A Surrogate Model for Vortex Generator Flows

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by

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to obtain the degree of Master of Science at Delft University of Technology, to be defended publicly on Thursday July 25, 2019 at 13:30.

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Abstract

A surrogate model for a single vortex generator was developed. The surrogate model generates a source-term distribution to be added to the momentum Navier-Stokes equation. This model has two crucial characteristics: it is independent of the mesh and it provides a more accurate flow field than the well-established jBAY model.

The model was developed by first generating an optimal source-term field on a uniform grid without a vortex generator. This optimal source-term is referred to as the optimized source-term (OST), and is based on the work of Florentie et al. (2018) [15]. The OST solution uses the flow field of a vortex generator simulation with a body-fitted mesh (BFM) as objective. While the OST solution generates an accurate flow field, it must be modified to be grid-independent and suitable for training a surrogate model. This was done by generating a reduced order approximation of the OST solution using a set of basis functions. The shape factor and flow field of the reduced order simulation yielded very similar results to the OST solution, irrespective of the refinement of the mesh. This indicates that a smooth and reduced-order source-term is sufficient for modeling a vortex generator.

The choice of orthogonal basis functions implies that all the scaling coefficients are uncorrelated. This allows using a simple single-output surrogate method. Kriging was used as it is a predictor that yields the lowest expected L_2 error. The model was trained by generating reduced order approximations for 24 OST solutions. The inflow angle and Reynolds number for the 24 OST solutions were chosen using a Latin hypercube sampling plan. The output of the Kriging model is a set of coefficients that are used to scale their corresponding basis functions. The dot product of the coefficients and basis functions vectors is the source-term used to simulate the vortex generator.

The surrogate model was validated by removing a training case, and by subsequently testing the source-term at the same Reynolds number and inflow angle as that of the removed training case. The sensitivity of the surrogate model was also tested by using only 10 training cases. For both the full and partial training simulations, the shape factor and velocity profile downstream of the vortex generator showed a close agreement with the BFM results. The surrogate model was able to replicate the Reynolds stresses of the OST method, however the OST method itself is not yet capable at accurately predicting the Reynolds stresses in the boundary layer.

Acknowledgements

This thesis would not have been possible without the unwavering support of my supervisor, Dr. S. J. Hulshoff, whose advice steered me on the right path from day one. Thank you for your enthusiasm and persistence, which helped me achieve a high standard of work.

Similarly, this thesis would not have been completed without the essential work of Dr. L. Florentie, *Aspects of Source-Term Modeling for Vortex-Generator Induced Flows*. If the model presented in this thesis were an aircraft, then Florentie's algorithm would be the wings making it fly.

I would like to thank my family, whose unconditional support I will always be grateful for. Finally I would also like to thank my close friends Baris, Dionisios, Thibault and Ruben, without whom my time in Delft would not have been the same.

Jan Dierickx

Nomenclature

Acronyms & Abbreviations

- BAY Bender–Anderson–Yagle
- BFM Body-Fitted Mesh
- CFD Computational Fluid Dynamics
- Cov Covariance
- DOF Degrees of Freedom
- IM Identification Mesh
- HM Host Mesh
- VG Vortex Generator
- jBAY Jirasek-Bender–Anderson–Yagle
- OST Optimized Source-Term
- OSTA OST with cell selection A
- OSTB OST with cell selection B
- RQ Research Question
- SST Shear Stress Transport
- TI Turbulence Intensity

Greek Symbols

- α_i Local angle of attack
- β Inflow angle
- **Γ** Circulation
- δ Boundary layer thickness
- ε Error
- λ Lagrange multiplier
- ρ Air density
- ϕ_i Basis function with index *i*
- $\Omega_{VG} \quad \text{Domain with source term} \quad$
- ω Vorticity

Latin Symbols

- *a_i* Basis function scaling coefficient
- **b** Vortex generator spanwise vector
- *C* Covariance matrix
- E Expected value
- **F** Source-term field
- **F** Resultant force
- **f** Source-term field
- *i* Basis function index
- *i* Local cell index
- J Cost function
- *k* Wave number
- *H* Shape factor
- H Grid spacing
- *h* Vortex generator height
- *L* Lagrangian function
- N Number of cells
- *n* Polynomial order
- *n* Maximum basis function order
- **n** Vortex generator wall-normal vector
- P_n Polynomial of degree n
- *p* Pressure
- p_0 Order of convergence
- r Grid spacing ratio
- Re Reynolds Number
- *S* Manufactured solution
- t Vortex generator chordwise vector
- **u** Mean velocity field
- U_{∞} Inflow velocity
- V Volume
- w Test function
- *w* Kriging weight
- *x*_{VG} Streamwise location of VG
- \hat{Y} Predicted value

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Introduction

Vortex generators are small vanes typically located on the suction side of a lifting surface with the intended effect of delaying separation. In essence, separation occurs when the boundary layer lacks the momentum to overcome an adverse pressure gradient. The main benefit of vortex generators is to mix the lower boundary layer with higher momentum flows farther from the surface. This mixing effectively re-energizes the boundary layer, delaying separation. Vortex Generators are not only installed on aircraft wings but are also common on wind turbine blades, racing cars and turbofan casings. Typical installations on an aircraft and on a wind turbine blade are shown in Figures 1.1 and 1.2 respectively.

Since vortex generators are popular in aerodynamic applications, it is common to include them in CFD simulations. To this end, one either chooses to fully resolve the geometrical details of the vortex generators, or to approximate their effects through the use of a model. A new type of model is the purpose of this work. It is motivated by the current drawbacks of both fully resolved and modeled simulations.

In fully resolved simulations, the length scales of the vortex generator geometry typically differ by at least an order of magnitude from the surface it is positioned on (this can be seen in Figure 1.1). This substantially increases the required number of grid cells to perform the simulation, thereby leading to a high computational cost. The need to mesh the vortex generator also adds complexity, time and cost to the mesh generation process.

Alleviating the prohibitive requirements on fully resolved CFD is the purpose of simulations with a model. Since vortex generators are installed to delay flow separation, variables of particular importance include the shape factor and the velocity profile at some distance behind the vortex generator. While several models have been proposed since the 1990's, at this stage not a single model is adequate to reproduce those key outputs. Currently, the most commonly applied models in industry include the BAY [3] and jBAY [19] models. In addition to the problems already stated, the first of these has the added drawbacks of being highly grid-dependent, as well as needing calibration. These two issues were to some extent alleviated with the jBAY model. The jBAY model defines a narrow region where the source-term is applied and sets a fixed value for its calibration. Both the BAY and jBAY models enforce flow tangency in the vortex generator region on a coarse mesh. Enforcing tangency on a coarse mesh while trying to replicate the effect of a vortex generator that is an order of magnitude smaller is almost an impossible task. Neither a small vortex generator domain nor flow tangency are requirements for obtaining good flow field predictions downstream of the vortex

generator, so an entirely new approach should be considered.

An extremely promising new approach was developed in the paper by Florentie et al. (2018) [15]. This approach provides the main building blocks for this thesis and will be referred to in the remainder of this work as the Optimized Source-Term (OST) method. Instead of attempting to reproduce the small scale flow features created by the vortex generator, the OST method selects a domain surrounding the region where the vortex generator would be located, and generates a source-term to be added to the Navier-Stokes momentum equation. The source-term is generated from an existing fully resolved simulation, and the resulting output is optimized to yield a flow field closely matching the fully resolved simulation. The OST approaches the problem from a different perspective, namely instead of predicting the effect of a vortex generator, it simply evaluates the best possible approximation a model can generate, using an existing fully resolved solution. The results showed that the optimized source-term does not match that of jBAY, but is rather arbitrary and not connected to the vortex generator geometry.

While the results obtained by the optimized source-term approach were substantial, it did not provide a workable model. Each solution generated by the OST method is only valid for one single simulation, which needs its own fully resolved custom solution; the results cannot be used for another case, with, for example, a different inflow angle. A second problem with the OST method is that the source-term is defined individually for each cell. As each cell has a source-term in three spatial dimensions, this means that the implementation has numerous degrees of freedom, namely the number of cells in the vortex generator domain multiplied by 3. So, to turn the OST approach into a workable model, first, the number of degrees of freedom should be reduced, and second, the reduced-order solution should somehow be modified to allow for model inputs. This is effectively a surrogate model. In contrast to the OST method, it should not require a customized fully resolved solution, as this would defeat the purpose of a model





Figure 1.1: Vortex generator installed on the wing of a Boeing 787 aircraft [8].

Figure 1.2: Illustration of vortex generator benefits on a wind turbine blade [13].

Objective

Although the OST method showed very promising results, the previous paragraph clarified the issue that it cannot be used as a model. Creating a model based on the OST method requires two additional steps. These are summarized below:

1. The high number of degrees of freedom makes generating a model impractical. As an illus-

tration, for an incompressible, single vortex generator case, the OST method outputs 1300 three-dimensional source-terms. While it is possible to model these separately, it unjustifiably complicates the process. Consequently, a first step to creating a workable model is to develop a reduced-order approximation of the optimized source-term.

2. If a reduced-order approximation is successful at generating the correct flow field, then the second step is to model the remaining degrees of freedom. This model should use as inputs the Reynolds number (U_{∞}) and the inflow angle.

To solve step 1, popular order reduction techniques such as Proper Orthogonal Decomposition (POD) or Dynamic Mode Decomposition appear suitable. However, for step 2, the same modes must be used across a range of different cases. This means that the dominant structures found with a method such as POD would not apply from one case to the other. Instead of looking for the dominant modes, it is more coherent to instead to choose a fixed basis (such as Fourier modes or Legendre Polynomials), and observe how the scaling of these modes changes across different cases.

The degree of smoothness (a smoother solution implies a lower order solution) can be varied by adjusting the number of modes used to approximate the solution. Fewer modes are likely to generate a simpler model in step two. Furthermore if the chosen modes are orthogonal to each other, then the coefficients scaling them are uncorrelated. Since the modes are uncorrelated, each coefficient scaling those modes can be modeled individually. For this task, Kriging can be used to yield the lowest expected value of the L_2 error.

Outline

The required background to the motivation behind this thesis is explained in Chapter 2. This leads to a set of objectives and research goals, which are the subject of Chapter 3. The work done for this Master's thesis can be broadly divided in two parts, each part reflecting steps 1 and 2 in the previous paragraph: first verifying if a reduced-order model can reproduce adequate flow-field results (item 1), and then generating a model (item 2). This is the subject of Chapters 4 and 5 respectively. Each of these two Chapters begins with a methodology section, followed by their respective results.

2

Vortex Generator Modeling: Theoretical Background

The purpose of this Chapter is to outline the current state of the art for the modeling of vortex generators. In the introduction, a preview of source-term models was given. For a wider overview, a tree structure outlining the work done on modeling and analyzing vortex generator induced flows is shown in Figure 2.1. In the introduction, only source-term models were mentioned, which are the focus of this thesis. However it is of interest to show a preview of other available methods as well. These are briefly introduced in the paragraph below. A detailed analysis of source-term models is then the subject of the subsequent Section 2.1, and the the Optimized Source-Term (OST) method is explained in Section 2.2.

The first branch in Figure 2.1 is the highest level division on simulating vortex generators, which, in decreasing level of accuracy can be broadly divided in experimental tests, CFD simulations and CFD simulations with model. The latter can again be subdivided in two main methods.

The first considers a force term in a region Ω_{VG} close to the vortex generator. This is the same as a "source-term" which was discussed in the introduction, and is also the focus of this work. The most well known source-term approach is the model introduced by Bender, Anderson and Yagle (1999) [3], and later improved by Jirasek (2005) [19]. These are commonly known as the BAY and jBAY models, and will be referred to by these names throughout this text. Considerable research on these two models has been done so far (see tree branch "Performance assessment"). The main references for this thesis are by Florentie et al. (2017) [14] and Florentie et al. (2018) [15]. More particularly, the latter publication proved that a very accurate representation of the vortex by using a source-term distribution is possible. This very accurate representation is found by generating an Optimized Source-Term (OST) in the vortex generator domain. The OST method may be considered as the starting point for this thesis. Source-term models are also grid dependent, which also is an area of improvement considered for this thesis.

The second approach is to model the vortex profile directly. This may be done by theoretical means, such as the work of May (2001) [23], or by semi-empirical methods, such as the initial work of Bray (1998) [5] or Wendt (2001) [32]. The latter model was further investigated in [9]. Eriksson (2006) [31] used a different approach in incorporating the vortex profile in the governing equations. Instead of adding a source-term in the momentum equation, the vortex profile is added to the Reynolds stresses of the turbulence model.

Author	$U_{\infty} [{ m m/s}]$	β [°]	h/δ [-]	Turb.	Shape	Misc.
Yao et al. [33]	34	10, 16, 23	1/5	Exp.		Single
Yao et al. [33]	34	10, 16, 23	1.0	Exp.		Single
Allan et al. [1]	34	10, 23	0.31-0.21	SA/SST	\square	Single
Waithe [30]	34	10	2.27	SST	\square	Single
Jirasek [19]	34	23	0.16	EARSM		Single
Brunet et al. [6]	30	18	0.37	See ¹	\frown	Pair
Dudek [10]	35	16	0.2	SST		Single
Fernández et al. [11]	1.0^{2}	20	1	SST		Single
Florentie et al. [12]	34	10	0.23	SST	\square	Single
Baldacchino et al. [2]	15	18	0.25	Exp.		Pair
Poole et al. [27]	34	16	0.2	SA		Pair
Florentie et al. [14]	15	18	1/3	See ³		Pair
Florentie et al. [15]	34	16	1/5	SST		Single

Table 2.1: Overview of Test Cases in Literature. "Exp" Refers to a Wind Tunnel Test Instead of a CFD Simulation.

¹ The authors performed a very extensive turbulence model investigation, including SA, k- ω SST, EARSM, and also limited LES runs.

² As stated in the publication. However this number seems very low.

³ SA, $k - \epsilon$, SST, RST of [21] and [17].

