega;\gamma(0,s;H,X_{\theta_G-\frac14\varepsilon}))}\\
&\quad\lesssim T_0^{\epsilon}\sup_{0\le s\le T_0}\|s\mapsto (s-u)^{-\alpha}[G(u,U(u))-G(u,U^{(n)}(u))]\|_{L^p(\Omega;\gamma(0,s;H,X_{\theta_G}))}\\
&\quad\lesssim T_0^{\epsilon}\|U-U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)},
\end{aligned}
\tag{6.1.15}
$$

where the last step used (5.2.4); the implied constants are independent of n and T_0.

Part 2. Substituting (6.1.6), (6.1.7), (6.1.10), (6.1.12), (6.1.14), (6.1.15) into (6.1.5) we obtain that there exists an exponent $\epsilon_0>0$ and a constant $C>0$, both of which are independent of n, x_0, and y_0, such that for all $T_0\in[0,T]$ we have:

$$\|U - U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \le C\big(\|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})\big) + CT_0^{\epsilon_0}\|U - U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}. \tag{6.1.16}$$

From now on we fix $T_0 := \min\{(2C)^{1/\epsilon_0}, T\}$. Note that T_0 is independent of n, x_0, y_0, and we have:

$$\begin{aligned}\|U - U^{(n)}\|&_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\ &\le 2C\|x_0 - y_0\|_{L^p(\Omega;X)} + 2Cn^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}).\end{aligned} \tag{6.1.17}$$

Part 3. Let us fix $n \in \mathbb{N}$ and pick $t_0 \in \{t_j^{(n)} : j = 0, 1, \ldots, n\}$. For $x \in L^p(\Omega, \mathscr{F}_{t_0}; X)$ we denote by $U(x, t_0, \cdot)$ the process in $\mathscr{V}_\infty^{\alpha,p}([t_0, t_0 + T] \times \Omega; X)$ satisfying, almost surely for all $s \in [t_0, t_0 + T]$:

$$\begin{aligned}U(x, t_0, s) = S(t - t_0)x &+ \int_{t_0}^t S(t - t_0 - s)F\big(s, U(x, t_0, s)\big)\, ds \\ &+ \int_{t_0}^t S(t - t_0 - s)G\big(s, U(x, t_0, s)\big)\, dW_H(s).\end{aligned}$$

By $U^{(n)}(x, t_0, \cdot)$ we denote the process obtained from the modified splitting scheme initiated in t_0 with initial value $x \in L^p(\Omega, \mathscr{F}_{t_0}; X)$. Thus, almost surely for $t \in [t_0, t_0 + T]$:

$$\begin{aligned}U^{(n)}(x, t_0, t) = S(\bar{t} - t_0)x &+ \int_{t_0}^t S(\bar{t} - t_0 - \underline{s})F\big(s, U^{(n)}(x, t_0, s)\big)\, ds \\ &+ \int_{t_0}^t S(\bar{t} - t_0 - \underline{s})G\big(s, U^{(n)}(x, t_0, s)\big)\, dW_H(s).\end{aligned}$$

From the proof of (6.1.17) it follows that for any $x \in L^p(\Omega, \mathscr{F}_{t_0}; X_\eta)$ and $y \in L^p(\Omega, \mathscr{F}_{t_0}; X)$ we have:

$$\begin{aligned}\|U(x, t_0, \cdot) - U^{(n)}(y, t_0, \cdot)&\|_{\mathscr{V}_\infty^{\alpha,p}([t_0,t_0+T_0]\times\Omega;X)} \\ &\le 2C\|x - y\|_{L^p(\Omega;X)} + 2Cn^{-\eta}(1 + \|x\|_{L^p(\Omega;X_\eta)}),\end{aligned} \tag{6.1.18}$$

with C as in (6.1.17).

Part 4. Let T_0 be as in part 2 and fix $N \in \mathbb{N}$ large enough such that $\frac{T}{N} \le T_0$. Let $M = \lceil 2T/T_0 \rceil$. Then $M \ge 2$ and $2T \le MT_0 \le 2T + T_0 \le 3T$.

Let us now fix $n \ge N$. Then $\frac{1}{2}T_0 \le \min\{T_0 - \frac{T}{n}, \frac{T}{n}\}$ and therefore $\underline{T}_0 \ge \frac{1}{2}T_0$. Hence, $T \le M\underline{T}_0 \le 3T$.

From now on we fix an integer $n \ge N$. By the uniqueness of the mild solution to (SDE) and by the definition of $U^{(n)}$ we have, for any $s_0, t_0 \in \{t_j^{(n)} : j = 0, 1, \ldots, M\}$, any $x \in L^p(\Omega, \mathscr{F}_{s_0}; X)$ and any $t \in [t_0, t_0 + T_0]$ that:

$$U(x, s_0, t) = U\big(U(x, s_0, t_0), t_0, t\big);$$

$$U^{(n)}(x, s_0, t) = U^{(n)}\big(U^{(n)}(x, s_0, t_0), t_0, t\big).$$

For $j \in \{1, \ldots, M\}$, from (6.1.18) (with $x = U(x_0, (j-1)\underline{T}_0)$ and $y = U^{(n)}(y_0, (j-1)\underline{T}_0))$ we obtain:

$$\begin{aligned}
&\|U(x_0, 0, j\underline{T}_0) - U^{(n)}(y_0, 0, j\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\quad = \big\|U\big(U(x_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \underline{T}_0\big) \\
&\qquad\qquad - U^{(n)}\big(U^{(n)}(y_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \underline{T}_0\big)\big\|_{L^p(\Omega;X)} \\
&\quad \lesssim \|U(x_0, 0, (j-1)\underline{T}_0) - U^{(n)}(y_0, 0, (j-1)\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\qquad + n^{-\eta}(1 + \|U(x_0, (j-1)\underline{T}_0)\|_{L^p(\Omega;X_\eta)}),
\end{aligned} \tag{6.1.19}$$

with implied constants independent of j, n, x_0, y_0.

By (5.2.7) we have

$$\begin{aligned}
\sup_{1\le j\le M} \|U(x_0, 0, j\underline{T}_0)\|_{L^p(\Omega;X_\eta)} &\le \sup_{s\in[0,3T]} \|U(x_0, 0, s)\|_{L^p(\Omega;X_\eta)} \\
&\lesssim 1 + \|x_0\|_{L^p(\Omega;X_\eta)},
\end{aligned} \tag{6.1.20}$$

and therefore, by (6.1.19):

$$\begin{aligned}
&\|U(x_0, 0, j\underline{T}_0) - U^{(n)}(y_0, 0, j\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\qquad \lesssim \|U(x_0, 0, (j-1)\underline{T}_0) - U^{(n)}(y_0, 0, (j-1)\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\qquad\quad + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),
\end{aligned}$$

with implied constants independent of j and n. By induction we obtain:

$$\begin{aligned}
&\sup_{1\le j\le M} \|U(x_0, 0, j\underline{T}_0) - U^{(n)}(y_0, 0, j\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\qquad \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),
\end{aligned} \tag{6.1.21}$$

with implied constants independent of j and n as M is independent of n.

The estimate (6.1.21) is precisely what we need to extend (6.1.17) to the interval $[0, T]$. To do so, we once again fix $j \in \{1, \ldots, M\}$. Set

$$x = U(x_0, 0, (j-1)\underline{T}_0) \quad \text{and} \quad y = U^{(n)}(y_0, 0, (j-1)\underline{T}_0)$$

in (6.1.18) to obtain, using (6.1.20) and (6.1.21):

$$\begin{aligned}
&\big\|U(x_0, 0, \cdot) - U^{(n)}(y_0, 0, \cdot)\big\|_{\mathscr{V}_\infty^{\alpha,p}([(j-1)\underline{T}_0, j\underline{T}_0]\times\Omega;X)} \\
&\quad = \big\|U\big(U(x_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \cdot\big) \\
&\qquad\qquad - U^{(n)}\big(U^{(n)}(y_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \cdot\big)\big\|_{\mathscr{V}_\infty^{\alpha,p}([(j-1)\underline{T}_0, j\underline{T}_0]\times\Omega;X)} \\
&\quad \lesssim \|U(x_0, 0, (j-1)\underline{T}_0) - U^{(n)}(y_0, 0, (j-1)\underline{T}_0)\|_{L^p(\Omega;X)} \\
&\qquad + n^{-\eta}(1 + \|U(x_0, 0, (j-1)\underline{T}_0)\|_{L^p(\Omega;X_\eta)})
\end{aligned}$$

$$\lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),$$

with implied constants independent of j and n.

Due to inequality (5.2.3) we thus obtain:

$$\begin{aligned}
&\|U - U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)} \\
&\quad \le \sum_{j=1}^{M} \big\| U\big(U(x_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \cdot\big) \\
&\qquad\qquad - U^{(n)}\big(U^{(n)}(y_0, 0, (j-1)\underline{T}_0), (j-1)\underline{T}_0, \cdot\big)\big\|_{\mathscr{V}_\infty^{\alpha,p}([(j-1)\underline{T}_0, j\underline{T}_0]\times\Omega;X)} \\
&\quad \lesssim \sum_{j=1}^{M} \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}) \\
&\quad \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})
\end{aligned}$$

since M is independent of n. This proves estimate (6.1.1). □

In the next subsection we shall need the following corollary of Theorem 6.2:

Corollary 6.3. *Let the setting be as in Theorem 6.2. Let $0 \le \delta < \eta_{\max}$ and $p \in [2, \infty)$, and assume that $y_0 \in L^p(\Omega; X_\delta)$. Then for all $\alpha \in [0, \frac{1}{2})$ one has:*

$$\sup_{n\in\mathbb{N}} \|U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X_\delta)} \lesssim 1 + \|y_0\|_{L^p(\Omega;X_\delta)}.$$

Proof. By assumption one can pick $\varepsilon > 0$ such that $\delta + \varepsilon < \eta_{\max}$. Because $\theta_F > \delta - 1 + (\frac{1}{\tau} - \frac{1}{2}) + \varepsilon$, the restriction of $F : [0,T] \times X \to X_{\theta_F}$ to $[0,T] \times X_\delta$ induces a mapping $F : [0,T] \times X_\delta \to X_{\delta-1+(\frac{1}{\tau}-\frac{1}{2})+\varepsilon}$ which satisfies **(F)** with $\tilde\theta_F = -1 + (\frac{1}{\tau} - \frac{1}{2}) + \varepsilon$. Similarly, from $\theta_G > \delta - \frac{1}{2} + \varepsilon$ we obtain a mapping $G : [0,T] \times X_\delta \to X_{\delta-\frac{1}{2}+\varepsilon}$ which satisfies **(G)** with $\tilde\theta_G = -\frac{1}{2} + \varepsilon$. The desired result is now obtained by combining Theorem 6.2 (with state space X_δ, initial conditions $x_0 = y_0$, and exponent $\eta = 0$) and Theorem 5.3. □

6.2 Convergence of the classical splitting scheme

We consider the stochastic differential equation (SDE) under the assumptions **(A)**, **(F)**, **(G)**, with initial value x_0, under the additional assumption that $\theta_F, \theta_G \ge 0$.

For $n \in \mathbb{N}$ let $(\tilde U_j^{(n)})_{j=0}^n$ be defined by (6.0.4) with initial value $\tilde y_0 \in L^p(\Omega, \mathscr{F}_0; X)$, $p \in [2, \infty)$, and set

$$\tilde U^{(n)}(t) := \sum_{j=1}^{n} 1_{I_j^{(n)}}(t)\, \tilde U_j^{(n)}(t), \quad t \in [0, T).$$

As before, U denotes the mild solution to (SDE) with initial value x_0.

Theorem 6.4. *Let $0 \leq \eta < \eta_{\max}$, and suppose $x_0 \in L^p(\mathscr{F}_0, X_\eta)$ and $\tilde{y}_0 \in L^p(\mathscr{F}_0, X)$ for some $p \in [2,\infty)$. Then for all $\alpha \in [0, \frac{1}{2})$ we have:*

$$\|U - \tilde{U}^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)} \lesssim \|x_0 - \tilde{y}_0\|_{L^p(\Omega;X)} + n^{-\eta}\big(1 + \|x_0\|_{L^p(\Omega;X_\eta)}\big), \tag{6.2.1}$$

with implied constants independent of n, x_0 and $\tilde{y}_0$.

Proof. Fix $T > 0$, $n \in \mathbb{N}$. For $\tilde{U}^{(n)}$ the following relation holds (see also (6.1.2)):

$$\tilde{U}^{(n)}(s) = S(\overline{s})\tilde{y}_0 + \int_0^s S(\underline{s} - \underline{u})F(u, \tilde{U}^{(n)}(u))\,du \tag{6.2.2}$$

$$+ \int_0^s S(\underline{s} - \underline{u})G(u, \tilde{U}^{(n)}(u))\,dW_H(u). \tag{6.2.3}$$

At first sight the processes $\tilde{U}^{(n)}$ and $U^{(n)}$ are very similar, and one would expect the proof of Theorem 6.4 to be entirely analogous to the proof of Theorem 6.2. However, there is a subtle difficulty when considering $\tilde{U}^{(n)}$: for the proof of Theorem 6.2 we make use of the fact that $\overline{s} - \underline{u} \geq s - u$ for all $0 \leq u \leq s$, $s \in [0,T]$. This allows us to write

$$S(\overline{s} - \underline{u}) = S(\overline{s} - s)S(s - u)S(u - \underline{u}) \tag{6.2.4}$$

and (see (6.1.8)):

$$S(s-u)-S(\overline{s}-\underline{u}) = (I-S(\overline{s}-s))S(s-u)+S(\overline{s}-s)S(s-u)(I-S(u-\underline{u})). \tag{6.2.5}$$

As a result, we can interpret the (deterministic and stochastic) integral terms in (6.1.5) as (stochastic) convolutions and use Lemmas A.7 and A.8 to obtain estimates for these terms.

For $\tilde{U}^{(n)}$ one of the difficulties lies in the fact that for

$$s \in [0,T] \setminus \{t_j^{(n)} : j = 1, \ldots, n\}$$

we have $\underline{s} - \underline{u} > s - u$ for some values of $u \in [0,s]$, but $\underline{s} - \underline{u} < s - u$ for other values of $u \in [0,s]$. Instead of (6.2.4) we have, for $s - u \geq \frac{T}{n}$:

$$S(\underline{s} - \underline{u}) = S(\overline{s} - s)S(s - \tfrac{T}{n} - u)S(u - \underline{u}). \tag{6.2.6}$$

Roughly speaking, this allows one to apply Lemmas A.7 and A.8 on the interval $[0, (s - \frac{T}{n})^+]$. However, an extra argument is needed for the remainder of the interval.

Another difficulty in dealing with $\tilde{U}^{(n)}$ is that for

$$s \in [0,T] \setminus \{t_j^{(n)} : j = 1, \ldots, n\}$$

and $u \in [\underline{s}, s]$ we have $S(\underline{s} - \underline{u}) = I$, and thus we cannot use the smoothing property of the semigroup there. Note that this occurs precisely in the aforementioned remainder.

Part 1. It is easier to deal with the remainder if we compare $\tilde{U}^{(n)}$ with $U^{(n)}$ instead of comparing $\tilde{U}^{(n)}$ with U: by Theorem 6.2 suffices to prove that

$$\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)} \lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega;X)} + n^{-\eta}\big(1 + \|y_0\|_{L^p(\Omega;X_\eta)}\big), \tag{6.2.7}$$

with implied constants independent of n, y_0 and $\tilde{y}_0$, where $U^{(n)}$ denotes the process obtained by applying the modified splitting scheme to (SDE) with initial value $y_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$.

By (6.1.2) and (6.2.2) we have:

$$\begin{aligned}
U^{(n)}(s) - \tilde{U}^{(n)}(s) = {}& S(\overline{s})(y_0 - \tilde{y}_0)\\
&+ (S(\tfrac{T}{n}) - I)\int_0^s S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\,du\\
&+ \int_0^s S(\underline{s} - \underline{u})\big[F(u, U^{(n)}(u)) - F(u, \tilde{U}^{(n)}(u))\big]\,du\\
&+ (S(\tfrac{T}{n}) - I)\int_0^s S(\underline{s} - \underline{u})G(u, U^{(n)}(u))\,dW_H(u)\\
&+ \int_0^s S(\underline{s} - \underline{u})\big[G(u, U^{(n)}(u)) - G(u, \tilde{U}^{(n)}(u))\big]\,dW_H(u).
\end{aligned} \tag{6.2.8}$$

As mentioned above, we can rewrite each of the (deterministic and stochastic) integrals above as a (deterministic or stochastic) convolution and a remainder term. Below, we will demonstrate this for the first deterministic integral term in (6.2.8). The convolutions can be dealt with in the same manner as in the proof of Theorem 6.2, and in part 2 of this proof we will demonstrate how to deal with the remainder.

For the first deterministic integral in (6.2.8) we have, by (6.2.6):

$$\begin{aligned}
&(S(\tfrac{T}{n}) - I)\int_0^s S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\,du\\
&\quad= (S(\tfrac{T}{n}) - I)S(\overline{s} - s)\int_0^{(s-\frac{T}{n})^+} S((s - \tfrac{T}{n})^+ - u)S(u - \underline{u})F(u, U^{(n)}(u))\,du\\
&\qquad+ (S(\tfrac{T}{n}) - I)\int_{(s-\frac{T}{n})^+}^s S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\,du.
\end{aligned}$$

Note that the first term on the right-hand side above involves the convolution of the process

$$u \mapsto S(u - \underline{u})F(u, U^{(n)}(u))$$

with the semigroup S, evaluated in $(s - \frac{T}{n})^+$. By arguments analogous to part 1c in the proof of Theorem 6.2 we can estimate this term, using Corollary 6.3 where in part 1c the estimate of Theorem 5.3 is applied:

$$\Big\| s \mapsto (S(\tfrac{T}{n}) - I)S(\overline{s} - s) \times \int_0^{(s-\frac{T}{n})^+} S((s - \tfrac{T}{n})^+ - u)S(u - \underline{u})F(u, U^{(n)}(u))\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim n^{-\eta}(1 + \|y_0\|_{L^p(\Omega;X_\eta)}. \tag{6.2.9}$$

For the remainder term we apply, for the time being, only the ideal property for γ-radonifying operators (2.3.2) to get rid of the term $S(\frac{T}{n}) - I$. We thus obtain, for all $T_0 \in [0, T]$:

$$\begin{aligned}
&\Big\| s \mapsto (S(\tfrac{T}{n}) - I) \int_0^s S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\quad \lesssim n^{-\eta}(1 + \|y_0\|_{L^p(\Omega;X)}) \\
&\qquad + n^{-(\theta_F\wedge 1)} \Big\| s \mapsto \int_{(s-\frac{T}{n})^+}^s S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\theta_F})}.
\end{aligned}$$

By applying similar arguments to the other three integral terms in (6.2.8) and by applying the argument of part 1b in the proof of Theorem 6.2 to the first term in (6.2.8), one obtains that there exists an $\varepsilon_0 \in (0, \frac{1}{2})$ such that for $T_0 \in [0, T]$ and α sufficiently large we have, setting $I_s := [(s - \frac{T}{n})^+, s]$:

$$\begin{aligned}
&\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\quad \lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega;X)} \\
&\qquad + n^{-\eta}\big(1 + \|y_0\|_{L^p(\Omega;X_\eta)}\big) + T_0^{\varepsilon_0}\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + n^{-(\theta_F\wedge 1)} \Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u})F(u, U^{(n)}(u))\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\theta_F})} && \text{(i)} \\
&\qquad + \Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u})\big[F(u, U^{(n)}(u)) - F(u, \tilde{U}^{(n)}(u))\big]\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} && \text{(ii)} \\
&\qquad + n^{-(\theta_G\wedge 1)} \Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u})G(u, U^{(n)}(u))\, dW_H(u) \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\theta_G})} && \text{(iii)} \\
&\qquad + \Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u})\big[G(u, U^{(n)}(u)) - G(u, \tilde{U}^{(n)}(u))\big]\, dW_H(u) \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}. && \text{(iv)}
\end{aligned}$$

Part 2. We now demonstrate how the terms (i) - (iv) above can be estimated using the following two claims. The proofs of the claims are postponed to parts 3 and 4.

Claim 1. *Let $\delta \in \mathbb{R}$, $\alpha \in [0, \frac{1}{2})$ and $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0, T] \times \Omega; X_\delta)$. Then for all $\varepsilon > 0$ and all $T_0 \in [0, T]$ we have, with implied constant independent of n and T_0:*

$$\Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u})\Phi(u)\, du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\delta)} \lesssim \big(\tfrac{T}{n} \wedge T_0\big)^{\frac{3}{2} - \frac{1}{\tau} - \varepsilon} \|\Phi\|_{L^\infty(0,T_0;L^p(\Omega,X_\delta))}.$$

Claim 2. *Let $\delta \in \mathbb{R}$, $\alpha \in [0, \frac{1}{2})$, and $\Phi \in \mathscr{V}_{\infty}^{\alpha,p}([0,T] \times \Omega; X_\delta)$. Then for all $T_0 \in [0,T]$ we have:*

$$\Big\| s \mapsto \int_{I_s} S(\underline{s} - \underline{u}) \Phi(u)\, dW_H(u) \Big\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,T_0]\times\Omega;X_\delta)} \lesssim \big(\tfrac{T}{n} \wedge T_0\big)^{\alpha} \sup_{0 \le t \le T_0} \| s \mapsto (t-s)^{-\alpha} \Phi(s) \|_{L^\infty(0,t;L^p(\Omega,X_\delta))},$$

with implied constant independent of n and T_0.

Pick $\varepsilon > 0$ such that

$$\varepsilon < \tfrac{3}{2} - \tfrac{1}{\tau} + \theta_F - \eta.$$

We shall apply Claim 1 with this choice of ε. To be precise, for (i) we apply Claim 1 with $\Phi = F(\cdot, U^{(n)}(\cdot))$ and $\delta = \theta_F$. For (ii) we apply Claim 1 with $\Phi = F(\cdot, U^{(n)}(\cdot)) - F(\cdot, \tilde{U}^{(n)}(\cdot))$ and $\delta = 0$.

Replacing $\alpha \in [0, \frac{1}{2})$ by a larger value if necessary, we may assume $\eta - \theta_G < \alpha < \frac{1}{2}$. For (iii) we apply Claim 2 with $\Phi = G(\cdot, U^{(n)}(\cdot))$ and $\delta = \theta_G$. Finally, for (iv) we apply Claim 2 with $\Phi = G(\cdot, U^{(n)}(\cdot)) - G(\cdot, \tilde{U}^{(n)}(\cdot))$ and $\delta = 0$. This gives:

$$\begin{aligned}
&\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\quad\lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega;X)} + n^{-\eta}\big(1 + \|y_0\|_{L^p(\Omega;X_\eta)}\big) \\
&\qquad + T_0^{\varepsilon_0} \|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + n^{-\eta} \|F(\cdot, U^{(n)}(\cdot))\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))} \\
&\qquad + T_0^{\frac{3}{2} - \frac{1}{\tau} - \varepsilon} \|F(\cdot, U^{(n)}(\cdot)) - F(\cdot, \tilde{U}^{(n)}(\cdot))\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\
&\qquad + n^{-\eta} \sup_{0 \le t \le T_0} \|s \mapsto (t-s)^{-\alpha} G(u, U^{(n)}(u))\|_{L^p(\Omega,\gamma(0,t;X_{\theta_G}))} \\
&\qquad + T_0^{\alpha - \frac{1}{p} - \varepsilon} \sup_{0 \le t \le T_0} \|s \mapsto (t-s)^{-\alpha} [G(u, U^{(n)}(u)) - G(u, \tilde{U}^{(n)}(u))]\|_{L^p(\Omega,\gamma(0,t;X))}.
\end{aligned}$$

Note that, as $\theta_F, \theta_G \ge 0$, we have continuous inclusions $X_{\theta_F} \hookrightarrow X$ and $X_{\theta_G} \hookrightarrow X$, so that the norms in $L^p(\Omega; X)$ and $\gamma(0,t;X)$ may be estimated by the norms in $L^p(\Omega; X_{\theta_F})$ and $\gamma(0,t;X_{\theta_G})$ in the third and fifth line, respectively.

Applying assumption **(F)** and the estimates (5.2.5) and (5.2.4), and then Corollary 6.3 (with $\delta = 0$), we obtain that there exists an $\tilde{\varepsilon}_0 > 0$ such that:

$$\begin{aligned}
\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,T_0]\times\Omega;X)} &\lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega;X)} + n^{-\eta}\big(1 + \|y_0\|_{L^p(\Omega;X_\eta)}\big) \\
&\quad + T_0^{\tilde{\varepsilon}_0} \|U^{(n)} - \tilde{U}^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,T_0]\times\Omega;X)}.
\end{aligned}$$

The remainder of the proof is entirely analogous to parts 3 and 4 in the proof of Theorem 6.2.

Part 3: proof of Claim 1. Fix $\varepsilon > 0$. Recall that $I_s = [(s - \frac{T}{n})^+, s]$. Observe that for $s \in [0, T_0]$ we have:

$$\int_{I_s} S(\underline{s}-\underline{u})\Phi(u)\,du = S(\tfrac{T}{n})\int_{(s-\frac{T}{n})^+}^{\underline{s}} \Phi(u)\,du + \int_{\underline{s}}^{s} \Phi(u)\,du. \tag{6.2.10}$$

Let $a, b : [0,T] \to \mathbb{R}$ be measurable and satisfy $a \leq b$. We shall prove that:

$$\begin{aligned} \Big\| s \mapsto \int_{I_s} 1_{\{a(s)\leq u \leq b(s)\}}\Phi(u)\,du \Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\delta)} \\ \lesssim \big(\tfrac{T}{n}\wedge T_0\big)^{\frac{3}{2}-\frac{1}{\tau}-\varepsilon}\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))}. \end{aligned} \tag{6.2.11}$$

The claim follows by applying the above estimate with $a(s) = (s-\frac{T}{n})^+$; $b(s) = \underline{s}$ to the first term in (6.2.10), and with $a(s) = \underline{s}$; $b(s) = s$ to the second term. (The term $S(\frac{T}{n})$ can be estimated away by (2.3.2).)

For $s \in [0,T_0]$ we have $|I_s| = s - (s-\frac{T}{n})^+ \leq \frac{T}{n}\wedge T_0$, and thus:

$$\Big\| s \mapsto \int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du \Big\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))} \lesssim \big(\tfrac{T}{n}\wedge T_0\big)\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))}. \tag{6.2.12}$$

For the estimate in the γ-radonifying norm we shall use the Besov embedding of Section 2.3.1 and Lemma A.3 in the Appendix. We begin with a simple observation. If $\Psi \in L^\infty(0,T_0;Y)$ for some Banach space Y and $\alpha \in (0,1]$, then:

$$\begin{aligned} &\Big\| s \mapsto \int_{I_s} \Psi(u)\,du \Big\|_{C^\alpha(0,T_0;Y)} \\ &\leq \sup_{s\in[0,T_0]} |I_s|\|\Psi\|_{L^\infty(0,T_0;Y)} + \sup_{0\leq s<t\leq T_0} (t-s)^{-\alpha}\Big\| \int_{I_t}\Psi(u)\,du - \int_{I_s}\Psi(u)\,du \Big\|_Y \\ &\leq (\tfrac{T}{n}\wedge T_0)\|\Psi\|_{L^\infty(0,T_0;Y)} + \sup_{0\leq s<t\leq T_0} (t-s)^{-\alpha}\Big\| \int_{I_t}\Psi(u)\,du - \int_{I_s}\Psi(u)\,du \Big\|_Y. \end{aligned}$$

If $t - s \geq \frac{T}{n}$, then

$$\begin{aligned} (t-s)^{-\alpha}\Big\| \int_{I_t}\Psi(u)\,du - \int_{I_s}\Psi(u)\,du \Big\|_Y &\leq 2(t-s)^{-\alpha}\sup_{s\in[0,T_0]}|I_s|\|\Psi\|_{L^\infty(0,T_0;Y)} \\ &\leq 2(\tfrac{T}{n}\wedge T_0)^{1-\alpha}\|\Psi\|_{L^\infty(0,T_0;Y)}. \end{aligned}$$

On the other hand, if $t - s \leq \frac{T}{n}$, then:

$$\begin{aligned} &(t-s)^{-\alpha}\Big(\Big\| \int_{I_t}\Psi(u)\,du - \int_{I_s}\Psi(u)\,du \Big\|_Y\Big) \\ &\qquad \leq (t-s)^{-\alpha}\Big(\Big\| \int_s^t \Psi(u)\,du \Big\|_Y + \Big\| \int_{(s-\frac{T}{n})^+}^{(t-\frac{T}{n})^+} \Psi(u)\,du \Big\|\Big) \\ &\qquad \leq 2(t-s)^{1-\alpha}\|\Psi\|_{L^\infty(0,T_0;Y)} \leq 2(\tfrac{T}{n}\wedge T_0)^{1-\alpha}\|\Psi\|_{L^\infty(0,T_0;Y)}. \end{aligned}$$

It follows that:

$$\Big\|s \mapsto \int_{I_s} \Psi(u)\,du\Big\|_{C^\alpha(0,T_0;Y)} \leq 3(\tfrac{T}{n} \wedge T_0)^{1-\alpha}\|\Psi\|_{L^\infty(0,T_0;Y)}. \tag{6.2.13}$$

Note that as $p \geq 2$ the type of $L^p(\Omega, X)$ is the same as the type τ of X. Without loss of generality we may assume that $\tau < 2$. Fix $q \geq 2$ such that $\frac{1}{q} < \frac{1}{\tau} - \alpha$. By isomorphism (2.3.3), the Besov embedding (2.3.7), and Lemma A.3 there exists an $\epsilon_0 > 0$ such that we have:

$$\begin{aligned}
&\sup_{t\in[0,T_0]} \Big\|s \mapsto (t-s)^{-\alpha}\int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{L^p(\Omega;\gamma(0,t;X_\delta))}\\
&\quad\eqsim \sup_{t\in[0,T_0]} \Big\|s \mapsto (t-s)^{-\alpha}\int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{\gamma(0,t;L^p(\Omega;X_\delta))}\\
&\quad\lesssim \sup_{t\in[0,T_0]} \Big\|s \mapsto (t-s)^{-\alpha}\int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(0,T_0;L^p(\Omega;X_\delta))}\\
&\quad\lesssim T_0^{\epsilon_0}\Big\|s \mapsto \int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))}\\
&\qquad + T_0^{\epsilon_0}\Big\|s \mapsto \int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T_0;L^p(\Omega;X_\delta))}\\
&\quad\lesssim T_0^{\epsilon_0}\Big\|s \mapsto \int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,du\Big\|_{C^{\frac{1}{\tau}-\frac{1}{2}+\varepsilon}(0,T_0;L^p(\Omega;X_\delta))}\\
&\quad\lesssim \big(\tfrac{T}{n} \wedge T_0\big)^{\frac{3}{2}-\frac{1}{\tau}-\varepsilon}\|\Phi\|_{L^\infty(0,T_0;X_\delta)},
\end{aligned}$$

where in the final estimate we used (6.2.13).

Part 4: proof of Claim 2. As before let $a, b : [0, T] \to \mathbb{R}$ be measurable and satisfy $a \leq b$. It suffices to prove that:

$$\begin{aligned}\Big\|s \mapsto \int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)\,dW_H(u)\Big\|_{\mathscr{V}^{\alpha,p}_\infty([0,T_0]\times\Omega;X_\delta)}&\\
\lesssim \big(\tfrac{T}{n} \wedge T_0\big)^{\alpha}\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))}.&\end{aligned} \tag{6.2.14}$$

Note that for any $s \in [0, T_0]$ we have:

$$\begin{aligned}
\Big\|\int_{I_s} 1_{\{a(s)\leq u\leq b(s)\}}\Phi(u)dW_H(u)\Big\|_{L^p(\Omega;X_\delta)} &\leq \Big\|\int_{I_s} \Phi(u)dW_H(u)\Big\|_{L^p(\Omega;X_\delta)}\\
&= \Big\|\int_0^s \Phi(u)dW_H(u) - \int_0^{(s-\frac{T}{n})^+} \Phi(u)dW_H(u)\Big\|_{L^p(\Omega;X_\delta)}\\
&\leq (s-(s-\tfrac{T}{n})^+)^\alpha\Big\|s \mapsto \int_0^s \Phi(u)dW_H(u)\Big\|_{C^\alpha(0,T_0;L^p(\Omega;X_\delta))}.
\end{aligned}$$

Thus by (2.4.6) we have:

$$\begin{aligned}\Big\| s \mapsto \int_{I_s} 1_{\{a(s)\le u\le b(s)\}} \Phi(u)\, dW_H(u)\Big\|_{L^\infty(0,T_0;L^p(\Omega;X_\delta))} \\ \lesssim \sup_{0\le r\le T_0} |I_r|^\alpha \Big\| s\mapsto \int_0^s \Phi(u)\, dW_H(u)\Big\|_{C^\alpha(0,T_0;L^p(\Omega;X_\delta))} \\ \lesssim \big(\tfrac{T}{n}\wedge T_0\big)^\alpha \sup_{0\le t\le T_0} \| s\mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega,\gamma(0,t;H,X_\delta))}.\end{aligned} \tag{6.2.15}$$

Let $t\in[0,T_0]$. We wish to apply Lemma A.2 with

$$f(r,u)(s) = (t-s)^{-\alpha}(t-u)^\alpha 1_{\{(s-\frac{T}{n})^+\le u\le s\}} 1_{\{a(s)\le u\le b(s)\}},$$

$R=[0,1]$, $S=[0,t]$ (both with the Lebesgue measure), $X_1=X_2=X_\delta$, $\Phi_2\equiv I$ and $\Phi_1(u)=(t-u)^{-\alpha}\Phi(u)$. Note that:

$$\begin{aligned}\sup_{(r,u)\in[0,1]\times[0,t]} \|f(r,u)\|^2_{L^2(0,t)} &\le \sup_{u\in[0,t]} (t-u)^\alpha \Big\| s\mapsto (t-s)^{-\alpha} 1_{\{(s-\frac{T}{n})^+\le u\le s\}}\Big\|^2_{L^2(0,t)} \\ &\le \sup_{u\in[0,t]} (t-u)^{2\alpha} \int_u^{(u+\frac{T}{n})\wedge t} (t-s)^{-2\alpha}\, ds.\end{aligned}$$

If $t-u\ge \frac{2T}{n}$, then $t-(u+\frac{T}{n})\ge \frac12(t-u)$ and

$$\int_u^{(u+\frac{T}{n})\wedge t} (t-s)^{-2\alpha}\, ds \le \tfrac{T}{n}(t-(u+\tfrac{T}{n}))^{-2\alpha} \le 2^{2\alpha}\tfrac{T}{n}(t-u)^{-2\alpha},$$

while if $t-u\le\frac{2T}{n}$, then

$$\int_u^{(u+\frac{T}{n})\wedge t} (t-s)^{-2\alpha}\, ds \le \int_u^t (t-s)^{-2\alpha}\, ds = \tfrac{1}{1-2\alpha}(t-u)^{1-2\alpha} \le \tfrac{1}{1-2\alpha}\tfrac{2T}{n}(t-u)^{-2\alpha}.$$

In both cases, we also have the estimate

$$\int_u^{(u+\frac{T}{n})\wedge t} (t-s)^{-2\alpha}\, ds \le \int_u^t (t-s)^{-2\alpha}\, ds \eqsim (t-u)^{1-2\alpha} \le T_0(t-u)^{-2\alpha}.$$

Combining this with the previous estimates we find:

$$\sup_{(r,u)\in[0,1]\times[0,t]} \|f(r,u)\|_{L^2(0,t)} \lesssim \big(\tfrac{T}{n}\wedge T_0\big)^{\frac12}. \tag{6.2.16}$$

Thus Lemma A.2 gives:

$$\begin{aligned}\Big\| s\mapsto (t-s)^{-\alpha} \int_{I_s} 1_{\{a(s)\le u\le b(s)\}}\Phi(u)\, dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X_\delta))} \\ \lesssim \big(\tfrac{T}{n}\wedge T_0\big)^\alpha \|u\mapsto (t-u)^{-\alpha}\Phi(u)\|_{L^p(\Omega,\gamma(0,t;H,X_\delta))}.\end{aligned} \tag{6.2.17}$$

Taking the supremum over $t\in[0,T_0]$ above and combining the result with (6.2.15) we obtain (6.2.14). This completes the proof of the claim. $\square$

6.3 Proof of Theorem 6.1

Let $T > 0$ and $n \in \mathbb{N}$ be fixed. Set $u := (U(t_j^{(n)}))_{j=0}^n$, where U is the solution to (SDE) with initial value $x_0 \in L^p(\Omega, \mathscr{F}_0, X_\eta)$ for some $p > 2$ and $\eta \geq 0$ such that $\frac{1}{p} < \eta < \eta_{\max}$, with $\eta_{\max}$ as defined in (7.2.1).

Set $u^{(n)} = (U^{(n)}(t_j^{(n)})_{j=0}^n$, where $U^{(n)}$ is defined by the modified splitting scheme with initial value $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$. Note that we consider the slightly more general case that $y_0 \neq x_0$. This does not require extra arguments and Theorem 6.1 follows immediately by taking $x_0 = y_0$, see also Remark 6.8.

The proof of Theorem 6.1 is based on the following version of Kolmogorov's continuity criterion (see, e.g., [121, Theorem I.2.1]):

Proposition 6.5 (Kolmogorov's continuity criterion). *Let Y be a Banach space. For all $\alpha > 0$, $q \in (\frac{1}{\alpha}, \infty)$ and $0 \leq \gamma < \alpha - \frac{1}{q}$ there exists a constant K such that for all $T > 0$, $k \in \mathbb{N}$, and $u, v \in c_\alpha^{(2^k)}([0,T]; L^q(\Omega; Y))$ we have:*

$$\|u - v\|_{L^q(\Omega; c_\gamma^{(2^k)}([0,T];Y))} \leq K\|u - v\|_{c_\alpha^{(2^k)}([0,T];L^q(\Omega;Y))}.$$

Proof (of Theorem 6.1). Upon replacing η by a smaller value, we may assume that $\gamma + \delta + \frac{1}{p} < \eta < \eta_{\max}$ with $\eta_{\max}$ as in (5.2.6).

Let $k \in \mathbb{N}$ be such that $2^{k-1} < n \leq 2^k$. Then $T \leq \frac{2^k T}{n} < 2T$. For $j \in \{0, \ldots, 2n\}$ set

$$d_j^{(n)} := U(t_j^{(n)}) - U^{(n)}(t_j^{(n)}),$$

using that there exists a unique mild solution U to (SDE) on $[0, 2T]$; the definition of $U^{(n)}$ on $[T, 2T]$ is straightforward.

By Theorem 6.2 applied to the interval $[0, 2T]$ with $2n$ time steps and with η as fixed above we have, because $|t_j^{(n)} - t_i^{(n)}| \geq \frac{T}{n}$,

$$\begin{aligned}
\sup_{0\leq i<j\leq 2n} \frac{\|d_j^{(n)} - d_i^{(n)}\|_{L^p(\Omega;X)}}{|t_j^{(n)} - t_i^{(n)}|^{\eta-\delta}} &\leq \big(\tfrac{n}{T}\big)^{\eta-\delta} \sup_{0\leq i<j\leq n} \big(\|d_j^{(n)}\|_{L^p(\Omega;X)} + \|d_i^{(n)}\|_{L^p(\Omega;X)}\big) \\
&\lesssim n^{\eta-\delta}\big(\|x_0 - y_0\|_{L^p(\Omega,X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})\big) \\
&= n^{\eta-\delta}\|x_0 - y_0\|_{L^p(\Omega,X)} + n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),
\end{aligned}$$

with implied constant independent of n, x_0 and y_0. In particular:

$$\begin{aligned}
\sup_{0\leq j\leq 2n} \|d_j^{(n)}\|_{L^p(\Omega;X)} &\lesssim \|d_0^{(n)}\|_{L^p(\Omega;X)} + n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}) \\
&= \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),
\end{aligned}$$

with implied constant independent of n, x_0 and y_0.

It follows that

$$\big\|\big(d_j^{(n)}\big)_{j=0}^{2^k}\big\|_{c_{\eta-\delta}^{(2^k)}([0,t_{2^k}^{(n)}];L^p(\Omega;X))}$$

$$\lesssim n^{\eta-\delta}\|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}),$$

with implied constant independent of n, x_0 and y_0. Thus, by Kolmogorov's criterion, using that $\eta - \delta > \gamma + \frac{1}{p}$, and the fact that $T \leq \frac{2^k T}{n} < 2T$;

$$\begin{aligned}
\|u - u^{(n)}\|_{L^p(\Omega;c_\gamma^{(n)}([0,T];X))} &= \big\|\big(d_j^{(n)}\big)_{j=0}^{n}\big\|_{L^p(\Omega;c_\gamma^{(n)}([0,T];X))} \\
&\leq \big\|\big(d_j^{(n)}\big)_{j=0}^{2^k}\big\|_{L^p(\Omega;c_\gamma^{(2^k)}([0,t_{2^k}^{(n)}];X))} \\
&\lesssim \big\|\big(d_j^{(n)}\big)_{j=0}^{2^k}\big\|_{c_{\eta-\delta}^{(2^k)}([0,t_{2^k}^{(n)}];L^p(\Omega;X))} \\
&\lesssim n^{\eta-\delta}\|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}).
\end{aligned}$$

Theorem 6.1 now follows from the fact that in there we assumed $y_0 = x_0$. $\square$

The corollary below is obtained from Theorem 6.1 by a Borel-Cantelli argument.

Corollary 6.6. *Let $\gamma, \delta \geq 0$, $\eta > 0$, and $p \in [2, \infty)$ be such that $\gamma + \delta + \frac{2}{p} < \min\{\eta_{\max}, \eta, 1\}$. Suppose that $x_0 = y_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$. Then there exists a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$:*

$$\|u - u^{(n)}\|_{c_\gamma^{(n)}([0,T];X)} \leq \chi n^{-\delta}. \tag{6.3.1}$$

Proof. We may assume that $\gamma + \delta + \frac{2}{p} < \eta < \min\{\eta_{\max}, 1\}$. Pick $\overline{\delta} > 0$ such that $\delta + \frac{1}{p} < \overline{\delta} < \eta_{\max} - \gamma - \frac{1}{p}$. By Chebychev's inequality and Theorem 6.1 (with $\overline{\delta}$ instead of δ) we have $\mathbb{P}(\Omega_n) \lesssim n^{-(\overline{\delta}-\delta)p}$, where

$$\Omega_n := \big\{\omega \in \Omega \,:\, \|u(\omega) - u^{(n)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} > n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})\big\}.$$

By assumption we have $\overline{\delta} - \delta > \frac{1}{p}$, and therefore $\sum_n \mathbb{P}(\Omega_n) < \infty$. The corollary now follows from the Borel-Cantelli lemma. $\square$

Remark 6.7. In Chapter 8, Corollary 8.1, we will see that in fact (6.3.1) holds for $\gamma + \delta < \min\{\eta_{\max}, \eta, 1\}$ provided $x_0 = y_0 \in L^0(\Omega, \mathscr{F}_0; X_\eta)$.

Remark 6.8. It is clear from the proof of Theorem 6.1 and Corollary 6.6 that the assertions remain valid if $U^{(n)}$ starts from an initial value $y_0^{(n)} \in L^p(\Omega, \mathscr{F}_0; X)$, provided that for all $n \in \mathbb{N}$ we have $\|x_0 - y_0^{(n)}\|_{L^p(\Omega;X)} \lesssim n^{-\eta}$.

6.4 The splitting scheme for SDEs with additive noise

Consider the following stochastic linear Cauchy problem:

$$\begin{cases} dU(t) = AU(t)\,dt + dW(t), & t \in [0,T], \\ U(0) = x_0, \end{cases} \tag{SCP}$$

where A is the generator of a C_0-semigroup $S = (S(t))_{t\geq 0}$ on an arbitrary (real) Banach space X, $W = (W(t))_{t\geq 0}$ is an X-valued Brownian motion on a probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, and $x_0 \in L^p(\Omega, \mathscr{F}_0; X)$ is the initial value. Note that we do not assume that X is a UMD space, this assumption is not needed in the additive noise case. Also, for the time being we do not assume that S is analytic. However, in Subsection 6.4.2 below we *also* consider the case that S is analytic, and in that case we may assume W to take values in X_θ, for some $\theta > -\frac{1}{2}$.

The relation between W and the H-cylindrical Brownian motion W_H in the previous sections is as follows: let H be the reproducing kernel Hilbert space associated with the Gaussian random variable $W(1)$ and let $i : H \hookrightarrow X$ be the natural inclusion mapping. Then W induces an H-cylindrical Brownian motion W_H by putting

$$W_H(f \otimes i^*x^*) := \int_0^T f \, d\langle W, x^*\rangle, \quad f \in L^2(0,T),\ x^* \in X^*. \tag{6.4.1}$$

On the other hand, (SCP) can be interpreted as a variant of (SDE) by taking $F \equiv 0$ and $G \equiv g \in \gamma(H,X)$, with $W(t) := gW_H(t)$.

Fix $T > 0$ and $n \in \mathbb{N}$. If one applies the modified splitting scheme as considered in Section 6.1, see also (6.0.1), to (SCP) one obtains the following formula for in the grid points $(t_j^{(n)})_{j=1}^n$ (i.e., $U_j^{(n)} = U^{(n)}(t_j^{(n)})$ in (6.0.1)):

$$\begin{aligned} U_0^{(n)} &:= x_0, \\ U_j^{(n)} &:= S(\tfrac{T}{n})(U^{(n)}(t_{j-1}^{(n)}) + \Delta W_j^{(n)}), \quad j = 1, \dots, n. \end{aligned} \tag{6.4.2}$$

For the classical splitting scheme one has $U_j^{(n)} = S(\frac{T}{n})U^{(n)}(t_{j-1}^{(n)}) + \Delta W_j^{(n)}$, but we shall not investigate this scheme any further. We have the following explicit formula for $U_j^{(n)}$ (compare to (6.1.2)):

$$U_j^{(n)} = S(t_j^{(n)})x_0 + \sum_{i=1}^{j} S(t_{j-i+1}^{(n)})\Delta W_i^{(n)}, \quad j = 0, \dots, n. \tag{6.4.3}$$

Assuming the existence of a (unique) mild solution U of the problem (SCP), we may ask for conditions ensuring the convergence of the splitting scheme to U in some sense.

Note that so far we have only fixed the value of the splitting scheme process at the grid points $(t_j^{(n)})_{j=1}^n$. In Sections 6.1 and 6.2 the intermediate values were fixed by the variation of constants formula supplying the solution to splitting scheme equations (6.0.1) and (6.0.4). However, if the noise is additive, then a priori intermediate values are not fixed. Arguing that when it comes to convergence we are mostly interested in what happens at the grid points $(t_j^{(n)})_{j=1}^n$, we pick an interpolation that is continuous and that makes proving convergence easily accessible with continuous time techniques. Thus instead of considering the analogue to (6.1.2), which would be:

$$U^{(n)}(t) = S(\overline{t})x_0 + \int_0^t S(\overline{t} - \underline{s})dW(s), \tag{6.4.4}$$

we consider:

$$U_c^{(n)}(t) := S(t)x_0 + \int_0^t S(\overline{t-s})\,dW(s), \quad t \in [0,T]. \tag{6.4.5}$$

Note that the interpolation given by equation (6.4.4) is *not* continuous, in fact, *both* terms on the right-hand side of (6.4.4) are not continuous (except for in the trivial case that $A \equiv 0$). Also, if (SDE) admits a mild solution, then (6.4.5) has the nice property that

$$\begin{aligned} U(t) - U_c^{(n)}(t) &= \int_0^t S(t-s) - S(\overline{t-s})\,dW(s) \\ &= \int_0^t \Phi(t-s) - \Phi^{(n)}(t-s)\,dW_H(s), \end{aligned} \tag{6.4.6}$$

where $\Phi(s) := S(s) \circ i$ and $\Phi^{(n)}(s) = S(\overline{s}) \circ i$, and $i : H \to X$ is the natural embedding of the reproducing kernel Hilbert space corresponding to $W(1)$ into X. In particular, the initial value x_0 is irrelevant for proving convergence of $U^{(n)}$ against U. Thus in order to prove

$$\sup_{0 \le t \le T} \mathbb{E}\|U(t) - U^{(n)}(t)\|_X^p \to 0, \quad n \to \infty,$$

for $p \in [1,\infty)$, it suffices, by the Itô isomorphism (2.4.3) to prove:

$$\sup_{0 \le t \le T} \|s \mapsto (\Phi(s) - \Phi^{(n)}(s))\|_{\gamma(0,t;H,X)} \to 0, \quad n \to \infty,$$

which, by covariance domination, see (2.3.4), is equivalent to proving

$$\|s \mapsto (\Phi(s) - \Phi^{(n)}(s))\|_{\gamma(0,T;H,X)} \to 0, \quad n \to \infty. \tag{6.4.7}$$

In Subsection 6.4.1 we shall discuss the relation between convergence of the splitting scheme and the validity of a Lie–Trotter product formula for the Ornstein-Uhlenbeck semigroup $\mathscr{P} = (\mathscr{P}(t))_{t\ge 0}$ associated with the problem (SCP).

In Subsection 6.4.2 we shall prove convergence of $U_c^{(n)}$ against U if either of the following hold:

(i) X has type 2;

(ii) S restricts to a C_0-semigroup on the reproducing kernel Hilbert space associated with W;

(iii) S is analytic.

We refer to [40, 110] for a proof that there exists a mild solution to (SCP) in all three of the above mentioned cases (see also Remark 6.16).

For the case that S is analytic we also provide convergence rates that correspond to the rates found in Section 6.1. However, as we consider the continuous process $U_c^{(n)}$ instead of $U^{(n)}$, we have convergence in the continuous Hölder norm $C_\gamma([0,T];X)$ instead of in the discrete norm $c_\gamma^{(n)}([0,T];X)$. Moreover, we need not assume that X is a UMD space and the proof is significantly shorter.

Finally, in Subsection 6.4.3 we give an example involving a shift semigroup (which is not analytic) that demonstrates that if X does not have type 2, then one may have that $\int_0^1 S(s)dW(s)$ is well-defined (and thus a mild solution to (SCP) exists) but $\int_0^1 S(\overline{s})dW(s) \to \infty$ as $n \to \infty$.

6.4.1 The relation between the splitting scheme and the Lie-Trotter formula

Theorem 6.9 below states the equivalence of the convergence given by (6.4.7) and convergence of the corresponding measures on X. This in turn is equivalent to the validity of a Lie–Trotter product formula for the Ornstein-Uhlenbeck semigroup associated with the problem (SCP), as we will explain shortly.

Theorem 6.9. *Suppose that* (SCP) *admits a mild solution U. The following assertions are equivalent:*

(i) $\sup_{0\le t\le T} \mathbb{E}\|U(t) - U_c^{(n)}(t)\|^p \to 0$ *for some (all)* $1 \le p < \infty$*;*

(ii) $\|\Phi_n - \Phi\|_{\gamma(0,T;H,X)} \to 0$*;*

(iii) $\lim_{n\to\infty} \mu^{(n)} = \mu$ *weakly.*

Proof. The equivalence of (i) and (ii) was demonstrated on page 109, see (6.4.7).

