ORBITAL DECAY IN A DYNAMIC ATMOSPHERE

by

J. de Lafontaine

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Postscript

During computer simulation since this report was written, it was found that this theory is accurate for zero and small eccentricity ($e^2 \sim 0$). In order to extend this range to $0 \leq e \leq 0.2$ and still ensure good agreement, it was found that additional terms in the expansions of Equations (4.8), (4.12), (4.14), (5.5) and (5.12) are necessary. The analytical details and supporting simulation results can be found in a UTIAS report to be published in the near future.
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Summary

After a brief review of the field, the pertinent elements of the lifetime prediction problem are assembled. A quite complete model of the atmosphere is constructed which includes all the important dynamic effects, and which possesses several novel features. The equations of motion for the satellite are then presented together with a discussion of methods for their solution. The equations are first solved for the drag-only dispersions in a stationary atmosphere, and then, based on this foundation, more 'unified' solutions are derived due to the combined effects of air drag and Earth oblateness. Both circular and elliptical orbits are considered.
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List of Symbols

Unless defined otherwise in the text, the following symbolism will be assumed throughout this work. In cases of multiple significance, the context suggest the proper choice.

Roman, Lower Case

\(a\)
- semi-major axis of the orbit

\(a_D\)
- drag deceleration

\(a_i, b_i\)
- coefficients defined in Eqs. (3.117) and (3.118)

\(b\)
- bulge centre position vector

\(c\)
- flattening parameter, defined in Eq. (3.109)

\(d\)
- typical diameter of a satellite

\(d_i\)
- flattening coefficients, defined in Eqs. (3.115)

\(e\)
- eccentricity of the orbit

\(f\)
- Earth's flattening

\(P_{\text{max}} / P_{\text{min}}\) (Eq. (2.77))

\(f\)
- specific force vector

\(g\)
- acceleration due to gravity

\(g_i, h_i\)
- density coefficients, defined in Eqs. (3.114)

\(h\)
- altitude

\(|\mathbf{h}|\)
- magnitude of \(\mathbf{h}\)

\(\mathbf{h}\)
- specific angular momentum vector

\(j_i\)
- coefficients defined in Eqs. (3.120)

\(k\)
- proportionality constant between \(P\) and \(\rho\) (Eq. (B-2))

\(l\)
- typical length of a satellite

\(m\)
- mass (of the satellite, when no subscript)

\(n\)
- rate of change of \(H\) with \(h\)

\(\rightangle n\)
- angular distance between the line of nodes and the i.r.1.

\(p_i, q_0\)
- coefficients defined in Eqs. (3.96) and (3.116)

\(r\)
- magnitude of \(\mathbf{r}\)

\(\mathbf{r}\)
- satellite position vector

\(s\)
- scale-u

\(t\)
- time

\(u\)
- reciprocal of \(r\)
<table>
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<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>v</td>
<td>magnitude of $\gamma$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>velocity vector (of the satellite, when no subscript)</td>
</tr>
<tr>
<td>$x_1, x_2$</td>
<td>random variables</td>
</tr>
<tr>
<td>z</td>
<td>dimensionless variable, defined in Eq. (3.113)</td>
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**Roman, Upper Case**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>A</td>
<td>projected area of the satellite</td>
</tr>
<tr>
<td>$A_p$ or $K_p$</td>
<td>geomagnetic planetary index</td>
</tr>
<tr>
<td>B</td>
<td>ballistic parameter, defined in Eq. (3.39)</td>
</tr>
<tr>
<td>A, B</td>
<td>coefficients in the expansion of $\cos \phi$</td>
</tr>
<tr>
<td>C</td>
<td>defined in Eq. (3.81)</td>
</tr>
<tr>
<td>$C_D$</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>D</td>
<td>semi-annual and 11-year solar cycle density variations factor</td>
</tr>
<tr>
<td>F</td>
<td>bulge factor</td>
</tr>
<tr>
<td>$F$</td>
<td>force vector</td>
</tr>
<tr>
<td>$F_{10.7}$ or $F$</td>
<td>solar flux index</td>
</tr>
<tr>
<td>$\overline{F}$</td>
<td>mean solar flux index</td>
</tr>
<tr>
<td>G</td>
<td>universal gravitational constant</td>
</tr>
<tr>
<td>H</td>
<td>scale-height</td>
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<tr>
<td>I</td>
<td>orbital inclination</td>
</tr>
<tr>
<td>$I_n$</td>
<td>Bessel functions of imaginary argument of order $n$</td>
</tr>
<tr>
<td>J</td>
<td>oblateness parameter, defined in Eq. (3.36)</td>
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<tr>
<td>$J_2$</td>
<td>second harmonic of the geopotential field</td>
</tr>
<tr>
<td>K</td>
<td>reciprocal of the semi-latus rectum</td>
</tr>
<tr>
<td>L</td>
<td>Sun's mean longitude</td>
</tr>
<tr>
<td>M</td>
<td>molecular weight of the atmosphere</td>
</tr>
<tr>
<td>P</td>
<td>atmospheric pressure</td>
</tr>
<tr>
<td>R</td>
<td>random density variations factor</td>
</tr>
<tr>
<td>$S$</td>
<td>variable defined in Eq. (3.119)</td>
</tr>
<tr>
<td>$S_T$</td>
<td>Sun's position vector</td>
</tr>
<tr>
<td>T</td>
<td>temperature</td>
</tr>
<tr>
<td>$T_\infty$</td>
<td>exospheric temperature</td>
</tr>
<tr>
<td>$T_t$</td>
<td>tropical year</td>
</tr>
<tr>
<td>U</td>
<td>angle between $\gamma$ and the line of nodes</td>
</tr>
</tbody>
</table>
Greek, Lower Case

\( \alpha \)  
right ascension  
matching coefficient, defined in Eq. (2.23)

\( \beta \)  
scale-\( u \) correction factor

\( \gamma \)  
Earth's gravitational parameter \( (Gm_{\odot}) \)

\( \delta \)  
declination  
rate of change of the bulge factor with \( u \)

\( \varepsilon \)  
scale-\( u \) difference, defined in Eq. (2.37b)
Sun's orbital inclination

\( \theta \)  
true anomaly

\( \theta_h \)  
angle between \( \zeta \) and the i.r.l.

\( \lambda \)  
angle between \( S \) and \( b \)

\( \mu \)  
maximum relative deviation in \( B \)

\( \nu \)  
reciprocal of \( \sigma \)

\( \xi \)  
ratio of the reference distance to the initial perigee distance, Eq. (3.77a)

\( \rho \)  
atmospheric density

\( \sigma \)  
radius of a constant-altitude surface

\( \phi \)  
angle between \( \zeta \) and \( b \)

\( \psi \)  
angle between the perigee position and the i.r.l.

\( \omega \)  
argument of perigee  
angular velocity (with subscript)

Greek, Upper Case

\( \Lambda \)  
angular velocity of the atmosphere in revolutions/day

\( T \)  
period (of the orbit, when no subscript)

\( \phi \)  
defined in Eq. (2.86)

\( \Omega \)  
right ascension of the ascending node

Script Letters

\( D \)  
defined in Eq. (6.11)

\( F \)  
atmospheric rotation factor, Eq. (3.43)  
'vectrices'

\( S_1, S_2 \)  
defined in Eqs. (2.48)
Others

i.r.l. inertial reference line

Δ₁ change in an orbital parameter after the first half-orbit

Δ₂ change in an orbital parameter after the second half-orbit

Subscripts

A refer to the atmosphere

Aₐvg average over one body revolution and one orbit

c denotes circular-orbits parameters

D atmospheric drag

drag component of a vector

E refer to Earth values

e denotes equatorial values

f variable reference altitude (fitting point)

g gravity component of a vector

h component of a vector along h

I inertial frame

max, min maximum, minimum values

o orbital frame

p refer to perigee values

r fixed reference distance

component of vector along r

R relative to the atmosphere

S solar frame

sc solar cycle density variations

sa semi-annual density variations

U component of a vector along h x r (transverse)

u denotes the u distance of the point (u, δ)

v variable reference altitude

o initial or reference values

1,2 denote fitting points

2π change in an orbital parameter after one orbit

↓ refer to semi-latus rectum values
Superscript

- denotes non-dimensional variables
  \(k\) changes in \(\bar{K}\)
  \(e\) changes in \(e\)
  \(\omega\) changes in \(\omega\)
  \(i\) changes in \(I\)
  \(\Omega\) changes in \(\Omega\)
  \(c\) denotes circular orbits.
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1. INTRODUCTION

In order to fulfil the various needs of the nations that launched them, today's satellites are subject to very diversified types of missions. In particular, some require relatively low-altitude orbits (perigee below 400 to 600 km). Satellites on such orbits, often referred to as near-Earth satellites, experience the influence of air drag which, in some cases, can have many times the magnitude of other orbital perturbations. Under the presence of such disturbances, the nominal elliptical (or circular) path of the satellite is perturbed. When atmospheric drag is involved, secular variations in the trajectory become rapidly important. This phenomenon has motivated many scientists to study trajectory prediction and dispersion analysis [1] in order to:

(i) accurately predict the position of a satellite at any future time when air drag is present;

(ii) provide the analytical tools necessary to derive the properties of the atmosphere (chiefly density and its variation in space and time);

(iii) help mission planners in defining and predicting the particular and sometimes unexpected behaviour of satellites subject to air drag (early re-entry, out-of-plane dispersions, etc.).

Along with the needs for accurate trajectory predictions, another particular orbital phenomenon was at the source of the newly born theory of satellites in an atmosphere. Since atmospheric drag is a non-conservative perturbation, a satellite constantly loses its energy through friction and consequently its orbit slowly contracts until re-entry occurs. Here, 're-entry' refers to the final plunge of the satellite into the lower atmosphere. Ablation, breakup and volatilization are likely to occur during this phase but ground impact is a probable outcome. As the density, weight, and number of spacecraft on low terrestrial orbits increase (an average of 1,000 new satellites catalogued each year by NORAD between 1974 and 1977) [2], this probability becomes more and more important. The fraction of the catalogued satellites that eventually decay is surprisingly constant: 57%. This represents an approximate average of 570 decays each year, or more than three decays every two days [2]. In 1968, it was estimated that about once every week a large object would decay and scatter small fragments into the lower atmosphere, representing non-negligible hazards to aircraft [3]. Clearly, Cosmos 954 and Skylab are not isolated cases. In [4], the reader can verify
that the number of decayed satellites is indeed high. Hence, the theory of lifetime estimation was gradually developed [1] in order to:

(i) compute in advance the expected lifetime of satellites, permitting realistic and adequate mission planning;

(ii) predict the re-entry time of an uncontrolled satellite so as to avoid aircraft hazards, body injuries, etc;

(iii) in cases where the satellite is still under attitude control, obtain re-entry predictions as a function of the attitude, allowing the control with reasonable accuracy of the re-entry point and the probable impact region and thus avoid populated areas.

Trajectory estimations and lifetime predictions of near-Earth satellites are not trivial tasks. The theorist is limited by the almost impossible task of modelling parameters, the inaccurate models he uses for the known perturbations and the still further assumptions and approximations to which he must have recourse in order to solve the problem. Added to these difficulties is the stochastic nature of the atmospheric properties. Consequently any theory owes its accuracy to proper models of the main factors influencing the orbital life of a satellite. These factors, defined in [1] as "dispersion parameters" are discussed in Appendix A. It is the purpose of this thesis to set out the theoretical grounds from which lifetime predictions can be obtained by considering approximate mathematical models for the above-mentioned perturbations. But let us first briefly review the available papers and reports on this subject.

1.1 Review of Previous Work

The brief survey that follows will be considered in three sections:

(a) Trajectory dispersion analysis and lifetime prediction
(b) Modelling of the atmosphere
(c) Modelling of the satellite.

Work will be cited chronologically.

1.1.1 Trajectory Dispersion Analysis and Lifetime Prediction

The first semi-analytical method for predicting the lifetime of a satellite was developed by Singer [5] and analytical considerations of elliptical orbits were investigated by Henry [6]. Later, when the first satellites were launched, the determination of air density became an important topic of research which was carried out by Groves [7] and Sterne [8]. Nonweiler [9], and King-Hele and Leslie [10] first obtained the variation of the orbital period, of the perigee height and of the eccentricity with time.
Perkins [11] used perturbation solutions to obtain lifetime estimates which were general enough to incorporate arbitrary density models and, in particular, results were given for the cases of exponential and power-law density variation. Parkyn [12] integrated analytically Lagrange's planetary equations and obtained the variations in $a$, $e$ and $r_p$ in terms of Bessel functions. The effects of a non-stationary atmosphere on the orbital inclination were considered by Bosanquet [13], Vinti [14] and also by Merson and Plimmer [15]. Variations over the whole lifetime of a satellite were derived by Michielsen [16], including a power-law model, while per-orbit changes were investigated by Parkyn [17], Ewart [18] and many others. Parkyn also considered the effects of atmospheric flattening. One of the first attempts to develop a "unified" theory included gravitational and air-drag effects for circular orbits and is due to Brouwer and Hori [19].

The rotational motion of the atmosphere was then studied in terms of its effects on $I$ and $\omega$ (Cook and Plimmer [20]) and its effect on $\Omega$, $I$ and $\omega$ in the presence of atmosphere flattening was later discussed by Cook [21]. An extensive investigation was accomplished by Cook, King-Hele and Walker in order to consider the influence of atmospheric rotation, flattening and varying scale-height. Highly eccentric orbits were also discussed [22,23,24,25]. During this period, Karrenberg et al [26] obtained a closed-form solution for the satellite trajectory which was then used to compute in-plane and out-of-plane dispersions due to the uncertainty in the mean atmospheric density. The particular case of intermediate-eccentricity orbits was discussed by Newton [27] considering a simple model atmosphere.

In 1962, one of the first surveys of the available theories on satellite lifetime was presented by Billik [28]. The following year, Zee [29] derived the closed-form expression for the mean path of a satellite subject to drag, and for the oscillatory motion about this mean, assuming an exponentially varying density model. King-Hele [30] presented a very thorough theory of satellites in an atmosphere in which the Lagrange planetary equations were first integrated over one orbit and then, over the entire lifetime of the satellites. A simple spherical exponential density model was first discussed, and then flattening effects, scale-height variations and the influence of atmospheric rotation on $I$ and $\omega$ were included as well as the variations in $\omega$. This was later extended from a spherical to a diurnal atmosphere by Cook and King-Hele [31]. Next, nearly circular orbits were considered [32]. Graphical methods for estimating lifetime were introduced by Ladner and Ragsdale [33] where diurnal and 11-year cyclic density
variations were considered along with the effects of mass loss and attitude stabilization. Similarly, Vlasov [34] devised a graphical method for predicting lifetime in which the 11-year solar cycle variation was considered. The prediction of satellite orbit decay and impact was numerically investigated by Gasley et al [35] and by Barry et al [36], where both special and general perturbation theory were used to consider the effects of Earth's flattening and atmospheric drag in a dynamic model atmosphere. This theory, which obtains semi-analytical solutions by an asymptotic method, represents one of the most complete in trajectory prediction. Brofman [37] also used asymptotic methods and obtained approximate analytical solutions for the orbit decay as a function of time. Mean and oscillatory paths were also depicted. The combined influence of Earth oblateness and air drag were investigated by Zee [38] and the combined effect of atmospheric rotation and diurnal density variations on I were obtained by King-Hele and Walker [39].

A simple relation giving the time required for a near-circular orbit to decay down to a given height was introduced by Egorov [40]. Morduchow and Volpe [41] developed exact analytical solutions for the orbits of bodies under drag influence but the atmospheric models considered were, in general, simplistic. Rapid estimates of satellite lifetime were treated in a semi-computational manner by Fuchs et al [42], and graphically by Perini [43]. The consideration of a quadratic scale-height can be found in Willey and Piscane's paper [44]. Santora [45] presented an analysis similar to that of King-Hele [30], but he used a global approach instead of considering the effects of the individual dispersion parameters alone. In [45], decay rates for $T$, $e$ and $\omega$ were obtained by a semi-analytical method; the atmosphere was assumed to be flattened and diurnally varying. Auxiliary equations for the gravitational perturbations in $r_p$ and $\omega$ and the proper selection of the diurnal scale-height were also included. Santora later [46] introduced gravitational perturbations directly into the equations to provide more realistic density profiles. The effects of atmospheric rotation and of geomagnetic and solar activity on the accuracy of prediction of satellite position are studied in [47]. Extending his own theory, King-Hele [48] has recently used graphical approximations to consider the dynamic density variations of the atmosphere.

1.1.2 Modelling of the Atmosphere

Since a large number of papers have been written on this subject, a complete review will not be attempted. Instead, a few of the main topics
directly related to orbital decay will be considered here.

King-Hele [30,48], Jacchia [49,50,51,52], Walker [53], King-Hele and Walker [54] and Martin and Priester [55] present a relatively complete picture of the atmospheric fluctuations which dominate the theory of near-Earth satellite dynamics.

In particular, atmospheric rotation is analysed by King-Hele and Scott [56,57], King-Hele, Scott and Walker [58] (where its dependence on altitude is considered), Ching [59], King-Hele [60] and Schuchardt and Blum [61]. A tentative explanation of the semi-annual density variation is given by Cook [62], and other considerations can be found in King-Hele and Walker [63], Jacchia [64] and Cook [65]. The day-to-night and 11-year cyclic density variations were discussed in most of the above papers and especially in Jacchia and Slavey [66], King-Hele and Quinn [67], Martin in Priester [68] and in Smith [69], where the unexpectedly high solar activity of 1976 is treated. The rapid density fluctuations of unpredicted nature that are correlated with the random variations in solar and geomagnetic activity are treated in Nonweiler [70] (solar flares), King-Hele [71,72] (27-day cyclic variations), King-Hele and Quinn [73], Cook [74] and Bencze [75]. Atmospheric flattening is discussed by Lee [76], King-Hele [30] and Santora [45,46].

### 1.1.3 Modelling of the Satellite

A theoretical determination of the projected area, A, and of the drag coefficient, $C_D$, is also a very involved field. The most relevant papers on the aerodynamic properties of satellites are those by Cook [77,79], Nocilla [78], and Jastrow and Pearse [80]. Drag coefficients were investigated in King-Hele [30], Cook [77,79], Williams [81,82] and Nocilla [78]. The effects of uncertainty and variation in $C_D$ are treated in Hunziker [83] and a discussion of the thermal accommodation coefficient and other related parameters is found in King-Hele [30], Ladner and Ragsdale [33] and Wachman [84].

### 1.2 State of the Art

To assess its merits, a theory must be evaluated in the following aspects [1]:

Step (a) Derivation;
Step (b) Utilization (replacement of the algebraic variable by numerical values);
Step (c) Comparison of results with actual case.
Now let us turn to the two fundamental methods into which the various theories of satellite lifetime can be classified and consider whether they are simple, cheap, and accurate.

The first method is the 'general perturbation method' or, equivalently, the 'analytical method'. In this case, Step (a) involves the use of approximations, series expansions and analytical integration, which result in closed-form solutions giving the state of the satellite as a function of time. Sometimes these solutions are in parametric form and computer aid may be required. However, no computer integration is involved. The main features of this method are [1]:

(i) long and tedious mathematical manipulations in its formulation and implementation are usually required;
(ii) usually implies simplistic models for the perturbing forces (many approximations) and restricted range of application (many assumptions) in order to obtain closed-form solutions;
(iii) provides a better visualization of the effects of the various perturbation in Step (a) because, even before the solutions are applied to a particular satellite, they already give insight into the physical behaviour of any satellite;
(iv) little or no computing time required to get the results;
(v) general by nature because analytical results are obtained before any particular satellite is considered;
(vi) relatively accurate, depending on the amount of effort spent in Step (a).

From the author's literature survey, it can be inferred that analytical theories tend to be complex if some accuracy is needed but they are usually relatively inexpensive to utilize. A good example is King-Hele's theory [30]. Analytic methods are therefore well suited to lifetime predictions where many evaluations are required at low cost. Trajectory estimations can be accurate for short time intervals and become less precise with time because of the approximations made in Step (a).

The second fundamental method is the 'special perturbation' method (the 'numerical integration' method). In this case, the equations of motion of the perturbed satellite are directly integrated using an appropriate numerical procedure. Step (a) results in a standard computer program. The solutions are obtained only after Step (b) is carried out, that is, after a particular satellite is considered. The characteristics of this method are:

(i) usually simple in its formulation;
may include very realistic and complex perturbation models with a large range of applicability;

visualization of the perturbation effects not readily obtained (many trial trajectories must be integrated each time a given parameter is varied so that its influence can be recorded);

large computer time requirements;

specific by nature (numerical values of the parameters of a particular satellite must be available before results can be obtained);

relatively accurate, although truncation and roundoff errors can be significant in long term predictions.

Such methods are usually simple but they can be expensive especially when high accuracy is required, (e.g. [35]). Numerical methods are well suited for short lifetime predictions, although the cost and inaccuracy increase as the lifetime grows. They are also very efficient in trajectory prediction.

From these considerations, it is clear that the existing theories are costly in either Step (a) or Step (b) and that the accuracy of their results are proportional to the resources invested in these two steps. Therefore, current lifetime prediction is limited by the available analytical methods, and by computer hardware and software. However, even if the analyst were given the best possible tools in these two fields, his theory would still be restricted by other constraints. First, no mathematical model can perfectly describe a physical phenomenon, especially when random processes and unpredictable fluctuations are present, as they are in the case of the atmospheric density variations. Second, when closed-form solutions are sought, the theorist must rely on various approximations and assumptions. Finally, there remain areas of ignorance in the mechanisms affecting the satellite's behaviours, and these make a perfect mathematical model an impossible task. For instance, scientists are still not agreed on how to predict the future activity of the Sun in an upcoming cycle [69].

Despite these limiting factors, the principal orbital perturbations have been properly identified (atmosphere, geopotential field, satellite properties), and many elegant models have been derived to describe day-to-night and semi-annual density variations, the influence of solar and geomagnetic activity, atmospheric rotation and flattening, and the gravitational field. Several attempts have also been made to include these disturbances in a "unified" theory. Santora [45] is worth noting because he uses a semi-analytical semi-numerical method; when the complexity of the
perturbation model makes the implementation of an analytical theory (Step (a)) too costly (or impossible) and since numerical methods are too expensive in lifetime predictions, a wise compromise is to simply "split" the costs between Steps (a) and (b): solve part of the problem analytically and part by computer integration. Such semi-analytical methods seem very promising and well suited to the problem of lifetime prediction.

In summary, the chief current problems in the theory of trajectory prediction and lifetime estimation are:

(i) the increasing theoretical and computational complexity of the theory as mathematical models become more and more realistic;

(ii) the difficulty in deriving more accurate mathematical models of known relevant physical phenomena;

(iii) the identification of, and models for, new relevant physical phenomena.