Going through the tree structure in Figure 2.1, we now move back one branch and examine experimental techniques. A large variety of experimental cases can be found in literature. Brunet et al. (2002) [6] for example considered five pairs of vortex generators on a flat plate. Bray (1998) [5] also performed a very extensive study at both low and high Mach numbers flow. Yao et al. (2002) [33] performed a detailed PIV analysis on a single vortex generator. The experimental setup may be considered as consisting of micro vortex generators, where the vortex generator height is smaller than the boundary layer thickness.

This work is used by Florentie et al. (2018) [15] as reference geometry to test their OST method, and also as reference geometry in [1, 10, 30]. Florentie et al. (2018) [15] also tested their OST method on experimental results by Baldachino et al. (2015) [2], consisting of a contra-rotating vortex generator pair. In this dissertation, the experimental results of Yao et al. (2002) [33] are used as test case for the new model.

Table 2.1 shows an overview of vortex generator test cases (limited to vortex generators positioned on a flat plate). The k- ω SST turbulence model is most commonly used with vortex generator flows, and is known to provide the best results in flows with separated regions.

2.1. Source-Term Models

Source-term models are the most common type of vortex generator models. The common trait of these models is that they add a source-term in the momentum equation (and energy equation, in case of compressible flow), in a certain domain around the vortex generator. A source-term model defines the magnitude, distribution, and direction in this domain. The most common methods are the BAY and jBAY models.

Vortex Generators Simulations Experimental _ Bray (1998) [5] _ Yao et al. (2002) [33] Brunet et al. (2006)[6] Baldacchino et al. (2015) [2] Fully gridded CFD Allan et al. (2002) [1] _CFD with Model _ Force term model BAY/jBAY Model Bender et al. (1999) [3] Jirasek (2005). [19] Performance assessment Waithe (2004). [30] _Brunet et al. (2006) [6] Iannelli et al. [18] Booker et al. (2009) [4] _ Dudek (2011) [10] _ Joubert et al. (2011) [20] _ Fernández et al. (2012) [11] Florentie et al. (2014) [12] Manolesos et al. (2016) [22] Florentie et al. (2017) [14] **Other Forcing Logic** Phenomenological Improvements Poole et al(2016) [27] Low aspect ratio correction Leading edge suction analogy Polhamus (1966) [26] Goal Oriented Optimization (OST method) Florentie et al. (2018) [15] This thesis: reduced-order approximation ____ This thesis: surrogate model _ Modeling of Vortex Profile Bray (1998) [5] May (2001) [23] Wendt (2001) [32] Dudek (2006) [9] Sofia Moreira Ribeiro (2017) [28] **Reynolds Stresses** ____ Törnblom and Johansson (2007) [29]



2.1.1. BAY Model

Bender et al. (1999) [3] developed the original BAY model with the aim of simulating the effects of vortex generators by adding a source-term to the momentum equation. Instead of modeling the vorticity directly such as in [5] or [32], the BAY model aims to generate vorticity by adding the following source-term to the momentum equation:

$$\mathbf{f}_{i} = cA \frac{V_{i}}{V_{\text{tot}}} \rho \left(\mathbf{u}_{i} \cdot \mathbf{n} \right) \left(\mathbf{u}_{i} \times \mathbf{b} \right) \left(\frac{\mathbf{u}_{i}}{|\mathbf{u}_{i}|} \cdot \mathbf{t} \right)$$
(2.1)



Figure 2.2: Vectors along vortex generator, as used in Equation 2.1 [14].

From Equation 2.1, it seems as if the model is independent of the inflow angle β . However the vortex generator orientation is described in the unit vectors **b**, **t** and **n**. These vectors are illustrated in Figure 2.2. *A* is the vortex generator planform area, $\frac{V_i}{V_{tot}}$ is the ratio of the local cell volume over the total volume where the model is applied, ρ is the air density and **u**_i is the local velocity vector.

Equation 2.1 is derived by first defining the lift force as:

$$\mathbf{f}_{i} = cA \frac{V_{i}}{V_{\text{tot}}} \alpha_{i} \rho |\mathbf{u}_{i}|^{2} \mathbf{l}$$
(2.2)

Where α_i is the local angle of attack (note that this is not the same as the inflow velocity β), **l** is the unit vector defining the direction of **f**_{*i*}, in this case written as:

$$\mathbf{l} = \frac{\mathbf{u}_i}{|\mathbf{u}_i|} \times \mathbf{b} \tag{2.3}$$

If the calibration constant *c* is sufficiently large, the local velocity vector becomes aligned with the vortex generator, and this means that the vector **l** is normal to the vortex generator surface. The angle of attack can thus be written as follows:

$$\alpha_i \approx \sin \alpha_i = \frac{\mathbf{u}_i \cdot \mathbf{n}}{|\mathbf{u}_i|} \tag{2.4}$$

Finally a term is added to model the loss of lift at high angles of attack:

$$\frac{\mathbf{u}_i}{|\mathbf{u}_i|} \cdot \mathbf{t} \tag{2.5}$$

These terms can then be substituted into Equation 2.2:

$$\mathbf{f}_{i} = cA \frac{V_{i}}{V_{\text{tot}}} \frac{\mathbf{u}_{i} \cdot \mathbf{n}}{|\mathbf{u}_{i}|} \rho |\mathbf{u}_{i}|^{2} \left(\frac{\mathbf{u}_{i}}{|\mathbf{u}_{i}|} \times \mathbf{b}\right) \left(\frac{\mathbf{u}_{i}}{|\mathbf{u}_{i}|} \cdot \mathbf{t}\right)$$
(2.6)

Which simplifies as Equation 2.1.

2.1.2. Improved BAY Model: the jBAY Modification

Reference [19] developed an improved model that would somewhat alleviate two major drawbacks of the BAY model:

- The BAY model does not provide clear guidelines on defining the vortex generator region, $\Omega_{\text{VG}}.$
- Dependence on calibration constant *c*.

The jBAY model replaces the vortex generator by a zero thickness plane at the location of the vortex generator centerline. The intersection of this plane with the cell edges is then considered as the location where the model is applied. The flow quantities at these points are interpolated from the grid nodes, and the resulting source-term is then redistributed to the neighboring grid nodes as per the original BAY model. This means that Equation 2.1 is left unchanged. The improved formulation imposes flow tangency at a consistent location (as opposed to the BAY model), which removes a source of grid dependence. This modification shows clear improvements over the original BAY model, as demonstrated by the results obtained by Florentie et al. (2017) [14].

According to Jirasek (2005) [19] the new formulation eliminates the dependence on the calibration constant c, as each vortex generator is defined by local control points. This is to some extent misleading, however, as by independent of c, the authors simply set at a constant value (c = 10) for all test cases. This does not mean that the constant is not present anymore, but merely that a single value is chosen.

Figure 2.5 illustrates the difference between using the BAY and jBAY cell selection, where the jBAY model has additional cells selected such that there always one cell center on each side of the vortex generator.



(a) BAY model.

(b) New method proposed in [18].

Figure 2.3: Cell selection procedure according to [18].

2.1.3. Cell Selection

The force term defined by Equation 2.1 should only be applied in a certain domain in the vicinity of the vortex generator. However the definition of this region and the cell selection procedure varies in literature. The original publication of [3] claims that the source-term should be applied in "a small group of cells in the grid that encloses the vortex generator to be modeled". Meanwhile [19] interprets this as selecting the points laying inside the vortex generator, which is then claimed to be a challenge in case of very thin vortex generators. According to the same authors, a force term model should be applied to a small portion of the grid to improve stability. Later [14] mentioned that the BAY model should be implemented with source-terms at several rows corresponding to the location of the vortex generators. Again this is in some measure arbitrary, and according to the same authors the number of rows can be altered to calibrate the model. In the original publication of the BAY model, the authors went even further and postulated that multiple vortex generators may be modeled by simply selecting a single region encompassing all vortex generators. Even though the individual vortices are not resolved, it was argued that the overall effect on the secondary flow is modeled. Figure 2.4 shows an example in three dimensions for the cell selection for the BAY model [10]. The region is simply a rectangular region encompassing the vortex generator which is itself at an inflow angle of 16°.

Another variation of the cell selection procedure is shown in Figure 2.3 [18]. Comparing Figures 2.5 and 2.3a one can notice the large variations in the cell selection methods for the BAY model. Using Figure 2.3, Ianelli et al. (2006) have argued that the original BAY model applies the source-term in an "axis aligned bounding box" (see Figure 2.3a), while the new proposition applies the model in an "object oriented bounding box" (Figure 2.3b) [18]. By selecting the cells in a way that is oriented with the vortex generator, the geometry is better represented. However, at this stage, the added value of this procedure compared to the jBAY model is unclear.



Figure 2.4: Example of cell selection for the BAY model in 3 dimensions [10]. The region has a rectangular shape with grid dimensions of $22 \times 9 \times 36$, encompassing the vortex generator at $\beta = 16^{\circ}$.



Figure 2.5: Cell selection for the BAY and jBAY model [14]. The BAY model cells are shown by the full cells while the additional cells for the jBAY model are illustrated by the striped pattern.

2.1.4. Review of Force Term Direction and Magnitude

This Section elaborates on a recent publication by Florentie et al. [14]. This work provides another level of analysis of the BAY and jBAY models and as such deserves a section of its own. This publication was the first to thoroughly investigate the resulting total force of the BAY and jBAY models. Several force configurations were tested, but no analysis was done on the actual required distribution of the force term. This was the subject of a second publication (the OST method, [15]), where the optimal force distribution was found with an adjoint optimization process. The OST method is the subject of Section 2.2.

In their first publication, Florentie et al. considered six different approaches to model the force of the vortex generator.

- 1. Body fitted mesh without model. This is the benchmark result for a any simulation and allows to isolate the model errors from other errors, such as experimental measurement errors or errors arising from the turbulence model.
- 2. The original BAY model, with cell selection as shown in Figure 2.5.
- 3. The jBAY model.
- 4. A variation of the BAY model, where the total force is equal to the one of the BAY model, but where the force distribution is uniform. This is mathematically formulated as:

$$\mathbf{f}_{i}^{UB} = \frac{V_{i}}{V_{\text{total}}} \mathbf{F} \quad \text{with} \quad \mathbf{F} = \sum_{i=1}^{N} \mathbf{f}_{i,\text{BAY}}$$
(2.7)

The main difference compared to the BAY model is that the force is no longer focused on the leading edge of the vortex generator.

5. Same approach as 4, but with the total force being computed from the fully gridded solution rather than the BAY model. This means that the magnitude and direction of the model are an exact match to the body fitted mesh solution. This can be written as:

$$\mathbf{f}_{i}^{UE} = \frac{V_{i}}{V_{\text{total}}} \mathbf{F} \quad \text{with} \quad \mathbf{F}_{\text{exact}} = \iint_{S_{\text{VG}}} \tilde{p} \mathbf{n} dS$$
(2.8)

Where \tilde{p} is the pressure and shear stress on the vortex generator surface from the body-fitted mesh and **n** is the wall-normal direction.

6. This solution is the closest physically matching source-term distribution that can possibly exist. The approach is to first compute the fully gridded solution for the force distribution on the vortex generator, and map this force as closely as possible to the cells where the BAY model would normally be used. The force for every cell is computed with:

$$\tilde{\mathbf{f}}_{i}^{E} = \iint_{S_{i}} \tilde{p}_{i} \mathbf{n}_{i} dS \tag{2.9}$$

Which is then corrected for interpolation errors with:



Figure 2.6: Vortex generator resultant force [14].

Figure 2.6 shows an illustration of the direction of the force term for a vortex generator on a flat plate and an airfoil. Considering the flat plate case, all models under-predict the resultant force. This is especially the case for the BAY model for which the force is 26% less than the gridded solution. This error reduces to 14% for the jBAY model. This under-prediction cannot be attributed to the absence of wall shear-stress modeling by the BAY model, as the force deficit for the inviscid simulation is less compared to the modeled cases. However the lack of viscous modeling by the BAY/jBAY model does have a very strong influence on the relative deviation in the force component parallel to the vortex generator surface. The model is unable to deal with crossflow and vortex roll-up. The latter phenomenon is especially important for low aspect ratio geometries, and, for example, was tackled in [27] by using Polhamus' theory [26] in their physics improved model. The influence of these secondary components is visible in Figure 2.7, where rows 5 and 6, which have an exact resultant force term, are the only two models capable of displaying the uplifting of the vortex from the surface. This effect is also visible in the shape factor profile (see Figure 2.8), where the peak locations are closer to the ones obtained with the body fitted mesh.

Although method 6 is a perfect mapping of the force field of a fully resolved mesh, results show that there is room for further improvement. The best possible solution on a coarse mesh is obtained using the OST method, and is the subject of the next section.



Figure 2.7: Flowfield by Florenie et al. (2017) [14] for a flat plate case. Row 1: experimental data, row 2: body fitted mesh, row 3: BAY model, row 4: uniform BAY model, row 5: distributed F_{exact} , row 6: uniform F_{exact} .



Figure 2.8: Shape factor profile for three different downstream stations [14]. From left to right: $\Delta x/h = 5$, $\Delta x/h = 10$, and $\Delta x/h = 15$.

2.2. Goal Oriented Optimization: the Optimized Source-Term Method

Up to this point, this chapter discussed the details and results vortex generator models. The OST method instead uses the results of a fully resolved simulation, and generates the force field that best matches that of the fully resolved simulation. Since it requires a fully resolved mesh to work, it is not a model; however it gives useful insights and how a model should behave.

