Non-hydrostatic modelling of waves in layered fluids

S. H. Balkema

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Delft University of Technology
Faculty of Civil Engineering and Geosciences
Section Fluid Mechanics

Graduation Committee
prof. dr. ir. G. S. Stelling
dr. ir. M. Zijlema
dr. ir. J. C. Winterwerp
ir. M. A. de Schipper
prof. dr. ir. J. A. Battjes
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Abstract

Layered fluids occur at numerous places in nature where two fluids of different density are involved and mixing is limited. The two fluids flow on top of each other with interactions at the interface only, but they can influence each other to a high degree regarding wave propagation and dissipation.

In this thesis a non-hydrostatic two-layer 2DV model, based on the scheme presented by Stelling and Zijlema (2003), is developed. The model is non-hydrostatic; the velocities are corrected with a pressure gradient following from the requirement of a divergence free flow field per computational cell. The model is a two-layer model; only layered systems with two layers of different but constant density are considered and the mixing is left out of consideration. The divide between the between upper and lower layer is considered a sharp interface. The density is discontinuous at the interface and the fluids are completely separated by the interface. To guarantee an exact representation of the interface, a boundary following grid is used in the vertical.

In several test cases the model has been validated. In a closed basin test the dispersion relation is shown to follow the linear dispersion relation for internal and external waves almost exactly when two layers are used. The model is also tested against analytical models with regard to dissipation of the waves by a viscous lower layer (mud) and showed comparable results.

The model however performed less well for highly advective processes with large gradients such as the sex-change test. Considering differences in flow velocity between the layers, the stability of the model was substantially lower than according to linear theory. The limit for this new model has been found to depend on the thickness of one layer only in stead of the total depth.

Finally the model was used to reproduce results of laboratory experiments found in literature. The steepening of internal waves and generation of solitons was reasonable well represented.

The result of this study is a non-hydrostatic 2DV two-layer model, which has the same attractive properties as the model by Stelling and Zijlema regarding wave dispersion of both internal and external waves.
Preface
These pages you see before you are the end result, finally, of my graduation project and concludes my Master of Science program at the faculty of Civil engineering and Geosciences at Delft University of Technology, the Netherlands.

Playing with waves -as I like to call my graduation project- has actually always been fun. The time I spent at the section of environmental fluid mechanics (almost one and a half year) however wasn't only a 'game'. From time to time I lost myself in tiny little details which, in the end, just weren't relevant, but at the time I had to see it through. Those were the moments I may have pestered my fellow graduates and they certainly helped me by listening to my silly stories.

More specifically I would like to thank Pieter for his help and insight with all the problems I encountered and Matthijs for his down-to-earth statements once in a while. The help of my family was of a different sort; without their support I would not be here. The steadfast confidence of my wife and her powerful prayers helped me through the difficult times and without her I would have been lost in the waves.

Finally, I would like to thank professor Stelling for his guidance, expertise and most importantly for his personality. He was the one who distracted me with interesting stories and kept me motivated. And the other members of my committee for their enthusiasm and advise. I would be honoured if my work indeed has added something to the world of non-hydrostatic modelling.

Delft, March 2009,

Sijbrand Harkes Balkema

"But if any of you lacks wisdom, he should ask God who gives to all generously and ungrudgingly, and he will be given it. But he should ask in faith, not doubting, for the one who doubts is like a wave of the sea that is driven and tossed about by the wind."

- James 1:5-6
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1 Introduction

1.1 Layered fluids

Different fluids have different densities. The density of a specific fluid largely depends on the temperature and concentration of dissolved matter. Small density differences like between fresh water from a river and salt water in a sea can already result in particular phenomena. The interactions between the fluids can be of practical importance, for instance:

- In estuaries a salt wedge of sea water can exist underneath the fresh river water;
- In sea locks particular exchange flows occur at opening of the doors;
- Exchange of river water and sea water can cause large deposits in sea harbours;
- Where warm cooling-water is discharged in colder surface water;
- Deep lakes can become stratified by heating of the sun;
- When oil is released into the ocean or coastal waters (marine oil spills);
- The seabed can be covered with a layer of fluid mud.

The situations mentioned above are stable because the density increases with increasing depth. The fluid is said to be ‘stratified’. The stratification however can be of different kinds: gradually increasing density or with abrupt changes. In the latter case the surface where the rapid change of density occurs is called a ‘pycnocline’ and the fluid can be considered a layered fluid.

![Figure 1: Increasing density with increasing depth, but different stratification.](image)

When the lower layer consists of liquefied mud, waves in the upper layer can get damped. In fact, the damping can be so considerable that there are places, used by fishing boats as emergency harbour during storms, such as in the Gulf of Mexico, known as the Mud Hole (Gade, 1958).

The waves in the upper layer cause stresses in the mud. Above a certain limit the mud will liquefy and start to respond to the waves (De Wit, 1995). The waves in the upper layer will move the mud and the movement of the movement of the mud in turn will be damped by bottom friction and internal friction due to viscosity. In short, the movement and therefore the damping depends on the wave frequency and the thickness of the layers.
1.2 Problem analysis

The phenomena mentioned above illustrate the importance of the research on stratified flows. Numerous models have been developed for stratified flow problems. A very common approach is the linear two-layer model. Different schematisations, each focussing on specific properties of the layers, have been developed. However, the need of several simplifications largely restricts their general validity. Furthermore, schematisations that take the non-shallow water equations into account are rather complex and not all models give explicit analytical solutions.

The governing equations of fluid mechanics can also be solved numerically without the assumptions needed for the linear models. By applying fully dynamic models wave propagation properties can be modelled accurately and non-linear waves can also be correctly represented.

Stelling and Zijlema (2003) presented a method to solve the Navier-Stokes equations numerically without the need of a high resolution in the vertical, which had been a well-known drawback for these kind of non-hydrostatic models. Because the pressure is calculated at the surface and the bottom, they are able to model the wave propagation properties very accurately. The error in the propagation velocity remains below 1% for waves with relative depth $kH < 7.8$ with only two computational layers, whereas usually something in the order of ten layers is needed (Smit, 2008).

Since the wave propagation properties are well represented by this model, it would be interesting to study the implementation in a two-layer context. In case of a two-layer model the pressure is also calculated at the interface, influencing both upper and lower layer as it is a ‘surface’ for the lower layer and a ‘bottom’ for the upper layer. Propagation properties of both external and internal waves are therefore expected to be modelled quite accurately regardless of the wavelength or depth.

1.3 Objective

The objective of this project is the construction of a non-hydrostatic two-layer model based on the scheme presented by Stelling and Zijlema (2003) for single-layer fluids and an investigation of the internal and external wave propagation properties.

The implementation of the scheme in a two-layer model gives some additional requirements for the new model:

- An exact representation of the interface is needed for the layers to be on top of each other;
- The bottom boundary condition for the upper layer has to be extended to include the movement of the interface;
- The inclusion of wave dissipation by viscosity in the lower layer;

These requirements result in the following sub objectives:

- Use of terrain following boundaries in the vertical;
- Extension of the boundary conditions at the interface;
- Validation with simple test cases and verification with other models;
- Implementation of a viscosity term;
- Verification of wave damping by fluid mud;
1.4 Readers guide

In this chapter a short introduction in layered flow has been given and after a short problem analysis, the project objective was formulated and divided into sub objectives.

A more detailed description of layer models is given in chapter 2. Differences between internal and external waves are explained. The dispersion relation of a simple two-layer model is derived and several viscous models are described.

In chapter 3 and 4 the development of the model is outlined. In chapter 3 first an implementation of the model in sigma layers for one layer is given, and then the model is extended into a two-layer model. Chapter 4 describes the numerical implementation and the solution method.

In chapter 5 the model is validated in several test cases. The linear dispersion relation is tested with a closed basin test, the advection with a change of sex test and the dissipation by comparison with viscous models. Furthermore the stability of internal waves under influence of flow differences between the layers is tested and the model is applied to a laboratory experiment found in literature.

Finally, conclusions and recommendations resulting from this project are discussed in chapter 6.
Two-layer models

Where two fluids meet each other, interaction will occur. Velocity differences will introduce a shear stress at the interface, which in turn can cause turbulence. The turbulence will tend to mix the fluids.

When the turbulence is not too large and the fluids differ in density, layered flow can develop. Under influence of gravitation the mixing will be limited. The heavier fluid will sink and the lighter fluid will form a layer on top.

The ratio of the density gradient to the velocity gradient is expressed in the Richardson number. Higher values are more stable, lower values are less stable. Generally a value of \( \frac{1}{4} \) is used as a lower limit for the stability:

\[
Ri = \frac{-g}{\rho} \left( \frac{d\rho}{dz} \right) > \frac{1}{4}
\]

When mixing is neglected, distinct layers of constant density (isopycnal) can be considered. The interface between the layers is considered a sharp divide, which implies a large gradient locally. This assumption is the basic idea behind most two-layer models. The only interaction between the two layers is at the interface. The movement of the interface and the pressures at the interface are the same for both layers.

The interface is the common boundary between the two layers; it is both the ‘surface’ for the lower layer and the ‘bottom’ for the upper layer at the same time. For each layer the basic equations for fluid mechanics apply, although the different properties of the layers may justify different simplifications per layer.

In layered flow a distinction can be made between two types of waves: external and internal waves. External waves are more or less the ‘normal’ waves of the free surface which have an impact over the whole depth (in case of long waves in regard to the depth). Because the interface is located at lower depth, the movement of the interface is smaller, but in the same direction. Therefore external waves are characterized by a displacement of interface and free surface in the same direction (see Figure 2, left). The interface is ‘bound’ to the wave in the surface.

![Figure 2: External (left) and internal waves in a two-layer system, taken from C. Kranenburg (1998).](image)
An internal wave can be seen as a wave in the lower layer, with a response in the upper layer. The displacement of the interface is larger than that of the surface and in opposite direction to the surface displacement (Figure 2, right). The surface is ‘bound’ to the wave in the interface.

The notation adopted here follows from Figure 2. The upper layer is called ‘layer 1’ and the lower layer is ‘layer 2’. The variables inside the layers have the layer number in subscript to distinguish them from the same variable in the other layer:

- $h_1 =$ thickness of the upper layer
- $h_2 =$ thickness of the lower layer
- $u_1 =$ horizontal velocity in upper layer
- $u_0 =$ horizontal velocity in lower layer
- $w_1 =$ vertical velocity in upper layer
- $w_2 =$ vertical velocity in lower layer
- $\zeta =$ elevation of the free surface
- $\xi =$ elevation of the interface
- $c_e =$ propagation speed of external wave
- $c_i =$ propagation speed of internal wave

### 2.1 Basic linear dispersion relation

An overview of different schematisations and derivations of dispersion equations based on two-layer models is given by W. Kranenburg (2008). Here as introduction a non-viscous two-layer model is described. The method used to find the dispersion relations is more or less the same for each schematisation. They all assume the displacement of the surface and the interface to be a harmonic function with small amplitude:

\[
\begin{align*}
\zeta(x,t) &= ae^{i(kx-\omega t)} \\
\xi(x,t) &= \xi_0 e^{i(kx-\omega t)} = be^{i\varphi}e^{i(kx-\omega t)}
\end{align*}
\]  

The amplitude of the interface, $\xi_0$, is complex to account for a phase shift between free surface and interface displacement. All waves consist of a linear summation of external and internal waves.

Based on layer-averaged properties and interaction at the interface a harmonic solution of the linearized problem can be derived for linear sinusoidal waves with small relative density differences. The derivation as given by C. Kranenburg (1998) is outlined in Appendix A. The propagation velocity, $c$, of a wave with wave number $k$ in a non-viscous layered system is found to obey:

\[
\begin{align*}
(c-u_1)^2 - (c-u_2)^2 - \frac{gT_2}{k}(c-u_1)^2 - \frac{gT_1}{k}(c-u_2)^2 + \\
\varepsilon \frac{g^2T_1T_2}{k^2} + (1-\varepsilon)(c-u_1)^4 T_1 T_2 &= 0
\end{align*}
\]  

where $u_n$ is the velocity in layer $n$, $T_n = \tanh kh_n$ with $h_n$ being the layer thickness and the relative density difference given by:

\[
\varepsilon = \frac{\rho_2 - \rho_1}{\rho_2}
\]
The external wave speed can be found by neglecting terms with epsilon in (2.3) and assuming equal flow velocity for both layers \(u_1 = u_2 = u\). The solution is the same as the dispersion relation for a one layer system with a total depth of \(a\):

\[
c_e = u \pm \sqrt{\frac{g}{k} \tanh kH}
\]  

(2.5)

The internal wave speed can be derived by assuming the propagation velocity, \(c\), and the flow velocities, \(u_1\) and \(u_2\), in the order of \(gH\) and neglecting the terms of order \(\epsilon^2\):

\[
c_i = \frac{T_1 u_2 + T_2 u_1}{T_1 + T_2} \pm \sqrt{\frac{T_1 T_2}{(T_1 + T_2)^2} \left(\frac{g}{k} \epsilon - (u_1 - u_2)^2\right)}
\]

(2.6)

If the expression under the root operator of (2.6) becomes negative (large velocity difference), the internal propagation velocity becomes complex. A part of the internal waves will grow exponentially and make the interface unstable. This is known as the Kelvin-Helmholtz instability. The stability limit is given by expression (2.7) and graphically in Figure 3.

\[
\frac{\Delta u^2}{\epsilon gH} = \frac{\tanh kh_1 + \tanh kh_2}{kH}
\]

(2.7)

![Figure 3: Stability limits for a two-layer system (taken from C. Kranenburg, 1998).](image)

This means that for \(\frac{\Delta u^2}{\epsilon gH} > 1\) at every wave length instability occurs. For \(\frac{\Delta u^2}{\epsilon gH} < 1\) there is always a wave number at which instability occurs. These instability-waves however will develop a non-linear behaviour and finally break. Because of that, local turbulence will occur which will be suppressed by the density difference. The growth of the instability will therefore be limited. As long as the root in (2.6) is not complex and internal Froude numbers stay small (see e.g. C. Kranenburg, 1998), the layer model is still usable.

### 2.2 Viscous models

When one of the two layers exists of mud or another viscous fluid, the viscosity of that layer cannot be neglected and waves will get dissipated due to internal friction. Various viscous models have been developed for a two layer system with a non- or low viscous water layer and a viscous lower layer with higher density (fluid mud).
The different dispersion relations which are described by W. Kranenburg are the dispersion relations according to: Gade (1958), Dalrymple and Liu (1978), De Wit (1995), Ng (2000) and Guo (2002). A new dispersion relation is also derived, based on the derivation by De Wit, which is called the ‘Delft’ dispersion relation and yields a more general formulation.

The Gade and Guo relations are derived for cases with hydrostatic pressure (shallow water and a thin mud layer). The vertical momentum however is not taken into account; the solutions are not valid at deeper water and larger thickness of the mud layer.

The relations according to De Wit and Ng do take the vertical momentum into account, but only in regard to the water layer whereas the pressure in the mud is assumed to be hydrostatic. The same assumptions apply to the Delft dispersion relation.

Dalrymple and Liu give their dispersion relation for a case with fully hydrodynamic waves in both layers. The viscosity in the upper layer is also taken into account, although the effect thereof will only be small. The validity is restricted to cases where the mud layer is thick with respect to the viscous boundary layer:

\[
H_{m0} \gg \sqrt{2 \nu_m / \omega}
\]  

With the harmonic models mentioned above, solutions for the wave number of the external wave can be found as function of the frequency. The wave number consists of a real and an imaginary part. The imaginary part of the wave number, \(k_i\), can be used to calculate the wave damping. When the wave number is constant in the domain (no change in bottom elevation and properties of the layers), the wave height at a certain point relative to the height at the starting position, \(x\), can be calculated by:

\[
H = H_0 e^{-k_i x}
\]

### 2.3 Numerical alternative

Instead of solving the differential equations by assuming harmonic waves as mentioned above, a solution can also be found numerically. Numerical models can solve the Navier-Stokes equations directly without the use of assumptions needed to derive the harmonic models. The application is no longer restricted to cases where (Kranenburg, 2008):

1. The wave is of sinusoidal form;
2. The wave is a free wave;
3. The amplitude can be considered small relative to depth and wave length;
4. The bottom is horizontal without motion;
5. Variations of surface pressure are neglected;
6. The layers are considered to be of infinite horizontal extent;
7. The mean current is zero;
8. Only plane waves are considered;
9. The density is constant in a layer;
10. The viscosity is constant in a layer;
11. The fluid mud is assumed to be a Newtonian fluid;
12. The motions are divergence free (both fluids are incompressible);
13. There is no mixing between the layers;
14. Effects of earth rotation are neglected;

However, in this study not all restrictions above will be dropped. The restrictions with number 8 to 14 will still hold for the model presented here, but most of the effects can be included in the model with some adaptations.

The model presented in the next chapters is expressed for two dimensional problems in a vertical plane only (2DV, following restriction 8). However, the basis of the model (as introduced by Stelling and Zijlema, 2003) can also be used in three dimensional space (e.g. Smit, 2008). By applying models for the variation of the density or viscosity the restrictions 9, 10 and 11 can also be included. The same holds for the effects of earth rotation and incompressibility.
3 The model

In the preceding chapter an introduction to two-layer modelling was given. Internal and external waves have been distinguished and several analytical models have been mentioned. In this chapter the mathematical formulation of the model will be presented.

The basis of the model, the governing equations in fluid mechanics, are the Navier-Stokes Equations (NSE, see e.g. Batchelor, 1967). In fact, the Navier-Stokes equations follow from the law of conservation of momentum. According to the second law of Newton the change in momentum, \( mv \), is the result of the resultant force on a body:

\[
\frac{Dmu}{Dt} = \bar{F}
\]  

(2.10)

Before the two-layer model is presented, the existing non-hydrostatic single layer model will be discussed. The model was introduced by Stelling and Zijlema (2003) and further elaborated in Zijlema and Steling (2005). The description given here is taken from the latter, as a terrain-following vertical grid will be used in the two-layer model.