Let R_Φ and $R_{\Phi^{(n)}}$ denote the operators in $\gamma(0,T;H,X)$ corresponding to Φ and $\Phi^{(n)}$. (Note that $R_\Phi \in \gamma(0,T;H,X)$ by the assumption that (SCP) allows for a mild solution.) We claim that $\lim_{n\to\infty} R^*_{\Phi^{(n)}} x^* = R^*_\Phi x^*$ in $L^2(0,T;H)$ for all $x^* \in X^*$. Once we have shown this, the equivalence (ii)$\Leftrightarrow$(iii) follows [59, Theorem 3.1 and Corollary 3.2] (or by using the argument of [113, page 18ff]). To prove the claim we fix $x^* \in X^*$ and note that in $L^2(0,T;H)$ we have

$$R^*_\Phi x^* = i^* S^*(\cdot)x^*, \quad R^*_{\Phi^{(n)}} x^* = \sum_{j=1}^n 1_{I_j^{(n)}}(\cdot) \otimes i^* S^*(t_j^{(n)})x^*.$$

The inclusion mapping $i : H \hookrightarrow X$ is γ-radonifying and hence compact. As a consequence, the weak*-continuity of $t \mapsto S^*(t)x^*$ implies that $t \mapsto i^* S^*(t)x^* = \Phi^*(t)x^*$ is continuous on $[0,T]$. It follows that $\lim_{n\to\infty} R^*_{\Phi^{(n)}}(\cdot)x^* = R^*_\Phi(\cdot)x^*$ in $L^\infty(0,T;H)$, and hence in $L^2(0,T;H)$. $\square$

The Ornstein-Uhlenbeck semigroup associated with the problem (SCP), $\mathscr{P} = (\mathscr{P}(t))_{t\geq 0}$, is defined on the space $C_{\mathrm{b}}(X)$ of all bounded real-valued continuous functions on X by the formula

$$\mathscr{P}(t)f(x) = \mathbb{E}f(U(t)), \qquad x \in X,\ t \geq 0,$$

where U is the solution to (SCP). In order to explain the precise result, let us denote by $\mathscr{S} = (\mathscr{S}(t))_{t\geq 0}$ and $\mathscr{T} = (\mathscr{T}(t))_{t\geq 0}$ the semigroups on $C_{\mathrm{b}}(X)$ corresponding to the drift term and the diffusion term in (SCP). Thus,

$$\begin{aligned} &\mathscr{S}(t)f(x_0) = f(S(t)x_0), \\ &\mathscr{T}(t)f(x_0) = \mathbb{E}f(x_0 + W(t)), \qquad t \geq 0,\ x_0 \in X. \end{aligned}$$

Each of the semigroups $\mathscr{P}$, $\mathscr{S}$ and $\mathscr{T}$ is jointly continuous in t and x_0, uniformly on $[0,T]\times K$ for all compact sets $K \subseteq X$. It was shown in [83] that if condition (iii) of Theorem 6.9 holds, then for all $f \in C_{\mathrm{b}}(X)$ we have the Lie–Trotter product formula

$$\mathscr{P}(t)f(x_0) = \lim_{n\to\infty} \big[\mathscr{T}(t/n)\mathscr{S}(t/n)\big]^n f(x_0) \tag{6.4.8}$$

with convergence uniformly on $[0,T]\times K$ for all compact sets $K \subseteq X$. Conversely it follows from the proof of this result that (6.4.8) with $x_0 = 0$ implies condition (iii) of Theorem 6.9. In the same paper it was shown that (6.4.8) holds if at least one of the next two conditions is satisfied:

(a) X is isomorphic to a Hilbert space;

(b) S restricts to a C_0-semigroup on H.

Thus, either of these conditions implies the convergence $\lim_{n\to\infty} U_c^{(n)}(t) = U(t)$ in $L^p(\Omega; X)$ for all $x_0 \in X$ and $t \in [0,T]$ of the splitting scheme. The proofs in [83] are rather involved. A simple proof for case (b) has been subsequently obtained by Johanna Tikanmäki (personal communication). In Theorems 6.10 and 6.12 below we shall give simple proofs for both cases (a) and (b), based on the Proposition 6.11 and an elementary convergence result for γ-radonifying operators from [108], respectively. Moreover, case (a) is extended to Banach spaces with type 2.

6.4.2 Convergence in the additive noise case

Theorem 6.10. *If X has type 2, then:*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\|U_c^{(n)}(t) - U(t)\|^p = 0, \quad 1 \leq p < \infty.$$

To prove this theorem we shall need the following simple continuity result.

Proposition 6.11. *Let (S,d) be a metric space and let $V : S \to \mathscr{L}(X,Y)$ be strongly continuous. Then for all $R \in \gamma(\mathscr{H},X)$ the function $VR : S \to \gamma(\mathscr{H},Y)$,*

$$(VR)(s) := V(s)R, \qquad s \in S,$$

is continuous.

Proof. Suppose first that R is a finite rank operator, say $R = \sum_{j=1}^{k} h_j \otimes x_j$ with $(h_j)_{j=1}^{k} \in \mathscr{H}$ orthonormal and $(x_j)_{j=1}^{k}$ a sequence in X. Suppose that $\lim_{n\to\infty} s_n = s$ in S. Then

$$\lim_{n\to\infty} \|V(s_n)R - V(s)R\|^2_{\gamma(\mathscr{H},Y)} = \lim_{n\to\infty} \mathbb{E}\Big\| \sum_{j=1}^{k} \gamma_j (V(s_n) - V(s))x_j \Big\|^2 = 0.$$

The general case follows from the density of the finite rank operators in $\gamma(\mathscr{H},X)$ and the norm estimate $\|V(s)R\|_{\gamma(\mathscr{H},Y)} \le \|V(s)\|\|R\|_{\gamma(\mathscr{H},X)}$. □

Proof (of Theorem 6.10). By Proposition 6.11 we have $\Phi \in C([0,T];\gamma(H,X))$. This clearly implies that $\lim_{n\to\infty} \Phi^{(n)} = \Phi$ in $L^\infty(0,T;\gamma(H,X))$, and hence in $L^2(0,T;\gamma(H,X))$. Since X has type 2, by embedding (2.3.5) it follows that $\lim_{n\to\infty} \Phi^{(n)} = \Phi$ in $\gamma(0,T;H,X)$. This suffices by Theorem 6.9. □

Theorem 6.12. *If S restricts to a C_0-semigroup on H, then we have*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\|U_c^{(n)}(t) - U(t)\|^p = 0, \quad 1 \le p < \infty.$$

Proof. Let S_H denote the restricted semigroup on H. From the identity $S(t)\circ i = i \circ S_H(t)$ we have $R_\Phi = i \circ T$ and $R_{\Phi^{(n)}} = i \circ T^{(n)}$, where T and $T^{(n)}$ are the bounded operators from $L^2(0,T;H)$ to H defined by

$$Tf := \int_0^T S_H(t)f(t)\,dt, \quad T^{(n)}f = \int_0^T S_H(\bar{t})f(t)\,dt.$$

Since $\lim_{n\to\infty}(T^{(n)})^* h = T^* h$ for all $h \in H$ by the strong continuity of the adjoint semigroup S_H^* (see [116]), it follows from [108, Proposition 2.4] that $\lim_{n\to\infty} \Phi^{(n)} = \Phi$ in $\gamma(0,T;H,X)$. This suffices by Theorem 6.9. □

Theorem 6.13. *Assume that the semigroup S is analytic on X and that W is a Brownian motion in X_θ for some $\theta \ge -\frac{1}{2}$. Then for all $0 \le \eta \le 1$ such that $\eta - \theta < \frac{1}{2}$, and all $t \in [0,T]$ we have, for all $1 \le p < \infty$:*

$$\big(\mathbb{E}\|U_c^{(n)}(t) - U(t)\|_X^p\big)^{\frac{1}{p}} \lesssim n^{-\eta} t^{\frac{1}{2}-(\eta-\theta)^+} \tag{6.4.9}$$

with implied constant independent of $n \ge 1$ and $t \in [0,T]$.

The proof of this theorem relies of course on the γ-boundedness estimates of Lemma 2.21 and the Kalton-Weis multiplier theorem 2.14. A minor difficulty arises when applying the multiplier theorem in arbitrary Banach spaces: as $\widetilde{M}$ maps into $\gamma_\infty(\mathscr{H}, X)$, one needs to check that the image is in fact in $\gamma(\mathscr{H}, X)$ when necessary. The following proposition is used for this purpose, it is a minor extension of a result due to Kalton and Weis [77]. We refer to [102, Section 13] for a detailed proof.

Proposition 6.14. *Let $\Phi : (a,b) \to \gamma(H,X)$ be continuously differentiable with*

$$\int_a^b (s-a)^{\frac{1}{2}} \|\Phi'(s)\|_{\gamma(H,X)}\, ds < \infty.$$

Then $\Phi \in \gamma(a,b;H,X)$ and

$$\|R_\Phi\|_{\gamma(a,b;H,X)} \le (b-a)^{\frac{1}{2}} \|\Phi(b)\|_{\gamma(H,X)} + \int_a^b (s-a)^{\frac{1}{2}} \|\Phi'(s)\|_{\gamma(H,X)}\, ds.$$

Remark 6.15. As S is assumed to be analytic, by (2.6.3) and the ideal property for γ-radonifying operators we obtain the following estimate for $\Phi(t) := S(t) \circ i$:

$$\begin{aligned}\|\Phi'(t)\|_{\gamma(H,X)} &\le \|AS(t)\|_{\mathscr{L}(X_\theta,X)} \|i\|_{\gamma(H,X_\theta)} \\ &= \|S(t)\|_{\mathscr{L}(X,X_{1-\theta})} \|i\|_{\gamma(H,X_\theta)} \lesssim t^{-(1-\theta)^+} \|i\|_{\gamma(H,X_\theta)}\end{aligned}$$

where $r^+ := \max\{0,r\}$ for $r \in \mathbb{R}$; the implied constant is independent of $t \in [0,T]$ and $i \in \gamma(H,X_\theta)$. If $\theta > -\frac{1}{2}$, it then follows from Proposition 6.14 that

$$\|R_\Phi\|_{\gamma(0,t;H,X)} \lesssim t^{\min\{\frac{1}{2}+\theta, \frac{3}{2}\}} \|i\|_{\gamma(H,X_\theta)},$$

with implied constant independent of $t \in [0,T]$ and $i \in \gamma(H,X_\theta)$. In particular, taking $\theta = 0$ we see that (SCP) admits a mild solution.

Remark 6.16. Suppose that $\delta \in [0, \frac{1}{2})$. We have

$$\begin{aligned}&\|s \mapsto s^{-\delta} S(t-s) i\|_{\gamma(0,t;H,E)} \\ &\qquad \le \|s \mapsto s^{-\delta} S(t-s) i\|_{\gamma(0,\frac{t}{2};H,E)} + \|s \mapsto (t-s)^{-\delta} S(s) i\|_{\gamma(0,\frac{t}{2};H,E)}.\end{aligned}$$

Applying Proposition 6.14 to both terms on the right-hand side it follows that

$$[s \mapsto s^{-\delta} S(t-s) i] \in \gamma(0,t;H,E)$$

for all $t \in [0,T]$.

Proof. Proof of Theorem 6.13 By rescaling time we may assume that $T = 1$. Let θ, η be as indicated. We begin by noting that the embedding $i : H \hookrightarrow X$ associated with W belongs to $\gamma(H, X_\theta)$.

Pick $(\eta - \theta)^+ < \delta < \frac{1}{2}$. Note that by definition of $\Phi^{(n)}$ we have, for all $n \ge 1$,

$$\Phi^{(n)}(s) - \Phi(s) = s^{\delta} S(s) \circ \big(S(n^{-1}\lceil ns\rceil - s) - I\big) \circ s^{-\delta} i. \tag{6.4.10}$$

Fix $t \in (0,1]$. By the first part of Lemma 2.21 the set

$$\mathscr{S}_\delta = \{s^\delta S(s) : \ s \in [0,t]\}$$

is γ-bounded in $\mathscr{L}(X_{\eta-\theta}, X)$ and we have

$$\gamma_{[X_{\eta-\theta},X]}(\mathscr{S}_\delta) \lesssim t^{\delta-(\eta-\theta)^+}. \tag{6.4.11}$$

By the final part of Lemma 2.21 the set

$$\mathscr{T}_{\eta,\frac{1}{n}} = \{S(s) - I : \ s \in [0, n^{-1}]\}$$

is γ-bounded in $\mathscr{L}(X_\theta, X_{\theta-\eta})$, and we have

$$\gamma_{[X_\theta,X_{\theta-\eta}]}(\mathscr{T}_{\eta,\frac{1}{n}}) \lesssim n^{-\eta}. \tag{6.4.12}$$

Using (6.4.10), Remark 6.16, Proposition 2.14, Proposition 2.5 together with the estimates (6.4.11), and (6.4.12), and noting that $n^{-1}\lceil ns\rceil - s \le n^{-1}$, one obtains

$$\begin{aligned}\|\Phi^{(n)} - \Phi\|_{\gamma(0,t;H,X)} &\le \gamma_{[X_{\eta-\theta},X]}(\mathscr{S}_\delta)\gamma_{[X_\theta,X_{\theta-\eta}]}(\mathscr{T}_{\eta,\frac{1}{n}})\|s^{-\delta} i\|_{\gamma(0,t;H,X_\theta)} \\ &\lesssim n^{-\eta} t^{\frac{1}{2}-(\eta-\theta)^+}\|i\|_{\gamma(H,X_\theta)}.\end{aligned}$$

□

Remark 6.17. The results of Theorems 6.10, 6.12 and 6.13 above also apply if we replace $U_c^{(n)}$ by $U^{(n)}$, where $U^{(n)}$ is the process defined by (6.4.4). However, the proofs are slightly longer as the equivalence (i)⇔(ii) in Theorem 6.9 fails to hold.

Under the assumptions that S is analytic on X and W is a Brownian motion on X, the solution U of (SCP) with $x_0 \equiv 0$ has a version with trajectories in $C^\gamma([0,T];X)$ for any $\gamma \ge 0$ such that $\gamma < \frac{1}{2}$ [40]. The following theorem asserts that for $x_0 \equiv 0$ the approximating processes $U_c^{(n)}$ also have trajectories in $C^\gamma([0,T];X)$, and the splitting scheme converges with respect to the $C^\gamma([0,T];X)$-norm, with a convergence rate depending on γ and the smoothness of the noise.

Theorem 6.18. *Let S be analytic on X and suppose that W is a Brownian motion in X_θ for some $\theta \ge 0$. If $\eta, \gamma \ge 0$ satisfy $\eta + \gamma < 1$ and $(\eta-\theta)^+ + \gamma < \frac{1}{2}$, then for all $1 \le p < \infty$ the solution U of* (SCP) *satisfies*

$$\big(\mathbb{E}\|U_c^{(n)} - U\|^p_{C^\gamma([0,T],X)}\big)^{\frac{1}{p}} \lesssim n^{-\eta},$$

with implied constant independent of $n \ge 1$.

Proof. By scaling we may assume $T = 1$. Put $D^{(n)} := U_c^{(n)} - U$. Let θ, γ and η be as indicated. Without loss of generality we assume that $\gamma > 0$. The main step in the proof is establishing the following claim.

Claim. *There exists a constant C such that for all $n \geq 1$, all $0 \leq s < t \leq 1$ satisfying $t - s < \frac{1}{2n}$ we have*

$$\big(\mathbb{E}\|D^{(n)}(t) - D^{(n)}(s)\|_X^2\big)^{\frac{1}{2}} \leq C n^{-\eta}(t-s)^{\gamma}.$$

Proof (of Claim). Fix $n \geq 1$ and $0 \leq s < t \leq 1$ such that $t - s < \frac{1}{2n}$. Clearly,

$$\begin{aligned}\big(\mathbb{E}\|D^{(n)}(t) - D^{(n)}(s)\|_X^2\big)^{\frac{1}{2}} &\leq \Big(\mathbb{E}\Big\|\int_s^t \Phi(t-r) - \Phi^{(n)}(t-r)\,dW(r)\Big\|^2\Big)^{\frac{1}{2}} \\ &\quad + \Big(\mathbb{E}\Big\|\int_0^s \Phi(t-r) - \Phi(s-r)\,dW(r)\Big\|^2\Big)^{\frac{1}{2}} \\ &\quad + \Big(\mathbb{E}\Big\|\int_0^s \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r)\,dW(r)\Big\|^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{6.4.13}$$

For the first term we note that by (2.4.1) (and the remark following it) and Theorem 6.13 one has

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\int_s^t \Phi^{(n)}(t-r) - \Phi(t-r)\,dW_H(r)\Big\|_X^2\Big)^{\frac{1}{2}} \\ &\qquad = \Big(\mathbb{E}\Big\|\int_0^{t-s} \Phi^{(n)}(r) - \Phi(r)\,dW_H(r)\Big\|_X^2\Big)^{\frac{1}{2}} \\ &\qquad \lesssim n^{-\eta}(t-s)^{\frac{1}{2}-(\eta-\theta)^+}\|i\|_{\gamma(H,X_\theta)} \\ &\qquad \leq n^{-\eta}(t-s)^{\gamma}\|i\|_{\gamma(H,X_\theta)}.\end{aligned} \tag{6.4.14}$$

The estimate for the second term is extracted from arguments in [109]; see also [101, Theorem 10.19]. Fix $\eta > 0$ such that $(\eta-\theta)^+ + \gamma < \eta < \frac{1}{2}$. Then the set $\{t^\eta S(t) : \ t \in (0,T)\}$ is γ-bounded in $\mathscr{L}(X_{-\eta+\theta-\gamma}, X)$ by the first part of Lemma 2.21, and therefore

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\int_0^s \Phi(t-r) - \Phi(s-r)\,dW_H(r)\Big\|_X^2\Big)^{\frac{1}{2}} \\ &\quad = \Big(\mathbb{E}\Big\|\int_0^s [(s-r)^\eta S(s-r)] \circ [(s-r)^{-\eta}(S(t-s)-I)\circ i]\,dW_H(r)\Big\|_X^2\Big)^{\frac{1}{2}} \\ &\quad \lesssim \Big(\mathbb{E}\Big\|\int_0^s (s-r)^{-\eta}(S(t-s)-I)\circ i)\,dW_H(r)\Big\|_{X_{-\eta+\theta-\gamma}}^2\Big)^{\frac{1}{2}} \\ &\quad = \Big(\int_0^s (s-r)^{-2\eta}\,dr\Big)^{\frac{1}{2}}\|(S(t-s)-I)\circ i\|_{\gamma(H,X_{-\eta+\theta-\gamma})} \\ &\quad \lesssim \|S(t-s)-I\|_{\mathscr{L}(X_\theta, X_{-\eta+\theta-\gamma})}\|i\|_{\gamma(H,X_\theta)} \\ &\quad \eqsim \|S(t-s)-I\|_{\mathscr{L}(X_{\gamma+\eta}, X)}\|i\|_{\gamma(H,X_\theta)}\end{aligned}$$

$$
\begin{aligned}
&\lesssim (t-s)^{\gamma+\eta}\|i\|_{\gamma(H,X_\theta)} \\
&\lesssim n^{-\eta}(t-s)^{\gamma}\|i\|_{\gamma(H,X_\theta)}.
\end{aligned} \tag{6.4.15}
$$

To estimate the third term on the right-hand side of (6.4.13), we first define sets B_0 and B_1 by

$$
\begin{aligned}
B_0 &:= \{r \in (0,s) :\ S(\overline{t-r}) = S(\overline{s-r})\} \\
&= \{r \in (0,s) :\ \lceil n(t-r)\rceil = \lceil n(s-r)\rceil\}, \\
B_1 &:= \{r \in (0,s) :\ S(\overline{t-r}) = S(n^{-1})S(\overline{s-r})\} \\
&= \{r \in (0,s) :\ \lceil n(t-r)\rceil = \lceil n(s-r)\rceil + 1\}.
\end{aligned}
$$

Both equalities follow from the identity $S(\overline{u}) = S(n^{-1}\lceil nu\rceil)$ for $u \in (0,T)$. By definition of B_0 and B_1 one has

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\int_0^s \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r)\, dW_H(r)\Big\|_X^2\Big)^{\frac12} \\
&\quad \le \Big(\mathbb{E}\Big\|\int_{B_0} \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r)\, dW_H(r)\Big\|_X^2\Big)^{\frac12} \\
&\qquad + \Big(\mathbb{E}\Big\|\int_{B_1} \Phi^{(n)}(t-r) - \Phi^{(n)}(s-r)\, dW_H(r)\Big\|_X^2\Big)^{\frac12} \\
&\quad = \Big(\mathbb{E}\Big\|\int_{B_1} S(\overline{s-r})(S(n^{-1})-I)i\, dW_H(r)\Big\|_X^2\Big)^{\frac12},
\end{aligned} \tag{6.4.16}
$$

noting that the integrand of the integral over B_0 vanishes.

Set $\delta := \eta+\gamma$. To estimate the right-hand side, observe that from $\delta < \frac12+\theta$ we may pick $\eta > 0$ such that $\delta - \theta < \eta < \frac12$. Using the identity $S(\overline{u}) = S(n^{-1}\lceil nu\rceil)$ and applying Theorem 2.14 and part (1) of Lemma 2.21, and then using the estimate $\|S(u) - I\|_{\mathscr{L}(X_\delta,X)} \lesssim u^\delta$ and Proposition 2.5, we obtain

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\int_{B_1} S(\overline{s-r})(S(n^{-1})-I)i\, dW_H(r)\Big\|_X^2\Big)^{\frac12} \\
&\quad \eqsim \Big(\mathbb{E}\Big\|\int_{B_1} (n^{-1}\lceil n(s-r)\rceil)^{\eta} S(n^{-1}\lceil n(s-r)\rceil) \\
&\qquad \times (n^{-1}\lceil n(s-r)\rceil)^{-\eta}(S(n^{-1})-I)i\, dW_H(r)\Big\|_X^2\Big)^{\frac12} \\
&\quad \lesssim \Big(\mathbb{E}\Big\|\int_{B_1} (n^{-1}\lceil n(s-r)\rceil)^{-\eta}(S(n^{-1})-I)i\, dW_H(r)\Big\|_{X_{\theta-\delta}}^2\Big)^{\frac12} \\
&\quad \lesssim n^{-\delta}\|(s-\cdot)^{-\eta}\|_{L^2(B_1)}\|i\|_{\gamma(H,X_\theta)}.
\end{aligned} \tag{6.4.17}
$$

In order to estimate the $L^2(B_1)$-norm of the function $f_s(r) := (s-r)^{-\eta}$ we note that $B_1 \subseteq \bigcup_{j=1}^n B_1^{(j)}$, where

$$
\begin{aligned}
B_1^{(j)} &= \{r \in (0,s) :\ s-r \le jn^{-1} < t-r\} \\
&= \{r \in (0,s) :\ jn^{-1} - t + s < s-r \le jn^{-1}\}.
\end{aligned} \tag{6.4.18}
$$

From this it is easy to see that $|B_1^{(j)}| \le t-s$ and that for $r \in B_1^{(j)}$ one has

$$(s-r)^{-2\eta} \le (jn^{-1}-t+s)^{-2\eta} \le n^{2\eta}(j-\tfrac{1}{2})^{-2\eta}$$

(the latter inequality following from $t-s < 1/2n$), and therefore

$$\|f_s\|_{L^2(B_1)}^2 = \int_{B_1} |f_s(r)|^2\,dr \le n^{2\eta}|B_1^{(j)}| \sum_{j=1}^{n} \frac{1}{(j-\frac{1}{2})^{2\eta}} \lesssim n(t-s).$$

As a consequence,

$$\|f_s\|_{L^2(B_1)} \lesssim n^{\frac{1}{2}}(t-s)^{\frac{1}{2}} = n^{\frac{1}{2}}(t-s)^{\frac{1}{2}-\gamma}(t-s)^{\gamma} \lesssim n^{\gamma}(t-s)^{\gamma}. \tag{6.4.19}$$

Combining the estimates (6.4.17) and (6.4.19) and estimating the non-negative powers of s by 1 we find

$$\begin{aligned}\Big(\mathbb{E}\Big\| \int_{B_1} S(\overline{s-r})(S(\overline{t-s})-I)i\,dW_H(r)\Big\|_X^2\Big)^{\frac{1}{2}} \\ \lesssim n^{-\eta}(t-s)^{\gamma}\|i\|_{\gamma(H,X_\theta)}.\end{aligned} \tag{6.4.20}$$

The claim now follows by combining (6.4.13), (6.4.14), (6.4.15), (6.4.16) and (6.4.20). □

We are now ready to finish the proof of the theorem. By the triangle inequality and Theorem 6.13, for all $0 \le s,t \le 1$ we have

$$\begin{aligned}\big(\mathbb{E}\|D^{(n)}(t)-D^{(n)}(s)\|_X^2\big)^{\frac{1}{2}} &\le \big(\mathbb{E}\|U_c^{(n)}(t)-U(t)\|_X^2\big)^{\frac{1}{2}} + \big(\mathbb{E}\|U_c^{(n)}(s)-U(s)\|_X^2\big)^{\frac{1}{2}} \\ &\lesssim n^{-\delta}\|i\|_{\gamma(H,X_\theta)}.\end{aligned}$$

Hence if $t-s \ge (2n)^{-1}$ one has

$$\big(\mathbb{E}\|D^{(n)}(t)-D^{(n)}(s)\|_X^2\big)^{\frac{1}{2}} \lesssim n^{-\delta}\|i\|_{\gamma(H,X_\beta)} \lesssim n^{-\eta}(t-s)^{\gamma}\|i\|_{\gamma(H,X_\theta)}. \tag{6.4.21}$$

The random variables $D^{(n)}(t)$ being Gaussian, from the claim and (6.4.21) combined with the Kahane-Khintchine inequalities we deduce that for all $1 \le q < \infty$ and $0 \le s < t \le 1$ one has

$$\big(\mathbb{E}\|D^{(n)}(t)-D^{(n)}(s)\|_X^q\big)^{\frac{1}{q}} \lesssim n^{-\eta}(t-s)^{\gamma}\|i\|_{\gamma(H,X_\theta)}. \tag{6.4.22}$$

Now fix any $0 < \gamma' < \gamma$ and take $1/\gamma' < q < \infty$. Then by (6.4.22) and the Kolmogorov-Chentsov criterion with L^q-moments (see [44, Theorem 5]),

$$\|U_c^{(n)} - U\|_{L^q(\Omega;C^{\gamma'-\frac{1}{q}}([0,T];X))} \lesssim \|U_c^{(n)} - U\|_{C^{\gamma}([0,T];L^q(\Omega;X))} \lesssim n^{-\eta}\|i\|_{\gamma(H,X_\theta)}.$$

This inequality shows that for all $0 < \bar\gamma < \gamma$ we have

$$\|U_c^{(n)} - U\|_{L^q(\Omega;C^{\bar\gamma}([0,T];X))} \lesssim n^{-\eta}\|i\|_{\gamma(H,X_\theta)}$$

for all sufficiently large $1 \le q < \infty$. It is clear that once we know this, this inequality extends to all values $1 \le q < \infty$. This completes the proof of the theorem (with $\bar\gamma$ instead of γ, which obviously suffices). □

The corollary below is obtained by a Borel-Cantelli argument; see also Corollary 6.6.

Corollary 6.19. *Suppose that S is analytic on X and that W is a Brownian motion in X_θ for some $\theta \geq 0$. Let $\gamma, \eta \geq 0$ satisfy $\eta+\gamma < 1$ and $(\eta-\theta)^+ + \gamma < \frac{1}{2}$. Then there exits a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$ we have:*

$$\|U_c^{(n)} - U\|_{C^\gamma([0,T];X)} \leq \chi n^{-\eta}.$$

6.4.3 A counterexample for convergence

We shall now present an example of a C_0-semigroup S on a Banach space X and an X-valued Brownian motion W such that the problem (SCP) admits a solution with continuous trajectories whilst the associated splitting scheme fails to converge. Although the actual construction is somewhat involved, the semigroup in this example is simply a translation semigroup on a suitable vector-valued Lebesgue space.

We take $X = L^q(0,1;\ell^p)$, with $1 \leq p < 2$ and $q \geq 2$. Note that X has type p. Consider the X-valued Brownian motion $W_f = w \otimes f$, where w is a standard real-valued Brownian motion and $f \in X$ is a fixed element. With this notation a function $\Psi : (0,1) \to \mathscr{L}(X)$ is stochastically integrable with respect to W_f if and only if $\Psi f : (0,1) \to X$ is is stochastically integrable with respect to w, in which case we have

$$\int_0^1 \Psi \, dW_f = \int_0^1 \Psi f \, dw.$$

Let $1 \leq p < 2$ and $u > \frac{2}{p}$ be fixed. For $k = 1, 2, \ldots$ and $j = 0, \ldots 2^{k-1} - 1$ define the intervals $I_{k,j} = (\frac{2j+1}{2^k}, \frac{2j+1}{2^k} + 2^{-uk}]$. As in particular $u > 1$, for all $k = 1, 2, \ldots$ the intervals $I_{k,i}$ and $I_{k,j}$ are disjoint for $i \neq j$. Let $0 < r < 1 - \frac{p}{2}$ and denote the basic sequence of unit vectors in ℓ^p by $(e_n)_{n\geq 1}$. Inspired by [125, Example 3.2] we define $f \in L^\infty(\mathbb{R}; \ell^p)$ by

$$f(t) := \sum_{k=1}^{\infty} \sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k} 1_{I_{k,j}}(t) e_{2^{k-1}+j}.$$

Observe that $f(t) = 0$ for $t \in \mathbb{R} \setminus (0,1)$ and f is well-defined: because $I_{k,j}$ and $I_{k,i}$ are disjoint for $i \neq j$ one has, for any $t \in (0,1)$,

$$\|f(t)\|_{\ell^p}^p \leq \sum_{k=1}^{\infty} 2^{-rk} < \infty.$$

For a given interval $I = (a,b]$, $0 \leq a < b < \infty$, we write $\Delta w_I := w(b) - w(a)$.

Claim. *The function f is stochastically integrable on $(0,1)$ and*

$$\begin{aligned}\int_0^1 f(t)\,dw(t) &= \sum_{n=1}^{\infty}\int_0^1 \langle f(t), e_n^*\rangle e_n\,dw(t)\\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k}\Delta w_{I_{k,j}} e_{2^{k-1}+j},\end{aligned} \tag{6.4.23}$$

where $(e_n^*)_{n\ge 1}$ *is the basic sequence of unit vectors in* $\ell^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. We shall deduce this from [110, Theorem 2.3, (3) $\Rightarrow$ (1)]. Define the ℓ^p-valued Gaussian random variable

$$X := \sum_{k=1}^{\infty}\sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k}\Delta w_{I_{k,j}} e_{2^{k-1}+j}.$$

This sum converges absolutely in $L^p(\Omega;\ell^p)$. Indeed, let γ denote a standard Gaussian random variable. Then by Fubini's theorem one has

$$\begin{aligned}\mathbb{E}\Big\|\sum_{k=1}^{\infty}\sum_{j=0}^{2^{k-1}-1} 2^{-\frac{r}{p}k}\Delta w_{I_{k,j}} e_{2^{k-1}+j}\Big\|_{\ell^p}^p &= \sum_{k=1}^{\infty}\sum_{j=0}^{2^{k-1}-1} 2^{-rk}2^{-\frac{u}{2}pk}\mathbb{E}|\gamma|^p\\ &= \sum_{k=1}^{\infty} 2^{k(1-r-\frac{u}{2}p)-1}\mathbb{E}|\gamma|^p < \infty.\end{aligned} \tag{6.4.24}$$

By the Kahane-Khintchine inequalities, the sum defining X converges absolutely in $L^q(\Omega;\ell^p)$ for all $1\le q<\infty$.

For any linear combination $a^* = \sum_{n=1}^N a_n e_n^* \in \ell^{p'}$ one easily checks that

$$\langle X, a^*\rangle = \int_0^1 \langle f(t), a^*\rangle\,dw(t).$$

Hence by [110, Theorem 2.3], f is stochastically integrable and (6.4.23) holds.□

By similar reasoning (or an application of [110, Corollary 2.7]), for all $s\in\mathbb{R}$ the function $t\mapsto f(t+s)$ is stochastically integrable on $(0,1)$ and

$$\int_0^1 f(t+s)\,dw(t) = \sum_{n=1}^{\infty}\int_0^1 \langle f(t+s), e_n^*\rangle e_n\,dw(t).$$

Let $q\ge 1$ and let $(S(t))_{t\in\mathbb{R}}$ be the left-shift group on $L^q(\mathbb{R};\ell^p)$ defined by

$$(S(t)g)(s) = g(t+s),\quad s,t\in\mathbb{R},\ g\in L^q(\mathbb{R};\ell^p).$$

Claim. *For any* $q\ge 1$ *the* $L^q(\mathbb{R};\ell^p)$*-valued function* $t\mapsto S(t)f$ *is stochastically integrable on* $(0,1)$ *and*

$$\Big(\int_0^1 S(t)f\,dw(t)\Big)(s) = \int_0^1 f(t+s)\,dw(t)$$

for almost all $s\in\mathbb{R}$ *almost surely.*

Proof. For $s \notin (-1,1)$ the function $t \mapsto f(t+s)$ is identically 0 on $(0,1)$, and for $s \in (-1,1)$ we have

$$\mathbb{E}\Big\| \int_0^1 f(t+s)\,dw(t)\Big\|_{\ell^p}^q \le \mathbb{E}\Big\| \int_0^1 f(t)\,dw(t)\Big\|_{\ell^p}^q.$$

As a consequence, $L^q(\Omega;\ell^p)$-valued function $s \mapsto \int_0^1 f(s+t)\,dw(t)$ defines an element of $L^q(\mathbb{R};L^q(\Omega;\ell^p))$. Under the natural isometry $L^q(\mathbb{R};L^q(\Omega;\ell^p)) \simeq L^q(\Omega;L^q(\mathbb{R};\ell^p))$ we may identify this function with $Y \in L^q(\Omega;L^q(\mathbb{R};\ell^p))$. To establish the claim, with an appeal to [110, Theorem 2.3] it suffices to check that for all $a^* \in \ell^{p'}$ and Borel sets $A \in \mathscr{B}(\mathbb{R})$ we have

$$\int_0^1 \langle S(t)f, 1_A \otimes a^*\rangle\,dw(t) = \langle Y, 1_A \otimes a^*\rangle.$$

By writing out both sides, this identity is seen to be an immediate consequence of the stochastic Fubini theorem (see, e.g., [110, Theorem 3.3]). $\square$

Similarly, one sees that for $t \ge 0$ the stochastic integrals $\int_0^t S(-s)f\,dw(s)$ are well-defined. Because the process $t \mapsto \int_0^t S(-s)f\,dw(s)$ is a martingale having a continuous version by Doob's maximal inequality, we also know that the convolution process

$$U(t) := \int_0^t S(t-s)f\,dw(s) = S(t)\int_0^t S(-s)f\,dw(s)$$

has a continuous version. However, as we shall see, the splitting scheme for U fails to converge.

For notational simplicity, we set $S^{(n)}(t) := S(\bar{t}) = S(n^{-1}\lceil nt\rceil)$ for $n \in \mathbb{N}$ and $t \ge 0$. Observe that for any $n \in \mathbb{N}$ and $s,t \in \mathbb{R}$

$$(S^{(n)}(t)f)(s) = \sum_{k=1}^n 1_{(\frac{k-1}{n},\frac{k}{n}]}(t) f\big(\tfrac{k}{n}+s\big). \tag{6.4.25}$$

Similarly to the above one checks that

$$\Big(\int_0^1 S^{(n)}(t)f\,dw(t)\Big)(s) = \sum_{k=1}^n f\big(\tfrac{k}{n}+s\big)\big[w\big(\tfrac{k}{n}\big) - w\big(\tfrac{k-1}{n}\big)\big]$$

for almost all $s \in \mathbb{R}$ almost surely.

The clue to this example is that for n fixed and $s \in (0, 2^{-un}]$ the function $t \mapsto (S^{(2^n)}(t)f)(s)$ always 'picks up' the values of f at the left parts of the dyadic intervals where f is defined to be non-zero. Thus for these values of s the function $t \mapsto (S^{(2^n)}(t)f)(s)$ it is nowhere zero and its stochastic integral blows up as $n \to \infty$. We shall make this precise. Our aim is to prove that for certain values of $q > 2$ (to be determined later on) one has

$$\mathbb{E}\Big\| \int_0^1 S^{(2^n)}(t) f \, dw(t) \Big\|^p_{L^q(\mathbb{R};\ell^p)} \to \infty \quad \text{as } n \to \infty. \tag{6.4.26}$$

By Minkowski's inequality we have, for any $n \geq 1$ and $q \geq p$,

$$\begin{aligned}
&\Big[\mathbb{E}\Big\| \int_0^1 S^{(2^n)}(t) f \, dw(t) \Big\|^p_{L^q(\mathbb{R};\ell^p)}\Big]^{\frac{1}{p}} \\
&\qquad \geq \Big[\int_{\mathbb{R}} \Big(\mathbb{E}\Big\| \Big(\int_0^1 S^{(2^n)}(t) f \, dw(t)\Big)(s) \Big\|^p_{\ell^p}\Big)^{\frac{q}{p}} \, ds\Big]^{\frac{1}{q}}.
\end{aligned}$$

Now fix $n \geq 1$. For any $1 \leq k \leq n$ and any $j = 0, \ldots, 2^{k-1} - 1$ there exists a unique $1 \leq i_{k,j} \leq 2^n - 1$ such that $\frac{i_{k,j}}{2^n} = \frac{2j+1}{2^k}$. Now observe that by definition of f one has for $s \in (0, 2^{-un}]$ that

$$\Big\langle f\big(\tfrac{i_{k,j}}{2^n} + s\big), e^*_{2^{k-1}+j} \Big\rangle = 2^{-k\frac{r}{p}}.$$

Using this and representation (6.4.25) one obtains that for $s \in (0, 2^{-un}]$, $1 \leq k \leq n$, $j = 0, \ldots, 2^{k-1} - 1$, and any $t \in \big(\frac{i_{k,j}-1}{2^n}, \frac{i_{k,j}}{2^n}\big] =: I^n_{k,j}$,

$$\big\langle (S^{(2^n)}(t) f)(s), e^*_{2^{k-1}+j} \big\rangle = 2^{-k\frac{r}{p}}. \tag{6.4.27}$$

Recall that γ denotes a standard Gaussian random variable. To prove (6.4.26), we now estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(\mathbb{E}\Big\| \Big(\int_0^1 S^{(2^n)}(t) f \, dw(t)\Big)(s) \Big\|^p_{\ell^p}\Big)^{\frac{q}{p}} \, ds \\
&\qquad \geq \int_0^{2^{-un}} \Big(\sum_{k=1}^n \sum_{j=0}^{2^{k-1}-1} \mathbb{E}\Big| \int_0^1 \big\langle (S^{(2^n)}(t) f)(s), e^*_{2^{k-1}+j} \big\rangle \, dw(t) \Big|^p \Big)^{\frac{q}{p}} \, ds \\
&\qquad \geq \int_0^{2^{-un}} \Big(\sum_{k=1}^n \sum_{j=0}^{2^{k-1}-1} 2^{-kr} \mathbb{E}|\Delta w_{I^n_{k,j}}|^p \Big)^{\frac{q}{p}} \, ds \\
&\qquad = \int_0^{2^{-un}} \Big(\sum_{k=1}^n 2^{k-1} 2^{-kr} 2^{-n\frac{p}{2}} \mathbb{E}|\gamma|^p \Big)^{\frac{q}{p}} \, ds \\
&\qquad \geq 2^{-un-1} 2^{n(1-r-\frac{p}{2})\frac{q}{p}} (\mathbb{E}|\gamma|^p)^{\frac{q}{p}},
\end{aligned}$$

where in the second inequality we use (6.4.27). Thus if $-u + (1 - r - \frac{p}{2})\frac{q}{p} > 0$, that is, if $q > up/(1 - r - \frac{p}{2})$ (recall that $r < 1 - \frac{p}{2}$), the left-hand side expression diverges as $n \to \infty$.

7

Implicit-linear Euler approximations

In this chapter we prove pathwise Hölder convergence with optimal rates for an abstract type of time discretizations for (SDE) under the assumptions **(A)**, **(F)**, **(G)** of Section 5.1 and the additional assumptions **(F′)** and **(G′)** below. The most important example of such a type of discretization is the implicit-linear Euler scheme, which is defined as follows. Fixing a finite time horizon $0 < T < \infty$ and an integer $n \in \mathbb{N}$, we set $V_0^{(n)} := x_0$ and, for $j = 1, \ldots, n$, define the random variables $V_j^{(n)}$ implicitly by the identity

$$V_j^{(n)} = V_{j-1}^{(n)} + \tfrac{T}{n}\big[AV_j^{(n)} + F\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big)\big] + G\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big)\Delta W_j^{(n)}.$$

Recall that

$$t_j^{(n)} = \tfrac{jT}{n} \quad \text{and, formally,} \quad \Delta W_j^{(n)} = W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)}).$$

The rigorous interpretation of the term $G\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big)\Delta W_j^{(n)}$ is explained in Section 7.2. Note that this scheme is implicit only in its linear part. As a consequence of this, and noting that for large enough $n \in \mathbb{N}$ we have $\frac{n}{T} \in \varrho(A)$, the resolvent set of A, this identity may be rewritten in the explicit format

$$V_j^{(n)} = (1 - \tfrac{T}{n}A)^{-1}\big[V_{j-1}^{(n)} + \tfrac{T}{n}F\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big) + G\big(t_{j-1}^{(n)}, V_{j-1}^{(n)}\big)\Delta W_j^{(n)}\big]. \tag{7.0.1}$$

Theorem 7.10 in Section 7.2 states the convergence of

$$V^{(n)} := \sum_{j=1}^{n} V_j^{(n)} 1_{I_j^{(n)}}$$

against U, where U is the mild solution to (SDE), in $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$ for p arbitrarily large. By the same Kolmogorov argument used in Section 6.3 to obtain Theorem 6.1 from Theorem 6.2, we can use Theorem 7.10 to obtain the following, where $u = (U(t_j^{(n)}))_{j=0}^n$, and $v^{(n)} = (V_j^{(n)})_{j=0}^n$. See also Remark 7.13 on page 135.

Theorem 7.1 (Hölder convergence of the Euler scheme). *Let X be a* UMD *Banach space with Pisier's property (α), and let $\tau \in (1,2]$ be the type of X. Suppose $p \in (2,\infty)$, $\gamma, \delta \in [0,\frac{1}{2})$ and $\eta > 0$ satisfy*

$$\gamma + \delta + \tfrac{1}{p} < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + (\theta_F \wedge 0),\, \tfrac{1}{2} + (\theta_G \wedge 0),\, \eta\},$$

and suppose that $x_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$. Then there is a constant C, independent of x_0, such that for all large enough $n \in \mathbb{N}$,

$$\big(\mathbb{E}\|u - v^{(n)}\|^p_{C^{(n)}_\gamma([0,T];X)}\big)^{\frac{1}{p}} \leq Cn^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}). \tag{7.0.2}$$

Note that in contrast to our result for the splitting scheme in Chapter 6, the convergence rate does not improve as θ_F and θ_G increase above 0. Also note that it is now necessary to assume that the UMD Banach space X has property (α).

This chapter contains two sections: in Section 7.1 we prove an abstract result concerning approximation of semigroup operators, and in Section 7.2 we prove convergence of $V^{(n)}$ against U in $\mathscr{V}^{\alpha,p}_\infty([0,T] \times \Omega; X)$ for p arbitrarily large.

The convergence of the Euler scheme is deduced from the convergence of the splitting scheme. A more streamlined proof for the Euler scheme would be possible, but we have chosen the indirect route for the following reason. In the splitting scheme, the semigroup is discretized, but not the noise. In the Euler scheme, both the semigroup and the noise are discretized. Because of this, it is not possible to derive the correct rates for the splitting scheme from those of the Euler scheme. The present arrangement gives the optimal rates for both schemes.

Related work on pathwise convergence

The literature on convergence rates for numerical schemes for stochastic evolution equations and SPDEs is extensive. For an overview we refer the reader to the excellent review paper [70].

To the best of our knowledge, our results are the first that concern pathwise convergence with respect to Hölder norms for numerical schemes for stochastic evolution equations with (locally) Lipschitz coefficients. The works [52,55,58,61,62,68,82,97,115,120,133] consider frameworks which are amenable to a comparison with Theorem 7.1; quite likely this list is far from complete. All these papers exclusively consider Hilbert spaces X, the only exception being [62] where X is taken to be of martingale type 2. In particular, in all these papers X has type 2. Let us also mention the paper [57], where convergence is proved, for $p = 2$ and X Hilbertian, for the implicit Euler scheme under monotonicity assumptions on the operator A.

Most of these papers cited above give endpoint convergence rates only. The first pathwise uniform convergence result of the implicit Euler scheme seems to be due to Gyöngy [55], who obtains convergence rate $n^{-\frac{1}{8}}$ for the 1D stochastic heat equation with multiplicative space-time white noise. This has been extended in [115] to the case of space-dependent dispersion, but without rates. Pathwise uniform convergence of the implicit Euler schemes (with convergence in probability) has been obtained by Printems [120] for the Burgers equation with rate $n^{-\gamma}$ for any $\gamma < \frac{1}{4}$.

Concerning the optimality of the rate

For end-point estimates (i.e., a weaker type of estimates than the type we consider, which are pathwise) it is proven in [34] that the critical convergence rate of a time discretization for the heat equation in one dimension with additive space-time white noise based on n equidistant time steps is $n^{-\frac{1}{4}}$ (see Remark 9.2). In that sense our results on the convergence for the heat equation are optimal, see Chapter 9, where the stochastic heat equation is considered.

For 'trace class noise' (which corresponds to taking $\theta_G = 0$ in our framework) it is known that the critical convergence rate $n^{-\frac{1}{2}}$ is in a sense the best possible even in the simpler setting of ordinary stochastic differential equations with globally Lipschitz coefficients. To be precise, it is shown in [22] that there exist examples of equations of the form

$$\begin{cases} dX(t) = f(X(t))\,dt + g(X(t))\,dW(t), \quad t \in [0,T], \\ X(0) = x_0, \end{cases}$$

with $x_0 \in \mathbb{R}^d$ and W a Brownian motion in $\mathbb{R}^d$, whose solution X satisfies the endpoint estimate

$$\big(\mathbb{E}\big|X(T) - \mathbb{E}(X(T)|\mathscr{P}_n)\big|^2\big)^{\frac{1}{2}} = n^{-\frac{1}{2}}\sqrt{\tfrac{1}{2}T}.$$

Here $\mathscr{P}_n = \sigma\{W(t_j^{(n)}) \,:\, j = 1,\dots,n\}$. Thus, if X is approximated by a sequence of processes $X^{(n)}$ whose definition only depends on knowing $\{W(t_j^{(n)}) \,:\, j = 1,\dots,n\}$ (such is the case for the implicit Euler scheme), the convergence rate cannot be better than $n^{-\frac{1}{2}}$.

In the special case $\theta_G = 0$, Theorem 7.1 can be applied with any $\gamma, \delta \geq 0$ such that $\gamma + \delta < \frac{1}{2}$, provided $\theta_F \geq -1 + \frac{1}{\tau}$, x_0 takes values in X_η with $\eta \geq \frac{1}{2}$, and p is taken large enough. In particular, by taking $\gamma = 0$, this leads to pathwise uniform convergence of order $n^{-\delta}$ for arbitrary $\delta \in [0, \frac{1}{2})$.

Our methods do not produce the critical convergence rate $n^{-\frac{1}{2}}$. We know of two examples in the literature where this rate is obtained, namely [55, 82]. In these examples the operator A has the property of 'stochastic maximal regularity' (see [106]) and the underlying space has type 2, and in neither of these results the convergence is pathwise. In the present work, we do obtain pathwise convergence under the weaker assumption that A generates an analytic semigroup, but in this more general framework we do not expect to attain the critical rate.

7.1 Approximating semigroup operators

In this section we prove a γ-boundedness result for families of operators defined in terms of the so-called Hille-Phillips functional calculus of an operator A that generates an analytic C_0-semigroup on a Banach space X. This result will be used in the next section, where we prove an abstract convergence result for time discretization schemes of (SDE).

Let $(\mu_n)_{n\geq N}$ be a sequence of non-negative finite measures on $[0,\infty)$ and let $R \geq 0$ be given. For $j \in \mathbb{N}$ let $\mu_n^{*j} = \mu_n * \cdots * \mu_n$ denote the j-fold convolution product of μ_n with itself. Consider the following properties:

(M1) For all $n \geq 1$ we have:

$$\int_0^\infty t \, d\mu_n(t) = \tfrac{1}{n};$$

(M2) There exists an $N \geq 1$ such that for all $n \geq N$, all $j = 1, \ldots, n$, and every $\alpha \in (-1, 1]$ we have:

$$\int_0^\infty t^\alpha e^{Rt} \, d\mu_n^{*j}(t) < \infty;$$

(M3) For every $\alpha \in (-1, 1]$ we have:

$$\sup_{n\geq N} \sup_{1\leq j\leq n} \Big| j \int_0^\infty \big[1 - \big(\tfrac{tn}{j}\big)^\alpha\big] e^{Rt} \, d\mu_n^{*j}(t) \Big| < \infty.$$

Let A be the generator of an analytic C_0-semigroup S on X and let $\omega \geq 0$ be such that $(e^{-\omega t} S(t))_{t\geq 0}$ is uniformly bounded. Fix $T > 0$. Let $(\mu_n)_{n\geq N}$ be a sequence of non-negative σ-finite measures on $[0,\infty)$ that satisfy **(M1)**, **(M2)**, **(M3)** for $R = \omega T$. Let N be as in **(M2)**. For $n \geq N$ define $E(\frac{T}{n}) \in \mathscr{L}(X)$ by

$$E(\tfrac{T}{n})x := \int_0^\infty S(t)x \, d\mu_n(t/T), \quad x \in X, \tag{7.1.1}$$

where for $n \in \mathbb{N}$, $j \in \mathbb{N}$ we define:

$$\int_0^\infty f(t) \, d\mu_n^{*j}(t/T) := \int_0^\infty f(tT) \, d\mu_n^{*j}(t).$$

By **(M2)**, the right-hand side of (7.1.1) is well defined as a Bochner integral in X. It is an easy consequence of the semigroup property that, for all $j \geq 1$,

$$E(t_j^{(n)})x := [E(\tfrac{T}{n})]^j x = \int_0^\infty S(t)x \, d\mu_n^{*j}(t/T), \quad x \in X. \tag{7.1.2}$$

We supplement these definitions by putting $E(0) := I$.

Example 7.2. In Section 7.1.1 below we will demonstrate that the family of measures

$$d\mu_n(t) = ne^{-nt}\,dt$$

satisfy **(M1)**, **(M2)**, **(M3)**. For these measures we have

$$E(t_j^{(n)})x = (I - \tfrac{T}{n}A)^{-j}x,$$

which means that $E(t_j^{(n)})$ is the j^{th} Euler approximation of $S(t_j^{(n)})$.

The following proposition and corollary give the γ-boundedness estimates for the differences $E(t_j^{(n)}) - S(t_j^{(n)})$ that play the same role in the proof of Theorem 7.10 as the estimates of Lemma 2.21 played in the proof of Theorem 6.2.