For modeling vortex generators, Florentie et al. (2018) [15] identify three quantities of interest: vorticity, (location and value of) peak vorticity, circulation and the shape factor. All these variables are a function of the (mean) velocity field, hence the OST method optimizes a source-term that minimizes the L^2 -norm of the deviation in the velocity field.

$$J(\mathbf{u}) = \int_{\Omega} |\mathbf{u} - \tilde{\mathbf{u}}|^2 d\Omega$$
(2.11)

Where $J(\mathbf{u})$ is the functional to be minimized, and $\tilde{\mathbf{u}}$ is a reference high fidelity solution (in this case the projection of the body fitted mesh solution onto the coarse mesh used for the simulation with a source-term). Using Equation 2.11, the optimal source-term distribution is found with:

$$\mathbf{f}^* = \operatorname*{argmin}_{\mathbf{u}} J(\mathbf{u}) \quad \text{subject to} \quad \mathbf{R}(\mathbf{u}, p, \mathbf{f}) = 0$$
 (2.12)

Where $\mathbf{R}(\mathbf{u}, p, \mathbf{f}) = (\mathbf{R}_u, R_p)^{\mathrm{T}}$ is the state equations and boundary conditions to be satisfied, namely the Reynolds Averaged Navier-Stokes Equations. R_p is the continuity equation and \mathbf{R}_u represents the momentum equation:

$$\mathbf{R}_{u} = (\mathbf{u} \cdot \nabla) \,\mathbf{u} + \nabla p - \nabla \cdot (2\nu D \,(\mathbf{u})) + \mathbf{f}$$
(2.13)

The source-term **f** is what the solution is optimized for, more specifically this is done by defining **f** as:

$$\mathbf{f} = C\mathbf{f}_0 \quad \text{with} \quad C = \text{diag}(\mathbf{c}) \tag{2.14}$$

C is a coefficient matrix used to vary the source-term in the cells. \mathbf{f}_0 is an initial estimate for the source-term. This source-term is only nonzero in Ω_{VG} :

$$\mathbf{f}_0 = \begin{cases} \mathbf{F}_0 / V_{\text{tot}} & \text{in } \Omega_{\text{VG}} \\ 0 & \text{in } \Omega \backslash \Omega_{\text{VG}} \end{cases}$$
(2.15)

Where V_{tot} is the total volume of Ω_{VG} . The optimized source-term is found by varying the vector **c**. According to Florentie et al. (2018) this indirect approach (instead of directly varying **f**) prevents problems arising from poor scaling of the system. A detailed explanation on how to obtain the vector **c** is available in [13].

2.2.1. Optimized Source-Term Results

The optimization problem above finds an optimum for the source-term in a domain restricted to Ω_{VG} , which automatically places an upper limit on the achievable accuracy. Theoretically, a force term could be applied in the entire computational domain, however this would be impractical in the scope of vortex generator models. Instead, two types of domains are used, shown in Figure 2.9. Two selection methods are used, labeled as OSTA (Optimal source-term A) and OSTB (Optimal source-term B). The shape of these two regions is similar to earlier work in [18], shown in Figure 2.3, where method A is labeled as an "axis aligned bounding box" and method B as an "object oriented bounding box". Theoretically OSTA should yield the best results, as it allows the most degrees of freedom for **f** in a larger Ω_{VG} , which allows compensating for numerical diffusion and is less related to the vortex generator geometry.





(a) Cell selection method A, rectangular domain encompassing the vortex generator.

(b) Cell selection method B, domain with cells aligned with the vortex generator.

Figure 2.9: Region of source-term optimization [15].

Two test cases are considered, the first consisting of a single sub-boundary layer vortex generator as defined by Yao et al. (2002) [33] with $h/\delta = 1/5$, an incidence angle of $\beta = 16^{\circ}$, and a freestream velocity of $U_{\infty} = 34$ m/s. This setup is used in various papers, such as [1], [30] and [10]. The second test case consists of the setup by Baldachino et al. (2015) [2]. This setup consists of a flat plate with submerged counter-rotating rectangular vortex generators. The vortex generator height is given as $h/\delta = 1/3$, the inflow angle is $\beta = 16^{\circ}$, and the free stream velocity $U_{\infty} = 15$ m/s.

For both cases, a source-term that significantly reduces the deviation $J(\mathbf{u})$ (see Equation 2.11) is obtained. For the single VG case, three meshes are considered. The first mesh is the body fitted mesh (i.e. fully resolved), which is assumed as the best possible solution with 1.7E6 nodes. The two other meshes are the so-called "coarse" meshes. These meshes are referred to as mesh M1 and M2, with 0.1E6 and 0.4E6 nodes respectively. Mesh M0 is the lowest mesh refinement. The double vortex generator case also uses a similar naming convention. An overview of the meshes is shown in Table 2.2.

	Single VG					VG Pair			
	N	N_z	<i>J</i> (u) (OSTA)	<i>J</i> (u) (OSTB)	N	N_z	<i>J</i> (u) (OSTA)	<i>J</i> (u) (OSTB)	
BFM	1.7E6	96	-	-	4.9E6	176	-	-	
M0	-	-	-	-	0.1E6	15	3.84E-6	3.89E-6	
M1	0.1E6	25	2.78E-5	3.01E-5	0.4E6	30	2.63E-6	6.68E-6	
M2	0.4E6	50	4.93E-5	5.54E-5	1.6E6	60	-	1.12E-5	

Table 2.2: Meshes used in the by Florentie et al. (2018) with the OST method [15] with their corresponding objective function values. N_z is the number of cells in the vertical direction, from the flat plate surface to the top of the domain.

On the single vortex generator case, the value of the objective function reduces from 1.34E-4 (using jBAY) for mesh M1 to 2.78E-5 for OSTA and 3.01E-5 for OSTB, also on mesh M1. These results yield two important conclusions: first this is almost an order of magnitude reduction of the objective function. Considering $J(\mathbf{U})$, this clearly supports the authors' claim that a source-term yielding a significant reduction in flow deviation is achievable. A second key observation is that OSTA consistently yields better results than OSTB. The better results of OSTA, combined with the fact that the larger region allows for a smoother reduced-order function, indicates that it is the best choice for the surrogate model.

The second test case (VG pair) also shows a clear improvement in the objective function. The authors mention that for OSTB, $J(\mathbf{U})$ increases with mesh refinement. This, of course, is counter-intuitive as mesh refinement should improve the results. This is explained by the fact that for OSTB,

upon mesh refinement, $J(\mathbf{U})$ is evaluated in a larger region. With increasing mesh refinement, Ω_{VG} is reduced and confined closer to the vortex generator. Meanwhile the region where $J(\mathbf{u})$ is evaluated increases in volume, and is closer to the vortex generator surface where a boundary layer is present, which is not well resolved by the coarse meshes. To avoid issues similar to those arising with the OSTB cell selection method, for the surrogate model proposed in this dissertation the consistent and fixed OSTA region is used for Ω_{VG} .

The improvements in the objective functions are clearly reflected in the velocity profile plots downstream of the vortex generator, shown in Figure 2.10. Even the coarsest mesh M0, on the more complex vortex generator pair case, shows a solution much closer to the projected body fitted mesh. Results are improved when moving further downstream, both for the jBAY and the optimized solution.



Figure 2.10: Streamwise and rotational velocity downstream of vortex generator pair case [15]. The crossflow location is given by the trailing edge of the vortex generator, $y = y_{\text{te}}$. From left to right: $\Delta x/h = 5$, $\Delta x/h = 10$, and $\Delta x/h = 15$.

The error in the shape factor (Figure 2.11) also shows improved results, especially when compared to the jBAY model. The shape factor is an integral quantity of the velocity field, so an improved velocity indeed results in an improved shape factor. The paper does not show any peak vorticity results, although peak vorticity results are commonly shown in other publications in the field [1, 10, 22]. However predicting peak vorticity with a coarse mesh (typically used with a source-term model) is very difficult, as it is an inherently small scale feature of the flow. This is further elaborated in Section 3.1.



Figure 2.11: Mean deviation in shape factor from BFM result downstream of vortex generator [15]. The error is normalized with with the deviation in *H* for an undisturbed boundary layer.

2.2.2. Characteristics of the Source-term

The aim of the thesis is to somehow provide a model leading to an accurate flow field solution. This model is to operate by finding an appropriate function for the forcing term **f** within Ω_{VG} . As such, Figures 2.12 and 2.13 are very useful as indications for the form of that function. The color map seems to indicate that, although the force is applied in a relatively large square shaped Ω_{VG} , the force components with the highest magnitude seem to be located close to the position of the actual vortex generator location. The force in the *x*-direction seems to be relatively spread out along the vortex generator, with the magnitude increasing towards the trailing edge of the vortex generator. In the *y* direction the force is more concentrated on the leading edge of the model. This is in line with the BAY model, which concentrates the force on the leading edge as well.

Figure 2.13 shows the resultant force for OSTA and OSTB on meshes M0 and M1. A first observation is the large difference between the OSTA and OSTB vector. This difference is mainly in magnitude for M0 and in direction for M1. This difference may be explained by the fact that for OSTA the domain is large. Hence a force may be present in cells which are relatively distant from the vortex generator, which does not allow for local physical interpretations.



Figure 2.12: Color map of source-term magnitude in x-, y-, and z- direction for a plane located at z = h/2 [15]. Due to the sign of **f**, the displayed effect of the body forces act in opposite direction as its effect on the velocity field.

Overall, while Figures 2.12 and 2.13 shed light on the force distribution and magnitude, these few data points are clearly not sufficient to attempt building a new model, and many more runs are likely to be needed in order to extract enough information. Furthermore, the force arrows and directions seem to indicate that finding an accurate resultant force is not necessary. This is also deduced by the authors of the OST method, who claim that "aiming for an exact representation of the resultant force is not the optimal approach when using an under-resolved mesh".



Figure 2.13: Illustration of magnitude and direction of source-term forcing [15].

The results obtained with the OST method should be compared with the results obtained in their earlier publication [14], where the conclusions are drawn from a different perspective. The authors first computed the reference force which was then applied in the vortex generator domain (Ω_{VG}). Using this reasoning, a clear improvement was obtained, especially by better modeling the uplifting of the boundary layer. However in their second publication, the authors first found the optimal solution, and from that solution concluded that the force does not need to match the true force from the gridded solution. From this we may conclude that even though improving the resultant force does improve the solution when compared to the jBAY model, an accurate resultant force is not a necessity for accurately representing the effects of the vortex generator downstream.

3

Thesis Objective

Figure 3.1 shows an overview of the error sources that arise when performing vortex generator CFD simulations. Traceability of the error source is important to understand where further work can add value. Trying to improve RANS errors would imply modifying or creating a new turbulence model and is out of the scope of this work. Discretization error is inherently grid-dependent, and is especially of interest since the jBAY model is typically applied to coarser meshes than its fully gridded counterpart. The jBAY model is not consistent when applied on different grids. This suggests the need for a grid-independent model in order to better differentiate discretization and vortex generator model error. The last major source of error is related to the vortex generator model itself. This is the error source that this dissertation is attempting to reduce using a surrogate model.

Section 3.1 uses an end goal approach to understand which variables are of interest in the scope of vortex generator simulations. This is followed in Section 3.2 by the research objective and the research questions of this thesis. Section 3.3 provides a high level methodological overview of the steps involved in addressing these research questions. An overview of the experimental setup is given in Section 3.4.

3.1. Variables of Interest

The literature suggests that the flow field **U**, vorticity and peak vorticity are paramount in understanding and assessing the quality of a vortex generator simulation. For the scope of this work, the variables of interest are established through the following two questions.

- 1. What are the relevant scales for modeling vortex generators? And are those scales compatible with the coarse grids used with vortex generator models?
- 2. What is the end goal of performing a vortex generator simulation, and which variables are needed to achieve this goal?

For question one, three length scales are considered:

• Largest geometrical feature of the vortex generator, i.e., the vortex generator height or chord. This length scale is typically of the same magnitude as the boundary layer thickness.



Figure 3.1: Error Source Diagram [14].

- Minimum grid size to resolve the flow this is somewhat vague and is elaborated upon below.
- Smallest geometrical feature of the vortex generator, i.e., the vortex generator leading edge radius.

The second and third bullet points are somewhat related depending on the variable of interest, and are of particular importance for predicting peak vorticity. The general consensus is that peak vorticity is under-predicted [1, 10, 14, 22]. More particularly, mesh refinement at the vortex generator tip showed somewhat improved results [10]. This implies that peak vorticity is inherently a small scale feature of the flow, and attempting to obtain accurate peak vorticity results on a coarse mesh combined with a model is not pertinent to the problem of modeling vortex generators. Instead integral properties of the flow that are less mesh dependent are more useful as comparison tools with fully gridded simulations. Alternatively, one may only consider the variables needed to predict separation. An obvious candidate would appear to be the downstream shape factor, since it gives an indication of the flow separation, and also is an integral property of the flow.

As to question two, the end goal of installing vortex generators is to delay flow separation. In the scope of RANS simulations, separation is modeled and predicted by the turbulence model, so in order to have an accurate prediction of separation, the inputs for the Reynolds stress model should be error-free. This suggests that computing the correct velocity field **U** and Reynolds stresses $\overline{u'_i u'_j}$ are the two variables of interest for this work. At this stage RANS models are not entirely adequate in cases involving separation. Assuming, however, that a near perfect RANS turbulence model would be available, a vortex generator model predicting **U** and $\overline{u'_i u'_j}$ accurately would be paramount for computing correct separation behavior. The hypothesis explained in this paragraph is illustrated in Figure 3.2, which shows an airfoil at high angle of attack with separated flow. Following the flow from left to right, first a boundary layer is formed along the smooth airfoil upstream of the vortex generator height. A vortex is generator, the boundary layer is typically thicker than the vortex generator height. A vortex is generated, and further downstream, for illustration purposes, the flow is separated. In order to predict this separation, a plane of interest is downstream of the vortex generator

and upstream of the separation point. At that plane the flow field and the Reynolds stresses should be accurate in order to obtain reliable separation results.



Figure 3.2: Relevant variables for separation prediction. At a location in front the separated area, an adequate model should supply the correct flow field and Reynolds stresses.

3.2. Research Objectives and Research Questions

The arguments summarized in the previous section lead to the following research objective:

The research objective of this thesis is to design a model yielding an improved flow field compared to the jBAY model by generating a reduced-order and grid-independent source-term in the vortex generator domain.