Inclusion of the non-hydrostatic pressure complicates the lateral boundary conditions. To reduce this problem before things get really complicated in the two-layer case, a new method to account for the non-hydrostatic pressures at the boundaries will be introduced. This is achieved by fading the influence of the non-hydrostatic pressures towards the boundary.

Finally, when the single layer model is ready, it will be applied to each density layer separately in the two-layer model. The ‘surface’ of the lower layer will be the ‘bottom’ of the upper layer. To account for internal waves, the boundary conditions will be slightly changed.

3.1 Single layer model

We consider a two-dimensional vertical plane that is bounded in vertical direction by the free surface and the bottom. In a Cartesian co-ordinate system the vertical co-ordinate, \( z \), of the free surface, \( \zeta \), is described by \( z = \zeta(x,t) \) in which \( x \) is the horizontal co-ordinate and \( t \) is time. The vertical co-ordinate of the fixed bottom is a function of \( x \) only: \( z = -d(x) \) as indicated in Figure 4.

![Figure 4: The two-dimension vertical plane bounded by bottom and surface.](image-url)
3.1.1 Governing equations

The flow is assumed to be uniform in the second horizontal direction, \( y \), which means that the properties of the flow do not change in that direction. The flow is now two-dimensional in a vertical plane (2DV) and the continuity equation is reduced to:

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.11}
\]

As the bottom and free surface are streamlines, fluid particles can only move along them; there is no transport of matter over streamlines. From these requirements follow the kinematic conditions:

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} & = 0 \\
\frac{\partial \zeta}{\partial t} + w \frac{\partial \zeta}{\partial z} & = -\frac{d}{dx}
\end{align*} \tag{2.12}
\]

Integration of (2.11) from bottom to free surface results in the free-surface condition:

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} \int_{d}^{\zeta} \rho u dz = 0 \tag{2.13}
\]

Gravity is the driving force for gravity waves. The influences from other external forces (e.g. wind) and body forces (e.g. Coriolis) are neglected. The effects of turbulence and viscosity are also left out for now. The equations of motion considered here are:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} & = 0 \\
\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} & = -g
\end{align*} \tag{2.14}
\]

3.1.2 Pressure splitting

In the above equations the pressure gradient is the only driving force of the flow. The pressure in a certain point can be considered existing of two parts. The hydrostatic part, \( p_h \), depends on the weight of the fluid above that point, whereas the hydrodynamic part, \( q \), depends on the flow itself:

\[
p = p_h + q \tag{2.15}
\]

The hydrostatic part is only related to the water surface, \( \zeta \), and its vertical derivative therefore equals the gravity acceleration. After this pressure-splitting the momentum equations read:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + g \frac{\partial \zeta}{\partial x} + \frac{1}{\rho} \frac{\partial q}{\partial x} & = 0 \\
\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{1}{\rho} \frac{\partial q}{\partial z} & = 0
\end{align*} \tag{2.16}
\]
3.1.3 Boundary conditions

The above system of equations holds everywhere in the domain, but flow properties enter at the boundaries. Boundary conditions have to be provided at all boundaries to get unique solutions. Four types of boundary conditions can be distinguished:

- In vertical direction:
  - Free surface boundary
  - Bottom boundary
- In lateral direction:
  - Reflecting boundary
  - Non-reflecting boundary

The two lateral boundaries can be divided into two sub-categories: water level and velocity boundary. In fact, velocity boundaries are mostly used, because imposing a water level at the boundary does require an additional condition for the non-hydrostatic pressure. For long waves a reasonable approximation is to assume hydrostatic pressure in the vertical, but for shorter waves this is not valid. The velocity of the waves in the domain will be smaller than implied at the boundary and conservation of energy will result in larger wave amplitudes than implied at the boundary. Using velocity boundaries, conditions for the non-hydrostatic pressure are no longer needed, since the velocity at the boundary is imposed and not computed by solving the momentum equation.

*Free surface boundary*

At the free surface any tangential stress (e.g. wind) is neglected in this study. Considering gravity waves, the surface tension can also be neglected. By assuming the atmospheric pressure to be zero, the condition reads:

\[ q_{z} = 0 \quad (2.17) \]

The vertical velocity, \( w \), follows from the solution of the momentum equation (2.16).

*Bottom boundary*

There is no velocity perpendicular to the bottom; the pressure gradient perpendicular to the bottom is zero and therefore the vertical velocity \( w \) can be imposed by the kinematic condition (2.12). The tangential stress (effects from bottom friction) is neglected.

*Reflecting boundary*

The reflecting boundaries are constructed by imposing a prescribed velocity normal to the boundary. When this velocity is zero or a constant value, the boundary is a *closed boundary* or an *open boundary*. The velocity can also be prescribed according to a certain function in time to account for a *wave generating boundary*.

The wave generating boundary is constructed by implying the velocity as some function in time. The function is chosen according to the linear wave theory for harmonic motions and superimposed on the mean flow in the layer.
where $\omega$ is the radial frequency, $k$ is the wave number and $a$ is the elevation amplitude of the imposed wave; $h$ is the local water depth and $z$ is the vertical location of $u_{\text{boundary}}$.

Because the domain is initially in rest, a certain start-up will be needed. To prevent initial short waves the velocity at the boundary is multiplied by a so-called ramp-function, $f_r$, during a certain period of time, $t_{\text{start}}$:

\[
\left. u \right|_{\text{boundary}} = u_{\text{layer}} + f_r^2 \frac{\alpha \omega}{kh} \frac{\cosh k(h+z)}{\sinh(kh)} \sin(\alpha x)
\]

where $f_r$ is defined as:

\[
f_r = \frac{1}{2} \left( \tanh \left( \frac{\sin(z(2y-1))}{1-(2y-1)^2} \right) + 1 \right)
\]

Additionally, to dampen any resulting short waves at start-up, the horizontal viscosity can initially be set to a relatively high value and then faded during a few periods by applying a similar function.

**Non-reflecting boundary**

Non-reflecting boundary conditions are needed where the waves should leave the domain. Wave absorbing boundaries can be constructed based on the Sommerfeld-type radiation condition:

\[
\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0
\]

where $c$ is the wave speed and $\phi$ can be the elevation of the surface, $\zeta$, or a velocity, $w$.

With this condition waves are exactly followed by the boundary and therefore leave the domain without reflection. For long, hydrostatic waves the wave speed is given by the hydrostatic approximation: $c = \sqrt{gh}$. For short, non-hydrostatic waves however, it is not known a priori. Short waves will still get reflected by these kind of absorbing boundaries. Other solutions have to be found to get rid of the reflection.

One alternative that can be applied is a sponge layer. The computational domain is extended with a length $L_{\text{sponge}}$ of a few wavelengths starting at $x_{\text{sponge}}$. The wave energy is damped by a gradually increasing friction as the waves travel over the sponge layer. The horizontal momentum equation is slightly modified to contain a friction factor, $C_f$:

\[
\frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} + C_f u = 0
\]

with:

\[
C_f = \left( \frac{x-x_{\text{sponge}}}{L_{\text{sponge}}} \right)^3
\]

At the end of the extended domain there is still need for a normal boundary condition. Remaining waves that reflect at this boundary have to travel through the sponge twice.
This sponge layer however is difficult to implement when a mean flow in the layer is considered. Damping of the mean flow velocity will result in a closed boundary. Although by subtracting the mean flow velocity before applying the friction factor this problem can be reduced, the sponge still causes errors in the water levels (see section 5.6).

### 3.2 Fading non-hydrostatic effects at boundaries

The problems with the boundaries are caused by the introduction of the non-hydrostatic pressure as unknown variable. In this section a method is presented to get rid of these effects of the non-hydrostatic pressure by fading the influence at the boundaries.

The computational domain is extended with a few wavelengths. In the extension a factor is applied to reduce the influence of the non-hydrostatic pressure towards the boundary. The horizontal momentum equation is slightly changed to include this factor:

\[
\frac{Du}{Dt} + g \frac{\partial \zeta}{\partial x} + \frac{1}{\rho} \frac{\partial q}{\partial x} (1 - C) = 0
\]

(2.22)

As the factor increases from 0 at the end of the domain of interest to 1 at the boundary at the end of the extension, the influence of the non-hydrostatic pressure on the horizontal momentum equation will decrease. The wave speed will increase until the speed equals the value according to the hydrostatic approximation. Any hydrostatic boundary condition can be applied to the boundary itself, e.g. a Sommerfeld-radiation type boundary will give the same absorbing results as in a hydrostatic case.

The hydrostatic factor should be smoothly increasing at the beginning of the extension, but should also be smoothly going to 1 at the end where the hydrostatic boundary is applied. A shape function is used in the extension from \(x_{\text{start}}\) to \(x_{\text{end}}\) (Dingemans, 1997):

\[
C = \frac{1}{2} \left( \tanh \left[ \frac{\sin \left( \frac{\pi}{2} \left( 2y - 1 \right) \right)}{1 - (2y - 1)^2} \right] + 1 \right)
\]

(2.23)

where \(y = \frac{x - x_{\text{start}}}{x_{\text{end}} - x_{\text{start}}}\)

This type of hydrostatic boundary condition can also be applied to the wave generating boundary, where the vertical distribution of horizontal velocities is unknown. The start of the domain is extended and the factor \(C\) is now decreased from 1 to 0 by a similar shape function. The wave generating boundary condition can now consist of any shallow water wave condition with velocities uniformly distributed over the depth:

\[
u_{\text{boundary}} = \bar{u} + \hat{u} \sin \left( \omega x \right)
\]

(2.24)

where \(\hat{u}\) is the amplitude and \(\bar{u}\) is the mean flow in the layer.

At deeper water, where the influence of the hydrodynamic pressure is larger, this will result in steepening of the generated waves as they travel through the extension from hydrostatic to non-hydrostatic calculation. Because the influence is introduced smoothly, no instabilities will occur.
\section*{3.3 Two-layer model}

The non-hydrostatic model as discussed above can now be extended into a two-layer model by introducing an interface between the two distinct layers, which is both the ‘bottom’ for the upper layer as well as the ‘surface’ for the lower layer. Each layer has its own system of equations and consequently requires its own set of boundary conditions.

The upper layer, indicated with subscript 1, is defined by the vertical distance between the free surface, $\zeta$, and the interface, $\xi$, whereas the lower layer, indicated with subscript 2, is defined by the vertical distances between the interface, $\xi$, and the bottom, $d$, see Figure 5.

![Figure 5: The two layers in a double layer system; layer 1 on top of layer 2.](image)

\subsection*{3.3.1 Governing equations}

The momentum equations are slightly different for each layer, because the densities differ and because the upper layer acts as an additional pressure on the lower layer (no atmospheric pressure at the interface). The viscosity of the upper layer is neglected, because this layer is considered to consist of water. The viscosity in the lower layer however cannot be neglected, because in some cases the lower layer will be considered to consist of viscous mud. Hence, the equations concerning the lower layer have a non-zero right-hand side to account for the viscosity, $\eta$:

\begin{align*}
\frac{\partial u_1}{\partial t} + \frac{\partial u_1^2}{\partial x} + \frac{\partial w_1 u_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial q}{\partial x} + g \frac{\partial (\zeta)}{\partial x} &= 0 \\
\frac{\partial w_1}{\partial t} + \frac{\partial w_1 u_1}{\partial z} + \frac{\partial w_1^2}{\partial z} + \frac{1}{\rho_1} \frac{\partial q}{\partial z} &= 0 \\
\frac{\partial u_2}{\partial t} + \frac{\partial u_2^2}{\partial x} + \frac{\partial w_2 u_2}{\partial x} + \frac{1}{\rho_2} \frac{\partial q}{\partial x} + g \left( 1 - \frac{\rho_1}{\rho_2} \right) \frac{\partial \xi}{\partial x} + g \frac{\rho_1}{\rho_2} \frac{\partial (\xi)}{\partial x} &= \eta \frac{\partial^2 w}{\partial x^2} + \frac{\eta}{\rho_2} \frac{\partial^2 u}{\partial z^2} + \frac{\eta}{\rho_2} \frac{\partial^2 w}{\partial z^2} \\
\frac{\partial w_2}{\partial t} + \frac{\partial w_2 u_2}{\partial z} + \frac{\partial w_2^2}{\partial z} + \frac{1}{\rho_2} \frac{\partial q}{\partial z} &= \eta \frac{\partial^2 w}{\partial x^2} + \frac{\eta}{\rho_2} \frac{\partial^2 w}{\partial z^2} + \frac{\eta}{\rho_2} \frac{\partial^2 w}{\partial z^2}
\end{align*}

The continuity equation (2.11) is the same for both layers and is not repeated here.

The interface is considered a sharp divide. No transport of matter can take place between the layers; the interface is in fact a streamline like the bottom and the free surface. In a double layer system therefore two additional kinematic conditions can be given at the interface:
just above the interface \[ w_1|\xi = \frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + u_1 \frac{\partial \xi}{\partial x} \] (2.26)

just below the interface \[ w_2|\xi = \frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + u_2 \frac{\partial \xi}{\partial x} \]

Indicating ‘just above the interface’ with ‘+’ and ‘just below the interface’ with ‘–’. The derivatives of the interface are the same in both layers, which is not necessarily true for the flow velocities:

\[ \frac{D\xi}{Dt}^+ \neq \frac{D\xi}{Dt}^- \] (2.27)

The above equations make clear that the vertical component of the flow velocity at the interface, \( w \), does not have to be the same for both layers. The layers do not share the properties of the flow velocity; they only share the same interface.

As with the free surface condition, (2.13), integration of the continuity equation, (2.11), results in a similar condition for the interface. The integration is now carried out over the lower layer only in stead of over the whole depth:

\[ \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} \int_{-d}^{0} u_2 \, dz = 0 \] (2.28)

### 3.3.2 Boundary conditions

The boundary conditions as given for the single-layer case still hold for the two-layer model. However, additional boundary conditions are needed due to the introduction of the interface.

The double layer system consists of two independent layers with each their own velocities, pressures and boundaries. By extension of the kinematic ‘bottom’ boundary of the upper layer to include the movement of the interface and by implying the ‘bottom’ pressure in the upper layer as ‘surface’ pressure on the lower layer, the two layers are coupled to interact with each other.

In the two-layer implementation, the boundary conditions in lateral direction are slightly more difficult. Two types of waves can be considered: external waves and internal waves. Using the fading non-hydrostatic effects, relatively simple boundary conditions using the shallow water approximation can be applied.

**Additional boundary conditions in the vertical**

The first additional boundary condition is just below the interface. For the lower layer, the interface is like a surface, but the pressure is not zero. It follows from the fact that the pressure in a fluid is non-directional and continuous. The pressure at the interface is the same in both layers:

\[ p|\xi = p|^\xi \] (2.29)

The hydrostatic pressure at the interface is the same for both layers (i.e. the weight of the upper layer) and therefore this condition applies for the non-hydrostatic pressures too:
The second additional boundary condition is just above the interface. The upper layer flows over the lower layer as if the lower layer is a (moving) bottom. To guarantee that the water particles stay in the interface, the pressure gradient perpendicular to the interface is zero in the upper layer and the vertical velocity \( w_1 \) is imposed by the kinematic condition at the interface (2.26). The tangential stress (effects from friction between the layers) is neglected.

**Wave generating boundary**

Concerning external waves, the condition applied at the boundary is not really different from (2.24):

\[
\begin{align*}
  u_1 & = \overline{u}_1 + f_r^2 \dot{u}_r \sin \left( \omega t \right) \\
  u_2 & = \overline{u}_2 + f_r^2 \dot{u}_r \sin \left( \omega t \right)
\end{align*}
\]

(2.31)

The frequency of these external waves, \( \omega_e \), follows from the propagation velocity of the waves in the non-hydrostatic part of the domain (see Appendix A):

\[
\omega_e = kc_i \approx k \frac{\overline{u}_1 h_1 + \overline{u}_2 h_2}{h} + \sqrt{gk \tanh kh}
\]

(2.32)

with total depth \( h = h_1 + h_2 \).

The condition for internal waves is only a little different, as it involves a difference in flow direction and different amplitudes in each layer:

\[
\begin{align*}
  u_1 & = \overline{u}_1 - f_r^2 \frac{\dot{u}_r}{h_1} \sin \left( \omega t \right) \\
  u_2 & = \overline{u}_2 + f_r^2 \frac{\dot{u}_r}{h_2} \sin \left( \omega t \right)
\end{align*}
\]

(2.33)

The frequency of the internal waves follows from the dispersion relation in the non-hydrostatic part of the domain (see Appendix A for the derivation according to C. Kranenburg):

\[
\omega = kc_i = k \left( \frac{T_2 \overline{u}_1 + T_1 \overline{u}_2}{T_1 + T_2} \right) + k \sqrt{\frac{T_1 T_2}{(T_1 + T_2)^2}} \left( \frac{\varepsilon g}{k} - \frac{\overline{u}_1 - \overline{u}_2}{2} \right)
\]

(2.34)

with \( T_n = \tanh kh_n \) where \( h_n \) is the thickness of layer \( n \).

**Wave absorbing boundaries**

Concerning wave absorbing boundary conditions for external waves, a radiation condition is applied to the surface elevation only and the interface is following the free surface:

\[
\frac{\partial \zeta}{\partial t} + c_r \frac{\partial \zeta}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \zeta}{\partial t} + c_r \frac{\partial \zeta}{\partial x} \left( \frac{h_2}{H} \right) = 0
\]

(2.35)

The external wave speed is given by the hydrostatic approximation:
\[ c_e = \frac{\bar{u}_1 h_1 + \bar{u}_2 h_2}{H} + \sqrt{gH} \]  \hspace{1cm} (2.36)

Conditions at boundaries which absorb the internal wave can be expressed in the same way as for external waves. The condition is applied to the interface with a small response in the surface:

\[
\frac{\partial \xi}{\partial t} + c_i \frac{\partial \xi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial t} + c_i \frac{\partial h_i}{\partial x} H = 0
\]  \hspace{1cm} (2.37)

The propagation velocity is again given by the hydrostatic approximation:

\[
c_i = \frac{h_1 u_2 + h_2 u_1}{H} + \sqrt{\frac{h_1 h_2}{H^2} \left[ \frac{\varepsilon gH - (u_1 - u_2)^2}{H} \right]}
\]  \hspace{1cm} (2.38)

Although with the above equations the incoming external or internal waves are quite well absorbed by the boundary conditions, simplifications in the derivation of the wave propagation velocity will introduce errors which result in (relatively small) reflections, especially in the case of internal waves.
4 Numerical implementation

In the preceding chapter the Navier-Stokes equations are elaborated for a case with one horizontal dimension and two separate layers with different densities. To make numerical simulation possible, the equations have to be discretised on certain points in time and space (a grid).