Future trends for further development would seem to depend on the increased use of semi-analytical methods and the development of more accurate models.

1.3 Present Study

This work is concerned with the prediction of re-entry initiation for uncontrolled, near-Earth satellites of "normal" shape, in moderate and low-eccentricity orbits. Prediction of re-entry initiation does not include consideration of the re-entry stage itself. Estimates will be sought of the time spent in Stage A only.

Uncontrollability, which is usually a property of a decaying satellite, permits simplifications of the theory regarding lift generation, some satellite properties (C_D, A), and the attitude motion. "Near-Earth" satellites suffer the influence of air drag during all or part of their orbit (perigee heights below 500 km.). "Normal" shape refers to cylindrical (1/2 < l/d > 8), near-spherical, and spherical satellites, of normal mean density. Balloon-type satellites (subject to strong solar pressure) and spacecraft subject to non-negligible gravity gradient torques are thus excluded.

1. Stage A and Stage B refer to the decaying and re-entry trajectories respectively. Physically, the "fast" and the "slow" motions of the satellite are separable in the first, and comparable in the second. See [1] for a more complete definition.
Moderate- and low-eccentricity orbits lie in the range

$0 \leq e \leq 0.2$

Combined with the condition $r_p \leq 500$ km., this condition renders luni-solar perturbations negligible compared to drag. Hence they are excluded in the formulation.

It is obvious from the literature survey given in Section 1.1 that only a few theories are "unified" (i.e. include more than one dispersion parameter). Moreover, these theories usually consider only two or three dispersion parameters in an analytical approach and computer integration must be resorted to if, for instance, all the (known) dynamic density variations are included. Also, a very limited number of theorists considered the effects of random density variations induced by the irregular solar flux variations (solar flares, Sun's rotation, active regions, etc.). Inclusion of all these factors can rapidly lead to mathematical intractibility or costly computer integration. The aim of the present work is to develop a model that possesses simplicity without too much loss in accuracy.

The method used combines the accuracy of the numerical integration method and the efficiency of the analytic method: it is a semi-analytic approach to the estimation of satellite lifetime. The "fast motion" of the satellite on its orbit is solved analytically in a manner similar to that of King-Hele and others. Once the dependence on the true anomaly is withdrawn, the resulting equations can be efficiently integrated on a computer with the smallest time scale being one orbital period.

In view of the existing problems described in Section 1.2, the following goals are set:

(a) Include a realistic model for the static and dynamic properties of the atmospheric density, namely the altitude, solar cycle, diurnal, semi-annual and flattening dependences. This deterministic formulation will represent a nominal atmosphere;

(b) Include random density variations about this nominal atmosphere so that the effects of unpredicted fluctuations due to solar flares and geomagnetic disturbances on the lifetime will be considered;

(c) Include atmospheric rotation;

(d) Include the possibility of analyzing the effects of random variations in the cross-sectional area $A$ and in the drag coefficient $C_D$ (see Appendix C);

(e) Include gravitational perturbations due to the oblate mass
distribution of the Earth;
(f) Develop analytical solutions suitable for efficient numerical
treatment without too much concession to either simplicity or
accuracy;
(g) Derive a method for obtaining the probability of re-entry within a
given time span;
(h) Derive solutions suitable for sensitivity analysis of the dispersion
parameters.
1.4 Outline of Thesis
The mathematical models for $C_D$, $A$, and the Earth's second-order
gravitational perturbation will be taken directly from the literature.
However, the atmospheric density model is discussed in detail in Section 2.

Section 3 gives the basic assumptions and the mathematical background
to the theory.

Section 4 considers the results of drag-induced planar dispersions,
and Section 5 generalizes this approach to include out-of-plane motion due to
the atmospheric rotation and the Earth's oblateness. In these two sections,
the variation of the orbital elements per orbit are obtained. These equations
give insight into the dispersive effects of drag and oblate mass. Circular
and equatorial orbits require a slightly different analysis and will thus be
considered separately in Sections 6 and 7.

A few comments on the above results will close the discussion
(Section 8).

Before discussing the atmospheric model, some definitions are
introduced to avoid possible misinterpretation.
1.5 Some Definitions

Even though the existence of a "satellite" may be said to cease at
volatilization in the lower atmosphere or at ground impact, the word
"lifetime" is used here to refer to the time spent in Stage A of its life.
It thus excludes the time required for the satellite to undergo its final
plunge towards the Earth, i.e. its re-entry (Stage B). This last stage is
relatively short [3], hence the distinction rarely needs to be considered.

The single word "oblateness" usually refers in the literature to
both the figure and the mass distribution of the Earth. However, a further
distinction saves lengthy descriptions. In the following, the term "oblateness"
will refer to the oblate mass distribution of the Earth while "flattening"
will suggest the flattened figure of the Earth.

The concept of "Perigee Approximation", denoted "PA" is also useful. In the course of the theoretical derivation, some parameters will have an unwanted dependence on the true anomaly. In other words, a variable \( v(\theta) \) complicates the theory, making closed-form solutions impossible. When these variables represent only a very small correction to the drag perturbation, it is relatively accurate and analytically simpler to evaluate this variable where the drag is dominant: at perigee (see Appendix B). Then, a "perigee approximation":

\[ v(\text{PA}) = v_p = v(\theta_p) = v(0) \]

would get rid of this dependence on the true anomaly although no appreciable accuracy is lost for small corrective parameters.

Finally, because the reciprocal of the radial distance, \( u \equiv 1/r \)

will be used throughout this work, the words 'altitude' and 'radial distance' will sometimes refer to the value of \( u \) at this altitude or distance. For instance, a \( u \)-distance of 6,578 km corresponds to a \( u \)-altitude of 200 km. (at the equator) and to the value \( u = 1.52 \times 10^{-4} \) km.\(^{-1} \).

The reader is referred to Appendix A for the definitions of environmental and specific perturbations as well as dispersion parameters, and to Appendix B for those of static, dynamic, theoretical, semi-empirical, empirical, flattened and spherical (equatorial) density models. Stage A and Stage B are briefly described in the footnote on page 8.

2. MODEL OF THE ATMOSPHERE

2.1 Theoretical and Empirical Considerations

The static density model will be derived by a semi-empirical technique. More specifically, the following steps will be considered:

(i) Under the assumptions of hydrostatic equilibrium and constant temperature, a theoretical formulation will be obtained.

(ii) This mathematical expression will then undergo small corrections to allow for the flattening of the atmosphere and for the unrealistic assumptions of homogeneity (hidden in the hydrostatic equilibrium assumption, see Appendix B for details) and constant temperature. In particular, two different expressions for the scale-height as a
function of the altitude will be derived.

(iii) Finally, the resulting expression will be fitted to a standard model of the atmosphere. Different numbers of fitting points will be available, depending on the particular scale-height expression used.

To this fundamental density model will be added the dynamic density variations induced by the Sun. The sophistication of this dynamic density model will be limited by the current knowledge in this field. The resulting model will include the following features, some of which are refinements not encountered in the available literature:

(a) Altitude dependence of the specific force due to gravity in the hydrostatic equation:
\[ g = \gamma / r^2 \; ; \; \gamma = G m_E \]  
(2.1)

(b) Consideration of flattening effects through the definition of a constant-altitude spheroid of reference.

(c) Dependence of scale-height on altitude to account for:
   - temperature variations
   - diffusive separation and variations in the molecular weight (heterogeneity)

(d) Consideration of 11-year solar cycle density fluctuations as well as their dependence on altitude

(e) Inclusion of day-to-night density variations.

(f) Altitude variation of the bulge factor \( F \) (See Appendix B for definition)

(g) Consideration of semi-annual density variations and their dependence on altitude.

(h) Random variations due to geomagnetic activity and solar flares, with the 27-day recurrence of high density induced by the Sun's rotation.

One might think that the consideration of so many sources of density variations would render the model analytically intractable, with mathematical simplicity traded for a more realistic density expression. Actually,

1. The hydrostatic equation is, in theory, no longer valid at the altitude where heterogeneity becomes dominant. Here, however, the hydrostatic equation is kept and the scale-height assumes an appropriate altitude variation to account for this fact.
although the analytic work load is nonnegligible, it will be shown that closed-form solutions for one orbital revolution are still possible. As a result, this density model possesses the following advantages:

1) More realistic theory than one based on the usual "exponential" model, since both $g$ and the scale-height are altitude-dependent here.

2) Better theoretical foundation than the power-law model which, strictly speaking, is an empirical model.

3) Mathematical simplicity when used in Lagrange's Planetary Equations. Bessel functions of imaginary argument can be obtained without the usual change of variable from the true to the eccentric anomaly (as in [30], [45]).

4) The mathematical expression of this model provides many matching coefficients so that many fitting points can be defined, thus increasing the accuracy of the static part of the model.

Finally, the rotation of the atmosphere with respect to an inertial frame is considered.

2.2 Static Model for the Density

It must be kept in mind that the static model represents the mean density of the atmosphere averaged over diurnal and semi-annual variations for moderate solar activity. The only dependence is on the altitude.

2.2.1 Altitude Dependence

The discussion will begin with the consideration of an equatorial (or spherical)\(^1\) density model. This assumption will be released in Section 2.2-2.

Using the hydrostatic equation (Equation B-1)

$$\frac{dp}{\rho} = -\frac{g}{k} \, dh$$

(2.2)

$\rho =$ Atmospheric Density

$h =$ Altitude

$k:$ defined in Equation B-2.

Let us perform the change of variable

$$h = r - r_r$$

(2.3)

where $r_r$ is the (constant) radius of a spherical surface of reference so that $dr = dh$ in (2.2). Keeping $k$ constant and substituting (2.1) into (2.2), the integration can be carried out to yield:

\(^1\)See Appendix B for definition of equatorial or spherical static density models.
\[ \rho(r) = \rho_r \exp \left[ \frac{\gamma}{k} \left( \frac{1}{r} - \frac{1}{r_r} \right) \right] \]

where \( \rho_r = \rho(r_r) \) is the density on the sphere of reference.

Defining
\[ u = \frac{1}{r}, \quad u_r = \frac{1}{r_r} \]

the expression becomes
\[ \rho = \rho_r \exp \left( \frac{\gamma}{k} (u - u_r) \right) \]  \hspace{1cm} (2.5)

Equation (2.5) represents the spherical static density model under the assumptions of constant \( k \) (constant temperature and molecular weight) and hydrostatic equilibrium.

Note that when
\[ \frac{\gamma}{r r_r} = \frac{\gamma}{r_r^2} = g_r \]

Equation (2.5) reduces to the exponential model. Under similar assumptions, the power-law form given by Egorov[40] is obtained:
\[ \rho \alpha = \rho_r \alpha_r \]

By analogy with the scale-height, \( H = -\rho/(d\rho/dr) \), a "scale-u", \( s \), is defined:
\[ s = \frac{1}{\rho} \frac{d\rho}{du} \]  \hspace{1cm} (2.6a)

so that
\[ s = \frac{\gamma}{k} = \frac{\gamma M}{RT} \] from Appendix B. \hspace{1cm} (2.6b)

Defined in this manner, \( s \) has the same units as \( H \). From (2.6) follows:
\[ \rho = \rho_r e^{s(u - u_r)} \]  \hspace{1cm} (2.7)

Equation (2.7) is merely the equation of a variable-\( g \) exponential density model.

As stated earlier, equation (2.7) assumes constant temperature \( T \) and constant molecular weight \( M \). However, it is recognized that above approximately 100 km, the atmospheric temperature increases with altitude [49] while the molecular-weight decreases. From (2.6b), it can be seen that \( s \) decreases with altitude. Consequently, further refinements are required. These will be discussed in Section 2.2-3. Let us first consider the flattening of the atmosphere.
2.2.2 Flattening Corrections

This correction is simply carried out by substituting a flattened spheroid of reference for the spherical surface of reference used in Section 2.2.1. The details of this transformation due to King-Hele [30] are given in [1] and reported in Appendix B for easy reference.

The geocentric radius of the spheroid of reference, \( \sigma_r \), is (Equation B-10),

\[
\sigma_r = \sigma_{re} (1 - f \sin^2 \delta) \quad (2.8)
\]

where \( \sigma_{re} \) = equatorial radius of the reference spheroid

\( f \) = Earth's flattening

\( \delta \) = geocentric latitude (declination) of the point for which the altitude is required.

The subscript "r" on \( \sigma_r \) means that the reference spheroid contains the point \((r_r, \delta_r)\), used as a reference point. Then,

\[
\sigma_{re} = \frac{r_r}{1 - f \sin^2 \delta_r} \quad (2.9)
\]

and

\[
\sigma_r = \left[ \frac{r_r}{1 - f \sin^2 \delta_r} \right] (1 - f \sin^2 \delta) \quad (2.10)
\]

Using definition (2.4), the constant altitude surface is expressed in the u-space. The following identities follow:

\[
v_r = \frac{1}{\sigma_r} \quad (2.11)
\]

\[
v_{re} = \frac{1}{\sigma_{re}} \quad (2.12)
\]

and consequently

\[
v_r = \frac{v_{re}}{(1 - f \sin^2 \delta)} \quad (2.13)
\]

where

\[
v_{re} = u_r (1 - f \sin^2 \delta_r) \quad (2.14)
\]

and so

\[
v_r = u_r \frac{(1 - f \sin^2 \delta_r)}{(1 - f \sin^2 \delta)} \quad (2.15)
\]

Inserting the flattening correction into (2.7) yields:

\[
\rho = \rho_r \exp [s(u - v_r)] \quad (2.16)
\]
where $\rho_{r}$ is the density on the spheroid $\sigma_{r}$, that is, the density at the point $(r_{r}, \delta_{r})$. From Equation (2.15), it is clear that $\nu_{r}$ is a function of $\delta$ and that $\nu_{r}$ describes a flattened surface when $\delta$ takes on all possible values. However, it is important to note that in equations like (2.16), $\nu_{r}$ now represents a distance since $\delta$ takes on the specific value of the declination of the point at which $\rho$ is required. Although this dependence of $\nu_{r}$ on the spatial position (through $\delta$) is not explicitly evident in the expression $(u - \nu_{r})$, it must always be kept in mind. When confusion is probable, the notation

$$\rho_{i} = \rho_{r} \exp [s(u_{i} - \nu_{r}(\delta_{u_{i}}))]$$

will be used for the density at any point $(u_{i}, \delta_{u_{i}})$. Note that $\nu_{r}(\delta_{u_{i}})$ still contains the reference point $(u_{r}, \delta_{r})$. From these considerations, it is clear that, choosing the reference surface so that it contains the point $(u, \delta)$, we get from (2.13) and (2.14)

$$\nu_{u} = \nu_{ue}/(1 - f \sin^{2}\delta)$$

where

$$\nu_{ue} = u (1 - f \sin^{2}\delta_{u})$$

so that

$$\nu_{u} = \frac{u (1 - f \sin^{2}\delta_{u})}{(1 - f \sin^{2}\delta)}$$

and, because $\delta_{u} = \delta$,

$$\nu_{u} = \nu_{u}(\delta_{u}) = u$$

(2.17a)

and

$$\nu_{ue} = \nu_{e} = u(1 - f \sin^{2}\delta)$$

(2.17b)

When the static model is to be fitted to the standard atmosphere, it is evident that the flattening correction is redundant. In fact, a standard atmosphere gives density as a function of the altitude. Therefore, a flattened static density model must be transformed in its equatorial (or spherical) form before the fit is performed. This transformation is performed by setting

$$f = 0$$

or, equivalently, by considering the density in the equatorial plane, i.e.

$$\delta = 0$$

It follows that $\nu_{re}$ is substituted for $\nu_{r}$ in an equational form. For
instance, the equatorial form of (2.16) is
\[ \rho = \rho_r e^{s(u_e - v_{re})} \]
or, from (2.17b),
\[ \rho = \rho_r e^{s(v_{ue} - v_{re})} \]
This notation for equatorial models will avoid confusion with the flattened static models.

2.2.3 Scale-u Variations

Because of the dependence of the molecular-weight \( M \) and of the temperature \( T \) on altitude, it is clear that \( s \) will also vary with altitude. The proper expression of this dependence will be obtained through the empirical technique of curve fitting. Two mathematical formulations will be derived.

The first of these consists in relating \( s \) to the usual scale-height, \( H \), for which a detailed dependence on altitude is given in CIRA 1972 (Cospar International Reference Atmosphere). The following relation is a good approximation within the range 200-600 km:
\[ H = H_1 + m(r - \sigma_i) \tag{2.18} \]
where
\[ m = \frac{dH}{dr} \approx 0.1 \quad \text{(in that altitude range)} \]
\( \sigma_i \) = reference spheroid where \( H = H_1 \).

Note that the flattening effect is taken into account in (2.18) through \( \sigma_i \).

From the definition
\[ H = -\rho/(d\rho/dr) \]
one obtains
\[ s = \frac{1}{u^2H} \tag{2.19} \]
or
\[ s = \left[ u^2(H_r - \frac{m}{v_r}) + mu \right]^{-1} \tag{2.20} \]
from (2.18) in which the reference spheroid \( \sigma_i \) is taken at "r", that is,
\[ \sigma_i = \sigma_r \quad \text{and} \quad H_i = H_r \quad (2.21) \]

Using equation (2.13) in (2.20) yields:

\[ s = \frac{\nu_{re}}{\mu[\nu_{re} - \alpha u]} \quad (2.22) \]

where

\[ \alpha = 1 - f \sin^2 \delta - \frac{H_r \nu_{re}}{m} \quad (2.23) \]

Using (2.23) into (2.16) gives the first static density model, labelled SM1 for easy reference. It is equivalent to a g- and H-varying exponential density model. Model SM1 is thus:

\[ \rho = \rho_r \exp \left[ \frac{\nu_{re}}{\mu(\nu_{re} - au)} (u - \nu_r) \right] \quad (2.24) \]

where \( \nu_{re} \) and \( \nu_{ue} \) are given in equations (2.13) and (2.14)

\[ \alpha = 1 - f \sin^2 \delta - \frac{H_r \nu_{re}}{m} \]

(\( u, \delta \)) = u-distance and declination of the point considered

\( H_r = H \) at \((u_r, \delta_r)\) i.e. on \( \nu_r \).

\( m = \frac{dH}{dr} \approx 0.1 \)

Equation (2.24) represents a convenient mathematical expression to be fitted to experimental data or any standard atmospheric density tables. Note that its equatorial form, SM1_e, has first to be obtained:

\[ \rho = \rho_r \exp [s_e(\nu_{ue} - \nu_{re})] \quad (2.25) \]

with

\[ s_e = \nu_{re} \left[ m \nu_{ue} (\nu_{re} - \alpha_e \nu_{ue}) \right]^{-1} \quad (2.26) \]

and

\[ \alpha_e = 1 - \frac{H_r \nu_{re}}{m} \quad (2.27) \]

Then, because

\[ \nu_{ue} = 1/(r_{ee} + h) \quad (2.28a) \]

\[ \nu_{re} = 1/(r_{ee} + h_r) \quad (2.28b) \]

SM1_e is directly related to the altitude, the required variable. Note that since
\[ \rho_r = \rho_{re} \]

the notation \( \rho_r \) is kept in (2.25).

The coefficients \( \alpha_e \) (given in (2.27)) and \( m \) (equal to \( dH/dr \)) could be evaluated from their definition and \( \text{SM}1_e \) would be completely defined, apart from the reference altitude which is left to the choice of the analyst. Approximate values are

\[
\begin{align*}
\alpha_e &\approx 0.9695 & v_{re} \text{ at 170 km.} \\
\alpha_e &\approx 0.8844 & v_{re} \text{ at 800 km.}
\end{align*}
\]

assuming

\[ H = 20 + 0.1(h - 170) \text{ km.} \]

However, it seems more attractive and probably more accurate to use \( m \) and \( \alpha_e \) as matching coefficients so that \( \text{SM}1_e \) closely fits any standard atmosphere. This will be discussed in Section 2.2.4.

Once \( m \) and \( \alpha_e \) are obtained, \( \text{SM}1 \) is recovered from (2.27) and (2.22):

\[ \alpha = \alpha_e - f \sin^2 \delta. \]

\( \text{SM}1 \) has the advantage of providing three matching coefficients \( (m, \alpha_e, v_{re}) \) and consequently three fitting points will result.

A second possible expression for the scale-u is an extension to Equation (2.26).

\[ \hat{s} = [a_0 + a_1 u + a_2 u^2 + \ldots + a_{n-1} u^{n-1}]^{-1} \]  
\[ (2.29) \]

This model, \( \text{SM}2 \), written

\[ \rho = \rho_r \exp[\hat{s} (u - v_{re})] \]  
\[ (2.30) \]

offers \( n-1 \) matching coefficients. Contrarily to \( \text{SM}1 \), \( \text{SM}2 \) does not allow the scale-u to vary with the latitude and consequently flattening effects in \( \hat{s} \) are ignored. However, it does provide a better fit since many fitting points can be defined, but the computational requirements increase proportionally.

The coefficients \( a_i \) can be solved for once the equatorial form of (2.30) is fitted to a standard atmosphere. This will be discussed again in the next section.

While the accuracy of an \( s \)-varying static model is very desirable it leads to difficulties when it is used in orbital theory. The scale-u
depends on the spatial position \((u, \delta)\) and consequently on the true anomaly of the satellite considered. Unless this dependence of \(s\) on the "fast" motion of the satellite is avoided, no closed-form solutions for the decay of the osculating orbits can be obtained. It will then be necessary to evaluate \(s\) at a given reference point, denoted by subscript "\(v\)". This reference point will vary with the "slow" motion of the satellite. In other words it will be fixed during each orbital revolution but will vary between them. Consequently, we have

\[
\begin{align*}
    s_v &= \frac{v_r}{\mu_v(v_{re} - \alpha_v u_v)} \quad (2.31) \\
    \alpha_v &= 1 - f \sin^2 \delta_v - v_{re} H_r/m \\ 
    (u_v, \delta_v) &= \text{current perigee location}
\end{align*}
\]

Since the drag deceleration is dominant at the perigee of the orbit, it is wise to make sure that \(s\) is accurate near or at perigee. Consequently, the variable reference point \(v\) is taken at the current perigee point (denoted by subscript "\(p\)"). Therefore,

\[
\begin{align*}
    s_p &= \frac{v_{re}}{\mu_p(v_{re} - \alpha_p u_p)} \quad (2.33) \\
    \alpha_p &= 1 - f \sin^2 \delta_p - v_{re} H_r/m \\
    (u_p, \delta_p) &= \text{current perigee location}
\end{align*}
\]

The reader familiar with Section 1.5 has already recognized that Equation (2.33) is the "perigee-approximated" equation of \(s\) given in (2.22). In (2.33), the unwanted dependence of \(s_p\) on the true anomaly is discarded. The perigee-approximated static model thus obtained, denoted by "SM1-s(PA)" is:

\[
\begin{align*}
    \rho &= \rho_r \exp \{s_p[u - v_r(\delta)]\} \\ 
    s_p &= \frac{v_{re}}{\mu_p(v_{re} - \alpha_p u_p)} \\ 
    \alpha_p &= 1 - f \sin^2 \delta_p - v_{re} H_r/m \\
    (u_p, \delta_p) &= \text{current perigee location}
\end{align*}
\]

Similarly, SM2 - s(PA) is

\[
\begin{align*}
    \rho &= \rho_r \exp \{s_p[u - v_r(\delta)]\} \\ 
    (u, \delta) &= \text{perigee location}
\end{align*}
\]

\[1. \quad \text{This is explained in Appendix B.}\]
\[ \hat{s}_p = [a_0 + a_1u_p + a_2u_p^2 + \ldots + a_{n-1}u_p^{n-1}]^{-1} \]  \hspace{1cm} (2.36b)

Note that when one speaks of a "perigee-approximated static model", one does not mean that the density is evaluated at perigee but rather the scale-only. This is why the letters "s(PA)" are attached to SM1 and SM2.