The OST method showed that the traditional approach of adding a source-term in a restricted domain, with a force term that matches the local physics of a fully resolved simulation is not a requirement for an effective model. Therefore this thesis deviates from this classical approach by answering the following two research questions:

RQ 1: Can a reduced-order source-term, defined by a set of assumed modes, generate a flow field similar to the discrete optimized source-term obtained by Florentie et al. (2018) [15] ?

- How effective is the reduced-order solution at generating an accurate flow field, shape factor and Reynolds stresses in the region downstream of the vortex generator and upstream of the point of separation?
- What is the reduction in degrees of freedom?

A positive outcome for the questions defined above is a requirement to solve the second part of this thesis, namely to use the reduced-order solution to make a surrogate model.

RQ 2: Are the amplitudes of the aforementioned assumed modes tunable for a set of input variables, leading to a new vortex generator model?

3.3. Methodology

This section aims to provide a high level overview of the steps to solve RQ1 and RQ2, shown in Figures 3.3 and 3.4 respectively.

In Figure 3.3, the discrete solution, $\mathbf{F}_{optimal}$ obtained from the OST method, is mapped onto a set of smooth modes ϕ , leading to a source-term function $\hat{\mathbf{F}}$. These modes are not defined on the grid nodes, but based on a local coordinate system. Using $\hat{\mathbf{F}}$, the simulations are run and particular attention is paid to two inputs:

- 1. The variation of the flow field with *n*.
- 2. The variation of the flow field with the source-term applied on different grids. The modes are defined on local coordinates instead of the grid nodes. This means that the reduced-order source-term is itself grid-independent. However the resulting flow field might vary if the reduced-order source-term is applied on different grids.



Figure 3.3: Diagram outlining procedure to solve research question one.

The diagram above only addresses RQ1. In order to build a model and answer RQ2, the reducedorder solution \hat{F} must be generalized to vary with respect to input variables. For the model, the input variables are chosen as inflow angle, β , and the vortex generator height-based Reynolds number, Re. This is done by modifying \mathbf{a}_i as \mathbf{a}_i (β , Re).

$$\hat{\mathbf{F}}_{\text{model}} = \hat{\mathbf{F}}_{\text{model}}\left(\beta, \text{Re}\right) = \sum_{i=0}^{N} \mathbf{a}_{i}\left(\beta, \text{Re}\right)\phi_{i}$$
(3.1)

In order to model how \mathbf{a}_i varies with model input variables, (β , Re), it is necessary to run a fully resolved simulation and an adjoint optimization for various cases (details on the sampling method are explained in Chapter 5). Using the obtained data from the runs, the surrogate model can output the required scaling variables \mathbf{a}_i . Using \mathbf{a}_i , the source-term can be reconstructed and used on a design (coarse) mesh simulation, which is the solution of the model.



Figure 3.4: Diagram outlining procedure to solve research question two.

3.4. Experimental Setup

The test case consists of a single vortex generator on a flat plate based on the experimental setup of Yao et al. (2002) [33]. Table 3.1 shows an overview of the main parameters used. The single vortex generator geometry avoids modeling complex vortex interaction effects and simplifies the problem. With a height of 35 mm, the vortex generator is completely submerged in the boundary layer, with a vortex generator height to boundary layer thickness ratio of 1/5. The freestream velocity is given as 34 m/s, yielding a vortex height based Reynolds number of 16400. All simulations are performed with the OpenFOAM CFD package, solving the incompressible, steady RANS equations using the SIMPLE algorithm. The $k - \omega$ SST turbulence model [24] is used for all simulations. This turbulence model is known to provide better results for flows with separated regions. The $k - \omega$ SST model also consistently provides the closest match with experimental data across literature [1, 12, 30]. A fully turbulent inflow profile was set up with a turbulence intensity of 1%. The inflow profile is obtained from a separate OpenFOAM simulation, which itself assumes free-stream values of the turbulent kinetic energy and specific dissipation rate of $k = 0.1734 \text{ m}^2/\text{s}^2$ and $\omega = 54.08 \text{ 1/s}$. Scaling these two parameters with respect to the free-stream velocity has a negligible effect on the boundary layer profile ($\Delta u/U_{\infty} = \mathcal{O} (10^{-3}, 10^{-2})$) and are thus kept constant across the range of inflow velocities.

Four grids are used throughout this work. The meshes are the same as the ones that were used for the OST method [15], and are labelled as BFM, M1, M2 and M3. BFM stands for body-fitted mesh, and is used to generate the reference solution throughout this text. M1, M2 and M3 are used as coarse meshes to test the reduced-order approximation and the surrogate model. As such they are uniform and do not contain the vortex generator geometry. The BFM mesh is shown in Figure 3.5. To train the surrogate model, twenty-three body-fitted meshes were created based on the reference BFM, with each mesh having a different inflow angle. An example of such a modified mesh, with an inflow angle of $\beta = 5.8^{\circ}$ is shown in Figure 3.6.

Validation of the Body-fitted Mesh

The body-fitted mesh used throughout this work is based on the work of Florentie (2018) [13], who validated the discretization by means of a Richardson's extrapolation [7] and by comparing the results with the experimental data of Yao et al. (2002) [33]. The Richardson's extrapolation was performed by using three body-fitted meshes with 1.2E6, 3.3E6 and 8.9E6 cells [13]. A maximum discretization error of 0.74% was found for the mesh with 1.2E6 cells. Throughout this work, the slightly more refined body-fitted mesh as shown in Table 3.2 is used, with 1.7E6 million cells.

For this study the geometry was modified in order to train the model. Simulations were performed at the same or lower Reynolds number as the default used in [15]. Similarly, simulations with a different inflow angle never exceeded the angle of 16° found in [15]. With smaller Reynolds

Parameter	Symbol	Value		
	0,110,01	Original Case [15]	Model	
Freestream Velocity	U_{∞}	34 m/s	10 - 34 m/s	
Turbulence Intensity	TI	1%	1%	
Boundary Layer Thickness	δ	35 mm	35 - 44 mm	
Vortex Generator Height	h	$\delta/5 \mathrm{mm}$	$\delta/5\mathrm{mm}$	
Inflow Angle	β	16°	5 - 16°	
VG Height based Reynolds Number	Re	4800-16400	16400	
Domain Length	L	600 mm	600 mm	
Domain Height	Н	105 mm	105 mm	
Domain Width	W	150 mm	150 mm	
Streamwise Location of Vortex Generator	x_{vg}	14.5h	14.5h	
Turbulence Model	-	$k - \omega$ SST	$k - \omega$ SST	
Shape	-			

Table 3.1: Detailed Overview of Simulations Performed for this Study.

Table 3.2: Grids overview, N_z is the number of cells in the wall-normal direction

	N	N_x	N_y	N_z
BFM	1.7E6	-	-	100
M1	1.0E5	80	25	50
M2	3.8E5	160	50	50
M3	1.5E6	320	100	50

numbers and inflow angles used throughout this work, the error obtained should not exceed that of Florentie et al. (2018) [15]. The discretization in wall-normal direction is kept unchanged for the meshes M1, M2 and M3. This stems from the observation that the vortex generator is completely submerged in the boundary layer, and in this region, the required mesh refinement is determined by the need to resolve the boundary layer rather the vortex generator itself. The only exception to this rule is in Section 4.2 of Chapter 4, where the grid spacing is modified in all dimensions to analyze the discretization error.


Figure 3.5: Reference body-fitted mesh.



Figure 3.6: Body-fitted mesh with modified vortex generator inflow angle, $\beta = 5.83^{\circ}$. This mesh is subsequently used as one of the training cases in Chapter 5.

4

Order Reduction of Optimal Source-Term

This Chapter addresses the first research question. The procedure for the order reduction was outlined in Chapter 3. A detailed explanation is given in Section 4.1. In Section 4.2 the reduced-order source-term is applied on different grids to analyze the discretization error. In Sections 4.3 to 4.5 the error arising from the order-reduction is analyzed by comparing the resulting flow field with the results obtained without any order reduction.

4.1. Methodology

The discrete source-term obtained by Florentie et al. (2018) [15] individually defines the three forcing components for each cell in which the source-term is applied. For the purpose of this work we seek to define the source-term in all cells in the vortex generator domain as a set of scaled basis functions. The reduced-order source-term is defined as:

$$\hat{\mathbf{F}} = \begin{cases} \hat{F}_{x} = \sum_{i=0}^{n} a_{x,i} \phi_{i} (x, y, z) \\ \hat{F}_{y} = \sum_{i=0}^{n} a_{y,i} \phi_{i} (x, y, z) , \quad (x, y, z) \in \Omega_{\mathrm{VG}} \\ \hat{F}_{z} = \sum_{i=0}^{n} a_{y,i} \phi_{i} (x, y, z) \end{cases}$$
(4.1)

Or more concisely as:

$$\hat{\mathbf{F}} = \sum_{i=0}^{n} \mathbf{a}_{i} \phi_{i} \left(x, y, z \right)$$
(4.2)

Where $\hat{\mathbf{F}}$ is an approximation of the optimized source-term [15]. The *x*, *y* and *z* directions correspond to the inflow, crossflow and wall-normal directions respectively. The coordinate system is illustrated at the bottom left of Figures 3.5 and 3.6. Once a basis for the functions $\phi_1, ..., \phi_n$ is chosen, a suitable method to define the amplitudes $\mathbf{a}_i, ..., \mathbf{a}_n$ is needed. The methodology used here is a spectral method since the basis functions $\phi_1, ..., \phi_n$ are defined on the entire vortex generator domain. A Galerkin method is then used to find:

$$\hat{\mathbf{F}} \approx \mathbf{F}_{\text{optimized}}$$
 (4.3)

The residual is defined as $\hat{\mathbf{F}} - \mathbf{F}_{optimal}$, and the Galerkin scheme requires the integral of the residual to be zero for a set of weighting functions. This can be written as:

$$\int_{\Omega_{\rm VG}} w(x, y, z) \left(\hat{\mathbf{F}} - \mathbf{F}_{\rm optimal} \right) d\Omega = \mathbf{0}$$
(4.4)

Which is also often referred to as the weak form. If an infinite set of test functions w is chosen, then theoretically the weak form should produce the exact solution to Equation 4.3. However, the purpose is to map the infinite-dimensional set of functions **F** and w to a finite dimensional set of degree n. This means that the weak form in Equation 4.4 only has to be tested for a finite set of w test functions. The finite dimensional equivalent of Equation 4.4 then becomes:

$$\int_{\Omega_{VG}} w_0(x, y, z) \left(\hat{\mathbf{F}} - \mathbf{F}_{optimal} \right) d\Omega = \mathbf{0}$$

$$\int_{\Omega_{VG}} w_1(x, y, z) \left(\hat{\mathbf{F}} - \mathbf{F}_{optimal} \right) d\Omega = \mathbf{0}$$

$$\vdots = \mathbf{0}$$

$$\int_{\Omega_{VG}} w_n(x, y, z) \left(\hat{\mathbf{F}} - \mathbf{F}_{optimal} \right) d\Omega = \mathbf{0}$$
(4.5)

Where Equation 4.2 is used as the finite dimensional solution of $\mathbf{F}_{\text{optimal}}$. Considering only the *x*-component of the force field, and by substituting Equation 4.2 for $\hat{\mathbf{F}}$, Equation 4.5 can be rewritten as the following system of equations:

The *a*-coefficients can be taken out of the integrals, and Equation 4.6 generates a $n \times n$ linear system of equations that can be solved for each coefficient a_i . This procedure must be repeated to find the force field in the *y* and *z* direction. Once the coefficient $a_1, ..., a_n$ are found, the smooth approximation of the force field can be constructed by taking the dot product of the *a* coefficients with their respective function ϕ . Considering again only the *x*-component of the force field, this can be written as:

$$\hat{F}_x = a_x \cdot \phi = \sum_{i=0}^n a_{x,i} \phi_i \tag{4.7}$$

4.1.1. Basis for Smooth Interpolation: Trigonometric Functions and Polynomials

The derivation in the previous section was kept as general as possible and did not yet consider the function basis for the interpolation. When generating the L_2 projection, the basis functions should

have certain properties:

- Resolvable on a coarse mesh, i.e, the gradients in the function should be small enough to avoid aliasing.
- A smooth transition towards zero on the boundary of the domain. This avoids having a jump on the boundary of the domain, and allows for a consistent solution on the boundary of Ω_{VG} when the source-term is used on different meshes.

The output of the OST method does not approach zero on the boundaries, so satisfying the second property will inevitably increase the error of the projection. However, it does have the benefit of making the source-term truly grid-independent.

Two kinds of basis functions are used: trigonometric basis functions that satisfy the homogeneous boundary condition requirement, and polynomial basis functions. The definitions shown below hold for $x \in (-1,1)$. This means that, when applied in Ω_{VG} , the basis function must be mapped to the correct physical coordinates.

Trigonometric Basis Functions

A set of basis functions that naturally go to zero on the (-1, 1) boundaries is as follows:

$$\phi_n(x) = \begin{cases} \cos\left(\frac{i+1}{L}\pi x\right) & i \in \{2k : k \in \mathbb{Z}\} \\ \sin\left(\frac{i+1}{L}\pi x\right) & i \in \{2k+1 : k \in \mathbb{Z}\} \end{cases}$$
(4.8)

Where *L* is the length of the domain. With the boundaries at (-1, 1), the length of the domain is given by L = 2. For even values of *i*, the cosine is used, and for odd values of *i*, the sine is used. An illustration of the basis functions up to i = 4 is shown in Figure 4.1.

Homogeneous boundary conditions imply that each successive basis function must have half of an extra wavelength. For example, the first two basis function are given by $\phi_0(x) = \cos(1/2\pi x)$ with i = 0, and by $\phi_1(x) = \sin(\pi x)$ with i = 1. Since the model is used on a coarse grid with only 5 cells in crossflow direction in Ω_{VG} , only $\phi_0(x)$ has a low enough order to be resolved in the crossflow direction. If a polynomial is used, five cells would allow resolution up to a second order polynomial. As will be shown in the next subsection, a second order polynomial is the equivalent of three Legendre basis functions. In other words, using polynomials without the homogeneous boundary conditions allows for two extra degrees of freedom without aliasing.