This chapter firstly will give a brief explanation of the grid that is used and the choice of the locations of variables. Then integration can take place to derive the space-discretised equations. Finally, after time discretisation, a solution method for this model can be given.

As a matter of fact, the numerical implementation of the double-layer model is almost completely the same as that of the single-layer model. Although in the previous chapter first the model has been given for the single layer case, this chapter will skip the single layer case and directly focuses on the double layer model to avoid unnecessary repetition of steps. The line of thought, however, is the same as in Zijlema and Stelling (2005).

Note that until now the term layer referred solely to density layers. Because of the vertical discretisation, the term computational layers will be introduced. The difference will become clear in the following section. The terms upper layer and lower layer will be used as abbreviations for upper and lower density layers.

4.1 Grid

Numerical computation requires a discrete domain. Horizontally the computational domain is split in a fixed number of steps with fixed length $\Delta x$. Using Cartesian co-ordinates the horizontal coordinate in this grid is given by $m$ steps of size $\Delta x$:

$$x_{m+\frac{1}{2}} = m \Delta x \quad (3.1)$$

Vertically a boundary following grid is used for each density layer separately. This is important to get an exact representation of the bottom, the interface and the surface in the grid. Each density layer (upper and lower layer numbered reps. 1 and 2) is divided into a fixed number ($K_1$ and $K_2$), of computational layers, see Figure 6. The computational layers are counted from bottom to top by the layer-index, $k$. The interface between two computational layers is given by:

$$z_{k+\frac{1}{2}} = z_{k+\frac{1}{2}}(x,t), \quad k = 0, ..., K_1 + K_2 \quad (3.2)$$

This implies that the bottom, interface and free surface are located at:

$$z_{-\frac{1}{2}} = -d$$
$$z_{K_1+\frac{1}{2}} = \xi$$
$$z_{K_1+K_2+\frac{1}{2}} = \zeta$$
Density layer 1 with its own computational layers

Density layer 2 with its own computational layers

Figure 6: Each density layer is subdivided in several computational layers.

The layer thickness, \( \Delta z_k \), is defined in a relative way (i.e. a constant part of the water depth), similar to the \( \sigma \)-co-ordinate system. The thickness of the layer is given by a fraction, \( \sigma_{1,k} \) or \( \sigma_{2,k} \), of the density layer thickness, \( h_1 \) or \( h_2 \), depending on the density layer under consideration:

**upper layer,** \( k > K_2 \)

\[
\zeta_{k+\frac{1}{2}} = \zeta_{k-\frac{1}{2}} + \sigma_{1,k} \left( h_k \right) \]  

**lower layer,** \( k \leq K_2 \)

\[
\zeta_{k+\frac{1}{2}} = \zeta_{k-\frac{1}{2}} + \sigma_{2,k} \left( h_k \right) \]  

Because the thickness of each density layer varies in space and time, the vertical grid is a function in space and time.

### 4.2 Variables

The equations (2.25) contain four different variables: horizontal and vertical velocities, a pressure and a pressure gradient. These variables are only evaluated at certain points in the grid.

Horizontal velocities are discretised at the centre of vertical cell borders and vertical velocities are located at the centre of horizontal cell borders. The grid is staggered in both horizontal and vertical direction with regard to the velocities. In Figure 7 the numbering of cells and variables is indicated.

Figure 7: Numbering of volumes and variables per cell.

The two water levels (surface and interface) are evaluated at the surface and at the interface at integer horizontal grid points (indicated by \( m \)). Furthermore, the dynamic pressures are evaluated at the vertical velocity points. As a result, the horizontal velocities are staggered relative to the pressure points, whereas in the vertical a compact approach is used.
It should be noted that only the cell centres are located at integer grid points and all other variables are in some way in between two adjacent cells:

- Horizontal velocities are horizontally staggered: they are located horizontally at half grid points (horizontal arrows in the figure, counted as \( m \pm \frac{1}{2}, k \)).
- Vertical velocities and pressure points are located vertically between two cells (vertical arrows and dots, at \( m, k \pm \frac{1}{2} \)).

When variables are required at grid points where they are not defined as above, they are computed by interpolation or using an upwind approach.

### 4.2.1 Surface and interface at u-points

Because the surface and interface are only evaluated at the \( w \)-points, they have to be approximated at the \( u \)-points. To guarantee positive water depths, the best way to determine the surface is to use an upwind-approach depending on the direction of the mean flow velocity:

\[
\begin{align*}
\zeta^{u}_{m+\frac{1}{2}} &= \zeta_{m} \quad \text{for} \quad \sum u_{1} h_{1} + \sum u_{2} h_{2} > 0 \\
\zeta^{u}_{m+\frac{1}{2}} &= \zeta_{m+1} \quad \text{for} \quad \sum u_{1} h_{1} + \sum u_{2} h_{2} < 0
\end{align*}
\]  
(3.5)

The displacement of the interface however does not only depend on the mean flow velocity over the whole depth, but also on the propagation of the internal wave. Positive layer thickness of the lower layer is guaranteed by applying a slightly different condition:

\[
\begin{align*}
\bar{z}^{u}_{m+\frac{1}{2}} &= \bar{z}_{m} \quad \text{for} \quad \sum u_{1} h_{1} + \sum u_{2} h_{1} > 0 \\
\bar{z}^{u}_{m+\frac{1}{2}} &= \bar{z}_{m+1} \quad \text{for} \quad \sum u_{1} h_{1} + \sum u_{2} h_{1} < 0
\end{align*}
\]  
(3.6)

With the above expressions no negative water levels are encountered (which happens when the same condition for the interface is applied as for the surface) and the internal waves does not become saw-toothed (which happens when only the discharge in the lower layer is taken into account).

### 4.2.2 Relative ‘vertical’ velocity

Due to the movement of the vertical grid, the relative vertical velocities need to be calculated. A new variable for the relative particle velocity, \( \omega \), is introduced:

\[
\omega_{k+\frac{1}{2}} = w_{k+\frac{1}{2}} \quad \frac{Dz_{k+\frac{1}{2}}}{Dt} = w_{k+\frac{1}{2}} \quad \frac{\partial z_{k+\frac{1}{2}}}{\partial t} - u_{k+\frac{1}{2}} \quad \frac{\partial z_{k+\frac{1}{2}}}{\partial x}
\]  
(3.7)

The location of this particle velocity is the same as for the vertical velocity, but the direction is perpendicular to the bounding surfaces of the computational layers and therefore not strictly vertical.

This variable is always zero at the bottom and the free surface, because the relative particle velocity perpendicular to the bottom and surface is zero, as expressed by (2.12). This holds also for the interface, as there is no transport of matter over the interface, as expressed by (2.26).


4.3 Integration of the equations

Because the equations derived in the previous chapter are valid at all points in the computational domain (in their own layer), they hold also for each sub-domain. For each velocity component a collection of a finite number of non-overlapping control volumes is defined such that the whole domain is covered. The centre of each cell coincides with the velocity component under consideration.

Integration of the momentum equations over the volume of their corresponding cells result in space discretised equations. Time discretisation is treated separately in section 4.5.

4.3.1 Discrete continuity equation

When the continuity equation (2.11) is integrated over a computational layer (using the Leibniz’ rule) the layer-averaged continuity equation is obtained:

\[
\int_{z^+}^{z^-} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = \frac{\partial \Delta z_k u_k}{\partial x} - u_{k+\frac{1}{2}}^* \frac{\partial z_{k+\frac{1}{2}}}{\partial x} + u_{k-\frac{1}{2}} \frac{\partial z_{k-\frac{1}{2}}}{\partial x} + w_{k+\frac{1}{2}} - w_{k-\frac{1}{2}} = 0
\]  

This can be used to find the values of omega for all computational layers after substitution of (3.7) and by calculating the values of the relative particle velocities in the system from the bottom to the interface and from the interface to the free surface:

\[
\omega_{k+\frac{1}{2}} = \omega_{k-\frac{1}{2}} = \frac{\partial \Delta z_k}{\partial t} - \frac{\partial \Delta z_k u_k}{\partial x}
\]  

The layer averaged continuity equation (3.8) still has to be discretised in horizontal direction by integrating over a horizontal cell:

\[
\Delta z_{m+\frac{1}{2},k} u_{m+\frac{1}{2},k} = \Delta z_{m-\frac{1}{2},k} u_{m-\frac{1}{2},k} - u_{m,k+\frac{1}{2}}^* \Delta z_{upper} + u_{m,k-\frac{1}{2}}^* \Delta z_{lower} + \left( w_{m,k+\frac{1}{2}} - w_{m,k-\frac{1}{2}} \right) \Delta x = 0
\]  

in which \( \Delta z_{upper} = z_{m+\frac{1}{2},k+\frac{1}{2}} - z_{m-\frac{1}{2},k+\frac{1}{2}} \) and \( \Delta z_{lower} = z_{m+\frac{1}{2},k-\frac{1}{2}} - z_{m-\frac{1}{2},k-\frac{1}{2}} \)

The different in- and outflow velocities and corresponding surfaces are shown in the next figure:

![Figure 8: In- and outflow in horizontal direction (left) and vertical direction (right).]
The horizontal velocities indicated by an asterix (the second row of equation (3.10)) are located at the surfaces bounding the volume in vertical direction. The horizontal velocity at these points is approximated using a midpoint rule depending on the slope of the boundary, by taking the average in horizontal direction of the two nearest horizontal velocities:

\[
\begin{align*}
    u_{m,k+1/2}^* &= \frac{1}{2} \left( u_{m-y_j^k,k+1} + u_{m+y_j^k,k} \right), \quad \text{if } \frac{z_{m+y_j^k,k} + z_{m-y_j^k,k}}{2} > 0 \\
    u_{m,k+1/2}^* &= \frac{1}{2} \left( u_{m-y_j^k,k} + u_{m+y_j^k,k+1} \right), \quad \text{if } \frac{z_{m+y_j^k,k} + z_{m-y_j^k,k}}{2} < 0
\end{align*}
\] (3.11)

At the bottom, the interface and the free surface, the kinematic conditions, (2.12) and (2.26), guarantee that there is no transport over these boundaries.

The free surface condition (2.13) and the condition of the interface (2.28) are discretised in a similar way, resulting in discretised versions of the conditions:

\[
\begin{align*}
    \frac{\partial \xi^m_k}{\partial t} + \sum_{k=1}^{k=k_m-k_n+1} \Delta z_{m+y_j^k,k} u_{m+y_j^k,k} - \Delta z_{m-y_j^k,k} u_{m-y_j^k,k} = 0 \\
    \frac{\partial \xi^m_k}{\partial t} + \sum_{k=1}^{k=k_m-k_n+1} \Delta z_{m+y_j^k,k} u_{m+y_j^k,k} - \Delta z_{m-y_j^k,k} u_{m-y_j^k,k} = 0
\end{align*}
\] (3.12) (3.13)

### 4.3.2 Horizontal momentum equation

The horizontal momentum equations, given by the first and third row of (2.25), can be written in the following way for each density layer, layer \(n\), as:

\[
\frac{\partial u_n}{\partial t} + \frac{\partial u_n^2}{\partial x} + \frac{\partial u_n w_n}{\partial z} + \frac{1}{\rho_n} \frac{\partial p_n}{\partial x} + \frac{1}{\rho_n} \frac{\partial q}{\partial x} = v_{isc,hor,n}
\] (3.14)

where for the upper layer, \(n = 1\), the terms for the pressure gradient and viscosity are given by

\[
\left. \frac{1}{\rho_1} \frac{\partial p_1}{\partial x} \right|_1 = \frac{\partial \xi}{\partial x} \quad \text{and} \quad v_{isc,hor,1} = 0
\] (3.15)

as for the lower layer, \(n = 2\), the terms for the pressure gradient and viscosity are given by

\[
\left. \frac{1}{\rho_2} \frac{\partial p_2}{\partial x} \right|_2 = \frac{\partial \left( \xi + \frac{\rho}{\rho_2} (\xi - \xi) \right)}{\partial x} \quad \text{and} \quad v_{isc,hor,2} = \frac{\eta}{\rho_2} \frac{\partial^2 u}{\partial x^2} + \frac{\eta}{\rho_2} \frac{\partial^2 u}{\partial z^2}
\] (3.16)

The right hand side consists of the terms representing viscosity, \(v_{isc}\). The viscosity of the upper layer, in most cases involving water, is considered to be zero, whereas the viscosity in the lower layer can have relatively large values in cases involving liquefied mud. The numerical implementation of the viscosity is separately in section 4.3.4.

The left hand side of the momentum equation exists of four parts, which will be treated separately: time derivative, advection and pressure gradient, which has been split in a hydrostatic and a non-hydrostatic part. Zijlema and Stelling (2005) first integrate each term over a layer and thereafter over a horizontal cell.
The *time derivative* in density layer $n$ is integrated using the Leibniz’ rule, in which $u^*$ indicates a horizontal velocity not located at an $u$-point:

$$
\int_{z_{n-1/2}}^{z_{n+1/2}} \frac{\partial u_m}{\partial t} \, dz = \frac{\partial \Delta z_k}{\partial t} u_{k+1/2} + u_{k-1/2} \frac{\partial \Delta z_{k-1/2}}{\partial t} + u_{k+1/2} \frac{\partial \Delta z_{k+1/2}}{\partial t} - u_{k-1/2} \frac{\partial \Delta z_{k-1/2}}{\partial t} 
$$

(3.17)

The *advection* terms after integration and after substitution of (3.7) read:

$$
\int_{z_{n-1/2}}^{z_{n+1/2}} \left( \frac{\partial u^2}{\partial x} + \frac{\partial v u}{\partial z} \right) \, dz = \frac{\partial}{\partial x} \int_{z_{n-1/2}}^{z_{n+1/2}} u^2 \, dz + u_{k+1/2} \left( \omega_{k+1/2} + \frac{\partial \Delta z_{k+1/2}}{\partial t} \right) - u_{k-1/2} \left( \omega_{k-1/2} + \frac{\partial \Delta z_{k-1/2}}{\partial t} \right) 
$$

(3.18)

The integral at the right hand side of equation (3.18) can be written as

$$
\int_{z_{n-1/2}}^{z_{n+1/2}} u^2 \, dz = h_k u_k^2 + \int_{z_{n-1/2}}^{z_{n+1/2}} (u - u_k)^2 \, dz 
$$

(3.19)

in which the integral at the right hand side is the dispersion due to vertical non-uniformities of the flow velocity. It is common practice to model this dispersion as diffusion and therefore it will be neglected here.

The *hydrostatic pressure* in the upper layer depends on the local surface elevation, $\zeta$, only; in the lower layer it depends on the local elevation of both interface, $\xi$, and surface, $\zeta$. The horizontal derivative of the hydrostatic pressure therefore does not depend on the vertical position and integration over the vertical is unnecessary.

The *non-hydrostatic pressure* depends on the properties of the flow, and therefore it cannot be expressed in local layer properties as with the hydrostatic part. Integration of this term reads:

$$
\int_{z_{n-1/2}}^{z_{n+1/2}} \frac{\partial q^*}{\partial x} \, dz = \frac{\partial}{\partial x} \int_{z_{n-1/2}}^{z_{n+1/2}} q^* \, dz - q_{k+1/2} \frac{\partial \Delta z_{k+1/2}}{\partial x} + q_{k-1/2} \frac{\partial \Delta z_{k-1/2}}{\partial x} 
$$

(3.20)

in which the integral at the right is approximated by

$$
\int_{z_{n-1/2}}^{z_{n+1/2}} q^* \, dz \approx \frac{1}{2} \Delta z_k \left( q_{k+1/2} + q_{k-1/2} \right) 
$$

(3.21)

With all these layer-integrated terms together, the horizontal momentum equation for layer $n$ reads:
The next step is the discretisation in horizontal direction by integration over a control volume with its cell centre at a \( u \)-point. Writing the different terms for each pressure point together gives for each layer the space discretised horizontal momentum equation:

\[
\frac{\partial \Delta z_k u_k}{\partial t} + \frac{\partial \Delta z_k u_k^2}{\partial x} + \frac{u_{k+j/2}^*}{\partial x} - \frac{u_{k-j/2}^*}{\partial x} + \frac{\partial p_{h.n}}{\partial x} + \frac{1}{\rho_n} \left( \frac{\partial \Delta z_k}{\partial x} \right) + \frac{q_{k+j/2}^*}{\partial x} - \frac{q_{k-j/2}^*}{\partial x} = 0
\]

(3.22)

In the above expression the horizontal velocity at the cell interface, \( \phi^* = \Delta z u \), and the cell-face value, \( \phi^* = \Delta z u \), have to be approximated at the bounding surfaces of the control volume, which can be done straightforwardly with a central differencing scheme or with higher order upwind schemes.