Proposition 7.3. *Let the setting be as described above.*

(i) For all $\delta \in (-1, 1]$ there exists a constant C such that for all $n \geq N$:

$$\sup_{j=1,\ldots,n} \big\|(t_j^{(n)})^{1-\delta}[E(t_j^{(n)}) - S(t_j^{(n)})]\big\|_{\mathscr{L}(X_\delta;X)} \leq Cn^{-1}.$$

(ii) For all $\delta \in (-1, 1]$, $0 \leq \beta \leq 1 - \delta$, and $\epsilon > 0$ there exists a constant C such that for all $n \geq N$:

$$\gamma_{[X_\delta,X]}\big\{(t_j^{(n)})^{\beta}[E(t_j^{(n)}) - S(t_j^{(n)})] : j = 1, \ldots, n\big\} \leq Cn^{-\beta-\delta+\epsilon}.$$

Remark 7.4. Stronger γ-boundedness estimates can be obtained by imposing stronger conditions on the measures μ_n. Such conditions would correspond to using higher-order numerical approximation schemes. However, this will not improve the overall convergence rates as provided by Theorem 7.10 because the bottle-neck for convergence rate is the noise discretization.

Remark 7.5. The first part of Proposition 7.3, concerning the uniform boundedness, has been known since the 1970's for the case that $\delta = 0$ and $E(t_j^{(n)})$ is the Euler approximation. Generally such results are proven by functional calculus methods. Our proof may be read as an extension of the approach taken by Bentkus and Paulauskas [8], which is of more probabilistic nature and seems the most suitable for our needs.

Before turning to the proof of Proposition 7.3, we state a simple corollary.

Corollary 7.6. *Let the setting be as described above.*

(i) For all $\delta \in (-1, 0]$ there exists a constant C such that for all $n \geq N$:

$$\sup_{j=1,\ldots,n} \big\|(t_j^{(n)})^{-\delta}E(t_j^{(n)})\big\|_{\mathscr{L}(X_\delta;X)} \leq C.$$

(ii) For all $\delta \in (-1, 0]$, $-\delta < \beta \leq 1-\delta$, and $0 < \epsilon < \beta+\delta$ there exists a constant C such that for all $n \geq N$ and all $k = 1, \dots, n$ we have

$$\gamma_{[X_\delta, X]}\big\{(t_j^{(n)})^\beta E(t_j^{(n)}) : j = 1, \dots, k\big\} \leq C(t_k^{(n)})^{\beta+\delta-\epsilon}.$$

Proof. By the first part of Proposition 7.3, for all $1 \leq j \leq n$

$$\big\|(t_j^{(n)})^{-\delta}\big(E(t_j^{(n)}) - S(t_j^{(n)})\big)\big\|_{\mathscr{L}(X_\delta;X)} \leq Cn^{-1}(t_j^{(n)})^{-1} \lesssim 1.$$

Moreover, by the analyticity of the semigroup S (i.e., estimate (2.6.3)), and the fact that $\delta \leq 0$,

$$\sup_{0 \leq j \leq n} (t_j^{(n)})^{-\delta}\|S(t_j^{(n)})\|_{\mathscr{L}(X_\delta;X)} \lesssim 1.$$

This proves the first part of the corollary.

By part (1) of Lemma 2.21 and part (ii) of Proposition 7.3, observing that $\beta > -\delta \geq 0$, we have:

$$\begin{aligned}
\gamma_{[X_\delta, X]}&\big\{(t_j^{(n)})^\beta E(t_j^{(n)}) \,:\, j = 1, \dots, k\big\} \\
&\lesssim \gamma_{[X_\delta, X]}\big\{(t_j^{(n)})^\beta S(t_j^{(n)}) \,:\, j = 1, \dots, k\big\} \\
&\quad + \gamma_{[X_\delta, X]}\big\{(t_j^{(n)})^\beta\big(E(t_j^{(n)}) - S(t_j^{(n)})\big) \,:\, j = 1, \dots, k\big\} \\
&\lesssim (t_j^{(n)})^{\beta+\delta} + n^{-\beta-\delta+\epsilon} \\
&\lesssim (t_k^{(n)})^{\beta+\delta-\epsilon},
\end{aligned}$$

with implied constants independent of k and n (although they may depend on T). $\square$

In order to prove Proposition 7.3 we shall make use the following simple observation. Suppose μ is a probability measure on $[0, \infty)$, $t = \int_0^\infty s\, d\mu(s)$, and $f : [0, \infty) \to X$ is twice continuously differentiable. By integration by parts one has:

$$\int_0^\infty f(s)\, d\mu(s) - f(t) = \int_0^\infty \int_t^s (s-r)f''(r)\, dr\, d\mu(s). \tag{7.1.3}$$

We substitute $f(s) = S(s)x$, $x \in X$, and $\mu = \mu_n^{*j}$ for $n > N$ and $j \in \{1, \dots, n\}$ in the above. From **(M1)** we have that $\int_0^\infty s\, d\mu_n^{*j}(s/T) = t_j^{(n)}$. Thus by setting $t = t_j^{(n)}$ in (7.1.3) we obtain, for $x \in X$:

$$\begin{aligned}
E(t_j^{(n)})x - S(t_j^{(n)})x &= \int_0^\infty S(s)x\, d\mu_n^{*j}(s/T) - S(t_j^{(n)})x \\
&= \int_0^\infty \int_{t_j^{(n)}}^s (s-r)A^2S(r)x\, dr\, d\mu_n^{*j}(s/T).
\end{aligned} \tag{7.1.4}$$

Proof (of Proposition 7.3.). We first prove the statement on γ-boundedness. Let N be as in assumption **(M2)**. Let $n > N$. Without loss of generality we may assume $\delta-\epsilon \neq 0$ and $\delta-\epsilon \neq -1$. For $j = 1, 2, \ldots, n$ define $\phi_j : [0,\infty)\times[0,\infty) \to \mathbb{R}$ by

$$\phi_j(s,r) := (t_j^{(n)})^{\beta} r^{-2+\delta-\epsilon}(s-r)e^{\omega r}(1_{\{t_j^{(n)}\leq r\leq s\}} - 1_{\{s\leq r\leq t_j^{(n)}\}}).$$

By equality (7.1.4) we have, for $x \in X$:

$$\begin{aligned}(t_j^{(n)})^{\beta}&[E(t_j^{(n)})x - S(t_j^{(n)})x] \\ &= \int_0^\infty \int_0^\infty \phi_j(s,r) r^{2-\delta+\epsilon} e^{-\omega r} A^2 S(r)x\, dr\, d\mu_n^{*j}(s/T),\end{aligned} \tag{7.1.5}$$

$j = 1, 2, \ldots, n$.

In a similar fashion as used for Lemma 2.21 in Section 2.6 one may prove that for $\delta \leq 2$ the set

$$\{r^{2-\delta+\epsilon}e^{-\omega r}A^2 S(r) : r \in [0,\infty)\}$$

is γ-bounded in $\mathscr{L}(X_\delta; X)$. By Proposition 2.13, Lemma 2.11, and equation (7.1.5) it follows that

$$\begin{aligned}\gamma_{[X_\delta,X]}&\Big\{(t_j^{(n)})^{\beta}[E(t_j^{(n)}) - S(t_j^{(n)})] \,:\, j = 1,\ldots,n\Big\} \\ &\lesssim \sup_{1\leq j\leq n} \|\phi_j\|_{L^1([0,\infty)\times[0,\infty),\, \mu_n^{*j}(\cdot/T)\times\lambda)},\end{aligned} \tag{7.1.6}$$

with implied constant independent of n, where λ is the Lebesgue measure on $[0,\infty)$.

Observe that for all $j = 1, 2, \ldots, n$ one has, because $\omega \geq 0$;

$$\begin{aligned}\|\phi_j\|_{L^1([0,\infty)\times[0,\infty),\, \mu_n^{*j}(\cdot/T)\times\lambda)} &= (t_j^{(n)})^{\beta} \int_0^\infty \int_{t_j^{(n)}}^s r^{-2+\delta-\epsilon}(s-r)e^{\omega r}\, dr\, d\mu_n^{*j}(s/T) \\ &\leq (t_j^{(n)})^{\beta} \int_0^\infty e^{(s\vee T)\omega} \int_{t_j^{(n)}}^s r^{-2+\delta-\epsilon}(s-r)\, dr\, d\mu_n^{*j}(s/T).\end{aligned}$$

As $\delta - \epsilon \neq 0$ and $\delta - \epsilon \neq -1$, basic calculus gives:

$$\int_{t_j^{(n)}}^s r^{-2+\delta-\epsilon}(s-r)\, dr = \frac{(t_j^{(n)})^{\delta-\epsilon}}{1-\delta+\epsilon}\Big[\frac{1}{\delta-\epsilon}\Big(1 - \Big(\frac{s}{t_j^{(n)}}\Big)^{\delta-\epsilon}\Big) + \frac{s}{t_j^{(n)}} - 1\Big]. \tag{7.1.7}$$

In the final estimate below we apply assumption **(M3)** (recall that we have $R = \omega T$). Due to that assumption there exists a constant C independent of n and $j \in \{1,\ldots,n\}$ such that:

$$
\begin{aligned}
&\|\phi_j\|_{L^1([0,\infty)\times[0,\infty),\,\mu_n^{*j}(\cdot/T)\times\lambda)} \\
&\leq \frac{T^{\delta-\epsilon}}{1-\delta+\epsilon}(t_j^{(n)})^{\beta+\delta-\epsilon}\int_0^\infty e^{(s\vee T)\omega}\big[\tfrac{1}{\delta-\epsilon}\big(1-\big(\tfrac{s}{t_j^{(n)}}\big)^{\delta-\epsilon}\big)+\tfrac{s}{t_j^{(n)}}-1\big]\,d\mu_n^{*j}(s/T) \\
&= \frac{T^{\delta-\epsilon}}{1-\delta+\epsilon}(t_j^{(n)})^{\beta+\delta-\epsilon}\int_0^\infty e^{(s\vee 1)\omega T}\big[\tfrac{1}{\delta-\epsilon}((\tfrac{sn}{j})^{\delta-\epsilon}-1)+\tfrac{sn}{j}-1\big]\,d\mu_n^{*j}(s) \\
&\lesssim (t_j^{(n)})^{\beta+\delta-\epsilon}j^{-1}\lesssim n^{-\beta-\delta+\epsilon}.
\end{aligned} \tag{7.1.8}
$$

This in combination with estimate (7.1.6) completes the proof, as $\beta+\delta-\epsilon<1$.

As for the statement concerning uniform boundedness, by (7.1.4) and analyticity (estimate (2.6.3)) we have, for $\delta\neq 0$:

$$
\begin{aligned}
\big(t_j^{(n)}\big)^{1-\delta}&\|E(t_j^{(n)})-S(t_j^{(n)})\|_{\mathscr{L}(X_\delta;X)} \\
&\lesssim \big(t_j^{(n)}\big)^{1-\delta}\int_0^\infty\int_{t_j^{(n)}}^s (s-r)\|A^{2-\delta}S(r)\|_{\mathscr{L}(X)}\,dr\,d\mu_n^{*j}(s/T) \\
&\lesssim \big(t_j^{(n)}\big)^{1-\delta}\int_0^\infty\int_{t_j^{(n)}}^s r^{-2+\delta}(s-r)e^{\omega r}\,dr\,d\mu_n^{*j}(s/T) \\
&\lesssim n^{-1},
\end{aligned}
$$

where the final estimate follows by similar arguments as used to obtain (7.1.8). In the case that $\delta=0$ or $\delta=1$ the evaluation of the integral in (7.1.7) will contain a logarithmic term, which can be estimated in a suitable manner by observing that $\ln x\leq x-1$ for all $x>0$. We leave the details to the reader.

□

7.1.1 Examples

We have two main examples in mind, which lead to a splitting scheme with discretized noise and the implicit Euler scheme, respectively.

Example 7.7 (Splitting with discretized noise). The simplest example obtained by taking

$$\mu_n := \delta_{\frac{1}{n}},$$

which correspond to the trivial choice

$$E(\tfrac{T}{n}) := S(\tfrac{T}{n}).$$

The conditions **(M1)**, **(M2)**, **(M3)** are trivially fulfilled for any $R\geq 0$.

Example 7.8 (Implicit Euler). We will show that the measures

$$d\mu_n(t) = ne^{-nt}\,dt$$

fulfill assumptions **(M1)**, **(M2)**, **(M3)** for any $R\geq 0$. These measures give rise to the operators

$$E(\tfrac{T}{n}) = (I - \tfrac{T}{n}A)^{-1}.$$

To start the proof, first note that by induction,

$$d\mu_n^{*j}(t) = \frac{(nt)^{j-1}}{(j-1)!} n e^{-nt}\, dt,$$

and thus, for all $\alpha > -j$ and all $n > \omega T$, one has:

$$\begin{aligned}\int_0^\infty t^\alpha e^{\omega T t}\, d\mu_n^{*j}(t) &= \frac{n^j}{(j-1)!(n-\omega T)^{j+\alpha}} \int_0^\infty u^{j+\alpha-1} e^{-u}\, du \\ &= \frac{n^j}{(n-\omega T)^{j+\alpha}} \frac{\Gamma(j+\alpha)}{\Gamma(j)}.\end{aligned} \tag{7.1.9}$$

This proves that **(M1)** and **(M2)** are satisfied with $N > \omega T$.

As for **(M3)**, by (7.1.9) we have for $\alpha \in (-1, 1]$ and $n \geq N \geq 2\omega T$:

$$\begin{aligned}&\int_0^\infty \big[1 - \big(\tfrac{tn}{j}\big)^\alpha\big] e^{\omega T t}\, d\mu_n^{*j}(t) \\ &\quad = \Big(\frac{n}{n-\omega T}\Big)^j \Big(1 - \Big(\frac{n}{n-\omega T}\Big)^\alpha\Big) + \Big(\frac{n}{n-\omega T}\Big)^{j+\alpha} \Big[1 - \frac{\Gamma(j+\alpha)}{j^\alpha \Gamma(j)}\Big].\end{aligned} \tag{7.1.10}$$

As $(\frac{n}{n-\omega T})^{n+1} \to e^{\omega T}$ as $n \to \infty$, there exists a constant M such that

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,n+1]} \Big|\Big(\frac{n}{n-\omega T}\Big)^s\Big| = \sup_{n\in\mathbb{N}} \Big(\frac{n}{n-\omega T}\Big)^{n+1} \leq M.$$

Moreover, for $n \geq 2\omega T$ we have:

$$\Big|1 - \Big(\frac{n}{n-\omega T}\Big)^\alpha\Big| = |\alpha| \int_1^{1+\frac{\omega T}{n-\omega T}} s^{\alpha-1}\, ds \leq |\alpha| \frac{\omega T}{n-\omega T} \leq 2|\alpha| \frac{\omega T}{n}.$$

From (7.1.10) and the above estimates we thus obtain:

$$\Big|\int_0^\infty \big[1 - \big(\tfrac{tn}{j}\big)^\alpha\big] e^{\omega T t}\, d\mu_n^{*j}(t)\Big| \leq 2M|\alpha| \frac{\omega T}{n} + M\Big[1 - \frac{\Gamma(j+\alpha)}{j^\alpha \Gamma(j)}\Big]. \tag{7.1.11}$$

For $j \geq 2$ define $g_j : [-1, 1] \to \mathbb{R}$ by:

$$g_j(x) = \begin{cases} \dfrac{1}{x}\Big(1 - \dfrac{\Gamma(j+x)}{j^x \Gamma(j)}\Big); & x \neq 0, \\ \ln j - \Psi(j); & x = 0, \end{cases}$$

where $\Psi = (\ln(\Gamma))'$ is the di-gamma function.

Assumption **(M3)** follows from (7.1.11) for $N \geq 2\omega T$ once the following claim is established:

Claim. *For $j \geq 2$ we have $g_j(x) \in [0, \frac{1}{j-1}]$ for all $x \in [-1, 1]$.*

Proof (of Claim). As $g_j(-1) = \frac{1}{j-1}$ and $g_j(1) = 0$ for all $j \geq 2$, it suffices to show that g_j is non-increasing on $[-1, 1]$.

For $j \geq 2$ define the function $h_j : [-1, 1] \to \mathbb{R}$ by $h_j(x) := 1 - j^{-x}\frac{\Gamma(j+x)}{\Gamma(j)}$. For $x \in [-1, 1]$ and $j \geq 2$ we have:

$$\begin{aligned} h_j'(x) &= \frac{\Gamma(j+x)}{j^x\Gamma(j)}\big[\ln j - \Psi(j+x)\big] = (1 - h_j(x))(\ln j - \Psi(j+x)); \\ h_j''(x) &= \frac{-\Gamma(j+x)}{j^x\Gamma(j)}\big[\big(\Psi(j+x) - \ln j\big)^2 + \Psi'(j+x)\big]. \end{aligned}$$

As the Γ-function is log-convex on $(0, \infty)$, we have that Ψ' is positive on that interval. As $j \geq 2$ and $x \in [-1, 1]$ we have that $j + x > 0$ and thus $h_j''(x) \leq 0$ for $x \in [-1, 1]$.

One may check that g_j is continuously differentiable and

$$g_j'(x) = \begin{cases} \dfrac{1}{x^2}(xh_j'(x) - h_j(x)); & x \neq 0, \\ -\frac{1}{2}\big[(\Psi(j) - \ln j)^2 + \Psi'(j)\big]; & x = 0. \end{cases}$$

To prove that g_j is non-increasing on $[-1, 1]$ it suffices to prove that

$$xh_j'(x) - h_j(x) \leq 0, \text{ for all } x \in [-1, 1].$$

Observe that $g_j'(0) \leq 0$, hence it suffices to prove that $x \mapsto xh_j'(x) - h_j(x)$ is non-decreasing on $[-1, 0]$ and non-increasing on $[0, 1]$. This follows from the fact that $\frac{d}{dx}[xh_j'(x) - h_j(x)] = xh_j''(x)$ and $h_j'' \leq 0$ on $[-1, 1]$. $\square$

7.2 An abstract time discretization

In this section we prove a convergence result for a general class of approximation schemes for (SDE) involving the operators $E(t_j^{(n)})$ as defined in (7.1.1) and discretized noise. In particular, the convergence result contains the implicit Euler scheme as the special case that $E(t_j^{(n)}) = (I - \frac{T}{n}A)^{-j}$ (see Example 7.8).

Throughout this section we consider the problem (SDE) under the assumptions **(A)**, **(F)**, **(G)**. On the part of X we shall assume that it is an UMD space with Pisier's property (α).

Set:

$$\zeta_{\max} := \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + (\theta_F \wedge 0), \tfrac{1}{2} + (\theta_G \wedge 0)\}, \tag{7.2.1}$$

where $\tau \in (1, 2]$ is the type of X. In addition to the assumptions **(F)**, **(G)** we shall assume:

(F′) There exists a constant C such that for all $x \in X$ and $s, t \in [0, T]$ we have:

$$\|F(t, x) - F(s, x)\|_{X_{\theta_F \wedge 0}} \leq C|t - s|^{\zeta_{\max}}(1 + \|x\|_X).$$

(G′) We have $G : [0,T] \times X \to \gamma(H, X_{\theta_G \wedge 0})$ and there exists a constant C such that for all $x \in X$ and $s,t \in [0,T]$ we have:

$$\|G(t,x) - G(s,x)\|_{\gamma(H,X_{\theta_G \wedge 0})} \le C|t-s|^{\zeta_{\max}+\frac{1}{\tau}-\frac{1}{2}}(1+\|x\|_X).$$

Remark 7.9. Clearly, condition **(F′)** is automatically satisfied if F is not time-dependent and satisfies **(F)**.

Condition **(G′)** is also automatically satisfied if G is not time-dependent and satisfies **(G)**. Indeed, in that case, from [109, Lemma 5.3] it follows that G takes values in $\gamma(H, X_{\theta_G \wedge 0})$ and the linear growth condition of **(G′)** is satisfied.

The reader will have noticed that the above assumptions are phrased in terms of $\theta_F \wedge 0$ and $\theta_G \wedge 0$. The reason for this is explained in Remark 7.11 below. Because of this, *for the rest of this section, without loss of generality we shall assume that* $\theta_F, \theta_G \ge 0$. The other assumptions on θ_F and θ_G remain in force. Explicitly, we assume

$$-1 + (\tfrac{1}{\tau} - \tfrac{1}{2}) < \theta_F \le 0, \quad -\tfrac{1}{2} < \theta_G \le 0.$$

Once this convention is in force, of course one has $\zeta_{\max} = \eta_{\max}$. In order to remind the reader of the convention, we shall continue the use of $\zeta_{\max}$.

Let us now introduce the discrete-time approximation scheme that will be studied in this section. Fix $T > 0$ and let $(\mu_n)_{n=1}^\infty$ be a family of measures satisfying **(M1)**, **(M2)**, **(M3)** for $R = \omega T$, where $\omega \ge 0$ is such that $e^{-\omega t}S(t)$ is uniformly bounded in $t \in [0,\infty)$. Let $E(t_j^{(n)})$ be defined by (7.1.2). We fix $p > 2$ and let U be the mild solution to (SDE) with initial value $x_0 \in L^p(\Omega, \mathscr{F}_0; X)$. We fix another initial value $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$ (in the applications below, the typical situation is that y_0 is a close approximation to x_0). Let $n \ge N$, where N is as in **(M2)**. Set $V_0^{(n)} := y_0$ and define $V_j^{(n)}$, $j = 1, \dots, n$, inductively as follows:

$$V_j^{(n)} := E(\tfrac{T}{n})[V_{j-1}^{(n)} + \tfrac{T}{n} F(t_{j-1}^{(n)}, V_{j-1}^{(n)}) + G(t_{j-1}^{(n)}, V_{j-1}^{(n)})\Delta W_j^{(n)}]. \tag{7.2.2}$$

Here,

$$\Delta W_j^{(n)} := W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)}).$$

The rigorous interpretation of the term $G(t_{j-1}^{(n)}, V_{j-1}^{(n)})\Delta W_j^{(n)}$ proceeds in three steps.

Step 1: Let us first fix an operator $R \in \gamma(H, X_{\theta_G})$. By standard results on γ-radonifying operators (see, e.g. [102]) may write

$$R = \sum_{k=1}^\infty h_k \otimes x_k$$

for some orthonormal sequence $(h_k)_{k=1}^\infty$ in H and a sequence $(x_k)_{k=1}^\infty$ in X_{θ_G} (the convergence of the sum being in $\gamma(H, X_{\theta_G})$). For sets $B \in \mathscr{F}_{j-1}^{(n)} := \mathscr{F}_{t_j^{(n)}}$ we now define

$$(1_B \otimes R)\Delta W_j^{(n)} := 1_B \sum_{k=1}^{\infty} W_H(h_k \otimes 1_{(t_{j-1}^{(n)}, t_j^{(n)}]}) \otimes x_k. \tag{7.2.3}$$

The sum on the right-hand side above converges in $L^p(\Omega; X_{\theta_G})$ since W_H extends to a bounded operator from $\gamma(L^2(0,T;H); X_{\theta_G})$ into $L^p(\Omega; X_{\theta_G})$ (see [102]). By the independence of $W_H(h_k \otimes 1_{(t_{j-1}^{(n)}, t_j^{(n)}]})$ and $\mathscr{F}_{j-1}^{(n)}$, the product of 1_B and this sum converges in $L^p(\Omega; X_{\theta_G})$ as well. Moreover, by the Kahane-Khintchine inequality,

$$\begin{aligned}
&\|(1_B \otimes R)\Delta W_j^{(n)}\|_{L^p(\Omega; X_{\theta_G})} \\
&\qquad \lesssim (\mathbb{E}(1_B))^{\frac{1}{p}} \Big\| \sum_{k=1}^{\infty} (h_k \otimes 1_{(t_{j-1}^{(n)}, t_j^{(n)}]}) \otimes x_k \Big\|_{\gamma(L^2(0,T;H); X_{\theta_G})} \\
&\qquad = (t_j^{(n)} - t_{j-1}^{(n)})^{\frac{1}{2}} (\mathbb{E}(1_B))^{\frac{1}{p}} \Big\| \sum_{k=1}^{\infty} h_k \otimes x_k \Big\|_{\gamma(H, X_{\theta_G})} \\
&\qquad = (\tfrac{T}{n})^{\frac{1}{2}} (\mathbb{E}(1_B))^{\frac{1}{p}} \|R\|_{\gamma(H, X_{\theta_G})}
\end{aligned}$$

with implied constants depending on p only.

Step 2: Now fix a simple random variable $\phi \in L^p(\Omega, \mathscr{F}_{j-1}^{(n)}; \gamma(H, X_{\theta_G}))$, say $\phi = \sum_{j=1}^{k} 1_{B_j} \otimes R_j$ with the sets $B_j \in \mathscr{F}_{j-1}^{(n)}$ disjoint. By the above,

$$\begin{aligned}
\|\phi \Delta W_j^{(n)}\|_{L^p(\Omega; X_{\theta_G})} &\lesssim (\tfrac{T}{n})^{\frac{1}{2}} \sum_{j=1}^{k} (\mathbb{E}(1_{B_j}))^{\frac{1}{p}} \|R_j\|_{\gamma(H, X_{\theta_G})} \\
&= (\tfrac{T}{n})^{\frac{1}{2}} \|\phi\|_{L^p(\Omega; \gamma(H, X_{\theta_G}))}.
\end{aligned} \tag{7.2.4}$$

By density, the above estimate holds for all $\phi \in L^p(\Omega, \mathscr{F}_{j-1}^{(n)}; \gamma(H, X_{\theta_G \wedge 0}))$.

Step 3: It remains to prove that $G(t_j^{(n)}, V_j^{(n)}) \in L^p(\Omega, \mathscr{F}_{j-1}^{(n)}; \gamma(H, X_{\theta_G \wedge 0}))$. This follows from $\mathbf{(G')}$ by the following argument:

$$\begin{aligned}
&\|G(t_j^{(n)}, V_j^{(n)})\|_{\gamma(H, X_{\theta_G \wedge 0})} \\
&\qquad \le \|G(t_j^{(n)}, V_j^{(n)}) - G(t_j^{(n)}, 0)\|_{\gamma(H, X_{\theta_G \wedge 0})} + \|G(t_j^{(n)}, 0)\|_{\gamma(H, X_{\theta_G \wedge 0})} \\
&\qquad \le (t_j^{(n)})^{\eta_{\max}} \|V_j^{(n)}\|_X + \|G(t_j^{(n)}, 0)\|_{\gamma(H, X_{\theta_G \wedge 0})} \quad a.s.
\end{aligned}$$

Returning to the abstract scheme (7.2.2), we have the following explicit expression for $V_j^{(n)}$:

$$\begin{aligned}
V_j^{(n)} = E(t_j^{(n)}) y_0 &+ \tfrac{T}{n} \sum_{k=1}^{j} E(t_{j-k+1}^{(n)}) F(t_{k-1}^{(n)}, V_{k-1}^{(n)}) \\
&+ \sum_{k=0}^{j} E(t_{j-k+1}^{(n)}) G(t_{k-1}^{(n)}, V_{k-1}^{(n)}) \Delta W_k^{(n)}, \quad j = 0, \ldots, n.
\end{aligned}$$

We define, for $s \in [0,T]$,

$$V^{(n)}(s) = \sum_{j=1}^{n} V_j^{(n)} 1_{I_j}(s), \tag{7.2.5}$$

where, as always, $I_j = [t_{j-1}^{(n)}, t_j^{(n)})$. Observing that $V^{(n)}(s) = V^{(n)}(\underline{s})$, this process satisfies the identity

$$\begin{aligned} V^{(n)}(s) = E(\overline{s})y_0 + \int_0^{\overline{s}} E(\overline{s} - \underline{u})F(\underline{u}, V^{(n)}(u))\, du \\ + \int_0^{\overline{s}} E(\overline{s} - \underline{u})G(\underline{u}, V^{(n)}(u))\, dW_H(u). \end{aligned} \tag{7.2.6}$$

The main result of this section reads as follows. Recall the definition of $\zeta_{\max}$ in (7.2.1) and let N be as in **(M2)**.

Theorem 7.10. *Let $\eta > 0$ be such that $0 \le \eta < \zeta_{\max}$. Then for all $p \in (2,\infty)$ and $\alpha \in [0, \frac{1}{2})$ there exists a constant C such that for all $x_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$, $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$, and $n \ge N$ we have*

$$\|U - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;X)} \le C\|x_0 - y_0\|_{L^p(\Omega;X)} + Cn^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)}). \tag{7.2.7}$$

Remark 7.11. Unlike the case in Theorem 6.2, the convergence rate of the Euler approximations does not improve if θ_F and θ_G increase above 0. This is mainly due to the time-discretization of the noise. More precisely, the estimates on the sixth and tenth term in (7.2.11) below do not improve if θ_F and θ_G increase above 0. The estimate on third term in (7.2.11) as presented here also does not improve if θ_F increases above 0, but we believe this is just an artifact of our proof.

Remark 7.12. The Hölder conditions of **(F′)** and **(G′)** can be weakened: in order to obtain convergence rate η in Theorem 7.10 it suffices that the Hölder exponent in **(F′)** is η instead of $\zeta_{\max}$, and that the exponent in **(G′)** is $\eta + \frac{1}{\tau} - \frac{1}{2}$ instead of $\zeta_{\max} + \frac{1}{\tau} - \frac{1}{2}$.

Remark 7.13. Given the above theorem, the proof of Theorem 7.1 is entirely analogous to the proof of Theorem 6.1 as presented in Section 6.3. In fact, it is clear that Theorem 7.1 holds for the more general case that $(V_j^{(n)})_{j=0}^n$ is defined by the abstract scheme described above, where $E(t_j^{(n)})$ is defined in terms of a family of measures $(\mu_n)_{n\ge\mathbb{N}}$ satisfying **(M1)**, **(M2)**, and **(M3)**.

By the same Borel-Cantelli argument used to obtain Corollary 6.6 from Theorem 6.1 we obtain the following from Theorem 7.1:

Corollary 7.14. *Let $\gamma, \delta \geq 0$, $\eta > 0$, and $p \in (2, \infty)$ be such that $\gamma + \delta + \frac{2}{p} < \min\{\zeta_{\max}, \eta\}$. Suppose that $x_0 = y_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta)$. Then there exists a random variable $\chi \in L^0(\Omega)$, independent of n such that for $n \geq N$:*

$$\|u - v^{(n)}\|_{c_\gamma^{(n)}([0,T];X)} \leq \chi n^{-\delta}. \tag{7.2.8}$$

Remark 7.15. As in the case of Corollary 6.6, we will see in Chapter 8, Corollary 8.2, that in fact (7.2.8) holds for $\gamma + \delta < \min\{\zeta_{\max}, \eta\}$ provided $x_0 = y_0 \in L^0(\Omega, \mathscr{F}_0; X_\eta)$.

Proof (of Theorem 7.10). We begin by observing that, due to the assumption $2 < p < \infty$, the spaces X and $L^p(\Omega; X)$ have the same type τ.

Part 1. The main issue is to prove that there exists $T_0 \in (0, T]$ and a constant C such that for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, n\}$ we have:

$$\begin{aligned}&\|U - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)} \\ &\qquad \leq C\|U(t_j^{(n)}) - V_j^{(n)}\|_{L^p(\Omega;X)} + Cn^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega;X_\eta)}).\end{aligned}$$

This statement is entirely analogous to the result obtained in part 3 of the proof of Theorem 6.2. Once it has been established, the extension to the interval $[0, T]$ can be obtained in precisely the same way as in part 4 of Theorem 6.2.

For $n \in \mathbb{N}$ let $(U_j^{(n)})_{j=1}^n$ be the modified splitting scheme as defined by (6.0.4) on page 88, with initial value x_0. Let $U^{(n)}$ be the corresponding process as defined by (6.0.2) on page 87. By Theorem 6.2 we have, for all $j \in \{1, \ldots, n\}$ and $T_0 \in (0, T]$:

$$\begin{aligned}&\|U - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)} \\ &\quad \leq \|U - U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)} + \|U^{(n)} - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)} \\ &\quad \lesssim n^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega;X_\eta)}) + \|U^{(n)} - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)},\end{aligned} \tag{7.2.9}$$

with implied constants independent of n and j. Thus it suffices to show that there exists a constant C such that for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, n\}$ we have:

$$\begin{aligned}&\|U^{(n)} - V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([t_j^{(n)}, t_j^{(n)}+T_0]\times\Omega;X)} \\ &\qquad \leq C\|U(t_j^{(n)}) - V_j^{(n)}\|_{L^p(\Omega;X)} + Cn^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega;X_\eta)}).\end{aligned} \tag{7.2.10}$$

Part 2. For simplicity we shall prove this for $j = 0$ (careful examination of the proof reveals that the other $t_j^{(n)}$ do not generate extra difficulties). In that case we have $U^{(n)}(t_j^{(n)}) = U(0) = x_0$ and $V_j^{(n)} = V_0^{(n)} = y_0$.

Until further notice we fix $n \geq N$ and $T_0 \in [0, T]$. From the integral representations (6.1.2) and (7.2.6) we have:

$$
\begin{aligned}
U^{(n)}(t) - V^{(n)}(t) &= (S(\bar{t}) - E(\bar{t}))x_0 + E(\bar{t})(x_0 - y_0) \\
&\quad + \int_0^{\bar{t}} [S(\bar{t} - \underline{s}) - E(\bar{t} - \underline{s})]F(s, U^{(n)}(s))\,ds \\
&\quad + \int_0^{\bar{t}} E(\bar{t} - \underline{s})[F(s, U^{(n)}(s)) - F(s, V^{(n)}(s))]\,ds \\
&\quad + \int_0^{\bar{t}} E(\bar{t} - \underline{s})[F(s, V^{(n)}(s)) - F(\underline{s}, V^{(n)}(s))]\,ds \\
&\quad + \int_t^{\bar{t}} S(\tfrac{T}{n})F(s, U^{(n)}(s))\,ds \\
&\quad + \int_0^{\bar{t}} [S(\bar{t} - \underline{s}) - E(\bar{t} - \underline{s})]G(s, U^{(n)}(s))\,dW_H(s) \\
&\quad + \int_0^{t} E(\bar{t} - \underline{s})[G(s, U^{(n)}(s)) - G(s, V^{(n)}(s))]\,dW_H(s) \\
&\quad + \int_0^{t} E(\bar{t} - \underline{s})[G(s, V^{(n)}(s)) - G(\underline{s}, V^{(n)}(s))]\,dW_H(s) \\
&\quad + \int_t^{\bar{t}} S(\tfrac{T}{n})G(s, U^{(n)}(s))\,dW_H(s).
\end{aligned} \tag{7.2.11}
$$

We shall estimate each of the ten terms on the right-hand side above separately. The implied constants in these estimates may depend on T, although this will not be stated explicitly. However, for the fourth, fifth, eighth and ninth term (part 2d and 2g below) it will be necessary to keep track of the dependence upon T_0.

Without loss of generality we may assume that $\tau \in (1, 2)$. We fix $0 < \varepsilon < \frac{1}{2}$ such that

$$\varepsilon < \min\{\zeta_{\max} - \eta, 1 - 2\alpha\}, \tag{7.2.12}$$

where $\zeta_{\max}$ is defined as in (7.2.1). As $\mathscr{V}^{\alpha,p}_{\infty} \hookrightarrow \mathscr{V}^{\beta,p}_{\infty}$ for $\alpha > \beta$, we may also assume that

$$\tfrac{1}{2} - \tfrac{2}{3}\varepsilon < \alpha < \tfrac{1}{2} - \tfrac{1}{2}\varepsilon. \tag{7.2.13}$$

Part 2a. For the first term on the right-hand side of (7.2.11) we have, by the uniform boundedness estimate of Proposition 7.3 with $\delta = \eta$, pointwise in $\omega \in \Omega$:

$$\|s \mapsto (S(\overline{s}) - E(\overline{s}))x_0\|_{L^\infty(0,T_0;X)} \lesssim n^{-1} \sup_{1 \le j \le n} (t_j^{(n)})^{1-\eta} \|x_0\|_{X_\eta} \lesssim n^{-\eta} \|x_0\|_{X_\eta}. \tag{7.2.14}$$

Let $t \in [0, T_0]$. By the γ-boundedness result of Proposition 7.3 with $\beta = \epsilon = \frac{1}{2}\varepsilon$ and $\delta = \eta$, the γ-multiplier theorem (Theorem 2.14), and (2.3.3) we have, pointwise in $\omega \in \Omega$:

$$\|s \mapsto (t-s)^{-\alpha}(S(\overline{s}) - E(\overline{s}))x_0\|_{\gamma(0,t;X)} \lesssim n^{-\eta} \|s \mapsto (t-s)^{-\alpha}\overline{s}^{-\frac{1}{2}\varepsilon} x_0\|_{\gamma(0,t;X_\eta)}$$

$$
\begin{aligned}
&= n^{-\eta}\|s \mapsto (t-s)^{-\alpha}\overline{s}^{-\frac{1}{2}\varepsilon}\|_{L^2(0,t)}\|x_0\|_{X_\eta} \\
&\eqsim n^{-\eta} t^{\frac{1}{2}-\alpha-\frac{1}{2}\varepsilon}\|x_0\|_{X_\eta},
\end{aligned}
$$

with implied constants independent of n, T_0 and x_0. As $\frac{1}{2}-\alpha-\frac{1}{2}\varepsilon > 0$, we have $t^{\frac{1}{2}-\alpha-\frac{1}{2}\varepsilon} \leq T^{\frac{1}{2}-\alpha-\frac{1}{2}\varepsilon}$. By taking the supremum over $t \in [0,T_0]$ in the above, combining the result with (7.2.14), and then taking p^{th} moments, one obtains:

$$
\|s \mapsto (S(\overline{s}) - E(\overline{s}))x_0\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim n^{-\eta}\|x_0\|_{L^p(\Omega;X_\eta)}, \tag{7.2.15}
$$

with implied constants independent of n, T_0, and x_0.

Part 2b. Concerning the second term on the right-hand side of (7.2.11) recall that as A generates an analytic C_0-semigroup, there exists a constant M such that

$$
\sup_{n\geq N}\sup_{k\in\{1,\ldots,n\}} \|E(t_k^{(n)})\|_{\mathscr{L}(X)} \leq M.
$$

Thus pointwise in $\omega \in \Omega$ we have:

$$
\|s \mapsto E(\overline{s})(x_0 - y_0)\|_{L^\infty(0,T_0;X)} \lesssim \|x_0 - y_0\|_X.
$$

Let $t \in [0,T_0]$. We apply the second part of Corollary 7.6 with $\beta = \frac{1}{2}\varepsilon$, $\delta = 0$, $\epsilon = \frac{1}{4}\varepsilon$, $k = n$. Arguing as in the previous estimate we obtain:

$$
\begin{aligned}
\|s \mapsto (t-s)^{-\alpha}E(\overline{s})(x_0 - y_0)\|_{\gamma(0,t;X)} &\lesssim \|s \mapsto (t-s)^{-\alpha}\overline{s}^{-\frac{1}{2}\varepsilon}\|_{L^2(0,t)}\|x_0 - y_0\|_X \\
&\lesssim \|x_0 - y_0\|_X,
\end{aligned}
$$

with implied constants independent of n, T_0, x_0 and y_0.

As $t \in [0,T_0]$ was arbitrary, by taking p^{th} moments it follows that:

$$
\|s \mapsto E(\overline{s})(x_0 - y_0)\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} \tag{7.2.16}
$$

with implied constants independent of n, T_0, x_0, and y_0.

Part 2c. By the uniform boundedness estimate of Proposition 7.3 with $\delta = \theta_F$ one has, for $s \in [0,T_0]$ fixed:

$$
\begin{aligned}
&\Big\| \int_0^{\overline{s}} [S(\overline{s} - \underline{u}) - E(\overline{s} - \underline{u})]F(u, U^{(n)}(u))\,du \Big\|_{L^p(\Omega;X)} \\
&\qquad \leq \int_0^{\overline{s}} \|[S(\overline{s} - \underline{u}) - E(\overline{s} - \underline{u})]F(u, U^{(n)}(u))\|_{L^p(\Omega;X)}\,du \\
&\qquad \lesssim \int_0^{\overline{s}} n^{-1}(\overline{s} - \underline{u})^{-1+\theta_F}\,du \|F(\cdot, U^{(n)})\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))} \\
&\qquad \leq \frac{1}{n}\sum_{j=1}^n (T/n)\cdot(jT/n)^{-1+\theta_F}\|F(\cdot, U^{(n)})\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}
\end{aligned}
$$

$$\lesssim n^{-1-\theta_F} \sum_{j=1}^{n} j^{-1+\theta_F} (1 + \|U^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))})$$
$$\lesssim n^{-1-\theta_F+\frac{1}{2}\varepsilon}(1 + \|x_0\|_{L^p(\Omega;X)}),$$

where we used the linear growth condition in **(F)** and that $\sum_{j=1}^n j^{-1+\theta_F} \lesssim 1$ if $\theta_F < 0$ and $\sum_{j=1}^n j^{-1+\theta_F} \lesssim \ln n \lesssim n^{\frac{1}{2}\varepsilon}$ if $\theta_F = 0$. In the last step we used Corollary 6.3 with $\delta = 0$. The implied constants are independent of n, T_0, and x_0.

By taking the supremum over $s \in [0, T_0]$ we obtain:

$$\Big\| s \mapsto \int_0^{\overline{s}} [S(\overline{s} - \underline{u}) - E(\overline{s} - \underline{u})] F(u, U^{(n)}(u))\, du \Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \lesssim n^{-1-\theta_F+\frac{1}{2}\varepsilon}(1 + \|x_0\|_{L^p(\Omega;X)}). \tag{7.2.17}$$

For the estimate in the weighted γ-space we shall use Lemma A.3. Define $\Psi : [0, T_0] \to L^p(\Omega; X)$ by

$$\Psi(s) = \int_0^{\overline{s}} [S(\overline{s} - \underline{u}) - E(\overline{s} - \underline{u})] F(u, U^{(n)}(u))\, du.$$

Let $t \in [0, T_0]$ and let $q = (\frac{1}{\tau} - \frac{1}{2} + \frac{1}{2}\varepsilon)^{-1}$ (so $\frac{1}{\tau} - \frac{1}{2} < \frac{1}{q} < \frac{1}{\tau} - \alpha$). By Lemma A.3 we have:

$$\sup_{t\in[0,T_0]} \|s \mapsto (t - s)^{-\alpha}\Psi(s)\|_{\gamma(0,t;L^p(\Omega;X))} \lesssim \|\Psi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}([0,T_0];L^p(\Omega;X))} + \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))}, \tag{7.2.18}$$

with implied constant independent of T_0.

Let $\rho \in [0, 1]$ and let $0 < |h| < \rho$. We have, with $I = [0, T_0]$,

$$\|T_h^I \Psi(s) - \Psi(s)\|_{L^p(\Omega;X)} \le \begin{cases} 0, & \overline{s+h} = \overline{s}, s + h \in [0, T_0]; \\ 2\|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))}, & \overline{s+h} \ne \overline{s} \text{ or } s + h \notin [0, T_0]. \end{cases}$$

Suppose $|h| < \frac{T}{n}$. Define $I_h = \{s \in [0, T_0] : \overline{s+h} \ne \overline{s}\}$ and observe that $|I_h| \le n|h|$. Thus by the definition of q and by (7.2.17):

$$\begin{aligned}\|T_h^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))} &\lesssim (n|h|)^{\frac{1}{q}} \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\ &\lesssim |h|^{\frac{1}{\tau}-\frac{1}{2}+\frac{1}{2}\varepsilon} n^{-\frac{3}{2}+\frac{1}{\tau}-\theta_F+\varepsilon}(1 + \|x_0\|_{L^p(\Omega;X)}).\end{aligned}$$

On the other hand, if $|h| \ge \frac{T}{n}$ then we have:

$$\begin{aligned}\|T_h^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))} &\lesssim \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\ &\lesssim n^{-1-\theta_F+\frac{1}{2}\varepsilon}(1 + \|x_0\|_{L^p(\Omega;X)}) \\ &= |h|^{\frac{1}{\tau}-\frac{1}{2}+\frac{1}{2}\varepsilon} n^{-\frac{3}{2}+\frac{1}{\tau}-\theta_F+\varepsilon}(1 + \|x_0\|_{L^p(\Omega;X)}).\end{aligned}$$

Combining the two cases and recalling that $\eta < \frac{3}{2} - \frac{1}{\tau} + \theta_F - \varepsilon$ by (7.2.12) we obtain:

$$\sup_{0<|h|\leq\rho} \|T_h^I \Psi(s) - \Psi(s)\|_{L^q(0,T_0;L^p(\Omega;X))} \leq \rho^{\frac{1}{\tau}-\frac{1}{2}+\frac{1}{2}\varepsilon} n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}).$$

By the definition of $B_{q,\tau}^{\frac{1}{\tau}-\frac{1}{2}}$ and equation (7.2.17), this gives:

$$\begin{aligned}
&\|\Psi\|_{B_{q,\tau}^{\frac{1}{\tau}-\frac{1}{2}}([0,T_0];L^p(\Omega;X))} \\
&\qquad \lesssim \|\Psi\|_{L^q(0,T_0;L^p(\Omega;X))} + n^{-\eta}\int_0^1 \rho^{\frac{1}{2}\varepsilon\tau - 1} d\rho (1 + \|x_0\|_{L^p(\Omega,X)}) \\
&\qquad \lesssim \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega,X)}) \\
&\qquad \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}),
\end{aligned}$$

with implied constant independent of n, T_0, and x_0. Inserting the above and (7.2.17) into (7.2.18) we obtain:

$$\sup_{t\in[0,T_0]} \|s \mapsto (t-s)^{-\alpha}\Psi(s)\|_{\gamma(0,t;L^p(\Omega;X))} \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}). \tag{7.2.19}$$

Finally, by combining (7.2.17) and (7.2.19) we obtain that

$$\|\Psi\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}), \tag{7.2.20}$$

with implied constants independent of n, T_0, and x_0.

Part 2d. In this part we provide estimates for the fourth and fifth term in (7.2.11). In order to do so, we shall prove that there exists an $\varepsilon_1 > 0$ such that for any $\Phi \in L^\infty(0,T;L^p(\Omega;X_{\theta_F}))$ we have:

$$\Big\|s \mapsto \int_0^{\overline{s}} E(\overline{s} - \underline{u})\Phi(u)\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \lesssim \overline{T_0}^{\varepsilon_1}\|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}, \tag{7.2.21}$$

with implied constants independent of n, T_0, and Φ.

Once (7.2.21) is obtained, we immediately get:

$$\begin{aligned}
&\Big\|s \mapsto \int_0^{\overline{s}} E(\overline{s} - \underline{u})[F(u,U^{(n)}(u)) - F(u,V^{(n)}(u))]\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad \lesssim \overline{T_0}^{\varepsilon_1}\|V^{(n)} - U^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))},
\end{aligned} \tag{7.2.22}$$

by **(F)**. Moreover, by **(F′)** we get:

$$\begin{aligned}
&\Big\|s \mapsto \int_0^{\overline{s}} E(\overline{s} - \underline{u})[F(u,V^{(n)}(u)) - F(\underline{u},V^{(n)}(u))]\,du\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad \lesssim \overline{T_0}^{\varepsilon_1} n^{-\zeta_{\max}}\big(1 + \|V^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))}\big) \\
&\qquad \leq \overline{T_0}^{\varepsilon_1}\big[\|V^{(n)} - U^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))} + n^{-\eta}\big(1 + \|U^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))}\big)\big] \\
&\qquad \lesssim \overline{T_0}^{\varepsilon_1}\big[\|V^{(n)} - U^{(n)}\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X))} + n^{-\eta}\big(1 + \|x_0\|_{L^p(\Omega;X)}\big)\big],
\end{aligned} \tag{7.2.23}$$

the last step being again a consequence of Corollary 6.3 with $\delta = 0$.

It remains to prove (7.2.21). We fix $\Phi \in L^\infty(0,T;L^p(\Omega;X_{\theta_F}))$. By the uniform boundedness estimate of Corollary 7.6 with $\delta = \theta_F$ we obtain:

$$\begin{aligned}
\Big\| s \mapsto \int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,du \Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\
&\le \sup_{0\le s\le T_0} \int_0^{\overline{s}} \|E(\overline{s}-\underline{u})\Phi(u)\|_{L^p(\Omega;X)}\,du \\
&\lesssim \sup_{0\le s\le T_0} \int_0^{\overline{s}} (\overline{s}-\underline{u})^{\theta_F}\,du \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))} \\
&\lesssim \overline{T_0}^{1+\theta_F} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))},
\end{aligned} \tag{7.2.24}$$

where the last step follows from a similar calculation as in (7.2.17), the difference being that now we consider the terms 'up to T_0'. The implied constants are independent of n, T_0, and Φ.

For the estimate in the weighted γ-space we shall again use Lemma A.3. Define $\Psi : [0,T_0] \to L^p(\Omega;X)$ by

$$\Psi(s) := \int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,du. \tag{7.2.25}$$

Let $t \in [0,T_0]$ and let $q = (\frac{1}{\tau} - \frac{1}{2} + \frac{1}{2}\varepsilon)^{-1}$ (so $\frac{1}{\tau} - \frac{1}{2} < \frac{1}{q} < \frac{1}{\tau} - \alpha$). Combining Lemma A.3 and (7.2.24) we obtain, for some $\varepsilon_0 > 0$:

$$\begin{aligned}
\sup_{t\in[0,T_0]} \|s \mapsto (t-s)^{-\alpha}\Psi(s)\|_{\gamma(0,t;L^p(\Omega;X))} \\
&\lesssim T_0^{\varepsilon_0}\big(\|\Psi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}([0,T_0];L^p(\Omega;X))} + \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))}\big) \\
&\lesssim T_0^{\varepsilon_0}\big(\|\Psi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}([0,T_0];L^p(\Omega;X))} + \overline{T_0}^{1+\theta_F} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\big),
\end{aligned} \tag{7.2.26}$$

with implied constant independent of T_0 and Ψ.