Figure 4.1: First four one-dimensional trigonometric basis functions.

Polynomial Basis Functions

Using the Bubnov-Galerkin method, the set of test functions w is the same as the set of base functions ϕ . This method allows taking advantage of a particular set of polynomials called *orthogonal polynomials*. In general, orthogonal polynomials of degree n have a very useful property, which can be defined (for a one-dimensional case) as follows:

$$\int_{\Omega} P_m(x) P_n(x) W(x) dx = 0, \quad m \neq n$$
(4.9)

Where $P_n(x)$ is a polynomial of order n, W(x) is a weighting function that enforces the validity of Equation 4.9, and Ω is the region for which the definition holds, and is typically defined as -1 < x < 1. There are several kinds of orthogonal polynomials, however *Legendre polynomials* have a particularly convenient property, namely that their weighting function is simply equal to 1. A particularly compact formulation for Legendre polynomials is given by *Rodrigues' formula*:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n \tag{4.10}$$

This formulation may be used for deriving lower order Legendre polynomials. However, if for ease of implementation one prefers to not generate the algebraic *n*-th derivative of $(x^2 - 1)^n$, Legendre polynomials have the other useful property of satisfying the three-term Bonnet recursion formula. This allows to express a Legendre polynomial of degree n + 1 explicitly using the previous terms P_n and P_{n-1} :

$$P_{n+1}(x) = \frac{(2n+1)xP_n(x) - nP_{n-1}(x)}{n+1}$$
(4.11)

An illustration of the first five Legendre polynomials is shown in Figure 4.2.



Figure 4.2: First five one-dimensional Legendre polynomials.

Basis functions in higher dimensions

Up to this point the discussion of basis functions was limited to the one-dimensional case. In Equation 4.2, the force field is defined over a three-dimensional domain, hence a method is needed to convert a *n*-th basis function to a three-dimensional function of the form $\phi(x, y, z)$. Formally, the higher dimensional basis functions can be described using the tensor product notation. Considering two vectors *a* and *b*, in the vector spaces V_x and V_y , with $a = (\phi_0(x), ..., \phi_n(x))$ and $b = (\phi_0(y), ..., \phi_n(y))$, then the one-dimensional basis function can be described as a second order tensor through the tensor product of *a* and *b*:

$$\phi(x, y) = a \otimes b \tag{4.12}$$

Likewise, using a third vector space $c = (\phi_0(z), ..., \phi_n(z))$, then the three-dimensional basis function becomes a third order tensor defined as:

$$\phi(x, y, z) = a \otimes b \otimes c \tag{4.13}$$

Or using a less formal description, the three-dimensional basis function can simply be generated as follows:

$$\phi_{i,j,k}\left(x, y, z\right) = \phi_i\left(x\right)\phi_j\left(y\right)\phi_k\left(z\right) \tag{4.14}$$

Where *i*, *j*, *k* refer to the order of the basis function in *x*, *y* and *z* directions respectively. Throughout this work, a particular source-term containing basis functions up to a certain order is referred to as n = a, b, c, with a, b, c being the highest order basis function allowed in *x*, *y* and *z* directions.

Benefits of orthogonality

Using orthogonal polynomials has several very useful properties for defining a smooth solution of

the form of Equation 4.2.

The left side of the system of equations shown in 4.6 can be separated into a matrix containing the integrals multiplied by the vector $a_i = (a_1, ..., a_n)^T$. For the *x*-direction, this then becomes:

$$\begin{bmatrix} \int \phi_{0}\phi_{0}d\Omega & \dots & \int \phi_{0}\phi_{n}d\Omega \\ \int _{\Omega_{VG}}\phi_{1}\phi_{0}d\Omega & \dots & \int \phi_{1}\phi_{n}d\Omega \\ \int _{\Omega_{VG}}\phi_{n}\phi_{0}d\Omega & \dots & \int \phi_{n}\phi_{n}\phi_{n}d\Omega \\ \int _{\Omega_{VG}}\phi_{n}\phi_{0}d\Omega & \dots & \int \phi_{n}\phi_{n}\phi_{n}d\Omega \end{bmatrix} \cdot \begin{vmatrix} a_{x,1} \\ \vdots \\ a_{x,n} \end{vmatrix} = \begin{vmatrix} \int \phi_{1}F_{x,optimal}d\Omega \\ \vdots \\ \int _{\Omega_{VG}}\phi_{n}F_{x,optimal}d\Omega \end{vmatrix}$$
(4.15)

As already mentioned, Legendre polynomials are zero when $n \neq m$. Since now the basis is defined to be the set of Legendre polynomials, *P* can be substituted for ϕ . Then using relation 4.9 it is clear that all off-diagonal terms are zero. This means that the coefficients a_i generated with orthogonal polynomials will have the following advantages:

- A diagonal system is much easier to solve.
- If for some reason more accuracy is needed, one can easily add extra coefficients without having to recompute all previous coefficients.
- In the next Chapter, the base functions coefficients are predicted as a function of the vortex generator inflow angle and Reynolds number. Since the model has two inputs (β and Re) and n outputs, this model is a multiple-input-multiple-output model. However, since the matrix is diagonal, all coefficients are independent of each other. This means that this effectively becomes a multiple-input-single-output model, where each coefficient is modeled individually.

A final useful characteristic of Legendre Polynomials is that the integral of Equation 4.9 has a closed-form solution:

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{2}{2m+1} \delta_{mn}, \quad m \neq n$$
(4.16)

Where $\delta_{i,j}$ is the Kronecker delta. $\delta_{i,j}$ is 1 when i = j and 0 for $i \neq j$. In a three-dimensional case, Equation 4.16 can be extended to also have a closed-form solution. This reduces the computational cost of evaluating the integrals on the left-hand side of Equation 4.15. Let $P_{i,j,k} = P_i(x) P_j(y) P_k(z)$ and $P_{l,m,n} = P_l(x) P_m(y) P_n(z)$ be two three-dimensional Legendre polynomials, then their orthogonality relationship can be written as:

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P_{i,j,k}(x, y, z) P_{l,m,n}(x, y, z) dx dy dz =$$
(4.17)

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P_{i}(x) P_{j}(y) P_{k}(z) \cdot P_{l}(x) P_{m}(y) P_{n}(z) dx dy dz =$$
(4.18)

$$\int_{-1}^{1} P_{i}(x) P_{l}(x) dx \cdot \int_{-1}^{1} P_{j}(y) P_{m}(y) dy \cdot \int_{-1}^{1} P_{k}(z) P_{n}(z) dz =$$
(4.19)

$$\frac{2}{2i+1}\delta_{il}\cdot\frac{2}{2k+1}\delta_{jm}\cdot\frac{2}{2k-1}\delta_{kn} \tag{4.20}$$

From the expression above, the integral of the polynomial is nonzero only if the *i*, *j*, *k* indices match exactly the *l*, *m*, *n* indices. This means that the resulting matrix for the system of equations is diagonal, as was the case in the one-dimensional formulation.

4.1.2. Equivalence with L₂ projection

If a Bubnov-Galerkin method is used, where $w = \phi$, then the system of equations 4.15 is equivalent to minimizing the square of the error. Defining the square of the error (for the *x*-direction force field) as:

$$\epsilon = \int_{\Omega} \left(\sum_{i=1}^{n} a_{x,i} \phi_i \left(x, y, z \right) - F_{x, \text{optimal}} \right)^2 d\Omega$$
(4.21)

Now the objective is to find the combination of $a_i = a_1, ..., a_n$ that minimizes Equation 4.21. In order to do this, the derivative with every single a_i is taken and set equal to zero. The derivative of Equation 4.21 with respect to a particular a_j present in the set $a_1, ..., a_n$ can be written as:

$$\frac{\partial}{\partial a_j} \left(\int_{\Omega} \left(\sum_{i=1}^n a_{x,i} \phi_i \left(x, y, z \right) - F_{x, \text{optimal}} \right)^2 d\Omega \right) = 0$$
(4.22)

Where the result is set to zero to find the minimum of the error. Since a_j is independent of (x, y, z), the derivative can be taken inside the integral, and using the chain rule:

$$\int_{\Omega} 2\left(\sum_{i=1}^{n} a_{x,i}\phi_i\left(x, y, z\right) - F_{x,\text{optimal}}\right) \frac{d}{da_j} \left(\sum_{i=1}^{n} a_{x,i}\phi_i\left(x, y, z\right) - F_{x,\text{optimal}}\right) d\Omega = 0$$
(4.23)

In the relation above, the derivative of the summation of $a_i\phi_i$ is equal to zero for every single value of $i \neq j$, so the derivative of the summation is simply $\phi_j(x, y, z)$. Equation 4.23 becomes:

$$\int_{\Omega} \left(\sum_{i=1}^{n} a_{x,i} \phi_i \left(x, y, z \right) - F_{x,\text{optimal}} \right) \phi_j \left(x, y, z \right) d\Omega = 0$$
(4.24)

Which, for the case of a Bubnov-Galerkin method where $w = \phi$, is the same as the system of equations 4.5.

4.1.3. Verification

In order to test the procedure, the reduced-order approximation with Legendre basis functions is tested on the unit cube with vertices located at (0, 1). The interval is deliberately shifted from the original definition of the Legendre polynomials (-1, 1) to test whether the scaling for an arbitrary domain is implemented correctly. The following expression is used as manufactured solution:

$$S(x, y, z) = \sin(2kx\pi)\cos(2ky\pi)\sin(2kz\pi), \quad (x, y, z) \in \Omega$$

$$(4.25)$$

Where *k* is the wave number, which indicates the number of wavelengths on the (0, 1) interval. The grid is a uniform $(60 \times 60 \times 60)$ mesh, sufficient to resolve the highest wave numbers and polynomials without aliasing. The approximation is tested for various wave numbers by increasing the order of the polynomial, *n*. The resulting error (scaled Euclidean norm) is shown in Figure 4.3. The relation for ϵ , given by Equation 4.29, is computed using the same method as for the flow field in Section 4.4. The reduced-order approximation shows a very small error for low wave numbers. The approximation filters out higher frequencies, which explains the larger error for higher wave numbers. Considering more specifically the line for k = 2, the error is more or less constant until the order of the polynomial approaches 6. This is the first polynomial for which the number of wavelengths is higher than the function we are trying to approximate, and therefore the function can be resolved.



Figure 4.3: Error of the reduced-order approximation. n is constant in all three dimensions.

Approximation of the source-term in the vortex generator domain

The resulting source-term is a force with 3 components (x, y, z directions) defined in an arbitrary three-dimensional domain encompassing the vortex generator. In previous papers such as the original BAY model [3], instead of a *domain*, the terminology *cell selection* is used as the BAY model source-term is typically applied in a narrow region surrounding the vortex generator. Although recent work shows that restricting the source-term to a narrow region around the vortex generator can

theoretically provide accurate results, no such restriction is imposed here. Instead, the source-term is defined in a larger rectangular region surrounding the vortex generator, as a small region makes it more difficult to satisfy the smoothness of the source-term requirement. This domain, or "cell selection approach" is borrowed from [15] and is illustrated in Figure 4.4. This large domain allows the use of smooth basis functions. Furthermore, because these functions are defined in Cartesian coordinates rather than per cell, the resulting source-term is insensitive to grid variations.

The resulting source-term is shown in Figure 4.5. The blue surface is the reducer-order approximation of the red optimal source-term. As expected, the quality of the fitted solution improves with increasing *n*. The quality of the resulting flow field using $\hat{\mathbf{F}}$ instead of $\mathbf{F}_{\text{optimal}}$ is investigated in the next section.



Figure 4.4: Vortex generator domain, Ω_{VG} [15].



Figure 4.5: Illustration of a reduced-order approximation. The red surface is the optimized source-term, and the blue surface its reduced-order approximation. The surfaces represent the source-term in *y*-direction on a plane 0.2 [mm] above the flat plate surface.

4.2. Vortex Generator Simulation with Reduced-Order Source-Term: Discretization Error

The source-term distribution is based on a set of basis functions, where each function is scaled according to a particular coefficient. In this section, the coefficients are kept constant and applied on differing levels of mesh refinement. The functions are defined as function of local Cartesian *x*, *y*, *z* coordinates, so applying the same source-term from one mesh to another is straightforward. The smooth transition towards zero on the boundaries of Ω_{VG} make trigonometric basis functions the most robust when the host mesh is different from the identification mesh, and as such are used

in this section to illustrate the mesh convergence. For clarity, the mesh labeling used so far (M1, M2...) is modified, and the distinction is made between:

- *Identification Mesh (IM)*. The mesh on which the optimized source-term is generated, and also the mesh on which the reduced-order approximation is computed.
- *Host Mesh (HM)*. The mesh on which the reduced-order source-term is applied.

Richardson's extrapolation method is used to test the convergence of the grid. Mesh IM1 is used as identification mesh, and mesh HM2 is used as host mesh. The trigonometric basis functions are chosen as n = 1,0,0 (the zero-th function corresponds to half a wavelength, and the first function to a full wavelength). The mesh spacing, using mesh HM2 as baseline, is reduced by a factor of 1.2, and increased by a factor of 1.2, yielding a constant mesh spacing ratio of r = 1.2. Using the shape factor as reference variable, where H_1 and H_3 refer to the shape factor on the coarsest and finest mesh respectively, the observed order of accuracy can be computed as:

$$p_0 = \frac{\ln\left(\frac{H_3 - H_2}{H_2 - H_1}\right)}{\ln\left(r\right)} \tag{4.26}$$

Using this method, the observed order of accuracy is $p_0 = 1.49$. In Figure 4.6 the shape factor appears to be fully converged. The boundary layer velocity profile is more sensitive to the grid spacing, as illustrated in Figure 4.7. Although the boundary layer converges more slowly, it shows monotonous improvements with mesh refinement. In contrast to the velocity profile, the shape factor is an integral quantity of the boundary layer, and as such is less sensitive to the grid refinement.