To reduce spurious flow phenomena which have been encountered in the long run during the closed basin test (section 5.1), an upwind scheme has been applied, as outlined in Appendix C. The flow velocity is considered as the concentration of momentum:

\[
\bar{u} = \frac{1}{\rho V} \int \rho u \, dV
\]

(3.24)

The advection of momentum over the cell boundaries is only taken into account when the flow is directed inward. The transport over a vertical boundary is the horizontal velocity times the vertical grid step size at the boundary. The horizontal velocity at the boundary can be approximated by taking the average of the velocity in the two adjacent cells. Outward directed transport can be left out of consideration, because it does not change the momentum in the cell:
\[ Q_{\text{left}} = u_{m,k} \Delta z_{m,k} \quad \text{where} \quad u_{m,k} = \frac{1}{2} \left( u_{m+j/2,k} + u_{m-j/2,k} \right) \]
\[ \quad \text{with} \quad u_{m,k} = 0 \quad \text{if} \quad u_{m,k} < 0 \]
\[ Q_{\text{right}} = -u_{m+1,k} \Delta z_{m+1,k} \quad \text{where} \quad u_{m+1,k} = \frac{1}{2} \left( u_{m+j/2,k} + u_{m+j/2,k} \right) \]
\[ \quad \text{with} \quad u_{m+1,k} = 0 \quad \text{if} \quad u_{m+1,k} > 0 \]

(3.25)

The relative vertical velocity is directed perpendicular to the local vertical grid. The area of the bounding surface depends on the local slope of the grid. As this relative velocity is located at the corners of the \( u \)-volume, two times half the surface is taken into account. Again, transport over the boundary is zero when directed outward, but now the contribution of each relative velocity component is checked separately to prevent instabilities due to averaging.

\[ Q_{\text{lower}} = \left( \omega_{m,k-j/2} + \omega_{m+1,k-j/2} \right) \frac{1}{2} \sqrt{\Delta z_{\text{lower}}^2 + \Delta x^2} \]
\[ \quad \text{with} \quad \omega_{m,k-j/2} = 0 \quad \text{if} \quad \omega_{m,k-j/2} < 0 \quad \text{and} \quad \omega_{m+1,k-j/2} = 0 \quad \text{if} \quad \omega_{m+1,k-j/2} < 0 \]
\[ Q_{\text{upper}} = -\left( \omega_{m,k+j/2} + \omega_{m+1,k+j/2} \right) \frac{1}{2} \sqrt{\Delta z_{\text{upper}}^2 + \Delta x^2} \]
\[ \quad \text{with} \quad \omega_{m,k+j/2} = 0 \quad \text{if} \quad \omega_{m,k+j/2} > 0 \quad \text{and} \quad \omega_{m+1,k+j/2} = 0 \quad \text{if} \quad \omega_{m+1,k+j/2} > 0 \]

(3.26)

The change of concentration of momentum in the cell under consideration depends on the difference in concentration between the inflow and the mean in the cell. The velocity is assumed to be homogeneous inside a cell and the concentration of inflow therefore equals the concentration of the adjacent cell. After summation over all bounding surfaces, the advection term of the momentum equations can be rewritten to give the space-discretised equations for each layer:

\[ \frac{\partial \Delta z_{m+j/2,k} u_{m+j/2,k}}{\partial t} \Delta x \]
\[ + Q_{\text{left}} \left( u_{m-j/2,k} - u_{m+j/2,k} \right) + Q_{\text{right}} \left( u_{m+j/2,k} - u_{m+j/2,k} \right) \]
\[ + Q_{\text{lower}} \left( u_{m+j/2,k-1} - u_{m+j/2,k} \right) + Q_{\text{upper}} \left( u_{m+j/2,k+1} - u_{m+j/2,k} \right) \]
\[ + \frac{\Delta z_{m+j/2,k}}{\rho_n} \left( p_{h,n,m+1} - p_{h,n,m} \right) \]
\[ + \frac{1}{2} q_{m+1,k-j/2} \left( z_{m+1,k+j/2} - z_{m,k-j/2} \right) + q_{m+1,k+j/2} \left( z_{m,k+j/2} - z_{m+1,k-j/2} \right) \]
\[ - \frac{1}{2} q_{m,k-j/2} \left( z_{m,k+j/2} - z_{m+1,k-j/2} \right) + q_{m+1,k+j/2} \left( z_{m+1,k+j/2} - z_{m,k-j/2} \right) = 0 \]

(3.27)

### 4.3.3 Vertical momentum equation

The vertical momentum equations, given by the second and fourth row of (2.25), is repeated here in the following way for each density layer \( n \) as:
\[
\frac{\partial w_n}{\partial t} + \frac{\partial w_n u_n}{\partial x} + \frac{\partial w_n^2}{\partial z} + \frac{1}{\rho_n} \frac{\partial q}{\partial z} = \text{visc}_{\text{vert},n}
\] (3.28)

where the right hand side represents the viscosity, which is non-zero for the lower layer only. Although the term is given here, the discretisation is treated later (in section 4.3.4).

\[
\text{visc}_{\text{vert},z} = \frac{\eta}{\rho_z} \frac{\partial^2 w}{\partial x^2} + \frac{\eta}{\rho_z} \frac{\partial^2 w}{\partial z^2}
\] (3.29)

Following Zijlema and Stelling (2005) again, the left hand side exists of three parts: time derivative, advection and hydrodynamic pressure. The control volume used is different from the control volume used in the previous section, because the w-points are located at integer m-points and half k-points. Integration is carried out over the control volumes as given in the figure below.

![Control volume](image)

**Figure 9: Control volume of a w-point.**

The volume of a certain cell is expressed as:

\[
V = \frac{1}{4} \left( \Delta z_{m-\frac{1}{2},k}^u + \Delta z_{m-\frac{1}{2},k+1}^u + \Delta z_{m+\frac{1}{2},k}^u + \Delta z_{m+\frac{1}{2},k+1}^u \right) \Delta x
\] (3.30)

where \( \Delta z_u \) is the vertical step size at u-points. The volumes of the cells at the bottom and top are only half of the size calculated in this way, as half of the volume is located outside the domain.

The vertical integral over a layer around a w-point is approximated using the midpoint rule:

\[
\int_{z_a}^{z_{a+1}} w_{m,k+\frac{1}{2},z} dz = \Delta z_{m,k+\frac{1}{2},z} w_{m,k+\frac{1}{2}} \quad \text{with} \quad \Delta z_{m,k+\frac{1}{2},z} = \frac{1}{2} \left( \Delta z_{m,k} + \Delta z_{m,k+1} \right)
\] (3.31)

The layer-integrated vertical momentum equation is derived in the same manner as done for the horizontal momentum equation except for the pressure gradient:

\[
\frac{\partial \Delta z_{k+\frac{1}{2},z} w_{k+\frac{1}{2},z}}{\partial t} + \frac{\partial \Delta z_{k+\frac{1}{2},z} u_{k+\frac{1}{2},z}}{\partial x} + w_{k+1} \dot{\alpha}_{k+1} - w_{k} \dot{\alpha}_k + \int_{z_a}^{z_{a+1}} \frac{\partial q}{\partial z} dz = 0
\] (3.32)
where the asterix indicates an approximation at the cell interface again. The integral of the vertical pressure gradient is approximated using the midpoint rule:

\[
\int_{z_2}^{z_4} \frac{\partial q}{\partial z} dz = \Delta z_{k+\frac{1}{2}} \left. \frac{\partial q}{\partial z} \right|_{k+\frac{1}{2}}
\]  \(3.33\)

where the vertical pressure gradient at a pressure point is approximated using “das Mehrstellenverfahren” (introduced by Collatz, 1966):

\[
\frac{\partial q}{\partial z}_{k+\frac{1}{2}} = 2 \frac{q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}}}{\Delta z} \left. \frac{\partial q}{\partial z} \right|_{k-\frac{1}{2}} \quad \text{with} \quad \Delta z_{m,k+\frac{1}{2}} = \frac{1}{2} \left( \Delta z_{m,k} + \Delta z_{m,k+1} \right)
\]  \(3.34\)

Zijlema and Stelling use a method equivalent to central differencing for the advection terms. However, as with the horizontal momentum, in this study an upwind approach is used.

The horizontal transport over the cell boundaries is given by the horizontal velocity times the vertical grid step size in that point. Because the horizontal velocity is vertically staggered in regard to the vertical velocity, there are two components at each boundary to be taken into account:

\[
Q_{\text{left}+} = u_{m-k+1,\frac{1}{2}} \Delta z_{m-k+1} \left. \frac{\partial q}{\partial z} \right|_{k+\frac{1}{2}} \quad \text{with} \quad Q_{\text{left}+} = 0 \quad \text{if} \quad Q_{\text{left}+} < 0
\]

\[
Q_{\text{left}+} = u_{m-k+1,\frac{1}{2}} \Delta z_{m-k+1} \left. \frac{\partial q}{\partial z} \right|_{k+\frac{1}{2}} \quad \text{with} \quad Q_{\text{left}+} = 0 \quad \text{if} \quad Q_{\text{left}+} < 0
\]  \(3.35\)

\[
Q_{\text{right}+} = -u_{m+\frac{1}{2},k} \Delta z_{m+\frac{1}{2}} \left. \frac{\partial q}{\partial z} \right|_{k+\frac{1}{2}} \quad \text{with} \quad Q_{\text{right}+} = 0 \quad \text{if} \quad Q_{\text{right}+} < 0
\]

\[
Q_{\text{right}+} = -u_{m+\frac{1}{2},k} \Delta z_{m+\frac{1}{2}} \left. \frac{\partial q}{\partial z} \right|_{k+\frac{1}{2}} \quad \text{with} \quad Q_{\text{right}+} = 0 \quad \text{if} \quad Q_{\text{right}+} < 0
\]  \(3.36\)

The vertical transport over the cell boundaries is given by the relative velocity perpendicular to the moving vertical grid. Because the vertical grid is a boundary following grid, the surfaces are not strictly horizontal, but have to be calculated each time step:

\[
Q_{\text{lower}} = \omega_{m,k} \sqrt{\Delta z_{\text{down}}^2 + \Delta x^2} \quad \text{with} \quad Q_{\text{lower}} = 0 \quad \text{if} \quad Q_{\text{lower}} < 0
\]

\[
Q_{\text{upper}} = -\omega_{m,k+1} \sqrt{\Delta z_{\text{up}}^2 + \Delta x^2} \quad \text{with} \quad Q_{\text{upper}} = 0 \quad \text{if} \quad Q_{\text{upper}} < 0
\]  \(3.37\)

where: \(\omega_{m,k} = \frac{1}{2} \left( \omega_{m,k-\frac{1}{2}} + \omega_{m,k+\frac{1}{2}} \right)\)

After integration over a horizontal cell the vertical momentum equations read:

\[
\frac{\partial \Delta z_{m,k+\frac{1}{2}} w_{m,k+\frac{1}{2}}}{\partial t} \Delta x
\]

\[
+ \left( Q_{\text{left}+} + Q_{\text{left}+} \right) \left( w_{m-k,\frac{1}{2}} - w_{m+k,\frac{1}{2}} \right) + \left( Q_{\text{right}+} + Q_{\text{right}+} \right) \left( w_{\text{right}} - w_{m+k,\frac{1}{2}} \right)
\]

\[
+ Q_{\text{lower}} \Delta z_{m-k,\frac{1}{2}} + Q_{\text{upper}} \Delta z_{m,k+\frac{1}{2}}
\]

\[
+ 2 \frac{q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}}}{\Delta z} \left. \frac{\partial q}{\partial z} \right|_{k-\frac{1}{2}} = 0
\]  \(3.38\)
The above equation is solved for each $w$-point, except the points at the bottom (in the lower layer) and the interface (in the upper layer). Just above the bottom and interface the vertical pressure gradient follows from the kinematic conditions for the vertical velocity component (2.12) and (2.26), which read in discretised form:

$$w_2 \bigg|_{m, y_2} = \frac{1}{2} \left( \frac{u_{m-1, y_2} + u_{m+1, y_2}}{\Delta x} \right) \frac{z_{m+1, y_2} - z_{m-1, y_2}}{\Delta x}$$

$$w_1 \bigg|_{m, K_1, y_2} = \frac{\partial \zeta}{\partial t} + \frac{1}{2} \left( \frac{u_{m-K_1, y_2} + u_{m+K_1, y_2+1}}{\Delta x} \right) \frac{z_{m+K_1, y_2+1} - z_{m-K_1, y_2}}{\Delta x}$$

By substitution of the horizontal momentum equations in the expressions above, the vertical pressure gradient at the bottom and at the interface are obtained. These pressure gradients are used in the vertical momentum equations in the next layers upwards.

### 4.3.4 Viscosity

Viscosity adds two extra terms to the right hand side of each momentum equation in (2.25). However, the viscosity in the upper layer can be neglected most of the time and this study, accordingly, only deals with viscosity in the lower layer.

The terms representing viscosity in the horizontal and vertical momentum equations of the lower layer are given by (3.16) and (3.29) and repeated here for convenience. They exist of two second derivatives of the flow velocity: one in horizontal and one in vertical direction:

$$visc_{\text{hor}, 2} = \frac{\eta}{\rho} \frac{\partial^2 u}{\partial x^2} + \frac{\eta}{\rho} \frac{\partial^2 u}{\partial z^2}$$

$$visc_{\text{vert}, 2} = \frac{\eta}{\rho} \frac{\partial^2 w}{\partial x^2} + \frac{\eta}{\rho} \frac{\partial^2 w}{\partial z^2}$$

The second derivative in horizontal direction is approximated using a central difference scheme:

$$\eta \frac{\partial^2 \phi}{\partial x^2} \bigg|_{m+1/2} \approx \frac{\phi_{m+1/2} - 2\phi_{m} + \phi_{m-1/2}}{\Delta x^2}$$

where $\phi$ may represent $u$ or $w$.

In vertical direction the second derivative is approximated in two steps. First, the shear stress, $\tau$, is calculated in between two adjacent velocity points. Second, the vertical derivative thereof is calculated. At the bottom a zero-slip condition is applied and at the interface (which is ‘as a surface’ for the lower layer) the shear stress is assumed to be zero:

$$\tau_{\text{bottom}} = \eta \frac{\phi_{m+1/2}}{\Delta z_{m,1}}$$

$$\tau_{\text{interface}} = 0$$

The viscosity in the intermediate points can now be approximated with a central difference scheme:
Normally, a zero-slip condition at the bottom would result in unacceptable strong effects of the viscosity, but in this case with only small movements in the lower layer this error does not have a large impact on the waves in the upper layer. A larger number of layers in the lower layer (e.g. here a number of 6 is used) reduces the error of the zero-slip condition.

4.4 Boundary conditions

As discussed in section 3.1.3, boundary conditions are needed at the boundaries of the computational domain. In vertical direction the two density layers are limited by vertical boundaries at bottom, interface and surface. In horizontal direction the domain is limited by lateral boundaries at both ends. The velocity at these boundaries is implied as described in section 3.3.2 and not repeated here, because the discretisation does not yield different expressions.

4.4.1 Boundary conditions in the vertical

The kinematic condition at the bottom is given by expression (2.12). To get the horizontal velocity at the $w$-points, the midpoint rule is used. The gradient of the bottom profile is approximated by a central difference method. This results in the discretised version of the condition at the bottom:

For layer 2:

$$
\tau_{m+\frac{1}{2},k} - \tau_{m-\frac{1}{2},k} \quad (3.43)
$$

At the interface, the kinematic condition for the vertical velocity in the upper layer, (2.26), is discretised in the same way. Yet since the interface acts as a moving bottom, there is an additional term for the time derivative of the interface:

For layer 1:

$$
\frac{\partial \xi}{\partial t} + \frac{1}{2} \left( u_{m+\frac{1}{2},k_2+1} + u_{m-\frac{1}{2},k_2+1} \right) \frac{\xi_{m+\frac{1}{2},k_2+1} - \xi_{m-\frac{1}{2},k_2+1}}{\Delta x} \quad (3.44)
$$

The expressions above, (3.43) and (3.44), can be substituted in the vertical momentum equations to get the vertical pressure gradient at respectively the bottom and the interface.

4.5 Time discretisation

The momentum equations still have to be discretised with respect to time. For each term a time discretisation method is chosen. For the hydrostatic pressures a special case of the Leapfrog scheme is used, introduced by Hansen (1956). A first order explicit scheme is used for the advection terms because it is easy to implement and advection is not the dominant factor with regard to wave propagation. The non-hydrostatic pressures, which act as a correction factor on the velocities, are discretised using a first order implicit method for stability reasons.

The free surface gradient, included in the horizontal momentum equation as hydrostatic pressure, is discretised using the Hansen scheme. This scheme evaluates the velocity at half time steps, while the
free surface is evaluated at whole time steps, whereas in the traditional leapfrog scheme this is done at the same time steps.

This scheme therefore has the advantage of the leapfrog scheme; dampening wave like solutions slowly, without the need to store the variables from the previous time step. The Hansen scheme therefore removes another drawback of the Leapfrog scheme: the need of an extra initial condition.

The explicit discretisations of the advection terms restrict the maximum time step size by the Courant-Friedrichs-Levy condition (CFL). In the case of simulation of short waves however, this condition will not be the limiting factor. Even for very small time steps the calculation time will be within reasonable limits, because the model has one horizontal dimension only.

Discretisation of the viscosity terms and the friction terms in the wave absorbing boundary conditions are discussed separately. As they are discretised using a first order explicit method, they only yield an adaptation to the right hand side of the frictionless case.

**4.5.1 Discretised equations**

First the horizontal velocities are approximated without non-hydrostatic pressures. The discretisation of the non-hydrostatic pressure terms is treated separately. The horizontal momentum equation (3.27) can be written as:

\[
\frac{u_{m}^{n+\frac{1}{2}} - u_{m}^{n-\frac{1}{2}}}{\Delta t} + ADV_{\text{hor}}^{n-\frac{1}{2}} + \frac{1}{\rho} \frac{p_{\text{hydro}}^{n+1} - p_{\text{hydro}}^{n}}{\Delta x} = 0
\]  

(3.45)

The different advection terms are taken together in one term, \( ADV \), for the sake of brevity. The advection is evaluated at the old time step. The third term represents the hydrostatic pressure and is evaluated at a half time step between the new and the old time step. The asterisk indicates that the effect of the hydrodynamic pressures is not included.

In the hydrostatic case only the above equations are solved for each horizontal velocity point. Substitution of the new velocities in the depth averaged local continuity equation results in new water levels. The vertical velocities are approximated with the local continuity equation at the new grid.