In order to estimate the Besov norm in the right-hand side, let us first fix $s \in [0,T_0]$ and $k \in \{1,\dots,n\}$ such that $s + t_k^{(n)} \le T_0$. We have, using (7.1.2),

$$\begin{aligned}
\Psi(s+t_k^{(n)}) - \Psi(s) &= \int_0^{\overline{s}} (E(t_k^{(n)}) - I)E(\overline{s}-\underline{u})\Phi(u)\,du \\
&\quad + \int_{\overline{s}}^{\overline{s}+t_k^{(n)}} E(\overline{s}+t_k^{(n)}-\underline{u})\Phi(u)\,du.
\end{aligned} \tag{7.2.27}$$

By Lemma 2.21 (3) and the uniform boundedness result of Proposition 7.3 with $\delta = 1 + \theta_F - \frac{1}{2}\varepsilon$ (note that $\delta > 0$ by (7.2.12)) we have:

$$\big\|E(t_k^{(n)}) - I\big\|_{\mathscr{L}(X_{1+\theta_F-\frac{1}{2}\varepsilon};X)} \lesssim (t_k^{(n)})^{1+\theta_F-\frac{1}{2}\varepsilon}.$$

Moreover, by the uniform boundedness result in Corollary 7.6 with $\delta = -1 + \frac{1}{2}\varepsilon$ we have:

$$\big\|E(t_k^{(n)})\big\|_{\mathscr{L}(X_{\theta_F};X_{1+\theta_F-\frac{1}{2}\varepsilon})} \lesssim (t_k^{(n)})^{-1+\frac{1}{2}\varepsilon},$$

and with $\delta = \theta_F$:

$$\big\|E(t_k^{(n)})\big\|_{\mathscr{L}(X_{\theta_F};X)} \lesssim (t_k^{(n)})^{\theta_F}.$$

Thus for the first term in (7.2.27) we have:

$$\begin{aligned}
&\Big\|\int_0^{\overline{s}} (E(t_k^{(n)}) - I)E(\overline{s}-\underline{u})\Phi(u)\,du\Big\|_{L^p(\Omega;X)} \\
&\quad\le \int_0^{\overline{s}} \big\|E(t_k^{(n)}) - I\big\|_{\mathscr{L}(X_{1+\theta_F-\frac{1}{2}\varepsilon};X)}\big\|E(\overline{s}-\underline{u})\big\|_{\mathscr{L}(X_{\theta_F};X_{1+\theta_F-\frac{1}{2}\varepsilon})} \\
&\qquad\qquad \times \big\|\Phi\big\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\,du \\
&\quad\lesssim (t_k^{(n)})^{1+\theta_F-\frac{1}{2}\varepsilon}\int_0^{\overline{s}} (\overline{s}-\underline{u})^{-1+\frac{1}{2}\varepsilon}\,du\big\|\Phi\big\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))} \\
&\quad\lesssim (t_k^{(n)})^{1+\theta_F-\frac{1}{2}\varepsilon}\big\|\Phi\big\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}.
\end{aligned}$$

For the second term in (7.2.27) we have:

$$\begin{aligned}
&\Big\|\int_{\overline{s}}^{\overline{s}+t_k^{(n)}} E(\overline{s}+t_k^{(n)}-\underline{u})\Phi(u)\,du\Big\|_{L^p(\Omega;X)} \\
&\qquad\lesssim \int_0^{t_k^{(n)}} (t_k^{(n)}-\underline{u})^{\theta_F}\,du\big\|\Phi\big\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))} \\
&\qquad\lesssim (t_k^{(n)})^{1+\theta_F}\|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}.
\end{aligned}$$

Combining the two estimates above we obtain:

$$\|\Psi(s+t_k^{(n)}) - \Psi(s)\|_{L^p(\Omega;X)} \lesssim (t_k^{(n)})^{1+\theta_F}\|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}. \tag{7.2.28}$$

This enables us to find the right estimate for the Besov norm in (7.2.26). Fix $\rho \in [0,1]$ and $0 < |h| < \rho$. Set $I = [0,T_0]$. Suppose first that $|h| \le \frac{T}{n}$. In that case we have, by (7.2.28):

$$\begin{aligned}
&\|T_h^I\Psi(s) - \Psi(s)\|_{L^p(\Omega;X)} \\
&\qquad\le \begin{cases} 0, & \overline{s+h} = \overline{s} \text{ and } s+h \in [0,T_0]; \\ \|\Psi(s+\frac{T}{n}) - \Psi(s)\|_{L^p(\Omega;X)}, & \overline{s+h} \ne \overline{s} \text{ and } s+h \in [0,T_0]; \\ \|\Psi(s)\|_{L^p(\Omega;X)}, & s+h \notin [0,T_0] \end{cases} \\
&\qquad\lesssim \begin{cases} 0, & \overline{s+h} = \overline{s} \text{ and } s+h \in [0,T_0]; \\ n^{-1-\theta_F}\|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}, & \overline{s+h} \ne \overline{s} \text{ and } s+h \in [0,T_0]; \\ \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}, & s+h \notin [0,T_0]. \end{cases}
\end{aligned}$$

In the above we used $\|\Psi(s)\|_{L^p(\Omega;X)} \leq \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))}$ and (7.2.24).

Define $I_h = \{s \in [0,T_0] : \overline{s+h} \neq \overline{s}\}$ and observe that $|I_h| \leq n|h|$. Moreover $|\{s \in [0,T_0] : s+h \notin [0,T_0]\}| \leq |h|$. Thus by the definition of q and by (7.2.28),

$$\begin{aligned}
&\|T_h^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))}\\
&\qquad \lesssim \big[(n|h|)^{\frac{1}{q}} n^{-1-\theta_F} + |h|^{\frac{1}{q}}\big] \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\\
&\qquad \lesssim |h|^{\frac{1}{\tau}-\frac{1}{2}+\varepsilon/2} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}.
\end{aligned}$$

Next let $|h| > \frac{T}{n}$. Then, $|h|/2 < \underline{|h|} \leq |h|$ and $|h| < \overline{|h|} \leq 2|h|$. Let us deal with the case $h > \frac{T}{n}$; the case $-h > \frac{T}{n}$ is dealt with entirely analogously. It follows from the definition of Φ in (7.2.25) that for each $s \in [0,T_0]$ we either have $\Psi(\overline{s+h}) = \Psi(\overline{s} + \overline{h})$ or $\Psi(\overline{s+h}) = \Psi(\overline{s} + \underline{h})$. Hence, by (7.2.28):

$$\begin{aligned}
&\|T_h^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))}\\
&\qquad \leq \|1_{\{s+h\notin[0,T_0]\}}\Psi\|_{L^q(0,T_0;L^p(\Omega;X))}\\
&\qquad\quad + \|T_{\overline{h}}^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))} + \|T_{\underline{h}}^I \Psi - \Psi\|_{L^q(0,T_0;L^p(\Omega;X))}\\
&\qquad \lesssim \big(h^{\frac{1}{q}} + \underline{h}^{1+\theta_F} + \overline{h}^{1+\theta_F}\big) \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\\
&\qquad \lesssim h^{\frac{1}{\tau}-\frac{1}{2}+\frac{1}{2}\varepsilon} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))},
\end{aligned}$$

where we use that $1 + \theta_F \geq \frac{1}{\tau} - \frac{1}{2} + \frac{1}{2}\varepsilon$ (by (7.2.12)).

Thus we have:

$$\sup_{|h|\leq\rho} \|T_h\Psi - \Psi\|_{L^q(0,t;L^p(\Omega;X))} \lesssim \rho^{\frac{1}{\tau}-\frac{1}{2}+\frac{1}{2}\varepsilon} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}.$$

With (7.2.24) it follows that

$$\begin{aligned}
\|\Psi\|_{B_{q,\tau}^{\frac{1}{\tau}-\frac{1}{2}}([0,t];L^p(\Omega;X))} &\lesssim \|\Psi\|_{L^q(0,T_0;L^p(\Omega;X))} + \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\\
&\lesssim \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} + \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}\\
&\lesssim \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))},
\end{aligned}$$

and thus, by (7.2.26),

$$\sup_{t\in[0,T_0]} \|s \mapsto (t-s)^{-\alpha}\Psi(s)\|_{\gamma(0,t;L^p(\Omega;X))} \lesssim T_0^{\varepsilon_0} \|\Phi\|_{L^\infty(0,\overline{T_0};L^p(\Omega;X_{\theta_F}))}. \tag{7.2.29}$$

This completes the proof of (7.2.21).

Part 2e For the sixth term we again use Besov embeddings. First of all observe that by (2.6.3), the linear growth condition on F in **(F)** and Corollary 6.3, for all $s \in [0,T_0]$ we have:

$$\Big\| \int_s^{\overline{s}} S(\tfrac{T}{n}) F(u, U^{(n)}(u))\, du \Big\|_{L^p(\Omega;X)} \lesssim (\tfrac{T}{n})^{\theta_F} \int_s^{\overline{s}} \|F(u, U^{(n)}(u))\|_{L^p(\Omega;X_{\theta_F})}\, du$$

$$
\begin{aligned}
&\lesssim (\tfrac{T}{n})^{\theta_F} \int_s^{\overline{s}} \|U^{(n)}(u)\|_{L^p(\Omega;X)}\,du \\
&\lesssim (\overline{s}-s)n^{-\theta_F}(1+\|x_0\|_{L^p(\Omega;X)}) \\
&\leq n^{-1-\theta_F}(1+\|x_0\|_{L^p(\Omega;X)}),
\end{aligned}
$$

and therefore

$$
\Big\| s\mapsto \int_s^{\overline{s}} S(\tfrac{T}{n})F(u,U^{(n)}(u))\,du \Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \lesssim n^{-1-\theta_F}(1+\|x_0\|_{L^p(\Omega;X)}). \tag{7.2.30}
$$

Similarly, for $h\in(0,\frac{T}{n}]$:

$$
\begin{aligned}
&\Big\| s\mapsto \int_s^{s+h} S(\tfrac{T}{n})F(u,U^{(n)}(u))\,du \Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\
&\qquad\lesssim hn^{-\theta_F}(1+\|x_0\|_{L^p(\Omega;X)}) \\
&\qquad\leq h^{\frac{1}{\tau}-\frac12+\frac12\varepsilon} n^{-\frac32+\frac1\tau-\theta_F+\frac12\varepsilon}(1+\|x_0\|_{L^p(\Omega;X)}).
\end{aligned} \tag{7.2.31}
$$

Define $\Psi:[0,T_0]\to L^p(\Omega;X)$ by

$$
\Psi(s) := \int_s^{\overline{s}} S(\tfrac{T}{n})F(u,U^{(n)}(u))\,du.
$$

Fix $\rho\in[0,1]$ and let $0\leq h<\rho$ (the case that $-\rho<h\leq 0$ is entirely analogous). Suppose first that $h\leq\frac{T}{n}$. Then, for $s\in I:=[0,T_0]$:

$$
\begin{aligned}
&\|T_h^I\Psi(s)-\Psi(s)\|_{L^p(\Omega;X)} \\
&\quad\leq \begin{cases} \Big\| s\mapsto \int_s^{s+h} S(\tfrac{T}{n})F(u,U^{(n)}(u))\,du \Big\|_{L^p(\Omega;X)}, & \overline{s+h}=\overline{s},\ s+h\in[0,T_0]; \\ 2\|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Recall that $|\{s\in[0,T_0]:\overline{s}\neq\overline{s+h}\}|\leq nh$ and $|\{s\in[0,T_0]:s+h\notin[0,T_0]\}|\leq h$. Let $q=(\frac1\tau-\frac12+\frac12\varepsilon)^{-1}$. By (7.2.30) and (7.2.31) we have:

$$
\begin{aligned}
&\|T_h^I\Psi-\Psi\|_{L^q(0,T_0;L^p(\Omega;X))} \\
&\qquad\lesssim (h^{\frac1\tau-\frac12+\frac12\varepsilon}n^{-\frac32+\frac1\tau-\theta_F+\frac12\varepsilon}+((n+1)h)^{\frac1q}n^{-1-\theta_F})(1+\|x_0\|_{L^p(\Omega;X)}) \\
&\qquad\lesssim h^{\frac1\tau-\frac12+\frac12\varepsilon}n^{-1-\theta_F+\frac1q}(1+\|x_0\|_{L^p(\Omega;X)}).
\end{aligned}
$$

On the other hand, if $h>\frac{T}{n}$, then by (7.2.30):

$$
\begin{aligned}
\|T_h^I\Psi-\Psi\|_{L^q(0,T_0;L^p(\Omega;X))} &\leq 2\|\Psi\|_{L^q(0,T_0;L^p(\Omega;X))} \\
&\lesssim n^{-1-\theta_F}(1+\|x_0\|_{L^p(\Omega;X)}) \\
&\lesssim h^{\frac1\tau-\frac12+\frac12\varepsilon}n^{-1-\theta_F+\frac1q}(1+\|x_0\|_{L^p(\Omega;X)}).
\end{aligned}
$$

As in part 2c, using that by (7.2.12) we have $\eta+\varepsilon < \frac{3}{2} - \frac{1}{\tau} + \theta_F < 1+\theta_F - \frac{1}{q} + \varepsilon$, this implies

$$\|\Psi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}([0,T_0];L^p(\Omega;X))} \lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}),$$

with implied constants independent of n, x_0 and T_0. By Lemma A.3 it now follows that

$$\begin{aligned}\sup_{t\in[0,T_0]} \Big\| s \mapsto (t-s)^{-\alpha} \int_s^{\overline{s}} S(\tfrac{T}{n}) F(u, U^{(n)}(u))\, du \Big\|_{\gamma(0,t;L^p(\Omega;X))} \\ \lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}),\end{aligned}$$

with implied constants independent of n, x_0 and T_0. Combining this with (7.2.30) we obtain:

$$\Big\| s \mapsto \int_s^{\overline{s}} S(\tfrac{T}{n}) F(u, U^{(n)}(u))\, du \Big\|_{\mathscr{V}^{\alpha,p}_{\infty}([0,T_0]\times\Omega;X)} \lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}), \tag{7.2.32}$$

with implied constants independent of n, x_0 and T_0.

Part 2f. By Theorem 2.7 we have, for any $s \in [0,T_0]$:

$$\begin{aligned}\Big\| \int_0^{\overline{s}} [S(\overline{s}-\underline{u}) - E(\overline{s}-\underline{u})] G(u, U^{(n)}(u))\, dW_H(u) \Big\|_{L^p(\Omega;X)} \\ \lesssim \big\| u \mapsto [S(\overline{s}-\underline{u}) - E(\overline{s}-\underline{u})] G(u, U^{(n)}(u)) \big\|_{L^p(\Omega;\gamma(0,\overline{s};H,X))}.\end{aligned}$$

By the second part of Proposition 7.3 with $\delta = \theta_G$, $\epsilon = \frac{1}{3}\varepsilon$, $\beta = \frac{1}{2} - \frac{2}{3}\varepsilon$, and Theorem 2.14 and (7.2.12) we have:

$$\begin{aligned}&\big\| u \mapsto [S(\overline{s}-\underline{u}) - E(\overline{s}-\underline{u})] G(u, U^{(n)}(u)) \big\|_{L^p(\Omega;\gamma(0,\overline{s};H,X))} \\ &\quad\lesssim n^{-\frac{1}{2}-\theta_G+\varepsilon} \big\| u \mapsto (\overline{s}-\underline{u})^{-\frac{1}{2}+\frac{2}{3}\varepsilon} G(u, U^{(n)}(u)) \big\|_{L^p(\Omega;\gamma(0,\overline{s};H,X_{\theta_G}))} \\ &\quad\lesssim n^{-\eta} \big\| U^{(n)} \big\|_{\mathscr{V}^{\frac{1}{2}-\frac{2}{3}\varepsilon,p}_{\infty}([0,\overline{T_0}]\times\Omega;X)} \\ &\quad\lesssim n^{-\eta} \big\| U^{(n)} \big\|_{\mathscr{V}^{\alpha,p}_{\infty}([0,\overline{T_0}]\times\Omega;X)} \\ &\quad\lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}),\end{aligned}$$

where we also used **(G)** in the sense of (5.2.5), the fact that $\alpha > \frac{1}{2} - \frac{2}{3}\varepsilon$ (whence $\mathscr{V}^{\alpha,p}_{\infty}([0,\overline{T_0}]\times\Omega;X) \hookrightarrow \mathscr{V}^{\frac{1}{2}-\frac{2}{3}\varepsilon,p}_{\infty}([0,\overline{T_0}]\times\Omega;X)$), and we used Corollary 6.3. Note that the implied constants are independent of n, T_0 and x_0. As $s \in [0,T_0]$ was arbitrary, it follows that

$$\begin{aligned}\Big\| s \mapsto \int_0^{\overline{s}} [S(\overline{s}-\underline{u}) - E(\overline{s}-\underline{u})] G(u, U^{(n)}(u))\, dW_H(u) \Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\ \lesssim n^{-\eta}(1+\|x_0\|_{L^p(\Omega;X)}).\end{aligned} \tag{7.2.33}$$

Next we estimate the part concerning the weighted γ-radonifying norm. We begin by recalling that, since X is a UMD Banach space, $\gamma(0,t;H,X)$ is a UMD Banach space for any $t>0$ (by noting that this space embeds into $L^2(\tilde{\Omega};X)$ isometrically whenever $(\tilde{\Omega},\tilde{\mathbb{P}})$ is a probability space supporting a Gaussian sequence; see, e.g., [102]). Thus, by Theorem 2.7 (applied with state space $\gamma(0,\overline{t};X)$) and isomorphism (2.3.6), for all $t\in[0,T_0]$ we obtain

$$\begin{aligned}
&\Big\| s\mapsto (t-s)^{-\alpha}\int_0^{\overline{s}} [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X))}\\
&\qquad = \Big\| \int_0^{\overline{t}} \big[s\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\big]\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X))}\\
&\qquad \eqsim \big\| u\mapsto \big(s\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\big)\big\|_{L^p(\Omega;\gamma(0,\overline{t};H,\gamma(0,t;X)))}\\
&\qquad \eqsim \big\| (s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X))}.
\end{aligned}$$

By the second part of Proposition 7.3 with $\delta=\theta_G$, $\epsilon=\frac{1}{2}\varepsilon$, $\beta=\frac12-\frac12\varepsilon$, Theorem 2.14, isomorphism (2.3.6), once again Theorem 2.14 combined with Theorem 2.15, Lemma A.6, Corollary 6.3 and (7.2.12) we have:

$$\begin{aligned}
&\big\| (s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X))}\\
&\qquad \lesssim n^{-\frac12-\theta_G+\varepsilon}\big\| (s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times (\overline{s}-\underline{u})^{-\frac12+\frac12\varepsilon}G(u,U^{(n)}(u)))\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X_{\theta_G}))}\\
&\qquad \lesssim n^{-\eta}\big\| u\mapsto \big(s\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}\\
&\qquad\qquad \times (\overline{s}-\underline{u})^{-\frac12+\frac12\varepsilon}G(u,U^{(n)}(u))\big)\big\|_{L^p(\Omega;\gamma(0,\overline{t};H,\gamma(0,t;X_{\theta_G})))}\\
&\qquad \lesssim n^{-\eta}\sup_{u\in[0,\overline{t}]}\Big\{(t+\tfrac{T}{n}-u)^{\alpha}\|s\mapsto (t-s)^{-\alpha}(\overline{s}-\underline{u})^{-\frac12+\frac12\varepsilon}\|_{L^2(\underline{u},t)}\Big\}\\
&\qquad\qquad \times \big\| u\mapsto (t+\tfrac{T}{n}-u)^{-\alpha}G(u,U^{(n)}(u))\big\|_{L^p(\Omega;\gamma(0,\overline{t};H,X_{\theta_G}))}\\
&\qquad \lesssim n^{-\eta}\big(1+\|U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{t}]\times\Omega;X)}\big)\\
&\qquad \lesssim n^{-\eta}\big(1+\|x_0\|_{L^p(\Omega;X)}\big).
\end{aligned}$$

The implied constants above are independent of x_0, n, t and T_0.

Combining the above with (7.2.33) above one obtains:

$$\begin{aligned}
&\Big\| s\mapsto \int_0^{\overline{s}} [S(\overline{s}-\underline{u})-E(\overline{s}-\underline{u})]G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}\\
&\qquad \lesssim n^{-\eta}\big(1+\|x_0\|_{L^p(\Omega;X)}\big),
\end{aligned} \tag{7.2.34}$$

with implied constant independent of n, T_0 and x_0.

Part 2g. The estimate for the eighth and ninth term in (7.2.11) is similar to part 2f, except that one needs to keep track of dependence on T_0.

We shall prove that for any $\Phi \in L^p(\Omega;\gamma(0,T;H,X_{\theta_G}))$ we have:

$$\begin{aligned}\Big\| \int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X_{\theta_G})} \\ \lesssim \overline{T_0}^{\frac12-\theta_G-\varepsilon} \sup_{0\le t\le \overline{T_0}} \big\| s\mapsto (t-s)^{-\alpha}\Phi(s)\big\|_{L^p(\Omega;\gamma(0,t;H,X_{\theta_G}))},\end{aligned} \tag{7.2.35}$$

with implied constant independent of n and T_0, provided the right-hand side above is finite.

The estimate for the eighth term in (7.2.11) follows immediately from (7.2.35) and (5.2.5) (i.e., **(G)**):

$$\begin{aligned}\Big\| s\mapsto \int_0^{\overline{s}} E(\overline{s}-\underline{u})[G(u,U^{(n)}(u))-G(u,V^{(n)}(u))]\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)} \\ \lesssim \overline{T_0}^{\frac12-\theta_G-\varepsilon}\|U^{(n)}-V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)}.\end{aligned} \tag{7.2.36}$$

The estimate for the ninth term in (7.2.11) follows immediately from (7.2.35) in combination with (7.2.12) and Lemma A.5 (i.e., **(G′)**), with $B_j = V_j^{(n)}$ noting that $V^{(n)}(u) = V^{(n)}(\underline{u}) = V^{(n)}_{\underline{u}n/T}$:

$$\begin{aligned}&\Big\| s\mapsto \int_0^{\overline{s}} E(\overline{s}-\underline{u})[G(u,V^{(n)}(u))-G(\underline{u},V^{(n)}(u))]\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)} \\ &\lesssim \overline{T_0}^{\frac12-\theta_G-\varepsilon} n^{-\eta}\big(1+\|V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)}\big) \\ &\lesssim \overline{T_0}^{\frac12-\theta_G-\varepsilon}\|U^{(n)}-V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)} + n^{-\eta}\big(1+\|U^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)}\big).\end{aligned} \tag{7.2.37}$$

It remains to prove (7.2.35). By Theorem 2.7 we have, for $s\in[0,T_0]$:

$$\Big\| \int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;X)} \lesssim \big\| u\mapsto E(\overline{s}-\underline{u})\Phi(u)\big\|_{L^p(\Omega;\gamma(0,\overline{s};X))}.$$

By the second part of Corollary 7.6 with $\delta=\theta_G$, $\epsilon=\frac13\varepsilon$, $\beta=\frac12-\frac23\varepsilon$, and Theorem 2.14 we have:

$$\begin{aligned}\big\| u\mapsto E(\overline{s}-\underline{u})\Phi(u)\big\|_{L^p(\Omega;\gamma(0,\overline{s};X))} \\ \lesssim \overline{T_0}^{\frac12+\theta_G-\varepsilon}\big\| u\mapsto (\overline{s}-\underline{u})^{-\alpha}\Phi(u)\big\|_{L^p(\Omega;\gamma(0,\overline{s};X_{\theta_G}))},\end{aligned}$$

where we used that $\alpha > \frac12-\frac23\varepsilon$. Note that the implied constants are independent of n and T_0. As $s\in[0,T_0]$ was arbitrary, it follows that:

$$\begin{aligned}\Big\| s \mapsto \int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,dW_H(u)\Big\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\ \lesssim \overline{T_0}^{\frac{1}{2}+\theta_G-\varepsilon} \sup_{0\le t\le \overline{T_0}} \big\| s\mapsto (t-s)^{-\alpha}\Phi(s)\big\|_{L^p(\Omega;\gamma(0,\overline{T_0};H,X_{\theta_G}))}.\end{aligned} \tag{7.2.38}$$

As for the part concerning the weighted γ-radonifying norm, as before we have, for $t\in[0,T_0]$:

$$\begin{aligned}\Big\| s\mapsto (t-s)^{-\alpha}\int_0^{\overline{s}} E(\overline{s}-\underline{u})\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X))} \\ \lesssim \big\|(s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}E(\overline{s}-\underline{u})\Phi(u)\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X))}.\end{aligned}$$

By the second part of Corollary 7.6 with $\delta=\theta_G$, $\epsilon=\frac{1}{2}\varepsilon$, $\beta=\frac{1}{2}-\frac{1}{2}\varepsilon$, Theorem 2.14, isomorphism (2.3.6), once again Theorem 2.14 combined with Theorem 2.15 and Lemma A.6 we have:

$$\begin{aligned}&\big\|(s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}(t-s)^{-\alpha}E(\overline{s}-\underline{u})\Phi(u)\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X))}\\ &\quad\lesssim \overline{T_0}^{\frac{1}{2}+\theta_G-\varepsilon}\big\|(s,u)\mapsto 1_{\{0\le u\le \overline{s}\}}\\ &\qquad\qquad \times (t-s)^{-\alpha}(\overline{s}-\underline{u})^{-\frac{1}{2}+\frac{1}{2}\varepsilon}\Phi(u)\big)\big\|_{L^p(\Omega;\gamma([0,t]\times[0,\overline{t}];H,X_{\theta_G}))}\\ &\quad\lesssim \overline{T_0}^{\frac{1}{2}+\theta_G-\varepsilon}\big\|u\mapsto (t+\tfrac{T}{n}-u)^{-\alpha}\Phi(u)\big\|_{L^p(\Omega;\gamma(0,\overline{t};X))}\\ &\quad\lesssim \overline{T_0}^{\frac{1}{2}+\theta_G-\varepsilon}\big\|u\mapsto (t-u)^{-\alpha}\Phi(u)\big\|_{L^p(\Omega;\gamma(0,\overline{t};X))}.\end{aligned}$$

Taking the supremum over $t\in[0,T_0]$ and combining the above with (7.2.38) one obtains (7.2.35).

Part 2h. As for the final term in (7.2.11), first observe that because $\theta_G\le 0$ we have, by (2.6.3):

$$\begin{aligned}\Big\| s\mapsto \int_s^{\overline{s}} S(\tfrac{T}{n})G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}\\ \lesssim n^{-\theta_G}\Big\| s\mapsto \int_s^{\overline{s}} G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\theta_G})},\end{aligned} \tag{7.2.39}$$

with implied constants independent of n and x_0.

By (2.4.6) (take $\tilde{\alpha}=\frac{1}{2}-\varepsilon/2$ and $\tilde{\varepsilon}=\varepsilon/2$), (5.2.5), and Corollary 6.3 we have, for $s\in[0,T_0]$:

$$\begin{aligned}&\Big\|\int_s^{\overline{s}} G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{L^p(\Omega;X_{\theta_G})}\\ &\quad\le (\overline{s}-s)^{-\frac{1}{2}+\varepsilon}\Big\| s\mapsto \int_0^s G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{C^{\frac{1}{2}-\varepsilon/2}(0,T_0;L^p(\Omega;X_{\theta_G}))}\\ &\quad\lesssim n^{-\frac{1}{2}+\varepsilon}\sup_{0\le t\le T_0}\|u\mapsto (t-u)^{-\frac{1}{2}+\frac{1}{2}\varepsilon}G(u,U^{(n)}(u))\|_{L^p(\Omega;\gamma(0,t;H,X_{\theta_G}))}\end{aligned}$$

$$\begin{aligned}
&\lesssim n^{-\frac{1}{2}+\varepsilon}\big(1+\|U^{(n)}\|_{V_\infty^{\frac{1}{2}-\frac{1}{2}\varepsilon,p}([0,T_0]\times\Omega;X)}\big)\\
&\lesssim n^{-\frac{1}{2}+\varepsilon}\big(1+\|x_0\|_{L^p(\Omega;X)}\big),
\end{aligned}\tag{7.2.40}$$

with implied constants independent of n, T_0, and x_0. We have shown that

$$\Big\|s\mapsto\int_s^{\overline{s}}G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_G}))}\lesssim n^{-\frac{1}{2}+\varepsilon}\big(1+\|x_0\|_{L^p(\Omega;X)}\big).\tag{7.2.41}$$

Next fix $t\in[0,T_0]$. By Lemma A.2 (with $R=(0,1)$ and $S=(0,t)$ with the Lebesgue measure, $f(r,u)(s)=(t-s)^{-\alpha}(t-u)^{\alpha}1_{\{s\le u\le\overline{s}\}}$, $\Phi_2\equiv I$ and $\Phi_1(u)=(t-u)^{-\alpha}G(u,U^{(n)}(u)))$ we obtain

$$\begin{aligned}
&\Big\|s\mapsto(t-s)^{-\alpha}\int_s^{\overline{s}}G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X_{\theta_G}))}\\
&\quad\lesssim\sup_{u\in[0,t]}(t-u)^{\alpha}\|s\mapsto(t-s)^{-\alpha}1_{\{\underline{u}\le s\le u\}}\|_{L^2(0,t)}\\
&\qquad\qquad\times\|u\mapsto(t-u)^{-\alpha}G(u,U^{(n)}(u))\|_{L^p(\Omega;\gamma(0,t;H,X_{\theta_G}))}\\
&\quad\lesssim n^{-\frac{1}{2}}\|u\mapsto(t-u)^{-\alpha}G(u,U^{(n)}(u))\|_{L^p(\Omega;\gamma(0,t;H,X_{\theta_G}))},
\end{aligned}$$

with implied constants independent of x_0, n and T_0.

From here we proceed as in (7.2.40) and take the supremum over $t\in[0,T_0]$ to arrive at the estimate

$$\begin{aligned}
\sup_{t\in[0,T_0]}&\Big\|s\mapsto(t-s)^{-\alpha}\int_s^{\overline{s}}G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;X_{\theta_G}))}\\
&\lesssim n^{-\frac{1}{2}}\big(1+\|x_0\|_{L^p(\Omega;X)}\big).
\end{aligned}\tag{7.2.42}$$

Combining (7.2.41) and (7.2.42) with (7.2.39) and recalling (7.2.12) we obtain:

$$\begin{aligned}
\Big\|s\mapsto\int_s^{\overline{s}}&S(\tfrac{T}{n})G(u,U^{(n)}(u))\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}\\
&\lesssim n^{-\frac{1}{2}-\theta_G+\frac{1}{p}+\varepsilon}\big(1+\|x_0\|_{L^p(\Omega;X)}\big)\\
&\le n^{-\eta}\big(1+\|x_0\|_{L^p(\Omega;X)}\big).
\end{aligned}\tag{7.2.43}$$

Part 3. By combining equations (7.2.11), (7.2.15), (7.2.16), (7.2.20), (7.2.22), (7.2.23), (7.2.32), (7.2.34), (7.2.36), (7.2.37) and (7.2.43), we obtain that there exist constants $C>0$ and $\epsilon>0$, independent of n, x_0 and y_0, such that for all $n\ge N$ and $T_0\in(0,T]$:

$$\begin{aligned}
\|U^{(n)}-V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)}&\le C\|x_0-y_0\|_{L^p(\Omega;X)}\\
+Cn^{-\eta}(1+\|x_0\|_{L^p(\Omega;X_\eta)})&+C\overline{T_0}^{\epsilon}\|U^{(n)}-V^{(n)}\|_{\mathscr{V}_\infty^{\alpha,p}([0,\overline{T_0}]\times\Omega;X)}.
\end{aligned}\tag{7.2.44}$$

Define $c_0 = \frac{1}{2}(2C)^{-\frac{1}{\epsilon}}$ and let $N_0 \in \mathbb{N}$ be such that $N_0 > \max\{N, T/c_0\}$, this implies that for $n \geq N_0$ we have $c_0 \leq \overline{c_0} \leq 2c_0$, and thus $\overline{c_0}^{\epsilon} \leq (2c_0)^{\epsilon} = (2C)^{-1}$. For $n \geq N_0$ we obtain, by taking $T_0 = \overline{c_0}$ in (7.2.44);

$$\begin{aligned}\|U^{(n)} - V^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,\overline{c_0}]\times\Omega;X)}\\ \leq 2C\big(\|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})\big),\end{aligned}$$

and thus there exists a constant $\tilde{C}$ such that for all $n \geq N$ we have:

$$\|U^{(n)} - V^{(n)}\|_{\mathscr{V}_{\infty}^{\alpha,p}([0,c_0]\times\Omega;X)} \leq \tilde{C}\big(\|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X_\eta)})\big),$$

which is precisely estimate (7.2.10). $\square$

8

Localization

The pathwise convergence results of Corollaries 6.6 and 7.14 on pages 107 and 136 remain valid if F and G are merely locally Lipschitz and satisfy linear growth conditions and $x_0 = y_0 \in L^0(\Omega, \mathscr{F}_0; X_\eta)$. See also Remarks 6.7 and 7.15.

To be precise, we consider (SDE) in a UMD Banach space X, where A satisfies **(A)** and F and G satisfy:

($\mathbf{F_{loc}}$) For some $\theta_F > -1 + (\frac{1}{\tau} - \frac{1}{2})$, where τ is the type of X, the function $F : [0,T] \times X \to X_{\theta_F}$ is measurable in the sense that for all $x \in X$ the mapping $F(\cdot, x) : [0,T] \to X_{\theta_F}$ is strongly measurable. Moreover, F is *locally* Lipschitz continuous and *uniformly* of linear growth in its second variable.

That is to say, for every $m \in \mathbb{N}$ there exists a constant $C_{0,m}$ such that for all $t \in [0,T]$, and all $x_1, x_2, \in X$ such that $\|x_1\|_X, \|x_2\|_X \leq m$:

$$\|F(t, x_1) - F(t, x_2)\|_{X_{\theta_F}} \leq C_{0,m} \|x_1 - x_2\|_X.$$

Moreover, there exists a constant C_1 such that for all $t \in [0,T]$ and all $x \in X$:

$$\|F(t, x)\|_{X_{\theta_F}} \leq C_1(1 + \|x\|_X).$$

($\mathbf{G_{loc}}$) For some $\theta_G > -\frac{1}{2}$, the function $G : [0,T] \times X \to \mathscr{L}(H, X_{\theta_G})$ is measurable in the sense that for all $h \in H$ and $x \in X$ the mapping $G(\cdot, x)h : [0,T] \to X_{\theta_G}$ is strongly measurable. Moreover, G is *locally* L^2_γ-Lipschitz continuous and *uniformly* of linear growth in its second variable.

That is to say, for every $m \in \mathbb{N}$ there exists a function $G_m : [0,T] \times X \to \mathscr{L}(H, X_{\theta_G})$ that satisfies **(G)** and for which one has, for all $x \in X$, $\|x\|_X \leq m$, that $G_m(t,x) = G(t,x)$ for all $t \in [0,T]$. Moreover, there exists a constant C_1 such that for all $\alpha \in [0, \frac{1}{2})$, all $t \in [0,T]$, and all simple functions $\phi : [0,T] \to X$ one has:

$$\begin{aligned}
&\|s \mapsto (t-s)^{-\alpha} G(s, \phi(s))\|_{\gamma(0,t;H,X_{\theta_G})} \\
&\qquad \leq C_1 \big(1 + \|s \mapsto (t-s)^{-\alpha} \phi(s)\|_{L^2(0,t;X) \cap \gamma(0,t;X)}\big).
\end{aligned}$$

Condition $\mathbf{(G_{loc})}$ is satisfied if X is a type 2 space and $G : [0,T] \times X \to \gamma(H, X_{\theta_G})$ is locally Lipschitz and uniformly of linear growth in the second variable; in that case one may take $G_m(t,x) := G\big(t, x(1 \wedge \frac{m}{\|x\|_X})\big)$. See also the examples given on page 84.

Following [109, Section 7], for $T > 0$, $p \in [1,\infty)$ and $\alpha \in [0,\frac{1}{2})$ we define $V_c^{\alpha,0}([0,T] \times \Omega; X)$ to be the space containing all continuous processes $\Phi \in L^0_{\mathscr{F}}(\Omega, \gamma(0,T;X))$ such that almost surely,

$$\|\Phi\|_{C([0,T];X)} + \sup_{0 \le t \le T} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(0,t;X)} < \infty.$$

The space $V_\infty^{\alpha,0}([0,T] \times \Omega; X)$ contains all processes $\Phi \in L^0_{\mathscr{F}}(\Omega, \gamma(0,T;X))$ such that almost surely,

$$\|\Phi\|_{L^\infty(0,T;X)} + \sup_{0 \le t \le T} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(0,t;X)} < \infty.$$

It has been proven in [109] that if one assumes $\mathbf{(F_{loc})}$ and $\mathbf{(G_{loc})}$ instead of $\mathbf{(F)}$ and $\mathbf{(G)}$, and moreover assumes that $x_0 \in L^0(\Omega, \mathscr{F}_0; X)$, then for every $p > 2$ satisfying $\frac{1}{p} < \frac{1}{2} + \theta_G$ equation (SDE) has a unique mild solution in $V_c^{\alpha,0}([0,T] \times \Omega; X)$ for all $T > 0$ and all $\alpha \in [0, \frac{1}{2})$. The solution is constructed by approximation; uniqueness is proven separately.

The approximations are obtained as follows. For $m \in \mathbb{N}$ let G_m be as in $\mathbf{(G_{loc})}$ and define $F_m(t,x) := F(t, (1 \wedge \frac{m}{\|x\|})x)$. Clearly F_m and G_m satisfy $\mathbf{(F)}$ and $\mathbf{(G)}$. By Theorem 5.3, for all $p \in (2,\infty)$ satisfying $\frac{1}{p} < \frac{1}{2} + \theta_G$ there exists, for all $\alpha \in [0,\frac{1}{2})$, a unique mild solution $U_m \in V_c^{\alpha,p}([0,T] \times \Omega; X)$ to:

$$\begin{cases} dU_m(t) = AU_m(t)\,dt + F_m(t, U_m(t))\,dt \\ \qquad\qquad\qquad + G_m(t, U_m(t))\,dW_H(t); \quad t \in [0,T], \\ U_m(0) = 1_{\{\|x_0\| \le m\}} x_0. \end{cases} \tag{8.0.1}$$

Fix $T > 0$ and set

$$\tau_m^T(\omega) := \inf\{t \ge 0 \,:\, \|U_m(t,\omega)\|_X \ge m\},$$

with the convention that $\inf(\varnothing) = T$. By a uniqueness argument one may show that for $m_1 \le m_2$ one has $U_{m_1}(t) = U_{m_2}(t)$ on $[0, \tau_{m_1}^T]$. Moreover, by [109, Section 8] we have, due to the linear growth conditions on F and G, that

$$\lim_{m\to\infty} \tau_m^T = T \quad \text{almost surely.}$$

In fact, because this holds for arbitrary $T > 0$, there exists a set $\Omega_0 \subseteq \Omega$ of measure one such that for all $\omega \in \Omega_0$ there exists an m_ω such that $\tau_m^T(\omega) = T$ for all $m \ge m_\omega$.

The mild solution U to (SDE) with F and G satisfying $\mathbf{(F_{loc})}$ and $\mathbf{(G_{loc})}$ is defined by setting:

$$U(t,\omega) := \lim_{m\to\infty} U_m(t,\omega), \quad t \in [0,T],\ \omega \in \Omega_0,$$

and $U(t,\omega) := 0$ for $t \in [0,T]$ and $\omega \in \Omega \setminus \Omega_0$.

For $m,n \in \mathbb{N}$ let $U^{(n,m)}$ denote the process obtained by applying the modified splitting scheme with step size $\frac{T}{n}$ to (8.0.1). Note that $U^{(n,m)}$ is precisely the same process as the one obtained by approximating the solution to (6.0.1) in the splitting scheme in the same way as we approximated the solution U to (SDE) by the processes U_m above. Thus for all $n \in \mathbb{N}$ there exists a $U^{(n)} \in V_\infty^{\alpha,0}([0,T]\times\Omega;X)$ such that almost surely we have $U^{(n)}(t) = \lim_{m\to\infty} U^{(n,m)}(t)$ for all $t \in [0,T]$. Set $u := (U(t_j^{(n)}))_{j=0}^n$ and $u^{(n)} = (U^{(n)}(t_j^{(n)}))_{j=0}^n$.

Corollary 8.1 (Localization of Corollary 6.6). *Let $\gamma,\delta \geq 0$ and $\eta > 0$ be such that $\gamma+\delta < \min\{\eta_{\max},\eta,1\}$, where $\eta_{\max}$ as defined by* (5.2.6)*. Suppose that $x_0 = y_0 \in L^0(\Omega,\mathscr{F}_0;X_\eta)$. Then there exists a random variable $\chi \in L^0(\Omega)$, independent of n, such that:*

$$\|u-u^{(n)}\|_{c_\gamma^{(n)}([0,T];X)} \leq \chi n^{-\delta}.$$

Before proving Corollary 8.1 we state the analogous result for the Euler scheme. Thus in addition to the above assumptions on A, F and G and the Banach space X we now assume that X has property (α) and that $\mathbf{(F')}$ and $\mathbf{(G')}$ are satisfied. Let R be such that $(R/T,\infty) \subset \varrho(A)$.

First of all we observe that the implicit-linear Euler scheme is well-defined for (SDE) under the assumptions $\mathbf{(A)}$, $\mathbf{(F_{loc})}$, $\mathbf{(F_{loc})}$, $\mathbf{(F')}$, and $\mathbf{(G')}$; by applying the scheme with step size $\frac{T}{n}$ pointwise in Ω one obtains a sequence $(V_j^{(n)})_{j=0}^{(n)} \subset L^0(\Omega,X)$. In order to see this, one need only check that $G(t_{j-1}^{(n)},V_{j-1}^{(n)})\Delta W_j^{(n)}$ is well-defined in $L^0(\Omega,X_{\theta_G\wedge 0})$ for all $n > R$ and $j = 1,\dots n$.

To this end fix $j \in \{1,\dots,n\}$. If $\phi \in L^0(\Omega,\mathscr{F}_{j-1}^{(n)};\gamma(H;X_{\theta_G\wedge 0}))$ is a finite rank simple function, then one easily checks that $\phi\Delta W_j^{(n)} \in L^0(\Omega,X_{\theta_G\wedge 0})$. Let $(\phi_i)_{i\in I}$ be a Cauchy net in $L^0(\Omega,\mathscr{F}_{j-1}^{(n)};\gamma(H;X_{\theta_G\wedge 0}))$ consisting of finite rank simple functions. From Step 2 on page 134 it follows that for $B \in \mathscr{F}_{j-1}^{(n)}$ we have, for $\varepsilon > 0$ and $\delta > 0$ given and $i,k \in I$;

$$\begin{aligned}
\mathbb{P}(\|(\phi_i-\phi_k)\Delta W_j^{(n)}\|_{X_{\theta_G\wedge 0}} \geq \varepsilon) &\leq \mathbb{P}(\|1_{B^c}(\phi_i-\phi_k)\Delta W_j^{(n)}\|_{X_{\theta_G\wedge 0}} \geq \varepsilon) + \mathbb{P}(B)\\
&\leq \tfrac{1}{\varepsilon}\mathbb{E}\|1_{B^c}(\phi_i-\phi_k)\Delta W_j^{(n)}\|_{X_{\theta_G\wedge 0}} + \mathbb{P}(B)\\
&\leq \tfrac{1}{\varepsilon}\mathbb{E}\|1_{B^c}(\phi_i-\phi_k)\|_{\gamma(H,X_{\theta_G\wedge 0})} + \mathbb{P}(B)\\
&\leq \tfrac{1}{\varepsilon}\sup_{\omega\in B^c}\|\phi_i(\omega)-\phi_k(\omega)\|_{\gamma(H,X_{\theta_G\wedge 0})} + \mathbb{P}(B).
\end{aligned}$$

Taking $B = \{\omega \in \Omega : \|\phi_i(\omega)-\phi_k(\omega)\|_{\gamma(H,X_{\theta_G\wedge 0})} > \frac{1}{2}\varepsilon\delta\}$ we arrive at:

$$\mathbb{P}(\|(\phi_i-\phi_k)\Delta W_j^{(n)}\|_{X_{\theta_G\wedge 0}} \geq \varepsilon) \leq \tfrac{\delta}{2} + \mathbb{P}(\|\phi_i-\phi_k\|_{\gamma(H,X_{\theta_G\wedge 0})} > \tfrac{1}{2}\varepsilon\delta).$$

As $(\phi_i)_{i\in I}$ is a Cauchy net in $L^0(\Omega;\gamma(H;X_{\theta_G\wedge 0}))$, it follows that there is an i_ε such that for $i,k\geq i_\varepsilon$ we have:

$$\mathbb{P}(\|(\phi_i-\phi_k)\Delta W_j^{(n)}\|_{X_{\theta_G\wedge 0}}\geq \varepsilon)\leq \delta.$$

As $\varepsilon,\delta>0$ were arbitrary, it follows that $(\phi_i\Delta W_j^{(n)})_{i\in I}$ is a Cauchy net in $L^0(\Omega,X_{\theta_G\wedge 0})$. It follows that $\phi\Delta W_j^{(n)}$ is well-defined in $L^0(\Omega,X_{\theta_G\wedge 0})$ for all $\phi\in L^0(\Omega,\mathscr{F}_{j-1}^{(n)};\gamma(H;X_{\theta_G\wedge 0}))$.

For $n\geq R$ set $v^{(n)}=(V_j^{(n)})_{j=0}^n$. Moreover, for $n\geq R$ and $m\in\mathbb{N}$ let $v^{(n,m)}=(V_j^{(n,m)})_{j=0}^n$, where $(V_j^{(n,m)})_{j=0}^n$ is defined by the implicit-linear Euler scheme of Chapter 7 applied to (8.0.1), with step size $\frac{T}{n}$.

Corollary 8.2 (Localization of Corollary 7.14). *Let $\gamma,\delta\geq 0$ and $\eta>0$ be such that $\gamma+\delta<\min\{\zeta_{\max},\eta\}$, where $\zeta_{\max}$ as defined by (7.2.1). Suppose that $x_0=y_0\in L^0(\Omega,\mathscr{F}_0;X_\eta)$. Then for almost all $\omega\in\Omega$ there exists an N_ω such that the following limit exists in $\ell^\infty_{n+1}(X)$ for all $n\geq N_\omega$:*

$$v^{(n)}=\lim_{m\to\infty}v^{(n,m)}.$$

Moreover, there exists a random variable $\chi\in L^0(\Omega)$, independent of n, such that for all $n\geq R$:

$$\|u-v^{(n)}\|_{c_\gamma^{(n)}([0,T];X)}\leq \chi n^{-\delta},$$

where $u:=(U(t_j^{(n)}))_{j=0}^n$.

Proof (of Corollaries 8.1 and 8.2). Let $\eta>0$, $\gamma,\delta\geq 0$ be as prescribed and fix $x_0\in L^0(\Omega;X_\eta)$. Set $u_m=(U_m(t_j^{(n)}))_{j=0}^n$, with U_m the solution to (8.0.1), and set $u^{(n,m)}=(U^{(n,m)}(t_j^{(n)}))_{j=0}^n$. Fix $\omega\in\Omega_0$, with Ω_0 as on page 152. Let m_ω be such that $\tau_m^T(\omega)=T$ for all $m\geq m_\omega$. Note that for all $m\geq m_\omega$ we have $\|U_m(\omega)\|_{L^\infty(0,T;X)}\leq m_\omega$ and thus $u_m(\omega)=u(\omega)$.

However, a priori this does not guarantee that $\|U^{(n,m)}(\omega)\|_{L^\infty(0,T;X)}\leq m_\omega$ for $m\geq m_\omega$ and $n\in\mathbb{N}$, nor that $\|v^{(n,m)}(\omega)\|_{\ell^\infty_{n+1}(X)}\leq m_\omega$ for $m\geq m_\omega$ and $n\geq R$; this requires an additional argument.

By Corollary A.16, with $p>2$ such that $\delta+\frac{2}{p}<\min\{\eta_{\max},\eta,1\}$, there exists a constant C_ω depending on ω (and m_ω), but independent of n, such that:

$$\|U_{2m_\omega}(\omega)-U^{(n,2m_\omega)}(\omega)\|_{L^\infty(0,T;X)}\leq C_\omega n^{-\delta}.$$

In particular, for large enough n, say $n\geq N_\omega$, we have:

$$\|U_{2m_\omega}(\omega)-U^{(n,2m_\omega)}(\omega)\|_{L^\infty(0,T;X)}\leq m_\omega.$$

As $\|U_{2m_\omega}(\omega)\|_{L^\infty(0,T;X)} \leq m_\omega$, it follows that $\|U^{(n,2m_\omega)}(\omega)\|_{L^\infty(0,T;X)} \leq 2m_\omega$ for $n \geq N_\omega$. Thus by definition of F_{2m_ω} and G_{2m_ω} and the uniqueness result of [109, Lemma 7.2] we have, for $m \geq m_\omega$, $n \geq N_\omega$ and $t \in [0,T]$;

$$U^{(n,m)}(\omega,t) = U^{(n)}(\omega,t).$$

By Corollary 6.6 applied to u_{2m_ω} and $u^{n,2m_\omega}$, with $p > 2$ such that $\gamma+\delta+\frac{2}{p} < \min\{\eta_{\max}, \eta, 1\}$, it follows that there exists a constant C_ω depending on ω, but independent of n, such that for $n \geq N_\omega$:

$$\|u(\omega) - u^{(n)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} = \|u_{2m_\omega}(\omega) - u^{(n,2m_\omega)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} \leq C_\omega n^{-\delta}.$$

This proves Corollary 8.1.

Corollary 8.2 is proven by an analogous argument using Corollary 7.14 first to guarantee that there exists an $N_\omega \in \mathbb{N}$ such that $\|v^{(n,m)}(\omega)\|_{\ell^\infty_{n+1}(X)} \leq 2m_\omega$ for $m \geq 2m_\omega$ and $n \geq N_\omega$. We then apply Corollary 7.14 to u_{2m_ω} and $v^{(n,2m_\omega)}$ (with $p > 2$ such that $\gamma+\delta+\frac{2}{p} < \min\{\zeta_{\max}, \eta\}$) again to obtain that there exists a constant C_ω depending on ω, but independent of n, such that for $n \geq N_\omega$:

$$\|u(\omega) - v^{(n)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} = \|u_{2m_\omega}(\omega) - v^{(n,2m_\omega)}(\omega)\|_{c_\gamma^{(n)}([0,T];X)} \leq C_\omega n^{-\delta}.$$

$\square$

9

Example: the stochastic heat equation in one space dimension

In this chapter we present an example of a stochastic partial differential equation that, when interpreted as a stochastic differential equation in a suitable Banach space X, fits in the framework of Chapters 6 and 7. We interpret the results of those chapters in this setting.

The equation we consider is the stochastic heat equation on $[0, 1]$, this is an extension of the example given in the introduction of this thesis:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) = a_2(\xi)\dfrac{\partial^2 u}{\partial \xi^2}(\xi,t) + a_1(\xi)\dfrac{\partial u}{\partial \xi}(\xi,t) & \\ \qquad\qquad + f(t,\xi,u(\xi,t)) + g(t,\xi,u(\xi,t))\dfrac{\partial w}{\partial t}(\xi,t); & \xi \in (0,1),\ t \in (0,T], \\ b_{1,\xi}\dfrac{\partial u}{\partial \xi}(\xi,t) = -b_{0,\xi}u(\xi,t); & \xi \in \{0,1\},\ t \in (0,T], \\ u(0,\xi) = u_0(\xi); & \xi \in [0,1]. \end{cases} \tag{9.0.1}$$

Here w denotes a space-time white noise on $[0,T]\times[0,1]$ and $b_{i,\xi}$ are real numbers ($i,\xi \in \{0,1\}$). It is assumed that $a_2 \in C[0,1]$ is bounded away from 0 and $a_1 \in C[0,1]$. Moreover, $f : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are jointly measurable and globally Lipschitz in the second variable, uniformly in the first variable. More precisely, there exist constants L_f and L_g such that for all $t \in [0,T]$, $\xi \in [0,1]$, and $x,y \in \mathbb{R}$ we have:

$$|f(t,\xi,x) - f(t,\xi,y)| \le L_f|x-y|, \qquad |g(t,\xi,x) - g(t,\xi,y)| \le L_g|x-y|.$$

We also impose the linear growth conditions

$$|f(t,\xi,x)| \le C(1+|x|), \qquad |g(t,\xi,x)| \le C(1+|x|),$$

with constant C independent of $\xi \in [0,1]$, $t \in [0,T]$, $x \in \mathbb{R}$.