Figure 4.6: Shape factor for different levels of grid spacing.



Figure 4.7: Velocity profile downstream of the vortex generator. The spacing ratio is varied in each simulation by a factor of $\sqrt{1.2}$.

4.3. Choice Between Trigonometric and Legendre Basis Functions

It was previously mentioned that trigonometric basis functions are more difficult to resolve on a coarse mesh. If the homogeneous boundary conditions requirement is excluded and Legendre polynomials are used instead, a considerably more accurate boundary layer is obtained. This is illustrated in Figure 4.8. The optimized boundary layer is the result we are trying to attain.

With only 5 cells in the cross flow direction in Ω_{VG} , three Legendre basis functions can be used without aliasing (up to second order polynomial), compared to only a single trigonometric basis function. The smaller number of allowable degrees of freedom clearly leads to a higher error. In the previous section the trigonometric basis functions were used as they allow to verify the convergence of the mesh. Using the Legendre basis functions and excluding the homogeneous boundary conditions requirement leads to a much better boundary layer prediction. As a result, Legendre basis functions are used for the remainder of this work.



Figure 4.8: Velocity profile with highest possible basis function order, using both a trigonometric and Legendre basis. The optimized result is the solution that the reduced-order source-term is trying to approximate.

4.4. Measuring the Order Reduction Error

The error arising from the order reduction in the source-term is directly related to the number of basis functions used to interpolate the OST results. To indicate the number of basis functions, the notation n = a, b, c is used in the subsequent sections, where a, b, c indicate the highest order of the

basis functions in *x*, *y* and *z*-directions respectively.

The maximum allowed value for n depends on the underlying mesh. A high-order function applied on a coarse mesh will show aliasing, leading to erroneous results. Aliasing is especially important if the mesh is modified, and this is subsequently discussed in Section 4.6. For the purpose of this work, a general rule of 10 cells per wavelength is used.

The OST method uses the following measure for its cost function [15]:

$$J(\mathbf{u}) = \int_{\Omega} |\mathbf{u}_{\rm obj} - \mathbf{u}|^2 \, d\Omega \quad \text{on} \quad \Omega / \Omega_{\rm VG}$$
(4.27)

Where U_{obj} is the BFM solution mapped on the coarse grid. Measuring the error as shown in Equation 4.27 is valid for optimization purposes. However, since the domain is very small ($\Omega_{VG} = \mathcal{O}^{-3}$) it leads to very small values of the objective function, which may be misleading. In order to get the complete picture, the objective function is also scaled as:

$$J_{\text{scaled}}\left(\mathbf{u}\right) = \frac{\int \Omega \left|\frac{\mathbf{u}_{\text{obj}} - \mathbf{u}}{U_{\infty}}\right|^2 d\Omega}{V_{\Omega}} \quad \Omega > x_{\nu g} + 5h$$
(4.28)

Where V_{Ω} is the volume of the integration region, and U_{∞} is added to make the expression nondimensional. The region where the vortex generator is located cannot be accurately predicted with only a source-term. Furthermore, as was mentioned in Chapter 3, only the region downstream of the vortex generator is of interest. For these reasons, in Equation 4.28 the measurement domain of *J* is pushed further backwards in the downstream direction.

The scaled Euclidean norm is also used to measure the error:

$$\epsilon \left(\mathbf{u} \right) = \sqrt{\frac{1}{N_{\text{cells}}} \sum_{n=1}^{N_{\text{cells}}} \left(\frac{\mathbf{u}_{\text{obj}} - \mathbf{u}}{U_{\infty}} \right)^2} \quad , \quad \left(x, y, z \right) \in \Omega > x_{vg} + 5h$$
(4.29)

Using Equation 4.29 is biased towards putting more emphasis on the error close to the flat plate surface. This is because the grid is much more refined in in this region. This could be solved by scaling the equation for the cell size. However for predicting separation the region close to the surface is the most important, hence Equation 4.29 automatically puts more emphasis on region.

An overview of all error measurements is shown in Table 4.1. The error is computed by comparing the flow field with the BFM solution mapped on mesh M1. All results in Table 4.1 are for mesh M1. As expected, the reduced-order solution approaches the error of the optimized solution as the degrees of freedom increase. The values of *n* indicate the maximum order of the Legendre polynomials in the *x*, *y* and *z* directions respectively. The case with n = 0, 0, 0 is equivalent to a constant source-term in the vortex generator domain. For n = 4, 4, 4, the difference in error (compared to the OST solution) is very small, although it represents a tenfold reduction in degrees of freedom (using mesh M1). If a more refined mesh such as mesh M2 is used, then the reduction in degrees of freedom is even higher, with a 1/40 reduction in degrees of freedom. With orthogonal polynomials the cost of computing the coefficients is irrelevant, and for n = 4, 4, 4 it simply involves 3 sets of 125 divisions.

One should keep in mind that for n = 4, 4, 4, the high-order polynomials are not resolved on mesh M1. While the algorithm can use the extra degrees of freedom to generate a better source-term

field on this specific mesh, aliasing occurs. This means that the same reduced-order distribution cannot be used on another mesh and will lead to spurious results, as will be shown in Section 4.5.

The jBAY model is of similar accuracy to a reduced-order approximation with n = 1, 1, 1.

In the subsequent sections, a more detailed analysis of several properties of the flow is made, namely the velocity profile, shape factor and vorticity.

Case	DOF	ϵ_x	ϵ_y	ϵ_z	$J_{x,scaled}$	$J_{y,\text{scaled}}$	$J_{z,scaled}$
Objective	-	-	-	-	-	-	-
Optimized	1300	7.75E-03	2.26E-03	1.13E-03	5.81E-06	1.86E-06	1.05E-06
n = 0, 0, 0	1	1.55E-02	6.29E-03	2.66E-03	7.93E-05	8.38E-06	5.19E-06
n = 1, 1, 1	8	1.15E-02	4.97E-03	2.06E-03	2.18E-05	4.93E-06	2.88E-06
n = 2, 1, 3	24	1.09E-02	4.13E-03	1.76E-03	2.55E-05	3.99E-06	2.44E-06
n = 3, 2, 4	60	8.81E-03	3.68E-03	1.26E-03	1.04E-05	3.88E-06	1.49E-06
n = 4, 4, 4	125	7.78E-03	2.36E-03	1.13E-03	6.02E-06	1.99E-06	1.07E-06
jBAY	-	1.15E-02	4.94E-03	2.27E-03	3.36E-05	8.42E-06	4.17E-06

Table 4.1: Error in the velocity field, measured in the domain $x > x_{vg} + 5h$ with mesh M1

4.5. Flow Field with Reduced-Order Source-Term

Boundary Layer Velocity Profile

Velocity profile plots at three stations downstream of the vortex generator are shown in Figure 4.9. For all models, results improve with increasing distance behind the vortex generator. The constant n = 0, 0, 0 source-term distribution does have an effect on the boundary layer; it shows that a simple uniform source-term is insufficient for modeling vortex generators. Results for n = 4, 4, 4 show a particularly good agreement with the optimized solution, with the boundary layer being almost indistinguishable from the OST solution. Although n = 4, 4, 4 yields the best results, the source-term is aliased. If the reduced-order source-term is to be used on a different mesh, the smaller n = 3, 2, 4 solution must be used. Using a more refined mesh would allow using more basis functions, at the expense of an increased computation cost.



(a) $\Delta x/h = 5$





(c) $\Delta x/h = 15$

Figure 4.9: Boundary layer profile at three stations downstream of the vortex generator leading edge.

Shape Factor

The shape factor is an integral quantity of the boundary layer velocity profile, and as such it is not as sensitive to the correct representation of the velocity profile shown above. This is clearly shown in Figure 4.10. While the jBAY model was unable to resolve the boundary layer, its shape factor results are much more accurate. Still, the reduced-order approximation shows more consistent results, with a smoother peak in the shape factor at y/h = 2. The peak of the jBAY model is likely a result of the jBAY model being applied in a very thin region encompassing the vortex generator.





Figure 4.10: Shape factor profile downstream of the vortex generator.

Vorticity

As discussed in Chapter 2, vorticity is known to be difficult to predict on a coarse mesh. As the vorticity contour plots show, the reduced-order solution closely reproduces the solution of the OST method. Although there is a minor discrepancy between the objective and optimized solutions, the overall shape of the reduced-order solution is superior to the jBAY model. Also, the inability to model peak vorticity in the vortex core can clearly be seen. This is an inherent feature of coarse meshes, and the only solution to model small features of the flow is to increase mesh refinement. Dudek (2011) [10] found that peak vorticity prediction is improved by refining the mesh at the vortex generator tip. This is at the cost of increased computational time and added complexity in the meshing process. Since peak vorticity is not relevant for predicting flow separation downstream of the vortex generator, no further attempt is made here at predicting peak vorticity correctly.



Figure 4.11: Vorticity contour plots at $\Delta x/h = 5$ downstream of the vortex generator.

4.6. Grid-Independence of Non-Homogeneous Legendre Basis Functions

Grid independence is essential in industrial applications. In a case such as a row of vortex generators on a wing, grid independence allows to arbitrarily position the vortex generators without having to reconsider the mesh. If the model is grid-independent, consistent results are obtained irrespective of the local grid topology. In this section the identification mesh is always the coarsest mesh, namely Mesh M1, which we shall now call mesh IM1. This is done for the following two reasons:

- The results of the OST solution indicate that the OST method is powerful enough such that there is no significant loss in the quality of the velocity field, even on the coarsest mesh IM1 [15].
- Running an optimization on a coarse mesh requires less computational time, so it makes sense to exploit the optimization fully by generating it right away on the coarsest mesh.

Given the sharp drop to zero on the boundary, a formal investigation of convergence, as was done for the trigonometric basis functions, is less relevant. Instead the resulting error is computed using the same measure as in Section 4.4, namely using Equations 4.29 and 4.28. The reference flow field is the BFM flow field mapped on each respective mesh. The resulting error is shown in Table 4.2 and Figure 4.12. When a Legendre basis is used, there still is a reduction in error with increasing

mesh refinement. This suggests that, using a host mesh different from the identification mesh with the Legendre basis is still possible, and in any case the error reduction with mesh refinement is still more consistent than that of the jBAY model (Figure 4.12).

Table 4.2: Error on grids M1, M2, and M3. All the reduced-order simulations are computed using Legendre polynomials of order n = 3, 2, 4.

Grid	ϵ_x	ϵ_y	ϵ_z	$J_{x,\text{scaled}}$	$J_{y,\text{scaled}}$	$J_{z,scaled}$
Objective	-	-	-	-	-	-
IM1	1.32E-02	7.10E-03	4.32E-03	2.83E-05	1.27E-05	1.16E-05
HM2	7.60E-03	5.49E-03	2.88E-03	1.64E-05	7.99E-06	6.27E-06
HM3	6.42E-03	4.16E-03	2.00E-03	1.28E-05	4.73E-06	3.13E-06



Figure 4.12: Error for the reduced-order and jBAY model as function of mesh refinement. $\bar{V}_{cell,VG}$ is the average cell volume in the vortex generator domain, $V_{\Omega_{VG}}$ is the total vortex generator volume. The mesh is only refined in *x* and *y*-directions, so the cell volume is used as indication of the grid refinement.

Results for the velocity profile and shape factor are shown in Figures 4.13 and 4.14 respectively. It was previously noted that mesh M1 cannot resolve the n = 4,4,4 case, which is visible in Figures 4.13b and 4.14b. The source-term distribution that was aliased on mesh M1 leads to an incorrect source-term distribution on grids M2 and M3. This can be seen in Figure 4.13b where grid M2 leads to an incorrect velocity profile, while the same distribution on mesh M1 shows a very good match with the optimized solution. The shape factor in Figure 4.14b also shows the same effect, with unexpected peaks for mesh M3.

The velocity profile for the case with n = 3, 2, 4 is remarkably consistent across the three different grids. The order of the polynomials is low enough to be fully represented on all three meshes, and the reduced-order solution is grid-independent.

While the jBAY model does show improving results with increasing mesh refinement (Figures 4.14c), the results for mesh M1 are considerably worse. Mesh M3 contains roughly the same number of cells as the body-fitted reference mesh (BFM). The fact that the jBAY model can only yield an accurate flow field if the mesh refinement is of the same order as that of a fully gridded vortex generator is inconvenient due to the related computational cost.

The velocity profile is plotted at a crossflow location aligned with the vortex generator leading

edge, which is equivalent to y/h = 0 in the shape factor plots of Figure 4.14. The good match in the velocity profile means that the shape factor curves are very accurate at y/h = 0. At the vortex generator trailing edge $(y/h \approx 2)$, the curves show considerably worse results. This stems from the sharp drop to zero of the source-term on the boundaries of Ω_{VG} . The different host meshes resolve this sharp drop differently, leading to differing results. Indeed, the shape factor with the trigonometric homogeneous basis functions on the boundaries, shown in Figure 4.6, do not exhibit this problem. This means that a choice between grid consistency or low model error must be made. However the shape factor in Figure 4.14 with the Legendre basis appears to be much closer to the optimized solution than the shape factor with the trigonometric basis functions. This indicates that, although there is a localized inconsistency in the shape factor, the Legendre order reduction still yields a much more accurate flow field than the trigonometric basis. For this reason the Legendre basis is used for the surrogate model, which is the subject of the subsequent Chapter 5.



(a) n = 3, 2, 4, boundary layer converges with increasing mesh refinement.



(b) n = 4, 4, 4, aliasing error is visible



Figure 4.13: Velocity profile for two different polynomial orders and the jBAY model, $\Delta x/h = 15$.



Figure 4.14: Shape factor for two different polynomial orders and the jBAY model, $\Delta x/h = 15$. The crossflow location y/h = 0 is aligned with the vortex generator leading edge.