In the non-hydrostatic case the non-hydrostatic pressures act as a correction factor on the approximated velocity given in (3.45):

\[
\frac{u_{n,m}^{n+\frac{1}{2}} - u_{n,m}^{n-\frac{1}{2}}}{\Delta t} + \frac{\frac{z_{m+1,k} - z_{m,k}}{\Delta z_{m,k} + \Delta z_{m+1,k}} - \frac{z_{m,k} - z_{m-1,k}}{\Delta z_{m,k} + \Delta z_{m-1,k}} - \frac{q_{m+1,k} - q_{m,k}}{\Delta z_{m,k} + \Delta z_{m+1,k}}}{\rho_n} = \frac{z_{m+1,k} - z_{m,k}}{\Delta z_{m,k} + \Delta z_{m+1,k}} - \frac{z_{m,k} - z_{m-1,k}}{\Delta z_{m,k} + \Delta z_{m-1,k}} + \frac{q_{m+1,k} - q_{m,k}}{\Delta z_{m,k} + \Delta z_{m+1,k}}
\]  

(3.46)

The non-hydrostatic pressures, \( q \), are taken at the new time level.

The vertical momentum equation (3.38) is also discretised in time. For the advection a first order explicit method is used again. The pressure gradient is expressed as the approximation given by the compact difference method (3.34):

36
The equation above is solved for every vertical velocity except for the vertical velocity at the bottom (for the lower layer) and the interface (for the upper layer). At these points the vertical velocity follows directly from the kinematic boundary conditions (2.12) and (2.26).

The space-discretised kinematic boundary at the bottom is given by (3.43). The momentum equation can be rewritten to solve the vertical pressure gradient at the bottom at the new time step:

\[
\frac{1}{\rho} \frac{\partial q}{\partial z}^{m+\frac{1}{2}} = -\frac{w_{j,x}^{m+\frac{1}{2}} - w_{j,x}^{m+\frac{1}{2}}}{\Delta t} - ADV_{vrr}^{m+\frac{1}{2}}
\]  

Substitution of the space-discretised boundary condition at the new time step into this expression above results in an expression for the vertical pressure gradient at the bottom:

\[
\frac{1}{\rho} \frac{\partial q}{\partial z}^{m+\frac{1}{2}} = \frac{1}{2} \left( u_{m+\frac{1}{2},K+1}^{n+\frac{1}{2}} + u_{m+\frac{1}{2},K+1}^{n+\frac{1}{2}} \right) \frac{d_{m+\frac{1}{2}} - d_{m-\frac{1}{2}}}{\Delta x} + w_{j,x}^{m+\frac{1}{2}} - ADV_{vrr}^{m+\frac{1}{2}}
\]  

The boundary at the interface (for the upper layer) is slightly different, because of the movement of the interface. A space-discretised version is given by (3.44) and repeated here:

\[
\frac{\partial \xi}{\partial t}^{m+\frac{1}{2}} = \frac{\partial Q_{m+\frac{1}{2}}}{\partial x} = \int_{\text{bottom}} \frac{\partial u}{\partial z} dz \approx \sum_{l=1}^{K_x} u_{m+\frac{1}{2},l}^{n+\frac{1}{2}} \Delta z_{m+\frac{1}{2},l} - u_{m+\frac{1}{2},l}^{n+\frac{1}{2}} \Delta z_{m+\frac{1}{2},l}
\]  

Using the old grid the time derivative of the interface can be replaced by the layer averaged local continuity equation of the lower layer, which is approximated as follows:

\[
\frac{\partial \xi^m}{\partial t}^{m+\frac{1}{2}} = \int_{\text{interface}} \frac{\partial u}{\partial z} dz \approx \sum_{l=1}^{K_x} u_{m+\frac{1}{2},l}^{n+\frac{1}{2}} \Delta z_{m+\frac{1}{2},l} - u_{m+\frac{1}{2},l}^{n+\frac{1}{2}} \Delta z_{m+\frac{1}{2},l}
\]  

This expression can be substituted into expression (3.50) which can than be used to write the vertical gradient of the non-hydrostatic pressure at the interface as:
non-hydrostatic modelling of waves in layered fluids

\[
\frac{1}{\rho} \frac{\partial q}{\partial z}
\bigg|_{K_2 + \frac{1}{2}}^{n+\frac{1}{2}} = - \sum_{j=1}^{K_2} u_{m+1/2,j}^{n+1/2} \frac{\Delta z_{m+1/2,j}^n - u_{m-1/2,j}^{n+1/2} \Delta z_{m-1/2,j}^n}{\Delta x} + \frac{1}{2} \left( u_{m+1/2,j}^{n+1/2} + u_{m+1/2,j}^{n+1/2} \right) \frac{\xi_{m+1/2,j}^n - \xi_{m-1/2,j}^n}{\Delta x} - w_{m,k_2 + \frac{1}{2}}^{n-1/2} - ADV_{vert}^{n-1/2} \Delta t
\]

(3.52)

This expression makes the vertical gradient of the non-hydrostatic pressure known and can be substituted in the vertical momentum equation for the upper layer.

### 4.5.2 Matrix notation

Matrix notation can be used to write the momentum equations in compact form. The final horizontal momentum equation can be written by combining (3.45) and (3.46):

\[
u_{m,k}^{n+1/2} + \sum_{o=1}^{1} \sum_{l=0}^{k} U_{m,k,o,l} q_{m,o,l}^{n+1/2} = [r_u]_{m,k}
\]

(3.53)

where:

\[
[r_u]_{m,k} = u_{m,k}^{n-1/2} - \Delta t ADV_{horizontal} - g \frac{\partial p_{stat}^n}{\partial x}
\]

\[
U_{m,k,0,k} = \Delta t \frac{z_{m+1,k} - z_{m,k-1}}{\Delta x \left( \Delta z_{m,k} + \Delta z_{m+1,k} \right)}
\]

\[
U_{m,k,0,k-1} = \Delta t \frac{z_{m,k} - z_{m+1,k-1}}{\Delta x \left( \Delta z_{m,k} + \Delta z_{m+1,k} \right)}
\]

\[
U_{m,k,1,k-1} = -U_{m,k,0,k-1} \quad \text{and} \quad U_{m,k,1,k-1} = -U_{m,k,0,k}
\]

(3.54)

The \(U\)-matrix couples the four surrounding hydrodynamic pressures at the new time step to the velocity at that certain point. The vertical coordinates are taken from the old time step. The right hand side, \(r_u\), consists of all explicit terms.

The vertical momentum equation can also be written in matrix notation:

\[
w_{m,k}^{n+1} + \sum_{o=1}^{1} \sum_{l=0}^{k} W_{m,k,o,l} q_{m,o,l}^{n+1} = [r_w]_{m,k}
\]

(3.55)

where:

\[
[r_w]_{m,k} = w_{m,k}^{n} - ADV_{vert} \Delta t - [r_{pz}]_{m,k} \Delta t
\]

\[
W_{m,k,0,l} = -PZ_{m,k,o,l} \Delta t
\]

(3.56)

In the notation above the vertical pressure gradients at bottom and interface are written in matrix notation too. The pressure gradients can be substituted from the bottom and interface upwards as:

\[
\frac{\partial q}{\partial z}_{m,k} + \sum_{l=0}^{k} PZ_{m,k,o,l} q_{m,l} = [r_{pz}]_{m,k}
\]

(3.57)
where:

\[
[r_{pc}]_{m,k} = -\frac{\partial q}{\partial z_{m,k-1}}
\]

\[
PZ_{m,k,o,l} = (-1)^{1-l} \frac{-2}{\Delta z_{m,k}}
\]

The velocities at the new time step are now known as function of the flow properties at the old time step and a factor of the (yet unknown) surrounding hydrodynamic pressures.

4.5.3 Viscosity and sponge layer

The discretisation used for the viscosity terms and the friction in the wave absorbing boundary is first order explicit. Therefore these terms are additional factors for the right hand side vector \([r]\). For the horizontal momentum equation this vector becomes:

\[
[r_{u}]_{m,k} = [r_{u}]_{m,k} + \Delta t\eta \left( \frac{u_{m-1,k} - 2u_{m,k} + u_{m+1,k}}{\Delta x^2} + \Delta t \left( \frac{\tau_{k+\frac{1}{2},m} - \tau_{k-\frac{1}{2},m}}{\Delta z_{m,k}} \right) \right)
\]

Implementation of the sponge layer friction is done in the same way (where the sponge-factor, \(C_f\), is zero everywhere except at u-points in the sponge layer; see section 3.1.3):

\[
[r_{u}]_{m,k} = [r_{u}]_{m,k} - \Delta t C_f u_m^u
\]

4.6 Substitution in continuity equation

The hydrodynamic pressures at the new time step can be found by substitution of the momentum equations in the continuity equation. Because the fluid is incompressible, the inflow in a cell has to equal the outflow. The only free variable after substitution is the hydrodynamic pressure, which therefore acts as a correction on the velocities to guarantee a divergence free flow field.

Each pressure point is coupled by the vertical momentum equation (3.55) with all other pressure points in the vertical. The horizontal momentum equation couples each vertical to the direct surrounding verticals. Each pressure therefore depends linearly on the other pressures in the same vertical and the neighbouring verticals. This results in a linear system with the pressures as unknowns. Because the expressions for the velocities at the new time step are expressed in matrix notation, the continuity equation for each cell can be expressed in matrix notation too:

\[
\sum_{o=1}^{1} \sum_{l=0}^{K} A_{m,k,o,l} q_{m+o,l}^{n+1} - [r]_{m,k} = 0
\]

The structure of matrix \(A\) is a particular form of a diagonal matrix with ‘blocks’ of dependent pressures on the diagonal. Each block couples the three neighbouring verticals together. With a certain number of pressure points in the vertical, \(K\), they have a width of \(3K\) and a height of \(K\) (Figure 10). Equation (3.61) can be solved by a simple Gauss-elimination. Because of the special structure of the matrix, the elimination can be optimized by sweeping the cells only in the neighbouring verticals.
When this matrix is solved, the hydrodynamic pressures are known and the velocities can be corrected with the pressure gradients. At bottom and surface the kinematic boundary conditions are substituted into the continuity equation, which guarantees that there is no transport over the boundary.

### 4.7 Solution method

The solution at the new time step can be obtained by following the sequence of computation:

1. Compute the advection in horizontal and vertical direction (3.25)-(3.26) and (3.35)-(3.37) with the velocities at the current time step.
2. Compute the hydrostatic pressure gradient (3.15) or (3.16) for each density layer.
3. Compute the new horizontal velocities by substitution of the hydrostatic pressure and the advection in the \( \mathbf{r}_u \) vector and compute the \( \mathbf{U} \) matrix (3.54) for both layers separately.
4. Substitute the horizontal velocities as function of \( \mathbf{U} \) and \( \mathbf{r}_u \) in the boundary condition at the bottom (3.43) and substitute that into the vertical momentum equation to get the vertical pressure gradient at the bottom (3.49).
5. Substitute that into the equation for the vertical pressure gradient (3.34) and compute the pressure matrix \( \mathbf{P}_Z \) (3.57) for the lower layer.
6. Substitute the horizontal velocities in the lower layer as function of \( \mathbf{U} \) and \( \mathbf{r}_u \) into the layer averaged continuity equation to get the time derivative of the interface.
7. Substitute that, together with the horizontal velocities as function of \( \mathbf{U} \) and \( \mathbf{r}_u \), in the boundary condition just above the interface (3.44).
8. Substitute the boundary condition just above the interface into the vertical momentum equation to get the vertical pressure gradient at the interface.
9. Repeat step 5, but now for the upper layer.
10. Substitute for each layer the \( \mathbf{P}_Z \) matrix (3.57) into the \( \mathbf{W} \) matrix (3.55).
11. Substitute the equations for the corrected velocities (3.53) and (3.55) into the local continuity equation(3.10) to get the continuity divergence per cell (3.61).
12. Compute the hydrodynamic pressures by solving the system of pressure equations following from the requirement of a divergence free flow field.
13. Correct the velocities \( w^{n+1} \) and \( u^{n+1} \) for both layers.
14. Compute the new vertical grid with the velocities at the new time step.
5 Test results

Several test cases have been performed to validate the new two-layer model. The model is compared to analytical solutions and results from experiments. The most urgent question was how the model would represent the dispersion of external and internal waves, which was tested using a closed basin test and compared with other models. Furthermore the stability of the interface is a factor of interest where two-layer models are involved and the model has been tested on a laboratory case found in literature. The advection terms are tested in a change of sex test.

The use of a wave generating boundary with viscous lower layer raised the question about the validity of the hydrostatic boundary extension. In a separate test the results of these boundary conditions are presented.

Most of the test cases somehow have a one-layer equivalent. The closed basin test and the viscosity and mean flow in a layer have also been tested in single layer models (by setting the thickness of one of the layers to zero). In this way the implementations of different aspects of the model have been validated, layer by layer, before the model was applied to a two-layer case. Not all single layer verifications are given below.

5.1 Standing wave in closed basin

5.1.1 Test setup

A simple and common method to verify the dispersion relation of a model is the closed-basin test. The closed basin consists of a domain with length 0.5\(L\) and constant depth \(d\) with zero-velocity boundaries at both ends \((u = 0)\), which fully reflect the wave. As initial condition a small surface elevation is given, which is a sinusoidal wave with an exact wavelength of \(L\), with a maximum and minimum elevation at the velocity boundaries. These conditions result in a standing wave in the basin with a wave number given by:

\[
k = \frac{2\pi}{L}
\] (4.1)

The wave can be considered a linear wave when the amplitude is small enough, and will be approximated well by the linear wave theory. The shape of the standing wave will stay intact and symmetrical: the surface at both ends will go up and down, but the centre will stay at the same height.

By measuring the elevation at one point as a function of time, the frequency of the wave can be obtained. Repeating this test for several depths or several wave lengths results in a relationship between the wavelength-depth \((kh)\) ratio and the relative frequency. This relationship can then be compared with the theoretical relationship according to the linear wave theory for a single layer system (with \(\omega = \sqrt{gk}\) being the frequency of the external wave at deep water):

\[
\frac{\omega^2}{\omega_0^2} = \tanh (kd)
\] (4.2)
For the layered system the interface between the two layers will get an initial displacement with smaller amplitude. Based on the test results the initial interface was given an elevation in opposite direction to make a clear distinction between internal and external waves. The initial conditions are given with enlarged amplitudes in Figure 11 below:

Figure 11: Initial conditions for the closed basin test (in this figure the amplitudes are enlarged).

If there is an internal wave in the basin, the time-signal of the elevation at a certain point will contain the influence of that internal wave. The signal is analysed using a Fast Fourier Transformation (FFT) to get the two most distinct frequencies. The low frequency belongs to the internal wave; the high frequency belongs to the external wave. These two frequencies can be compared to the linear expression for a layered system as outlined in Appendix A:

$$\frac{\omega^2}{\omega_0^2} = \frac{(T_1 + T_2) \pm \sqrt{(T_1 + T_2)^2 - 4(1 + (1 - \varepsilon)T_1T_2)eT_1T_2}}{2(1 + (1 - \varepsilon)T_1T_2)}$$

(4.3)

with relative density $\varepsilon = \frac{\rho_2 - \rho_1}{\rho_2}$, $T_n = \tanh{(kh_n)}$

One solution of the equation above will give the frequency of the internal wave; the other will contain the solutions for the external wave. For each relative density difference the dispersion relation of the model can be obtained and compared with this linear expression.

From now on, the configuration of the model is included in the figure title. The first line of the title gives the wave length, $L$, (two times the basin length), the notation next to it refers to the number of computational layers in each density layer (upper – lower). The second line gives the densities of both layers and the ratio of the layer thicknesses. For the tests in this section each time a relatively small horizontal grid step size of 0.5 is adopted, which corresponds to 100 steps.
5.1.2 Results – single layer system

The test is done first with one (single) density layer with only one computational layer. Since there is no difference between the two density layers except density, there is no difference in using the ‘upper’ layer or the ‘lower’ layer; both give the same results as the existing model by Stelling and Zijlema.

To evaluate the dispersion relation of the model, the results for $kh$ values (i.e. the relative depth) in the range $0 – 6$ are plotted in one figure together with the expected values according to the linear wave theory. A straight line is also shown, which represents the dispersion according to the shallow-water approximation, see Figure 12:

![Figure 12: Dispersion relation of the single layer model with one computational layer.](image)

In Figure 12 the model (numerical calculation) clearly shows better results than the shallow-water approximation (hydrostatic approximation), but the results are not yet satisfying for larger depths. The explanation can be found in the discretisation of the equations, as the discretisation introduces a certain error in the dispersion of the model. The theoretical dispersion relation for the model with one computational layer can be derived by linearization and equating the determinant to zero (see Appendix B). This theoretical dispersion of the model has been plotted in the figure too (vertically discretised equations), which shows that the model follows the expected dispersion relation exactly.

The error of the discretisation can be reduced by using more computational layers. With two computational layers the dispersion of the model already follows the relation of the linear wave theory almost exactly, see Figure 13 below:
non-hydrostatic modelling of waves in layered fluids

Figure 13: Dispersion relation of single layer model with two computational layers.

In the range under consideration here, two computational layers are clearly enough, but at larger depths and shorter waves, more layers can be needed. By choosing the thickness of the upper computational layer small enough with regard to the wave length or depth the number of two computational layers is sufficient in most cases.

5.1.3 Results – layered system

When the model uses two sigma grid layers, each consisting of one computational layer and both with the same density (∊ = 0), the results should be the same as with one sigma grid consisting of two computational layers, because there is no difference in the physics. The results (Figure 14) show that this is indeed the case; the internal wave has a frequency of zero.

Figure 14: Dispersion relation of two-layer model without density differences between the layers. Each density layer exists of only one computational layer.
The theoretical line for the dispersion of the model is now omitted as the straightforward linearization as used with one single density layer is not possible due to the interactions at the interface.

When the densities start to differ, an internal wave at the interface with its own propagation velocity and dispersion relation can exist separately from the external wave at the surface. Starting from the situation where both densities are the same, the density of the lower layer will be increased. As a result the internal wave speed will increase too.