One may object that the time spent in deriving an accurate expression of \( s \) as a function of \( h \) is wasted when the perigee approximation is carried out. This is not so because, first, \( s \) still varies with altitude as the perigee height decays and oscillates during the satellite lifetime. Secondly, since perigee is the most important region for drag considerations, the s(PA)-models are very simple and legitimate approximations. Discrepancies in \( \rho \) at altitudes well above perigee height are comparatively less susceptible of influencing the decay of the orbit. Finally, for circular and very low eccentricity orbits, these models are accurate. But, for the sake of accuracy, an improvement to these s(PA)-models will be derived. Let us first find out the effects of perigee approximation.

Let \( \rho_R \) denote the density given by SM1 (or SM2; it does not matter). Subscript "R" will here refer to "real" or "actual" density, assuming, for a moment, that the model is a perfect fit. Let \( \rho^* \) denote its perigee-approximated form. Then, it is easy to show that,

\[ \rho^* = \rho_R e^{\varepsilon(u-v_r)} \]  \hspace{1cm} (2.37a)

where

\[ \varepsilon = s_p - s \]  \hspace{1cm} (2.37b)

It is clear from (2.37) that the s(PA)-model is accurate for a small range of altitude around the perigee height and around the reference height where \( \varepsilon(u-v_r) \) is nearly (or equal to) zero, so that \( \rho^* = \rho_R \). Moreover, since \( s \) decreases with altitude and the perigee is the lowest point on an orbit, \( \varepsilon \) is always positive and monotone increasing with altitude. From these remarks, two situations are recognized:

(i) Reference Altitude Lower Than, or At, Perigee (\( v_r \leq u \), for all \( u \))

In this case, \( (u-v_r) \) is always negative and monotone decreasing. This, coupled with the properties of \( \varepsilon \), renders \( \rho^* \) always smaller than \( \rho_R \) by a factor \( \exp[\varepsilon(u-v_r)] \leq 1 \). Then \( \rho^* \) is accurate near perigee but diverges rapidly from \( \rho_R \) as the altitude increases.
Reference Altitude Higher Than Perigee

In this case, \((u-v_r)\) is still monotone decreasing but it is positive for altitudes between perigee and reference height \((u>v_r)\) and negative for \(u<v_r\). Then, as the altitude increases, \(\rho^*\) starts from \(\rho_R (u=u_p)\), increases to a maximum above \(\rho_R (v_r<u<u_p)\), decreases and equals \(\rho_R \) again \((v_r=u<u_p)\) and finally decreases below \(\rho_R\) for all subsequent increases in altitude.

This divergence of \(\rho^*\) with respect to \(\rho_R\) will be shown graphically in the next section where numerical considerations for \(SM1\) are involved. It is pointed out here just to stress the fact that the perigee-approximated static model correctly fits the original model at two points: \(u_p\) and \(u_r\). Now suppose this original model (\(SM1\) or \(SM2\)) was obtained by curve fitting with a standard atmosphere. Then, while we are certain that \(SM1\) (or \(SM2\)) will be exact at \(u_r\) (since \((\rho_r, u_r)\) is a fitting point), it cannot be exact for all perigee heights. Therefore, in general, \(SM1 - s(PA)\) gives the same density as the standard atmosphere at only one point, the reference altitude \(v_r\) (its accuracy at perigee being proportional to the degree of success of the curve fitting). Then, one of the properties of the \(s(PA)\)-models is that the density they give at \(u_r\) is always exact (relative to the standard atmosphere). This property is useless if \(u_r\) is such that the influence of the atmosphere on the decay is either nonexistent or negligible. However, one could profit from this situation if an additional parameter were inserted to "control" this "exact-density altitude" so that it is always close to the altitude where \(\rho^*\) begins to diverge significantly from \(\rho_R\). Not only would one benefit from this property of the \(SM1 - s(PA)\) model, but one would increase the accuracy of this model. Such a parameter is introduced as follows.

Let us keep the notations \(\rho^*\) and \(\rho_R\) to represent the models \(SM1 - s(PA)\) and \(SM1^1\) respectively. Then, \(\rho^*\) evaluated at perigee yields,

\[
\rho_p^* = \rho_p \exp[s_p(u_p - v_r)] = \rho_p \exp[s_p(v_p - v_r)]
\]

(2.38)

From this we get,

\[
\rho_r = \rho_p^* \exp[s_p(v_r - v_p)]
\]

(2.39)

and substitution back into \(\rho^*\) gives,

\[
\rho^* = \rho_p^* e^{s_p(u-v_p)}
\]

(2.40)

1. This analysis also applies to the \(SM2\) model.
This is just an alternate form for SM1 - s(PA). Now, inspired by King-Hele [30], let us introduce the "scale-u correction factor", \( \beta_f \), in the following way,

\[
\rho = \rho^* \left[ 1 + \beta_f (u - \nu_p)^2 \right] \tag{2.41a}
\]

or

\[
\rho = \rho^* \left[ 1 + \beta_f (u - \nu_p)^2 \right] \exp[s_p (u - \nu_p)] \tag{2.41b}
\]

Then, \( \beta_f \) may be thought of as a new matching coefficient that enables the analyst to choose a fitting point in the altitude range relevant to the orbit. In particular, when \( \beta_f = 0 \), this point is at \( \nu_r \). Subscript "f" in \( \beta_f \) has the same meaning as "p" in \( s_p \); \( \beta_f \) is constant over one orbital revolution but varies in between and, in particular, as the perigee height and the orbit shape and size slowly vary during the satellite life. For easy reference, \( \tilde{\rho} \) in equation (2.41) will be called SM3 or SM4 depending on the model, SM1 or SM2, respectively, that it reduces to when \( \beta_f \) is set to zero.

The evaluation of \( \beta_f \) is the next problem. Suppose we want \( \tilde{\rho} \) to give exactly the same approximation of the standard atmosphere as \( \rho_R \) did (the original SM1 model). Then, if \( \tilde{\rho} = \rho_R \), substituting (2.37a) into (2.41a) and cancelling \( \tilde{\rho} \) and \( \rho_R \) give a condition on \( \beta_f \) for \( \tilde{\rho} \) to be exact, namely,

\[
\beta_f(u) = \frac{\exp[\varepsilon(\nu_r - u)] - 1}{(u - \nu_p)^2} \tag{2.42}
\]

Since we are dealing with curve fitting here, we must be consistent with our notation and write

\[
\beta_f(u) = \frac{\exp[\varepsilon(\nu_{re} - \nu_{ue})] - 1}{(\nu_{ue} - \nu_{pe})^2} \tag{2.43}
\]

From (2.43) it is clear that \( \beta_f(u) \) is undefined at perigee (where \( \varepsilon = 0 \)) and also that \( \beta_f \) must be dependent on altitude for \( \tilde{\rho} \) to be equal to \( \rho_R \). However, as stated earlier, \( \beta_f \) will be evaluated at a specific height, \( u_f \), most probably near perigee, and it will then be constant over each revolution. This specific height, which will vary between revolutions with \( u_f \) will be defined later. Then, \( \beta_f \) will be given by,

\[
\beta_f = \beta_f(u_f) = \frac{\exp[\varepsilon_f(\nu_{re} - \nu_{fe})] - 1}{(\nu_{fe} - \nu_{pe})^2} \tag{2.44a}
\]

23
\[ \varepsilon_f = S_{pe} - S_{fe} \]  

The same two situations that were previously discussed are now carried out for \( \beta_f \).

(i) Reference Altitude Lower Than, or At, Perigee \((v_{re} \geq v_p), (r_r < r_p)\)

Since, by definition, \( u_p > v_{fe} \), then it is clear that

\[ S_{pe} > S_{fe} \Rightarrow \varepsilon_f > 0 \]

\[ v_{re} > v_{fe} \]

and consequently \( \beta_f \) is always positive.

(ii) Reference Altitude Higher than Perigee

Two situations may occur:

\[ v_{re} < v_{fe} < v_p \quad (r_r > r_f > r_p) \]

\[ v_{fe} < v_{re} < v_p \quad (r_f > r_r > r_p) \]

In the first case, \( \beta_f \) is negative and in the second, it is positive. It is obviously zero when \( v_{fe} = v_{re} \) (i.e. \( u_f = u_r \)).

While case (i) is very desirable, case (ii) causes a problem.

Suppose we have

\[ v_{fe} < v_{re} < v_p \quad (r_f > r_r > r_p) \]

so that \( \beta_f \) is positive. Then, because \( \beta_f \) is fixed over the orbit, it will still be positive when the satellite is in the altitude region between the reference altitude and the perigee, i.e. when

\[ v_{re} < u < v_p \quad (r_r > r > r_p) \]

But we know that in this region, \( \rho^* \) is greater than \( \rho_R \) because \( \beta_f \) must be negative (see equation (2.41a) and case (ii) above). Then, having \( \beta_f \) positive in a region when it should be negative causes the model SM3 to be worse than SM1 - s(PA). Therefore, situations where the reference altitude is between perigee and the fitting point \( f \) should be avoided. The reference altitude must then be taken at the highest or the lowest possible altitude. Since the former is not well defined, we will take \( v_r \) at the lowest perigee, in our case at 150 km. These theoretical considerations will be illustrated in
the next section where numerical examples are given.

The static density model that will be used in the orbital theory will then be SM3 or SM4, given by

\[ \rho = \rho_p \left[ 1 + \beta_f (u - u_p)^2 \right] \exp[s_p(u - v_p)] \]  

(2.45a)

where

\[ \rho_p \] is given by SM1 or SM2

and

\[ \beta_f = \frac{\exp[(s_p - s_f)(u_r - u_f)] - 1}{(u_f - u_p)^2} \]  

(2.45b)

if flattening effects are neglected in \( \beta_f \).

2.2.4 Curve Fitting

Let us first consider model SM1 in its equatorial form, SM1_e:

\[ \rho = \rho_r \exp[s_e(v_{re} - v_{re})] \]  

(2.46a)

where

\[ s_e = v_{re}[\ln(v_{re} - \alpha_e v_{ue})]^{-1} \]  

(2.46b)

Since three matching coefficients are available, namely \( \rho_r, \alpha_e \), and \( m \), three fitting points can be chosen,

i) \( \rho_r \) at \( v_{re} \)  
   (reference altitude, 150 km.)

ii) \( \rho_1 \) at \( v_{le} \)

iii) \( \rho_2 \) at \( v_{2e} \)

Substituting into (2.46) yields 3 equations to be solved for \( \alpha_e \) and \( m \):

\[ \alpha_e = \frac{v_{re}(S_2 - S_1)}{(S_2 v_{le} - S_1 v_{2e})} \]  

(2.47a)

\[ m = \frac{v_{re}(v_{ie} - v_{re})}{v_{ie}(v_{re} - \alpha_e v_{ie}) \ln(\rho_i/\rho_r)} \  \text{for} \ i = 1 \text{ or } 2 \]  

(2.47b)

\[ S_1 \triangleq (v_{le} - v_{re})v_{2e} \ln(\rho_2/\rho_r) \]  

(2.48a)

\[ S_2 \triangleq (v_{2e} - v_{re})v_{le} \ln(\rho_1/\rho_r) \]  

(2.48b)
It is important to note that \( m \) can be evaluated at either \((\rho_1, v_{1e})\) or \((\rho_2, v_{2e})\).

The United States Standard Atmosphere 1976 (USSA 76) is used as the source for the density data. Since SM1 is to provide perigee density only (in the SM3 model), consequently, according to the conditions we assumed for the perigee height (Section 1.3), the curve fitting has to be performed only in the altitude range.

\[ 150 \text{ km} \leq h \leq 500 \text{ km} \]

within which the perigee height of a near-Earth satellite will lie during its lifetime. When \( r_p < 150 \text{ km} \), the satellite enters the transition region between Stage A and Stage B, which is out of the scope of this present work.

By trial and error, the best fit is obtained with the following points,

\[
\begin{align*}
\rho_r &= 2.076 \times 10^{-9} \text{ kg/m}^3 \quad \text{at} \quad 150 \text{ km} \\
\rho_1 &= 1.184 \times 10^{-12} \text{ kg/m}^3 \quad \text{at} \quad 450 \text{ km} \\
\rho_2 &= 6.073 \times 10^{-11} \text{ kg/m}^3 \quad \text{at} \quad 250 \text{ km}
\end{align*}
\]

This gives the following values for \( \alpha_e \) and \( m \):

\[
\alpha_e = 0.9489 \quad \quad m = 0.06631
\]

Note that \( m \) was evaluated for \( i=2 \) in (2.47b). Figure 2.1 gives the curve of model SM1 versus altitude compared with the standard table (USSA 76). Since it is hard to appreciate the discrepancies between these two curves on a semi-log graph, Figure 2.2 gives the percent error of (2.46) with respect to the USSA 76 as a function of the altitude. The percent error is defined as,

\[
\% \text{ error} = \frac{\rho(\text{model}) - \rho(\text{USSA 76})}{\rho(\text{USSA 76})} \times 100 \quad (2.49)
\]

From Figure 2.2, one can see that the error is below 14% for all altitudes except near the upper limit of the interval, where it reaches 16.6%. If one considers the important random variations in density, it is clear that model SM1 represents a relatively accurate density model, sufficiently so to give the perigee density \( \rho_p \) in SM3.

Even though the better precision inherent to SM2 is not really required (since SM1 is sufficient and simpler) the curve fitting method for SM2 is briefly described here but no numerical results will be given. Model SM2 e. 
provides \( n+1 \) matching coefficients \((\rho_{\text{r}}, a_0, a_1, \ldots, a_{n-1})\). Once the reference altitude is specified (150 km here), \( n \) fitting points, \((\rho_i, v_{ie}) i=1, \ldots, n\) can be defined. In order to solve for the \( a_i \), one forms the matrix equation

\[
\begin{bmatrix}
1 & v_{1e} & v_{1e}^2 & \cdots & v_{1e}^{n-1} \\
1 & v_{2e} & v_{2e}^2 & \cdots & v_{2e}^{n-1} \\
1 & v_{3e} & v_{3e}^2 & \cdots & v_{3e}^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & v_{ne} & v_{ne}^2 & \cdots & v_{ne}^{n-1}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n-1}
\end{bmatrix}
= \begin{bmatrix}
\frac{v_{\text{re}} - v_{ie}}{\ln(\rho_{\text{r}}/\rho_i)} \\
\frac{v_{\text{re}} - v_{2e}}{\ln(\rho_{\text{r}}/\rho_2)} \\
\frac{v_{\text{re}} - v_{3e}}{\ln(\rho_{\text{r}}/\rho_3)} \\
\frac{v_{\text{re}} - v_{ne}}{\ln(\rho_{\text{r}}/\rho_n)}
\end{bmatrix}
\] (2.51)

which could, in theory, be solved for the \( a_i \) since all fitting points are distinct. One of the drawbacks of SM2 is that the \( v_{ie} \) are very small numbers and one may run into computational problems if \( n \) is relatively large since the \( v_{ie}^{n-1} \) become so small that one of the columns of the coefficient matrix may be nearly zero.

Now, one can obtain model SM1 - s(PA) directly from SM1_e described earlier. First, \( \alpha \) is given by

\[\alpha = \alpha_e - f \sin^2 \delta \]

and then, \( s \) is evaluated at perigee,

\[s_{up} = s_p = v_{re}[mu_p(v_{re} - \alpha_p u_p)]^{-1} \] (2.52a)

\[\alpha_p = \alpha_e - f \sin^2 \delta_p \] (2.52b)

From (2.37a), the factor that is required to bring SM1 - s(PA),\( \rho^* \), equal to SM1,\( \rho_R \), is

\[\exp[\varepsilon(v_r - u)] = \exp[(s_p - s)(v_r - u)] = \rho_R/\rho^* \]

This factor is plotted in Figure 2.3 as a function of altitude for different reference altitudes \( v_r \). Note that when \( v_r \) is at the perigee (150 km on the graph) situation (i) is present. SM1 - s(PA) is always lower than SM1 so that the factor is always greater than one. When \( v_r \) is higher than the perigee,
the factor is first a fraction \((\rho^* > \rho_R)\) and, above \(v_r\), it becomes greater than one, (situation (i)). Figure 2.4 shows how quickly \(SM1 - s(PA)\) diverges from \(SM1\). In this case, \(v_r\) is taken at perigee (150 km) so that \(\rho^* < \rho_R\) always except \(\rho^* = \rho_R\) at perigee. These figures show \(SM1 - s(PA)\) is theoretically valid for only a restricted altitude range. The required change in the sign of \(\beta_f\) is also evident from figure 2.3 and Eq. (2.45b), when \(v_r < u_p\) \((r_r > r_p)\).

Finally, \(SM3\) is now discussed:

\[
\rho = \rho_p [1 + \beta_f (u - u_p)^2] \exp[s_p (u - v_p)]
\]

(2.53)

where \(\rho_p\) is given by \(SM1\), i.e.

\[
\rho_p = \rho_r \exp[s_p (v_p - v_r)]
\]

(2.54)

Substituting (2.54) into (2.53) gives the fundamental form of \(SM3\),

\[
\rho = \rho_r [1 + \beta_f (u - u_p)^2] \exp[s_p (u - v_r)]
\]

(2.55)

where it is evident that \(\rho\) is exact at \(v_r\) when \(\beta_f = 0\) and \(\rho\) at perigee has the accuracy of \(SM1\). The scale-\(u\) correction factor, \(\beta_f\),

\[
\beta_f = \frac{\exp[(s_p - s_f)(u_r - u_f)] - 1}{(u_f - u_p)^2}
\]

(2.56)

is evaluated at some altitude \(u_f\) that we must now determine. But, first of all, the primary condition is that \(v_r\) must be at a lower altitude than the perigee for reasons given in the preceding section. These are well illustrated in Figures 2.5 and 2.6 where \(v_r < u_p\) \((r_r > r_p)\). In the first figure, it is clear that the density profile given by \(SM3\) is less accurate than that obtained with \(SM1 - s(PA)\) in the altitude range between \(u_p\) and \(v_r\). This is due to the fact that \(\beta_f\) is evaluated above \(v_r\), and is therefore positive, while \(SM1 - s(PA)\) requires a negative \(\beta_f\) for heights in the internal \([u_p, v_r]\). Obviously, if \(\beta_f\) were evaluated below \(v_r\), it would then be negative and the density profile would then be worsened above \(v_r\). The same situation is present when the perigee is higher (Figure 2.6). Again, it is meaningful to choose \(v_r\) at or below perigee (situation (i)) so that the required correction to \(SM1 - s(PA)\) is always in the same direction and slowly amplified by the factor \((u - u_p)^2\) in \([1 + \beta_f (u - u_p)^2]\).

However, the situation is not free of complications even when the altitude \(v_r\) is smaller than perigee \((r_r < r_p)\). In fact, although the
required correction is always in the same direction ($\beta_f$ always positive), problems similar to those already discussed may arise. Referring to Figure 2.4, (perigee at 150 km), suppose we need the density to be accurate at 400 km altitude. Then, taking $u_f$ at 400 km will result in a relatively large value for $\beta_f$, so large that at lower altitudes, near perigee, this correction factor is such that the density profile SM3 diverges significantly from the desired profile. What actually happens can be seen from Eq. (2.56): when $(u_f - u_p)$ is large, the exponential factor in the numerator starts to overcome the increase in the denominator $(u_f - u_p)^2$, and $\beta_f$ increases significantly as $u_f$ is taken further away from $u_p$. In conclusion, $u_f$ must be taken close enough to the perigee so that density profiles are not worsened in this critical region. In practice it is observed that taking $u_f$ at the semi-latus rectum, $u_\perp$, fulfills the required conditions. Then,

$$u_f = u_\perp = \frac{u_p}{1+e} \quad (2.57)$$

where $e$ is the eccentricity of the orbit. However, it was also observed that when $e > 0.02$, $u_\perp$ was too far away from perigee and the problem was again present. Consequently, we take

$$u_f = \max \left\{ \frac{u_p}{1+e}, \frac{u_p}{1+0.02} \right\} = \begin{cases} u_\perp & \text{for } e \leq 0.02 \\ \frac{u_p}{1.02} & \text{for } e > 0.02 \end{cases} \quad (2.58)$$

Equation (2.58) makes sense because when $e$ becomes greater than 0.02, the perigee region is gradually more critical than any other region of the orbit as far as drag is concerned. Therefore, it is, in practice, better to have accurate density profiles in the interval $[u_p, u_p/1.02]$ than in higher intervals where drag is not as dominant.

Figures 2.7, 2.8 and 2.9 show model SM3 compared with SM1 - s(PA) in terms of their divergence from the USSA 1976. Different perigee heights are considered and $u_f$ is in each case evaluated near $u_p/1.02$. Note that the accuracy of the model increases with perigee height. When $u_p$ is at 500 km (Fig. 2.9), SM3 is closer to USSA 1976 than the corresponding SM1. At low perigee height (Fig. 2.7), SM3 is still acceptable but more divergent. However, when the perigee comes at these low altitudes, the eccentricity has usually decayed below 0.02 so that very often $u_f$ will be lower than $u_p/(1+0.02)$ and the density profile given by SM3 will be more accurate near perigee that it is on Fig. 2.7.
As a last remark, when $s_f$ has to be evaluated above 500 km (as in Fig. 2.9), one requires the value of $s_f$ at these high altitudes. A high altitude fit ($500 \text{ km} < r < 1000 \text{ km}$) needs to be performed in order to get these $s_f$'s. SM1 is again used. A good approximation is obtained when:

\[
\begin{align*}
\rho_r &= 1.184 \times 10^{-12} \text{kg/m}^3 \text{ at } 450 \text{ km} \\
\rho_1 &= 7.824 \times 10^{-15} \text{kg/m}^3 \text{ at } 850 \text{ km} \\
\rho_2 &= 1.137 \times 10^{-13} \text{kg/m}^3 \text{ at } 600 \text{ km} \\
\alpha_e &= 0.8957, \quad m = 0.07589
\end{align*}
\]

Figures 2.10 and 2.11 compare this fit to USSA 1976. Obviously, no further corrections are required to this model since only the values for $s$ are needed. They are given by

\[
s_f = v_{re} \left[ m u_f (v_{re} - \alpha_f u_f) \right]^{-1}
\]

where $v_{re}$ is evaluated at 450 km, $m = 0.07589$, and $\alpha_f = 0.8957$.