5

Surrogate Model

In this chapter, the reduced-order source-term is used to train a surrogate model. In was previously shown in Chapter 4 that the Legendre basis functions form an orthogonal set, meaning that the coefficients scaling these bases are uncorrelated. This characteristic makes Kriging the most suitable method for this task, as each coefficient can use a separate Kriging estimator, and Kriging is proven to be an estimator that minimizes the expected error. A derivation of the Kriging method is shown in Section 5.1. Details of its implementation are described in Section 5.2. The flow field results of the surrogate model are shown in Section 5.3.

5.1. Kriging Model

A Kriging estimator is based on a weighted average of sampling points. Considering a set of sampled values $y_i, ..., y_N$, a prediction at an unknown point **s**₀ is defined as:

$$\hat{Y}(\mathbf{s_0}) = \hat{Y}_0 = \sum_{i=1}^n w_i y_i$$
 (5.1)

Now the weights w_i have to be computed such that:

- The solution is unbiased: the expected value of \hat{Y}_0 is the same as the true value Y_0 , or $\mathbb{E}(\hat{Y}_0) = \mathbb{E}(Y_0)$.
- The expectation of the error squared $\mathbb{E}\left[\left(Y_0 \hat{Y}_0\right)^2\right]$ is minimized.

In order to minimize the prediction error, the term inside the bracket is expanded:

$$\mathbb{E}\left[\left(Y_{0} - \hat{Y}_{0}\right)^{2}\right] = \mathbb{E}\left[Y_{0}^{2} - 2Y_{0}\hat{Y}_{0} + \hat{Y}_{0}^{2}\right]$$
(5.2)

$$= \mathbb{E}\left[Y_0^2\right] - \mathbb{E}\left[\hat{Y}_0 Y_0\right] + \mathbb{E}\left[\hat{Y}_0^2\right]$$
(5.3)

We introduce the variance $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ and the covariance: $Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y]$. Doing a linear combination of both:

$$\operatorname{Var}\left(Y_{0}\right) + \operatorname{Var}\left(\hat{Y}_{0}\right) - 2\operatorname{Cov}\left(Y_{0}, \hat{Y}_{0}\right) = \mathbb{E}\left[Y_{0}^{2}\right] - \left(\mathbb{E}\left[Y_{0}\right]\right)^{2} + \mathbb{E}\left[\hat{Y}_{0}^{2}\right] - \left(\mathbb{E}\left[\hat{Y}_{0}\right]\right)^{2} - 2\mathbb{E}\left[Y_{0}\hat{Y}_{0}\right] + 2\mathbb{E}\left[Y_{0}\right]\mathbb{E}\left[\hat{Y}_{0}\right] \quad (5.4)$$

As previously mentioned, the solution must be unbiased, i.e. $\mathbb{E}(\hat{Y}_0) = \mathbb{E}(Y_0)$. This means that $\mathbb{E}(\hat{Y}_0) - \mathbb{E}(Y_0) = 0$, which can be extended to $(\mathbb{E}(\hat{Y}_0) - \mathbb{E}(Y_0))^2 = 0$ and finally that $2\mathbb{E}[Y_0\hat{Y}_0] = (\mathbb{E}[Y_0])^2 + (\mathbb{E}[\hat{Y}_0])^2$. This means that in Equation 5.4 three of the terms cancel out, and that Equation 5.4 is the same as Equation 5.3. This can be written as:

$$\mathbb{E}\left[\left(Y_0 - \hat{Y}_0\right)^2\right] = \operatorname{Var}\left(Y_0\right) + \operatorname{Var}\left(\hat{Y}_0\right) - 2\operatorname{Cov}\left(Y_0, \hat{Y}_0\right)$$
(5.5)

$$= \sigma^{2} + \text{Cov}(\hat{Y}_{0}, \hat{Y}_{0}) - 2\text{Cov}(Y_{0}, \hat{Y}_{0})$$
(5.6)

Now the expression for \hat{Y}_0 in Equation 5.1 can be substituted in Equation 5.6:

$$\mathbb{E}\left[\left(Y_0 - \hat{Y}_0\right)^2\right] = \sigma^2 + \operatorname{Cov}\left(\sum_{i=1}^n w_i y_i, \sum_{j=1}^n w_j y_j\right) - 2\operatorname{Cov}\left(Y_0, \sum_{i=1}^n w_i y_i\right)$$
(5.7)

Taking out the summations then yields the following expression:

$$\mathbb{E}\left[\left(Y_{0}-\hat{Y}_{0}\right)^{2}\right] = \sigma^{2} + \sum_{i=1}^{n} \sum_{j=1}^{n} w_{i} w_{j} \operatorname{Cov}(y_{i}, y_{j}) - 2 \sum_{i=1}^{n} w_{i} \operatorname{Cov}(y_{i}, y_{0})$$
(5.8)

The aim is to minimize the above expression under the unbiasedness constraint that $\mathbb{E}(\hat{Y}_0) = \mathbb{E}(Y_0)$. The unbiasedness constraint is met if the sum of the weights is equal to 1: $\sum w_i = 1$. In order to do this, the method of Lagrange multipliers is used and the following Lagrangian is set up:

$$L = \mathbb{E}\left[\left(Y_0 - \hat{Y}_0\right)^2\right] + 2\lambda\left(\sum w_i - 1\right)$$
(5.9)

$$= \sigma^{2} + \sum_{i=1}^{n} \sum_{j=1}^{n} w_{i} w_{j} \operatorname{Cov}(y_{i}, y_{j}) - 2 \sum_{i=1}^{n} w_{i} \operatorname{Cov}(y_{i}, y_{0}) + 2\lambda \left(\sum w_{i} - 1\right)$$
(5.10)

To minimize Equation 5.10 the derivative with respect to w_i is taken and set equal to 0. For simplicity we define $C_{i,j} = \text{Cov}(y_i, y_j)$.

$$\frac{\partial L}{\partial w_i} = 2\sum_{j=1}^n w_j C_{i,j} - 2C_{i,0} + 2\lambda \tag{5.11}$$

The covariances are then put in a matrix, with one extra row of 1's to add the $\sum w_1 = 1$ constraint to the system. Solving the system yields the solution for the weights w_i and is called *Ordinary Kriging*. Note the extra factor of 2 present in Equation 5.9 in front of the unbiasedness constraint. This is merely for convenience, and when considering Equation 5.11 being set equal to 0, this allows to have all the 2's vanishing from the system. The methodology shown above has been implemented using the available Python package pyKriging [25]. Implementation details are outlined in the next section.

5.2. Implementation

In Equation 4 the details of a smooth approximation of the source-term in the vortex generator domain have been explained. The smooth approximation was based on a set of coefficients. Each dimension can go up to order n (the number of basis functions for the interpolation), and in three dimensions this means that one has n^3 coefficients to solve for. The source-term is also defined in the x, y and z directions, which means that, in total the model has to predict $3n^3$ coefficients. The surrogate model must also be trained with a certain set of sampled values, which is the subject of the next section.

5.2.1. Sampling Points

The model is first trained with 20 reference cases. Those cases are chosen according to the Latin hypercube method. The infill points are added after the first 20 data points are computed, and are added at the locations where the mean square of the error is the highest.

The inflow velocity is implemented by modifying the relevant OpenFOAM dictionary. The inflow angle is modified by rotating the vortex generator in the mesh. To accomplish this, a dictionary is generated using a set of vertices, and those vertices are rotated using a standard rotation matrix around the *z*-axis. Once this is done, the mesh can be generated using OpenFOAM's included blockMesh utility.



Figure 5.1: Overview of all training points for the surrogate model.

Case Number	β [°]	Re _h [-]		
		Re _h [-]	$U_{\infty} [m/s]$	
1	10.78	14386.21	29.80	
2	14.07	10910.34	22.60	
3	8.57	10331.03	21.40	
4	15.18	14965.52	31.00	
5	11.32	12068.97	25.00	
6	10.22	8593.10	17.80	
7	7.47	7434.48	15.40	
8	15.72	8013.79	16.60	
9	9.68	6275.86	13.00	
10	13.53	13227.59	27.40	
11	12.97	6855.17	14.20	
12	9.12	12648.28	26.20	
13	5.83	9172.41	19.00	
14	8.03	15544.83	32.20	
15	14.62	5696.55	11.80	
16	12.43	16124.14	33.40	
17	6.38	13806.90	28.60	
18	5.28	11489.66	23.80	
19	11.88	9751.72	20.20	
20	6.92	5117.24	10.60	
21	14.90	10620.69	22.00	
22	6.10	12937.93	26.80	
23	12.70	5986.21	12.40	

Table 5.1: Overview of all training points for Kriging Model

5.2.2. Procedure

Figure 5.2 shows an overview of the procedure required to generate a vortex generator model. Building the surrogate model is divided in three steps. First one has to generate the required cases to train the Kriging model. This is shown in the first top loop in Figure 5.2. The second step involves generating the $3(n + 1)^3$ coefficients for each corresponding case. The variable *n* is incremented by 1 due to the numbering of the basis functions starting from zero. This is assuming constant *n* in *x*, *y*, and *z*-directions. For this chapter, a combination defined by n = 3, 2, 4 is used, meaning that a total of 180 coefficients are modeled. Once the coefficients are generated, for every single coefficient the program generates a Kriging model that takes as input the same coefficient of all cases. This means that $3(n + 1)^3$ Kriging models are generated. To generate a source-term model, every single Kriging model is used to each output the corresponding a_i coefficient. Finally those coefficients are used to generate a source-term distribution as per Equation 4.2.

From those three steps, the first loop is by far the most computationally intensive, with each case taking about 2 days for the BFM and adjoint optimization to solve. While the code is not parallelized, the process can be sped up by running multiple single core cases at the same time. This is actually better than running multiple core cases sequentially, as it allows true linear scaling with the number of cores.



Figure 5.2: General model procedure.

5.2.3. Example Model Response

Figure 5.3 shows the variation of two coefficients a_i in the Re_h – β space. a_{000} is the lowest order coefficient. It is equivalent to a constant added in the Ω_{VG} domain. Observing Figure 5.3a, this low order coefficient is intuitively understandable. At a lower Reynolds number, its value is lower, and it increases as the Reynolds number increases, which agrees with the logic that the vortex generator force scales with the inflow velocity squared. The same observation is made with inflow angle. The coefficient increases with the inflow angle, which is consistent with the theory that the vortex generator force should scale linearly with the inflow angle.

Higher order coefficients such as a_{121} show less obvious behavior. The higher frequencies in the source-term cannot be predicted using empirical or dimensional analysis, hence a surrogate model is highly valuable to help evaluate them.



(a) $a_{0,0,0}$ for x-direction force

(b) $a_{1,2,1}$ for x-direction force

Figure 5.3: Example of model response for two coefficients. The surface is the model response in the Re, β space, and the red dots are its training coefficients.

5.3. Surrogate Model Results

After outlining the procedure to generate the modeled source-term, this section investigates the effectiveness of the model for predicting the flow field. Figure 5.4 shows an overview of the various errors arising across the modeling steps. The reference solution is the body fitted mesh (BFM) solution. This solution is then mapped onto the design mesh (coarse grid), and becomes the objective of the OST computation. The objective is used as a benchmark throughout this section. The next step is to obtain an adjoint optimized solution (the OST method). Unfortunately the optimized source-term cannot perfectly reproduce the objective mapped solution, and this is a source of error. The optimized solution is obtained using a source-term with $3N_c$ degrees of freedom, with N_c being the number of cells in the vortex generator domain. Reducing the degrees of freedom to a Legendre basis adds another layer of error (the differences between the two source-terms can easily be seen in Figure 5.3). The last source of error is the surrogate model itself. Since the surrogate model is trained from the reduced-order source-term, it is limited by the accuracy of the OST approach and by the accuracy the order reduction method. Accordingly, if one wants to evaluate the error arising from the surrogate model only, the reduced-order solution must be used as benchmark.



Figure 5.4: Diagram outlining error layers when generating U_{smooth} .

The model is generated using a set of 23 test cases. In order to assess the model's quality, one case at the time is removed from the training function. The removed case's velocity objective (i.e., BFM result mapped on the design mesh) is then used as the benchmark for the model. The order of the cases has no particular meaning regarding the value of inflow angle or Reynolds number, and as such may be regarded as a random sample of observations. In the next sections, the first two cases are used to assess the quality of the model. Looking back at Figure 5.4, the surrogate model is compared with each processing step, up to the mapped solution. In this section the jBAY model is shown as well, so in total five results are displayed: the objective, optimized source-term, reduced-order, surrogate and jBAY models. The aim is to investigate which of the processing steps show the most error, and how they compare with the jBAY model.

5.3.1. Boundary layer velocity profile

The boundary layer velocity profiles for cases 1 and 2 are shown in Figures 5.5 and 5.6 respectively. The (small) difference with the optimized source-term reduces for increasing $\Delta x/h$, and given that the deviation is already minimal, little improvement to the surrogate model is expected if more training points are added. Both the optimized and surrogate models are markedly superior to the jBAY solution.





Figure 5.5: Boundary layer profile at three stations downstream of vortex generator for case 1.



Figure 5.6: Boundary layer profile at three stations downstream of vortex generator for case 2.

5.3.2. Shape Factor

As opposed to the boundary layer velocity profile, the shape factor solutions (shown in Figures 5.7 and 5.8) show little difference with respect to each other. The shape factor is an integral quantity of the flow, hence it is less sensitive to the lack of modeling of small scales. More particularly, the small deficit in velocity of the optimized solution does not appear to have any effect on the shape factor prediction.

Overall the jBAY model also shows good results, especially in case 1. If the purpose of vortex generator simulations is only to predict the shape factor, then the jBAY model appears to be sufficient. However, the surrogate model appears to be more consistent, and does not show the unexpected peak in shape factor that the jBAY model shows for case 2.



Figure 5.7: Shape factor profile downstream of vortex generator, case 1.



Figure 5.8: Shape factor profile downstream of vortex generator, case 2.