The dispersion relation is tested at several density differences and with several layer thicknesses. The influence of the density difference will be treated later, but first a relative density difference, $\varepsilon$, of 0.5 is chosen and the influence of the layer thickness is investigated. At this relative density difference the internal wave has a relative high velocity with respect to cases with small density differences and the influence of the layer thickness is therefore more distinct. The relative frequencies of the external and internal wave with regard to the deep water frequency of the external wave are plotted along with the theoretical relations according to equation (4.3) against the relative water depth:

![Figure 15: Dispersion relations for internal and external waves. The density of the lower layer is two times the density of the upper layer. The layer thickness of both layers is equal.](image)

In the figure above no line for the theoretical dispersion of the model is given, because the derivation thereof is not as straightforward as in the single layer case. The linearization used to derive the theoretical dispersion of the model is no longer possible due to non-linear interaction at the interface. However, the error in the calculation results of the dispersion relation of the internal wave (the deviation of the dots from the lower line in Figure 15) looks like what can be expected for a model with one single density layer with one computational layer (as in Figure 12). The error in the dispersion relation of external waves is almost the same as with the one layer model with two computational layers (as in Figure 13). This can be explained by the fact that there is only one computational layer available for the internal wave (which propagates primarily in the lower layer) and two for the external wave (which propagates in both upper and lower layer).
When the thickness of the lower layer decreases, the deviation from the dispersion relation of external waves will increase, because the system behaves more like the single layer system with one computational layer. The deviation from the dispersion of the internal wave will stay more or less the same, as the influence of the upper layer on the internal wave is only small:

![Figure 16: Two density layers both with one computational layer. The lower layer is thin in regard to the upper layer.](image)

When in both density layers two computational layers are used, the deviation from the analytical dispersion relation disappears again. The dispersion relations of both external and internal waves are correct, as can be seen in Figure 17 below:

![Figure 17: Accurate computation of both internal and external wave with two computational layers per density layer.](image)

To guarantee correct dispersion relations of internal and external waves under all circumstances, in all computations from now on at least two computational layers per density layer will be used.
Changes in density difference will also strongly influence the ratio between the propagation velocities of the waves. Smaller density differences will give results which are closer to the single layer solution for the external wave and closer to zero for the internal wave. Here the result of a test with a relative density difference of ±5% is given:

![Dispersion relation for small relative density difference](image)

**Figure 18:** Dispersion relation for small relative density difference ($\varepsilon = 5\%$) with equal layer thickness.

As mentioned before the model is still stable for very small density differences between the layers and even without a density difference. When the density of the lower layer is smaller than that of the upper layer (negative relative density difference, $\varepsilon < 0$), it can be expected from physics that the two layers are unstable. The amplitude of the internal wave will increase (Figure 19, left) until steep gradients will break the calculation. This is indeed the behaviour of the two-layer model and after a few periods (depending on the relative depth) the oscillation becomes unstable (Figure 19, right).

![Standing wave in closed basin](image)

**Figure 19:** Negative relative density differences (here -5%) are physically not stable, which is correctly modelled by the two-layer model. After one period the amplitude of the interface is already enlarged, the initial surface coincides with the current (left); in the long run the computation breaks down (right).
5.1.4 Discussion

From the test results it can be concluded that the numerical model is capable of accurately representing the propagation velocity of external and internal linear waves. The configuration of two computational layers per density layer is not needed for all cases, but guarantees good approximations under a wide range of ratios of layer thicknesses and density differences. For extremely large depths however (e.g. $kh > 8$) the dispersion is not correct. In that case the thickness of the two computational layers can be taken different, such that the upper computational layer is small in regard to the wave length.

Although not explicitly mentioned, the basin test is also carried out with other initial conditions e.g. a disturbance in the interface only. Other linear initial conditions did not give other results for the dispersion relation.

For situations with no density difference the model is still stable. In these cases there is no propagation of the internal wave and the two-layer model gives identical results to the one-layer model. However, when the density of the upper layer is larger than the density of the lower layer (negative relative density difference) the model becomes unstable within a few periods.

5.2 Viscous models

In this section the numerical model is tested against the analytical models mentioned in section 2.2. The analytical models give solutions for the wave number as a function of the frequency and the properties of the layers. Firstly, only the dispersion relation of the model is compared with the analytical solutions for a non- or low-viscous case and secondly, the wave damping due to viscosity is examined. In all test cases two computational layers per density layers have been used.

The analytical models are developed for specific cases. They all assume the lower layer to be viscous whereas the numerical model can deal with a non-viscous lower layer. The calculation of the wave numbers according to the analytical models is done with a script by Gerben J. de Boer and Wouter M. Kranenburg, 2008. The input consist of the wave period (and frequency) of the external wave and the properties of the two layers (thickness, density and viscosity). The script returns the corresponding values of the wave number according to:

1. Guo
2. Gade (real and imaginary part)
3. The starting value of iteration for the Delft dispersion relation
4. De Wit (real and imaginary)
5. Delft (real and imaginary)
6. Dalrymple and Liu (real and imaginary)
7. Ng (real and imaginary)

The solutions given by 2 will only be valid under hydrostatic circumstances (small thickness of the layers). The approximations given by 3, 4 and 7 will be limited by the layer thickness of the lower (mud) layer, because they assume hydrostatic pressures in the lower layer. The waves in the upper (water) layer can be short with respect to the depth. The approximation of 6 takes the full momen-
tum equations into account for both layers. The applicability is limited by the thickness of the viscous boundary layer:

$$H_{m0} \gg \sqrt{2v_m / \nu}$$  \hspace{1cm} (4.4)

5.2.1 Dispersion

In absence of viscosity, the dispersion relation of the model is obtained as described in section 5.1 with a standing wave in a closed basin with two computational layers per density layer. In this way the eigenfrequencies belonging to the geometry of the basin and the layers are obtained. The eigenfrequencies are used to calculate the corresponding wave numbers according to several analytical models with no or very small viscosity.

By plotting the different relations for one specific case, the differences can be shown. This is repeated for a number of different mud layer thicknesses and different density differences. First, a case is run without density differences, the results are shown in Figure 20 below. The configuration of the numerical model is indicated in the figure title: each line corresponds to one layer.

![Figure 20: Comparison of the new model with existing analytical models (Guo and Ng) for a case without viscosity and no density difference. Equal layer thickness (left) and at thin lower layer (right).](image)

For cases without density differences such as these, only Guo and Ng give solutions. The dispersion relation according to Guo was derived for cases with only one density layer. Because it is applied here to a case with two density layers, Guo neglects the existence of the lower layer. The values for $k$ are therefore too high. Ng gives values for $k$ which are too low. When the thickness of the lower layer is smaller, the solutions are all close to the linear wave theory.

In cases with density differences results are found for the other analytical models too. The Delft dispersion relation (as is the case with Dalrymple and De Wit too) is already close to the linear wave theory at small relative density differences (e.g. $\varepsilon = \pm 2.5\%$, see Figure 21), whereas Ng comes close to the linear theory at large relative density differences only (e.g. $\varepsilon = 50\%$, see Figure 22).

With a thin lower layer all dispersion relations are close to the linear wave theory, except for Gade. The results of the analytical models are in line with what can be expected from theory, as outlined in the previous section.
Figure 21: At small relative density difference ($\varepsilon = 2.5\%$) the results of the numerical two-layer model and the Delft dispersion relation are close to the linear theory regardless of the layer thickness, whereas Ng still needs a thin lower layer. Equal layer thickness (left) and small lower layer (right).

Figure 22: Larger relative density difference ($\varepsilon = 50\%$); equal layer thickness (left) and small lower layer (right). The Ng dispersion relation is now also close to the linear theory, but Gade is still only valid at shallower relative depth (e.g. $kH < 1$).

As the figures above show, the numerical model performs well for all these linear cases. The results of the analytical models can be reproduced. The numerical model gives for each situation solutions close to the linear theory, whereas none of the analytical models is valid at each case.

5.2.2 Damping of waves by a viscous layer

When one of the layers has a viscosity that cannot be neglected, damping of the waves by internal friction can occur. The analytical models include this damping by the imaginary part of the wave number, $k_i$. As stated in section 2.2 (with constant wave number in the domain) the wave height at a certain point, relative to the starting position, $x$, can be calculated by means of:

$$H = H_0 e^{-k_i x}$$    (4.5)
The results of the numerical model can be compared with this wave damping. A wave basin is used as test case. Waves enter the basin at one boundary and leave at the other end of the domain. The wave generating boundary consists of the horizontal velocity due to the imposed wave according to the linear wave theory as described in section 3.1.3. The mean velocity in both layers is zero.

Due to the high viscosity, the amplitude of the wave in the lower layer will be smaller than in the non-viscous case. To generate waves at the boundary, the wave is therefore imposed on the upper layer only. The downstream boundary is a wave-absorbing boundary, as described in section 3.2.

The viscosity is implemented in the numerical model as described in section 4.3.4 with a zero-slip condition at the bottom. It can be expected that there is too much dissipation in the model in comparison with the analytical models. This error can be reduced by increasing the number of computational layers in the lower layer (here 6 layers were used).

For this test, a case is chosen comparable to what is used in Kranenburg (2008). The lower layer consists of mud with a density of 1750 kg/m\(^3\) and has a viscosity of 0.5 m\(^2\)/s. The upper layer consists of water with a density of 1025 kg/m\(^3\). The results of a computation with 6 computational layers in the lower layer are compared to the results according to the analytical models of Gade, Dalrymple and the Delft dispersion relation model in Figure 23.

The damping calculated with De Wit and Ng is comparable to damping calculated with the Delft model and therefore not included in the figure. Dalrymple clearly overestimates the damping, because this case is beyond the applicability limit given by (4.4). At larger depth also Gade overestimates the damping, as Gade actually is a shallow water model.

As the figures above show, the dissipation of waves calculated by the numerical model at small depths (Figure 23, left panel) is slightly too much in comparison with the Gade and Delft models. This can be explained by the fact that the zero-slip boundary at the bottom introduces an error. At intermediate depth however (Figure 23, right panel), the dissipation calculated by the numerical model seems to be too little compared to the Delft model.
5.2.3 Dispersion with viscosity

At (very) shallow conditions it is difficult to obtain the dispersion relation of the model, because the wave gets dissipated before a good estimate of the period can be obtained. However, at slightly larger depths it is no problem and the dispersion relation with influence of viscosity (0.5 m²/s) seems to correspond with earlier results without viscosity (see Figure 24).

![Figure 24: Comparison of dispersion relations with (left) and without viscosity (right).](image)

When the lower layer is thick in regard to the upper layer, the results of the numerical calculation are really close to the linear theory. The analytical models however assume a small thickness of the mud layer and are deviate from the linear theory at intermediate depths, as is shown in Figure 25.

![Figure 25: With a relatively thick mud layer, the analytical models deviate more from the linear theory at intermediate depth. Viscosity (left) has only a small influence compared to non-viscous cases (right).](image)

5.2.4 Discussion

In comparison with analytical models the results of the numerical model are closer to the linear wave theory, which is correct for these linear cases. The numerical model still gives results close to the linear wave theory for cases where the lower layer is thicker than the upper layer, whereas most of the analytical models only come close to the linear wave theory for a shallow lower (mud) layer.
The comparison of dissipation shows small differences between the analytical models and the numerical model. However, due to the straightforward implementation of the viscosity, a relatively large number of computational layers is needed.

### 5.3 Stability

In case of a difference in flow velocity between the layers two-layer flow becomes unstable for waves with wave number \( k \) above a certain limit. According to the stability condition, longer waves are more stable than shorter waves. As stated in section 2.1, the Kelvin-Helmholtz stability condition is given by (which is graphically shown in Figure 26):

\[
\frac{\Delta u^2}{\varepsilon g H} < \frac{\tanh kh_1 + \tanh kh_2}{kH}
\]

\[\text{(4.6)}\]

Figure 26: Stability limits for a two-layer system (taken from C. Kranenburg).

For cases where this condition is met, there is a certain wave number above which the layered flow is unstable and a part of the internal waves will grow exponentially. These unstable waves however will develop non-linear behaviour and turbulence, which will then be suppressed by the density difference.

This stability limit was derived for infinitesimal perturbations and it would be interesting to have some insight in how the new model, which includes non-linear wave propagation, represents this limit. The inclusion of the hydrodynamic pressure may result in suppression of the non-linear behaviour of the unstable waves, which could involve a higher stability limit for the model.

#### 5.3.1 Test setup

For this test case three things are needed: a velocity difference, a density difference and internal waves with a certain wave number.

Waves enter the domain at the left boundary and they leave it at the right boundary. The computational domain is extended at both ends. The influence of the hydrodynamic pressures has been faded in the extension (section 3.2). The boundaries are therefore fully hydrostatic. A wave following boundary of the Sommerfeld-type is applied at the end of the domain, as described in section 3.1.3.

Before the test cases were carried out, the model was tested for stability in absence of differences in flow velocity between the layers. It followed that the waves get damped at higher flow velocities due to the first order implementation of the advection terms, which is shown in Figure 27. Therefore, relatively small spatial grid step sizes have to be used (e.g. 100 points per wave length) and the flow...
velocities are kept small. Because of this effect, the model is expected to give more stable results than according to the limit given by equation (4.6).

Remark: the elevation of the surface due to the internal wave is only very small. Because of that, the figures in this section show almost no elevation of the free surface. The elevation of the interface at a certain moment (labelled with ‘Interface $\xi$’) and the maximum thereof over a certain period of time, the envelope, (labelled with ‘max $\xi$’) are given relative to the mean elevation of the interface.

Figure 27: Long waves propagating through the domain from left to right (above in absence of flow) get dissipated when a mean flow exists in the layer ($u=0.15m/s$) and resolution is too low (middle, 25 points per wavelength). This effect is less, but still clearly present, when a finer grid is used (below, 100 points).
In all test cases a depth of 2m is chosen with two layers of equal thickness. Because the flow velocities have to stay small to keep the dissipation low, the density difference is taken small too, e.g. 0.1%. All cases are inviscous and only small-amplitude, linear waves are used. The length of the domain itself is not very important; a few wave lengths suffice to show that the internal waves are growing and become unstable.

5.3.2 Test results

Shallow water conditions are chosen for the first test case (e.g. \( kH = 0.02 \)). The stability condition reduces to:

\[
\frac{\Delta u^2}{\varepsilon gH} < 1
\]  

(4.7)

With a relative density difference of 0.1% and a total depth of 2m this condition is still met at a velocity of 0.14m/s in one of the layers only. The results however show internal waves growing as they propagate through the domain (in Figure 28 they travel from left to right). After a certain time the waves become highly non-linear, which can be seen when the domain is a few times longer.

![Propagating internal waves become non-linear as they travel through the domain under influence of a flow velocity in the lower layer (\( u_2 = 0.14 \text{m/s}, kH = 0.02 \)).](image)

Although the highly non-linear waves do not grow anymore, they generate errors at the outflow boundary which get reflected in both layers, adding errors to the inflow boundary, and finally they
end the calculation. When there is a mean flow velocity in both layers the model is more stable, but
the condition given by (4.7) seems too high, even with relatively large velocities.

The test is repeated at larger depth (e.g. \( kH = 2 \), \( \varepsilon = 0.1\% \) and a velocity difference of 0.12 m/s. This
is well below the stability limit as given by (4.6) and therefore a stable solution is to be expected, but
again the internal waves increase and get unstable as is shown in Figure 29.

Figure 29: Internal waves steepen and become non-linear (\( kH=2 \)). Instantaneous elevation and envelope.

### 5.3.3 Another stability limit

By lowering the velocity difference a limit is found for which the waves do not increase. In the shal-
low water case, with equal layer thicknesses, the limit value was 0.1 m/s (see Figure 30). For several
other layer thicknesses a limit was found. The results of the calculations with a total depth of 2 meter
are presented in table 1. By changing the grid step size, the stabilizing effect of the advection terms
was investigated, but smaller grid step sizes did not yield other limit values.

Table 1: Limit values for the flow velocity with flow in the upper layer and a total depth of 2m.

<table>
<thead>
<tr>
<th>( h_2 ) [m]</th>
<th>Clearly stable conditions ( u_1 ) [m/s]</th>
<th>Clearly unstable conditions ( u_1 ) [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \varepsilon = 0.1% )</td>
<td>( \varepsilon = 0.2% )</td>
</tr>
<tr>
<td>1.5</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td>1</td>
<td>0.099</td>
<td>0.14</td>
</tr>
<tr>
<td>0.5</td>
<td>0.07</td>
<td>0.095</td>
</tr>
<tr>
<td>0.25</td>
<td>0.048</td>
<td>0.07</td>
</tr>
</tbody>
</table>
The thickness of the upper layer was not of any importance in regard to the stability. This was the reason to check whether the stability was more accordingly to a different expression, given by (4.8):

\[
\frac{u_1^2}{\varepsilon gh_2} < 1
\]  

(4.8)

The model seemed to follow this limit in stead of the limit given by the theory of the Kelvin-Helmholtz instability. With flow in the lower layer in stead of the upper layer, the stability seemed to depend on the thickness of the upper layer and the following expression holds:

\[
\frac{u_2^2}{\varepsilon gh_1} < 1
\]  

(4.9)

Figure 30: Stable internal propagating waves (upper panel) and unstable internal waves (lower panel). The amplitude of the internal waves is clearly growing as the waves travel from left to right.
Although it is not clear why the model follows this stability limit, it was used in cases with flow in one layer only to predict whether the results would be stable or not. With flow in both layers the stability was found to be larger than given by the expressions above, but no expression was found.

In regard to short waves, the same adaption was made to the left hand side of the stability limit given by (4.6). Again depending on flow in the lower resp. upper layer, the condition for the model reads:

\[
\frac{\Delta u^2}{\varepsilon g h_i} < \frac{\tanh kh_i + \tanh kh_j}{kH} \quad \text{or} \quad \frac{\Delta u^2}{\varepsilon g h_j} < \frac{\tanh kh_i + \tanh kh_j}{kH}
\]  

(4.10)

This limit has been tested for different ratios of layer thickness and several (relatively small) relative density differences. It can be concluded that the stability of the present model depends on this condition. However, this condition holds only for cases where flow in one of the layers can be neglected.