2.3 Dynamic Model for the Density

In this subsection, the asymmetry and time-dependence of the density distribution induced by the Sun and the geomagnetic field are considered. The 11-year solar cycle and the diurnal and semi-annual density variations will be considered, as well as unpredictable random variations.

2.3.1 The 11-Year Solar Cycle Variations

The details of this solar-induced fluctuation in the density are summarized in [1,30] and Appendix B. The important feature here is the direct influence of the average solar activity, $F_{10.7}$ (or simply $\bar{F}$)\(^1\) on the exospheric temperature, $T_\infty$, which, in turn, is directly proportional to the density. We will first relate $T_\infty$ to $\bar{F}$, and then $\rho$ to $T_\infty$.

Jacchia [49] obtained the following empirical relation

\[
T_\infty = 5.48 \bar{F}^{0.8} + 101.8 \bar{F}^{0.4}
\]

where $T_\infty$ is the arithmetic mean of the global extrema of the diurnal variation in the exospheric temperature under quiet geomagnetic conditions ($K_p = 0$) (in°K), and $\bar{F}$ is the average solar flux over six solar rotations (weighted mean) in units of $10^{-22} \text{W/m}^2 \text{Hz}$, and $F$ is the solar flux (in $10^{-22} \text{W/m}^2 \text{Hz}^{-1}$).

---

\(^1\) See Appendix B for the definition of this mean flux $\bar{F}_{10.7}$. Note that only in this particular case does the bar "\_" denote something other than a non-dimensional variable.
If the rapid fluctuations in $F$ are neglected here, we obtain an average value for $T$,

$$T_\infty = 5.48 F^{0.8} + 101.8 F^{0.4}$$  \hspace{1cm} (2.61)

With the solar flux related to the exospheric temperature with this equation, the density $\rho$ must now be expressed as a function of $T_\infty$. Jacchia [49] was able to derive such an expression but in a form which is not suitable in an analytic approach to the theory of decaying satellite orbits. Since there is no simple mathematical relation between $T_\infty$ and $\rho$, we must create one.

Let us first look at Fig. 2.12, which is taken from [49]. The variation of $\rho$ with $T_\infty$ is shown for various altitudes. It should be possible to generate these curves with a simple mathematical expression. The empirical technique, which was successful in the static model, is used again here. However, the fit must now be to two variables (Fig. 2.12) i.e., a surface:

$$\rho = \rho(h, T_\infty)$$  \hspace{1cm} (2.62)

One can see that the influence of the Sun (on the density) varies with altitude as well. From this last remark, it can be inferred that the scale-$u$, $s$, will be directly affected and it will then vary during the solar cycle. However it has been noted in the discussion of the static density model that a complex expression for $s$ is not desirable since it renders intractable further mathematical manipulation. It would be much simpler if a factor multiplying the static model could give the required 11-year modulation of the density. This factor, denoted by $D_{sc}$ ("sc" for solar cycle), would then be defined as,

$$D_{sc} \triangleq \rho_{sc}(h, T_\infty)/\rho(\text{STATIC})$$  \hspace{1cm} (2.63)

where $\rho_{sc}$ denotes the atmospheric density when the only dynamic effect considered is the solar cycle. In our case, $\rho(\text{STATIC})$ is given by SM3 which is a curve-fitted expression of USSA 1976. Since this standard atmosphere assumed an exospheric temperature of 1000 K, (2.63) is rewritten,

$$D_{sc} = \rho_{sc}(h, T_\infty)/\rho(h, 1000 \degree K)$$  \hspace{1cm} (2.64)

---

1. They will be considered in Section 2.3.4.
Values of $D_{sc}$ can easily be obtained from [49] where tables of $\rho_{sc}(h, T_\infty)$ as a function of $h$ and $T_\infty$ are given. These tables generate Fig. 2.12 from which one can infer what type of curve should be used to fit these data. Because of the similarity between the function $T_\infty = T_\infty(\rho)$ for a given height and a parabola, a tentative relation could be

$$T_\infty = a(h)D_{sc}^2 + b(h)D_{sc} + c(h) \quad (2.65)$$

Before expressions for $a(h)$, $b(h)$ and $c(h)$ are deduced from the tables, the range of applicability of Eq. (2.65) is first narrowed down. The static density model is already quite complex and the dynamic variations in the density usually represent an uncertain and/or a small sophistication to the model. Therefore these time-dependent variations will be evaluated at perigee only. Otherwise, the increased complication induced in the derivation would not be worth the small refinement to the theory. With this assumption, Eq. (2.65) need only be fitted for the altitude range $150 \text{ km} \leq h \leq 500 \text{ km}$. However, because the effects of the Sun are negligibly small below 200 km, the range $200 \text{ km} \leq h \leq 500 \text{ km}$ will be used.

Unless an exceptionally high solar activity is present, $F$ will usually vary between 70 and 200. Thus the temperature interval $700^\circ K \leq T_\infty \leq 1200^\circ K$ will prove sufficient for this analysis. Boundaries on $h$ and $T_\infty$ have now been defined.

Let us rewrite (2.65) in a normalized form,

$$\bar{T}_\infty = A(h)D_{sc}^2 + B(h)D_{sc} + C(h) \quad (2.66)$$

where $\bar{T}_\infty = T_\infty/1000^\circ K$. Three matching coefficients are available for each given altitude. From (2.64), they must satisfy the identity

$$1 = A(h) + B(h) + C(h) \quad , \forall h. \quad (2.67)$$

Consequently, two fitting points can be chosen while (2.67) gives the third fitting point at $T_\infty = 1000^\circ K$. Once $A$, $B$ and $C$ are evaluated at each altitude, they can also be expressed as empirical functions of altitude. Performing this task yields:

$$A(h) = 0.4393 e^{-0.0341(h-200.0)} - 0.0843 \quad (2.68a)$$

$$B(h) = 0.5687 - 4.8189 \times 10^{-4}(h-200.0)
- 0.3987 e^{-0.0788(h-200.0)} \quad (2.68b)$$

$$C(h) = 0.4671 + 6.9128 \times 10^{-4}(h-200.0) \quad (h \text{ in km}) \quad (2.68c)$$
Since we wish to express $D_{sc}$ as a function of $h$ and $T_\infty$, (2.66) is rewritten

$$D_{sc}(h, T_\infty) = -B(h) + \left\{ B(h)^2 - 4A(h)[C(h) - T_\infty] \right\}^{1/2} \tag{2.69}$$

Finally, the density during the solar cycle is directly obtained from (2.63), (2.69) and (2.68):

$$\rho_{sc}(h, T_\infty) = \rho(\text{STATIC})D_{sc}(h, T_\infty) \tag{2.70}$$

Figure (2.13) compares $\rho_{sc}(h, T_\infty)$ given above and the data available in [49].

The model loses its accuracy at low solar activity (low $T_\infty$) as the altitude increases. Of course, the error would be significantly reduced if a third order term were added in (2.66) and if some more terms were also introduced in (2.68).

If maximum and minimum solar activity of 200 and 70 are assumed, $T_\infty$ varies between 721 9K and 1227 9K and consequently the model is accurate to within 29.9% and -15.5% for all altitudes, and within ±18% below 350km.

In view of the large imprecision induced by random density variations\(^1\), this deterministic approach is quite valid and somewhat simple.

From (2.6a) and 2.70), it is clear that the scale-$u$ is indirectly affected by solar activity through $D_{sc}$. This verifies an observation made previously. Note also that because of the assumed perigee-approximation Eq. (2.70) is rewritten:

$$\rho_{sc}(h, T_\infty) = \rho(\text{STATIC})D_{scp} \tag{2.71}$$

where

$$D_{scp} = D_{sc}(h_p, T_\infty) = \frac{-B(h_p) + \left\{ B(h_p)^2 - 4A(h_p)[C(h_p) - T_\infty] \right\}^{1/2}}{2A(h_p)}$$

Now that the mathematical path from $F$ to $\rho_{sc}$ has been derived in a relatively accurate model, we come to the most unpredictable and inaccurate part of this section on the solar cycle: the dependence of $F$ on time.

While $F$ can easily be measured, it is hardly predictable. Scientists usually rely on past cycles to extrapolate for future values of $F$. This technique is fallible. Skylab was a victim of this inaccurate method, among other factors. Fig. 2.14 shows the average activity of the Sun, $\overline{F}$, for $\sim$1.5 cycles. These data were computed by Jacchia [49]. Note that the 1958 solar

\(^1\) These sporadic variations may represent a factor of 6 at 600 km. It decreases to 2 near 200 km, [48].
Since $\bar{F}$ is nearly constant over periods of 30 to 60 days and varies by only a few units over a few months, a constant value of $\bar{F}$ (equal to that measured at the time of the prediction) can be used in the equations for the estimation of lifetimes of that order. For longer lifetimes, an appropriate average value, based on previous cycles, can be used. This includes predictions of a few years. For lifetimes extending over many solar cycles, a logical value for $\bar{F}$ is around 135, [48].

The theoretical approach is here limited by our inability to predict the future activity of the Sun. Until more research generates a proper deterministic equation relating $\bar{F}$ to time, the sinusoidal approximation [33,48] is sometimes adequate and always easy to handle.

From Eq. (2.61), a mean exospheric temperature of 1000 K roughly corresponds to a solar index of 135. From Fig. 2.14, the average peak value is 55 and the time between two successive minima is around 4200 days. These considerations yield

$$\bar{F} = 135 + 55 \cos\left[\frac{2\pi(t-t_0)}{T_{SC}}\right]$$  \hspace{1cm} (2.72)

where $t_0 = 36,500$ (in M.J.D.) and $T_{SC} = 4200$ days.

Equation (2.72) is plotted in Figure 2.14. It roughly represents the mean of measured data except that the actual solar minimum is more extended than that given by a sinusoidal function.

From (2.72) and (2.61), the minimum and maximum exospheric temperatures are,

$$T_{\infty}(\text{min}) = 770 \text{ K}, \quad T_{\infty}(\text{max}) = 1195 \text{ K}$$  \hspace{1cm} (2.73)

This restricts further the previous boundaries. Conditions (2.73) are shown in Fig. 2.13.

Since the variation of $\bar{F}$ in time is slow compared to an orbital period, $D_{scp}$ can be taken as constant over one orbit, but will vary from one orbit to the next.

The first dynamic density variation (solar cycle) has now been completely described. It must be remembered that it is subject to the unpredictable behaviour of nature and the resulting lifetime estimates must be interpreted accordingly.
2.3.2 Day-to-Night Variations

Being constantly heated by the Sun, the air on the sunlit side of
the Earth expands in a big hump and, under, the hydrostatic equilibrium
conditions, the density at a given height increases as one approaches the
centre of this bulge, which lags some 25 to 30° behind the subsolar point
due to atmospheric rotation. The shape of this bulge is not perfectly
symmetric with respect to the line between the Earth and the bulge centre
[66,1]. However, the simple sinusoidal variation introduced by Cook and
King-Hele [31] is sufficiently accurate for our purposes:

\[ \rho = \rho_o (1 + F \cos \phi) \]  

(2.74)

where \( \rho_o \) = diurnally averaged density, \( F \) = bulge factor \( \phi \) = angular distance
from the bulge centre and the point considered (satellite's position in our
case). Contrarily to [31], where \( F \) was constant, we will assume the bulge factor
depends on altitude, as pointed out by Harris & Priester [68], according to
the linear approximation,

\[ F = F_v + \delta (u - u_v) \]  

(2.75)

where subscript "v" represents some "slowly varying" reference altitude.
Obviously, for consistency and simplicity, this reference altitude will be at
perigee so that

\[ F = F_p + \delta (u - u_p) \]  

(2.76)

We now require some data to evaluate \( F_p \) and \( \delta \). Let us first define
\( F \) such that the density is accurate at its minimum point (night, \( \phi = 180° \))
and at its maximum (day, \( \phi = 0° \)). Then

\[ F = \frac{f - 1}{f + 1} \]  

(2.77)

where

\[ f = \frac{\rho_{\text{max}}}{\rho_{\text{min}}} = \frac{\rho_{\text{DAY}}}{\rho_{\text{NIGHT}}} \]

A graph of \((f-1)/(f+1)\) due to Harris and Priester, is reproduced in [31].
For average solar activity, one could use the approximation

\[ F = F_o + \delta_o (u - u_o) \quad (200 \text{ km} \leq h \leq 400 \text{ km}) \]  

(2.78)

1. See Appendix B.
\[ F_0 = 0.19 \]
\[ \delta_0 = -1.147 \times 10^5 \text{ km} \]
\[ u_0 = 1.51 \times 10^{-4} \text{ km}^{-1} \text{(i.e. at 245 km)} \]

for altitudes in the range [200 km, 400 km]. The maximum error is approximately ±5%. Above 400 km a second expression could be,

\[ F = F_1 + \delta_1 (u - u_1) \quad h \geq 400 \text{ km} \quad (2.79) \]

\[ F_1 = 0.59 \]
\[ \delta_1 = -5.009 \times 10^4 \text{ km} \]
\[ u_1 = 1.475 \times 10^{-4} \text{ km}^{-1} \text{(at 400 km)} \]

From these two equations, \( F_p \) and \( \delta \) are evaluated at each revolution as follows.

(i) If \( u_p \geq 1.475 \times 10^{-4} \text{ km}^{-1} \) (perigee below 400 km), then

\[ F_p = 0.19 - 1.147 \times 10^5 (u_p - 1.51 \times 10^{-4}) \text{ from Eq. (2.78)} \]

(ii) If \( u_p < 1.475 \times 10^{-4} \text{ km}^{-1} \) (perigee above 400 km), then

\[ F_p = 0.59 - 5.009 \times 10^4 (u_p - 1.475 \times 10^{-4}) \text{ from Eq. (2.79)}. \]

From (2.76) we have,

\[ \delta = \frac{F - F_p}{(u - u_p)} \quad (2.80) \]

Similarly to the technique used in the static density model, \( \delta \) will be evaluated so that \( F \) is accurate at some altitude \( u_f \) above perigee. Then,

\[ \delta = \delta_f = \frac{F_f - F_p}{(u_f - u_p)} \quad (2.81) \]

where \( F_f \) is evaluated using Eq. (2.78) or (2.79) depending on whether \( u_f \) is greater or smaller than \( 1.475 \times 10^{-4} \text{ km}^{-1} \) (i.e. below or above 400 km). The fitting point \( u_f \) is obtained from (2.58).

It must be noted here that the size of the day-to-night effect depends on the degree of solar activity, and hence the bulge factor, \( F \), varies with the 11-year cycle of the Sun. However, this dependence is here assumed negligible and all solar cycle variations are considered in the factor \( D_{SCP} \) only. The only coupling between the two is of the form

Note that when \( u_f \) and \( u_p \) are in the same altitude range (given in (2.78) and (2.79)), then \( \delta = \delta_0 \) or \( \delta_1 \) depending on the particular range.
Finally, Eqs. (3.78) and (3.79) were derived for a solar index of $F = 150$ corresponding to an exospheric temperature of about $1057^\circ K$. This discrepancy from the nominal value $1000^\circ K$ in the static model is assumed unimportant.

### 2.3.3 Semi-Annual Variations

It is pointed out in [64,1] that semi-annual density variations are correlated not to the variations in solar activity but to the shape of the magnetosphere. From this one may suppose the semi-annual fluctuations to be independent of both the 11-year cycle and the diurnal bulge but that they alter a nominal density $\rho_n$ (which contains all the other variations) in a relation described by Jacchia [49]:

$$\rho_{sa} = \rho_n \exp[(\ln 10)f(h)g(t)] \quad (2.82)$$

where

$$f(h) = [0.04(h/200)^2 + 0.05]\exp(-0.25h/100) \quad (h \text{ in km}) \quad (2.83)$$

$$g(t) = 0.0284 + 0.382[1 + 0.467 \sin(2\pi t + 4.14)] \sin(4\pi t + 4.26) \quad (2.84)$$

$$\tau = \phi + 0.0954\{[0.5 + 0.5\sin(2\pi \phi + 6.04)]^{1.65} - 0.5\} \quad (2.85)$$

$$\phi = (t - t_0)/T_t \quad (2.86)$$

where $T_t$ is the length of the tropical year and $t_0 = 1.0$ January.

Similarly to the solar cycle density variations, we define the semi-annual density factor $D_{sa}$ as

$$D_{sa} = \exp[(\ln 10)f(h)g(t)] \quad (2.87)$$

Since $D_{sa}$ is a function of altitude, it is clear from this equation that the scale-$u$ will be affected by the semi-annual effect.

However, since the semi-annual effect is a small correction to the density variation, its perigee-approximated form will be used. The equations would otherwise be unnecessarily complex and long without much gain in accuracy. Similarly, since the time variation is very slow, $t$ can be assumed fixed over one orbit and, for all practical purposes, it could even be fixed over a few days.

$$D_{sap} = \exp[(\ln 10)f(h_p)g(t)] \quad (2.88)$$
2.3.4 Random Variations

As explained in Appendix B, these fluctuations are mainly due to unpredictable solar activity variations ($F_{10.7}$) and the influence of the solar wind on the magnetosphere. Graphs of densities and $F_{10.7}$ are given in Fig. 2.15.

Let's approximate these variations by straight line segments,

\[ R = \rho / \rho_N \]

where $\rho$ is the actual density and $\rho_N$ the nominal density (at that altitude) given by all the other static and dynamic variations. Writing

\[ \rho = \rho_N R \]  \hspace{1cm} (2.89)

then $R$ could be plotted as

where not only the magnitude of $R$ but also its frequency vary randomly with time. However, it is possible to recognize a maximum and a minimum ratio
\[ R_{\text{max}} \triangleq \frac{\rho_{\text{max}}}{\rho_{\text{N}}} \]
\[ R_{\text{min}} \triangleq \frac{\rho_{\text{min}}}{\rho_{\text{N}}} \]

which are dependent on altitude and on time. Also a maximum and a minimum time span between two successive peaks may be defined:

so that the time span between two neighbouring peaks lies randomly in the interval \([t_{\text{min}}, t_{\text{max}}]\) where \(t_{\text{min}}\) and \(t_{\text{max}}\) is of the order of a few days. With these definitions, it is possible to model random fluctuations by linear approximations with the following algorithm:

1° Define two random variables \(x_1, x_2\) where \(x_1\) is uniformly distributed in \([R_{\text{min}}, R_{\text{max}}]\), and \(x_2\) is uniformly distributed in \([t_{\text{min}}, t_{\text{max}}]\).

2° At initial time \(t_i\), start with an initial value of \(R, R_i\), picked up at random in its domain of definition.

3° Compute the final point \((R_f, t_f)\), where \(R_f = x_1\) and \(t_f = t_i + x_2\).

4° Now \(R\) is given by
\[ R = R_i + g(t - t_i), \quad t_i \leq t \leq t_f \]  
where
\[ g = \frac{\frac{R_f - R_i}{t_f - t_i}}{x_2} = \frac{x_1 - R_i}{x_2} \]  
(2.92)

5° When \(t = t_f\), let \(t_i = t_f\), \(R_i = R_f\) and, taking a new sample for \(x_1\), and \(x_2\), go back to step 3° and proceed with these new values for \(t_i, R_i, x_1\) and \(x_2\).
Note that $R$ may be dependent on altitude through the parameters $R_{\text{max}}$, $R_{\text{min}}$, $t_{\text{max}}$ and $t_{\text{min}}$. However, this altitude variation is a very small refinement and the perigee-approximated values for these four parameters will suffice. Then we have

$$\rho = R_p \rho_N$$  \hspace{1cm} (2.93)

Also, $x_1$ and $x_2$ may not be uniformly distributed. Until more research is done in this area, we will however assume it is a uniform distribution.

During one orbit, $R$ may be taken as fixed because the time variation is very slow. Between orbits, $R$ is updated to its new value. This method could be used in a Monte-Carlo simulation where many trial trajectories are integrated with each one having a different series of random variables $x_1$ and $x_2$. Then, some conclusions on the effects of the day-to-day random fluctuations in density could, in theory, be obtained.

Finally, the 27-day recurrence of high peaks in $R$ could be modelled by offsetting the domain of definition of $x_1$ to a higher interval ($R_{\text{min}}$, $R_{\text{max}}$) every 27 days.

From [48], $R_{\text{max}}$ may be as large as 6 at 600 km but is $\sim 2$ at 200 km. From Fig. 2.15, $t_{\text{max}}$ could be taken as 15 days and $t_{\text{min}}$ as 2 or 3 days, or even less.