Following the approach of [109, Section 10] we may rewrite this equation to fit in the functional-analytic framework of Section 5.1. For $\theta > 0$ and $1 < q < \infty$ we define $H_B^{\theta,q} := H^{\theta,q}(0,1)$ for $0 < \theta < 1 + \frac{1}{q}$, and

$$H_B^{\theta,q} := \big\{u \in H^{\theta,q}(0,1)\colon b_{1,\xi}\tfrac{\partial u}{\partial \xi}(\xi,t) + b_{0,\xi}u(\xi,t) = 0;\ \xi \in \{0,1\}\big\}$$

for $1 + \frac{1}{q} < \theta < \infty$.

The operator $A : H_B^{2,q} \to L^q(0,1)$ defined by

$$Au := a_2\frac{\partial^2 u}{\partial \xi^2} + a_1\frac{\partial u}{\partial \xi}$$

generates an analytic C_0-semigroup on $L^q(0,1)$, see [92, Section 3.1], which is based on [1].

From now on we take $q \in (2,\infty)$ and fix $\beta \in (\frac{1}{2q}, \frac{1}{4})$. The part of A in the space

$$X := H_B^{2\beta,q} = H^{2\beta,q}(0,1)$$

generates an analytic C_0-semigroup in X and $-A$ has bounded imaginary powers in X by [93, Example 4.2.3]. By abuse of notation we shall denote the operator by A again. As a consequence of [93, Theorem 4.2.6] and reiteration of the complex interpolation method, for $\theta \in (0,1)$, $2\beta + 2\theta \neq 1 + \frac{1}{q}$, we have, for $0 < \theta < 1$,

$$X_\theta = [X, D(A)]_\theta = \big[H_B^{2\beta,q}, H_B^{2\beta+2,q}\big]_\theta = H_B^{2\beta+2\theta,q}.$$

The reason for picking the space X as our state space is two-fold. Firstly, we need a certain amount of space-regularity ($\beta > \frac{1}{2q}$) in order to prove that the Nemytskii operators F and G induced by f and g satisfy **(F)** and **(G)**. Secondly, as we shall see in Theorem 9.1, there is a trade-off between the regularity of the space in which we consider convergence in which we consider convergence and the convergence rate: as β increases to $\frac{1}{4}$, the convergence rate decreases. Beyond this critical value we are no longer able to prove convergence.

Observe that X is a UMD space, and since we assume $q > 2$ the type of X equals $\tau = 2$. Set $H := L^2 := L^2(0,1)$. For $t \in [0,T]$, $u \in X$, and $h \in H$ we define the Nemytskii operators

$$F(t,u)(\xi) := f(t,\xi,u(\xi));$$
$$(G(t,u)h)(\xi) := g(t,\xi,u(\xi))h(\xi).$$

Set $\theta_F := -\beta$ and pick $\varepsilon > 0$ sufficiently small such that $\theta_G := -\frac{1}{4} - \beta - \varepsilon > -\frac{1}{2}$. Under the above assumptions on f and g, it was shown in the proof of [109, Theorem 10.2] (here we use that $\beta > \frac{1}{2q}$) that F defines a mapping from $[0,T]\times X$ to $X_{\theta_F} = L^q(0,1)$ that satisfies **(F)** and G defines a mapping from $[0,T]\times X$ to $\gamma(H, X_{\theta_G}) = \gamma(L^2, H_B^{-\frac{1}{2}-2\varepsilon,q})$ that satisfies **(G)**; the measurability conditions are satisfied due to the measurability of f and g (in the notation of [109] we take $E = L^q(0,1)$ and $\eta = \beta$, so that $E_\eta = X$).

Furthermore, the part of the A in X satisfies **(A)**. Modeling the space-time white noise as an H-cylindrical Brownian motion W_H, we may rewrite (9.0.1) as follows:

$$\begin{cases} dU(t) = AU(t)\,dt + F(t,U(t))\,dt + G(t,U(t))\,dW_H(t); \quad t \in [0,T], \\ U(0) = u_0. \end{cases} \tag{9.0.2}$$

In order to obtain convergence of the Euler scheme for U, we must ensure that **(F′)** and **(G′)** are satisfied. This requires extra assumptions on f and g. Noting that $\eta_{\max} = \zeta_{\max} = \frac{1}{4} - \beta - \varepsilon$, we assume that there exists a constant C such that for all $s \in [0,1]$ and all $x \in \mathbb{R}$ we have:

$$\|t \mapsto f(t,s,x)\|_{C^{\frac{1}{4}-\beta}([0,T])} \leq C(1+|x|);$$

and

$$\|t \mapsto g(t,s,x)\|_{C^{\frac{1}{4}-\beta}([0,T])} \leq C(1+|x|).$$

By similar arguments that were used in [109, Section 10] to prove that F and G satisfy **(F)** and **(G)**, one can use the above to prove that F and G satisfy **(F′)** and **(G′)**.

Fix $T > 0$ and $n \in \mathbb{N}$, let U be the mild solution to (9.0.2) on $[0,T]$ and set $u := (U(t_j^{(n)})_{j=0}^n$ with $t_j^{(n)} = jT/n$.

Theorem 9.1. *Let $p > 4$, $q > 2$, $\alpha > 0$, $\beta \in (\frac{1}{2q}, \frac{1}{4})$, and γ, $\delta \geq 0$ satisfy*

$$\beta + \gamma + \delta + \tfrac{1}{p} < \min\{\tfrac{1}{4}, \alpha\}.$$

Fix $T > 0$. Let $U^{(n)}$ be defined by the modified splitting scheme with initial value $u_0 \in L^p(\Omega, \mathscr{F}_0; H^{2\alpha,q}(0,1))$, and let $V^{(n)}$ be defined by the implicit Euler scheme. Let $u^{(n)} := (U^{(n)}(t_j^{(n)})_{j=0}^n$ and $v^{(n)} = (V_j^{(n)})_{j=0}^n$. Then:

$$\big(\mathbb{E}\|u - u^{(n)}\|^p_{c_\gamma([0,T];H^{2\beta,q}(0,1))}\big)^{\frac{1}{p}} \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega;H^{2\alpha,q}(0,1))});$$
$$\big(\mathbb{E}\|u - v^{(n)}\|^p_{c_\gamma([0,T];H^{2\beta,q}(0,1))}\big)^{\frac{1}{p}} \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega;H^{2\alpha,q}(0,1))});$$

with implied constant independent of n.

Proof. This follows from Theorems 6.1 and 7.1 with $X = H^{2\beta,q}$ and $\eta = \alpha - \beta$. □

By Corollaries 8.1 and 8.2, almost sure convergence in $c_\gamma([0,T];H^{2\beta,q}(0,1))$ with rate $n^{-\delta}$ holds for $x_0 = y_0 \in L^{(0)}(\Omega, \mathscr{F}_0; H^{2\alpha,q}(0,1))$ under the weaker assumption

$$\beta + \gamma + \delta < \min\{\tfrac{1}{4}, \alpha\}. \tag{9.0.3}$$

(Again assuming $\beta \in (\frac{1}{2q}, \frac{1}{4})$ and $\gamma, \delta > 0$.)

For $2\beta > \lambda + \frac{1}{q}$, the Sobolev embedding theorem provides a continuous embedding $H^{2\beta,q}(0,1) \hookrightarrow C^\lambda[0,1]$. Hence, for $\lambda, \gamma, \delta > 0$ such that $\lambda + 2\gamma + 2\delta + \frac{1}{q} < \min\{\frac{1}{2}, 2\alpha\}$, we have:

$$\|u - u^{(n)}\|_{c_\gamma([0,T];C^\lambda[0,1])} \lesssim n^{-\delta} \quad \text{a.s.}$$

Let us now take $\gamma = 0$ and suppose that

$$\lambda + 2\delta < \tfrac{1}{2}. \tag{9.0.4}$$

Suppose $u_0 \in L^0(\Omega, \mathscr{F}_0; H^{\frac{1}{2},q}(0,1))$, i.e. we take $\alpha = \frac{1}{4}$. By picking q large enough, we have $\lambda + 2\delta + \frac{1}{q} < \frac{1}{2} = \min\{\frac{1}{2}, 2\alpha\}$. By the above we then obtain almost sure uniform convergence (with respect to the grid points $t_j^{(n)}$) in the space $C^\lambda[0,1]$ with rate δ:

$$\sup_{0\le j\le n} \|u(t_j^{(n)}) - u_j^{(n)}\|_{C^\lambda[0,1]} \lesssim n^{-\delta} \quad \text{a.s.}$$

Remark 9.2. It is proven in [34] that the optimal convergence rate of a time discretization for the heat equation in one dimension with additive space-time white noise based on n equidistant time steps is $n^{-\frac{1}{4}}$. This is under the assumption that the noise approximation of the n^{th} approximation is based only on linear combinations of $(W_H(t_j^{(n)}))_{j=0}^n$. In the theorem above we obtain convergence rate $n^{-\frac{1}{4}+\varepsilon}$ for $\varepsilon > 0$ arbitrarily small by taking $\gamma = 0$, β sufficiently small and p, q sufficiently large.

In [34] the authors also provide optimal convergence rates for the heat equation in one dimension with multiplicative space-time white noise, but these results concern simultaneous discretizations of time and space and are therefore not applicable to our situation.

10

A perturbation result for SDEs

This chapter contains the perturbation result as presented in [26]. We consider the effect of perturbations of A on the solution to (SDE), in the setting of Section 5.1.

The main motivation to study the effect of perturbations of A on solutions to (SDE) is the desire to prove convergence of certain numerical space approximation schemes. In the next chapter we demonstrate how the perturbation result can be used to obtain convergence, as $n \to \infty$, of the solution to (SDE) with A replaced by its n^{th} Yosida approximation $A_n = nAR(n : A)$. Moreover, for the case that X is a Hilbert space, we prove pathwise convergence of Galerkin and finite element methods for (SDE).

With applications to numerical approximations in mind, we assume the perturbed equation to be set in a (possibly finite dimensional) closed subspace X_0 of X. We assume that there exists a bounded projection $P_0 : X \to X_0$ such that $P_0(X) = X_0$. Let i_{X_0} be the canonical embedding of X_0 in X and A_0 be a generator of an analytic C_0-semigroup S_0 on X_0. In the setting of numerical approximations, A_0 would be a suitable restriction of A to the finite dimensional space X_0.

The perturbed equation we consider is the following stochastic evolution equation:

$$\begin{cases} dU^{(0)}(t) = A_0 U^{(0)}(t)\,dt + P_0 F(t, U^{(0)}(t))\,dt + P_0 G(t, U^{(0)}(t))\,dW_H(t), \; t > 0; \\ U^{(0)}(0) = P_0 x_0. \end{cases} \tag{SDE$_0$}$$

In the upcoming section, we shall prove the following theorem:

Theorem 10.1. *Let $\omega \geq 0$, $\theta \in (0, \frac{\pi}{2})$ and $K > 0$ be such that A and A_0 are both of type (ω, θ, K). Suppose there exist $\delta \in [0,1]$ and $p \in (2, \infty)$ satisfying*

$$0 \leq \delta < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F, \, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G\}$$

such that for some $\lambda_0 \in \varrho(A)$ we have:

$$D_\delta(A, A_0) := \|R(\lambda_0 : A) - i_{X_0} R(\lambda_0 : A_0) P_0\|_{\mathscr{L}(X^A_{\delta-1}, X)} < \infty. \tag{10.0.1}$$

Suppose $x_0 \in L^p(\Omega, \mathscr{F}_0; X_\delta^A)$ *and* $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$. *Then for all* $\alpha \in [0, \frac{1}{2})$ *there exists a unique process* $U^{(0)} \in V_c^{\alpha,p}([0,T_0] \times \Omega; X_0)$ *that is a mild solution to* (SDE$_0$) *with initial value* $P_0 y_0$. *Moreover:*

$$\begin{aligned} &\|U - i_{X_0} U^{(0)}\|_{V_c^{\alpha,p}([0,T]\times\Omega;X)} \\ &\qquad \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega;X_\delta^A)}), \end{aligned} \tag{10.0.2}$$

with implied constant depending on X_0 *only in terms of* $\|P_0\|_{\mathscr{L}(X,X_0)}$, *on* A *and* A_0 *only in terms of* $1 + D_\delta(A, A_0)$, ω, θ *and* K, *and depending on* F *and* G *only in terms of their Lipschitz constants Lip*(F) *and Lip*$_\gamma(G)$, *and the linear growth bounds* $M(F)$ *and* $M_\gamma(G)$.

As a corollary of Theorem 10.1 we obtain an estimate in the Hölder norm provided we compensate for the initial values (see Corollary 10.4 on page 172).

A natural question to ask is how the type of perturbation studied here relates to the perturbations known in the literature. In [39], [72], and [123] (see also [43, Chapter III.3]) one has derived conditions for perturbations of A that lead to an estimate of the type $\|S(t) - S_0(t)\|_{\mathscr{L}(X)} = \mathscr{O}(t)$. In light of Proposition 10.2 below these results are comparable to our results if we were to take $\alpha = -1$. In particular, [43, Theorem III.3.9] gives precisely the same results as Proposition 10.2, but then for the case $\alpha = -1$ and $\beta = 0$.

Theorem 10.1 implies that if $(A_n)_{n\in\mathbb{N}}$ is a family of generators of analytic semigroups such that the resolvent of A_n converges to the resolvent of A in $\mathscr{L}(X_{\delta-1}^A, X)$ for some $\delta \in [0,1]$ (and $(A_n)_{n\in\mathbb{N}}$ is uniformly analytic), then the corresponding solution processes U_n converge to the actual solution in $L^p(\Omega; C([0,T]; X))$ and the convergence rate is given by $D_\delta(A, A_n)$. Recently, it was proven in [85] that if $(A_n)_{n\in\mathbb{N}}$ is a family of generators of analytic semigroups such that the resolvent of A_n converges to the resolvent of A in the strong operator topology, then the corresponding solution processes U_n converge to the actual solution in $L^p(\Omega; C([0,T]; X))$. However, the approach taken in that article does not provide convergence rates and requires $\theta_F, \theta_G \geq 0$.

Another article in which approximations of solutions to (SDE) are considered in the context of perturbations on A is [14]. In that article, it is assumed that X is a UMD space with martingale type 2. In Section 5 of that article the author considers approximations of A, F, G and of the noise. Translated to our setting, the author assumes the perturbed operator A_0 to satisfy $X_{\theta_F}^{A_0} = X_{\theta_F}^A$ and $X_{\theta_G}^{A_0} = X_{\theta_G}^A$ (in particular, X_0 cannot be finite-dimensional).

The proof of Theorem 10.1 requires regularity results for stochastic convolutions. As the convolution under consideration does not concern a semigroup, the celebrated factorization method of [32] fails (see also Theorem 4.6). Therefore we prove a new result on the regularity of stochastic convolutions, see Lemma A.9 in Appendix A.2. This lemma in combination with some randomized boundedness results on $S - S_0 P_0$ form the key ingredients of the proof Theorem 10.1.

10.1 Proof of the perturbation result

For $t \geq 0$ define $\tilde{S}_0 \in \mathscr{L}(X)$ by $\tilde{S}_0(t) := i_{X_0} S_0(t) P_0$, this defines a *degenerate* C_0-semigroup, i.e. $\tilde{S}_0$ satisfies the semigroup property but $\tilde{S}_0(0) = i_{X_0} P_0$ (which is clearly not the identity unless $X_0 = X$).

Let $x_0 \in L^p(\Omega, \mathscr{F}_0; X)$ (where $p > 2$ satisfies $\frac{1}{p} \leq \frac{1}{2} + \theta_G$) and let U be the solution to (SDE) with operator A and initial data x_0 as provided by Theorem 5.3. To prove Theorem 10.1 we need a proposition concerning the γ-boundedness of $S - \tilde{S}_0$. The proof of this proposition is postponed to the end of this section.

Proposition 10.2. *Let A, A_0 be as introduced above, i.e., A generates an analytic semigroup on X and A_0 generates an analytic semigroup on X_0. Let $\omega \geq 0, \theta \in (0, \frac{\pi}{2})$ and $K > 0$ be such that A and A_0 are of type (ω, θ, K). Suppose there exists a $\lambda_0 \in \mathbb{C}$, $\mathscr{R}e(\lambda_0) > \omega$, and $\delta \in \mathbb{R}$ such that $D_\delta(A, A_0) < \infty$, where $D_\delta(A, A_0)$ is as defined in (10.0.1). Set*

$$\omega' = \omega + |\lambda_0 - \omega|(\cos\theta)^{-1}.$$

Then for all $\beta \in \mathbb{R}$ such that $\beta \in [\delta - 1, \delta]$ one has:

$$\sup_{t \in [0,\infty)} t^{\delta-\beta} e^{-\omega' t} \|S(t) - \tilde{S}_0(t)\|_{\mathscr{L}(X^A_\beta, X)} \lesssim D_\delta(A, A_0), \tag{10.1.1}$$

and

$$\sup_{t \in [0,\infty)} t^{\delta-\beta+1} e^{-\omega' t} \|\tfrac{d}{dt} S(t) - \tfrac{d}{dt} \tilde{S}_0(t)\|_{\mathscr{L}(X^A_\beta, X)} \lesssim D_\delta(A, A_0), \tag{10.1.2}$$

with implied constants depending only on $\|P_0\|_{\mathscr{L}(X,X_0)}$, ω, θ, K, $\delta - \beta$.

Moreover, for all $\alpha > \delta - \beta$ we have, for $t \in [0, T]$:

$$\gamma_{[X^A_\beta, X]}\left(\left\{s^\alpha [S(s) - \tilde{S}_0(s)];\, 0 \leq s \leq t\right\}\right) \lesssim t^{\alpha+\beta-\delta} D_\delta(A, A_0),$$

with implied constant depending only on $\|P_0\|_{\mathscr{L}(X,X_0)}$, ω, θ, K, $\delta - \beta$, and T.

Proof (of Theorem 10.1.). We split the proof into several parts.

Part 1. In order to prove existence and uniqueness of a mild solution $U^{(0)} \in V^{\alpha,p}_{\mathrm{c}}([0, T_0] \times \Omega; X_0)$ to (SDE$_0$) it suffices, by Theorem 5.3, to prove that there exist $\eta_F > -\frac{3}{2} + \frac{1}{\tau}$ and $\eta_G > -\frac{1}{2} + \frac{1}{p}$ such that $P_0 F : [0, T] \times X \to X^{A_0}_{0,\eta_F}$ is Lipschitz continuous and of linear growth and $P_0 G : [0, T] \times X \to \gamma(H, X^{A_0}_{0,\eta_G})$ is L^2_γ-Lipschitz continuous and of linear growth. If $\theta_F \geq 0$ then clearly we may take $\eta_F = 0$, and we have $\mathrm{Lip}(P_0 F) \leq \|P_0\|_{\mathscr{L}(X,X_0)} \mathrm{Lip}(F)$, $M(P_0 F) \leq \|P_0\|_{\mathscr{L}(X,X_0)} M(F)$. The same goes for $\theta_G \geq 0$.

Now suppose $\theta_F < 0$. Recall the following representation of negative fractional powers of an operator A generating an analytic semigroup S of type (ω, θ, K) (see page 27):

$$(\lambda I - A)^{\eta} = \frac{1}{\Gamma(-\eta)} \int_0^{\infty} t^{-\eta-1} e^{-\lambda t} S(t) dt, \quad \eta < 0, \ \mathscr{R}e(\lambda) > \omega.$$

Let $\bar{\omega} > \omega'$, where ω' is as in Proposition 10.2. From the representation above and Proposition 10.2 it follows that for $\beta \in [\delta - 1, \delta]$, $\eta < \beta - \delta$ and $x \in X$ we have:

$$\begin{aligned}
\|P_0 x\|_{X_{0,\eta}^{A_0}} &\eqsim \|((\bar{\omega} I - A_0)^{\eta} P_0 x\|_X = \Big\| \frac{1}{\Gamma(-\eta)} \int_0^{\infty} t^{-\eta-1} e^{-\bar{\omega} t} \tilde{S}_0(t) x dt \Big\|_X \\
&\leq \frac{1}{\Gamma(-\eta)} \int_0^{\infty} t^{-\eta-1} e^{-\bar{\omega} t} \|(S(t) - \tilde{S}_0(t)) x\|_X dt \\
&\quad + \frac{1}{\Gamma(-\eta)} \Big\| \int_0^{\infty} t^{-\eta-1} e^{-\bar{\omega} t} S(t) x dt \Big\|_X \\
&\lesssim D_{\delta}(A, A_0) \int_0^{\infty} t^{-\eta-1+\beta-\delta} e^{-(\bar{\omega} - \omega')t} dt \|x\|_{X_{\beta}^{A}} + \|(\bar{\omega} I - A)^{\eta} x\|_X,
\end{aligned}$$

with implied constants depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X, X_0)}$ and on A and A_0 only in terms of ω, θ, and K. Thus for $\beta \in [\delta - 1, \delta]$, $\eta < \beta - \delta$ we have:

$$\begin{aligned}
\|P_0 x\|_{X_{0,\eta}^{A_0}} &\eqsim \|(\bar{\omega} I - A_0)^{\eta} P_0 x\|_X \\
&\lesssim (1 + D_{\delta}(A, A_0)) \|(\bar{\omega} I - A)^{\beta} x\|_X \eqsim (1 + D_{\delta}(A, A_0)) \|x\|_{X_{\beta}^{A}},
\end{aligned} \tag{10.1.3}$$

with implied constants depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X, X_0)}$ and on A and A_0 only in terms of ω, θ, and K.

Note that by assumption we have $\theta_F > -\frac{3}{2} + \frac{1}{\tau} + \delta \geq \delta - 1$. Hence one can pick η_F such that $-\frac{3}{2} + \frac{1}{\tau} < \eta_F < \theta_F - \delta$. By (10.1.3) it follows that $P_0 F : [0, T] \times X \to X_{0,\eta_F}^{A_0}$ is Lipschitz continuous and

$$\mathrm{Lip}(P_0 F) \lesssim (1 + D(A, A_0)) \mathrm{Lip}(F); \qquad M(P_0 F) \lesssim (1 + D(A, A_0)) M(F), \tag{10.1.4}$$

with implied constant depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X, X_0)}$ and on A and A_0 only in terms of ω, θ, and K.

Similarly, if $\theta_G < 0$ there exists a η_G such that $-\frac{1}{2} + \frac{1}{p} < \eta_G < \theta_G - \delta$ such that $P_0 G : [0, T] \times X \to \gamma(H, X_{0,\eta_G}^{A_0})$ is L_{γ}^2-Lipschitz continuous and

$$\mathrm{Lip}_{\gamma}(P_0 G) \lesssim (1 + D(A, A_0)) \mathrm{Lip}_{\gamma}(G); \quad M_{\gamma}(P_0 G) \lesssim (1 + D(A, A_0)) M_{\gamma}(G), \tag{10.1.5}$$

with implied constant depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X, X_0)}$ and on A and A_0 only in terms of ω, θ, and K.

Part 2. Define $\tilde{U}^{(0)} = i_{X_0} U^{(0)}$ and observe that if $U^{(0)}$ is a mild solution to (SDE$_0$), then $\tilde{U}^{(0)}$ satisfies:

$$\tilde{U}^{(0)}(t) = \tilde{S}_0(t - s) y_0 + \int_0^T \tilde{S}_0(t - s) F(s, \tilde{U}^{(0)}(s)) \, ds$$

$$+\int_0^t \tilde{S}_0(t-s)G(s,\tilde{U}^{(0)}(s))\,dW_H(s), \quad \text{a.s.}$$

Let $T_0 \in [0,T]$ be fixed. By the above we have:

$$\begin{aligned}
&\|U-\tilde{U}^{(0)}\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\quad \le \|(S-\tilde{S}_0)x_0\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} + \|\tilde{S}_0(x_0-y_0)\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + \Big\|t\mapsto \int_0^t \tilde{S}_0(t-s)[F(s,U(s))-F(s,\tilde{U}^{(0)}(s))]\,ds\Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + \Big\|t\mapsto \int_0^t [S(t-s)-\tilde{S}_0(t-s)]F(s,U(s))\,ds\Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + \Big\|t\mapsto \int_0^t \tilde{S}_0(t-s)[G(s,U(s))-G(s,\tilde{U}^{(0)}(s))]\,dW_H(s)\Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad + \Big\|t\mapsto \int_0^t [S(t-s)-\tilde{S}_0(t-s)]G(s,U(s))\,dW_H(s)\Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)}.
\end{aligned} \tag{10.1.6}$$

Let η_F and η_G be as defined in part 1. Let $\varepsilon>0$ be such that

$$\begin{aligned}
&\varepsilon \le 1-2\alpha; \\
&\varepsilon < \min\{\tfrac{3}{2}-\tfrac{1}{\tau}+\eta_F, \tfrac{1}{2}-\tfrac{1}{p}+\eta_G\}.
\end{aligned}$$

It follows that $\varepsilon+\delta<\min\{\frac{3}{2}-\frac{1}{\tau}+\theta_F, \frac{1}{2}-\frac{1}{p}+\theta_G\}$. By equation (5.2.2) we may assume, without loss of generality, that $\alpha=\frac{1}{2}-\varepsilon/2$.

We will estimate each of the six terms on the right-hand side of (10.1.6) in parts 2a-2f below. In part 2c and 2e we keep track of the dependence on T_0, for the other parts this is not necessary.

Part 2a. By Proposition 10.2 with $\beta=\delta$ there exists an $\mathscr{M}>0$ depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, and on A and A_0 only in terms om ω, θ and K, such that

$$\begin{aligned}
\sup_{t\in[0,T_0]}\|S(t)-\tilde{S}_0(t)\|_{\mathscr{L}(X_\delta^A,X)} &\le \mathscr{M}D_\delta(A,A_0); \\
\gamma_{[X_\delta^A,X]}\{t^{\varepsilon/2}(S(t)-\tilde{S}_0(t))\,:\,t\in[0,T_0]\} &\le \mathscr{M}D_\delta(A,A_0).
\end{aligned}$$

Thus by the multiplier theorem 2.14 we have

$$\begin{aligned}
&\|(S-\tilde{S}_0)x_0\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\qquad \le \mathscr{M}D_\delta(A,A_0) \\
&\qquad\quad \times\Big[\sup_{t\in[0,T_0]}\|s\mapsto(t-s)^{-\alpha}s^{-\varepsilon/2}x_0\|_{L^p(\Omega;\gamma(0,t;X_\delta^A))}+\|x_0\|_{L^p(\Omega;X_\delta^A)}\Big].
\end{aligned}$$

For $f\in L^2(0,t)$ and $x\in L^p(\Omega;X_\delta^A)$ we have

$$\|f\otimes x\|_{L^p(\Omega;\gamma(0,t;X_\delta^A))}=\|f\|_{L^2(0,t)}\|x\|_{L^p(\Omega;X_\delta^A)}.$$

Thus, recalling that $\alpha = \frac{1}{2} - \varepsilon/2$, we have:

$$\begin{aligned}\sup_{t\in[0,T_0]} &\|s \mapsto (t-s)^{-\alpha} s^{-\varepsilon/2} x_0\|_{L^p(\Omega;\gamma(0,t;X_\delta^A))} \\ &\leq \|s \mapsto (1-s)^{-\alpha} s^{-\varepsilon/2}\|_{L^2(0,1)} \|x_0\|_{L^p(\Omega;X_\delta^A)} \leq C_\varepsilon \|x_0\|_{L^p(\Omega;X_\delta^A)},\end{aligned}$$

where C_ε is a constant depending only on ε, and we used that $\alpha = \frac{1}{2} - \varepsilon/2$. Hence

$$\|(S - \tilde{S}_0)x_0\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \leq \mathscr{M} D_\delta(A, A_0)(1 + C_\varepsilon)\|x_0\|_{L^p(\Omega;X_\delta^A)}. \tag{10.1.7}$$

Part 2b. By assumption (see Remark 2.19) there exists an $\mathscr{M}$ depending only on $\|P_0\|_{\mathscr{L}(X,X_0)}$, ω, θ, and K and T such that we have that

$$\sup_{t\in[0,T]} \|\tilde{S}_0(t)\|_{\mathscr{L}(X,X_0)} \leq \mathscr{M}.$$

Moreover, by Lemma 2.21 we may pick $\mathscr{M}$ such that in addition we have that $\gamma_{[X,X]}\{t^{\varepsilon/2}\tilde{S}_0(t) : t \in [0,T_0]\} \leq \mathscr{M}$. Thus by the same argument as in part 2a we have:

$$\|\tilde{S}_0(x_0 - y_0)\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \leq \mathscr{M}(1 + C_\varepsilon)\|x_0 - y_0\|_{L^p(\Omega;X)}. \tag{10.1.8}$$

Part 2c. Recall that $\eta_F \leq 0$. By equation (2.6.3) there exists an $\mathscr{M}$ depending only on ω, θ, K and T such that for all $t \in [0,T]$ we have:

$$\begin{aligned}t^{-\eta_F+\varepsilon} &\|\tfrac{d}{dt} S_0(t)x\|_{\mathscr{L}(X_{0,\eta_F}^{A_0},X)} + (\varepsilon - \eta_F)t^{-\eta_F+\varepsilon-1}\|S_0(t)x\|_{\mathscr{L}(X_{0,\eta_F}^{A_0},X)} \\ &= t^{-\eta_F+\varepsilon}\|\tfrac{d}{dt} S_0(t)x\|_{\mathscr{L}(X_{0,\eta_F}^{A_0},X_0)} + (\varepsilon - \eta_F)t^{-\eta_F+\varepsilon-1}\|S_0(t)x\|_{\mathscr{L}(X_{0,\eta_F}^{A_0},X_0)} \\ &\leq \mathscr{M} t^{-1+\varepsilon}.\end{aligned}$$

By Corollary A.13 with $Y_1 = X_{0,\eta_F}^{A_0}$, $Y_2 = X$,

$$\Phi(s) = P_0[F(s, U(s)) - F(s, \tilde{U}^{(0)}(s))],$$

$\Psi(s) = S_0(s)$, $\theta = -\eta_F + \varepsilon$ and $g(v) = \mathscr{M} v^{-1+\varepsilon}$, it follows that:

$$\begin{aligned}\Big\|t \mapsto &\int_0^t \tilde{S}_0(t-s)[F(s,U(s)) - F(s,\tilde{U}^{(0)}(s))]\,ds\Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\ &\lesssim T_0^\varepsilon \|P_0[F(\cdot,U) - F(\cdot,\tilde{U}^{(0)})]\|_{L^p(\Omega;L^\infty(0,T_0;X_{0,\eta_F}^{A_0}))} \\ &\leq T_0^\varepsilon \mathrm{Lip}(P_0F)\|U - \tilde{U}^{(0)}\|_{L^p(\Omega;L^\infty(0,T_0;X))} \\ &\lesssim T_0^\varepsilon(1 + D_\delta(A, A_0))\mathrm{Lip}(F)\|U - \tilde{U}^{(0)}\|_{L^p(\Omega;L^\infty(0,T_0;X))},\end{aligned} \tag{10.1.9}$$

where the second-last estimate follows by Lipschitz-continuity of P_0F and the final estimate follows by (10.1.4). Note that the implied constants are independent of T_0, and depend on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$ and on A_0 only in terms of ω, θ and K.

Part 2d. By Proposition 10.2 with $\beta = \theta_F \wedge \delta \in [\delta-1, \delta]$ we have that there exists a constant $\mathscr{M}$ depending only on $\|P_0\|_{\mathscr{L}(X,X_0)}$, ω, θ, K, $(\delta - \theta_F) \vee 0$ and T such that for all $t \in [0,T]$ we have:

$$\begin{aligned} t^{(\delta-\theta_F)^+ + \varepsilon} &\| \tfrac{d}{dt}[S(t) - \tilde{S}_0(t)]\|_{\mathscr{L}(X^A_{\theta_F\wedge\delta}, X)} \\ &+ ((\delta-\theta_F)^+ + \varepsilon) t^{(\delta-\theta_F)^+ + \varepsilon - 1} \|S(t) - \tilde{S}_0(t)\|_{\mathscr{L}(X^A_{\theta_F\wedge\delta}, X)} \\ &\quad \leq \mathscr{M} D_\delta(A, A_0) t^{-1+\varepsilon}. \end{aligned}$$

Thus by Corollary A.13 with $Y_1 = X^A_{\theta_F\wedge\delta}$, $Y_2 = X$, $\Phi(s) = F(s, U(s))$, $\Psi(s) = S(s) - \tilde{S}_0(s)$, $\theta = (\delta - \theta_F)^+ + \varepsilon$, $g(v) = \mathscr{M} D_\delta(A, A_0) v^{-1+\varepsilon}$, we obtain:

$$\begin{aligned} \Big\| t \mapsto \int_0^t & [S(t-s) - \tilde{S}_0(t-s)] F(s, U(s))\, ds \Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\ &\lesssim D_\delta(A, A_0) \|F(\cdot, U)\|_{L^p(\Omega; L^\infty(0,T_0; X^A_{\theta_F\wedge\delta}))} \\ &\leq D_\delta(A, A_0) M(F) \|U\|_{L^p(\Omega; L^\infty(0,T_0;X))} \\ &\lesssim D_\delta(A, A_0) M(F) (1 + \|x_0\|_{L^p(\Omega;X)}), \end{aligned} \tag{10.1.10}$$

where the penultimate estimate follows by the linear growth condition on F and the final estimate by (5.2.7). Note that the implied constants are independent of T_0, and depend on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, and on A and A_0 only in terms of ω, θ and K.

Part 2e. Recall that $\eta_G \leq 0$. By equation (2.6.3) there exists an $\mathscr{M}$ depending only on ω, θ, K and T such that for all $t \in [0,T]$ we have:

$$\begin{aligned} t^{-\eta_G + \varepsilon/2} \| \tfrac{d}{dt} S_0(t)\|_{\mathscr{L}(X^A_{0,-\eta_G}, X_0)} &+ (\varepsilon/2 - \eta_G) t^{-\eta_G + \varepsilon/2 - 1} \|S_0(t)\|_{\mathscr{L}(X^A_{0,-\eta_G}, X_0)} \\ &\leq \mathscr{M} t^{-1+\varepsilon/2}. \end{aligned}$$

By applying Proposition A.11 with $Y_1 = X^{A_0}_{0,\eta_G}$, $Y_2 = X$, $\Psi(s) = S_0(s)$, $\eta = \alpha$, $\alpha = \alpha$, $\theta = -\eta_G + \varepsilon/2$ and $g(v) = \mathscr{M} v^{-1+\varepsilon/2}$ and

$$\Phi(s) = P_0[G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))]$$

we obtain:

$$\begin{aligned} \Big\| t \mapsto \int_0^t & \tilde{S}_0(t-s) [G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))]\, dW_H(s) \Big\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)} \\ &\lesssim T_0^{\frac{1}{2}\varepsilon} \sup_{0\leq t\leq T_0} \|s \mapsto (t-s)^{-\alpha} P_0[G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))]\|_{L^p(\Omega; \gamma(0,t; X^{A_0}_{0,\eta_G}))} \\ &\lesssim T_0^{\frac{1}{2}\varepsilon} (1 + D_\delta(A, A_0)) \mathrm{Lip}_\gamma(G) \|U - \tilde{U}^{(0)}\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;X)}, \end{aligned} \tag{10.1.11}$$

where the final estimate follows from estimates (5.2.4) and (10.1.5). Note that the implied constants are independent of T_0, and depend on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, and on A and A_0 only in terms of ω, θ and K.

Part 2f. By Proposition 10.2 with $\beta = \theta_G \wedge \delta \in [\delta - 1, \delta]$ we have that there exists a constant $\mathscr{M}$ depending only on $\|P_0\|_{\mathscr{L}(X,X_0)}$, ω, θ, K, $(\delta - \theta_G) \vee 0$ and T such that for all $t \in [0,T]$ we have:

$$\begin{aligned}
t^{(\delta-\theta_G)^+ + \varepsilon/2} &\| \tfrac{d}{dt}[S(t) - \tilde{S}_0(t)] \|_{\mathscr{L}(X^A_{\theta_G\wedge\delta}, X)} \\
&\quad + ((\delta-\theta_G)^+ + \varepsilon/2) t^{(\delta-\theta_G)^+ + \varepsilon/2 - 1} \|S(t) - \tilde{S}_0(t)\|_{\mathscr{L}(X^A_{\theta_G\wedge\delta}, X)} \\
&\le \mathscr{M} D_\delta(A, A_0) t^{-1+\varepsilon/2}.
\end{aligned}$$

Thus by Proposition A.11 with $Y_1 = X^A_{\theta_G\wedge\delta}$, $Y_2 = X$, $\Phi(s) = G(s, U(s))$, $\Psi(s) = S(s) - \tilde{S}_0(s)$, $\eta = \alpha$, $\alpha = \alpha$, $\theta = (\delta - \theta_G)^+ + \varepsilon/2$ and $g(v) = \mathscr{M} D_\delta(A, A_0) v^{-1+\varepsilon/2}$ we obtain:

$$\begin{aligned}
\Big\| t \mapsto \int_0^t & [S(t-s) - \tilde{S}_0(t-s)] G(s, U(s))\, dW_H(s) \Big\|_{V^{\alpha,p}_c([0,T_0]\times\Omega;X)} \\
&\lesssim D_\delta(A, A_0) \sup_{0\le t\le T_0} \|s \mapsto (t-s)^{-\alpha} G(s, U(s))\|_{L^p(\Omega;\gamma(0,t;X^A_{\theta_G\wedge\delta}))} \\
&\le D_\delta(A, A_0) M_\gamma(G) \|U\|_{V^{\alpha,p}_c([0,T_0]\times\Omega;X)} \\
&\lesssim D_\delta(A, A_0) M_\gamma(G) \big(1 + \|x_0\|_{L^p(\Omega;X)}\big),
\end{aligned} \tag{10.1.12}$$

where the penultimate line follows by estimate (5.2.5). Note that the implied constants are independent of T_0, and depend on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$ and on A and A_0 only in terms of ω, θ and K.

Part 2g. Inserting (10.1.7), (10.1.8), (10.1.9), (10.1.10), (10.1.11), and (10.1.12) in (10.1.6) we obtain that there exists a constant $C > 0$ independent of x_0 and y_0, depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, on A and A_0 only in terms of $1 + D_\delta(A, A_0)$, ω, θ and K, and on F and G only in terms of their Lipschitz and linear growth constants $\mathrm{Lip}(F)$, $\mathrm{Lip}_\gamma(G)$, $M(F)$ and $M_\gamma(G)$, such that for all $T_0 \in [0,T]$ one has:

$$\begin{aligned}
\|U - \tilde{U}^{(0)}&\|_{V^{\alpha,p}_c([0,T_0]\times\Omega;X)} \\
&\le C T_0^{\frac{1}{2}\varepsilon} \|U - U^{(0)}\|_{V^{\alpha,p}_c([0,T_0]\times\Omega;X)} \\
&\quad + C\Big(\|x_0 - y_0\|_{L^p(\Omega;X)} + D_\delta(A, A_0)\big(1 + \|x_0\|_{L^p(\Omega;X^A_\delta)}\big)\Big).
\end{aligned}$$

Setting $T_0 = [2C]^{-2/\varepsilon}$ we obtain:

$$\begin{aligned}
\|U - \tilde{U}^{(0)}&\|_{V^{\alpha,p}_c([0,T_0]\times\Omega;X)} \\
&\le 2C\Big(\|x_0 - y_0\|_{L^p(\Omega;X)} + D_\delta(A, A_0)\big(1 + \|x_0\|_{L^p(\Omega;X^A_\delta)}\big)\Big).
\end{aligned} \tag{10.1.13}$$

Part 3. Let $t_0 \ge 0$, $z \in L^p(\Omega, \mathscr{F}_{t_0}; X)$, $T > 0$ and $\alpha \in [0, \frac{1}{2})$. By $U(z, t_0, \cdot)$, we denote the (unique) process in $V^{\alpha,p}_c([t_0, t_0 + T] \times \Omega; X)$ satisfying, for $s \in [t_0, t_0 + T]$:

$$U(z,t_0,s) = S(t-t_0)z + \int_{t_0}^{t} S(t-t_0-s)F\big(U(z,t_0,s)\big)\,ds \\ + \int_{t_0}^{t} S(t-t_0-s)G\big(U(z,t_0,s)\big)\,dW_H(s) \quad \text{a.s.}$$

The process $U^{(0)}(z,t_0,\cdot)$ is defined analogously.

From the proof of (10.1.13) it follows that for any $x \in L^p(\Omega,\mathscr{F}_{t_0};X_\delta^A)$ and $y \in L^p(\Omega,\mathscr{F}_{t_0};X)$ we have:

$$\begin{aligned} &\|U(x,t_0,\cdot) - U^{(0)}(y,t_0,\cdot)\|_{V_c^{\alpha,p}([t_0,t_0+T_0]\times\Omega;X)} \\ &\qquad \le 2C\Big(\|x-y\|_{L^p(\Omega;X)} + D_\delta(A,A_0)[1+\|x\|_{L^p(\Omega;X_\delta^A)}]\Big), \end{aligned} \tag{10.1.14}$$

with C as in (10.1.13). The remainder of the proof is entirely analogous to part 4 of the proof of Theorem 6.2. □

It remains to provide a proof for Proposition 10.2. For that purpose, we first prove the following lemma. Given the lemma, the proof of Proposition 10.2 basically follows the lines of known results concerning comparison of semigroups, see [43, Chapter III.3.b]. For notational simplicity we define the pseudo-resolvent

$$R(\lambda:\tilde{A}_0) := i_{X_0}R(\lambda:A_0)P_0, \quad \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}, \tag{10.1.15}$$

(we leave it to the reader to verify the resolvent identity).

Lemma 10.3. *Let the setting be as in Proposition 10.2. Then for all $\lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\theta}$ we have:*

$$\begin{aligned} &\|R(\lambda:A) - R(\lambda:\tilde{A}_0)\|_{\mathscr{L}(X_\beta^A,X)} \\ &\qquad \le C_{\omega,\theta,K,P_0}|\lambda-\omega|^{\delta-\beta-1}\|R(\lambda_0:A) - R(\lambda_0:\tilde{A}_0)\|_{\mathscr{L}(X_{\delta-1}^A,X)}, \end{aligned}$$

where C_{ω,θ,K,P_0} is a constant depending only on ω,θ,K and $\|P_0\|_{\mathscr{L}(X,X_0)}$.

Proof. Using only the resolvent identity and the definition of $R(\lambda:\tilde{A}_0)$ (see (10.1.15)) one may verify that the following identity holds:

$$\begin{aligned} &R(\lambda:A) - R(\lambda:\tilde{A}_0) \\ &\qquad = [I+(\lambda_0-\lambda)R(\lambda:\tilde{A}_0)][R(\lambda_0:A) - R(\lambda_0:\tilde{A}_0)](\lambda_0-A)R(\lambda:A). \end{aligned} \tag{10.1.16}$$

Moreover, one may check that:

$$\omega' + \Sigma_{\frac{\pi}{2}+\theta} \subset \big(\omega + \Sigma_{\frac{\pi}{2}+\theta}\big) \bigcap \big\{\lambda \in \mathbb{C} : |\lambda-\omega| \ge |\lambda_0-\omega|\big\}.$$

Therefore one has, for $\lambda \in \omega' + \Sigma_{\frac{\pi}{2}+\theta}$,

$$\|I+(\lambda_0-\lambda)R(\lambda:\tilde{A}_0)\|_{\mathscr{L}(X)} \le 1 + \tfrac{|\lambda_0-\lambda|}{|\lambda-\omega|}K\|P_0\|_{\mathscr{L}(X,X_0)} \le 1 + 2K\|P_0\|_{\mathscr{L}(X,X_0)}.$$

From (10.1.16) one obtains:

$$\|R(\lambda:A)-R(\lambda:\tilde{A}_0)\|_{\mathscr{L}(X^A_\beta,X)} \leq (1+2K\|P_0\|_{\mathscr{L}(X,X_0)}) \\ \times \|R(\lambda_0:A)-R(\lambda_0:\tilde{A}_0)\|_{\mathscr{L}(X^A_{\delta-1},X)}\|(\lambda_0-A)R(\lambda:A)\|_{\mathscr{L}(X^A_\beta,X^A_{\delta-1})}. \tag{10.1.17}$$

Let $\bar{\lambda}\in\mathbb{C}$, $\Re e(\bar{\lambda})>\omega$ be such that $|\bar{\lambda}-\lambda_0|\leq 2|\bar{\lambda}-\omega|$ (if $\Re e(\lambda_0)>\omega$ one may simply pick $\bar{\lambda}=\lambda_0$). For $\eta\in\mathbb{R}$ and $x\in X^A_\eta$ set $\|x_0\|_{X^A_\eta}:=\|(\bar{\lambda}-A)^\eta x\|_X$. Then:

$$\begin{aligned}\|(\lambda_0-A)R(\lambda:A)\|_{\mathscr{L}(X^A_\beta,X^A_{\delta-1})} &= \|(\bar{\lambda}-A)^{\delta-\beta-1}(\lambda_0-A)R(\lambda:A)\|_{\mathscr{L}(X)}\\ &\leq \big(1+\tfrac{|\bar{\lambda}-\lambda_0|}{|\bar{\lambda}-\omega|}K\big)\|(\bar{\lambda}-A)^{\delta-\beta}R(\lambda:A)\|_{\mathscr{L}(X)}\\ &\leq (1+2K)\|(\bar{\lambda}-A)^{\delta-\beta}R(\lambda:A)\|_{\mathscr{L}(X)}.\end{aligned}$$

If $\delta-\beta=1$ then:

$$\|(\bar{\lambda}-A)^{\delta-\beta}R(\lambda:A)\|_{\mathscr{L}(X)} = \|(\bar{\lambda}-A)R(\lambda:A)\|_{\mathscr{L}(X)} \leq 1+2K.$$

If $\delta-\beta=0$ then:

$$\|(\bar{\lambda}-A)^{\delta-\beta}R(\lambda:A)\|_{\mathscr{L}(X)} = \|R(\lambda:A)\|_{\mathscr{L}(X)} \leq K|\lambda-\omega|^{-1}.$$

For $\delta-\beta\in(0,1)$ we have, by Theorem 2.20,

$$\begin{aligned}\|(\bar{\lambda}-A)^{\delta-\beta}R(\lambda:A)\|_{\mathscr{L}(X)} &\leq 2(1+K)\|R(\lambda:A)\|^{1+\beta-\delta}_{\mathscr{L}(X)}\|(\bar{\lambda}-A)R(\lambda:A)\|^{\delta-\beta}_{\mathscr{L}(X)}\\ &\leq 2(1+K)(1+2K)^{\delta-\beta}K^{1+\beta-\delta}|\lambda-\omega|^{\delta-\beta-1}\\ &\leq 2(1+2K)^2|\lambda-\omega|^{\delta-\beta-1}.\end{aligned}$$

Substituting this into (10.1.17) one obtains:

$$\begin{aligned}&\|R(\lambda:A)-R(\lambda:\tilde{A}_0)\|_{\mathscr{L}(X^A_\beta,X)}\\ &\quad\leq 2(1+2K)^4\|P_0\|_{\mathscr{L}(X,X_0)}|\lambda-\omega|^{\delta-\beta-1}\|R(\lambda_0:A)-R(\lambda_0:\tilde{A}_0)\|_{\mathscr{L}(X^A_{\delta-1},X)}.\end{aligned}$$

□

Proof (of Proposition 10.2). Let ω' be as defined in Lemma 10.3. For brevity set $\varepsilon=\delta-\beta$. First of all observe that

$$\lim_{s\downarrow 0} s^\varepsilon\|[S(s)-\tilde{S}_0(s)]\|_{\mathscr{L}(X^A_\beta,X)}=0.$$

Fix $\theta'\in(0,\theta)$. It follows from [114, Theorem 1.7.7], that one has, for all $t>0$:

$$S(t)=\tfrac{1}{2\pi i}\int_{\omega'+\varGamma_{\theta'}} e^{\lambda t}R(\lambda:A)d\lambda;$$

where $\Gamma_{\theta'}$ is the path composed from the two rays $re^{i(\frac{\pi}{2}+\theta')}$ and $re^{-i(\frac{\pi}{2}+\theta')}$, $0 \leq r < \infty$, and is oriented such that $\Im m(\lambda)$ increases along $\Gamma_{\theta'}$. As $\omega' \geq \omega$, the integral is well-defined as $\mathscr{L}(X)$-valued Bochner integral, and for $t > 0$ one has:

$$\tfrac{d}{dt}S(t) = \tfrac{1}{2\pi i}\int_{\omega'+\Gamma_{\theta'}} \lambda e^{\lambda t} R(\lambda : A)\,d\lambda;$$

the integral again being well-defined as $\mathscr{L}(X)$-valued Bochner integrals (see also the proof of [114, Theorem 2.5.2]). Analogous identities hold for $\tilde{S}_0$ and $R(\lambda : \tilde{A}_0)$.