### 5.3.4 Discussion

With a mean flow equal in both layers the internal waves are stable, but get dissipated rather quickly. A fine spatial grid has to be used in order to reduce the dissipation caused by the first order implementation of the advection terms.

The test results show that the internal waves increase under influence of a difference in flow velocity above a certain limit. The limit for the present numerical model was found to be not according to the expression found in literature, (4.6), but follows a slightly different line. An expression has been found, which can be used to predict whether the model will be unstable or not, depending on the flow in one of the layers:

- no flow in upper layer:
  \[
  \frac{u_2^2}{\varepsilon g h_1} = \frac{\tanh kh_1 + \tanh kh_2}{kH}
  \]
  
  (4.11)

- no flow in lower layer:
  \[
  \frac{u_1^2}{\varepsilon g h_2} = \frac{\tanh kh_1 + \tanh kh_2}{kH}
  \]
  
  (4.12)

When the velocity difference is larger than indicated by this limit, the wave amplitude starts to increase, becomes non-linear after a while and finally breaks the calculation. With a mean flow in both layers, the stability is enhanced considerably in regard to the expressions above (4.11) and (4.12).

Non-hydrostatic computation did not result in suppression of the non-linear behaviour of the unstable waves. Like in the hydrostatic case, once the waves increase, they increase until they become non-linear and end the calculation. For hydrostatic cases (shallow water), the results of hydrostatic and non-hydrostatic calculation are the same.

### 5.4 Change of sex test

#### 5.4.1 Test setup

To test the properties of the advection terms in the two-layer model a test case called the ‘change of sex test’ is used. The test deals with a closed basin with an initially horizontal surface. The test is
performed for inviscid fluids only. In this test there is a considerable displacement of volume of the fluids and advection is therefore a dominant factor.

The initial condition for the interface is the driving force (Figure 31); the initial interface has a big hump in the middle of the domain. After the test is started the interface will sag due to the relatively high density and the lower layer spreads out in horizontal direction.

![Figure 31: Initial conditions for change of sex test.](image)

The upper layer will react to the movement of the lower layer and starts to flow towards the centre of the domain, which enhances the pressure in the centre. The downward speed of the centre of the interface is enlarged by this additional pressure (Figure 32, left).

![Figure 32: Higher pressures in the upper layer accelerate the peak (left) and reflection of the internal wave at the boundaries (right)](image)

As soon as the peak comes close to the bottom it slows down because the thickness of the lower layer is getting smaller. There will now be a translating wave in the interface which reflects at the boundaries. As can be seen in the figure above, the lower layer does not disappear completely and quite some energy has gone to the surface.

After reflection at the boundaries (Figure 32, right), the opposite process takes place. Because energy has been transferred from the interface to the surface this does not result in the initial state again (shown in Figure 33, left), but also after each following cycle energy is lost. In the long run
however, the elevation of surface and interface becomes so small that it can be considered linear; the linear wave is not damped anymore (Figure 33, right).

![Figure 33: Conditions after one cycle (left) and in the long run (right).](image)

When the initial height of the hump is larger (the peak of the interface comes closer to the surface), the surface seems to ‘stick’ to the downward moving interface. Smaller horizontal step sizes, involving more points with small distance between surface and interface, seem to enlarge this problem. When this effect is too strong, the calculation breaks. The first order implementation of the advection terms seems to be an over-simplification for situations like this.

### 5.4.2 Discussion

The change of sex test showed that the small grid step sizes needed with this implementation of the advection terms gives rise to problems with steep gradients. The dissipation of the wave is obviously too large, as most of the wave energy is lost after only a few periods.

With a better implementation of the advection terms it would be possible to model a lock exchange test, which has even more extreme gradients than the test presented here. The initial condition would then consist of only salt water on one side of the domain and only fresh water at the other side.

### 5.5 Non-linear internal wave steepening

In this section the numerical model is tested on a laboratory case found in literature. Horn et al. (2001) did research on internal wave degeneration in stratified lakes. Wind stress acting at the surface tilts the interface of the lakes, generating large-scale gravity waves with relatively low frequency. Observed decay times are smaller than can be explained by internal dissipation. Other mechanisms are therefore needed to explain the transfer of energy from these low frequency basin scale waves to higher frequencies or turbulent scales.

The results and analysis of the experiments conducted by Horn et al. (2001) suggest that the generation of solitons by nonlinear steepening is an important mechanism for the degeneration of large scale internal waves. Although observations have shown that nonlinear effects become significant, most analytical studies have been based on linear theory.
Their final conclusion is that the evolution, propagation and shoaling of solitons appear to be important processes in most lakes, yet these processes are not covered captured by numerical models that use the hydrostatic approximation.

The hydrodynamic model presented in this study does include the evolution, propagation and shoaling of solitons and therefore is a good candidate to study these large scale internal waves. Because Horn et al. describe their laboratory experiments with all dimensions; it is easy to implement their test in our new model.

5.5.1 The experiments

The description of the experiments given here is taken from Horn et al. (2001). The experiments were carried out in a fully enclosed clear acrylic tank 600 cm long, 29 cm deep and 30 cm wide. The tank could rotate about a horizontal axis. A schematic diagram of the tank is shown in Figure 34.

The initial thickness of the interface was less than 1 cm. The tank was tilted slowly to the initial angle and than quickly returned to a horizontal position. The resulting flow was then recorded and the interface displacement was measured by ultrasonic wave gauges.

![Experimental set-up with dimensions (a), the tank immediately before (b) and directly after the start (c). Wavegaugs were located at the positions A, B and C.](image)

The variables considered, being of importance for the simulation, were: the angle of tilt, $\theta$ and the interface depth, $h$. The density difference was kept constant at $\Delta \rho = 20 \text{ kg/m}^3$. The angle of tilt was varied between $\theta = 0.125^\circ$ and $2.77^\circ$. The depth ratios ranged between $h/H = 0.2$ and $h/H = 0.5$.

At small initial tilt angles and a thickness of the lower layer close to half the total depth the interface oscillated until it was damped by viscous effects. At increasing non-linearity (by increasing the initial angle or decreasing the lower depth) the initial wave was observed by Horn et al. to steepen and to evolve into a train of solitons.

For the numerical model an angle of tilt is applied to the interface as initial condition. In this way the same internal waves are generated. However, the effect of the closed tank was not included, as the present model uses the free surface condition in stead of a rigid lid. Because the waves under consideration are internal waves, they have only small effects in the free surface and therefore the influence of the rigid lid could be neglected.
5.5.2 Results

Starting at an angle of $\theta = 0.25^\circ$ the angle of initial tilt was increased up to $\theta = 1.5^\circ$. At the first angle no large wave is recorded, as the measurements were done in the centre of the tank (location B) and no steepening occurs. At an increasing angle the steepening increases and more solitons are generated.

The time series for $h/H = 0.3$ as given in the paper (Figure 35, left) are reproduced by the numerical model presented in this study (Figure 35, right). A horizontal grid step size of 0.02m and two computational layers per layer have been used. The time step was chosen according to the Courant condition of external waves $\Delta t = \frac{1}{2} \Delta x / \sqrt{gH} = 0.0059s$.

To study the influence of the depth ratio Horn et al. decreased the layer thicknesses from 0.5 to 0.2 at a constant initial angle of $\theta = 0.5^\circ$. It was observed that the non-linear effects increase from almost zero to clearly evolving solitons at larger differences in layer thickness (Figure 36, left).

The numerical model was also applied to this test case found in the paper (Figure 36, right). The similarity can be seen clearly from the time records of the elevation of the interface at location B. Again the non-linear effects seem to be slightly overpredicted by the model.
Figure 36: Increasing non-linearity at increasing difference in layer thickness. Laboratory results (left) are compared with results of the numerical model (right).

A model based on the Korteweg-de Vries equations has been derived by Horn et al. (2002) for further investigation of the measurements mentioned above. They describe how they applied their model to two test cases. In their test cases a grid step size of $\Delta x = 0.023\text{m}$ and a time step of $\Delta t = 0.001\text{s}$ is used.

In the first case the depth ratio was $h/H = 0.2$, the initial angle of tilt $\theta = 0.5^\circ$ and the density difference was $20\text{ kg/m}^3$. The comparison of this test and their model is shown below (left) at two locations (B and C). The numerical model presented here is applied to this test case with a grid step size of $\Delta x = 0.05\text{m}$. The results are shown in the right panel of Figure 37.

Figure 37: Comparison of simulated (---) and observed (—) interface displacements, taken from Horn et al., 2002 (left), and the results of the new numerical non-hydrostatic two-layer model (right).

From the figures above it can be concluded that the numerical model represents the laboratory experiments fairly well, possibly better than the model used by Horn et al. However, the non-linear effects are slightly over predicted again.
Another test performed by Horn et al. was closely the same as previously described for the increasing initial tilt on page 62. The thickness of the lower layer was taken constant $h = 8.5 \text{ cm}$ and the tilt was increased from $\theta = 0.25^\circ$ to $\theta = 1.0^\circ$. The present model is applied to this test case with a grid step size of $\Delta x = 0.02 \text{ m}$, the results are presented in Figure 38.

![Figure 38: Comparison of the model (—) and experiments (---) by Horn et al. (left, taken from Horn et al., 2002) and the present numerical model with $\Delta x = 0.02 \text{ m}$ (right).](image)

### 5.5.3 Discussion

The results of the test cases show that the non-linear effects according to the new non-hydrostatic two-layer model are slightly larger than observed and modelled by Horn et al. (2001 and 2002). This can be explained by the fact that the present model neglects all sorts of friction, which are present in the laboratory experiments at, for instance, the bottom, the walls and the rigid lid. Moreover, as they mention in their paper, the thickness of the interface increases slightly during the experiments, whereas the numerical model is based on a strictly two-layer approach.

Although the similarities between the experiments and the newly derived model are clear, there is an important difference in the setup. The experiments are conducted in a fully enclosed tank whereas the numerical test case consisted of a closed basin with a free surface. Because the waves under consideration are internal waves with only very small effects on the free surface, the movement of the surface stays small. This justifies the neglecting of the rigid lid.

### 5.6 Boundary conditions

The horizontal boundaries are a problem for non-hydrostatic numerical models, especially in this case of a two-layer model involving propagating internal waves. In this paragraph it is shown that use of fading hydrodynamic effects can help solving the problems at in- and outflow boundaries.

Use of an additional friction term as sponge layer is not possible where a mean flow in the layers is considered, because it will prevent the flow from leaving the domain. By applying the friction to deviations of the mean flow only, the sponge layer has shown reasonable results. In a two-layer model however, it still resulted in errors of the elevation of the interface (Figure 39).
Figure 39: Errors arising from the sponge layer (right of 20m line) due to flow in one of the layers. Thin lines are the instantaneous elevations, thick lines are the envelopes.

At the inflow boundary the vertical distribution of the horizontal velocities is unknown when hydrodynamic pressures play a role (short waves or deep water). By smoothly increasing the hydrodynamic effects a hydrostatic boundary can be used. In the extension the velocities will get redistributed over the vertical.

5.6.1 Fading pressure influence

By applying a fading factor as described in section 3.2, the influence of the pressure on the horizontal momentum is reduced towards the boundaries. The phase speed at the wave generating boundary is therefore exactly according to the hydrostatic approximation and decreases until the full dispersion relation is met. As the wave speed decreases, the wave steepens and the amplitude increases. The increase in amplitude is compensated by a redistribution of the velocities over the vertical.

Exactly the opposite happens at the outflow boundary: the waves get longer and the speed increases until the waves travel with the propagation velocity according to the hydrostatic approximation again. By applying a radiation condition at the outflow boundary external waves can leave the domain, but internal waves still get partly reflected in the surface.

The effects of the fading pressure influence can be clearly seen in Figure 40. As propagating waves travel from left to right. The wave length is larger towards the boundaries both at inflow (left) and outflow (right).
non-hydrostatic modelling of waves in layered fluids

Figure 40: External (above) and internal waves (below) travelling from left to right. The boundaries of the domain are hydrostatic; between x=0 and x=20m the computation is non-hydrostatic. Thin lines are the instantaneous elevations, thick lines are the envelopes.

The fading of hydrodynamic effects at the boundaries introduces an important change in the matrix. Because the influence is faded in the horizontal momentum equation only, the horizontal dependence of the pressures is fading. The blocks with size $k \times 3k$ become more like blocks of size $k \times k$.

Solving the matrix in fact yields changing the hydrodynamic pressures such that the resulting flow field is divergence free. Close to the boundaries however, the influence of the horizontal pressure gradient is much smaller. To guarantee divergence free flow with fading horizontal effects, the horizontal gradient will increase. As the pressure at the surface is set to zero, the vertical pressure gradient also increases.

Although the horizontal dependence is very small or completely absent at the boundaries, the pressures will not tend to infinity because the pressure at the surface is set to zero. The divergence free flow field is guaranteed by correction of the vertical velocities only.

It is shown in Figure 41 that the pressures get larger at the boundaries, but there are no extremely large gradients. The waves in the non-hydrostatic part of the domain are not erroneously influenced by the fading pressure influence.
Figure 41: The pressures (caused by external waves at each computational layer interface) are larger in the hydrostatic parts of the domain, but they are bounded by the pressure at the free surface.

The hydrostatic boundaries constructed for this case result in a propagating wave which is fully hydrodynamic inside the domain of interest. Although the vertical distribution of the horizontal velocities imposed at the boundary is according to the hydrostatic approximation and therefore not valid at deep water, the smoothly increasing hydrodynamic effect results in the correct distribution at the start of the non-hydrostatic part of the domain.

Due to the start-up there will be very small initial short waves in front of the wave train generated at the boundary. Normally, this would not result in problems, as the amplitudes are only very small. In case of the stability test however, the amplitudes of short waves will increase as they travel through the domain. To suppress these short waves, viscosity is set to a non-zero value at the first time step and fading away during the first few periods of the generated wave.

5.6.2 Discussion

The sponge layer caused problems under influence of a mean flow. Especially for a two-layer model the sponge resulted in an erroneous distortion of the interface.

The fading hydrodynamic pressure effects have shown to be a good solution for the problem of unknown vertical distributions of horizontal velocities at the inflow boundary. At the outflow boundary it made application of a radiation condition possible.

The generation of internal waves causes also external waves, which are, due to their higher velocity, much earlier at the outflow boundary. Moreover, the radiation condition at the outflow boundary partly reflects internal waves in the surface. The boundary is therefore less effective for internal waves, but still much more effective than the sponge layer.
6 Conclusions and recommendations

6.1 Recapitulation of project objectives

The main objective of this project was the construction of a non-hydrostatic two-layer model based on the scheme presented by Stelling and Zijlema (2003) and an investigation of internal and external wave propagation properties.

Additional requirements for this model were formulated which resulted in the following sub-objectives:

- Use of $\sigma$-layers instead of $k$-layers;
- The bottom boundary condition of the upper layer has to be extended;
- Validation with simple test cases and verification with other models;
- Implementation of a viscosity term;
- Verification of wave damping by fluid mud;

6.2 Conclusions

In this study the question was whether the non-hydrostatic model presented by Stelling and Zijlema (2003) could be used in an efficient way to simulate waves in layered fluids. The answer is yes, and without large adaptations. In fact, the only adaptation made, is that the model is applied to each layer separately and an interface is introduced as common boundary.

In linear cases, the dispersion relation of the model is according to the linear wave theory. Using the closed basin test the properties of the dispersion relation were tested, thereby showing that in cases with equal thickness of the density layers and one computational layer per density layer, the dispersion relation of the external wave followed the same line as in the case of a single layer model with two computational layers. This is not the case with a relatively small thickness of the lower layer, for which the dispersion relation becomes more like that for the case of a single layer model with only one computational layer.

The number of computational layers in the lower layer defines the error in the dispersion relation of the internal wave. The error is similar to that in a single layer model with the same number of computational layers as the lower density layer.

For cases without density difference the dispersion in the closed basin test is exactly equal to that for a one layer model. When the lower layer is lighter than the upper layer, the two-layer model is unstable, which is consistent with the theory.

In comparison with analytical models the dispersion relation of the numerical model is closer to the linear wave theory, and also holds for cases where the lower layer is thicker than the upper layer. The simple and straightforward implementation of viscosity in the lower layer gives results for the dissipation of external waves which are comparable to the analytical models, but with the present implementation a relatively high number of computational layers is needed.
Boundary conditions for the non-hydrostatic model are still problematic. The smoothly increasing effects of the non-hydrostatic pressures in combination with a shallow-water boundary, as applied and tested in this study, have shown to be a good solution to this problem.

The stability of the model has been tested with different flow velocities in the layers. In most cases instability occurs much earlier than can be expected from the Kelvin-Helmholtz condition, depending on the thickness of the layer without flow. However, in cases with flow in both layers, the stability is enhanced. With relatively large flow velocities in both layers the limit of this model is still not according to the expression found in literature.

The simple first order implementation of the advection terms causes dissipation of waves in cases where advection is dominant, such as the stability test and the change of sex test. Small grid step sizes have to be used to reduce this effect, but very fine grids result in instabilities, which is a common problem when trying to conserve momentum.

Non-linear wave propagation of internal waves is simulated reasonably well by the new model. Steepening of internal waves and generation of internal solitons are well represented, as was shown in a test case with non-linear internal waves and in comparison with laboratory experiments performed by Horn et al (2001).

Summarizing, it can be concluded that the non-hydrostatic two-layer model performs well with regard to wave propagation and non-linear behaviour, whereas highly advective processes are still problematic due to the need of relatively small grid steps.

6.3 Recommendations

The propagation velocity of internal waves is far lower than that of external waves. Because the Courant condition has to be fulfilled, the time step size is determined by the fastest (i.e. external) wave. By applying an implicit method for the free surface the time step can be increased. This would make model a lot more efficient. (The interface probably does not need an implicit treatment, because when external waves are concerned, it follows the movement of the free surface).