2.4 Atmospheric Rotation

As discussed in Appendix B, the atmosphere is rotating at an angular velocity $\omega$ in revolutions per day. Then,

$$\omega_A = \omega_E \quad \text{(in the polar direction)}$$  \hspace{1cm} (2.94)

$$\omega_A = \text{angular velocity vector of the atmosphere}$$

$$\omega_E = \text{angular velocity vector of the Earth}$$

The inertial velocity at an atmospherically fixed point $\mathbf{r}$ then is:

$$\mathbf{v}_A = \omega_A \times \mathbf{r} \hspace{1cm} (2.95)$$

The velocity of a satellite (inertial velocity $\mathbf{v}$) with respect to the atmosphere is,

$$\mathbf{v}_R = \mathbf{v} - \mathbf{v}_A$$

or, from the above equation,
\( X_R = X + \mathbf{T} \times \mathbf{\Phi}_A \)  

(2.96)

### 2.5 Complete Atmosphere Model

From Eqs. (2.45), (2.71), (2.74), (2.76), (2.88) and (2.93), we obtain

\[
\rho = \rho_p R_p(t) D_{scp}(t) D_{sap}(t) \{1 + [F_p + \delta_f(u - u_p)]\cos\phi[1 + \beta_f(u - u_p)^2] \exp[s_p(u - v_p)] \}
\]

(2.97)

and substituting equation (2.38) and the following definition

\[
D_{p}(t) \triangleq (D_{sc} D_{sa})_p
\]

(2.98)

\[
\rho = \rho_p R_p(t) D_{p}(t) \{1 + [F_p + \delta_f(u - u_p)]\cos\phi[1 + \beta_f(u - u_p)^2] \exp[s_p(u - v_p)] \}
\]

(2.99)

One of the major assumptions made in (2.99) was the perigee approximation of the dynamic factor \( D_{p}(t) \). This assumption can be "corrected" by taking \( \beta_f \) as a function of the dynamic variations of the atmosphere. In particular, if we require the dynamical properties of the density to be accurate at some point \( u_f \) above perigee as well as at perigee itself, we just have to define \( \beta_f \) as

\[
\beta_f = \frac{(D_f/D_p)\exp[\epsilon_f(v_r - v_f)] - 1}{(v_f - v_p)^2}
\]

(2.100)

where \( \beta_f = \beta_f(t), D_f \triangleq D(h_f, T_f), h_f = v_f^{-1} - r_E, \) and \( v_f \) is again obtained from (2.58). When \( h_f \) is above 500 km, Eqs. (2.68) and (2.69) become very inaccurate and they must be redefined for a larger altitude range by adding new terms to the given empirical formulae or by deriving new ones. This is not considered here. Therefore, Eq. (2.100) can be used whenever \( u_f \) lies in the range \([200 \text{ km}, 500 \text{ km}]\).

### 3. EQUATIONS OF MOTION

#### 3.1 Assumptions

In Section 1.3, it was stated that the object of this work is centred on satellites with the following properties

(i) not controlled
(ii) near-Earth orbit
(iii) "normal" shape and density
(iv) moderate and low eccentricity orbit
(v) in Stage A only (i.e. no re-entry)

These conditions on the state of the satellite, which are the ones usually encountered in the study of decaying orbits, allow the following assumptions,

(a) Since the satellite is uncontrolled, it is assumed to tumble about its axis of greatest inertia and, from Appendix C, it can be concluded that
   - there is no average lift to affect the trajectory,
   - the product $C_D A$ is given by King-Hele's averaging technique, $(C_D A)_A$ (see Appendix C). Deviations from $(C_D A)_A$ are assumed random and not to exceed ±22%.
   - no attitude motion equations are required in the formulation of the problem.

(b) The perigee height lies in the altitude range [150 km - 500 km]

(c) Near-spherical, spherical and cylindrical shapes of satellite are considered. Balloon-type satellites and large extent spacecraft are excluded. Consequently, no solar-pressure and gravity-gradient perturbations are taken into account.

(d) The orbits considered are of eccentricity $0 \leq e \leq 0.2$
and this combined with assumption (b) makes luni-solar perturbations negligible with respect to drag.

(e) The re-entry stage itself (Stage B) is outside the scope of this work, and no breakup, ablation or other mass variations are assumed to take place. The orbital motion of the satellite is thus more rapid that the rate of orbit contraction and therefore "slow" and "fast" motions are separable.

To these five assumptions are added a few others that will render the mathematical manipulations less tedious.

(f) Only the $J_2$ term will be considered in the oblate mass perturbation equations.

(g) Earth-centred vectors pointing to the vernal equinox and the North Pole are assumed inertially fixed.
From these assumptions, it is clear that most of the specific perturbations are neglected apart from some considerations of the random variations in the product \((C_{DA})_{AV}\). The two main environmental perturbations will be included: air drag and Earth's oblateness.

3.2 Reference Frames

Three reference frames are now discussed. They are all Earth-centred.

The inertial frame \(F_I\), with "vectrix" (see [87] for a discussion of vectrices)

\[
F_I = [I_x \ I_y \ I_z]^T
\]

is shown in Fig. 3.1. \(I_x\) points to the vernal equinox (\(\tau\)), \(I_z\) to the Earth's North Pole and \(I_y\) completes the right-handed triad.

The orbital frame \(F_0\), with vectrix

\[
F_0 = [Q_r \ 0 \ Q_h]^T
\]

is shown on the same figure. \(Q_r\) points to the satellite (\(m\)), \(Q_h\) is taken along the angular momentum of the satellite (\(\mathbf{h}\)) and \(Q_u\) forms the right-handed triad.

Finally, the solar frame \(F_S\), with vectrix

\[
F_S = [S_r \ S_L \ S_h]^T
\]

is shown on Fig. 3.2. It is similar to frame \(F_0\) except that now the Sun ("orbiting" the Earth) replaces the satellite. The orbital frame is obtained from the inertial frame through a "3-1-3" Euler rotation sequence with angles (\(\Omega, I, U\)), as in Figure 3.3:

1. \(\Omega\) about \(I_z\)
2. \(I\) about the line of nodes, \(\mathbf{N}\)
3. \(U\) about \(Q_h\)

The rotation matrix is thus,

\[
R_{0I} = F_0 \cdot F_I^T = \begin{bmatrix}
C_{u\Omega} - S_{u\Omega} I_{c\Omega} & C_{u\Omega} S_{u\Omega} + S_{u\Omega} I_{c\Omega} & S_{u\Omega} S_I \\
-S_{u\Omega} C_{c\Omega} & -S_{u\Omega} S_{c\Omega} + C_{u\Omega} I_{c\Omega} & C_{u\Omega} S_I \\
S_{Ic\Omega} & -S_{Ic\Omega} & C_I
\end{bmatrix}
\]
where \( c \) and \( s \) denote the cosine and the sine functions. The angular velocity \( \omega \) of \( F_0 \) with respect to \( F_I \) (in \( F_0 \)) is easily obtained as

\[
\begin{bmatrix}
\omega_r \\
\omega_U \\
\omega_h
\end{bmatrix} =
\begin{bmatrix}
s_I s_U & c_U & 0 \\
s_I c_U & -s_U & 0 \\
c_I & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\Omega} \\
\dot{\iota} \\
\dot{\theta}
\end{bmatrix}
\]

(3.5)

where \((\cdot) = \frac{d}{dt}\). Since we have

\[
I_{\text{sun}} = \varepsilon = 23.44^\circ
\]

\[
U_{\text{sun}} = L \quad \text{(mean longitude)}
\]

\[
\Omega_{\text{sun}} = 0
\]

then \( R_{SI} = F_S \cdot F_I^T \) is readily obtainable from (3.4):

\[
R_{SI} =
\begin{bmatrix}
C & S & S \\
-C & C & S \\
0 & 0 & 1
\end{bmatrix}
\]

(3.7)

From (3.4) and (3.7), the inertial position of the satellite, \( \mathbf{Q}_r \), and the Sun, \( \mathbf{S}_r \), are:

\[
\mathbf{Q}_r = (c_U c_\Omega - s_U c_I s_\Omega)\mathbf{i}_x + (c_U s_\Omega + s_U c_I c_\Omega)\mathbf{i}_y + (s_U s_I)\mathbf{i}_z
\]

(3.8)

\[
\mathbf{S}_r = (c_L)\mathbf{i}_x + (s_L c_\varepsilon)\mathbf{i}_y + (s_L s_\varepsilon)\mathbf{i}_z
\]

(3.9)

When the right ascension \( \alpha \) and the declination \( \delta \) are used to locate the satellite \( m \), the spherical triangle \( N-m-A \) (Fig. 3.1) gives the following relations

\[
\sin \delta = \sin I \sin U
\]

(3.10a)

\[
\cos U = \cos(\alpha - \Omega) \cos \delta
\]

(3.10b)

\[
\cos I \sin U = \sin(\alpha - \Omega) \cos \delta
\]

(3.10c)

Similarly, the spherical triangle \( T-D-S \) in Fig. 3.2 gives:

\[
\sin \delta_S = \sin \varepsilon \sin L
\]

(3.11a)

\[
\cos L = \cos \alpha_S \cos \delta_S
\]

(3.11b)

\[
\cos \varepsilon \sin L = \sin \alpha_S \cos \delta_S
\]

(3.11c)

where \((\alpha_S, \delta_S)\) locates the Sun.
From the change of variables given in (3.10) and (3.11), Eqs. (3.8) and (3.9) may be rewritten as

\[ Q_r = (\cos \delta \cos \alpha) T_x + (\cos \delta \sin \alpha) T_y + (\sin \delta) T_z \] (3.12)

\[ S_r = (\cos \delta_s \cos \alpha_s) T_x + (\cos \delta_s \sin \alpha_s) T_y + (\sin \delta_s) T_z \] (3.13)

These two vectors will be useful to locate the satellite with respect to the Sun.

### 3.3 Equations of Motion

If \( \mathbf{r} \) is the satellite's position vector and \( \mathbf{F} \) the resultant force applied to the satellite, Newton's second law gives

\[ \mathbf{F} = m \ddot{\mathbf{r}} \]

or simply,

\[ \ddot{\mathbf{r}} = \mathbf{f} \] (3.14)

where (\( \cdot \)) denotes a time derivative with respect to the inertial frame and \( \mathbf{f} \) is the total force per unit mass of the satellite (\( \mathbf{F}_T/m \)).

#### 3.3.1 Equations in Terms of \( r, U, I, \Omega, h \)

Expanding \( \ddot{\mathbf{r}} \),

\[ \ddot{\mathbf{r}} = \ddot{\mathbf{r}} + 2\dot{\omega} \times \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} \] (3.15)

where (\( \cdot \)) denotes time derivative with respect to \( F_0 \) and using the notation

\[ \mathbf{r} = [r \ o \ o]^T = F_0^T \cdot \mathbf{r} \] (3.16a)

\[ \mathbf{f} = [f_r \ f_U \ f_h]^T = F_0^T \cdot \mathbf{f} \] (3.16b)

\[ \omega = [\omega_r \ \omega_U \ \omega_h]^T = F_0^T \cdot \omega \] (3.16c)

we obtain

\[ \ddot{\mathbf{r}} = F_0^T \begin{bmatrix} \dot{\mathbf{r}} - (\omega_U^2 + \omega_h^2) \mathbf{r} \\ 2\omega_h \dot{\mathbf{r}} + \omega_r \omega_U \mathbf{r} + \omega_h \dot{\mathbf{r}} \\ -2\omega_U \dot{\mathbf{r}} + \omega_r \omega_h \mathbf{r} - \dot{\omega}_U \mathbf{r} \end{bmatrix} \] (3.17)

Writing (3.14) in scalar form using (3.16b) and (3.17) yields:
\[
\ddot{r} - (\omega^2 + \omega^2_r) r = f_r \\
2\omega_r r + \omega_r^2 r + \dot{\omega}_r r = f_U \\
-2\omega_r^2 r + \omega_r \omega_r r - \dot{\omega}_r r = f_h
\]

(3.18a)  
(3.18b)  
(3.18c)

The inertial velocity of the satellite, \(\mathbf{v}\), and its angular momentum per unit mass, \(h\), are given by:

\[
\mathbf{v} = \dot{\mathbf{r}} = \mathbf{\dot{r}} + \omega \times \mathbf{r} \\
h = \mathbf{r} \times \mathbf{\dot{r}} = \mathbf{r} \times (\mathbf{\dot{r}} + \omega \times \mathbf{r})
\]

(3.19)  
(3.20)

Expressed in scalar form,

\[
\mathbf{v} = [\dot{r} \quad \omega r \quad -\omega_U r]^T \\
h = [0 \quad r^2 \omega_U \quad r^2 \omega_h]^T
\]

(3.21)  
(3.22)

In defining the orbital frame (Section 3.2), a constraint was specified on \(\mathbf{h}\), namely that \(Q_h\) should always be parallel to \(\mathbf{h}\) so that \(\mathbf{v}\) would always lie in the plane generated by \(Q_r\) and \(Q_U\). Consequently we must have

\[
h_r = h_U = 0; \quad v_h = 0
\]

(3.23)

which gives the condition,

\[
\omega_U = 0
\]

(3.24)

Then (3.18), (3.21) and (3.22) are rewritten

\[
\ddot{r} - \omega^2 h r = f_r \\
2\omega_h r + \omega_h^2 r = f_U \\
\omega_r \omega_h r = f_h
\]

(3.25a)  
(3.25b)  
(3.25c)

\[
\mathbf{v} = [\dot{r} \quad \omega r \quad 0]^T \\
h = [0 \quad 0 \quad r^2 \omega_h]^T
\]

(3.26)  
(3.27)

and one gets
\[ v = |\mathbf{x}| = (r^2 + r^2 \omega_h^2)^{1/2} \]  

\[ h = |\mathbf{y}| = r^2 \omega_h \]  

Rewriting Eq. (3.5) as

\[ \omega_r = \dot{n} \sin I \sin U + \dot{I} \cos U \]
\[ \omega_I = \dot{n} \sin I \cos U - \dot{I} \sin U \]
\[ \omega_h = \dot{n} \cos I + \dot{I} \]

we obtain from condition (3.24) and the second of the above equations,

\[ \dot{\omega} = \frac{\tan U}{\sin I} \quad (\sin I \neq 0) \]

With this relation, equations (3.5) are rewritten

\[ \omega_r = \frac{\dot{I}}{\cos U} \]  

\[ \dot{\omega} = (\tan U/\sin I) \dot{I} \]  

\[ \omega_h = \dot{n} \cos I + \dot{I} \]

Equations (3.25) and (3.31) completely describe the kinematics of the problem.

Using Eqs. (3.29) and (3.31a) in (3.25), the dependence on \( \omega_r \) and \( \omega_h \) is discarded,

\[ \ddot{r} - \frac{\dot{h}^2}{r^3} = f_r \]  

\[ \dot{h} = rf_u \]  

\[ \dot{I} = r \cos U \quad f_h/h \]

with

\[ h = r^2(\dot{n} \cos I + \dot{I}) \]

\[ \dot{n} = (\tan U/\sin I) \dot{I} \]

The equations of motion are now expressed as four first-order and one second-order differential equations in the five variables \( r, U, I, \omega, h \), with the velocity given by

\[ \mathbf{v} = [\dot{r} \quad \dot{h}/r \quad 0]^{T}; \quad v = (\dot{r}^2 + \dot{h}^2/r^2)^{1/2} \]
The specific forces \( f_r \), \( f_v \) and \( f_h \) must now be determined. For simplicity, they are written

\[
\begin{align*}
  f_r &= f_N + f_{gr} + f_{Dr} \\
  f_U &= f_{gU} + f_{DU} \\
  f_h &= f_{gh} + f_{Dh}
\end{align*}
\] (3.34)

where \( f_N \) represents the usual Newtonian inverse-square force:

\[
f_N = \frac{-\gamma}{r^2}, \quad \gamma = \frac{Gm_E}{r^2}
\] (3.35)

and subscripts \( g \) and \( D \) stand for gravity perturbations due to the oblateness of the Earth and the drag perturbations respectively.

### 3.3.2 Gravity Forces (Oblateness)

From [36] and the following definition

\[
J \triangleq \frac{3}{2} J_2 r_E > 0
\] (3.36)

the gravity perturbations due to the oblate mass of the Earth are written:

\[
\begin{align*}
  f_{gr} &= \frac{-\gamma J}{r^4} (1 - 3 \sin^2 I \sin^2 U) \\
  f_{gU} &= \frac{-\gamma J}{r^4} (\sin^2 I \sin 2U) \\
  f_{gh} &= \frac{-\gamma J}{r^4} (\sin 2I \sin U)
\end{align*}
\] (3.37)

### 3.3.3 Drag Forces

The drag deceleration is usually given as

\[
f_D = a_D = -B \rho \frac{v_R}{v} \frac{v}{v}
\] (3.38)

where

\[
B \triangleq C_D A / 2 m
\] (3.39)

is called the "ballistic parameter". From Eq. (2.96), \( v_R \) is:

\[
v_R = v + \frac{\omega}{-\omega} \\
\] (3.39)

where
Using Eqs. (3.33), (3.16a), (3.4) and (3.40), one gets:

\[
\begin{bmatrix}
\dot{r} \\
h/r - r\omega_A \cos I \\
r\omega_A \cos U \sin I
\end{bmatrix}
\]

(3.41)

The magnitude of \( v_R, v_R \), is readily obtained from (3.41),

\[
v_R^2 = r^2 + \frac{h^2}{r^2} - 2h\omega_A \cos I + r^2 \omega_A^2 \cos^2 I + r^2 \omega_A^2 \cos^2 U \sin^2 I
\]

(3.42)

Using (3.10a) and (3.33) yields,

\[
v_R = F v
\]

(3.43)

where

\[
F = \left[ (1 - \frac{r^2 \omega_A^2}{h^2} \cos I) + \frac{r^2 \omega_A^2}{V^2} (\cos^2 \delta - \frac{r^2 v^2}{h^2} \cos^2 I) \right]^{1/2}
\]

(3.44)

As King-Hele pointed out [30], the atmospheric rotation is so small compared to the uncertainty of the atmospheric density that (3.43) may undergo further approximations. The perigee-approximation is first applied and, noting that \( v_p = h_p/r_p \) we get,

\[
F_p = \left[ (1 - \frac{r^2 \omega_A^2}{h_p^2} \cos I) + \frac{r^4 \omega_A^2}{h_p^4} (\cos^2 \delta_p - \cos^2 I) \right]^{1/2}
\]

(3.45)

The \( F \) factor could be used as it is now. However the transverse velocity of the satellite, \( v_U = h/r \), is always much larger than the rotational velocity of the atmosphere, \( r\omega_A \), so that the following is true:

\[
\frac{r\omega_A}{v_U} = \frac{r\omega_A}{h/r} = \frac{r^2 \omega_A}{h} \ll 1
\]

(3.46)

especially at perigee, where \( h/r \) is the largest. Thus the term involving \( (r_p^2 \omega_A/h_p) \) in (3.44) may be neglected. Then

\[
F_p = 1 - (r_p^2 \omega_A/h_p) \cos I
\]

(3.47)

This result is similar to that of King-Hele (although the derivation is slightly
different), except that in his case, he evaluated this factor at initial perigee height and \( F \) was taken as constant. Here, \( F \) is given at the current perigee height and the current orbital inclination \( I \). From (3.42) and (3.46), one gets the following approximation:

\[
y_R = F
\]

To return to Eq. (3.38), the drag deceleration can now be expanded,

\[
F \cdot \mathbf{f}_D = -B \rho \mathbf{v} (F \cdot \mathbf{v}_R)
\]

Then, using (3.47) and (3.41)

\[
f_{Dr} = -B \rho F \mathbf{v} \mathbf{r} \quad (3.48a)
\]

\[
f_{Du} = -B \rho F \mathbf{v} (h/r - r_w \cos I) \quad (3.48b)
\]

\[
f_{Dh} = -B \rho F \mathbf{v} (r_w \cos I \sin I) \quad (3.48c)
\]

Again, the rotational motion of the atmosphere is neglected in view of the large magnitude of \( h/r \). Thus, from (3.45), the complete drag perturbation equations are:

\[
f_{Dr} = -B \rho \mathbf{v} \mathbf{r} \quad (3.49a)
\]

\[
f_{Du} = -B \rho \mathbf{v} \mathbf{h}/r \quad (3.49b)
\]

\[
f_{Dh} = -B \rho \mathbf{v} \mathbf{r} r_w \cos I \sin I \quad (3.49c)
\]

where \( \mathbf{v} \) and \( \rho \) are given in (3.33) and (2.99) respectively.

3.4 Change of Variable

Let \( \theta_h \) denote the angular distance of the satellite from any inertially fixed reference line (to be defined later) so that,

\[
\dot{\theta}_h \Delta h/r^2 = \dot{\Omega} \cos I + \dot{U} \quad (3.50)
\]

where ('') now refers to time derivatives and ('') will denote derivatives with respect to \( \theta_h \),

\[
(') = d/d\theta_h
\]

\[
(\cdot) = d/dt \quad (3.51)
\]
A new variable, $K$, related to the angular momentum is defined as,

$$K \triangleq \frac{\gamma}{h^2}$$

(3.52)

Substituting these new variables and the one defined in (2.4) into (3.32) gives

$$\begin{align*}
\dot{u} - \frac{\ddot{u}K}{2K} + u &= \frac{-fr}{\gamma u^2} K \\
\dot{K} &= \frac{-2K^2}{\gamma u^3} f_U \\
\dot{I} &= \frac{K}{\gamma u^3} \cos U f_h \\
\dot{\Omega} &= \frac{K}{\gamma u^3} \sin U \sin I f_h \\
\ddot{\theta} &= 1 - \dot{\Omega} \cos I
\end{align*}$$

(3.53a)

(3.53b)

(3.53c)

(3.53d)

(3.53e)

where the last one is readily obtained from (3.50). The five equations of motion are now expressed in terms of the five variables: $u$, $U$, $K$, $I$, $\Omega$. The independent variable, the inertial angle $\theta_h$, can be expressed as a function of time by (3.50) and (3.52),

$$\dot{\theta}_h = \sqrt{\frac{\gamma}{K}} \ u^2$$

(3.54)

Hence,

$$t - t_0 = \int_{\theta_{ho}}^{\theta_h} \sqrt{\frac{K}{\gamma}} \ u^{-2} \ d\theta_h$$

(3.54)

for the satellite at $\theta_{ho}$ at time $t_0$.