5.3.3. Reynolds Stresses

In Chapter 3, it was explained that the vortex generator surrogate model should allow for correct prediction of separation. Predicting separation downstream of the vortex generator is largely dependent on the turbulence model. While at this stage RANS turbulence models are inadequate in predicting separation, here we make the assumption that an improved turbulence model will be available in the future. So in order to have an accurate prediction of separation, the inputs for the Reynolds stress model should be error-free. This means that both the mean flow and turbulent stresses should be modeled accurately. In Chapter 2, it was mentioned that the OST method optimizes the source-term by reducing the cost function:

$$J(\mathbf{u}) = \int_{\Omega} |\mathbf{u} - \tilde{\mathbf{u}}|^2 \, d\Omega \tag{5.12}$$

Figure 5.9 shows a color-map of the u'v' stresses. Unfortunately, reducing the error in the mean

flow field still leads to error in the Reynolds stresses. Since the surrogate model is based on the data obtained by the OST method, it cannot predict the Reynolds stresses correctly. At this stage it is not clear whether the source of this error is due to the non-perfect reproduction of the boundary layer by the OST method, or simply due to the lack of representation in the cost function. In the latter case, the governing adjoint equations need to be re-derived in order to take into account of the Reynolds stresses in Equation 5.12. Reconsidering the velocity profile in Figures 5.5 and 5.6, the prediction is (slightly) incorrect in the same region as the incorrect Reynolds stresses. If instead the error is simply due to the non-perfect boundary layer flow field, weighting the cost function to increase sensitivity in the boundary layer may be sufficient.



Figure 5.9: Reynolds stress downstream of vortex generator for case 1, $\Delta x/h = 5$.

5.3.4. Vorticity

Extensive analyses on vorticity are available in the literature [1, 10, 14, 22]. There is a consensus that vorticity, and especially peak vorticity, is very difficult to correctly predict on a coarse mesh. As opposed to the shape factor, vorticity is not an integral quantity of the flow, but a first order derivative. As such, it is more sensitive to the mesh resolution, especially at the vortex center, where it is difficult to compute high gradients on a coarse mesh. This can be seen in Figures 5.10 and 5.11. Although

the optimized and model results show a strong improvement compared to the jBAY model, moving from objective to optimized, and from optimized to model solution, a reduction in the quality of the shape of the vortex can be observed. The center of the vortex is also less concentrated. These poorer results are also later reflected in the analysis of the circulation in Section 5.4.1. Circulation is computed by integrating the vorticity in the region where the vorticity is positive. If the vortex shape and strength are not accurate, this inevitably leads to errors in the computation of circulation.





(a) Solution objective



(b) Optimized solution



(c) Model solution

(d) jBAY solution

2.00

1.75

1.50

1.25

1.00

0.75

0.50

0.25

0.00

z/h







(b) Optimized solution

-2.7e-01

-1.5

-1.0

-0.5 0.0

4.0e-03

y/h

 $\omega_x h/U_\infty$

0.5

1.0

1.5

3.4e-01



Figure 5.11: Vorticity contour plots at $\Delta x/h = 5$ downstream of vortex generator for case 2.

5.4. Surrogate Model with Reduced Training

The analysis so far has shown that when the surrogate model is well-trained, the velocity profile is very close to the training points results, namely the optimized source-term. In order to evaluate the sensitivity of the model to its number of training points, the results of cases 1 and 2 are investigated with 10 instead of 23 training points. The new training points are taken from the original 20 points, Latin hypercube sampling method. Figures 5.12a and 5.12b show the training points of the original and of the reduced-training surrogate model respectively. In order for this test to be rigorous, training points should be removed evenly across the domain. Also, the Kriging method is a weighted average that puts more importance on the training points close to the desired output, so when selecting a test case, there should be no nearby training points left.



(a) All training points

(b) Reduced training points

Figure 5.12: Data for surrogate model training.

In Figure 5.12b, both case 1 and case 2 lost close neighboring points, especially case 2. The removal of the point closest to case 2 shows that the model deviates from the optimized solution. Interestingly, this deviation moves the velocity profile closer to the objective solution. So by reducing the model training, the solution has improved. It is possible that the optimization script of the OST method cannot find a good optimized source-term in this Re_h , β region, and that when these points are removed, the model relies on more distant, yet more accurate data points, yielding an overall reduction in error.

It has already been shown in Figure 5.3 that some coefficients behave linearly, but that obvious non-linearities are also possible (Figure 5.3b). So overall it is remarkable that, even with such a drastic reduction in training points, the surrogate model is able to predict an accurate flow field.



(b) Case 2

Figure 5.13: Boundary layer profile at $\Delta x/h = 15$.



Figure 5.14: Shape factor downstream of the vortex generator, $\Delta x/h = 15$.

5.4.1. Circulation and Vorticity

It was previously mentioned that, although much better than the jBAY model, the surrogate model is not yet fully capable of correctly predicting the vorticity. The circulation is computed from the vorticity field, and the inaccurate vorticity field appears to be visible in the circulation results. Figure 5.15 shows the circulation for the surrogate model with full and partial training. Considering case 1, the surrogate model is more accurate than the jBAY model, however for case 2 the results are inconclusive. Furthermore, case 2 indicates that the entire training set is needed for consistency of the circulation results. The circulation and vorticity field being directly related, the inferior circulation prediction is also reflected in the vorticity field shown in Figure 5.16b, where the shape of the vortex is different from its objective solution.



Figure 5.15: Circulation downstream of vortex generator. $\Delta x/h = 0$ corresponds to the streamwise location of the vortex generator trailing edge. The circulation is computed by integrating the vorticity in a region where $\omega > 0$.



Figure 5.16: Vorticity downstream of vortex generator for case 2 at $\Delta x/h = 15$. The poor prediction of vorticity is reflected by the inaccurate vortex circulation in Figure 5.15b.
6

Conclusion

A new surrogate model for vortex generator flows was developed. This model yielded improved results compared to the well-established jBAY model and appears to be adequate for predicting separation behind vortex generators. Also, unlike the jBAY model, the proposed surrogate model is consistent when used on different meshes. The goal-oriented optimization by Florentie et al. (2018) [15] proved three fundamental characteristics of a source-term model. First, the source-term does not have to be representative of the physical force applied by the vortex generator. Second, current models have not yet reached the best possible flow field when only a coarse mesh is used. Third, the source-term can be relatively smooth compared to the mesh. Building on these last three critical points, this thesis was an attempt to create a new surrogate vortex generator model.

Reduced-order source-term

The first step in building such a new model involved reducing the order of the goal oriented optimized source-term. The reduced-order source-term is not a function of the vortex generator domain cells; instead, it is defined using a set of basis functions that can be applied on differing meshes. The order reduction was achieved by fitting a set of three-dimensional trigonometric and Legendre polynomials to the discrete source-term. This changed the definition of the source-term from a forcing term for each cell to a set of dimensionless coefficients scaling the basis functions.

The order reduction (or "smoothness") is measured by the highest order of the basis function. The lower the value of *n*, the smoother the solution. The higher order trigonometric basis functions with homogeneous boundary conditions were found to be difficult to resolve on a coarse mesh, and only a maximum of two basis functions could be used. The reduced-order approximation with a Legendre basis yielded results similar to the optimized source-term, and differences in the velocity profile, shape factor and vorticity were minimal. After testing the approximation for increasing values of *n*, a combination of 3, 2 and 4 maximum order of Legendre polynomials was chosen for the order reduction of the source-term.

The reduced-order source-term, now defined as function of local coordinates rather than on a per cell basis, could then be applied on meshes with differing refinement. Together with the improved flow field prediction, mesh-independence is the other main advantage of the new model developed in this thesis. Unlike the jBAY model, the surrogate model is independent of the mesh refinement, and can be expected to yield similar results regardless of the local mesh topology. The trigonometric basis functions showed the best convergence with increasing host mesh refinement,

which is due to the fact that they were chosen to smoothly reach zero at the boundary of the vortex generator domain. The Legendre basis functions also showed consistent results when the host mesh was refined. However, the nonzero boundary in the vortex generator domain led to small deviations in the boundary layer at the vortex generator trailing edge.

Surrogate model

Using the promising results of the reduced-order solution, a surrogate model was built. Its purpose is to predict the variation of the scaling coefficients as function of the vortex generator height-based Reynolds number and inflow angle. Twenty-three training cases were run to train the surrogate model. The sampling plan was generated according to the Latin hypercube method. Using this training data, each coefficient was modeled individually with a Kriging estimator. Since the basis functions interpolating the discrete optimized source-term are orthogonal, the coefficients are uncorrelated and a single output estimator can be used. Kriging was used as it is an estimator that minimizes the L_2 norm of the deviation. The results obtained with twenty-three training points provided accurate results for the mean flow field and the shape factor downstream of the vortex generator. The boundary layer velocity profile downstream of the vortex generator proved to match the OST solution particularly well. As a consequence, the deviation in the velocity profile between the model and the solution was less than the error between the OST solution and the objective solution. The OST method already proved that its velocity profile was a substantial improvement over the jBAY model, and the surrogate model was able to match this result.

After evaluating the velocity profile, the quality of the shape factor was investigated. Given the surrogate model's accurate prediction of the velocity profile, one would expect it to produce an accurate shape factor. This was indeed the case, with the shape factor of the model showing a very close match with the OST solution and objective solution. Somewhat surprisingly, the jBAY model also proved quite capable at predicting the shape factor, despite its poor velocity profile results.

Vorticity prediction from the model matched the optimized solution relatively well, being better than the jBAY model but with some deviation in the overall vortex shape. The vorticity in the core was also slightly more diffused than the objective, however this is to be expected when computing vorticity on a coarse mesh.

The model was initially trained with a rather arbitrary starting point of 23 cases. This was sufficient for the considered range of Reynolds number and inflow angle. To investigate the results of the model with fewer training points, the model was re-trained with only 10 sampling points. While the velocity profile was again very accurate, vorticity and circulation showed inconsistent results on the two test cases. For the first case, the circulation curve downstream of the vortex generator coincided with the optimized and objective function, and was much more accurate than the jBAY model. However for the second case this was no longer the case, with the results being slightly worse than the jBAY model. This may be explained by the strong non-linearities of higher order coefficients in the Re – β domain. With fewer training points, the Kriging estimator can no longer accurately predict the higher order coefficients.

With the aim of correctly predicting all the inputs required for a turbulence model, the quality of the Reynolds stresses was evaluated. While the surrogate model was able to replicate the results of the OST method, the OST solution itself did not match the results of the objective field. The surrogate model's poor prediction is limited to the boundary layer region, and coincides with the region where the velocity profile of the OST solution does not perfectly match that of the objective. This suggests that this small error in the velocity field may lead to the erroneous Reynolds stresses.

6.1. Recommendations

While the surrogate model's results are promising, there are several avenues for future research to further improve the model.

Modifying the Legendre polynomial order reduction to achieve smoothness on the boundary of the vortex generator domain

While in this thesis grid independence was proven using the trigonometric basis functions, the nonhomogeneous Legendre polynomials on the vortex generator domain make it more difficult to distinguish the model error from the discretization error. This could be modified by connecting a local linear function with C^0 smoothness in a small region extending the vortex generator domain. Alternatively, the vortex generator domain could be extended and the homogeneous boundary conditions could be implemented in the L_2 projection. Increasing the vortex generator domain can be done as long as the model does not overlap with another nearby vortex generator. For the common case of counter-rotating vortex generators, the system of two vortex generators can be considered as one single Ω_{VG} .

Modifying the OST method to optimize for basis functions instead of individual cell source-terms

Currently the adjoint optimization and order reduction steps are independent of each other. Since with the OST method each cell represents a degree of freedom, the optimization is rather costly. An alternative would be to merge the optimization and order reduction into a single step by implementing the optimization algorithm such that it only uses the basis functions coefficients as degrees of freedom. This would drastically reduce the number of parameters to optimize for, and as such reduce the computational cost.

Reynolds stresses

At this stage, the model's prediction of the Reynolds stresses are erroneous in the boundary layer. Current turbulence models are inadequate in predicting separation. Furthermore, if the turbulence model cannot compute the correct Reynolds stresses with the vortex generator model, then prediction of separation is more difficult. Improving the Reynolds stresses would involve evaluating whether a small improvement in the boundary layer velocity profile has any impact on the Reynolds stresses. If this is not the case, then modifying the OST method is required. This would involve modifying the cost function and re-deriving the adjoint equations.

Ease of implementation

The surrogate model is more challenging to run than the jBAY model. In OpenFOAM, the current implementation of jBAY only requires the vortex generator position and geometry as inputs. On the other hand, the surrogate model required running roughly twenty training cases. At this stage, most of the sub-tasks are automated. The creation of a training case, for example, can be started with a single command line. However the entire process of generating the training points, training the model, generating the corresponding source-term distribution and starting OpenFOAM could be streamlined as maybe two or three OpenFOAM executables.

Extending the number of inputs for the surrogate model Currently the model is only trained to predict variations in the (β , Re) space. While the non-dimensionalization through the Reynolds number already allows variation in both the inflow velocity and the vortex generator height, more

variables such as the vortex generator height to chord ratio could be added. This however comes at the cost of increased sampling points. If a one-variable model is trained with *n* samples, achieving the same sampling density in a *k*-dimensional space would require n^k samples [16]. This is commonly referred to as the *curse of dimensionality*.

While those limitations may make the new surrogate model less practical than the well-established jBAY model, results have shown that it consistently delivers better results. The consistency of the source-term on different meshes, and its ability to provide accurate results on very coarse meshes are key benefits. Using the entire set of training points, the jBAY model does not match the accuracy of the surrogate model by any measure. Other sources of errors, which include the limitations of the OST method and the turbulence model, are likely the dominant sources of error.

The fact that the model is tunable for a range of inflow angles and inflow velocities make it particularly valuable for modeling a row of vortex generators on a wind turbine blade or on an aircraft wing (illustrated in the introduction: a row of vortex generators on a Boeing 787 wing). At every spanwise location of such a setup, the inflow angle and velocity are likely to change, in part due to wingtip vortices. For such a case, the surrogate model allows to reproduce an accurate flow field for the entire wing without the need of meshing and resolving the small vortex generators.

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