First let us assume that $\varepsilon \in (0,1)$. Below we shall apply Lemma 10.3, observing that for $r \in [0,\infty)$ we have

$$|\omega' + re^{\pm i(\frac{\pi}{2}+\theta')} - \omega| \geq K_\theta r,$$

where K_θ is a constant depending only on θ. Note that we use the coordinate transform $\lambda = \omega' + re^{\pm i(\frac{\pi}{2}+\theta')}$. For $s > 0$ we have:

$$\begin{aligned}
\|S(s) - \tilde{S}_0(s)\|_{\mathscr{L}(X^A_\beta,X)} &= \Big\| \tfrac{1}{2\pi i}\int_{\Gamma_{\theta'}} e^{\lambda s}[R(\lambda : A) - R(\lambda : \tilde{A}_0)]\,d\lambda \Big\|_{\mathscr{L}(X^A_\beta,X)}\\
&\leq \tfrac{1}{2\pi}\int_0^\infty \big|e^{-i(\frac{\pi}{2}+\theta')+(\omega'+re^{-i(\frac{\pi}{2}+\theta')})s}\big|\\
&\qquad \times \big\|R(\omega' + re^{-i(\frac{\pi}{2}+\theta')} : A) - R(\omega' + re^{-i(\frac{\pi}{2}+\theta')} : \tilde{A}_0)\big\|_{\mathscr{L}(X^A_\beta,X)}\,dr\\
&\quad + \tfrac{1}{2\pi}\int_0^\infty \big|e^{i(\frac{\pi}{2}+\theta')+(\omega'+re^{i(\frac{\pi}{2}+\theta')})s}\big|\\
&\qquad \times \big\|R(\omega' + re^{i(\frac{\pi}{2}+\theta')} : A) - R(\omega' + re^{i(\frac{\pi}{2}+\theta')} : \tilde{A}_0)\big\|_{\mathscr{L}(X^A_\beta,X)}\,dr\\
&\leq \tfrac{1}{\pi}C_{\omega,\theta,K,P_0}K_\theta D_\delta(A,A_0)e^{\omega' s}\int_0^\infty r^{\varepsilon-1}e^{-rs\sin\theta'}\,dr\\
&= \tfrac{1}{\pi}C_{\omega,\theta,K,P_0}K_\theta D_\delta(A,A_0)[s\sin\theta']^{-\varepsilon}e^{\omega' s}\int_0^\infty u^{\varepsilon-1}e^{-u}du\\
&= \tfrac{\Gamma(\varepsilon)}{\pi}[\sin\theta']^{-\varepsilon}C_{\omega,\theta,K,P_0}D_\delta(A,A_0)s^{-\varepsilon}e^{\omega' s}.
\end{aligned}$$

For $\varepsilon = 0$ one may avoid the singularity in 0 in the usual way: for $s > 0$ given we integrate over

$$\omega' + \Gamma_{\theta',s} = (\omega' + \Gamma^{(1)}_{\theta',s}) \cup (\omega' + \Gamma^{(2)}_{\theta',s}) \cup (\omega' + \Gamma^{(3)}_{\theta',s}),$$

where $\Gamma^{(1)}_{\theta',s}$ and $\Gamma^{(2)}_{\theta',s}$ are the rays $re^{i(\frac{\pi}{2}+\theta')}$ and $re^{-i(\frac{\pi}{2}+\theta')}$, $s^{-1} \leq r < \infty$, and $\Gamma^{(3)}_{\theta',s} = s^{-1}e^{i\phi}$, $\phi \in [-\frac{\pi}{2} - \theta', \frac{\pi}{2} + \theta']$. This leads to the following estimate:

$$\begin{aligned}
\|S(s) - \tilde{S}_0(s)\|_{\mathscr{L}(X^A_\beta,X)} &= \Big\| \tfrac{1}{2\pi i}\int_{\omega'+\Gamma_{\theta',s}} e^{\lambda s}[R(\lambda : A) - R(\lambda : \tilde{A}_0)]\,d\lambda \Big\|_{\mathscr{L}(X^A_\beta,X)}\\
&\leq C_{\omega,\theta,K,P_0}K_\theta D_\delta(A,A_0)e^{\omega' s}\Big[\tfrac{1}{\pi}\int_{s^{-1}}^\infty r^{-1}e^{-rs\sin\theta'}\,dr + e\Big]
\end{aligned}$$

$$\begin{aligned}
&\le C_{\omega,\theta,K,P_0} K_\theta D_\delta(A,A_0)\big[[\pi \sin\theta']^{-1} e^{-\sin\theta'} + e\big] e^{\omega' s} \\
&\le 2[\sin\theta']^{-1} C_{\omega,\theta,K,P_0} K_\theta D_\delta(A,A_0) e^{\omega' s}.
\end{aligned}$$

Recalling that $\varepsilon = \delta - \beta$ this proves the uniform boundedness estimate of (10.1.1).

Similarly to the above, for $\varepsilon \in [0,1]$ and $s > 0$ we have:

$$\begin{aligned}
&\|\tfrac{d}{ds} S(s) - \tfrac{d}{ds}\tilde{S}_0(s)\|_{\mathscr{L}(X_\beta^A,X)} \\
&\quad = \Big\| \tfrac{1}{2\pi i} \int_{\Gamma_{\omega'+\theta'}} \lambda e^{\lambda s}[R(\lambda : A) - R(\lambda : \tilde{A}_0)] d\lambda \Big\|_{\mathscr{L}(X_\beta^A,X)} \\
&\quad \le \tfrac{1}{2\pi} e^{\omega' s} \int_0^\infty r e^{-rs\sin\theta'} \big\| R(re^{-i(\frac{\pi}{2}+\theta')} : A) - R(re^{-i(\frac{\pi}{2}+\theta')} : \tilde{A}_0)\big\|_{\mathscr{L}(X_\beta^A,X)} dr \\
&\qquad + \tfrac{1}{2\pi} e^{\omega' s} \int_0^\infty r e^{-rs\sin\theta'} \big\| R(re^{i(\frac{\pi}{2}+\theta')} : A) - R(re^{i(\frac{\pi}{2}+\theta')} : \tilde{A}_0)\big\|_{\mathscr{L}(X_\beta^A,X)} dr \\
&\quad = \tfrac{1}{\pi} C_{\omega,\theta,K,P_0} K_\theta D_\delta(A,A_0)[s\sin\theta']^{-1-\varepsilon} e^{\omega' s} \int_0^\infty u^\varepsilon e^{-u} du \\
&\quad = \tfrac{\varepsilon\Gamma(\varepsilon)}{\pi}[\sin\theta']^{-1-\varepsilon} C_{\omega,\theta,K,P_0} K_\theta D_\delta(A,A_0) s^{-1-\varepsilon} e^{\omega' s}.
\end{aligned}$$

Recalling that $\varepsilon = \delta - \beta$ this proves the uniform boundedness estimate of (10.1.2).

Concerning the γ-boundedness estimates, fix $\alpha > \varepsilon$. By Proposition 2.12 one has:

$$\begin{aligned}
\gamma_{[X_\beta^A,X]}\Big(\{s^\alpha[S(s) - \tilde{S}_0(s)] : s \in [0,t]\}\Big) &\le \int_0^t \big\|\tfrac{d}{ds}\big(s^\alpha[S(s) - \tilde{S}_0(s)]\big)\big\|_{\mathscr{L}(X_\beta^A,X)} ds \\
&\le \int_0^t \alpha s^{\alpha-1}\big\|S(s) - \tilde{S}_0(s)\big\|_{\mathscr{L}(X_\beta^A,X)} ds \\
&\quad + \int_0^t s^\alpha \big\|\tfrac{d}{ds}S(s) - \tfrac{d}{ds}\tilde{S}_0(s)\big\|_{\mathscr{L}(X_\beta^A,X)} ds.
\end{aligned}$$

Substituting (10.1.1) and (10.1.2) into the above one obtains that there exists a constant C depending only on ω, θ, K, $\varepsilon = \delta - \beta$, and $\|P_0\|_{\mathscr{L}(X,X_0)}$ such that:

$$\begin{aligned}
\gamma_{[X_\beta^A,X]}\Big(\{s^\alpha[S(s) - \tilde{S}_0(s)] : s \in [0,t]\}\Big) &\le CD_\delta(A,A_0)\int_0^t e^{\omega' s} s^{\alpha-1-\varepsilon} e^{\bar{\omega} t} ds \\
&\le Ce^{(\omega' T)\vee 0} t^{\alpha-\varepsilon} D_\delta(A,A_0),
\end{aligned}$$

as $\alpha > \varepsilon$. □

Corollary 10.4. *Let the setting be as in Theorem 10.1. Let $\lambda \in [0,\frac{1}{2})$ satisfy*

$$0 \le \lambda < \min\{1 - (\delta - \theta_F) \vee 0, \tfrac{1}{2} - \tfrac{1}{p} - (\delta - \theta_G) \vee 0\}.$$

Suppose $x_0 \in L^p(\Omega, \mathscr{F}_0; X_\delta^A)$ and $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$, then:

$$\|U - Sx_0 - i_{X_0}(U^{(0)} - S_0 P_0 y_0)\|_{L^p(\Omega; C^\lambda([0,T];X))}$$

$$\lesssim \|x_0 - y_0\|_{L^p(\Omega,X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega;X_\delta^A)}),$$

with implied constant depending on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, on A and A_0 only in terms of $1+D_\delta(A, A_0)$, ω, θ and K, and on F and G only in terms of their Lipschitz and linear growth constants $Lip(F)$, $Lip_\gamma(G)$, $M(F)$, and $M_\gamma(G)$.

Proof. As before, we write:

$$\begin{aligned}
&\|U - Sx_0 - i_{X_0}(U^{(0)} - S_0P_0y_0)\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\quad = \Big\|t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s,U(s)) - F(s,\tilde{U}^{(0)}(s))]\,ds\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\quad + \Big\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s,U(s))\,ds\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\quad + \Big\|t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s,U(s)) - G(s,\tilde{U}^{(0)}(s))]\,dW_H(s)\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\quad + \Big\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]G(s,U(s))\,dW_H(s)\Big\|_{L^p(\Omega;C^\lambda([0,T];X))}.
\end{aligned} \tag{10.1.18}$$

For the first and second term on the right-hand side of (10.1.18) we apply Proposition A.12. Note that as before we may pick $\eta_F, \eta_G \le 0$ such that $\eta_F < \theta_F - \delta$ and $\eta_G < \theta_G - \delta$ and

$$\lambda < \min\{1 + \eta_F, \tfrac{1}{2} - \tfrac{1}{p} + \eta_G\}.$$

Our choice of Y_1, Y_2, Φ, Ψ is the same as in part 2c, respectively 2d, of the proof of Theorem 10.1, whereas we set $\theta = 1 - \lambda$. This leads to the following estimates:

$$\begin{aligned}
&\Big\|t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s,U(s)) - F(s,\tilde{U}^{(0)}(s))]\,ds\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\qquad \lesssim \|U - \tilde{U}^{(0)}(s)\|_{L^p(\Omega;L^\infty(0,T;X))} \le \|U - \tilde{U}^{(0)}(s)\|_{V_c^{\alpha,p}([0,T]\times\Omega;X)},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s,U(s))\,ds\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\qquad \lesssim D_\delta(A, A_0)\|U\|_{L^p(\Omega;L^\infty(0,T;X))} \lesssim D_\delta(A, A_0)\big(1 + \|x_0\|_{L^p(\Omega,X)}\big).
\end{aligned}$$

For the third and fourth term on the right-hand side of (10.1.18) we apply Corollary (A.10) with $\beta = \lambda$ and $\alpha \in (0, \frac{1}{2})$ such that $\alpha > \lambda + \frac{1}{p} + \eta_G$. The choice of Y_1, Y_2, Φ and Ψ is as in parts 2e and 2f of the proof of Theorem 10.1. This leads to:

$$\begin{aligned}
&\Big\|t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s,U(s)) - G(s,\tilde{U}^{(0)}(s))]\,dW_H(s)\Big\|_{L^p(\Omega;C^\lambda([0,T];X))} \\
&\qquad \lesssim \|U - \tilde{U}^{(0)}(s)\|_{V_c^{\alpha,p}([0,T]\times\Omega;X)},
\end{aligned}$$

and

$$\begin{aligned}\Big\| t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)] G(s, U(s))\, dW_H(s) \Big\|_{L^p(\Omega; C^\lambda([0,T];X))} \\ \lesssim D_\delta(A, A_0) \|U\|_{V_c^{\alpha,p}([0,T]\times\Omega;X)} \lesssim D_\delta(A, A_0)\big(1 + \|x_0\|_{L^p(\Omega,X)}\big).\end{aligned}$$

Combining these estimates with Theorem 10.1 gives the desired result. It goes without saying that all the implied constants above depend on X_0 only in terms of $\|P_0\|_{\mathscr{L}(X,X_0)}$, on A and A_0 only in terms of $1 + D_\delta(A, A_0)$, ω, θ and K, and on F and G only in terms of their Lipschitz and linear growth constants $\mathrm{Lip}(F)$, $\mathrm{Lip}_\gamma(G)$, $M(F)$, and $M_\gamma(G)$. $\square$

11

Space approximations

In this chapter we use the perturbation theorem of the previous chapter, i.e., Theorem 10.1, to prove pathwise convergence of the Yosida approximation of (SDE), and, for the case that X is a Hilbert space, pathwise convergence of certain Galerkin and finite element schemes.

Due to the fact that we obtain pathwise convergence of the approximations (see Corollaries 11.2, 11.6 and 11.10), it is possible to obtain convergence results for the case that the non-linear terms F and G are merely locally Lipschitz, as was demonstrated for time discretizations in Chapter 8. For space discretizations this was demonstrated by Jentzen [68] for equations with additive noise, with convergence rates that compare to what we obtain. Another pathwise convergence result for Galerkin approximations of equations with additive noise is presented in [80]. The convergence rates obtained there correspond to the ones we obtain.

Concerning pathwise convergence for equations with multiplicative noise, the only result known to us is due to Gyöngy [54], who considers convergence of the finite difference method for the one-dimensional heat equation with space-time white noise. To be precise, his work considers the example of Chapter 9 with $a_2 \equiv 1$, $a_1 \equiv 0$, and obtains pathwise convergence in probability, but without convergence rates.

Results concerning pointwise convergence of the Galerkin method have been obtained in [62] and extended to a setting comparable to the setting we study in [133].

An important issue to keep in mind when using Theorem 10.1 to prove convergence of approximations, is that the constant in (10.0.2) depends on ω, θ, and K, where ω, θ, and K are such that A and A_0 are both of type (ω, θ, K). Thus given a sequence of operators $(A_n)_{n\in\mathbb{N}}$ approximating A it is necessary to find $\omega \geq 0$, $\theta \in (0, \frac{2}{\pi}]$, and $K > 0$ such that A_n is of type (ω, θ, K) for all $n \in \mathbb{N}$. We call this property *uniform analyticity of* $(A_n)_{n\in\mathbb{N}}$. In the case that X is a Hilbert space, it is known that uniform analyticity is easy to check for many practical purposes, see Lemma 11.4 below.

11.1 Yosida approximations

Consider (SDE) under the assumptions **(A)**, **(F)**, and **(G)** with the additional assumption that $\theta_F, \theta_G \geq 0$. We define $A_n := nAR(n : A)$ to be the n^{th} Yosida approximation of A. We will use U to denote the solution to (SDE) with operator A and initial data $x_0 \in L^p(\Omega, \mathscr{F}_0; X)$ and $U^{(n)}$ to denote the solution to (SDE) with operator A_n, $n \in \mathbb{N}$, and initial data $y_0 \in L^p(\Omega, \mathscr{F}_0; X)$.

Theorem 11.1. *For any $\eta > 0$ and $p \in (2, \infty)$ such that*

$$\eta < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G\}$$

and any $\alpha \in [0, \frac{1}{2})$ we have, assuming $y_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta^A)$:

$$\|U - U^{(n)}\|_{V_c^{\alpha,p}([0,T]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega;X_\eta^A)}),$$

with implied constants independent of n, x_0 and y_0.

The following corollary is a direct consequence of the Borel-Cantelli lemma and the above theorem (see Corollary (6.6) or [81, Lemma 2.1]):

Corollary 11.2. *Let $\eta > 0$ and $p \in (2, \infty)$ be such that*

$$\eta + \tfrac{1}{p} < \min\{1 - (\tfrac{1}{\tau} - \tfrac{1}{2}) + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G, 1\}$$

and assume $y_0 = x_0 \in L^p(\Omega, \mathscr{F}_0; X_\eta^A)$. Then there exists a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$:

$$\|U - U^{(n)}\|_{C([0,T];\mathscr{H})} \leq \chi n^{-\eta}.$$

To prove Theorem 11.1 we shall need the following lemma:

Lemma 11.3. *Let $\beta \in [0, 1]$. Then there exists a constant K' such that for all $n \geq 2\omega$ and all $x \in X_\beta^A$ one has:*

$$\|(2\omega I - A_n)^\beta x\| \leq K'\|(2\omega I - A)^\beta x\|.$$

Proof. Observe that

$$\begin{aligned} 2\omega I - A_n &= (n + 2\omega)(\tfrac{2\omega n}{n+2\omega} I - A)R(n : A) \\ &= [(n + 2\omega)I - 4\omega^2 R(2\omega : A)](2\omega I - A)R(n : A). \end{aligned} \tag{11.1.1}$$

Thus for $x \in D(A)$ and $n \geq 2\omega$ we have:

$$\begin{aligned} \|(2\omega I - A_n)x\| &\leq \|[(n + 2\omega)I - 4\omega^2 R(2\omega : A)]R(n : A)\|_{\mathscr{L}(X)}\|(2\omega I - A)x\| \\ &\leq [K\tfrac{n+2\omega}{n-\omega} + K^2\tfrac{4\omega}{n-\omega}]\|(2\omega I - A)x\| \leq 4K(1 + K)\|(2\omega I - A)x\|. \end{aligned}$$

This proves the lemma for $\beta = 1$. For $\beta = 0$ the lemma is trivial. For $\beta \in (0, 1)$ we need two extra observations.

Firstly, let $\mu, \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}$. We have:

$$\|e^{-(\lambda I - A)R(\mu:A)t}\|_{\mathscr{L}(X)} = e^{-t}\|e^{(\mu-\lambda)R(\mu,A)t}\|_{\mathscr{L}(X)} \leq e^{-t+\frac{|\mu-\lambda|}{|\mu-\omega|}Kt}.$$

Secondly, for $s > \omega$ and $\beta \in (0,1)$ we have, by definition, (see (2.6.2)):

$$(sI - A)^{-\beta}x = \tfrac{\sin(\pi\beta)}{\pi}\int_0^\infty t^{-\beta}((t+s)I - A)^{-1}xdt,$$

and hence

$$\begin{aligned}\|(sI - A)^{-\beta}\|_{\mathscr{L}(X)} &\leq K\tfrac{\sin(\pi\beta)}{\pi}\int_0^\infty t^{-\beta}(t+s-\omega)^{-1}dt \\ &\leq K\tfrac{\sin(\pi\beta)}{\pi}\Big[(s-\omega)^{-1}\int_0^{s-\omega} t^{-\beta}dt + \int_{s-\omega}^\infty t^{-1-\beta}dt\Big] \\ &= K\tfrac{\sin(\pi\beta)}{\pi\beta(1-\beta)}(s-\omega)^{-\beta}.\end{aligned} \tag{11.1.2}$$

Now suppose $n \geq 2\omega(1+4K)$, $\lambda = 2\omega$, $\mu = \frac{n\lambda}{\lambda+n} = \frac{2\omega n}{2\omega+n}$. In that case one may check that $\frac{|\mu-\lambda|}{|\mu-\omega|}K \leq \frac{1}{2}$, and thus that for $\beta \in (0,1)$:

$$\begin{aligned}&\|[-(2\omega I - A)R(\tfrac{2\omega n}{2\omega+n} : A)]^{-\beta}\|_{\mathscr{L}(X)} \\ &\qquad = \Big\|\tfrac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-(2\omega I - A)R(\frac{2\omega n}{2\omega+n}:A)t}dt\Big\|_{\mathscr{L}(X)} \leq 2^\beta.\end{aligned}$$

As $[-(2\omega I - A)R(\frac{2\omega n}{2\omega+n} : A)]^{-\beta} \in \mathscr{L}(X)$ for any $n \geq 2\omega$, it follows that there exists a constant $M > 0$ such that for all $n \geq 2\omega$:

$$\|[-(2\omega I - A)R(\tfrac{2\omega n}{2\omega+n} : A)]^{-\beta}\|_{\mathscr{L}(X)} \leq M. \tag{11.1.3}$$

For $\beta \in (0,1)$ and $x \in X_\beta^A$ we have, by standard theory on functional calculus (see [60]), equation 11.1.1, and the estimates (11.1.2) and (11.1.3):

$$\begin{aligned}\|(2\omega I - A_n)^\beta x\| &= \|(n+2\omega)^\beta(\tfrac{2\omega n}{2\omega+n}I - A)^\beta(nI - A)^{-\beta}x\| \\ &\leq (n+2\omega)^\beta\|(nI - A)^{-\beta}\|_{\mathscr{L}(X)} \\ &\quad \times \|\big[-(2\omega I - A)R(\tfrac{2\omega n}{2\omega+n} : A)\big]^{-\beta}\|_{\mathscr{L}(X)}\|(2\omega I - A)^\beta x\| \\ &\leq 4^\beta KM\tfrac{\sin(\pi\beta)}{\pi\beta(1-\beta)}\|(2\omega I - A)^\beta x\|.\end{aligned}$$

□

Proof (of Theorem 11.1.). Without loss of generality we may assume $\omega \geq 0$. In order to apply Theorem 10.1, we must prove that A_n, $n \geq 2\omega$, are of uniform type, i.e., that there exist $\bar{\omega} \in \mathbb{R}, \bar{\theta} \in (0, \frac{\pi}{2})$ and $\bar{K} > 0$ such that A_n is of type $(\bar{\omega}, \bar{\theta}, \bar{K})$ for all $n \geq 2\omega$. Fix $n \geq 2\omega$. One checks that:

$$R(\lambda : A_n) = (n+\lambda)^{-1}(n - A)R(\tfrac{\lambda n}{n+\lambda} : A) \tag{11.1.4}$$

whenever $\frac{\lambda n}{n+\lambda} \in \omega + \Sigma_{\frac{\pi}{2}+\theta}$. Define $f : \mathbb{C} \to \mathbb{C}$; $f(z) = \frac{nz}{n-z}$. From (11.1.4) it follows that $\lambda \in \varrho(A)$ if and only if $f(\lambda) \in \varrho(A_n)$. By standard theory on Möbius transforms we have that

$$f\big(\{\omega + \Sigma_{\frac{\pi}{2}+\theta}\}\big) = \mathbb{C} \setminus \big(D_1 \cap D_2\big),$$

where D_1 and D_2 are both closed disks with radius $\frac{n^2}{2(n-\omega)\cos\theta}$; the center of D_1 is in $\frac{n}{2(n-\omega)}(2\omega - n, \tan(\theta))$ and the center of D_2 is is in $\frac{n}{2(n-\omega)}(2\omega - n, -\tan(\theta))$. The boundaries of these disks intersect each other on the real axis at the points $-n$ and $\frac{n\omega}{n-\omega}$. The angle at intersection is $\pi - 2\theta$. As $n \geq 2\omega$ we have $\frac{n\omega}{n-\omega} \leq 2\omega$ and thus $\varrho(A_n) \subset 2\omega + \Sigma_{\frac{\pi}{2}+\theta}$. It remains to prove the desired estimate on the resolvent.

Using (11.1.4) one may check that for $\lambda \in 2\omega + \Sigma_{\frac{\pi}{2}+\theta}$ we have:

$$R(\lambda : A) - R(\lambda : A_n) = -(\lambda + n)^{-1} A^2 R(\tfrac{\lambda n}{n+\lambda} : A) R(\lambda : A). \tag{11.1.5}$$

Thus by (2.6.1) we have, for $\lambda \in \omega(1 + 2(\cos\theta)^{-1}) + \Sigma_{\frac{\pi}{2}+\theta}$:

$$\|R(\lambda : A) - R(\lambda : A_n)\|_{\mathscr{L}(X)} \leq (1 + 2K)^2 |\lambda + n|^{-1} \leq (1 + 2K)^2 |\lambda - \omega|^{-1}.$$

The final estimate follows from the fact that by standard theory on Möbius transforms we have that $\frac{|\lambda-\omega|}{|\lambda+n|} \leq 1$ for $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}$. In conclusion we have, for $\lambda \in \omega(1 + 2(\cos\theta)^{-1}) + \Sigma_{\frac{\pi}{2}+\theta}$:

$$\begin{aligned} \|R(\lambda : A_n)\|_{\mathscr{L}(X)} &\leq \|R(\lambda : A) - R(\lambda : A_n)\|_{\mathscr{L}(X)} + \|R(\lambda : A)\|_{\mathscr{L}(X)} \\ &\leq [K + (1 + 2K)^2] |\lambda - \omega|^{-1}. \end{aligned}$$

This proves that A_n is of type $(\omega(1 + 2(\cos\theta)^{-1}), \theta, K(1 + 2K)^2)$ for all $n \geq 2\omega$.

It also follows from (11.1.5) that if we take, for example, $\lambda_0 = \omega(1 + 2(\cos\theta)^{-1})$, then we have, for $n \geq 2\omega$:

$$D_1(A, A_n) = \|R(\lambda_0 : A) - R(\lambda_0 : A_n)\|_{\mathscr{L}(X)} \leq (1 + 2K)^2 n^{-1}.$$

In other words, for all $n \in \mathbb{N}$ condition (10.0.1) in Theorem 10.1 is satisfied with $\delta = 1$ and $\lambda_0 = \omega(1 + 2(\cos\theta)^{-1})$. Thus we can apply Theorem 10.1 to obtain the desired result for the case $\theta_F > -\frac{1}{2} + \frac{1}{\tau}$, where τ is the type of X, and $\theta_G > \frac{1}{2} + \frac{1}{p}$. Concerning the implied constant in (10.0.2) we use that $1 + D_1(A, A_n)$ is uniformly bounded in n, both from above and away from 0.

In order to get the desired result for general $\theta_F, \theta_G \geq 0$ we seek an estimate for $\|R(\lambda_0 : A) - R(\lambda_0 : A_n)\|_{\mathscr{L}(X^{A_n}_{\delta-1}, X)}$. (Note that $R(\lambda_0 : A) - R(\lambda_0 : A_n) \notin \mathscr{L}(X^{A}_{\delta-1}, X)$ for any $\delta < 1$.) For $n \geq \omega(1 + 2(\cos\theta)^{-1})$ we have, by estimate (2.6.1), that $\|(2\omega I - A_n)\|_{\mathscr{L}(X)} \leq 2n(1 + K)$. Thus by Theorem 2.20 we have, for $\delta \in (0, 1)$:

$$\|(2\omega I - A_n)^{1-\delta} x\| \leq 2(1 + 2K) \|x\|^{\delta} \|A_n x\|^{1-\delta}$$

$$\leq 2^{2-\delta}(1+2K)^{2-\delta}n^{1-\delta}\|x\|.$$

It follows that for $\delta \in [0,1)$ we have:

$$D_\delta(A, A_n) = \|R(\lambda_0 : A) - R(\lambda_0 : A_n)\|_{\mathscr{L}(X^{A_n}_{\delta-1}, X)} \leq 2^{2-\delta}(1+2K)^{4-\delta}n^{-\delta}.$$

In particular, $1 + D_\delta(A, A_n)$ is uniformly bounded in n. We are now ready to apply Theorem 10.1. First of all observe that by Lemma 11.3 we have that $F : [0,T] \times X \to X^{A_n}_{\theta_F}$ is Lipschitz continuous and of linear growth for all $n \geq 2\omega$ with Lipschitz and growth constants independent of n, and $G : [0,T] \times X \to \gamma(H, X^{A_n}_{\theta_G})$ is L^2_γ-Lipschitz continuous and of linear growth for all $n \geq 2\omega$ with Lipschitz and growth constants independent of n.

Fix $\eta \in [0,1]$ such that $\eta < \min\{\frac{3}{2} - \frac{1}{\tau} + \theta_F, \frac{1}{2} - \frac{1}{p} + \theta_G\}$ and suppose $y_0 \in L^p(\Omega, \mathscr{F}_0; X^A_\eta)$. It follows from Theorem 10.1 with $\delta = \eta$, but with A_n playing the role of A and A playing the role of A_0, that:

$$\|U - U^{(n)}\|_{V^{\alpha,p}_c([0,T]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega;X^A_\eta)}),$$

with implied constants independent of n, x_0 and y_0. □

11.2 The Hilbert space case: Galerkin and finite element methods

Consider (SDE) in the case that $X = \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space. Technically there is no reason why one cannot assume that this Hilbert space is the same as the one that the Brownian motion W_H in (SDE) takes values in (i.e., one may assume $H = \mathscr{H}$). However, for notational clarity we choose to distinguish between the two spaces. By $\mathscr{H}^A_{\theta_F}$ we denote the fractional domain space/extrapolation space of $A : D(A) \subset \mathscr{H} \to \mathscr{H}$. By the examples on page 84 and equation (2.3.1) the assumptions $(\mathbf{A}_H)$, $(\mathbf{F}_H)$, and $(\mathbf{G}_H)$ reduce to

$(\mathbf{A}_H)$ A generates an analytic C_0-semigroup on $\mathscr{H}$.

$(\mathbf{F}_H)$ For some $\theta_F > -1$, the function $F : [0,T] \times \mathscr{H} \to \mathscr{H}^A_{\theta_F}$ is measurable in the sense that for all $x \in \mathscr{H}$ the mapping $F(\cdot, x) : [0,T] \to \mathscr{H}^A_{\theta_F}$ is strongly measurable. Moreover, F is uniformly Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants C_0 and C_1 such that for all $t \in [0,T]$ and all $x, x_1, x_2 \in \mathscr{H}$:

$$\begin{aligned}\|F(t,x_1) - F(t,x_2)\|_{\mathscr{H}^A_{\theta_F}} &\leq C_0\|x_1 - x_2\|_{\mathscr{H}},\\ \|F(t,x)\|_{\mathscr{H}^A_{\theta_F}} &\leq C_1(1 + \|x\|_{\mathscr{H}}).\end{aligned}$$

$(\mathbf{G}_H)$ For some $\theta_G > -\frac{1}{2}$, the function $G : [0,T] \times X \to \mathscr{L}_2(H, \mathscr{H}^A_{\theta_G})$ is measurable in the sense that for all $h_1 \in \mathscr{H}$ and $h_2 \in H$ the mapping $G(\cdot, h_1)h_2 : [0,T] \to \mathscr{H}^A_{\theta_G}$ is strongly measurable. Moreover, G is uniformly Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants C_0 and C_1 such that for all $s,t \in [0,T]$, and all $x, x_1, x_2 \in \mathscr{H}$ one has:

$$\begin{aligned}\|[G(s,x_1) - G(s,x_2)]\|_{\mathscr{L}_2(H,\mathscr{H}^A_{\theta_G})} &\leq C_0\|x_1 - x_2\|_{\mathscr{H}};\\ \|G(s,x)\|_{\mathscr{L}_2(H,\mathscr{H}^A_{\theta_G})} &\leq C_1\big(1 + \|x\|_{\mathscr{H}}\big).\end{aligned}$$

We shall use the following lemma to identify uniformly analytic families of operators.

Lemma 11.4. *Let $-A : D(A) \subset \mathscr{H} \to \mathscr{H}$ be m-θ-accretive for some $\theta \in [0, \frac{\pi}{2})$; i.e., $1 \in \varrho(A)$ and for all $x \in D(A)$ with $\|x\| = 1$ we have:*

$$\langle -Ax, x\rangle \in \overline{\Sigma_\theta}.$$

Suppose moreover that A is injective. Then for any $\omega > 0$ the operator $A + \omega$ is analytic of type

$$(\omega(1 + 2(\cos\theta')^{-1}), \theta', (4 + \tfrac{4}{\sqrt{3}})(1 - \sin(\theta + \theta'))^{-1})$$

for all $\theta' \in (0, \frac{\pi}{2} - \theta)$. More specifically, if A is self-adjoint and $A \leq 0$ (i.e., $\langle Ax, x\rangle \leq 0$ for all $x \in \mathscr{H}$), then $A + \omega$ is analytic of type $(\omega(1 + 2(\cos\theta')^{-1}), \theta', 2(1 - \sin(\theta'))^{-1})$ for all $\theta' \in (0, \frac{\pi}{2})$.

Proof. We first prove this Lemma for the case that $\omega = 0$. Suppose $-A$ is m-θ-accretive and injective. By [60, Proposition 7.1.1] the operator A generates an analytic C_0-semigroup on $\Sigma_{\theta'}$ for all $\theta' \in (0, \frac{\pi}{2} - \theta)$. Moreover, by [60, Corollary 2.1.17] we have that for $\lambda \in \Sigma_{\pi-\theta}$ the following estimate holds:

$$\begin{aligned}\|\lambda R(\lambda : A)\| &\leq (2 + \tfrac{2}{\sqrt{3}}) \sup_{z \in -\overline{\Sigma_\theta}} \left\| \frac{z}{\lambda - z} \right\| \\ &\leq (2 + \tfrac{2}{\sqrt{3}}) \cdot \begin{cases} 1; & |\mathrm{Arg}(\lambda)| \leq \frac{\pi}{2} - \theta; \\ \big(1 + \cos(\theta - |\mathrm{Arg}(\lambda)|)\big)^{-1}; & |\mathrm{Arg}(\lambda)| > \frac{\pi}{2} - \theta. \end{cases}\end{aligned}$$

Thus for $\theta' \in (0, \frac{\pi}{2} - \theta)$ fixed we find that for all $\lambda \in \Sigma_{\frac{\pi}{2}+\theta'}$ we have

$$\|\lambda R(\lambda : A)\| \leq (2 + \tfrac{2}{\sqrt{3}})(1 - \sin(\theta + \theta'))^{-1}.$$

If A is self-adjoint and $A \leq 0$ then by [60, Corollary 7.1.6] we have that A generates an analytic semigroup and

$$\|\lambda R(\lambda : A)\| \leq \sup_{t \in (-\infty,0)} \left|\tfrac{\lambda}{\lambda - t}\right| \leq \begin{cases} 1; & |\mathrm{Arg}(\lambda)| \leq \frac{\pi}{2}; \\ \big(1 + \cos(\mathrm{Arg}(\lambda))\big)^{-1}; & |\mathrm{Arg}(\lambda)| > \frac{\pi}{2}. \end{cases}$$

Now assume $\omega > 0$. As A generates an analytic semigroup S, it follows that $A + \omega$ generates the analytic semigroup $(e^{\omega t}S(t))_{t \geq 0}$ with the same angle of analyticity, and clearly $\omega(1+2(\cos\theta')^{-1}) + \Sigma_{\frac{\pi}{2}+\theta'} \subset \varrho(A+\omega)$ for any $\theta' \in (0, \frac{\pi}{2}-\theta)$. As for the estimate on the resolvent; fix $\theta' \in (0, \frac{\pi}{2}-\theta)$. One may check that for $\lambda \in \omega(1+2(\cos\theta')^{-1}) + \Sigma_{\theta'}$ we have $|\lambda| \geq 2\omega$, whence $\frac{|\lambda|}{|\lambda-\omega|} \leq 2$. Let $\lambda \in \omega(1+2(\cos\theta')^{-1}) + \Sigma_{\frac{\pi}{2}+\theta'}$, then:

$$\begin{aligned}\|\lambda R(\lambda : A+\omega)\| &= \tfrac{|\lambda|}{|\lambda-\omega|}\|(\lambda-\omega)R(\lambda-\omega : A)\| \\ &\leq (4+\tfrac{4}{\sqrt{3}})(1-\sin(\theta+\theta'))^{-1},\end{aligned}$$

or, for the self-adjoint case:

$$\begin{aligned}\|\lambda R(\lambda : A+\omega)\| &= \tfrac{|\lambda|}{|\lambda-\omega|}\|(\lambda-\omega)R(\lambda-\omega : A)\| \\ &\leq 2(1-\sin(\theta+\theta'))^{-1}.\end{aligned}$$

□

11.2.1 The spectral Galerkin method

Consider (SDE) under the assumptions $(\mathbf{A}_H)$, $(\mathbf{F}_H)$ and $(\mathbf{G}_H)$, with the additional assumption that A is a self-adjoint operator generating an eventually compact semigroup on a Hilbert space $\mathscr{H}$. By [43, Corollary V.3.2] it follows that the spectrum of A consists only of eigenvalues, and these eigenvalues lie in $(-\infty, \omega]$ for some $\omega \in \mathbb{R}$. We denote the eigenvalues by $(\lambda_n)_{n\in\mathbb{N}}$, and assume $(\lambda_n)_{n\in\mathbb{N}}$ is ordered such that $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. Let $(\phi_n)_{n\in\mathbb{N}}$ be the eigenfunctions corresponding to $(\lambda_n)_{n\in\mathbb{N}}$ and define $\mathscr{H}_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$. Let $P_n \in \mathscr{L}(\mathscr{H}, \mathscr{H}_n)$ be given by $P_n x = \sum_{k=1}^n \langle x, \phi_k\rangle \phi_k$ for $x \in \mathscr{H}$ (thus P_n is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_n$).

Let U be the solution to (SDE) with A as described above and initial data $x_0 \in L^p(\Omega, \mathscr{F}_0; \mathscr{H})$. Let $U^{(n)}$ be the n^{th} Galerkin approximation; i.e., $U^{(n)}$ is the solution to the finite-dimensional problem in $\mathscr{H}_n$:

$$\begin{cases} dU^{(n)}(t) = P_n A U^{(n)}(t)dt + P_n F(t, U^{(n)}(t))dt \\ \qquad\qquad\quad +P_n G(t, U^{(n)}(t))dW_H(t), \qquad\qquad t > 0; \\ U^{(n)}(0) = P_n x_0. \end{cases}$$

Note that by taking $W_H = W_{\mathscr{H}} = \sum_{k=1}^\infty W_k(t)\phi_k$, with $(W_k)_{k=1}^\infty$ independent standard Brownian motions, this reduces to:

$$\begin{cases} dU^{(n)}(t) = \sum_{k=1}^n \lambda_k \langle U^{(n)}(t), \phi_k\rangle \phi_k dt + \sum_{k=1}^n \langle F(U^{(n)}(t)), \phi_k\rangle \phi_k dt \\ \qquad\qquad\quad + \sum_{k=1}^n P_n[G(t, U^{(n)}(t))\phi_k]dW_k(t),\ t > 0; \\ U^{(n)}(0) \ = \sum_{k=1}^n \langle x_0, \phi_k\rangle \phi_k. \end{cases} \tag{11.2.1}$$

Theorem 10.1 leads to the following convergence result:

Theorem 11.5. *For any $\eta > 0$ and $p \in (2, \infty)$ such that*

$$\eta < \min\{1 + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G, 1\}$$

we have, assuming $x_0 \in L^p(\Omega, \mathscr{F}_0; \mathscr{H}^A_\eta)$:

$$\|U - U^{(n)}\|_{L^p(\Omega, C([0,T];\mathscr{H}))} \lesssim |\lambda_{n+1}|^{-\eta}(1 + \|x_0\|_{L^p(\Omega;\mathscr{H}^A_\eta)}),$$

with implied constants independent of n, $(\lambda_n)_{n\in\mathbb{N}}$ and x_0.

Again, by the Borel-Cantelli lemma we have:

Corollary 11.6. *Suppose there exists an $\alpha > 0$ and a constant C such that for all $n \in \mathbb{N}$ we have:*

$$|\lambda_n| \leq Cn^\alpha.$$

Let $\eta > 0$ and $p \in (2, \infty)$ be such that

$$\eta + \tfrac{1}{\alpha p} < \min\{1 + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G, 1\}$$

and assume $x_0 \in L^p(\Omega, \mathscr{F}_0; \mathscr{H}^A_\eta)$. Then there exists a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$:

$$\|U - U^{(n)}\|_{C([0,T];\mathscr{H})} \leq \chi n^{-\alpha\eta}.$$

Remark 11.7. Consider the example from Chapter 9 set in $\mathscr{H} = L^2(0, 1)$, with $a_2 \equiv 1$ and $a_1 \equiv 0$, with either Neumann or Dirichlet boundary conditions, and with f and g independent of the space parameter ξ. By [120] this fits into the setting of (SDE) if we take $X = L^2$, $\theta_F = 0$ and $\theta_G = -\frac{1}{4} - \varepsilon$ for some $\varepsilon > 0$. By the choice of a_2, a_1 and the boundary conditions we have $\lambda_n = \pi^2 n^2$. Thus the convergence rate for the Galerkin scheme is $n^{-\frac{1}{2}+\varepsilon_0}$ for ε_0 arbitrarily small, both in $L^p(\Omega; C([0, T]; \mathscr{H}))$ and almost surely, provided $u_0 \in L^p(\Omega, \mathscr{F}_0; H^{1,2})$ for p sufficiently large.

Proof (of Theorem 11.5.). Let $\omega \geq 0$ be such that $\sigma(A) \subset (-\infty, \omega]$. Then $A - \omega$ is self-adjoint and $A - \omega \leq 0$. Thus by Lemma 11.4 A is analytic of type $(\omega(1 + 2(\cos\theta')^{-1}), \theta', 2(1 - \sin\theta')^{-1})$ for all $\theta' \in (0, \frac{\pi}{2})$. Define $A_n : \mathscr{H}_n \to \mathscr{H}_n$ by:

$$A_n = \sum_{k=1}^{n} \lambda_k \langle \cdot, \phi_k \rangle \phi_k,$$

i.e., $A_n = P_n A i_{\mathscr{H}_n}$, where $i_{\mathscr{H}_n}$ is the canonical embedding of $\mathscr{H}_n$ into $\mathscr{H}$. Clearly $A_n - \omega$ is again self-adjoint and $A_n - \omega \leq 0$. Thus by Lemma 11.4 A_n is analytic of type $(\omega(1 + 2(\cos\theta')^{-1}), \theta', 2(1 - \sin\theta')^{-1})$ for all $\theta' \in (0, \frac{\pi}{2})$. It follows that $(A_n)_{n\in\mathbb{N}}$ is uniformly analytic.

Note that the process $U^{(n)}$ satisfying (11.2.1) is precisely the mild solution to (SDE_0) if we take $\mathscr{H}_0 = \mathscr{H}_n$, $P_0 = P_n$ and $A_0 = A_n$.

In order to apply Theorem 10.1, we must prove that condition (10.0.1) holds for some appropriate δ. Fix $\lambda \in \varrho(A)$ such that $\Re e(\lambda) > \omega$. We have:

$$R(\lambda : A) - i_{\mathscr{H}_n} R(\lambda : A_n) P_n = \sum_{k=n+1}^{\infty} \frac{\langle \cdot, \phi_k \rangle}{\lambda - \lambda_k} \phi_k$$

and for $\delta \geq 0$ and $x \in \mathscr{H}_\delta^A$ we have:

$$(\lambda I - A)^\delta x = \sum_{k=1}^{\infty} (\lambda - \lambda_k)^\delta \langle x, \phi_k \rangle.$$

As $|\lambda - \lambda_{i+1}| \geq |\lambda - \lambda_i|$ for all $i \in \mathbb{N}$ we have, for all $\delta \in [0, 1)$,

$$\begin{aligned} &\| R(\lambda : A) - i_{\mathscr{H}_n} R(\lambda : A_n) P_n \|_{\mathscr{L}(\mathscr{H}_{\delta-1}^A, \mathscr{H})} \\ &\eqsim \Big\| \sum_{i=n+1}^{\infty} (\lambda - \lambda_i)^{-\delta} \langle \,\cdot\,, \phi_i \rangle \phi_i \Big\|_{\mathscr{L}(\mathscr{H})} \leq |\lambda - \lambda_{n+1}|^{-\delta}, \end{aligned}$$

with implied constants depending on A only in terms of ω, θ, and K.

Fix $\eta < \min\{1 + \theta_F, \frac{1}{2} - \frac{1}{p} + \theta_G, 1\}$ and fix $T > 0$. The desired result now follows by applying Theorem 10.1 with $\delta = \eta$, $\mathscr{H}_0 = \mathscr{H}_n$, $P_0 = P_n$, $A = A$, $A_0 = A_n$, $U = U$, and $U^{(0)} = U^{(n)}$. □

11.2.2 The finite element method

It is not our intention to go into great detail concerning the question how a finite element method is constructed, nor to state convergence results for finite element methods in general. Instead, we wish to demonstrate by means of an example that the estimate necessary for the application of Theorem 10.1, namely estimate (10.0.2), is precisely the type of estimate sought after when trying to prove convergence of finite element methods for time-independent problems. The example we consider is the case that A is an second order differential operator. We follow the approach of [12, Sections 5.6 and 5.7] where finite element methods are treated for the problem $Au = f$ for such an operator A.

Thus consider (SDE) for the case that $\mathscr{H} = L^2(D)$, $D \subset \mathbb{R}^n$ open and bounded, and $A : H^{2,2}(D) \to L^2(D)$ is a second-order elliptic operator defined by:

$$Au := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \Big(a_{ij} \frac{\partial u}{\partial x_i} \Big), \quad u \in H^{2,2}(D), \tag{11.2.2}$$

where $a_{ij} \in L^\infty(D)$ satisfy the ellipticity condition, i.e., there exists an $\alpha > 0$ such that for all $x \in D$ and all $\xi \in \mathbb{R}^n$ we have:

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^{n} |\xi_i|^2.$$

Moreover, we assume boundary conditions as posed in [12, Section 5.6]; i.e., we assume $Bu \equiv 0$ on ∂D, where we define B by

$$Bu(x) := \sum_{i,j=1}^{n} a_{ij}(x) \frac{du}{dx_i}(x) \nu_j(x),$$

where $x \in \partial D$ and $\nu(x)$ is the normal of ∂D in x. This is well-defined provided $u \in H^{2s,2}(D)$ for $s > \frac{3}{4}$.

To incorporate these boundary conditions in the domain of A we define the space

$$H_B^{2,2}(D) := \{u \in H^{2,2}(D) \,:\, Bu = 0\}.$$

Note that $A : H_B^{2,2}(D) \to L^2(D)$ is self-adjoint and $A \leq 0$, whence by Lemma 11.4 A is analytic of type $(0, \theta, 2(1 - \sin\theta)^{-1})$ for all $\theta \in (0, \frac{\pi}{2})$.

We define the intermediate spaces $H_B^{2s,2}(D)$, $s \geq 0$ and $s \neq \frac{3}{4}$, as follows:

$$H_B^{2s,2}(D) := \begin{cases} H^{2s,2}(D), & 0 \leq s < \frac{3}{4}; \\ \{u \in H^{2s,2}(D) \,:\, Bu = 0\}, & s > \frac{3}{4}. \end{cases}$$

As A is self-adjoint it follows from [93, Theorem 4.3.11] and [53, Section 8] that one has, for $s \in (0,1)$ and $s \neq \frac{3}{4}$:

$$\mathscr{H}_s^A(D) \simeq [L^2(D), H_B^{2,2}(D)]_s \simeq H_B^{2s,2}(D), \tag{11.2.3}$$

where $[\mathscr{H}_1, \mathscr{H}_2]_\theta$ denotes the complex interpolation space of the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with parameter $\theta \in (0,1)$.

Let $(V_h)_{h \in I}$ be a set of finite-element spaces, where $I \subset (0, \infty)$ is an index set. We assume $V_h \subset H_B^{1,2}(D)$, and as usual h indicates the maximal diameter of the components of D that the finite elements of V_h are defined on. Let $a : H_B^{1,2}(D) \times H_B^{1,2}(D) \to \mathbb{R}$ be the form associated with A (i.e., $a(u,v) = \langle Au, v\rangle$ for all $u \in D(A)$ and $v \in H_B^{1,2}(D)$). For $h \in I$ fixed the finite-element approximation $U^{(h)}$ of U, the solution to (SDE) with initial condition x_0, is the element of $L^p(D; C([0,T]; V_h))$ satisfying, for all $t \in [0,T]$:

$$\begin{aligned} \langle U^{(h)}(t), v_h\rangle &= \int_0^t a(U^{(h)}(s), v_h)ds + \int_0^t \langle F(U^{(h)}(s)), v_h\rangle ds \\ &\quad + \int_0^t B^*(U^{(h)}(s)) v_h dW_H(s), \qquad \text{a.s. for all } v_h \in V_h, \\ \langle U^{(h)}(0), v_h\rangle &= \langle x_0, v_h\rangle, \end{aligned} \tag{11.2.4}$$

almost surely for all $v_h \in V_h$ (of course that it suffices to check the above for $(v_h^{(k)})_{k=1}^N$ a basis of V_h).

We prove convergence of $U^{(h)}$ against U under the following assumption on the finite elements:

(V) There exists a constant C such that for any $h \in I$ and $u \in H^{2,2}(D)$ we have:

$$\inf_{v \in V_h} \|u - v\|_{H^{1,2}(D)} \leq Ch\|u\|_{H^{2,2}(D)}.$$

Remark 11.8. Condition **(V)** is standard when proving convergence results for finite element methods (see [12, Theorem 5.7.6]). See also [12, Theorem 4.4.4] for an example of a finite element that satisfies **(V)** (in our case this example concerns approximation by first-order polynomials).

Theorem 11.9. *Consider* (SDE) *in the Hilbert space $L^2(D)$ with A as defined in* (11.2.2) *and F, G satisfying* $(\mathbf{F}_H)$ *and* $(\mathbf{G}_H)$. *Suppose that the family of finite elements $(V_h)_{h\in I}$ satisfies* **(V)**. *Let $p \in (2,\infty)$ and $\eta > 0$ satisfy*

$$\eta < \max\{1 + \theta_F, \tfrac{1}{2} + \theta_G - \tfrac{1}{p}, 1\}.$$

Let $x_0 \in H_B^{2\eta,2}(D)$. Then there exists an $h_0 > 0$ such that for every $h \in I$, $h \leq h_0$, there exists a unique $U^{(h)} \in L^p(\Omega; C([0,T];\mathscr{H}))$ satisfying (11.2.4). *Moreover, for all $h \in I, h \leq h_0$ we have:*

$$\|U - U^h\|_{L^p(\Omega;C([0,T];\mathscr{H}))} \lesssim h^{2\eta}(1 + \|x_0\|_{H_B^{2\eta,2}(D)}),$$

with implied constant independent of h and x_0.

Again we obtain, as a direct consequence of the Borel-Cantelli lemma:

Corollary 11.10. *Let $\eta > 0$ and $p \in (2,\infty)$ be such that*

$$\eta + \tfrac{1}{p} < \min\{1 + \theta_F, \tfrac{1}{2} - \tfrac{1}{p} + \theta_G, 1\}$$

and assume $x_0 \in L^p(\Omega, \mathscr{F}_0; \mathscr{H}_\eta^A)$. Suppose the index set I is given by $I = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then there exists a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$:

$$\|U - U^{(\frac{1}{n})}\|_{C([0,T];\mathscr{H})} \leq \chi n^{-2\eta}.$$

Proof (of Theorem 11.9.). We need to rewrite the setting into the setting of Theorem 10.1. Fix $h \in I$. We define $A_h : V_h \to V_h$ by $\langle A_h u, v\rangle = a(u,v)$ for all $v \in V_h$. Note that A_h is self-adjoint because A is self-adjoint, hence by Lemma 11.4 the operator A_h is of type $(0, \theta, 2(1-\sin\theta)^{-1})$ for all $\theta \in (0, \frac{\pi}{2})$. In other words, the family of operators $(A_h)_{h\in I}$ is uniformly analytic. We also define $P_h : H^{-1,2}(D) \to V_h$ to be the orthogonal projection of $H^{-1,2}(D)$ onto V_h, i.e., for $u \in L^2(D)$ we let $P_h u$ be the unique element of V_h satisfying

$$\langle P_h u, v_h\rangle = \langle u, v_h\rangle, \quad \text{for all } v_h \in V_h.$$

(This is well-defined due to the fact that $V_h \subset H_B^{1,2}(D)$.)

By equivalence of strong and mild solutions in finite dimensions, a process $U^{(h)} \in L^p(\Omega; C([0,T]; V_h))$ satisfies (11.2.4) if and only if it satisfies, for all $t \in [0,T]$:

$$\begin{aligned}U^{(h)}(t) &= e^{tA_h}x_0 + \int_0^t e^{(t-s)A_h}U^{(h)}(s)ds + \int_0^t e^{(t-s)A_h}P_hF(U^{(h)}(s))ds \\ &\quad + \int_0^t e^{(t-s)A_h}P_hG(U^{(h)}(s))dW_H(s).\end{aligned}$$

In other words, $U^{(h)} = U^{(0)}$ in Theorem 10.1 with $A_0 = A_h$, $X_0 = V_h$ and $P_0 = P_h$.