In terms of the new variables, the inertial velocity is now

$$v = \begin{bmatrix} -\sqrt{\frac{\gamma}{K}} \ \dot{u} \\ \sqrt{\frac{\gamma}{K}} \ u \\ 0 \end{bmatrix}; \quad \sqrt{\frac{K}{\gamma}} (\dot{u}^2 + u^2)^{1/2}$$

(3.55)

The same alterations are made on the perturbation equations (3.39) and (3.49)

$$f_{gr} = -\gamma Ju^4 (1 - 3 \sin^2 I \sin^2 U)$$

(3.56a)

$$f_{gU} = -\gamma Ju^4 (\sin^2 I \sin 2U)$$

(3.56b)
\( f_{gh} = -Ju^4 \sin 2I \sin U \) \hspace{1cm} (3.56c)

\( f_{Dr} = BFp^\rho(\gamma/K)\dot{u}(u^2 + u'^2)^{1/2} \) \hspace{1cm} (3.56d)

\( f_{DU} = -BFp^\rho(\gamma/K)\dot{u}(u^2 + u'^2)^{1/2} \) \hspace{1cm} (3.56e)

\( f_{Dh} = -BFp^\rho\sqrt{fY/K}(1/u)(u^2 + u'^2)^{1/2}w_A \cos U \sin I \) \hspace{1cm} (3.56f)

\( f_N = -\gamma u^2 \) \hspace{1cm} (3.56g)

and, from (3.34),

\( f_r = -\gamma u^2 + \gamma[BFp^\rho K^\rho(\dot{u}^2 + u'^2)^{1/2} - Ju^4(1 - 3 \sin^2 I \sin^2 U)] \) \hspace{1cm} (3.57a)

\( f_u = -\gamma[BFp^\rho K^\rho(\dot{u}^2 + u'^2)^{1/2} + Ju^4 \sin^2 I \sin 2U] \) \hspace{1cm} (3.57b)

\( f_h = -\gamma \left[ BFp^\rho K^\rho \frac{1}{\sqrt{YK}} u (u^2 + u'^2)^{1/2} w_A \cos U \sin I + Ju^4 \sin^2 I \sin U \right] \) \hspace{1cm} (3.57c)

3.5 The Unperturbed Solution

When the perturbation forces due to air drag and Earth's oblateness are excluded, the only remaining external force applied to the satellite is \( f_N \) and the equations of motion (3.53) reduce to:

\[ \ddot{u} + u = K \] \hspace{1cm} (3.58a)

\[ \dot{K} = 0 \] \hspace{1cm} (3.58b)

\[ \dot{i} = 0 \] \hspace{1cm} (3.58c)

\[ \dot{\Omega} = 0 \] \hspace{1cm} (3.58d)

\[ \dot{U} = 1 \] \hspace{1cm} (3.58e)

The unperturbed solution is easily obtained from (3.58),

\[ u = K_0[1 + e_0 \cos(\theta_h - \psi_0)] \] \hspace{1cm} (3.59a)

\[ \dot{u} = -K_0 e_0 \sin(\theta_h - \psi_0) \] \hspace{1cm} (3.59b)

\[ K = K_0 = \text{constant} \] \hspace{1cm} (3.59c)

\[ I = I_0 = \text{constant} \] \hspace{1cm} (3.59d)
\[
\Omega = \Omega_0 = \text{constant} \quad (3.59e)
\]

\[
U = \theta_h - \eta_0 \quad (3.59f)
\]

with the 6 constants of integration: \(e_0, \psi_0, K_0, I_0, \Omega_0\) and \(n_0\) where subscript "0" denotes initial, constant values (at \(t = t_0\)). Once the initial conditions are specified, all six constants may be solved for. It is clear that (3.59a) represents the equation of an ellipse of semi-latus rectum \(1/K_0\) and eccentricity \(e_0\). Also, \(\psi_0\) is the angle between the inertial reference line (i.r.l.) and the current perigee (point of closest approach) of the orbit. The angles \(I_0\) and \(\Omega_0\) represent the orbit inclination and the right ascension of the ascending nodes (N). Finally, (3.59f) shows that \(n_0\) represents the angle between the line of nodes N and the i. r. l., in the orbital plane. Figure (3.4a) shows these angular relations where the initial position of the satellite (at \(t = t_0\)) at an angle \(\theta_{ho}\) from the i.r.l. and \(U_0\) from N is included. This initial position can be taken at the initial perigee without loss of generality so that, initially, \(\psi_0\) would be zero. It remains so in the unperturbed case. Similarly, one can choose the inertial reference line to coincide with this initial position so that \(\theta_{ho}\) is always zero. Then, the new angular relations are shown in (3.4b). Finally, the initial time \(t_0\) can also be set to zero without loss of generality.

The change of independent variables from \(\theta_h\) to \(t\) becomes:

\[
t = \sqrt{\frac{K_0}{\gamma}} \int_{0}^{\theta_h} \frac{dx}{u^2(x)} \quad (3.60)
\]

The time required for one orbital revolution with respect to the i. r. l. \(T_0\), is thus (setting \(\theta_h = 2\pi\)):

\[
T_0 = 2\pi \sqrt{\frac{\gamma K_0^3 (1 - e_0^2)^3}{\mathbf{3}}} \quad (3.61)
\]

The angular distance from the current satellite position to its current perigee is called the true anomaly, denoted \(\theta\),

\[
\theta \triangleq \theta_h - \psi_0 = U - \omega_0 \quad (3.62)
\]

where \(\omega_0\) is the angle between the current perigee and the line of nodes (Fig. 3.5-1a).
\[ \omega_0 \triangleq \psi_0 - n_0 = \text{argument of perigee} \] (3.63)

and in our case, \( \omega_0 = U_0 \).

The complete unperturbed solution can then be written

\[ u = K_0 (1 + e_0 \cos \theta) \] (3.64a)

\[ \dot{u} = -K_0 e_0 \sin \theta \] (3.64b)

\[ K = K_0 \] (3.64c)

\[ I = I_0 \] (3.64d)

\[ \Omega = \Omega_0 \] (3.64e)

\[ U - \theta_h - n_0 \] (3.64f)

\[ \theta = \theta_h - \psi_0 \] (3.64g)

where \( e_0, \psi_0, K_0, I_0, \Omega_0, \) and \( n_0 \) are the six integration constants.

For the degenerate case of a circular orbit \( (e_0 = 0) \), the unperturbed solution is, simply:

\[ U = K_0 \] (3.65a)

\[ \dot{u} = 0 \] (3.65b)

\[ K = K_0 \] (3.65c)

\[ I = I_0 \] (3.65d)

\[ \Omega = \Omega_0 \] (3.65e)

\[ U = \theta_h - n_0 \] (3.65f)

and \( \psi_0 \) is no longer defined. The orbital period becomes

\[ T_0 = \frac{2\pi}{\sqrt{\frac{\gamma}{K_0^3}}} \]

and the initial condition can be taken at an angle \( U_0 \) from \( N \) and \( \theta_h \) from the i. r. 1. When \( e \) becomes very small \( (e \approx 0.0005) \), the orbit can be considered circular and the last position of the perigee, \( U_0 \), from \( N \) is then taken as the initial position. Then \( U \) becomes the independent variable.

3.6 The Perturbed Solution and the Variational Equations

The perturbed solution is obtained by the method of variation of
parameters. This method is useful when the unperturbed solution is available and the perturbing forces are small compared to the forces considered in the unperturbed case. These two conditions are satisfied here.

The perturbed solution was the same mathematical form as that in the unperturbed case except that the integration coefficients are now functions of the independent variable ($\theta_h$ in our case). Therefore the perturbed solution represents an ellipse which is continually varying in size, shape, and orientation. At time $t + \delta t$, an orbit, different from that at $t$, describes the trajectory of the satellite. We then speak of the osculating orbit, i.e. the orbit that would result were all the perturbing forces set to zero at a given instant.

The equations for the osculating orbit are thus (perturbed solution):

\[
\begin{align*}
u &= K(l + e \cos \theta) \quad (3.66a) \\
\dot{u} &= -Ke \sin \theta \quad (3.66b) \\
U &= \theta_h - n \quad (3.66c) \\
\theta &= \theta_h - \psi \quad (3.66d)
\end{align*}
\]

where $K$, $e$, $\psi$, are now functions of $\theta_h$. These relations are obtained by applying the method to (2.66) and solving the resulting variational equations (V.E.'s). Substituting (3.66) into the differential equations (3.53) and solving for the variable parameters give the six V.E.'s.

\[
\begin{align*}
\dot{K} &= -\frac{2K^2}{\gamma u^3} f_U \\
\dot{e} &= -(e + \cos \theta) \frac{\dot{K}}{K} + \frac{\sin \theta}{\gamma u^3} (\ddot{u}_U + u_{fU}'') \\
\dot{\psi} &= -\frac{\sin \theta}{e} \frac{\dot{K}}{K} - \frac{\cos \theta}{e\gamma u^3} (\ddot{u}_U + u_{fU}') \\
\dot{I} &= \frac{K}{\gamma u^3} \cos U f_h \\
\dot{\omega} &= \frac{K}{\gamma u^3} \frac{\sin U}{\sin I} f_h \\
\dot{n} &= \dot{\omega} \cos I
\end{align*}
\]

where
Solving (3.67) will give the complete evolution of the osculating orbit expressed in the perturbed solution (3.66). It must be noted in connection with the integration of (3.67), that the parameters are assumed fixed, and consequently, from (3.66f), we have \( \dot{\theta} = d/d\theta h \), because, with \( \psi \) constant,

\[
d\theta h = d(\theta + \psi) = d\theta
\]

and integration with respect to the true anomaly can be equivalently performed.

When the satellite has gone through an angle \( \theta h = 2\pi \) (with respect to the i.r.l.), it is not back to the perigee since the latter has moved through an angle \( \Delta \psi \). Therefore, the total orbital period becomes,

\[
T = T_0 \left( \frac{2\pi + \Delta \psi}{2\pi} \right)
\]

From (3.63), an ancillary equation can be obtained, namely that of the argument of perigee:

\[
\dot{\omega} = \dot{\psi} - \dot{n}
\]

\[
= \left( \frac{\sin \theta}{e} \right) \frac{K}{e\gamma u^3} (\dot{u}_U + u f_U') - \dot{n} \cos I
\]

For the degenerate case, the osculating orbit is defined by:

\[
u = K
\]

\[
\dot{u} = 0
\]

\[
U = \theta h - n
\]

and the V.E.'s are:

\[
\dot{u} = \ddot{K} = \frac{-2f_U}{\gamma u}
\]

\[
\dot{I} = \cos U f_h / \gamma u^2
\]

\[
\dot{\Omega} = \frac{\sin U}{\sin I} \frac{1}{\gamma u^2} f_h
\]

\[
\dot{n} = \dot{\Omega} \cos I
\]

where, from (3.72c), \( \dot{\theta} = d/d\theta h = d/dU \) since

1. Equatorial orbits present a different approach and will be dealt with in Section 7.
\[ d_\theta h = d(U + n) = dU \]  
(3.74)

with \( n \) constant.

The orbital period becomes
\[ T = T_0 \left( \frac{2\pi + \Delta n}{2\pi} \right) \]  
(3.75)

where \( \Delta n \) is the angular displacement of the line of modes when \( U \) varies from 0 to \( 2\pi \).

### 3.7 Initial Conditions and Non-Dimensional V.E.'s

As stated in Section 3.6, the initial conditions are taken at the initial perigee point \((u_{po}, \delta_{po})\), so that,

\[ u_o = u_{po}; \quad \delta_o = \delta_{po} \]

where "o" denotes initial conditions.

In Section 2, the reference altitude \( v_r \) was to be 150 km for better accuracy in the static density model. Consequently, we have

\[ v_r = \frac{v_{re}}{(1 - f \sin^2 \delta)} \]  
(3.76)

with \( v_{re} \) evaluated at 150 km. In terms of the initial perigee height, this equatorial distance \( v_{re} \) can be written as

\[ v_{re} = \xi u_{po}; \quad \xi \triangleq \frac{v_{re}}{u_{po}} \]  
(3.77a)

or

\[ v_{re} = \xi u_{po} (1 - f \sin^2 \delta_{po}) \]  
(3.77b)

from (2.14). Consequently, (3.76) is rewritten,

\[ v_r = \frac{\xi u_{po} (1 - f \sin^2 \delta_{po})}{(1 - f \sin^2 \delta)} \]  
(3.78)

where \( \xi \) is defined by equation (3.77a) and we must then have

\[ \xi \triangleq \frac{v_{re}}{u_{po}} \geq 1 \]

When the orbit is initially circular, the initial conditions are taken at a \( u \)-distance \( u_o \) (initial radius) in the equatorial plane. In this case, (3.78) still applies with \( \delta_{po} = \delta_o = 0 \). The nondimensional variables,
denoted by an overbar are now defined:

\[ \tilde{u} \triangleq \frac{u}{u_p} \]  
(3.79a)

\[ \tilde{K} \triangleq \frac{K}{K_0} \]  
(3.79b)

The perturbed solution becomes

\[ \tilde{u} = \tilde{C}\tilde{K}(1 + e \cos \theta) \]  
(3.80a)

\[ \dot{\tilde{u}} = -\tilde{C}\tilde{K} e \sin \theta \]  
(3.80b)

and the other equations are the same as (3.66c), (3.66d). In (3.80), we have used,

\[ \tilde{C} \triangleq \frac{K_0}{u_p} = \frac{1}{1 + e_0} \]  
(3.81)

The nondimensional V.E.'s are:

\[ \dot{\tilde{K}} = -2\tilde{C} \frac{\tilde{K}^2}{u_p} \tilde{f} \]  
(3.82a)

\[ \dot{e} = -(e + \cos \theta)\frac{\dot{\tilde{K}}}{\tilde{K}} + \frac{\sin \theta}{u_p} \frac{1}{3} \left[ \tilde{u}\tilde{f}_U + \tilde{u}\tilde{f}_r \right] \]  
(3.82b)

\[ \dot{\psi} = -\left( \frac{\sin \theta}{e} \right) \frac{\dot{\tilde{K}}}{\tilde{K}} - \frac{\cos \theta}{ug} \frac{1}{3} \left[ \tilde{u}\tilde{f}_U + \tilde{u}\tilde{f}_r \right] \]  
(3.82c)

\[ \dot{\theta} = \frac{\tilde{C}\tilde{K}}{u_p^2} \cos U \tilde{f}_h \]  
(3.82d)

\[ \dot{\tilde{\Omega}} = \frac{\tilde{C}\tilde{K}}{u_p^3} \frac{\sin U}{\sin \tilde{I}} \tilde{f}_h \]  
(3.82e)

\[ \dot{\tilde{\Omega}} = \dot{\tilde{\Omega}} \cos \tilde{I} \]  
(3.82f)

where

\[ g_{po} \triangleq \gamma u_p^2 \]  
(3.83)

For circular orbits, (3.79) becomes,

\[ \bar{u} = \bar{K} \triangleq u/u_0 = K/K_0 \]  
(3.84)

and (3.82a), (3.82d), (3.82e), (3.82f) follow with \( e = 0, u_p = u_0 = K_0 \) and \( \bar{u} = \bar{K} \).
The perturbing forces (3.57) are also expressed as functions of the nondimensional variables. The following definitions simplify this change of variables:

\[
\frac{\Delta \dot{r}}{\rho} = \frac{\rho}{\rho_r}(150 \text{ km}) \quad (3.85)
\]

\[
\frac{\Delta \dot{v}}{v_{r0}} = \frac{BF \rho_r}{K_0} \quad (3.86)
\]

\[
\frac{\Delta \omega_A}{\omega_A/\omega_0} \quad (3.87)
\]

where

\[
\omega_0 = \frac{v_{r0}}{r_{r0}} = (g_{r0}^2/\gamma K_0)^{1/2}
\]

\[
J = Ju^2_{r0} = \frac{3}{2}Ju^2_{r0} \quad (3.88)
\]

Then,

\[
\overline{f}_r = -J \overline{u}^4(1 - 3 \sin^2 I \sin^2 U) + B \frac{\dot{u}}{K} \overline{u}^2 + \overline{u}^2 \overline{\omega}_A \cos U \sin I \quad (3.89a)
\]

\[
\overline{f}_u = -J \overline{u}^4(\sin^2 I \sin 2U) - B \frac{\dot{u}}{K} \overline{u}^2 + \overline{u}^2 \overline{\omega}_A \quad (3.89b)
\]

\[
\overline{f}_h = -J \overline{u}^4(\sin 2I \sin U) - B \frac{\dot{u}}{K} \left( \frac{\overline{u}^2 + \overline{u}^2}{K} \right)^{1/2} \overline{\omega}_A \cos U \sin I \quad (3.89c)
\]

where

\[
\overline{f}_r = \frac{f_r}{g_{r0}} \quad (3.90a)
\]

\[
\overline{f}_u = \frac{f_u}{g_{r0}} \quad (3.90b)
\]

\[
\overline{f}_h = \frac{f_h}{g_{r0}} \quad (3.90c)
\]

Substituting (3.90) into (3.82) yields the complete set of variational equations:

\[
\frac{\dot{K}}{K} = -2C \frac{\overline{u}^2}{u^3} \overline{f}_u \quad (3.91a)
\]

\[
\dot{e} = -(e + \cos \theta) \frac{\dot{K}}{K} + \frac{\sin \theta}{u^3} [\overline{u} \overline{f}_U + \overline{u} \overline{f}_r'] \quad (3.91b)
\]
\[ \dot{\psi} = -\left(\frac{\sin \theta}{e}\right) \frac{\dot{K}}{K} - \cos \theta \frac{\dot{u}}{eu^3} \left(\bar{F}_u + \bar{u} \bar{F}_u^\prime\right) \]  

(3.91c)

\[ i = \frac{ck}{u^3} \cos U \bar{F}_h \]  

(3.91d)

\[ \dot{i} = \frac{ck}{u^3} \sin U \bar{F}_h \]  

(3.91e)

\[ \dot{n} = \dot{\Omega} \cos I \]  

(3.91f)

and we also have the auxiliary equation \( \dot{\omega} = \dot{\psi} - \dot{n} \).

The perturbed solution (for the osculating orbit) is:

\[ \bar{u} = CK(1 + e \cos \theta) \]  

(3.92a)

\[ \dot{\bar{u}} = -CK e \sin \theta \]  

(3.92b)

\[ U = \theta_h - n \]  

(3.92c)

where

\[ \theta = \theta_h - \psi \]  

(3.93)

and also

\[ \omega = \psi - n \]  

(3.94)

3.8 Perturbation models in Terms of Orbital Elements

The perturbing forces, \( \bar{F}_g \) and \( \bar{F}_D \), are now expressed in terms of the osculating orbital elements.

3.8.1 Gravity Perturbations (Oblateness)

The three components of the gravity force were given earlier as functions of the angle \( U \) between the satellite and the line of nodes, and the \( u \)-distance, \( \bar{u} \), (3.89).

Combining (3.92c), (3.93) and (3.94), \( U \) is expressed as a function of the true anomaly and the argument of perigee,

\[ U = \theta + \omega \]  

(3.95)

Defining the following variables,

\[ p_0 \equiv -\sin 2\omega \]  

(3.96a)

\[ p_1 \equiv \cos 2\omega \]  

(3.96b)

\[ q_0 \equiv \cos^2 \omega \]  

(3.96c)
allows one to write:
\[
\sin 2U = \sin 2(\theta + \omega) = \rho_0 + 2\rho_1 \cos \theta \sin \theta - 2\rho_0 \cos^2 \theta \quad (3.97)
\]
\[
\sin^2 U = \sin^2(\theta + \omega) = \rho_0 - \rho_1 \cos^2 \theta - \rho_0 \sin \theta \cos \theta \quad (3.98)
\]
Using (3.92a), it is easy to see that \( \overline{u^4} \) is
\[
\overline{u^4} = C^4(1 + e \cos \theta)^4 \quad (3.99)
\]
and the substitution of (3.97), (3.98) and (3.99) into (3.89) completely describes the gravity forces in terms of the true anomaly. Note that the term in \( \sin U \) (equation (3.89c)) need not be expanded at this point because it will be multiplied by \( \cos U \) or \( \sin U \) into the complete equations and, consequently, Eqs. (3.97) and (3.98) will suffice.

3.8.2 Drag Perturbations

The same operations are now carried out for the drag force. The density model, and then the complete expression, will be considered. A few comments on \( (CDA) \) will close Section 3.

(a) Density

From (2.99), we have,
\[
\tilde{\rho} = \frac{\rho}{\rho_r} = R_p(t)D_p(t)\{1 + [F_p + \delta_p(u - u_p)] \cos \phi][1 + \beta_P(u - u_p)^2]
\]
\[
\times \exp[s_p(u - v_r)] \quad (3.100)
\]

In terms of nondimensional variables, one may define
\[
\tilde{\delta}_f \triangleq \delta_f u_p \quad (3.101a)
\]
\[
\tilde{\beta}_f \triangleq \beta_f u_p^2 \quad (3.101b)
\]

so that
\[
\tilde{\delta}_f(u - u_p) = \tilde{\delta}_f(\overline{u} - \overline{u}_p) \quad (3.102a)
\]
\[
\tilde{\beta}_f(u - u_p)^2 = \tilde{\beta}_f(\overline{u} - \overline{u}_p)^2 \quad (3.102b)
\]

Now, \( \cos \phi \) will be expressed in terms of the true anomaly. It is easy to see that
\[
\cos \phi = \frac{b \cdot 0_r}{4}
\]
where \( \mathbf{b} \) is a unit vector pointing to the centre of the bulge. If the bulge lags by an angle \( \lambda \) relative to the sub-solar point, and if its declination is taken as that of the Sun,
\[
\mathbf{b} = \cos \delta_s \cos (\alpha_s + \lambda) \mathbf{i}_x \\
+ \sin (\alpha_s + \lambda) \cos \delta_s \mathbf{i}_y \\
+ \sin \delta_s \mathbf{i}_z
\]  
(3.104)

where \((\alpha_s, \delta_s)\) define the Sun's position. Using \( \varphi \), given by (3.8), (3.103) is expanded to yield:
\[
\cos \varphi = A \cos \theta + B \sin \theta
\]  
(3.105)

\[
A = [\cos \frac{2e}{2} \cos (\omega + \Omega - \lambda - L) + \sin \frac{2e}{2} \cos (\omega + \Omega - \lambda + L)] \cos \frac{2I}{2}
+ [\cos \frac{2e}{2} \cos (\omega - \Omega + \lambda + L) + \sin \frac{2e}{2} \cos (\omega - \Omega + \lambda - L)] \sin \frac{2I}{2}
+ [\cos (\omega - L) - \cos (\omega + L)] \frac{\sin I \sin e}{2}
\]  
(3.106a)

\[
B = [-\cos \frac{2e}{2} \sin (\omega + \Omega - \lambda - L) + \sin \frac{2e}{2} \sin (\omega + \Omega - \lambda + L)] \cos \frac{2I}{2}
- [\cos \frac{2e}{2} \sin (\omega - \Omega + \lambda + L) + \sin \frac{2e}{2} \sin (\omega - \Omega + \lambda - L)] \sin \frac{2I}{2}
- [\sin (\omega - L) - \sin (\omega + L)] \frac{\sin I \sin e}{2}
\]  
(3.106b)

Equations (3.105) and (3.106a) were obtained by King-Hele [31]. The exponential factor is now expanded. We have
\[
\exp[s_p(u - v_r)] = \exp[s_p(u - \xi) \frac{(1 - f \sin^2 \delta_p)}{\sin I \sin e}]
\]  
(3.107)

The denominator \((1 - f \sin^2 \delta)\) is now expanded in a power series in \( f^n \) and because \( f \) is small, terms of the order \( f^n, n \geq 2 \), are neglected.