A better implementation of the advection terms would make it possible to model more practical cases involving density differences. Modelling of, for instance, a lock exchange test where one side of the domain contains only salt water and the other side only fresh water would become better possible.

In the present study, turbulence has been left out of consideration. With an appropriate turbulence model, the model would be capable of simulating phenomena of practical importance like internal lee waves and bores.

The current implementation of viscosity has only been tested with regard to the damping of external waves over a mud layer with a relatively large viscosity. Under these conditions without flow, the zero slip boundary at the bottom gives reasonable results. A more accurate implementation of the viscosity is required to make the model more generally applicable. The amount of computational
layers currently needed is disproportionate in comparison with the amount needed for the dispersion relation.

The stability of the interface is still not fully understood. The large dissipation caused by the advection terms is a stabilizing factor, but it cannot fully explain the enhanced stability with larger flow in both layers. Further investigation is needed for better quantification of the stability limit and the sources thereof.
7 References

non-hydrostatic modelling of waves in layered fluids


# 8 List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
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<td>elevation of free surface</td>
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<tr>
<td>( \xi )</td>
<td>elevation of interface</td>
<td>m</td>
</tr>
<tr>
<td>( d )</td>
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Appendices

Appendix A  Linear two-layer dispersion relation

Here, the derivation of the dispersion relation according to the linear theory is outlined as given by C. Kranenburg (1998). Under assumption of hydrostatic pressure in the vertical, the governing equations are the layer averaged continuity equations and the horizontal momentum equations:

\[
\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x} h_1 u_1 = 0
\]
\[
\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x} h_2 u_2 = 0
\]  
(A.1)

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + g \frac{\partial}{\partial x} (z_{bot} + h_1 + h_2) = S_1
\]
\[
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{\rho_1}{\rho_2} g \frac{\partial h_1}{\partial x} + g \frac{\partial (z_{bot} + h_2)}{\partial x} = S_2
\]  
(A.2)

where \(u_n\) and \(a_n\) are the horizontal velocity and layer thickness of layer \(n\), \(z_{bot}\) is the vertical location of the bottom and \(S_n\) is the force acting on the layer.

By considering each of the independent variables \(a_1, a_2, u_1\) and \(u_2\) as linear functions in time and space a phase speed, \(c\), is introduced such that for each variable, \(F\), holds:

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + c \frac{\partial F}{\partial t} 
\]  
(A.3)

With this expression for the partial derivatives of each variable, there are now eight expressions and eight partial derivatives. By equating the determinant of this system to zero, the propagation velocity, \(c\), can be found:

\[
\left[ (c - u_1)^2 - gh_1 \right] \left[ (c - u_2)^2 - gh_2 \right] - (1 - \varepsilon) g^2 h_1 h_2 = 0
\]  
(A.4)

with \(\varepsilon = \frac{\rho_2 - \rho_1}{\rho_2}\).

The phase speed of internal waves can be found by neglecting the terms in the order of \((\varepsilon gH)^2\) and assuming the relative phase speed in the order of \(\sqrt{\varepsilon gH}\):

\[
c_i = \frac{h_1 u_1 + h_2 u_2}{H} \pm \frac{h_2 h_2}{a^2} \left[ \varepsilon gH - (u_1 - u_2)^2 \right]
\]  
(A.5)

The phase speed of external waves can be found by neglecting terms in the order of \(\varepsilon (gH)^2\), assuming the relative phase speed in the order of \(\sqrt{gH}\) and the velocity difference in the order of \(\sqrt{\varepsilon gH}\):
\[ c_e = \frac{h_1 u_1 + h_2 u_2}{H} \pm \sqrt{gH} \]  

(A.6)

With the two equations above, the phase speeds of external and internal waves have been found for shallow water conditions. For deep water the derivation is different. Here a short description is given. The governing equations are the continuity equation and two momentum equations per layer (\( n \) indicates the layer: 1 is the upper layer, 2 is the lower layer):

\[ \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial z} = 0 \]  

(A.7)

\[ \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} + w_n \frac{\partial u_n}{\partial z} + \frac{1}{\rho_n} \frac{\partial p}{\partial x} = 0 \]

\[ \frac{\partial w_n}{\partial t} + u_n \frac{\partial w_n}{\partial x} + w_n \frac{\partial w_n}{\partial z} + \frac{1}{\rho_n} \frac{\partial p}{\partial z} = -g \]  

(A.8)

Under assumption of vortex free flow a potential function can be introduced:

\[ u_n = \frac{\partial \phi_n}{\partial x}, w_n = \frac{\partial \phi_n}{\partial z} \]  

(A.9)

The continuity equation can now be written as:

\[ \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial^2 \phi_n}{\partial z^2} = 0 \]  

(A.10)

After integration of the momentum equations with respect to \( x \) and \( z \) they read:

\[ \frac{\partial \phi_n}{\partial t} + \frac{p_n}{\rho} + gz + \frac{1}{2} \left( u_n^2 + w_n^2 \right) = F_n(t) \]  

(A.11)

where \( F(t) \) is a yet unknown function in time.

A fluid particle at a boundary (bottom, interface or surface) will stay in the boundary. This results in kinematic conditions at the boundaries:

\[ w_n = \frac{\partial h_n}{\partial t} + u_n \frac{\partial h_n}{\partial x} \]  

(A.12)

where \( h \) is the vertical distance of the specific boundary to the bottom.

This condition applies at four places: at the bottom (\( n = 2, h = 0 \) which gives \( w = 0 \)), just below (\( n = 2 \)) and above (\( n = 1 \)) the interface (\( h = h_2 \)) and at the free surface (\( n = 1, h = h_1 \)).

Furthermore, the shear stress at the boundaries is neglected, which results in the condition that the pressure at both sides of the interface is the same and at the free surface it is zero.

The equations (A.11) and (A.12) contain product terms of variables. To make an analytical solution possible the problem has to be simplified. In both layers a mean flow velocity is introduced and a wave with small amplitude is superimposed on that.
\[ u_n' = u_n - \bar{u}_n = \frac{\partial \phi_n'}{\partial x} \]
\[ w_n' = w_n = \frac{\partial \phi_n'}{\partial z} \]
\[ \phi_n' = \phi_n - \bar{u}_n x \]
\[ h_0' = h_0 - H \]
\[ h_i' = h_i - h_z \]

in which \( h_n \) is the thickness of layer \( n \) and \( H = h_1 + h_2 \).

The new variables \( u_n' \) and \( w_n' \) are the velocities due to the wave relative to the mean flow velocity. The new variables \( h_0' \) and \( h_i' \) are the vertical disturbances of the free surface and the interface relative to their mean position.

These new variables are substituted into the equations (A.10), (A.11) and (A.12). Product terms are neglected to linearise the problem.

\[ \frac{\partial^2 \phi_n'}{\partial x^2} + \frac{\partial^2 \phi_n'}{\partial z^2} = 0 \]  
(A.14)

\[ \frac{\partial \phi_n'}{\partial t} + \bar{u}_n \frac{\partial \phi_n'}{\partial x} + \frac{p_{n-1}'}{\rho} + gh_{n-1}' = 0 \]  
(A.15)

\[ w_n'(x, h_m, t) \approx \frac{\partial h_m'}{\partial t} + \bar{u}_n \frac{\partial h_m'}{\partial x} \quad m = n - 1 \text{ or } m = n \]  
(A.16)

At every boundary there is a kinematic boundary condition. Dynamic boundary conditions apply where the pressure is known, i.e. at the free surface (pressure is zero by assuming zero atmospheric pressure) and at the interface (pressure just below equals pressure just above).

kinematic \[ \frac{\partial \phi'}{\partial z} = \frac{\partial h_0'}{\partial t} + u_i \frac{\partial h_0'}{\partial x} \]

The free surface

dynamic \[ \frac{\partial \phi'}{\partial t} + u_i \frac{\partial \phi'}{\partial x} + gh_0' = 0 \]
kinematic just above \( \frac{\partial \phi'}{\partial z} = \frac{\partial h'}{\partial t} + u_1 \frac{\partial h'}{\partial x} \)

kinematic just below \( \frac{\partial \phi'}{\partial z} = \frac{\partial h'}{\partial t} + u_2 \frac{\partial h'}{\partial x} \)

The interface

\[
\rho_2 \left( \frac{\partial \phi'}{\partial t} + u_2 \frac{\partial \phi'}{\partial x} + gh' \right) - \rho_1 \left( \frac{\partial \phi'}{\partial t} - u_1 \frac{\partial \phi'}{\partial x} + gh' \right) = p_2 - p_1 = 0
\] (A.17)

The bottom

kinematic \( \frac{\partial \phi'}{\partial z} = 0 \)

By assuming sinusoidal waves a general solution can be expressed in sinusoidal wave equations as function of location \((x\text{ and }z)\) and time \((t)\):

\[
h_{n-1}'(x,t) = A_n e^{ik(x-t)}
\]

\[
\phi'_n(x,z,t) = [B_n \cosh k(z-a_2) + C_n \sinh k(z-a_2)]e^{ik(x-t)}
\] (A.18)

In which \( c \) is the propagation velocity of waves with wave number \( k \) (= 2\pi / wave length).

The propagation velocity and the coefficients \( A_n, B_n, \) and \( C_n \) have to be chosen such that the boundary conditions are fulfilled. Substitution of the general solution (A.18) into the boundary conditions (A.17) results in a homogenous system of equations for \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2. \)

Non-trivial solutions are only found if the determinant of the system is zero. This results in an expression for the propagation velocity:

\[
(c - u_1)^2 \left( c - u_2 \right)^2 - \frac{gT_2}{k} \left( c - u_1 \right)^2 - \frac{gT_1}{k} \left( c - u_2 \right)^2 +
\]

\[
\epsilon \frac{g^2 T_1 T_2}{k^2} + (1 - \epsilon)(c - u_1)^4 T_1 T_2 = 0
\] (A.19)

in which \( T_n = \tanh kh_n \) with \( h_n \) the layer thickness of layer \( n \) and \( \epsilon = \frac{\rho_n - \rho_2}{\rho_2} \) the relative density difference.

The wave propagation velocities can be found by neglecting terms with epsilon in (A.19) and assuming equal flow velocity for both layers \( u_1 = u_2 = u \). The solution is the same as the dispersion relation for a one layer system with a total depth of \( H \):

\[
c_e = u \pm \frac{g}{k} \tanh kH
\] (A.20)
The internal wave speed can be derived by assuming the propagation velocity, \( c \), and the flow velocities, \( u_1 \) and \( u_2 \), in the order of \( \sqrt{\varepsilon g H} \) and neglecting the terms of order \( \varepsilon^2 \):

\[
\frac{T_2 u_1 + T_1 u_2}{T_1 + T_2} \pm \sqrt{\frac{T_1 T_2}{(T_1 + T_2)^2} \left( \frac{g}{k} \varepsilon - (u_1 - u_2)^2 \right)}
\]

(A.21)

In absence of a difference in flow velocity the above assumptions are not needed. The expression for the propagation velocity (A.19) can be rewritten into a common quadratic equation in \( c^2 \), which results in two values for \( c^2 \): the larger belongs to external waves, the smaller belongs to internal waves.

\[
\left( 1 + (1 - \varepsilon)T_1 T_2 \right)(c - u)^4 - \frac{g}{k}(T_1 + T_2)(c - u)^2 + \varepsilon \frac{g^2 T_1 T_2}{k^2} = 0
\]

(A.22)

The propagation velocities have now been derived without the need of neglecting terms with \( \varepsilon \) or assuming the velocities in the order of \( \varepsilon g a \). With expression (A.22) the dispersion relation can be computed for different values of \( k \) and with different layer properties. For cases without a mean flow velocity this dispersion relation reads:

\[
\frac{\omega^2}{\omega_0^2} = \frac{(T_1 + T_2) \pm \sqrt{(T_1 + T_2)^2 - 4 \left( 1 + (1 - \varepsilon)T_1 T_2 \right) \varepsilon T_1 T_2}}{2 \left( 1 + (1 - \varepsilon)T_1 T_2 \right)}
\]

(A.23)

with relative density \( \varepsilon = \frac{\rho_2 - \rho_1}{\rho_2} \) and \( T_n = \tanh(kh_0) \).
Appendix B  Dispersion relations of discretised equations

The simulations showed that the model has a different dispersion relation in regard to the linear theory. It can be shown that the error is caused by the vertical discretisation by comparing the results of the computations with the theoretical dispersion relation of the vertically discretised equations.

In cases where the disturbance of the surface is small compared with depth and wavelength, the waves can be considered linear and the non-linear terms in the equations can be neglected. The governing equations reduce to:

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial q}{\partial x} + g \frac{\partial \zeta}{\partial x} = 0
\]

\[
\frac{\partial w}{\partial t} + \frac{1}{\rho} \frac{\partial q}{\partial z} = 0
\]  

\[
\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0
\]  

The last equations, the continuity equation, can be integrated over the whole depth. By assuming a flat bottom the depth-intergrated continuity equation reads:

\[
\frac{\partial \zeta}{\partial t} + \sum \Delta z \frac{\partial u}{\partial x} = 0
\]  

These equations are discretised for the vertical direction only. The variables are located as indicated in the figure below.

Figure 42: Locations of variables for one computational layer.

Because the horizontal momentum equation is evaluated between two pressure points, the average of the pressures above and below is used. The vertical pressure gradient is discretised using the Hansen scheme:

\[
\frac{\partial q}{\partial z} \bigg|_{k+\frac{1}{2}} + \frac{\partial q}{\partial z} \bigg|_{k-\frac{1}{2}} = 2 \frac{q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}}}{\Delta z_k}
\]

Which results in a linear system of three equations per computational layer and an additional depth averaged continuity equation:
Each variable can be written as a complex variable with a certain amplitude and phase. The amplitude can be taken complex to account for the phase difference:

\[
\begin{align*}
\hat{u}_k &= \hat{u}_k e^{i(kx - \omega t)} \\
\hat{w}_{k+\frac{1}{2}} &= \hat{w}_{k+\frac{1}{2}} e^{i(kx - \omega t)} \\
\hat{q}_{k-\frac{1}{2}} &= \hat{q}_{k-\frac{1}{2}} e^{i(kx - \omega t)} \\
\hat{\zeta} &= \hat{\zeta} e^{i(kx - \omega t)}
\end{align*}
\]  

At the free surface the pressure is zero by definition. By assuming a horizontal bed, the vertical velocity at the bottom is zero. The system in case of one computational layer therefore can be written as:

\[
\begin{bmatrix}
-i\omega & ik \frac{1}{2\rho} & ikg \\
-i\omega & -2 \frac{1}{\rho \Delta z} & e^{i(kx - \omega t)} \\
iki \frac{1}{\Delta z} & -i\omega
\end{bmatrix}
\begin{bmatrix}
\hat{u}_i \\
\hat{w}_i \\
\hat{\zeta}
\end{bmatrix}
= 0
\]

This system has only non-trivial solutions if the determinant of the matrix is zero, which results in the dispersion relation of the system:

\[
\omega = k \sqrt{\frac{g \Delta z}{1 + \frac{x}{2} k^2 \Delta z^2}}
\]

In the same way the dispersion relation with two computational layers of equal thickness, \(h\), is found to be:

\[
\omega = k \sqrt{\frac{gh \left(1 + k^2 h^2\right)}{16 + 6k^2 h^2 + \frac{1}{16} k^4 h^4}}
\]
Appendix C  Volume integration

Integration over the volume of a cell results in a discretised momentum equation, which states that the change of momentum of the cell is the change due to inflow and outflow plus the change due to the resultant force. In principle, it is back to the basic second law of Newton:

\[ \bar{F} = \frac{d\bar{m}v}{dt} = \bar{v} \frac{dm}{dt} + m \frac{d\bar{v}}{dt} \quad \text{(C.1)} \]

This is also the basis of the Navier Stokes equations. In each cell the mean concentration of momentum is the mean flow velocity in the cell:

\[ \bar{U} = \frac{1}{\rho V} \int \rho \bar{U} dV \quad \text{(C.2)} \]

The volume that enters and leaves the cell due to inflow, \( Q_{\text{in}} \), and outflow, \( Q_{\text{out}} \), has a certain concentration of momentum, \( U_{\text{in}} \) and \( U_{\text{out}} \). The total momentum of the cell changes because of this transport over the boundaries. The momentum of a cell changes also because of resultant force:

\[ \frac{d\bar{U}}{dt} = \bar{U}_{\text{in}} Q_{\text{in}} - \bar{U}_{\text{out}} Q_{\text{out}} + \frac{1}{\rho} \bar{F}_{\text{res}} \quad \text{(C.3)} \]

The change of volume of a specific cell due to the total inflow and outflow can be multiplied by the mean velocity of the cell and substituted in (C.3) after applying the product rule. Now the momentum equation for the cell is obtained:

\[ V \frac{d\bar{U}}{dt} = \frac{d\bar{U}V}{dt} - \bar{U} \frac{dV}{dt} = Q_{\text{in}} \bar{U}_{\text{in}} - Q_{\text{out}} \bar{U}_{\text{out}} - \bar{U} (Q_{\text{in}} - Q_{\text{out}}) + \frac{1}{\rho} \bar{F}_{\text{res}} \quad \text{(C.4)} \]

By assuming the concentration of the volume that leaves the cell to have the same concentration of momentum as the mean of the cell the outflow of momentum cancels out, which gives the differential equation for the change of mean velocity in the cell:

\[ \frac{d\bar{U}}{dt} - \frac{Q_{\text{in}} (\bar{U}_{\text{in}} - \bar{U})}{V} = \frac{1}{\rho} \sum \bar{F} \quad \text{(C.5)} \]

In the above expression \( Q_{\text{in}} \) is the inflow over the boundaries of the cell. The inflow has a certain concentration of momentum, \( U_{\text{in}} \), which is the mean concentration of momentum in the adjacent cell. Because the outward flux in the cells is not included in the equations, this method is not exactly momentum-conservative, but it is first-order accurate.