It remains to prove an estimate of the type (10.0.1). Following the notation of [12] we define $u = A^{-1}f$ and $u_h = A_h^{-1}P_hf$, for $f \in L^2(D)$ given. Theorem [12, Theorem 5.7.6] states that for A the second-order elliptic operator defined above, under the assumption **(V)**, there exist constants C and h_0 such that for $h \le h_0$ and $u \in H^{2,2}(D)$ we have:

$$\|u - u_h\|_{L^2(D)} \le Ch^2|u|_{H^{2,2}(D)},$$

where $|u|_{H^{2,2}(D)} = \big(\sum_{|\alpha|=2} \|D^\alpha u\|^2_{L^2(D)}\big)^{\frac{1}{2}}$. This may be read as:

$$\|A^{-1}f - A_h^{-1}P_hf\|_{L^2(D)} \le Ch^2\|A^{-1}f\|_{D(A)} \eqsim h^2\|f\|_{L^2(D)},$$

which is precisely estimate (10.0.1) with $\delta = 1$. Moreover, by definition of u and u_h we have:

$$\begin{aligned}\|u - u_h\|^2_{L^2(D)} &= \langle u - u_h, u - u_h\rangle = \langle u - u_h, u\rangle \\ &\le \|u - u_h\|_{L^2(D)}\|u\|_{L^2(D)},\end{aligned}$$

which may be read as

$$\|A^{-1}f - A_h^{-1}P_hf\|_{L^2(D)} \le \|A^{-1}f\|_{L^2(D)} \eqsim h^2\|f\|_{\mathscr{H}^A_{-1}}.$$

By interpolation (see (11.2.3)) we obtain, for any $\delta \in [0,1] \setminus \{\frac{1}{4}\}$:

$$\begin{aligned}\|A^{-1} - A_h^{-1}P_h\|_{\mathscr{L}(\mathscr{H}^A_{\delta-1},\mathscr{H})} &\eqsim \|A^{-1} - A_h^{-1}P_h\|_{\mathscr{L}([\mathscr{H},D(A)]_{\delta-1},\mathscr{H})} \\ &\le C^{2\delta}h^{2\delta}.\end{aligned}$$

Thus we may apply Theorem 10.1 with $\delta = \eta$ (or, if $\eta = \frac{1}{4}$, take $\delta = \eta + \varepsilon$ for sufficiently small ε) to obtain the desired result. □

A

Appendix

A.1 Technical lemmas

Here we state and prove with two lemmas which give estimates for the γ-radonifying norm of stochastic and deterministic integral processes.

Lemma A.1. *Let $q \in [1, \infty]$, $\frac{1}{q} + \frac{1}{q'} = 1$, and let $(R, \mathscr{R}, \mu)$ be a finite measure space and $(S, \mathscr{S}, \nu)$ a σ-finite measure space. Let Y_1 and Y_2 be Banach spaces, and suppose $\Psi_1 \in L^q(R, \gamma(S; Y_1))$ and $\Psi_2 \in L^{q'}(R, \mathscr{L}(Y_1, Y_2))$ such that $(r, s) \mapsto \Psi_2(r)\Psi_1(r, s)$ defines an element of $L^1(R \times S; Y_2)$. Then:*

$$\Big\| s \mapsto \int_R \Psi_2(r)\Psi_1(r, s)\, d\mu(r) \Big\|_{\gamma(S;Y_2)} \leq \|\Psi_2\|_{L^{q'}(S;\mathscr{L}(Y_1,Y_2))} \|\Psi_1\|_{L^q(R,\gamma(S;Y_1))}.$$

Proof. We first consider the case $q \in [1, \infty)$. The L^1-assumption guarantees that the integral on the left-hand side exists as a Bochner integral in Y_2 for ν-almost all $s \in S$. By (2.3.3) and the fact that $q < \infty$ we may identify Ψ_1 with an element in $\gamma(S; L^q(R; Y_1))$, and by the Hölder inequality Ψ_2 induces a bounded operator from $L^q(R; Y_1)$ to Y_2. Under these identifications, the expression inside the norm at left-hand side equals the operator $\Psi_2 \circ \Psi_1 \in \gamma(S; Y_2)$ and the desired estimate is noting but the right ideal property for the γ-radonifying norm.

The case $q = \infty$ now follows by an approximation argument. Suppose first $\Psi_1 \in L^1 \cap L^\infty(R, \gamma(S; Y_1))$ and $\Psi_2 \in L^1 \cap L^\infty(R, \mathscr{L}(Y_1, Y_2))$. By the above we have:

$$\begin{aligned}\Big\| s \mapsto \int_R \Psi_2(r)\Psi_1(r, s)\, d\mu(r) \Big\|_{\gamma(S;Y_2)} &\leq \lim_{q\uparrow\infty, q'\downarrow 1} \|\Psi_2\|_{L^{q'}(S;\mathscr{L}(Y_1,Y_2))} \|\Psi_1\|_{L^q(R,\gamma(S;Y_1))} \\ &= \|\Psi_2\|_{L^\infty(S;\mathscr{L}(Y_1,Y_2))} \|\Psi_1\|_{L^1(R,\gamma(S;Y_1))}.\end{aligned}$$

The result for general $\Psi_1 \in L^\infty(R, \gamma(S; Y_1))$ and $\Psi_2 \in L^1(R, \mathscr{L}(Y_1, Y_2))$ follows by approximation. $\square$

The above lemma can be applied to prove the following generalization of [109, Proposition 4.5].

Lemma A.2. *Let X_1 and X_2 be* UMD *Banach spaces. Let $(R,\mathscr{R},\mu)$ be a finite measure space and $(S,\mathscr{S},\nu)$ a σ-finite measure space. Let $\Phi_1 : [0,T]\times\Omega \to \mathscr{L}(H,X_1)$, let $\Phi_2 \in L^1(R;\mathscr{L}(X_1,X_2))$, and let $f\in L^\infty(R\times[0,T];L^2(S))$. If Φ_1 is L^p-stochastically integrable for some $p\in(1,\infty)$, then*

$$\begin{aligned}\Big\| s\mapsto \int_0^T \int_R f(r,u)(s)\Phi_2(r)\Phi_1(u)\,d\mu(r)\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(S;X_2))} \\ \lesssim \operatorname*{ess\,sup}_{(r,u)\in R\times[0,T]} \|f(r,u)\|_{L^2(S)}\|\Phi_2\|_{L^1(R,\mathscr{L}(X_1,X_2))}\|\Phi_1\|_{L^p(\Omega;\gamma(0,T;H,X_1))},\end{aligned}$$

with implied depending only on p, X_1, X_2, provided the right-hand side is finite.

Proof. By [84, Corollary 2.17], for almost all $s\in S$ the family $\{T_{s,u} :\ u\in[0,T]\}$ is γ-bounded in $\mathscr{L}(X_1,X_2)$, where

$$T_{s,u}x = \int_R f(r,u)(s)\Phi_2(r)x\,d\mu(r).$$

Hence, by the γ-multiplier theorem (Theorem 2.14), for almost all $s\in S$ the function $u\mapsto \int_R f(r,u)(s)\Phi_2(r)\Phi_1(u)\,d\mu(r)$ belongs to $L^p_{\mathscr{F}}(\Omega;\gamma(0,T;H,X_2))$.

Moreover, by Theorem 2.14 in combination with Theorem 2.15 (note that UMD Banach spaces have non-trivial cotype) we have, for almost all $r\in R$;

$$u\mapsto (s\mapsto f(r,u)(s)\Phi_1(u)) \in L^p_{\mathscr{F}}(\Omega;\gamma(0,T;\gamma(S,X_1))).$$

By the stochastic Fubini theorem, the isomorphism (2.3.3) and Lemma A.1, with $q=\infty$, $Y_1 = L^p(\Omega;X_1)$ and $Y_2 = L^p(\Omega,X_2)$, $\Psi_2 = \Phi_2$, and

$$\Psi(r,s) = \int_0^T f(r,u)(s)\Phi_1(u)\,dW_H(u),$$

we have:

$$\begin{aligned}&\Big\| s\mapsto \int_0^T \int_R f(r,u)(s)\Phi_2(r)\Phi_1(u)\,d\mu(r)\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(S;X_2))} \\ &\eqsim \Big\| s\mapsto \int_R \Phi_2(r)\int_0^T f(r,u)(s)\Phi_1(u)\ dW_H(u)d\mu(r)\Big\|_{\gamma(S;L^p(\Omega;X_2))} \\ &\lesssim \|\Phi_2\|_{L^1(R,\mathscr{L}(X_1,X_2))}\Big\| s\mapsto \int_0^T f(r,u)(s)\Phi_1(u)\,dW_H(u)\Big\|_{L^\infty(R,\gamma(S;L^p(\Omega;X_1)))}. \end{aligned} \tag{A.1.1}$$

By isomorphism (2.3.3), Theorem 2.7, and Theorem 2.14 in combination with Theorem 2.15 we have, for almost all $r\in R$ with implicit constants independent of r:

$$\begin{aligned}\Big\| s\mapsto \int_0^T f(r,u)(s)\Phi_1(u)\,dW_H(u)\Big\|_{\gamma(S;L^p(\Omega;X_1))} \\ \eqsim \|u\mapsto (s\mapsto f(r,u)(s)\Phi_1(u))\|_{L^p(\Omega;\gamma(0,T;\gamma(S;X_1)))}\end{aligned}$$

$$\leq \operatorname*{ess\,sup}_{u\in[0,T]} \|f(r,u)\|_{L^2(0,T)} \|\Phi_1\|_{L^p(\Omega;\gamma(0,T;X_1))}.$$

The result now follows by inserting the above estimate into (A.1.1). □

We proceed with two lemmas and a corollary on Besov embeddings. The proof of the first lemma is closely related to the proof of [109, Lemma 3.1].

Lemma A.3. *Suppose Y is a Banach space with type $\tau \in [1,2)$, and let $\alpha \in [0, \frac{1}{2})$ and $q \in (2,\infty)$ satisfy $\frac{1}{q} < \frac{1}{\tau} - \alpha$. Let $\Phi \in B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;Y) \cap L^\infty(0,T;Y)$ and, for $t \in [0,T]$, define $\Phi_{\alpha,t} : (0,t) \to Y$ by*

$$\Phi_{\alpha,t}(s) = (t-s)^{-\alpha}\Phi(s).$$

Then there exists an $\varepsilon_0 > 0$ such that for all $T_0 \in [0,T]$:

$$\sup_{0\leq t\leq T_0} \|\Phi_{\alpha,t}\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(0,t;Y)} \lesssim T_0^{\varepsilon_0} \|\Phi\|_{L^\infty(0,T_0;Y)\cap B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T_0;Y)}. \tag{A.1.2}$$

An immediate consequence of the above and embedding 2.3.7 is the following Corollary (see also [109, Lemma 3.3]):

Corollary A.4. *Let X be a Banach space with type τ and let $T > 0$. Then for any $T_0 \in [0,T]$, $\varepsilon > 0$ and $\alpha \in [0, \frac{1}{2})$ there there exists and ε_0 such that one has:*

$$L^p(\Omega; C^{\frac{1}{\tau}-\frac{1}{2}+\varepsilon}([0,T_0];X)) \hookrightarrow V_c^{\alpha,p}([0,T_0]\times\Omega;X),$$

with embedding constant $CT_0^{\varepsilon_0}$), where C may depend on T but not on T_0.

Proof (of Lemma A.3). We shall in fact prove the following stronger result, namely that there exists an $\varepsilon_0 > 0$ such that for all $T_0 \in [0,T]$:

$$\sup_{0\leq t\leq T_0} \|\Phi_{\alpha,t}\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(\mathbb{R};Y)} \lesssim T_0^{\varepsilon_0} \|\Phi\|_{L^\infty(0,T_0;Y)\cap B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T_0;Y)}. \tag{A.1.3}$$

On the left-hand side above, we think of $\Phi_{\alpha,t}$ as being extended identically zero outside the interval $(0,t)$.

Let $q' \in (1,\infty)$ be such that $\frac{1}{q} + \frac{1}{q'} = \frac{1}{\tau}$. As we assumed $\frac{1}{q} < \frac{1}{\tau} - \alpha$ it follows that $\alpha q' < 1$. Thus we can pick $\varepsilon > 0$ such that $\varepsilon < \min\{\frac{1}{2} - \alpha, 1 - \alpha q'\}$.

Fix $t \in [0,T_0]$. Let $\rho \in (0,1]$ and let $0 < h < \rho$ (we only consider the case $h > 0$; the case $h < 0$ can be dealt with by observing that $\|T_h^{\mathbb{R}} f - f\|_{L^p(\mathbb{R},Y)} = \|T_{-h}^{\mathbb{R}} f - f\|_{L^p(\mathbb{R},Y)}$). First we consider the case that $h \leq t$. In that case we have:

$$\begin{aligned}
&\|T_h^{\mathbb{R}}(\Phi_{\alpha,t}) - \Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)} \\
&\quad\leq \|s \mapsto [(t-s-h)^{-\alpha} 1_{[-h,t-h]}(s) - (t-s)^{-\alpha} 1_{[0,t-h]}(s)]\Phi(s+h)\|_{L^\tau(\mathbb{R},Y)} \\
&\qquad + \|s \mapsto (t-s)^{-\alpha} 1_{[0,t]}(s)[\Phi(s+h)1_{[0,t-h]}(s) - \Phi(s)]\|_{L^\tau(\mathbb{R},Y)} \\
&\quad\leq \|s \mapsto [(t-s-h)^{-\alpha} 1_{[-h,t-h]}(s) - (t-s)^{-\alpha} 1_{[0,t-h]}(s)]\|_{L^\tau(\mathbb{R})} \|\Phi\|_{L^\infty(0,T_0;Y)}
\end{aligned}$$

$$+ \|s \mapsto (t-s)^{-\alpha} 1_{[0,t]}(s)\|_{L^{q'}(\mathbb{R},Y)} \|T_h^{\mathbb{R}}(\Phi) 1_{[0,t-h]} - \Phi\|_{L^q(0,t;Y)}.$$

As $\alpha q' < 1$ we have:

$$\|s \mapsto (t-s)^{-\alpha} 1_{[0,t]}(s)\|_{L^{q'}(\mathbb{R},Y)} \lesssim T^{\frac{1}{q'}-\alpha}.$$

For $p \geq 1$ and $0 \leq b \leq a$ one has $(a-b)^p \leq a^p - b^p$ and thus:

$$\begin{aligned}
&\|s \mapsto [(t-s-h)^{-\alpha} 1_{[-h,t-h]}(s) - (t-s)^{-\alpha} 1_{[0,t-h]}(s)]\|_{L^\tau(\mathbb{R})} \\
&\quad = \Big(\int_{-h}^{t-h} \big|(t-s-h)^{-\alpha} - (t-s)^{-\alpha} 1_{[0,t-h]}(s)\big|^\tau \, ds \Big)^{\frac{1}{\tau}} \\
&\quad \leq \Big(\int_{-h}^{t-h} [(t-s-h)^{-\alpha\tau} - (t-s)^{-\alpha\tau} 1_{[0,t-h]}(s)] \, ds \Big)^{\frac{1}{\tau}} \\
&\quad = (1-\alpha\tau)^{-\frac{1}{\tau}} h^{\frac{1}{\tau}-\alpha} \lesssim h^{\frac{1}{\tau}-\alpha-\varepsilon} T^\varepsilon,
\end{aligned}$$

where the last inequality uses $h \leq t \leq T_0$. Putting together these estimates,

$$\begin{aligned}
&\|T_h^{\mathbb{R}}(\Phi_{\alpha,t}) - \Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)} \\
&\qquad \leq h^{\frac{1}{\tau}-\alpha-\varepsilon} T_0^\varepsilon \|\Phi\|_{L^\infty(0,T_0;Y)} + T_0^{\frac{1}{q'}-\alpha} \|T_h^{\mathbb{R}}(\Phi) 1_{[0,t-h]} - \Phi\|_{L^q(0,t;Y)}. \\
&\qquad = h^{\frac{1}{\tau}-\alpha-\varepsilon} T_0^\varepsilon \|\Phi\|_{L^\infty(0,T_0;Y)} + T_0^{\frac{1}{q'}-\alpha} \|T_h^{I}(\Phi) - \Phi\|_{L^q(0,t;Y)}.
\end{aligned}$$

Next suppose $h > t$. In that case:

$$\begin{aligned}
\|T_h^{\mathbb{R}}(\Phi_{\alpha,t}) - \Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)} &= 2\|\Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)} \\
&\lesssim t^{\frac{1}{\tau}-\alpha} \|\Phi\|_{L^\infty(0,T_0;Y)} \leq h^{\frac{1}{\tau}-\alpha-\varepsilon} T_0^\varepsilon \|\Phi\|_{L^\infty(0,T_0;Y)},
\end{aligned}$$

this time using $t^{\frac{1}{\tau}-\alpha} \leq t^{\frac{1}{\tau}-\alpha-\varepsilon} T_0^\varepsilon$ and $t \leq h$. It follows that

$$\begin{aligned}
&\|\Phi_{\alpha,t}\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(\mathbb{R},Y)} \\
&\quad = \|\Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)} + \Big(\int_0^1 \rho^{-1+\frac{\tau}{2}} \sup_{|h|<\rho} \|T_h^{\mathbb{R}}(\Phi_{\alpha,t}) - \Phi_{\alpha,t}\|_{L^\tau(\mathbb{R},Y)}^\tau \frac{d\rho}{\rho} \Big)^{\frac{1}{\tau}} \\
&\quad \lesssim T_0^{\frac{1}{\tau}-\alpha} \|\Phi\|_{L^\infty(0,T_0;Y)} + T_0^{\frac{1}{q'}-\alpha} \Big(\int_0^1 \rho^{-1+\frac{\tau}{2}} \sup_{|h|<\rho} \|T_h^I(\Phi) - \Phi\|_{L^q(0,t;Y)}^\tau \frac{d\rho}{\rho} \Big)^{\frac{1}{\tau}} \\
&\qquad + T_0^\varepsilon \Big(\int_0^1 \rho^{\tau(\frac{1}{2}-\alpha-\varepsilon)} \frac{d\rho}{\rho} \Big)^{\frac{1}{\tau}} \|\Phi\|_{L^\infty(0,T_0;Y)} \\
&\quad \lesssim (T_0^{\frac{1}{\tau}-\alpha} \vee T_0^{\frac{1}{q'}-\alpha} \vee T_0^\varepsilon) \big(\|\Phi\|_{L^\infty(0,T_0;Y)} + \|\Phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,t;Y)} \big).
\end{aligned}$$

This gives the result, noting that $T_0^\alpha \vee T_0^\beta \leq T_0^{\min\{\alpha,\beta\}}(1 \vee T^{|\alpha-\beta|})$ for all $T_0 \in [0,T]$. $\square$

This lemma will be used to deduce the following estimate.

Lemma A.5. *Let X be a* UMD *Banach space, H a Hilbert space, and suppose $G : [0,T] \times \Omega \to \mathscr{L}(H,X)$ satisfies* **(G′)** *of Section 7.2. For all $0 \le \alpha < \frac{1}{2}$ and $\varepsilon > 0$ there for any $T > 0$ there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$, and any sequence $(B_j)_{j=0}^n$ in $L^p(\Omega; X)$, $p \in [2,\infty)$, we have:*

$$\begin{aligned}\sup_{0\le t\le T} &\|s \mapsto (t-s)^{-\alpha}\big[G(s, B_{\underline{s}n/T})) - G(\underline{s}, B_{\underline{s}n/T}))\big]\|_{L^p(\Omega;\gamma(0,t;H,X_{\theta_G}))} \\ &\le Cn^{-\zeta_{\max}+\varepsilon}\big(1 + \sup_{0\le j\le n} \|B_j\|_{L^p(\Omega;X)}\big).\end{aligned}$$

Proof. Without loss of generality we may assume $\varepsilon < \frac{1}{2} - \alpha$. Set $q = (\frac{1}{\tau} - \frac{1}{2} + \varepsilon)^{-1}$, so that $\frac{1}{\tau} - \frac{1}{2} < \frac{1}{q} < \frac{1}{\tau} - \alpha$. For $s \in [0,T)$ define

$$\Phi(s) := G(s, B_{\underline{s}n/T})) - G(\underline{s}, B_{\underline{s}n/T}).$$

Note that as $p \ge 2$, the type of $L^p(\Omega, X_{\theta_G})$ is the same as the type of X. By embedding (2.3.7), Lemma A.3 and isomorphism (2.3.3) we have:

$$\begin{aligned}\sup_{0\le t\le T} &\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(0,t;H,L^p(\Omega;X_{\theta_G}))} \\ &\lesssim \|\Phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))} + \|\Phi\|_{L^\infty(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))}.\end{aligned} \tag{A.1.4}$$

By the Hölder assumption of **(G′)** we have:

$$\begin{aligned}\|\Phi\|_{L^\infty(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))} &\lesssim n^{-\zeta_{\max}-\frac{1}{\tau}+\frac{1}{2}}\big(1 + \sup_{0\le j\le n} \|B_j\|_{L^p(\Omega;X)}\big) \\ &= n^{-\zeta_{\max}-\frac{1}{q}+\varepsilon}\big(1 + \sup_{0\le j\le n} \|B_j\|_{L^p(\Omega;X)}\big).\end{aligned} \tag{A.1.5}$$

In order to estimate the Besov norm on the right-hand side of (A.1.4) we fix $\rho \in (0,1)$, and let $|h| < \rho$. We have, with $I = [0,T]$,

$$\begin{aligned}&\|T_h^I\Phi(s) - \Phi(s)\|_{L^p(\Omega;\gamma(H,\theta_G))} \\ &\le \begin{cases} \|G(s+h, B_{\underline{s}n/T}) - G(s, B_{\underline{s}n/T})\|_{L^p(\Omega;\gamma(H,X_{\theta_G}))}, & \underline{s+h} = \underline{s},\ s+h \in [0,T], \\ 2\|\Phi\|_{L^\infty(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))}, & \text{otherwise.}\end{cases}\end{aligned}$$

For $|h| \ge \frac{T}{n}$ one never has $\underline{s+h} = \underline{s}$ and thus it follows from the above and (A.1.5) that

$$\begin{aligned}\|T_h^I\Phi - \Phi\|_{L^q(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))} &\lesssim n^{-\zeta_{\max}-\frac{1}{q}+\varepsilon}\big(1 + \sup_{0\le j\le n} \|B_j\|_{L^p(\Omega;X)}\big) \\ &\lesssim |h|^{\frac{1}{q}} n^{-\zeta_{\max}+\varepsilon}\big(1 + \sup_{0\le j\le n} \|B_j\|_{L^p(\Omega;X)}\big).\end{aligned}$$

On the other hand, for $h < \frac{T}{n}$ and $\underline{s+h} = \underline{s}$ we obtain, by **(G′)**:

$$\|G(s+h, B_{\underline{s+h}n/T}) - G(s, B_{\underline{s}n/T})\|_{L^p(\Omega;\gamma(H,X_{\theta_G}))}$$

$$\begin{aligned}
&\lesssim |h|^{\zeta_{\max}+\frac{1}{\tau}-\frac{1}{2}}(1+\|B_{\underline{s}n/T}\|_{L^p(\Omega;X)})\\
&\le |h|^{\frac{1}{q}}(\tfrac{T}{n})^{\zeta_{\max}+\frac{1}{\tau}-\frac{1}{2}-\frac{1}{q}}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big)\\
&\lesssim |h|^{\frac{1}{q}}n^{-\zeta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big).
\end{aligned}$$

For $|h|<\frac{T}{n}$ observe that $|\{s\in[0,T]:\underline{s+h}\neq\underline{s}\}|=n|h|$. Thus for $|h|<\frac{T}{n}$ we have, by the above estimate and (A.1.5):

$$\begin{aligned}
&\|T_h^I\Phi-\Phi\|_{L^q(0,T;L^p(\Omega;\gamma(H,X_{\theta_G}))}\\
&\quad\lesssim (T-n|h|)^{\frac{1}{q}}|h|^{\frac{1}{q}}n^{-z\eta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big)\\
&\qquad+(n|h|)^{\frac{1}{q}}n^{-\zeta_{\max}-\frac{1}{q}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big)\\
&\quad\lesssim |h|^{\frac{1}{q}}n^{-\zeta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big).
\end{aligned}$$

Collecting these estimates we find:

$$\sup_{|h|<\rho}\|T_h^I\Phi-\Phi\|_{L^q(0,T;L^p(\Omega;\gamma(H,X_{\theta_G}))}\lesssim \rho^{\frac{1}{q}}n^{-\zeta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big).$$

Because $\frac{1}{q}>\frac{1}{\tau}-\frac{1}{2}$ it follows that

$$\begin{aligned}
&\|\Phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;L^p(\Omega;\gamma(H,X_{\theta_G})))}\\
&\quad\lesssim \|\Phi\|_{L^q(0,T;\gamma(H,L^p(\Omega;X_{\theta_G})))}+n^{-z\eta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big)\\
&\quad\lesssim n^{-z\eta_{\max}+\varepsilon}\big(1+\sup_{0\le j\le n}\|B_j\|_{L^p(\Omega;X)}\big).
\end{aligned}$$

Inserting the above and (A.1.5) into (A.1.4) gives the required result. ☐

The final lemma is an elementary calculus fact.

Lemma A.6. *For all $0\le\delta,\theta<\frac{1}{2}$ there exists a constant C, depending only on δ and θ, such that for all $0\le u\le t$, all $T>0$ and all $n\in\mathbb{N}$:*

$$\int_{\underline{u}}^{t}(t-s)^{-2\theta}(\overline{s}-\underline{u})^{-2\delta}\,ds\le C^2(t+\tfrac{T}{n}-u)^{1-2\delta-2\theta}.$$

Proof. If $t-\underline{u}\le\frac{T}{n}$, then for $s\in[\underline{u},t)$ one has $\overline{s}-\underline{u}=\overline{u}-\underline{u}=\frac{T}{n}$ so

$$\int_{\underline{u}}^{t}(t-s)^{-2\theta}(\overline{s}-\underline{u})^{-2\delta}\,ds=(1-2\theta)^{-1}(\tfrac{T}{n})^{-2\delta}(t-\underline{u})^{1-2\theta}.$$

Note that $\frac{T}{n}\ge\frac{1}{2}(t+\frac{T}{n}-u)$ and $(t-\underline{u})^{1-2\theta}\le(t+\frac{T}{n}-u)^{1-2\theta}$. Thus:

$$\int_{\underline{u}}^{t}(t-s)^{-2\theta}(\overline{s}-\underline{u})^{-2\delta}\,ds \leq 2^{2\delta}(1-2\theta)^{-1}(t+\tfrac{T}{n}-u)^{1-2\delta-2\theta}.$$

On the other hand if $t-\underline{u} > \frac{T}{n}$ then $t-\underline{u} < t+T/n-u < 2(t-\underline{u})$. Moreover, $\overline{s}-\underline{u} \geq s-\underline{u}$, and the substitution $v=(s-\underline{u})/(t-\underline{u})$ gives:

$$\begin{aligned}\int_{\underline{u}}^{t}(t-s)^{-2\theta}(\overline{s}-\underline{u})^{-2\delta}\,ds &\leq \int_{\underline{u}}^{t}(t-s)^{-2\theta}(s-\underline{u})^{-2\delta}\,ds\\ &\leq (t-\underline{u})^{1-2\delta-2\theta}\int_0^1(1-v)^{-2\theta}v^{-2\delta}\,dv\\ &\leq 2^{(2\delta+2\theta-1)^+}(t+\tfrac{T}{n}-u)^{1-2\delta-2\theta}\int_0^1(1-v)^{-2\theta}v^{-2\delta}\,dv.\end{aligned}$$

□

A.2 Estimates for (stochastic) convolutions

A.2.1 Convolutions with an analytic semigroup

We shall present two estimates for stochastic convolutions. Throughout this section, Y is a UMD Banach space and $\tau\in(1,2]$ denotes its type. Moreover, S is an analytic semigroup on Y.

Roughly speaking, Lemma A.7 is contained in Step 2 of the proof of [109, Proposition 6.1], but there the space $V_{\mathrm{c}}^{\alpha,p}([0,T]\times\Omega;X)$ is considered. For completeness we give the proof below.

Lemma A.7. *Let $\delta\in(-\frac{3}{2}+\frac{1}{\tau},\infty)$, $\alpha\in[0,\frac{1}{2})$, and $p\in[2,\infty)$. For all $\Phi\in L^\infty(0,T;L^p(\Omega;Y_\delta))$, the convolution $S*\Phi$ belongs to $\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;Y)$, and for all $T_0\in[0,T]$ we have:*

$$\|S*\Phi\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;Y)} \lesssim (T_0^{1+(\delta\wedge 0)}+T_0^{\frac{1}{2}-\alpha})\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;Y_\delta))}.$$

Proof. By analyticity of the semigroup (equation (2.6.3)) we have, for $t\in[0,T_0]$:

$$\begin{aligned}\|(S*\Phi)(t)\|_{L^p(\Omega;Y)} &\lesssim \int_0^t (t-s)^{\delta\wedge 0}\,ds\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;Y_\delta))}\\ &\leq T_0^{1+(\delta\wedge 0)}\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;Y_\delta))}.\end{aligned}$$

Taking the supremum over $t\in[0,T_0]$ gives the estimate in $L^\infty(0,T_0;L^p(\Omega,Y))$.

It remains to prove the estimate in the weighted γ-norm. Fix $t\in[0,T_0]$. As $p\geq 2$, it follows that $L^p(\Omega,Y)$ has type $\tau\in[1,2]$ whenever Y has type τ. Moreover, if we interpret A as an operator on $L^p(\Omega,Y)$ acting pointwise, then $(L^p(\Omega,Y))_\delta = L^p(\Omega,Y_\delta)$. Thus by [109, Proposition 3.5] with $E=L^p(\Omega,Y)$, $\eta=0$, and $\theta=-\delta$ we have, as $\delta > -\frac{3}{2}+\frac{1}{\tau}$;

$$\|s \mapsto (t-s)^{-\alpha}(S * \Phi)(s)\|_{\gamma(0,t;L^p(\Omega,Y))} \lesssim T_0^{\frac{1}{2}-\alpha}\|\Phi\|_{L^\infty(0,T_0;L^p(\Omega;Y_\delta))}.$$

Taking the supremum over $t \in [0, T_0]$ gives the desired estimate. □

We proceed with the second Lemma.

Lemma A.8. *Let $\delta \in (-\frac{1}{2}, \infty)$ and $\alpha \in [0, \frac{1}{2})$. Suppose $\Phi : [0,T] \times \Omega \to \mathscr{L}(H, Y_\delta)$ is strongly measurable and adapted and satisfies*

$$\sup_{0 \le t \le T} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,T;H,Y_\delta))} < \infty, \tag{A.2.1}$$

for some $p \in (1, \infty)$.

(i) If $0 \le \beta < \min\{\frac{1}{2}-\alpha, \frac{1}{2}+\delta\}$, then there exists an $\epsilon > 0$ such that for all $T_0 \in [0,T]$:

$$\sup_{0\le t\le T_0} \Big\|s \mapsto (t-s)^{-\alpha-\beta}\int_0^s S(s-u)\Phi(u)\,dW_H(s)\Big\|_{L^p(\Omega;\gamma(0,t;Y))} \lesssim T_0^\epsilon \sup_{0\le t\le T_0} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;H,Y_\delta))}.$$

(ii) If, moreover, $\alpha > -\delta$, then there exists an $\epsilon > 0$ such that for all $T_0 \in [0,T]$:

$$\Big\|s \mapsto \int_0^s S(s-u)\Phi(u)\,dW_H(u)\Big\|_{\mathscr{V}_\infty^{\alpha+\beta,p}([0,T_0]\times\Omega;Y)} \lesssim T_0^\epsilon \sup_{0\le t\le T_0} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;H,Y_\delta))}.$$

Proof. Fix $t \in [0, T_0]$. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2} - \delta^- - \beta$. Here $\delta^- = (-\delta) \vee 0$. We apply Lemma A.2 with $X_1 = Y_\delta$, $X_2 = Y$, $R = S = [0,t]$, and the functions $\Phi_1(u) = (t-u)^{-\alpha}\Phi(u)$, $\Phi_2(r) = \frac{d}{dr}[r^{\delta^-+\epsilon}S(r)]$, and $f(r,u)(s) = (t-s)^{-\alpha-\beta}(s-u)^{-\delta^--\epsilon}(t-u)^\alpha 1_{\{0\le r\le s-u\}}$. By (2.6.3) we have $\|\Phi_2(r)\|_{\mathscr{L}(X_\delta,X)} \lesssim r^{-1+\varepsilon}$ for $r \in [0,T]$. From the lemma it follows that:

$$\Big\|s \mapsto (t-s)^{-\alpha-\beta}\int_0^s S(s-u)\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;Y))} \lesssim t^{\frac{1}{2}-\beta-\delta^-}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;H,Y_\delta))}.$$

Taking the supremum over $t \in [0, T_0]$ we obtain (i).

For the estimate in $\mathscr{V}_\infty^{\alpha+\beta,p}$-norm it remains, by part (i), to prove the estimate in $L^\infty(0,T_0;L^p(\Omega,Y_\delta))$. Let $\epsilon < \min\{\alpha+\delta, \frac{1}{2}-\delta^- -\beta\}$. By Lemma 2.21 (apply part (1) if $\delta \in (-\frac{1}{2}, 0]$ and part (2) if $\delta \in [0,\infty)$) the operators $r^\alpha S(r)$, $r \in [0,t]$, are γ-bounded from Y_δ to Y, with γ-bound at most $Ct^{\alpha+\delta}$ with C independent of $t \in [0,T]$. Hence, by the γ-multiplier theorem, for all $t \in [0,T]$,

$$\Big\|\int_0^t S(t-s)\Phi(s)\,dW_H(s)\Big\|_{L^p(\Omega;Y)} \lesssim t^{\alpha+\delta}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;H,Y_\delta))}.$$

The norm estimate in $L^\infty(0,T;L^p(\Omega;Y_\delta))$ is obtained by taking the supremum over $t \in [0,T]$.

□

A.2.2 General convolutions

In this section we provide the estimates for (stochastic) convolutions that are necessary to derive the perturbation result given in Theorem 10.1. In order to avoid confusion when applying these lemmas, we shall use Y_1 and Y_2 to denote UMD Banach spaces in this section.

Lemma A.9. *Let $T > 0$, $p \in [1, \infty)$ and $\eta > 0$. Let $q \in [1, \infty]$ and $q' \in [1, \infty]$ be such that $\frac{1}{q} + \frac{1}{q'} = 1$ (with the usual convention that $\frac{1}{\infty} = 0$). Suppose the process $\Phi \in L^p(\Omega; \gamma(0, T, H; Y_1))$ is adapted and satisfies:*

$$\big\| s \mapsto \| u \mapsto (s-u)^{-\eta} \Phi(u) \|_{L^p(\Omega;\gamma(0,s,H;Y_1))} \big\|_{L^q(0,T)} < \infty.$$

Let $\Psi : [0, T] \to \mathscr{L}(Y_1, Y_2)$ be such that Ψx is continuously differentiable on $(0, T)$ for all $x \in Y_1$. Suppose moreover there exists a $g \in L^{q'}(0, T)$ and $0 \leq \theta < \eta$ such that for all $v \in (0, T)$ we have:

$$\| v^{\theta} \tfrac{d}{dv} \Psi(v) x \|_{Y_2} + \theta \| v^{\theta - 1} \Psi(v) x \|_{Y_2} \leq g(v) \| x \|_{Y_1}, \quad \textit{for all } x \in Y_1.$$

Then the stochastic convolution process

$$s \mapsto \int_0^s \Psi(s-u) \Phi(u) \, dW_H(u)$$

is well-defined and

$$\begin{aligned} &\Big\| s \mapsto \int_0^s \Psi(s-u) \Phi(u) \, dW_H(u) \Big\|_{C^{\eta-\theta}([0,T];L^p(\Omega;Y_2))} \\ &\quad \leq (3 + 2^{\eta}) \bar{C}_p \| g \|_{L^{q'}(0,T)} \big\| s \mapsto \| u \mapsto (s-u)^{-\eta} \Phi(u) \|_{L^p(\Omega;\gamma(0,s,H;Y_1))} \big\|_{L^q(0,T)}, \end{aligned}$$

where $\bar{C}_p$ is the constant in the Burkholder-Davis-Gundy inequality for the p^{th} moment, for the space Y_1, see equation (2.4.5).

Before proving the Lemma, observe that the corollary below follows directly from Kolmogorov's continuity criterion (see Theorem I.2.1 in Revuz and Yor).

Corollary A.10. *Let the setting be as in Lemma A.9 and assume in addition that $\frac{1}{p} < \eta - \theta$. Let $0 < \beta < \eta - \theta - \frac{1}{p}$. There exists a modification of the stochastic convolution process $s \mapsto \int_0^s \Psi(s-u) \Phi(u) \, dW_H(u)$, which we shall denote by $\Psi \diamond \Phi$, such that:*

$$\begin{aligned} &\| \Psi \diamond \Phi \|_{L^p(\Omega;C^{\beta}([0,T];Y_2))} \\ &\qquad \leq \tilde{C} \| g \|_{L^{q'}(0,T)} \big\| s \mapsto \| u \mapsto (s-u)^{-\eta} \Phi(u) \|_{L^p(\Omega;\gamma(0,s,H;Y_1))} \big\|_{L^q(0,T)}, \end{aligned}$$

where $\tilde{C}$ depends only on η, β and p and $\bar{C}_p$.

Proof (of Lemma A.9). By Proposition 2.12 and assumption it follows that $\{s^\theta \Psi(s) : s \in [0,T]\}$ is γ-bounded. Thus by the Kalton-Weis multiplier theorem, see Theorem 2.14, and the fact that

$$\big\| s \mapsto \|u \mapsto (s-u)^{-\eta}\Phi(u)\|_{L^p(\Omega;\gamma(0,T,H;Y_1))} \big\|_{L^q(0,T)} < \infty,$$

it follows that $u \mapsto \Psi(s-u)\Phi(u)1_{\{u\in[0,s]\}} \in L^p(\Omega;\gamma(0,s,H;Y_2))$ for almost all $s \in [0,T]$. By Theorem 2.7 this process is stochastically integrable.

In what follows we let $\frac{d}{dv}$ denote the derivative with respect to the strong operator topology. By the triangle inequality we have:

$$\begin{aligned}
&\Big\| \int_0^t \Psi(t-u)\Phi(u)\,dW_H(u) - \int_0^s \Psi(s-u)\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)} \\
&\qquad \le \Big\| \int_0^s [\Psi(t-u)-\Psi(s-u)]\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)} \\
&\qquad\quad + \Big\| \int_s^t \Psi(t-u)\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)} \\
&\qquad = \Big\| \int_0^s \int_{s-u}^{t-u} \tfrac{d}{dv}\Psi(v)\,dv\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)} \\
&\qquad\quad + \Big\| \int_s^t (t-u)^{-\theta} \int_0^{t-u} \tfrac{d}{dv}[v^\theta \Psi(v)]\,dv\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)}.
\end{aligned} \tag{A.2.2}$$

Let us begin with the case that $q=\infty$. We wish to apply the stochastic Fubini theorem (i.e., Lemma 2.9). Consider $\Upsilon : [0,s]\times[0,t] \to \mathscr{L}(H,Y)$ defined by $\Upsilon(u,v) = 1_{\{s-u\le v\le t-u\}}\frac{d}{dv}\Psi(v)\Phi(u)$. As $\frac{d}{dv}\Psi$ is strongly continuous and Φ is H-strongly measurable, we have that Υ is H-strongly measurable. Moreover, as Φ is adapted it follows that $\Upsilon_v := \Upsilon(\cdot,v)$ is adapted for almost all $v\in[0,t]$. Finally, we have that $\Upsilon \in L^1(0,t;\gamma(0,s,H;Y_2))$ by assumption:

$$\|\Upsilon(\cdot,v)\|_{\gamma(0,s,H;Y_2)} \le v^{\eta-\theta}g(v)\|u\mapsto (s-u)^{-\eta}\Phi(u)\|_{\gamma(0,s,H;Y_1)},$$

where we use that $v \ge s-u$ on $\operatorname{supp}(\Upsilon)$. It follows that the conditions necessary to apply the stochastic Fubini theorem, as stated in Lemma 2.9, are satisfied, and we have:

$$\begin{aligned}
&\Big\| \int_0^s \int_{s-u}^{t-u} \tfrac{d}{dv}[\Psi(v)\Phi(u)]\,dv\,dW_H(u)\Big\|_{L^p(\Omega;Y_2)} \\
&\quad = \Big\| \int_0^t \int_{(s-v)\vee 0}^{(t-v)\wedge s} \tfrac{d}{dv}[\Psi(v)\Phi(u)]\,dW_H(u)\,dv\Big\|_{L^p(\Omega;Y_2)} \\
&\quad \le \int_0^t v^{-\theta}g(v)\Big\| \int_{(s-v)\vee 0}^{(t-v)\wedge s} \Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;Y_1)} dv \\
&\quad \le \bar{C}_p \int_0^t v^{-\theta}g(v)\big\|1_{[(s-v)\vee 0,(t-v)\wedge s]}\Phi\big\|_{L^p(\Omega;\gamma(0,t;Y_1))} dv
\end{aligned}$$

$$
\begin{aligned}
&\le \bar{C}_p \int_0^t v^{-\theta} g(v) [(t-s)\wedge v]^{\eta} \big\| u \mapsto ([((t-v)\wedge s) - u]^{-\eta} \Phi(u)) \big\|_{L^p(\Omega;\gamma(0,t;Y_1))} dv \\
&\le \bar{C}_p (t-s)^{\eta-\theta} \int_0^t g(v)\, dv \sup_{t\in[0,T]} \big\| u \mapsto (t-u)^{-\eta} \Phi(u) \big\|_{L^p(\Omega;\gamma(0,t;Y_1))}. \quad \text{(A.2.3)}
\end{aligned}
$$

For the final term in (A.2.2) one may also check that the conditions of the stochastic Fubini hold and thus:

$$
\begin{aligned}
&\Big\| \int_s^t (t-u)^{-\theta} \int_0^{t-u} \tfrac{d}{dv}[v^{\theta}\Psi(v)\Phi(u)]\, dv\, dW_H(u) \Big\|_{L^p(\Omega;Y_2)} \\
&\qquad \le \int_0^{t-s} g(v) \Big\| \int_s^{t-v} (t-u)^{-\theta} \Phi(u)\, dW_H(u) \Big\|_{L^p(\Omega;Y_1)} dv \\
&\qquad \le \bar{C}_p \int_0^{t-s} g(v) \big\| u \mapsto 1_{[s,t-v]}(u) (t-u)^{-\theta} \Phi(u) \big\|_{L^p(\Omega;\gamma(0,t,H;F_1))} dv \\
&\qquad \le \bar{C}_p (t-s)^{\eta-\theta} \|g\|_{L^1(0,T)} \sup_{t\in[0,T]} \big\| u \mapsto (t-u)^{-\eta} \Phi(u) \big\|_{L^p(\Omega;\gamma(0,t;Y_1))}.
\end{aligned}
$$

By inserting the two estimates above in (A.2.2) we obtain that

$$
\begin{aligned}
&\Big\| \int_0^t \Psi(t-u)\Phi(u)\, dW_H(u) - \int_0^s \Psi(s-u)\Phi(u)\, dW_H(u) \Big\|_{L^p(\Omega;Y_2))} \\
&\qquad \le 2\bar{C}_p (t-s)^{\eta-\theta} \|g\|_{L^1(0,T)} \sup_{t\in[0,T]} \big\| u \mapsto (t-u)^{-\eta} \Phi(u) \big\|_{L^p(\Omega;\gamma(0,t;Y_1))},
\end{aligned}
$$

which completes the proof as $0 \le s < t \le T$ where chosen arbitrarily.

Now let us consider the case that $q \in (1,\infty)$. We present the proof starting from (A.2.2). For the penultimate term in (A.2.2) we have, by the stochastic Fubini theorem and (2.4.5):

$$
\begin{aligned}
&\Big\| \int_0^s \int_{s-u}^{t-u} \tfrac{d}{dv}[\Psi(v)\Phi(u)] dv dW_H(u) \Big\|_{L^p(\Omega,Y)} \\
&\qquad = \Big\| \int_0^t \int_{(s-v)\vee 0}^{(t-v)\wedge s} \tfrac{d}{dv}[\Psi(v)\Phi(u)] dW_H(u) dv \Big\|_{L^p(\Omega,Y)} \\
&\qquad \le \int_0^t v^{-\theta} g(v) \Big\| \int_{(s-v)\vee 0}^{(t-v)\wedge s} \Phi(u) dW_H(u) \Big\|_{L^p(\Omega,X)} dv \\
&\qquad \le \bar{C}_p \int_0^t v^{-\theta} g(v) \Big\| 1_{[(s-v)\vee 0,(t-v)\wedge s]} \Phi \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv.
\end{aligned}
\quad \text{(A.2.4)}
$$

We proceed as follows:

$$
\begin{aligned}
&\int_0^t v^{-\theta} g(v) \Big\| 1_{[(s-v)\vee 0,(t-v)\wedge s]} \Phi \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&\qquad = \int_0^{t-s} v^{-\theta} g(v) \Big\| 1_{[(s-v)\vee 0,s]} \Phi \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{t-s}^{t} v^{-\theta} g(v) \Big\| 1_{[(s-v)\vee 0, t-v]} \Phi \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&\leq \int_0^{s\wedge(t-s)} 2^\eta v^{\eta-\theta} g(v) \Big\| u \mapsto 1_{[s-v,s]}(u)(s+v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&+ \int_{s\wedge(t-s)}^{t-s} v^{\eta-\theta} g(v) \Big\| u \mapsto 1_{[0,s]}(u)(v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&+ (t-s)^\eta \int_{t-s}^{t} v^{-\theta} g(v) \\
&\qquad \times \Big\| u \mapsto 1_{[(s-v)\vee 0, t-v]}(u)(t-v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv.
\end{aligned} \tag{A.2.5}
$$

We consider the three terms on the right-hand side separately. For the first term we have, by Hölder's inequality:

$$
\begin{aligned}
&\int_0^{s\wedge(t-s)} 2^\eta v^{\eta-\theta} g(v) \Big\| u \mapsto 1_{[s-v,s]}(u)(s+v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&\quad \leq 2^\eta (t-s)^{\eta-\theta} \Big(\int_0^{s\wedge(t-s)} g^{q'}(v) dv \Big)^{\frac{1}{q'}} \\
&\qquad \times \Big(\int_0^{s\wedge(t-s)} \Big\| u \mapsto (s+v-u)^{-\eta} \Phi(u) \Big\|^q_{L^p(\Omega,\gamma(0,s,X))} dv \Big)^{\frac{1}{q}} \\
&\quad \leq 2^\eta (t-s)^{\eta-\theta} \Big(\int_0^{s\wedge(t-s)} g^{q'}(v) dv \Big)^{\frac{1}{q'}} \\
&\qquad \times \Big(\int_0^{T} \Big\| u \mapsto (v-u)^{-\eta} \Phi(u) \Big\|^q_{L^p(\Omega,\gamma(0,v,X))} dv \Big)^{\frac{1}{q}}.
\end{aligned}
$$

For the second term we have:

$$
\begin{aligned}
&\int_{s\wedge(t-s)}^{t-s} v^{\eta-\theta} g(v) \Big\| u \mapsto 1_{[0,s]}(u)(v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&\quad \leq (t-s)^{\eta-\theta} \Big(\int_{s\wedge(t-s)}^{t-s} g^{q'}(v) dv \Big)^{\frac{1}{q'}} \\
&\qquad \times \Big(\int_0^{T} \Big\| u \mapsto (v-u)^{-\eta} \Phi(u) \Big\|^q_{L^p(\Omega,\gamma(0,v,X))} dv \Big)^{\frac{1}{q}}.
\end{aligned}
$$

For the third term we have:

$$
\begin{aligned}
&(t-s)^\eta \int_{t-s}^{t} v^{-\theta} g(v) \Big\| u \mapsto 1_{[(s-v)\vee 0, t-v]}(u)(t-v-u)^{-\eta} \Phi(u) \Big\|_{L^p(\Omega,\gamma(0,t,X))} dv \\
&\quad \leq (t-s)^{\eta-\theta} \Big(\int_{t-s}^{t} g^{q'}(v) dv \Big)^{\frac{1}{q'}} \\
&\qquad \times \Big(\int_{t-s}^{t} \Big\| u \mapsto (t-v-u)^{-\eta} \Phi(u) \Big\|^q_{L^p(\Omega,\gamma(0,t-v,X))} dv \Big)^{\frac{1}{q}}
\end{aligned}
$$

$$\leq (t-s)^{\eta-\theta}\Big(\int_{t-s}^{t} g^{q'}(v)dv\Big)^{\frac{1}{q'}} \times \Big(\int_0^T \Big\|u \mapsto (v-u)^{-\eta}\Phi(u)\Big\|^q_{L^p(\Omega,\gamma(0,v,X))} dv\Big)^{\frac{1}{q}}.$$

Substituting these three estimates in (A.2.5) and substituting the resulting inequality in (A.2.4) we obtain:

$$\begin{aligned}&\Big\|\int_0^s \int_{s-u}^{t-u} \tfrac{d}{dv}[\Psi(v)\Phi(u)]dvdW_H(u)\Big\|_{L^p(\Omega,Y)}\\ &\leq (2+2^\eta)\bar{C}_p(t-s)^{\eta-\theta}\|g\|_{L^{q'}(0,T)}\Big(\int_0^T\Big\|u\mapsto (v-u)^{-\eta}\Phi(u)\Big\|^q_{L^p(\Omega,\gamma(0,v,X))} dv\Big)^{\frac{1}{q}}.\end{aligned} \tag{A.2.6}$$

For the final term in (A.2.2) observe that:

$$\begin{aligned}&\Big\|\int_s^t (t-u)^{-\theta}\int_0^{t-u} \tfrac{d}{dv}[v^\theta\Psi(v)\Phi(u)]dvdW_H(u)\Big\|_{L^p(\Omega,Y)}\\ &\leq \int_0^{t-s} g(v)\Big\|\int_s^{t-v}(t-u)^{-\theta}\Phi(u)dW_H(u)\Big\|_{L^p(\Omega,X)}dv\\ &\leq \bar{C}_p\int_0^{t-s} g(v)\big\|u\mapsto 1_{[s,t-v]}(u)(t-u)^{-\theta}\Phi(u)\big\|_{L^p(\Omega,\gamma(0,t,H;X))}dv\\ &\leq \bar{C}_p(t-s)^{\eta-\theta}\Big(\int_0^t g^{q'}(v)dv\Big)^{\frac{1}{q'}}\Big(\int_0^t \big\|u\mapsto (t-u)^{-\eta}\Phi(u)\big\|^q_{L^p(\Omega,\gamma(0,t,X))}dv\Big)^{\frac{1}{q}}.\end{aligned} \tag{A.2.7}$$

By inserting (A.2.6) and (A.2.7) in (A.2.2) we obtain that

$$\begin{aligned}&\Big\|\int_0^t \Psi(t-u)\Phi(u)dW_H(u) - \int_0^s \Psi(s-u)\Phi(u)dW_H(u)\Big\|_{L^p(\Omega,Y))}\\ &\leq (3+2^\eta)\bar{C}_p(t-s)^{\eta-\theta}\|g\|_{L^{q'}(0,T)}\Big(\int_0^T \big\|u\mapsto (t-u)^{-\eta}\Phi(u)\big\|^q_{L^p(\Omega,\gamma(0,t,X))}dt\Big)^{\frac{1}{q}},\end{aligned}$$

which completes the proof as $0 \leq s < t \leq T$ where chosen arbitrarily.