Introducing
\[
\bar{s}_p \triangleq s_p u_{po}
\]  
(3.108)

\[
c \triangleq \frac{s_p f}{2} \xi \sin^2 I
\]  
(3.109)

the exponential factor becomes:
\[
\exp[s_p(u - v_r)] = \exp[\bar{s}_p(\bar{u} - \xi) + c \cos 2(\theta + \omega) - c \cos 2\omega_0]
\]  
(3.110)

Following King-Hele [30], the factor \( \exp[c \cos 2(\theta + \omega)] \) is expanded
to second order in $c$ (Section 4.2).

\[
\exp[c \cos 2(\theta + \omega)] = 1 + c \cos 2(\theta + \omega) + \frac{c^2}{2} \cos 2(\theta + \omega) \quad (3.111)
\]

With (3.101), (3.105), (3.110) and (3.111), Eq. (3.100) becomes,

\[
\bar{\rho} = \left[ R_p(t)D_p(t)e^{-c \cos 2 \omega_0} \left\{ 1 + \left[ F_p + \bar{\alpha}_f(\bar{u} - \bar{u}_p) \right] (A \cos \theta + B \sin \theta) \right\} \right. \\
* \left. \left[ 1 + \bar{\beta}_f(\bar{u} - \bar{u}_p)^2 \right] [1 + c \cos 2(\theta + \omega) + \frac{c^2}{2} \cos 2(\theta + \omega)] e^{p_\delta} (\bar{u} - \xi) \right)
\]

Transforming in terms of $\cos^n\theta$ and $\sin^n\theta$, and substituting $\bar{u}$ from (3.92a), one gets the form

\[
\bar{\rho} = \left[ R_p(t)D_p(t) e^{-c \cos 2 \omega_0 + \bar{s}_p(C\bar{K} - \xi)} \right] e^z \cos \theta \\
* \left[ g_0 + \sum_{i=1}^{8} \left( g_i \cos^i \theta + h_i \sin \theta \cos^{i-1} \theta \right) \right]
\]

where

\[
z \triangleq \bar{s}_p C\bar{K}e \quad (3.113)
\]

and the coefficients of the cosine and sine functions are:

\[
g_i \triangleq \left[ \sum_{n=0}^{2} d_n a_{i-2n} \right] + d_3 (b_i - b_{i-2}) + d_4 (b_{i-2} - b_{i-4}) \quad (3.114a)
\]

\[
h_i \triangleq \left[ \sum_{n=1}^{2} (d_n b_{i-2n} + d_{n+2} a_{i-2n}) \right] + d_0 b_i \quad (3.114b)
\]

where

\[
d_0 \triangleq 1 + \left( \frac{c^2}{4} \right) (1 + p_2) - cp_1 \quad (3.115a)
\]

\[
d_1 \triangleq 2cp_1 - d_2 \quad (3.115b)
\]

\[
d_2 \triangleq 2c^2 p_2 \quad (3.115c)
\]

\[
d_3 \triangleq 2cp_0 - d_4 / 2 \quad (3.115d)
\]

\[
d_4 \triangleq -2c^2 p_3 \quad (3.115e)
\]

$p_0$, $p_1$, $c$ given in (3.96a), (3.96b), (3.109), respectively

\[
p_2 = \cos 4\omega \quad (3.116a)
\]

\[
p_3 = \sin 4\omega \quad (3.116b)
\]
\[ a_1 = Se(j_1 + 2j_2 S) + \frac{A}{B} b_1 \]  
(3.117a)

\[ a_2 = S^2 e^{2j_2} + \frac{A}{B} b_2 \]  
(3.117b)

\[ a_3 = \frac{A}{B} b_3 \]  
(3.117c)

\[ a_4 = \frac{A}{B} b_4 \]  
(3.117d)

\[ a_0 = j_0 + j_1 S + j_2 S^2 \]  
(3.117e)

\[ a_i = 0 \text{ for } i \neq 0,1,2,3,4 \]  
(3.117f)

\[ b_1 = B(j_3 + j_4 S + j_5 S^2 + j_6 S^3) \]  
(3.118a)

\[ b_2 = Be(Sj_4 + 2j_5 S^2 + 3j_6 S^3) \]  
(3.118b)

\[ b_3 = Be^2(j_5 S^2 + 3j_6 S^3) \]  
(3.118c)

\[ b_4 = Be^3(j_6 S^3) \]  
(3.118d)

\[ b_i = 0 \text{ if } i \neq 1,2,3,4 \]  
(3.118e)

\[ S = cK \]  
(3.119)

\[ J_0 = 1 + \bar{\beta}_f \bar{u}_p \]  
(3.120a)

\[ J_1 = -2\bar{\beta}_f \bar{u}_p \]  
(3.120b)

\[ J_2 = \bar{\beta}_f \]  
(3.120c)

\[ J_3 = (1 + \bar{\beta}_f \bar{u}_p^2)(F_p - \bar{\delta}_f \bar{u}_p) = j_0(F_p - \bar{\delta}_f \bar{u}_p) \]  
(3.120d)

\[ J_4 = \bar{\delta}_f - 2\bar{\beta}_f \bar{u}_p F_p + 3\bar{\beta}_f^2 \bar{u}_p^2 = j_0 \bar{\delta}_f + j_1(F_p - \bar{\delta}_f \bar{u}_p) \]  
(3.120e)

\[ J_5 = -3\bar{\beta}_f \bar{\delta}_f \bar{u}_p + \bar{\beta}_f \bar{u}_p = j_1 \bar{\delta}_f + j_2(F_p - \bar{\delta}_f \bar{u}_p) \]  
(3.120f)

\[ J_6 = \bar{\beta}_f \bar{\delta}_f = j_2 \bar{\delta}_f \]  
(3.120g)
(b) Expansion of the Drag Forces

Using (3.92), the factor \((\bar{u}^2 + \bar{u}^2)^{1/2}\) in the drag forces is written as:

\[
(\bar{u}^2 + \bar{u}^2)^{1/2} = C \bar{K}(e^2 + 2e \cos \theta + 1)^{1/2}
\]

(3.121)

Inserting (3.121) into the drag part of Eqs. (3.89) gives:

\[
\overline{f_{Dr}} = \overline{BCp} \overline{u} (e^2 + 2e \cos \theta + 1)^{1/2}
\]

(3.122a)

\[
f_{DU} = -\overline{BCp} \overline{u}(e^2 + 2e \cos \theta + 1)^{1/2}
\]

(3.122b)

\[
f_{Dh} = -\overline{BC} \frac{\overline{\bar{\rho}} \overline{K}}{\overline{u}} (e^2 + 2e \cos \theta + 1)^{1/2} \overline{\omega_A} \cos \theta \sin I
\]

(3.122c)

(c) Drag Coefficient and Projected Area

Because the product \((CD)\) is given by an averaging technique, with errors of up to 20%, the coefficient \(B\) in (3.122),

\[
\overline{B} = \frac{CD A \overline{F_{pr}}}{2m \overline{K_0}}
\]

varies randomly, and this effect is now considered. We modify \(\overline{B}\) to have the form:

\[
\overline{B} = \overline{B}_0 (1 + x \mu)
\]

(3.123)

where

\[
\overline{B}_0 \triangleq \frac{CD A \overline{F_{pr}}}{2m \overline{K_0}}
\]

(3.124)

Here, \(x\) is a random variable, and \(\mu\) is a measure of the variation of \(\overline{B}\) about \(\overline{B}_0\). Since \(CD\) is an average over one orbit \(x\) will be constant within one orbit but will vary randomly from one orbit to the next. The most appropriate probability distribution function for \(x\) is not discussed here; we simply assume all orientations to be equally probable so that \(x\) is uniformly distributed over the interval \([-1, +1]\). In this case, \(\mu\) is given by

\[
\mu = \frac{\overline{B} - \overline{B}_0}{\overline{B}_0} \text{ max} > 0
\]

i.e. \(\mu\) is the maximum relative variation in \(\overline{B}\), so that \(\mu = 0.22\).
4. DRAG-INDUCED DISPERSION IN A STATIONARY ATMOSPHERE

For short-term approximate lifetime predictions, it is customary to neglect the secular effects of both Earth's oblateness and atmospheric rotation. The decay of the orbit is experienced in a fixed plane with respect to an inertial frame and, consequently, only in-plane motion is considered. Although lifetime predictions have some meaning under these assumptions, the actual trajectory is inaccurate.

4.1 Simplification of the V.E.'s

Setting \( \overline{f}_{gr} \), \( \overline{f}_{gu} \), \( \overline{f}_{gh} \) and \( \overline{w}_A \) to zero in (3.91) yields:

\[
\dot{K} = -2C \frac{K^2}{U^3} \overline{F}_{DU} \tag{4.1a}
\]

\[
\dot{e} = -(e + \cos \theta) \frac{K}{K} + \frac{\sin \theta}{U^3} \left[ \overline{u} \overline{F}_{DU} + \overline{u} \overline{F}_{Dr} \right] \tag{4.1b}
\]

\[
\dot{\psi} = -\left( \frac{\sin \theta}{e} \right) \frac{K}{K} - \frac{\cos \theta}{e U^2} \left[ \overline{u} \overline{F}_{DU} + \overline{u} \overline{F}_{Dr} \right] \tag{4.1c}
\]

\[
\dot{1} = 0 \tag{4.1d}
\]

\[
\dot{\Omega} = 0 \tag{4.1e}
\]

\[
\dot{n} = 0 \tag{4.1f}
\]

\[
\dot{\omega} = \dot{\psi} \tag{4.1g}
\]

where \( \overline{F}_{Dr} \) and \( \overline{F}_{DU} \) are given in (3.122a), (3.122b). From these two equations, it can be shown that the terms in the square brackets of (4.1b), (4.1c) vanish. Substituting \( \overline{F}_{DU} \) from (3.122b) and \( \overline{u} \) from the perturbed solution (3.92) gives, after simplification,

\[
\dot{K} = 2 \overline{B} \frac{\rho}{\rho} \frac{(e^2 + 2e \cos \theta + 1)^{1/2}}{(1 + e \cos \theta)^2} \tag{4.2a}
\]

\[
\dot{e} = -(e + \cos \theta) \frac{K}{K} \tag{4.2b}
\]

\[
\dot{\psi} = \omega = -\left( \frac{\sin \theta}{e} \right) \frac{K}{K} \tag{4.2c}
\]

\[
\dot{1} = \dot{\Omega} = \dot{n} = 0 \tag{4.2d}
\]

The solution is still given by (3.92) except that \( \Omega, I \) and \( n \) are
constant and equal to their initial values,

\[ I = I_0 \quad ; \quad \Omega = \Omega_0 \quad ; \quad n = n_0 \]  \hspace{1cm} (4.3)

However, \( \overline{K} \), \( e \) and \( \omega \) are variable due to drag. Before solving for these quantities, some assumptions are required.

4.2 Assumptions

To obtain closed-form solutions for the per-revolution change in \( \overline{K} \), \( e \), and \( \omega \), the following assumptions are made.

Assumption 4.2.1

We are concerned here with \( e < 0.2 \). Therefore, factors of the form \( [1 + f(e)]^n \), where \( f(e) \) is any function of the eccentricity with \( |f(e)| < 1 \) will be expanded as:

\[ [1 + f(e)]^n = 1 + nf(e) + n(n - 1)f^2(e)/2 + \ldots \]  \hspace{1cm} (4.4)

This series is then truncated by neglecting terms of order \( e^4 \) (i.e. of size 0.0016). Therefore, only terms in \( e^3 \), \( e^2 \), \( e^1 \), \( e^0 \) will be kept, producing an error < 0.2%.

The next two assumptions were discussed previously but are reproduced here for clarity.

Assumption 4.2.2

Because Stage A only is considered, the variation of \( \theta \) is much faster than that of \( \overline{K} \), \( e \) and \( \omega \), so that, when integrating (4.2), the latter three variables will be assumed constant over \( 0 \leq \theta \leq 2\pi \) (one orbit). This assumption is compatible with the concept of osculating orbits, where the perturbed solution is valid only for a small interval of time. This interval is thus taken as the orbital period, \( T \).

Assumption 4.2.3

The flattening correction factor, \( c = (s_p f \rho \sin^2 I)/2 \), is neglected when the order \( n \geq 3 \):

\[ c^n = 0 \quad \text{when} \quad n \geq 3 \]

Since the flattening effect is itself a small influence, terms in \( c^2 \) may be viewed as already third order (similar to \( e^3 \)) and are therefore the highest powers of \( c \) kept.
Assumption 4.2.4

Combining Assumptions 4.2.1 and 4.2.3, terms of order $e^3 c$, $c^2 e$, $c^3 e^4$, and smaller, are neglected, while terms of the order $e^2 c, c e, c^2 e^3$, and larger, are kept.

In view of these considerations, only the first five terms in the summation in (3.112) need to be kept:

$$
\dot{p} = [R_p(t)D_p(t)e^{c \cos \omega - 5 k \cos \theta} e^{\gamma_{p}(C^\infty)}] e^z \cos \theta
$$

* $[g_0 + \sum_{i=1}^{5} (g_i \cos^i \theta + h_i \cos^{i-1} \theta \sin \theta)]$ 

4.3 Solutions

Substituting (4.5) into (4.2a), (4.2b), (4.2c), multiplying out the terms, expanding according to Assumption 4.2.1, simplifying according to Assumptions 4.2.3 and 4.2.4, transforming the terms in $\cos^n \theta$ into terms like $\cos n \theta$ and using the properties

$$
\int_0^{2\pi} e^z \cos \theta \cos n \theta \sin \theta = 0 \quad (4.7)
$$

and, finally, keeping in mind Assumption 4.2.2, the change in $K$, $e$ and $\omega$ per orbit can be obtained in closed-form.

To begin, $\Delta K_{2\pi}$ may be written:

$$
\Delta K_{2\pi} = [4BnR_pD_p e^{c \cos \omega - 5 k \cos \theta} e^{\gamma_{p}(C^\infty)}] \sum_{n=0}^{5} \lambda_n I_n(z) \quad (4.8)
$$

where $I_n(z)$ is the Bessel function of imaginary argument (order $n$), and
\[ \lambda_0^k = \alpha_0^k + \frac{\alpha_2^k}{2} + \frac{3\alpha_4^k}{8} \]
\[ \lambda_1^k = \alpha_1^k + \frac{3\alpha_3^k}{4} + \frac{5\alpha_5^k}{8} \]
\[ \lambda_2^k = \frac{\alpha_2^k + \alpha_4^k}{2} \]
\[ \lambda_3^k = \frac{\alpha_3^k}{4} + \frac{5\alpha_5^k}{16} \]
\[ \lambda_4^k = \frac{\alpha_4^k}{8} \]
\[ \lambda_5^k = \frac{\alpha_5^k}{16} \]
\[ k_0 = e_0^g_0 \]
\[ k_1 = e_0^g_1 + e_1^g_0 \]
\[ k_2 = e_0^g_2 + e_1^g_1 + e_2^g_0 \]
\[ k_3 = e_0^g_3 + e_1^g_2 + e_2^g_1 + e_3^g_0 \]
\[ k_4 = e_0^g_4 + e_1^g_3 + e_2^g_2 + e_3^g_1 \]
\[ k_5 = e_0^g_5 + e_1^g_4 + e_2^g_3 + e_3^g_2 \]
\[ e_0 = 1 + e^2/2 \]
\[ e_1 = -e - 3e^3/2 \]
\[ e_2 = e^2/2 \]
\[ e_3 = e^3/2 \]

The \( g_i \) were given in (3.114a). The subscript \( 2\pi \) refers to a change occurring during a complete orbit and superscript \( k \) identifies coefficients associated with \( \Delta K_{2\pi} \).

With the same method, \( \Delta e_{2\pi} \) is obtained:

\[ \Delta e_{2\pi} = -\left[4\pi R D e^{\frac{e^2}{2}} \overline{S}_P (C - \varepsilon) \right] \sum_{n=0}^{5} \frac{(e\lambda_n^k + \lambda_n^e)I_n(z)}{K} \]

where the \( \lambda_n^k \) were given in (4.9) and
\[
\begin{align*}
\lambda_0^e &= \alpha_1^k / 2 + 3\alpha_3^k / 8 + 5\alpha_5^k / 16 = \lambda_1^k / 2 \\
\lambda_1^e &= \alpha_0^k + 3\alpha_2^k / 4 + 5\alpha_4^k / 8 \\
\lambda_2^e &= \alpha_1^k / 2 + \alpha_3^k / 2 + 15\alpha_5^k / 32 \\
\lambda_3^e &= \alpha_2^k / 4 + 5\alpha_4^k / 16 \\
\lambda_4^e &= \alpha_3^k / 8 + 3\alpha_5^k / 16 \\
\lambda_5^e &= \alpha_4^k / 16 \\
\end{align*}
\]

and the \( a_n^k \) are available in (4.10).

Finally,

\[
\Delta \omega_{2\pi} = \Delta \psi_{2\pi} = -\frac{c \cos 2\omega_0}{4\pi} \frac{e^{-p(CK-x)}}{e^{K}} \sum_{n=0}^{5} \eta_n^\omega \Lambda_n(z) 
\]

where

\[
\begin{align*}
\eta_0^\omega &= \beta_0^\omega + \beta_2^\omega / 2 + 3\beta_4^\omega / 8 + 5\beta_6^\omega / 16 \\
\eta_1^\omega &= \beta_1^\omega + 3\beta_3^\omega / 4 + 5\beta_5^\omega / 8 \\
\eta_2^\omega &= \beta_2^\omega / 2 + \beta_4^\omega / 2 + 15\beta_6^\omega / 32 \\
\eta_3^\omega &= \beta_3^\omega / 4 + 5\beta_5^\omega / 16 \\
\eta_4^\omega &= \beta_4^\omega / 8 + 3\beta_6^\omega / 16 \\
\eta_5^\omega &= \beta_5^\omega / 16 \\
\end{align*}
\]

Note that Eqs. (4.15) would be similar to Eqs. (4.9) if \( \beta_6^\omega \) were negligible.

Next,
\[ \beta_0^\omega = e_0 h_1 \]
\[ \beta_1^\omega = e_0 h_2 + e_1 h_1 \]
\[ \beta_2^\omega = e_0 h_3 + e_1 h_2 + e_2 h_1 - e_0 h_1 \]
\[ \beta_3^\omega = e_0 h_4 + e_1 h_3 + e_2 h_2 + e_3 h_1 - e_0 h_2 - e_1 h_1 \]
\[ \beta_4^\omega = e_0 h_5 + e_1 h_4 + e_2 h_3 + e_3 h_2 - e_0 h_3 - e_2 h_1 \]
\[ \beta_5^\omega = e_1 h_5 + e_2 h_4 + e_3 h_3 - e_0 h_4 - e_1 h_3 - e_2 h_2 - e_3 h_1 \]
\[ \beta_6^\omega = e_2 h_5 + e_3 h_4 - e_0 h_5 - e_1 h_4 - e_2 h_3 - e_3 h_2 \]

The \( e_i \) and \( h_i \) are shown in (4.11) and (3.114b) respectively.

The orbital period is then given by,
\[ T = (1 + \frac{\Delta \psi_{2\pi}}{2\pi}) T_0 = (1 + \frac{\Delta \omega_{2\pi}}{2\pi}) T_0 \]  

where \( T_0 \) is given in (3.61), since the satellite must go through an additional angle \( \Delta \psi_{2\pi} \) in order to come back to the perigee.

5. DISPERSION DUE TO DRAG AND OBLATE PRIMARY MASS

For long-term lifetime predictions and accurate trajectory estimation, the secular effects of Earth's oblateness and atmospheric rotation have to be considered. The orbital inclination and the right ascension vary, and the resulting trajectory is now three-dimensional.

5.1 Simplification of the V.E.'s

Complete expressions for the disturbing forces (3.89) are now considered. Since the drag parts of \( \tau_r \) and \( \tau_U \) have already been developed and solved in Section 4, these will not be repeated here.

Using (3.34),
\[ \dot{K} = \ddot{K}_D + \ddot{K}_g = \dot{K}_D - 2C \frac{K^2}{u^3} \tau \]
\[ \dot{e} = e_D + \dot{e}_g = e_D - (e + \cos \theta) \frac{\dot{q}}{K} + \frac{\sin \theta}{u^3} \left[ \frac{\tau}{u} \tau_{GU} + \frac{\tau}{u} \tau_{GR} \right] \]
\[ \dot{\psi} = \dot{\psi}_D + \dot{\psi}_g = \dot{\psi}_D - \frac{\sin \theta}{e} \frac{\dot{q}}{K} - \frac{\cos \theta}{e u^3} \left[ \frac{\tau}{u} \tau_{GU} + \frac{\tau}{u} \tau_{GR} \right] \]
\[ \ddot{i} = \ddot{i}_D + \ddot{i}_g = \frac{CK}{u^3} \cos U f D + \frac{CK}{u^3} \cos U f g \]  

\[ \ddot{\omega} = \ddot{\omega}_D + \ddot{\omega}_g = \frac{CK}{u^3} \sin U f D + \frac{CK}{u^3} \sin U f g \]  

\[ \ddot{n} = \ddot{n}_D + \ddot{n}_g = \ddot{\omega}_D \cos I + \ddot{\omega}_g \cos I \]  

\[ \ddot{\omega} = (\ddot{\psi}_D - \ddot{\omega}_D \cos I) + (\ddot{\psi}_g - \ddot{\omega}_g \cos I) \]  

Since \( \ddot{K}_D, \ddot{e}_D, \) and \( \ddot{\psi}_D \) were solved for in Section 4, we will here be concerned with \( \ddot{K}_g, \ddot{e}_g, \ddot{\psi}_g, \ddot{f}_g, \ddot{n}_g \) and then \( \ddot{f}_D \) and \( \ddot{n}_D \). Equations for \( \ddot{n} \) and \( \ddot{\omega} \) will follow from \( \ddot{\omega} \) and \( \ddot{\psi} \).

Substituting the perturbing forces (3.89), and incorporating the changes discussed in Section 3.8, we get:

\[ \ddot{K}_g = 2CK^2J \ddot{u}(\sin^2 I \sin 2U) \]  

\[ \ddot{e}_g = -(e + \cos \theta) \ddot{K}_g - \ddot{\omega} \ddot{e} \ddot{u}(\sin^2 I \sin 2U) + J \ddot{u}^2(1 - 3 \sin^2 I \sin^2 2U) \]  

\[ \ddot{\psi}_g = -\left(\frac{\sin \theta}{e}\right) \ddot{K}_g + \frac{\cos \theta}{e} \left[J \ddot{u}^2(\sin^2 I \sin 2U) + \ddot{J} \ddot{u}^2(1 - 3 \sin^2 I \sin^2 2U)\right] \]  

\[ \ddot{f}_g = -(CK \ddot{J} \ddot{u} \sin 2I \sin 2U)/2 \]  

\[ \ddot{\omega}_g = -2CK \ddot{J} \ddot{u} \cos I \sin^2 2U \]  

\[ \ddot{f}_D = -CK \sin \frac{B_0}{u^4} \left(\frac{\ddot{u}^2 + \ddot{u}^2}{K}\right)^{1/2} \omega A(1 - \sin^2 U) \]  

\[ \ddot{\omega}_D = \frac{CK B_0}{2} \frac{\ddot{u}^2 + \ddot{u}^2}{K} \left(\frac{\ddot{u}^2 + \ddot{u}^2}{K}\right)^{1/2} \omega A \sin 2U \]

Some additional assumptions are now made prior to solution of these equations.