The proof in the case $q = 1$ follows by simple adaptations of the above. □

Based on the above two lemmas, we obtain the following result for stochastic convolutions in the $V_{\mathrm{c}}^{\alpha,p}$-norm:

Proposition A.11. *Let the setting be the same as in Lemma A.9 with $q = \infty$, and assume in addition that $\frac{1}{p} \leq \eta - \theta$. Let $\alpha \in [0, \frac{1}{2})$. Then $\Psi \diamond \Phi \in V_{\mathrm{c}}^{\alpha,p}([0,T]\times \Omega; Y_2)$. Moreover, there exists a constant C such that for all $T_0 \in [0,T]$ we have:*

$$\|\Psi\diamond\Phi\|_{V_{\mathrm{c}}^{\alpha,p}([0,T_0]\times\Omega;Y_2)} \leq C\|g\|_{L^1(0,T)} \sup_{0\leq t\leq T_0}\|s\mapsto (t-s)^{-\eta}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;Y_1))}.$$

Proof. For the norm estimate in $L^p(\Omega; C([0,T_0];Y_2))$ one may apply Corollary A.10 with $\beta > 0$ such that $0 \le \beta + \theta < \eta - \frac{1}{p}$. For the estimate in the weighted γ-norm we first fix $t \in [0,T_0]$. We apply Lemma A.2 with $\Phi_1(u) = (t-u)^{-\eta}\Phi(u)1_{\{0\le u<t\}}$, $\Phi_2(r) = \frac{d}{dr}[r^\theta \Psi(r)]$, $R = [0,t]$ and $f(r,u)(s) = (t-s)^{-\alpha}(s-u)^{-\theta}(t-u)^{\eta}1_{\{0\le r<s-u\}}1_{\{0\le u<t\}}$. From the lemma it follows that:

$$\begin{aligned}\Big\| s \mapsto (t-s)^{-\alpha}\int_0^s \Psi(s-u)\Phi(u)\,dW_H(u)\Big\|_{L^p(\Omega;\gamma(0,t;Y_2))} \\ \lesssim t^{\frac12+\eta-\alpha-\theta}\|g\|_{L^1(0,T)}\|s\mapsto (t-s)^{-\eta}\Phi(s)\|_{L^p(\Omega;\gamma(0,t;Y_1))}.\end{aligned}$$

As $\frac12+\eta-\alpha-\theta > 0$ (because $\theta < \eta$ and $\alpha < \frac12$) we have $t^{\frac12+\eta-\alpha-\theta} \le T^{\frac12+\eta-\alpha-\theta}$. Taking the supremum over $t \in [0,T_0]$ we arrive at the desired result. □

For deterministic convolutions we have the following:

Proposition A.12. *Suppose $\Phi \in L^p(\Omega; L^\infty(0,T;Y_1))$ for some $p \in [1,\infty)$. Let $\Psi : [0,T] \to \mathscr{L}(Y_1,Y_2)$ be such that Ψx is continuously differentiable on $(0,T)$ for all $x \in Y_1$. Suppose moreover there exists a $g \in L^1(0,T)$ and a $\theta \in [0,1]$ such that for all $v \in (0,T)$ we have:*

$$\big\|v^\theta \tfrac{d}{dv}\Psi(v)x\big\|_{Y_2} + \big\|v^{\theta-1}\Psi(v)x\big\|_{Y_2} \le g(v)\|x\|_{Y_1}, \quad \textit{for all } x \in Y_1.$$

Then there exists a constant C such that:

$$\|\Psi * \Phi(\omega)\|_{C^{1-\theta}([0,T_0];Y_2)} \le C\|g\|_{L^1(0,T_0)}\|\Phi(\omega)\|_{L^\infty(0,T_0;Y_1)}$$

for almost all $\omega \in \Omega$.

By Corollary A.4, with $\varepsilon = \frac32 - \frac1\tau - \theta$, we obtain the following corollary:

Corollary A.13. *Let the setting be as in Proposition A.12. Assume in addition that Y_2 has type τ, and that $0 \le \theta < \frac32 - \frac1\tau$. Then for $\alpha \in [0,\frac12)$ and $p \in [1,\infty)$ there exists a constant C such that for $T_0 \in [0,T]$ one has:*

$$\|\Psi * \Phi\|_{V_c^{\alpha,p}([0,T_0]\times\Omega;Y_2)} \le C\|g\|_{L^1(0,T_0)}\|\Phi\|_{L^p(\Omega;L^\infty(0,T_0;Y_1))}.$$

Proof (of Proposition A.12). Observe that we have, for $0 \le s < t \le T_0$:

$$\begin{aligned}\Big\|\int_0^t \Psi(t-u)\Phi(u,\omega)\,du - \int_0^s \Psi(s-u)\Phi(u,\omega)\,du\Big\|_{Y_2} \\ \le \Big\|\int_0^s\int_{s-u}^{t-u} \tfrac{d}{dv}[\Psi(v)\Phi(u,\omega)]\,dv\,du\Big\|_{Y_2} \\ + \Big\|\int_s^t (t-u)^{-\theta}\int_0^{t-u} \tfrac{d}{dv}[v^\theta\Psi(v)\Phi(u,\omega)]\,dv\,du\Big\|_{Y_2}.\end{aligned} \tag{A.2.8}$$

Now

$$\Big\| \int_0^s \int_{s-u}^{t-u} \tfrac{d}{dv}[\Psi(v)\Phi(u,\omega)]\,dv\,du \Big\|_{Y_2}$$
$$\le \int_0^s \int_{(s-v)\vee s}^{(t-v)\wedge s} du (s-v)^{-\theta} g(v)\,dv \|\Phi(\omega)\|_{L^\infty(0,t;Y_1)}$$
$$\le (t-s)^{1-\theta} \int_0^t g(v)\,dv \|\Phi(\omega)\|_{L^\infty(0,t;Y_1)},$$

where we used that $\int_{(s-v)\vee s}^{(t-v)\wedge s} du \le (t-s)\wedge v$. Furthermore, we have

$$\Big\| \int_s^t (t-u)^{-\theta} \int_0^{t-u} \tfrac{d}{dv}[v^\varepsilon \Psi(v)]\Phi(u,\omega)\,dv\,du \Big\|_{Y_2}$$
$$\le (1-\theta)^{-1}(t-s)^{1-\theta} \Big\| \int_0^t g(v)\,dv \|\Phi(\omega)\|_{L^\infty(0,T;Y_1)}.$$

Inserting these two estimates in (A.2.8) completes the proof. □

A.3 Existence and uniqueness

The aim of this section is to outline the proof of Theorem 5.3. The setting is always that of Section 5.1.

Assume first that $\alpha \in [0,\frac{1}{2})$ is so large that $\alpha + \theta_G > \eta$. Let $p \in [2,\infty)$ and $T_0 \in [0,T]$ be fixed. For $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)$ define

$$L(\Phi)(t) := S(t)x_0 + \int_0^t S(t-s)F(s,\Phi(s))\,ds + \int_0^t S(t-s)G(s,\Phi(s))\,dW_H(s).$$

Copying Step 1 of the proof of [109, Proposition 6.1] without changes, and substituting Steps 2 and 3 by the Lemmas A.7 and A.8 above, we find that there exists an $\varepsilon_0 > 0$ and a $C > 0$ such that $L : \mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta) \to \mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)$ and

$$\|L(\Phi)\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)}$$
$$\le C\|x_0\|_X + CT^{\varepsilon_0}\|F(\cdot,\Phi(\cdot))\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))}$$
$$+ CT^{\varepsilon_0} \sup_{0\le t\le T_0} \|s\mapsto (t-s)^{-\alpha}G(s,\Phi(s))\|_{L^p(\Omega;\gamma(0,t;X_{\theta_G}))}$$
$$\le C\|x_0\|_X + C(M(F)+M(G))T_0^{\varepsilon_0}(1+\|\Phi\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)}),$$

where in the last line we used **(F)** and (5.2.5). Moreover,

$$\|L(\Phi_1) - L(\Phi_2)\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)}$$
$$\le C(\mathrm{Lip}(F) + \mathrm{Lip}_\gamma(G))T_0^{\varepsilon_0}\|\Phi_1-\Phi_2\|_{\mathscr{V}_\infty^{\alpha,p}([0,T_0]\times\Omega;X_\eta)},$$

where in the last line we used **(F)** and (5.2.4).

Thus by a fixed-point argument, for sufficiently small T_0 there exists a unique process $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0,T_0] \times \Omega; X_\eta)$ satisfying (5.2.1) on the interval $[0,T_0]$. By repeating this construction a finite number of times, each time taking the final value of the previous step as the initial value of the next, we obtain a solution on $[0,T]$.

So far, we have proved existence and uniqueness under the additional assumption $\alpha + \theta_G > \eta$. Existence in $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X_\eta)$ for arbitrary $\alpha \in [0, \frac{1}{2})$ follows by (5.2.2). It remains to prove uniqueness for arbitrary $\alpha \in [0, \frac{1}{2})$.

Let $\alpha \in [0, \frac{1}{2})$ and suppose $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X_\eta)$. Viewing F as a mapping from $[0,T] \times X_\eta$ to X_{θ_F} (as $\eta \geq 0$), we have $F(\cdot, \Phi(\cdot)) \in \mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X_{\theta_F})$. Then, by Lemma A.7 with $\delta = \theta_F - \eta$ and $Y = X_\eta$ (and $\tilde{\alpha} = \alpha + \beta$), we find $S * F(\cdot, \Phi(\cdot)) \in \mathscr{V}_\infty^{\alpha+\beta,p}([0,T] \times \Omega; X_\eta)$ for all $\beta \in [0, \frac{1}{2} - \alpha)$.

By part (i) of Lemma A.8 (with $Y = X_\eta$ and $\delta = \theta_G - \eta$) and (5.2.5) we have, for all $\beta \in [0, \frac{1}{2} - \alpha)$ such that $\beta < \frac{1}{2} + \theta_G - \eta$:

$$
\begin{aligned}
\sup_{0 \leq t \leq T} & \Big\| s \mapsto (t-s)^{-\alpha-\beta} \int_0^s S(s-u) G(u, \Phi(u)) \, dW_H(u) \Big\|_{L^p(\Omega, \gamma(0,t;X_\eta))} \\
&\lesssim \sup_{0 \leq t \leq T} \| s \mapsto (t-s)^{-\alpha} G(s, \Phi(s)) \|_{L^p(\Omega; \gamma(0,t;H,X_{\theta_G}))} \\
&\lesssim \| \Phi \|_{\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X_\eta)}.
\end{aligned}
$$

Since also $S(\cdot)x_0 \in \mathscr{V}_\infty^{\alpha+\beta,p}([0,T] \times \Omega; X_\eta)$ for all $\beta \in [0, \frac{1}{2} - \alpha)$, we see that if $\alpha \in [0, \frac{1}{2})$ and $\Phi \in \mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X_\eta)$ satisfies (5.2.1), then $\Phi \in \mathscr{V}_\infty^{\alpha+\beta,p}([0,T] \times \Omega; X)$ for all $\beta \in [0, \frac{1}{2} - \alpha)$ such that $\beta < \frac{1}{2} + \theta_G - \eta$. Repeating this argument a finite number of steps if necessary, we obtain that $\Phi \in \mathscr{V}_\infty^{\alpha+\beta,p}([0,T] \times \Omega; X)$ for all $\beta \in [0, \frac{1}{2} - \alpha)$. As uniqueness of a process in $\mathscr{V}_\infty^{\alpha,p}([0,T] \times \Omega; X)$ satisfying (5.2.1) has been established for $\alpha > \eta - \theta_G$ this completes the proof for arbitrary $\alpha \in [0, \frac{1}{2})$.

Finally, it remains to prove that the solution U is in fact continuous if $\frac{1}{p} < \frac{1}{2} + \theta_G$. This follows from [109, Theorem 6.2]. □

Remark A.14. Inspection of the proofs of the main theorems reveals that uniqueness is only used for large $\alpha \in [0, \frac{1}{2})$. As a consequence, the last part of the above proof is not needed for our purposes. It has been included for completeness reasons.

A.4 A density counterexample

The example below demonstrates that the finite-rank step processes are not dense in the spaces $V_\infty^{\alpha,p}([0,T] \times \Omega; X)$ and $\mathscr{V}_\infty^{\alpha,p}([a,b] \times \Omega; X)$, this follows by the fact that for V_α as defined below we have $V_\alpha \hookrightarrow V_\infty^{\alpha,p}([0,1] \times \Omega; X)$ and $V_\alpha \hookrightarrow \mathscr{V}_\infty^{\alpha,p}([0,1] \times \Omega; X)$.

Let $\alpha \in (0, \frac{1}{2})$ and let V_α be the subspace of L^2 consisting of the functions f for which the following norm is finite:

$$\|f\|_{V_\alpha} = \sup_{0\le t\le 1} \|s \mapsto (t-s)^{-\alpha} f(s)\|_{L^2(0,t)}.$$

Claim. For all $\alpha \in (0, \frac{1}{2})$ the simple functions are not dense in V_α.

Proof. Fix $\alpha \in (0, \frac{1}{2})$. Let $\beta > 1 - 2\alpha$ be such that:

$$\sqrt{(1+2\beta)^2 + 8} - (1+2\beta) < 4\alpha$$

(this is possible due to the fact that $\alpha > 0$). Set $p := \frac{2\beta}{1-2\alpha}$. By our choice of β we have $p > 1$ and it is possible to pick $q \in \mathbb{R}$ such that:

$$1 < q < \frac{2\alpha(1+\beta) - 1}{\alpha(1-2\alpha)}.$$

Let p', q' be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. For $n \in \mathbb{N}$ define

$$\begin{aligned} a_n &:= 1 - \tfrac{1}{2} \sum_{j=1}^{n} (\tfrac{1}{p'} j^{-p} + \tfrac{1}{q'} j^{-q}); \\ b_n &:= a_n - \tfrac{1}{2p'}(n+1)^{-p}. \end{aligned}$$

It follows that

$$\begin{aligned} a_n - b_n &:= \tfrac{1}{2p'}(n+1)^{-p}; \\ b_n - a_{n+1} &:= \tfrac{1}{2q'}(n+1)^{-q}. \end{aligned} \tag{A.4.1}$$

Moreover, for all $n \in \mathbb{N}$:

$$a_n \ge \tfrac{1}{2} \sum_{j=1}^{\infty} (\tfrac{1}{p'} j^{-p} + \tfrac{1}{q'} j^{-q}) \ge 1 - \tfrac{1}{2}\big(\tfrac{p'}{p'} + \tfrac{q'}{q'}\big) = 0.$$

Define $f : [0,1] \to \mathbb{R}$ by:

$$f(s) := \sum_{j=1}^{\infty} (j+1)^{\beta} 1_{[b_j, a_j]}(s).$$

Observe:

$$\begin{aligned} \|f\|_{V_\alpha} &= \sup_{0\le t\le 1} \|s \mapsto (t-s)^{-\alpha} f(s)\|_{L^2(0,t)} \\ &= \sup_{j\in\mathbb{N}} \sup_{t\in[b_j,a_j]} \|s \mapsto (t-s)^{-\alpha} f(s)\|_{L^2(0,t)}. \end{aligned}$$

Fix $j \in \mathbb{N}$. For $t \in [b_j, a_j]$ we have:

$$\begin{aligned} &\sup_{t\in[b_j,a_j]} \|s \mapsto (t-s)^{-\alpha} f(s)\|_{L^2(0,t)} \\ &\quad \le \|s \mapsto (a_j - s)^{-\alpha} f(s)\|_{L^2(b_j,a_j)} + \|s \mapsto (b_j - s)^{-\alpha} f(s)\|_{L^2(0,b_j)} \end{aligned}$$

$$\leq (\tfrac{1}{2}-\alpha)^{-\frac{1}{2}}(j+1)^{\beta}(a_j-b_j)^{\frac{1}{2}-\alpha} + \sum_{k=j+1}^{\infty}(k+1)^{\beta}\|s\mapsto (b_j-s)^{-\alpha}\|_{L^2(b_k,a_k)}$$
$$\leq (\tfrac{1}{2}-\alpha)^{-\frac{1}{2}}(j+1)^{\beta}(a_j-b_j)^{\frac{1}{2}-\alpha} + \sum_{k=j+1}^{\infty}(k+1)^{\beta}(a_k-b_k)^{\frac{1}{2}}(b_j-a_k)^{-\alpha}.$$

By (A.4.1) and the fact that $b_j - a_k \geq b_{k-1} - a_k$ it follows that:

$$\begin{aligned}\sup_{t\in[b_j,a_j]} &\|s\mapsto (t-s)^{-\alpha}f(s)\|_{L^2(0,t)}\\ &\leq (\tfrac{1}{2}-\alpha)^{-\frac{1}{2}}(\tfrac{1}{2p'})^{\frac{1}{2}-\alpha}(j+1)^{\beta}(j+1)^{-p(\frac{1}{2}-\alpha)}\\ &\quad + 2^{\alpha-\frac{1}{2}}(p')^{-\frac{1}{2}}(q')^{\alpha}\sum_{k=j+1}^{\infty}(k+1)^{\beta+\alpha q-\frac{p}{2}}.\end{aligned}$$

By choice of p and q we have $p(\frac{1}{2}-\alpha)=\beta$ and $\beta+\alpha q - \frac{p}{2} < -1$ and thus there exists a constant M_α such that for all $j\in\mathbb{N}$ we have:

$$\sup_{t\in[b_j,a_j]}\|s\mapsto (t-s)^{-\alpha}f(s)\|_{L^2(0,t)} \leq M_\alpha,$$

whence $f\in V_\alpha$.

Let $g:[0,1]\to\mathbb{R}$ be a simple function and let $N\in\mathbb{N}$ be such that $g(s)\leq N$ for all $s\in[0,1]$. Let $j\in\mathbb{N}$ be such that $(j+1)^{\beta}\geq 2N$. We have:

$$\begin{aligned}\|f-g\|_{V_\alpha} &\geq \|s\mapsto (a_j-s)^{-\alpha}(f(s)-g(s))\|_{L^2(b_j,a_j)}\\ &\geq \tfrac{1}{2}(j+1)^{\beta}\|s\mapsto (a_j-s)^{-\alpha}\|_{L^2(b_j,a_j)}\\ &= 2^{-\frac{1}{2}-\alpha}(\tfrac{1}{2}-\alpha)^{-\frac{1}{2}}(p')^{-\frac{1}{2}+\alpha\alpha}.\end{aligned}$$

□

A.5 Convergence of the splitting scheme (part 2)

As announced at the beginning of Chapter 6, it is possible to obtain truly pathwise convergence result for the (modified or classical) spitting scheme, in the sense that one does not only consider the grid points. We will outline the proof below.

Recall that by [109, Theorem 6.2] (see also Remark 5.4) there exists, in the space $V_{\mathrm{c}}^{\alpha,p}([0,T]\times\Omega;X)$, a unique solution to (SDE) under the conditions **(A)**, **(F)**, **(G)**, for $\alpha\in[0,\frac{1}{2})$ and $p>2$ such that $\frac{1}{p}<\frac{1}{2}+\theta_G$. It follows that the processes $U_j^{(n)}$ defined by the modified splitting method (6.0.1) or the classical splitting method (6.0.4) are elements of $V_{\mathrm{c}}^{\alpha,p}(I_j^{(n)}\times\Omega;X)$ for such α and p. However, the process $U^{(n)}$ defined by (6.0.2) is not continuous and therefore does not define an element of this space. Thus we consider convergence in $V_{\infty}^{\delta,p}([a,b]\times\Omega;X)$ as defined on page 85.

Theorem A.15. *Consider the stochastic differential equation* (SDE) *under the assumptions* **(A)**, **(F)**, **(G)**. *For $n \in \mathbb{N}$ let $U^{(n)}$ be defined by* (6.0.2) *with $U_j^{(n)}$ the solution to the modified splitting method* (6.0.1) *or, if θ_F and θ_G are non-negative, the classical splitting method* (6.0.4). *Let $\eta \geq 0$ and $p \in (2, \infty)$ satisfy:*

$$0 \leq \eta < \min\{\tfrac{3}{2} + \theta_F - \tfrac{1}{\tau}, \tfrac{1}{2} + \theta_G - \tfrac{1}{p}, 1\}.$$

If $x_0 \in L^p(\mathscr{F}_0, X_\eta)$, $y_0 \in L^p(\mathscr{F}_0, X)$, then one has, for all $\alpha \in [0, \frac{1}{2})$,

$$\begin{aligned}\|U - U^{(n)}\|_{V_\infty^{\alpha,p}([0,T]\times\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} \\ + n^{-\eta}\big(1 + \|x_0\|_{L^p(\Omega;X_\eta)}\big),\end{aligned}$$

with implied constants independent of n, x_0 and y_0.

The following corollary is obtained by a Borel-Cantelli argument, see also Corollary 6.6.

Corollary A.16. *Let the setting be as in Theorem A.15 with the additional assumption that $\eta \geq 0$ and $p \in (2, \infty)$ satisfy:*

$$0 \leq \eta + \tfrac{1}{p} < \min\{\tfrac{3}{2} + \theta_F - \tfrac{1}{\tau}, \tfrac{1}{2} + \theta_G - \tfrac{1}{p}, 1\}.$$

If $x_0 = y_0 \in L^p(\mathscr{F}_0, X_\eta)$, then there exists a random variable χ such that:

$$\|U - U^{(n)}\|_{L^\infty(0,T;X)} \lesssim \chi n^{-\eta}.$$

Basically, the proof of Theorem A.15 is identical to the proof of Theorem 6.2 and Theorem 6.4, except that in parts 1c and 1d of Theorem 6.2 one applies Lemma A.18 below instead of Lemma A.7, and in 1e and 1f one applies Lemma A.19 below instead of Lemma A.8. Moreover, at the instances where (2.4.6) is used in the proof of Theorem 6.4, we need the following regularity result (which follows by Kolmogorov from (2.4.6)):

Lemma A.17. *Let $p \in (1, \infty)$ and $\alpha, \beta \in (0, 1)$ satisfy $\beta < \alpha - \frac{1}{p}$. Suppose $\Phi \in L^p(\Omega; \gamma(0, t; H, X))$ is adapted. Then*

$$\Big\| \int_0^\cdot \Phi \, dW_H \Big\|_{L^p(\Omega;C^\beta([0,T];X))} \lesssim \sup_{0\leq t\leq T} \|u \mapsto (t-u)^{-\alpha}\Phi(u)\|_{L^p(\Omega;\gamma(0,t;H,X))},$$

with an implied constant depending only on α, β and p.

Finally, one may check that Corollary 6.3 remains valid for the space $V_\infty^{\alpha,p}$, which is necessary to extend the proof of Theorem 6.4 to the space $V_\infty^{\alpha,p}$.

The proof of Lemma A.18 can be derived from Step 2 of the proof of [109, Proposition 6.1] (see also Lemma A.7). The proof of Lemma A.19 can be found in Step 3 of the proof of [109, Proposition 6.1] for the space $V_{\mathrm{c}}^{\alpha,p}$ (i.e., the space with continuous paths). The proof remains valid if one considers $V_\infty^{\alpha,p}$ instead of $V_{\mathrm{c}}^{\alpha,p}$.

The condition $\eta < \frac{1}{2}+\theta_G-\frac{1}{p}$ (instead of $\eta < \frac{1}{2}+\theta_G$ as in Theorem 6.2) is due to the fact that the conditions on δ in Lemma A.19 are stronger than in Lemma A.8, and due to the fact that Lemma A.17 involves Kolmogorov's theorem. For the sake of completeness, we present the alternative arguments for the parts 1e and 1f below.

Lemma A.18. *Let $\delta \in \mathbb{R}$ be such that $\delta > -\frac{3}{2}+\frac{1}{\tau}$ and let $\alpha \in [0, \frac{1}{2})$. Assume $\Phi \in L^\infty(0,T;L^p(\Omega;Y_\delta))$ for some $T > 0$ and some $p \in [1,\infty]$. Then $t \mapsto \int_0^t S(t-s)\Phi(s)\,ds \in V_\infty^{\alpha,p}([0,T]\times\Omega;Y)$ and:*

$$\Big\|t \mapsto \int_0^t S(t-s)\Phi(s)\,ds\Big\|_{\mathscr{V}_\infty^{\alpha,p}([0,T]\times\Omega;Y)} \lesssim (T^{1-\delta^-}+T^{\frac{1}{2}-\alpha})\|\Phi\|_{L^\infty(0,T;L^p(\Omega;Y_\delta))}.$$

Lemma A.19. *Let $\delta > -\frac{1}{2}$, $p > 2$ and $\alpha \in [0,\frac{1}{2})$ be such that $\alpha > -\delta+\frac{1}{p}$. Assume $\Phi \in L^p(\Omega;\gamma(0,T;H,Y_\delta))$ for some $T > 0$ and*

$$\sup_{0\le t\le T}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,T;H,Y_\delta))} < \infty. \tag{A.5.1}$$

Then $t \mapsto \int_0^t S(t-s)\Phi(s)\,dW_H(s) \in V_\infty^{\alpha,p}([0,T]\times\Omega;Y)$ and there exists an $\epsilon > 0$ such that

$$\begin{aligned}\Big\|t \mapsto \int_0^t & S(t-s)\Phi(s)\,dW_H(s)\Big\|_{V_\infty^{\alpha,p}([0,T]\times\Omega;Y)}\\ &\lesssim (T^\epsilon+T^{\frac{1}{2}-\delta^-})\sup_{0\le t\le T}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L^p(\Omega;\gamma(0,T;H,Y_\delta))}.\end{aligned}$$

Proof (of Theorem A.15).

Part 1e'. As before, we obtain for every $T_0 \in [0,T]$:

$$\begin{aligned}&\Big\|s \mapsto \int_0^s [S(s-u)-S(\overline{s}-\underline{u})]G(u,U(u))\,dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}\\ &\lesssim n^{-\eta}\Big\|s \mapsto \int_0^s S(s-u)G(u,U(u))\,dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{2}+\theta_G-\frac{1}{p}-\frac{2}{3}\varepsilon})}\\ &\quad+\Big\|s \mapsto \int_0^s S(s-u)(I-S(u-\underline{u}))G(u,U(u))\,dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{3}\varepsilon})}.\end{aligned} \tag{A.5.2}$$

Now we apply Lemma A.19 to the last two terms. For the penultimate term in (A.5.2) we apply Lemma A.19 with $Y = X_{\frac{1}{2}+\theta_G-\frac{1}{p}-\frac{2}{3}\varepsilon}$, $\delta = -\frac{1}{2}+\frac{1}{p}+\frac{2}{3}\varepsilon$ and $\Phi(u) = G(u,U(u))$, using that $\alpha > \frac{1}{2}-\frac{2}{3}\varepsilon$, to obtain, for every $T_0 \in [0,T]$,

$$\begin{aligned}\Big\|s \mapsto \int_0^s & S(s-u)G(u,U(u))\,dW_H(u)\Big\|_{V_\infty^{\frac{1}{2}-\frac{1}{2}\varepsilon,p}([0,T_0]\times\Omega;X_{\frac{1}{2}+\theta_G+\frac{1}{p}-\frac{2}{3}\varepsilon})}\\ &\lesssim \sup_{0\le s\le T}\|u \mapsto (s-u)^{-\alpha}G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;H,X_{\theta_G}))}\end{aligned}$$

$$\lesssim 1 + \|U\|_{V_\infty^{\alpha,p}([0,T]\times\Omega;X)},$$

with implied constants independent of n, x and T_0. In the final line we use (5.2.5).

For the final term we take $Y = X_{\frac{1}{3}\varepsilon}$, $\delta = -\frac{1}{2} + \frac{1}{p} + \frac{2}{3}\varepsilon$ and $\Phi(u) = (I - S(u - \underline{u}))G(u, U(u))$. Note that (A.5.1) is satisfied due to the fact that $U \in V_\infty^{\alpha,p}([0,T]\times\Omega;X)$, and due to (5.2.5) and the γ-boundedness of $\{I - S(t) : t \in [0, \frac{T}{n}]\}$ in $\mathscr{L}(X_{\theta_G}, X_{-\frac{1}{2}+\frac{1}{p}+\varepsilon})$. Thus from Lemma A.19 one obtains, for every $T_0 \in [0,T]$,

$$\begin{aligned}
&\Big\| s \mapsto \int_0^s S(s-u)(I - S(u-\underline{u}))G(u,U(u))\, dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{3}\varepsilon})} \\
&\quad\lesssim \sup_{0\le s\le T} \|u \mapsto (s-u)^{-\alpha}(I - S(u-\underline{u}))G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;H,X_{-\frac{1}{2}+\frac{1}{p}+\frac{1}{3}\varepsilon}))} \\
&\quad\lesssim n^{-\min\{\frac{1}{2}+\theta_G-\frac{1}{p}-\varepsilon,1\}} \sup_{0\le s\le T} \|u \mapsto (s-u)^{-\frac{1}{2}+\frac{1}{3}\varepsilon} G(u,U(u))\|_{L^p(\Omega;\gamma(0,s;H,X_{\theta_G}))} \\
&\quad\lesssim n^{-\eta}(1 + \|U\|_{V_\infty^{\alpha,p}([0,T]\times\Omega;X)}),
\end{aligned}$$

with implied constants independent of n, x and T_0.

Combining these estimates and applying (5.2.7) we obtain, for every $T_0 \in [0,T]$,

$$\begin{aligned}
\Big\| s \mapsto \int_0^s [S(s-u) - S(\overline{s}-\underline{u})]G(u,U(u))\, dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X)}& \\
\lesssim n^{-\eta}(1 + \|x\|_{L^p(\Omega;X)}),&
\end{aligned}$$

with implied constants independent of n, x and T_0.

Part 1f'. We again apply first Theorem 2.14 in combination with Lemma 2.21 (2) with $\alpha = \epsilon = \frac{1}{6}\varepsilon$ to get rid of the term $S(\overline{s} - s)$. Then we apply Lemma A.19 with $Y = X_{\frac{1}{3}\varepsilon}$, $\delta = \theta_G - \frac{1}{3}\varepsilon$, and $\Phi(u) = S(u-\underline{u})G(u,U(u)) - G(u,U^{(n)}(u))$. Note that Φ satisfies condition (A.5.1) because G is L_γ^2-Lipschitz and $U^{(n)} \in V_\infty^{\alpha,p}([0,T_0]\times\Omega;X)$ and $U \in V_{\mathrm{c}}^{\alpha,p}([0,T_0]\times\Omega;X)$. We also use that $\alpha > \frac{1}{2} - \frac{2}{3}\varepsilon$. Finally, we apply Theorem 2.14 again in combination with Lemma 2.21 (2) with $\alpha = \epsilon = \frac{1}{6}\varepsilon$ to get rid of the term $S(u - \underline{u})$. We thus obtain that there exists an $\epsilon > 0$ such that for every $T_0 \in [0,T]$ one has:

$$\begin{aligned}
&\Big\| s \mapsto \int_0^s S(\overline{s}-\underline{u})[G(u,U(u)) - G(u,U^{(n)}(u))]\, dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X)} \\
&\quad\lesssim \Big\| s \mapsto \int_0^s S(s-\underline{u})[G(u,U(u)) - G(u,U^{(n)}(u))]\, dW_H(u)\Big\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X_{\frac{1}{6}\varepsilon})} \\
&\quad\lesssim T_0^{\epsilon} \sup_{0\le s\le T_0} \|s \mapsto (s-u)^{-\alpha} S(u-\underline{u}) \\
&\qquad\qquad \times [G(u,U(u)) - G(u,U^{(n)}(u))]\|_{L^p(\Omega;\gamma(0,s;H,X_{\theta_G-\frac{1}{6}\varepsilon}))} \\
&\quad\lesssim T_0^{\epsilon}\|U - U^{(n)}\|_{V_\infty^{\alpha,p}([0,T_0]\times\Omega;X)},
\end{aligned}$$

with implied constants independent of n, x, y and T_0. □

Summary

This thesis deals with various aspects of the study of stochastic partial differential equations driven by Gaussian noise. The approach taken here is functional analytic rather than probabilistic, which means that the equation is interpreted as an ordinary stochastic differential equation in a Banach space X.

The major part of this thesis, namely Chapters 5-11, concerns convergence of numerical schemes for stochastic differential equations. Apart from that, there is a chapter on delay equations and a chapter on decoupling.

Decoupling is a concept that is used when defining the stochastic integral with respect to a Brownian motion of a stochastic process Φ taking values in a Banach space X. When X is a Hilbert space, the stochastic integral of Φ can be defined using the finite-dimensional stochastic integration theory and orthogonality. For general Banach spaces, the stochastic integral of an X-valued *function* can be defined using an Itô isomorphism in which the L^2-norm is replaced by a suitable Gaussian norm. In order to extend this definition to adapted X-valued stochastic *processes*, one needs to be able to 'decouple' the process Φ from the Brownian motion - specifically, one wishes to be able to replace the Brownian motion by a copy that is independent of Φ.

It is known that Banach spaces with the UMD property allow for such a decoupling. All Hilbert spaces have the UMD property, however, a Banach space can only have the UMD property if it is reflexive. In Chapter 3, a weaker decoupling property is studied, which still allows for the definition of the stochastic integral of an X-valued stochastic process. The space $L^1(0,1)$ is an example of a space that does not have the UMD property, but does have the decoupling property. Our most important result is the so-called 'p-independence' of the decoupling property.

Delay equations are used to describe processes for which the development of the current state depends on previous states (e.g. population models and control systems). Such equations can be treated as an abstract Cauchy problem by considering the state space to be a function space over the relevant history interval. We take this approach to treat delay equations with multiplicative noise, in a UMD Banach space X with type 2. The state space for the corresponding

Cauchy problem is taken to be $L^p(0,r;X)$ for $p \in [1,\infty)$. We prove that such equations have a unique continuous solution.

The stochastic differential equation, for which convergence rates of approximations are considered in this thesis, is of the following type:

$$\begin{cases} dU(t) = AU(t)\,dt + F(t,U(t))\,dt + G(t,U(t))\,dW_H(t); \quad t \in [0,T], \\ U(0) = x_0. \end{cases} \tag{A.5.3}$$

Here U is a stochastic process taking values in a UMD Banach space X. For example, one may have $X = L^p(D)$ with $D \subset \mathbb{R}^d$ open and $p \in (1,\infty)$. By W_H we denote an H-cylindrical Brownian motion, where H is a Hilbert space. The operator A is assumed to be an unbounded operator generating a bounded analytic C_0-semigroup, for example: $A = \Delta$, the Laplacian. As A generates an analytic semigroup, it is possible to define the fractional domain space $D((-A)^\theta)$, for $\theta \in \mathbb{R}$. In the case that $A = \Delta$ and $X = L^p(D)$, the fractional domain spaces can be expressed in terms of Sobolev spaces.

The non-linear term F is assumed to take values in $D((-A)^{\theta_F})$, for some $\theta_F > -\frac{3}{2} + \frac{1}{\tau}$. By τ we denote the type of X; for UMD spaces we have $\tau \in (1,2]$. It is assumed that the non-linear term G takes values in $\mathscr{L}(H, D((-A)^{\theta_G}))$ where $\theta_G > -\frac{1}{2}$. Finally, we assume certain (global) Lipschitz and linear growth conditions on F and G.

We consider both time and space discretizations for equation (A.5.3). In Chapter 7 it is proven that for the sequence of random variables $(V_j^{(n)})_{j=1}^n$ obtained by applying the implicit Euler method to equation (A.5.3) over the interval $[0,T]$ with step size $\frac{T}{n}$, we have:

$$\Big(\mathbb{E} \sup_{0\le j\le n} \big\| U\big(\tfrac{jT}{n}\big) - V_j^{(n)} \big\|_X^p \Big)^{\frac{1}{p}} \lesssim n^{-\eta}\big(1 + \|x_0\|_{D((-A)^\eta)}\big),$$

provided that $\eta + \frac{1}{p} < \min\{1 - (\frac{1}{\tau} - \frac{1}{2}) + (\theta_F \wedge 0), \frac{1}{2} + (\theta_G \wedge 0)\}$. This convergence rate is optimal.

Concerning space discretizations, in Chapter 10 we quantify the effect on the solution to (A.5.3) of a perturbation of the operator A. This can be used to prove convergence of Galerkin and finite element schemes for (A.5.3) in the case that X is a Hilbert space. This is demonstrated in Chapter 11.

Our results lead to pathwise convergence of the approximations, which allows us to obtain convergence also for the case that F and G are locally Lipschitz continuous instead of globally Lipschitz continuous (see Chapter 8).

However, some open problems remain. First of all, it is unknown whether the space discretizations converge if the Banach space X is not a Hilbert space. Secondly, since the noise discretization is the bottleneck for the convergence rate, employing recent developments in noise discretization may produce higher-order convergence rates. Finally, there are indications that the critical convergence rates for the approximation schemes can be obtained if one assumes that A has 'stochastic maximal regularity'.

Samenvatting

Dit proefschrift beschouwt verschillende aspecten van de theorie voor stochastische partiële differentiaalvergelijkingen met normaal verdeelde ruis. De insteek is functionaal-analytisch, hetgeen betekent dat stochastische partiële differentiaalvergelijkingen worden opgevat als gewone stochastische differentiaalvergelijkingen in een Banachruimte.

De hoofdmoot van dit proefschrift, de hoofdstukken 5-11, betreft convergentie van numerieke schema's voor stochastische differentiaalvergelijkingen. Daarnaast bevat dit proefschrift een hoofdstuk over ontkoppeling in Banachruimten (*decoupling*) en een hoofdstuk over differentiaalvergelijking met vertraging (*delay equations*).

Ontkoppeling is een concept dat een rol speelt in de definitie van de stochastische integraal tegen een Brownse beweging van een stochastisch proces Φ met waarden in een Banachruimte X. Wanneer X een Hilbertruimte is, kan de stochastische integraal van Φ worden gedefinieerd door gebruik te maken van eindig-dimensionale theorie en orthogonaliteit. Voor het algemene geval kan de stochastische integraal van een X-waardige *functie* worden gedefinieerd met behulp van een Itô-isomorphisme waarin de L^2-norm is vervangen door een Gausische norm. Om een uitbreiding naar aangepaste X-waardige *stochastische processen* te verkrijgen moet men het proces kunnen 'ontkoppelen' van de Brownse beweging – men moet de Brownse beweging kunnen vervangen door een kopie die onafhankelijk is van het proces Φ.

Het is bekend dat de Banachruimten met de UMD eigenschap een dergelijke ontkoppeling toestaan. Alle Hilbertruimten hebben de UMD eigenschap, maar een Banachruimte kan deze eigenschap alleen hebben als zij reflexief is. In hoofdstuk 3 wordt een zwakkere eigenschap dan de UMD eigenschap bestudeerd, waarmee alsnog de stochastische integraal van een X-waardig stochastisch proces kan worden gedefinieerd. De ruimte $L^1(0,1)$ is een voorbeeld van een ruimte die wel deze eigenschap, maar niet de UMD eigenschap heeft. Als belangrijkste resultaat wordt de zogenaamde 'p-onafhankelijkheid' van de ontkoppelingseigenschap aangetoond.

Differentiaalvergelijkingen met vertraging worden gebruikt om processen te beschrijven waar de verandering van de huidige toestand afhangt van de toestand in het verleden (bijvoorbeeld populatiemodellen, regelsystemen). Door als toestandsruimte een functieruimte te kiezen over het tijdsinterval wat van invloed is, kan een dergelijke vergelijking worden opgevat als een abstract Cauchy probleem. In dit proefschrift wordt dit idee toegepast op differentiaalvergelijkingen met vertraging *en* multiplicatieve ruis, in een UMD Banachruimte X met type 2. De toestandsruimte voor het bijbehorende Cauchy probleem is $L^p(0,r;X)$, $p \in [1,\infty)$. We tonen aan dat deze vergelijkingen een unieke continue oplossing hebben.

De stochastiche differentiaalvergelijking waarvoor in dit proefschrift convergentieresultaten voor numerieke schema's is verkregen, is van de volgende vorm:

$$\begin{cases} dU(t) = AU(t)\,dt + F(t,U(t))\,dt + G(t,U(t))\,dW_H(t); \quad t \in [0,T], \\ U(0) = x_0. \end{cases} \tag{A.5.4}$$

In deze vergelijking is U een stochastisch proces dat waarden aanneemt in een Banachruimte X met de UMD eigenschap (bijvoorbeeld $X = L^p(D)$ met $D \subset \mathbb{R}^d$ open en $p \in (1,\infty)$). Met W_H wordt een H-cylindrische Brownse beweging aangeduid, waarbij H een Hilbertruimte is. Verder is A een onbegrensde operator op X die een begrensde analytische C_0-halfgroep genereert (bijvoorbeeld $A = \Delta$; de Laplaciaan). Omdat A analytisch is, is het mogelijk betekenis te geven aan de gebroken machten $(-A)^\theta$ voor $\theta \in \mathbb{R}$. Voor het geval $A = \Delta$ en $X = L^p(D)$ kan $D((-A)^\theta)$ uitgedrukt worden in termen van een Sobolev ruimte.

Er wordt aangenomen dat F waarden aanneemt in $D((-A)^{\theta_F})$, met $\theta_F > -\frac{3}{2} + \frac{1}{\tau}$. Met τ wordt het type van X aangeduid, voor UMD Banachruimten geldt $\tau \in (1,2]$. Tenslotte wordt aangenomen dat G waarden aanneemt in $\mathscr{L}(H, D((-A)^{\theta_G}))$ waarbij $\theta_G > -\frac{1}{2}$. Op F en G worden tevens – in eerste instantie globale – Lipschitz- en lineaire groei-voorwaarden aangenomen.

We beschouwen zowel tijds- als ruimte-discretisaties voor de vergelijking (A.5.4). Zij $(V_j^{(n)})_{j=1}^n$ de rij toevalsvariabelen die wordt verkregen door de impliciete Eulermethode toe te passen op (A.5.4) over het interval $[0,T]$ met stapgrootte $\frac{T}{n}$. In hoofdstuk 7 wordt bewezen dat:

$$\Big(\mathbb{E} \sup_{0\le j\le n} \big\|U\big(\tfrac{jT}{n}\big) - V_j^{(n)}\big\|_X^p\Big)^{\frac{1}{p}} \lesssim n^{-\eta}\big(1 + \|x_0\|_{D((-A)^\eta)}\big),$$

mits $\eta + \frac{1}{p} < \min\{\frac{3}{2} - \frac{1}{\tau} + (\theta_F \wedge 0), \frac{1}{2} + (\theta_G \wedge 0)\}$. Deze convergentiesnelheid is optimaal.

Wat ruimte-discretisaties betreft wordt in hoofdstuk 10 van dit proefschrift een perturbatieresultaat bewezen voor (A.5.4) dat aangeeft hoe de oplossing van deze vergelijking verandert als A verandert. In hoofdstuk 11 wordt met behulp van dit perturbatieresultaat convergentie van de Galerkinmethode en de eindige elementenmethode verkregen mits X een Hilbertruimte is.

Onze resultaten leiden tot padsgewijze convergentie van de approximaties, waardoor het mogelijk is tevens convergentie aan te tonen voor het geval dat F en G slechts lokaal Lipschitz zijn in plaats van globaal Lipschitz (zie hoofdstuk 8).

Er zijn nog verscheidene openstaande vragen. Ten eerste is nog onbekend of de ruimte-discretisaties ook convergeren in Banachruimten die geen Hilbertruimten zijn. Verder zou het interessant zijn te kijken naar de recente ontwikkelingen op het gebied van ruis-discretisatie, aangezien dit nu de beperkende factor is voor de convergentiesnelheid. Tenslotte is er reden om aan te nemen dat de kritieke convergentiesnelheden voor de approximatiemethoden verkregen kunnen worden als men aanneemt dat A zogenaamde 'stochastische maximale regulariteit' heeft.

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Index

Notation

General mathematics

$\varnothing$	the empty set
$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\Re e(z)$	the real part of $z \in \mathbb{C}$
$\Im m(z)$	the complex part of $z \in \mathbb{C}$
$\mathrm{Arg}(z)$	the argument of $z \in \mathbb{C}$
Σ_θ	$\{z \in \mathbb{C} \setminus \{0\} : \mathrm{Arg}(z) < \theta\}$
$\frac{\partial}{\partial x}$	the partial derivative w.r.t. x
x^+	$\max\{x, 0\}$ $(x \in \mathbb{R})$
x^-	$x^+ - x$ $(x \in \mathbb{R})$
$a \vee b$	$\max\{a, b\}$
$a \wedge b$	$\min\{a, b\}$
$a \lesssim b$, $a \gtrsim b$, $a \eqsim b$	p. 13

Spaces

H, $\mathscr{H}$	(real) Hilbert spaces
X, Y	(real) Banach spaces
$X \simeq Y$	X and Y are isomorphic as Banach spaces
$X \hookrightarrow Y$	X embeds isomorphically into Y
X^*	the dual Banach space of X
$\mathscr{L}(X, Y)$	the space of bounded linear operators from X to Y
$\mathscr{L}(X)$	$\mathscr{L}(X, X)$
$\mathscr{L}_2(H, X)$	the space of Hilbert-Schmidt operators from H to X
X_δ, X_δ^A	the fractional domain space/extrapolation space of $A : D(A) \subset X \to X$, see p. 27
$\mathscr{H}_\delta$, $\mathscr{H}_\delta^A$	the fractional domain space/extrapolation space of

	$A : D(A) \subset \mathscr{H} \to \mathscr{H}$ ($\mathscr{H}$ a Hilbert space), see p. 27
$\gamma_\infty(H, X)$	the space of γ-summing operators from H to X, see p. 15
$\gamma(H, X)$	the space of γ-radonifying operators from H to X, see p. 16
$\gamma(0, T; H, X)$	$\gamma(L^2(0, T; H), X)$
$\gamma(S, H; X)$	$\gamma(L^2(S; H), X)$
$L^p_{\mathscr{F}}(\Omega; \gamma(0, T, H; X))$	see p. 22
$L^0(S; X)$	the space of strongly measurable X-valued functions endowed with the topology of convergence in measure
$C(S; X)$	the space of continuous X-valued functions on S
$C_b(S; X)$	the space of continuous bounded X-valued functions on S
$C^\alpha([a, b]; X)$	the space of α-Hölder continuous functions on $[a, b]$
$H^{\alpha,p}(S)$	Sobolev space
$B^s_{q,r}(I; X)$	Besov space, see p. 17
$\mathscr{E}^p(X)$	$L^p(-1, 0; X) \times X$, see p. 63
$V^{\alpha,p}_{\mathrm{c}}([0, T] \times \Omega; X)$	see p. 85
$V^{\alpha,p}_\infty([0, T] \times \Omega; X)$	see p. 85
$\mathscr{V}^{\alpha,p}_\infty([0, T] \times \Omega; X)$	see p. 85
$V^{\alpha,0}_c([0, T] \times \Omega; X)$	see p. 152
$V^{\alpha,0}_\infty([0, T] \times \Omega; X)$	see p. 152
$c^{(n)}_\gamma([0, T]; X)$	see p. 87

Constants

$\mathscr{T}_p(X)$	the type p constant of X, see p. 13
$\mathscr{C}_q(X)$	the cotype q constant of X, see p. 13
$\beta_p(X)$	the UMD constant of X, see p. 14
$C_p(X), D_p(X)$	decoupling constants, see p. 35
$\beta^+_p(X), \beta^-_p(X)$	randomized UMD constants, see p. 35
$\mathrm{Lip}(F)$, $M(F)$	see p. 84
$\mathrm{Lip}_\gamma(G)$, $M_\gamma(G)$	see p. 84

Measure and probability

a.e.	almost everywhere
a.s.	almost surely
$\mathbb{E}$	expectation
$\mathbb{E}(\cdot\vert\mathscr{F})$	the conditional expectation with respect to $\mathscr{F}$
$\mathbb{P}$	a probability measure
$(\mathscr{F}_n)_{n\geq 1}$	a filtration (discrete)

$(\mathscr{F}_t)_{t\geq 0}$	a filtration (continuous)
$\mathscr{B}(X)$	the Borel σ-algebra on X
w	real-valued Brownian motion or space-time white noise
W	Brownian motion
W_H	H-cylindrical Brownian motion, see p. 18
ξ^*	$\sup_{j\in I} \lVert\xi\rVert$, see p. 38
ξ_n^*	$\sup_{j\leq n} \lVert\xi\rVert$, see p. 38

Operators

I	the identity operator
I_X	the identity operator on X
$\varrho(A)$	the resolvent set of A
$\sigma(A)$	the spectrum of A
$D(A)$	the domain of A
$R(\lambda : A)$	the resolvent of A in λ
A^*	the adjoint operator of A
$\mathscr{G}r(A)$	the graph of A, i.e., $\{(x, Ax) : x \in D(A)\}$

Notation for approximations (n, T given)

$t_j^{(n)}$	$\frac{jT}{n}$
$I_j^{(n)}$	$[t_{j-1}^{(n)}, t_j^{(n)})$
$\mathscr{F}_j^{(n)}$	$\mathscr{F}_{t_j^{(n)}}$
$\Delta W_j^{(n)}$	$W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)})$ (formally, see also p. 133)
$\underline{t}$	$t_{j-1}^{(n)}$; $t \in I_j^{(n)}$ (see p. 89)
$\bar{t}$	$t_j^{(n)}$; $t \in I_j^{(n)}$ (see p. 89)
$U^{(n)}$, $U_j^{(n)}$	the approximation obtained by the modified splitting scheme, see p. 87
$\tilde{U}^{(n)}$, $\tilde{U}_j^{(n)}$	the approximation obtained by the classical splitting scheme, see p. 88
$V_j^{(n)}$	an abstract time discretization of Chapter 7, e.g., the implicit-linear Euler scheme
u	$(U(t_j^{(n)}))_{j=0}^n$
$u^{(n)}$	$(U^{(n)}(t_j^{(n)}))_{j=0}^n$
$v^{(n)}$	$(V_j^{(n)})_{j=0}^n$
$E(\frac{T}{n})$, $E(t_j^{(n)})$	see p. 126

Miscellaneous

$\eta_{\max}$	see p. 86

$\zeta_{\max}$	see p. 132
$\Phi \diamond \Psi$	the stochastic convolution of Φ with Ψ, see p. 195
F_q	see p. 46

Curriculum Vitae

Sonja Gisela Cox was born in Vancouver, Canada, on July 3rd, 1983, to a German mother and a Canadian father. In 1992, her parents decided to move the family to the Netherlands. Sonja completed secondary education at the Rijnlands Lyceum Wassenaar in 2001. In September 2001, she enrolled for both Applied Mathematics at the Delft University of Technology and English Language and Culture at the University of Leiden.

Sonja only completed the first three semesters of her studies in English Language and Culture. Mathematics captured her more and in December 2006 she successfully defended her MSc thesis in Applied Mathematics (cum laude). The results on the decoupling inequality presented in this thesis are a follow-up project to Sonja's MSc thesis, which dealt with randomized UMD spaces.

In September 2007 Sonja began as a PhD student in Applied Mathematics in Delft with Dr. Birgit Jacob. Birgit Jacob soon accepted a professorship at the University of Paderborn, therefore Sonja continued her PhD research under the supervision of her former MSc advisor, Prof. Dr. Jan van Neerven.

At the Internet Seminar Jan van Neerven organized in the academic year 2007-2008 Sonja embarked on a project with Mariusz Górajski, which lead to the results in this thesis concerning delay equations. In 2009 Sonja spent three months visiting Prof. Dr. Ben Goldys at the University of New South Wales, Australia, where she extended earlier results with Jan van Neerven on the splitting scheme to the case of multiplicative noise. From December 2008 until December 2010 Sonja was a member of the work council of the Delft University of Technology.

After defending her PhD, Sonja will continue her research at the University of Innsbruck.