5.2 Assumptions

All the assumptions described in Section 4.2 are applicable here. It is important to note that although Earth's ablateness induces a variation in \( K, e \) and \( \psi \) within one orbit the net variation over the complete
orbit is zero. However, the intra-orbital variation implies slight changes in the density profile encountered by the satellite, especially for low altitude circular orbits [46], but this is ignored in the integration over one orbit.

5.3 Solutions (Oblateness and Atmospheric Rotation)

Integrating Eqs. (5.2) from $\theta = 0$ to $\theta = 2\pi$ yields:

$$\Delta K = 0 \quad \Delta e = 0$$  \hfill (5.4a, 5.4b)

$$\Delta \psi = 2C^2 K^2 J_\pi \left( \frac{3}{2} \sin^2 I - 1 \right)$$  \hfill (5.4c)

$$\Delta I = 0 \quad \Delta \Omega = -2C^2 K^2 J_\pi (\cos I)$$  \hfill (5.4d, 5.4e)

From (5.1f) and (5.1g) we get

$$\Delta \Omega = -2C^2 K^2 J_\pi (\cos I)$$  \hfill (5.4f)

$$\Delta \psi = C^2 K^2 J_\pi (5 \cos^2 I - 1)$$  \hfill (5.4g)

The influence of the atmospheric rotation on $I$ and $\Omega$ is obtained from (5.3). Substituting (3.97), (3.98) and (3.121) into (5.3), and integrating over $[0, 2\pi]$ yields:

$$\Delta I_D = \left[ \frac{2B\omega A}{K^2 c^2} \sin I - c \cos \omega \bar{\omega} \right] \sum_{n=0}^{\infty} \pi_n I_n(z)$$  \hfill (5.5)

$$\pi_0 = \psi_0^i + 1/2(\psi_2^i + \phi_2^i) + 3/8(\psi_4^i + \psi_4^i) + 5/16(\psi_6^i + \phi_6^i)$$

$$\pi_1 = (\psi_1^i + \phi_1^i) + 3/4(\psi_3^i + \phi_3^i) + 5/8(\psi_5^i + \psi_5^i) + 35/64(\psi_7^i + \phi_7^i)$$

$$\pi_2 = 1/2(\psi_2^i + \phi_2^i) + 1/2(\psi_4^i + \phi_4^i) + 15/32(\psi_6^i + \phi_6^i)$$

$$\pi_3 = 1/4(\psi_3^i + \phi_3^i) + 5/16(\psi_5^i + \phi_5^i) + 21/64(\psi_7^i + \phi_7^i)$$

$$\pi_4 = 1/8(\psi_4^i + \phi_4^i) + 3/16(\psi_6^i + \phi_6^i)$$

$$\pi_5 = 1/16(\psi_5^i + \phi_5^i) + 7/64(\psi_7^i + \phi_7^i)$$  \hfill (5.6)
\[
\psi_0 = g_0^i \\
\psi_1 = g_0^i + g_0 \sigma_1 \\
\psi_2 = g_0^i + g_1 \sigma_1 + g_0 \sigma_2 \\
\psi_3 = g_0^i + g_2 \sigma_1 + g_1 \sigma_2 + g_0 \sigma_3 \\
\psi_4 = g_0^i + g_3 \sigma_1 + g_2 \sigma_2 + g_1 \sigma_3 + g_0 \sigma_4 \\
\psi_5 = g_0^i + g_4 \sigma_1 + g_3 \sigma_2 + g_2 \sigma_3 + g_1 \sigma_4 + g_0 \sigma_5 \\
\psi_6 = g_5 \sigma_1 + g_4 \sigma_2 + g_3 \sigma_3 + g_2 \sigma_4 + g_1 \sigma_5 \\
\psi_7 = g_5 \sigma_1 + g_4 \sigma_2 + g_3 \sigma_3 + g_2 \sigma_4 + g_1 \sigma_5 \\
\phi_1 = h_1 \xi_2 \\
\phi_2 = h_1 \xi_3 + h_2 \xi_2 \\
\phi_3 = h_1 \xi_4 + h_3 \xi_2 + \phi_1 \\
\phi_4 = h_1 \xi_5 + h_2 \xi_4 + h_3 \xi_3 + h_4 \xi_2 - \phi_1 \\
\phi_5 = h_2 \xi_5 + h_3 \xi_4 + h_4 \xi_3 + h_5 \xi_2 - \phi_3 - \phi_1 \\
\phi_6 = h_3 \xi_5 + h_4 \xi_4 + h_5 \xi_3 - \phi_4 - \phi_2 \\
\phi_7 = h_4 \xi_5 + h_5 \xi_4 + \phi_5 - \phi_3 - \phi_1 \\
\sigma_0 = (1 - q_0) \xi_0^i \\
\sigma_1 = (1 - q_0) \xi_1^i \\
\sigma_2 = (1 - q_0) \xi_2^i + p_1 \xi_0^i \\
\sigma_3 = (1 - q_0) \xi_3^i + p_1 \xi_1^i \\
\sigma_4 = p_1 \xi_2^i \\
\sigma_5 = p_1 \xi_3^i 
\]

(5.7)

(5.8)

(5.9)
\[ \xi_2 = p_0 \xi_0 \]
\[ \xi_3 = p_0 \xi_1 \]
\[ \xi_4 = p_0 \xi_2 \]
\[ \xi_5 = p_0 \xi_3 \]

(5.10)

\[ \xi_0 = 1 + e^2 / 2 \] (5.11a)
\[ \xi_1 = -(3e + 5e^3 / 2) \] (5.11b)
\[ \xi_2 = (11/2)e^2 \] (5.11c)
\[ \xi_3 = -(15/2)e^3 \] (5.11d)

Expressions for \( q_0, p_0 \) and \( p_1 \) are given in (3.96) while the \( h_i \) and \( g_i \) are as in (3.114). It may also be noted that, since the product \( B_0 \omega_A \) is very small, perhaps some of the \( \pi_n^2 \) can be neglected and the expression for \( \Delta I_{D,2\pi} \) would be greatly simplified.

Performing the same derivations on (5.3b) yields

\[ \Delta \Omega_{D,2\pi} = -\left[\frac{B_0 \omega_A \pi}{K^2 / 2 C^2} R_D p_0 e^{-c \cos 2\omega_0 s} (C \bar{K} - \xi) \right] \sum_{n=0}^{5} \pi_n^\Omega I_n(z) \] (5.12)

The definitions of \( \pi_n^\Omega, \psi_n^\Omega \) and \( \varphi_n^\Omega \) are exactly the same as (5.6)-(5.8) except that superscript \( i \) is now replaced by superscript \( \Omega \). The other equations are slightly different:

\[ \phi_0^\Omega = p_0 \xi_0^i \]
\[ \phi_1^\Omega = p_0 \xi_1^i \]
\[ \phi_2^\Omega = p_0 \xi_2^i - 2p_0 \xi_0^i \]
\[ \phi_3^\Omega = p_0 \xi_3^i - 2p_0 \xi_1^i \]
\[ \phi_4^\Omega = -2p_0 \xi_2^i \]
\[ \phi_5^\Omega = -2p_0 \xi_3^i \] (5.13)
\[ \zeta_2^\Omega = p_1 \xi_0 \]
\[ \zeta_3^\Omega = p_1 \xi_1 \]
\[ \zeta_4^\Omega = p_1 \xi_2 \]
\[ \zeta_5^\Omega = p_1 \xi_3 \]  \hspace{1cm} (5.14)

where the \( \xi_n \) are given in (5.11) and \( p_0 \) and \( p_1 \) in (3.96a) and (3.96b) respectively.

From (5.1f) and (5.1g), we get the change in the line of nodes
\[ \Delta n_{D,2\pi} = \Delta \Omega_{D,2\pi} \cos I \]  \hspace{1cm} (5.15)

where \( \Delta \Omega_{D,2\pi} \) is given in (5.12). The auxiliary equation is
\[ \Delta \omega_{D,2\pi} = \Delta \psi_{D,2\pi} - \Delta \Omega_{D,2\pi} \cos I \]
\[ = \Delta \psi_{D,2\pi} - \Delta n_{D,2\pi} \]  \hspace{1cm} (5.16)

where \( \Delta \psi_{D,2\pi} \) and \( \Delta n_{D,2\pi} \) are given in (4.14) and (5.15) respectively.

5.4 Complete Solutions

\[ \Delta K_{2\pi} = \Delta K_{D,2\pi} \]  \hspace{1cm} (5.17a)
\[ \Delta e_{2\pi} = \Delta e_{D,2\pi} \]  \hspace{1cm} (5.17b)
\[ \Delta \psi_{2\pi} = \Delta \psi_{D,2\pi} + \Delta \psi_{g,2\pi} \]  \hspace{1cm} (5.17c)
\[ \Delta I_{2\pi} = \Delta I_{D,2\pi} \]  \hspace{1cm} (5.17d)
\[ \Delta \Omega_{2\pi} = \Delta \Omega_{D,2\pi} + \Delta \Omega_{g,2\pi} \]  \hspace{1cm} (5.17e)
\[ \Delta n_{2\pi} = (\Delta \Omega_{D,2\pi} + \Delta \Omega_{g,2\pi}) \cos I \]  \hspace{1cm} (5.17f)
\[ \Delta \omega_{2\pi} = \Delta \psi_{2\pi} - \Delta n_{2\pi} = (\Delta \psi_{D,2\pi} - \Delta \psi_{D,2\pi} \cos I) + (\Delta \psi_{g,2\pi} - \Delta \psi_{g,2\pi} \cos I) \]  \hspace{1cm} (5.17g)

where the individual variations are given in the following equations:
Finally, the orbital period is

\[ T = (1 + \frac{\Delta\psi_{2\pi}}{2\pi})T_0 \]  

(5.18)

6  CIRCULAR ORBITS

When the eccentricity or the inclination of an orbit go to zero, the equations so far described must undergo some changes. Circular orbits are treated here. Equatorial orbits will be dealt with in the next section.

From (3.91) and (3.72), we have

\[ \dot{K} = -2fU/K \]  

(6.1a)

\[ \dot{\iota} = \frac{\cos U_F}{K^2} \dot{h} \]  

(6.1b)

\[ \dot{\Omega} = \frac{1}{K^2} \frac{\sin U_F}{\sin \iota} \dot{h} \]  

(6.1c)

\[ \ddot{n} = \dot{\Omega} \cos \iota \]  

(6.1d)

with the perturbed solution,

\[ \ddot{u} = \ddot{K} \]  

(6.2a)

\[ \ddot{\iota} = 0 \]  

(6.2b)

\[ K, \iota, \Omega \text{ all given by (6.1)} \]  

(6.2c)

\[ U = \theta_h - n \]  

(6.2d)

Since the perigee is no longer defined, the motion will then be referenced to the line of nodes. From equation (6.2d) and the fact that no perturbation exists during an osculating orbit, we must have \( dU = d\theta_h \) so that
The variational equations (6.1) will then be integrated with respect to U instead of 6. Since the integration was from perigee to perigee in Sections 4 and 5, we will here integrate from \(\omega_0\) to \(\omega_0 + 2\pi\). When the eccentricity of the orbit becomes small \((0 < e < 0.001)\), the last argument of perigee is taken for \(\omega_0\), e is set to zero, and the solutions developed here apply. When the orbit is initially circular, \(\omega_0 = 0\) because initial conditions for circular orbits are taken in the equatorial plane (Section 3.7).

Substituting (3.89) into (6.1) and making use of (6.2a) and (6.2b) yields

\[
\dot{K}_D = 2B \rho \cos U 
\]

\[
\dot{I}_D = \frac{-B_0 A}{K^{5/2}} \sin I \cos^2 U 
\]

\[
\dot{\Omega}_D = \frac{-B_0 A}{K^{5/2}} \sin U \cos U 
\]

\[
\dot{\eta}_D = \dot{\Omega}_D \cos I 
\]

Here, \(K_g, I_g, \dot{\Omega}_g, \dot{\eta}_g\) are not considered because the results are the same whether \(e = 0\) or not. Then \(\Delta I_g, 2\pi, \Delta \Omega_g, 2\pi\) and \(\Delta n_g, 2\pi\) are as given in Eqs. (5.4d), (5.4e), (5.4f) and \(\Delta K_g, 2\pi = 0\).

In the present case, \(\tilde{\varphi}\) need not be expressed in terms of \(\theta\) but in terms of \(U\). Noting that \(\tilde{\varphi} = \tilde{\varphi}_p = \tilde{K}\), we get,

\[
\tilde{\varphi} = [R_D e_p \frac{c \cos 2 \omega}{\rho} \frac{5}{(K-\xi)} [1 + \frac{c}{2} \cos 2 U + \frac{c^2}{2} \cos^2 2 U][1 + F_p \cos \phi] 
\]

\[
\cos \phi = A_c \cos U + B_c \sin U 
\]

\[
A_c = \cos^2(\epsilon/2) \cos(\Omega - \lambda - L) + \sin^2(\epsilon/2) \cos(\Omega + \lambda + L) 
\]

\[
B_c = -[\cos^2(\epsilon/2) \sin(\Omega - \lambda - L) + \sin^2(\epsilon/2) \sin(\Omega + \lambda + L)] \cos I + \sin I \sin \epsilon \sin L 
\]

Expanding, one gets

\[
\tilde{\varphi} = \varphi \left[ \sum_{n=1}^{5} \left( \pi^c_n \cos^n U + \epsilon^c_n \sin U \cos^{n-1} U \right) \right] 
\]

where
\[ \pi_0 = \delta_0 \]
\[ \pi_1 = A_c p \delta_0 \]
\[ \pi_2 = \delta_1 \]
\[ \pi_3 = A_c p \delta_1 \]
\[ \pi_4 = \delta_2 \]
\[ \pi_5 = A_c p \delta_2 \]
\[ \xi_1 = B_c p \delta_0 \]
\[ \xi_2 = 0 \]
\[ \xi_3 = B_c p \delta_1 \]
\[ \xi_4 = 0 \]
\[ \xi_5 = B_c p \delta_2 \]
\[ \delta_0 = 1 - c + c^2/2 \]
\[ \delta_1 = 2c - 2c^2 \]
\[ \delta_2 = 2c^2 \]

and
\[ \varphi = R_p D e \]  
\[ \frac{-c \cos 2\omega}{p p} \exp[s_p(K - \xi)] \]

and \( c \) is given in (3.109). When the oblateness correction is neglected, i.e. \( c = 0 \), we have:
\[ \bar{\rho} = \varphi'[1 + A_c p \cos U + B_c p \sin U] \]

where
\[ \varphi' = R_p D e \]  
\[ \frac{s_p(K - \xi)}{p p} \]

We now integrate (6.3a) over \([U = \omega_0, U = \omega_0 + 2\pi]\) to get:
\[ \Delta K_{2\pi} = 4\pi BDY_0^{c} \]
where
\[ \gamma_0^c = \pi_0^c + \frac{\pi_2^c}{2} + \frac{3\pi_4^c}{8} = 1 + \frac{c^2}{4} \]  
(6.13)

However, the effect of the bulge vanishes when one integrates over \( 2\pi \).
Furthermore, the effect of drag on a circular orbit is great and the
variation in \( \overline{K} \) is great over one revolution. For these two reasons, it
is more appropriate to find the changes in \( \overline{K} \) every half-orbit. If \( \Delta_1 \overline{K} \)
represents the change in \( \overline{K} \) during the first half and \( \Delta_2 \overline{K} \) during the second
half, one gets,
\[ \Delta_1 \overline{K} = 2BD[\gamma_0^c \pi - 2\gamma_1^c \sin \omega_0 - 2\gamma_3^c \sin 3\omega_0 - 2\gamma_5^c \sin 5\omega_0 \]
\[ + 2\xi_1^c \cos \omega_0 + \frac{\xi_3^c}{3} \cos 3\omega_0 + \frac{2\xi_5^c}{5} \cos 5\omega_0] \]  
(6.14a)
\[ \Delta_2 \overline{K} = 2BD[\gamma_0^c \pi + 2\gamma_1^c \sin \omega_0 + 2\gamma_3^c \sin 3\omega_0 + 2\gamma_5^c \sin 5\omega_0 \]
\[ - 2\xi_1^c \cos \omega_0 - \frac{\xi_3^c}{3} \cos 3\omega_0 - \frac{2\xi_5^c}{5} \cos 5\omega_0] \]  
(6.14b)

where
\[ \gamma_0^c = \pi_0^c + \frac{\pi_2^c}{2} + \frac{3\pi_4^c}{8} \]
\[ \gamma_1^c = \pi_1^c + \frac{3\pi_3^c}{4} + 5\pi_5^c/8 \]
\[ \gamma_2^c = \frac{\pi_2^c}{4} + \frac{\pi_4^c}{4} \]
\[ \gamma_3^c = \frac{\pi_3^c}{12} + \frac{5\pi_5^c}{48} \]
\[ \gamma_4^c = \frac{\pi_4^c}{32} \]
\[ \gamma_5^c = \frac{\pi_5^c}{80} \]  
(6.15)

Obviously, adding (6.14a) to (6.14b) without updating would give exactly
(6.12). In practice, \( \overline{K} \) and \( I \) are updated after the first half-orbit;
hence the values of \( D \) and of the coefficients \( \gamma_i^c \) and \( \xi_i^c \) change because of
their dependence on \( I \) and \( \overline{K} \). Consequently, the sum of (6.14a) and (6.14b)
will slightly differ from (6.12).

Near the poles, the flattening effects are greater, but the
atmospheric rotation is very slow so that the flattening factor has little
effect on the changes in \( I \) and \( \omega \) due to atmospheric rotation. Therefore,
for simplicity, Eq. (6.10) (with \( c = 0 \)) will be used for \( \overline{P} \). Then, solving
for (6.3b) and (6.3c) yields:
\[ \Delta_1 I = \frac{-B \omega A}{K^{5/2}} \sin I D' \left[ \frac{\pi}{2} - \frac{3}{2} A_c F_p \sin \omega_0 - \frac{1}{6} A_c F_p \sin 3\omega_0 + \frac{2}{3} B c F_p \cos^3 \omega_0 \right] \] (6.16a)

\[ \Delta_2 I = \frac{-B \omega A}{K^{5/2}} \sin I D' \left[ \frac{\pi}{2} + \frac{3}{2} A_c F_p \sin \omega_0 + \frac{1}{6} A_c F_p \sin 3\omega_0 - \frac{2}{3} B c F_p \cos^3 \omega_0 \right] \] (6.16b)

and

\[ \Delta_1 \Omega = \frac{-B \omega A}{K^{5/2}} D' \left[ \frac{-B c F_p}{2} \sin \omega_0 + \frac{B c F_p}{6} \sin 3\omega_0 + \frac{2}{3} A_c F_p \cos^3 \omega_0 \right] \] (6.17a)

\[ \Delta_2 \Omega = \frac{-B \omega A}{K^{5/2}} D' \left[ \frac{B c F_p}{2} \sin \omega_0 - \frac{B c F_p}{6} \sin 3\omega_0 - \frac{2}{3} A_c F_p \cos^3 \omega_0 \right] \] (6.17b)

Finally, we have from (6.3d),

\[ \Delta n_D = \Delta \Omega D \cos I \] (6.18)

and the complete solutions when \( e = 0 \) are,

\[ \Delta \bar{K}_{2\pi} = \Delta_1 \bar{K} + \Delta_2 \bar{K} \] (6.19a)

\[ \Delta I_{2\pi} = \Delta_1 I + \Delta_2 I \] (6.19b)

\[ \Delta \Omega_{2\pi} = \Delta_1 \Omega - \left( \pi \bar{K}^2 J \cos I \right)_1 + \Delta_2 \Omega - \left( \pi \bar{K}^2 J \cos I \right)_2 \] (6.19c)

\[ \Delta n_{2\pi} = \left[ \Delta_1 \Omega \cos I - \left( \pi \bar{K}^2 J \cos^2 I \right)_1 \right] + \left[ \Delta_2 \Omega \cos I - \left( \pi \bar{K}^2 J \cos^2 I \right)_2 \right] \]

\[ = \Delta_1 n + \Delta_2 n \] (6.19d)

where subscript "1" denotes the variation after the first half-orbit and subscript "2" the change after the second half, with updated elements after the first half. Thus,

\[ \theta E(\pi) = E(0) + \Delta_1 E(0) \]

\[ 2\theta E(2\pi) = E(\pi) + \Delta_2 E(\pi) \]

\[ = E(0) + \Delta_1 E(0) + \Delta_2 E(\pi) \]

\[ = E(0) + \Delta E_{2\pi} \]

for any element \( E \).

Noting that the integration is carried out with respect to \( U \), i.e. referenced to the line of nodes, the orbital period is
where subscripts 1 and 2 have the meaning given above.

7. EQUATORIAL ORBITS

When the orbit inclination is zero, the solutions so far described present some difficulties because the line of nodes no longer exists, and \( n, \Omega \) and \( \omega \) are not defined. Some modifications are in order. The following discussion will consider elliptical and circular orbits separately.

7.1 Elliptical Orbits \((e \neq 0)\)

From (3.57c), we have \( f_h = 0 \) when \( I = 0 \). Therefore, from (3.67d), (3.67e) and (3.67f), we have

\[
\dot{I} = 0 \quad (7.1)
\]

\[
\dot{\Omega} = \text{undefined} \quad (7.2)
\]

\[
\dot{n} = \text{undefined} \quad (7.3)
\]

and since

\[
\dot{U} = 1 - \dot{n} \quad (7.4)
\]

\( \dot{U} \) is also undefined.

From (7.1), it is clear that when the orbit becomes equatorial, it remains so. The fact that \( \dot{\Omega} \) and \( \dot{n} \) are undefined is not of great importance since \( I \) will remain zero and no future line of nodes will have to be defined. However, while the oblate mass perturbations all vanish from (3.37), except for \( f_{gr} \) (which is well defined) the atmospheric drag perturbations are not completely defined. From (7.3) and (3.63),

\[
\omega = \psi - n \quad (7.5)
\]

it is clear that \( \omega \) is not defined and consequently, from (3.96) and (3.116), \( q_0 \) and the \( p_i \) are not defined. However, this does not affect the density model, as these coefficients are all multiplied by \( c \) which is zero when \( I = 0 \) (from (3.109)). The fact that \( \rho \) is not completely defined does not come from \( q_0 \) or the \( p_i \) but from \( A \) and \( B \) in (3.106). When \( I = 0 \), the bulge centre cannot be situated with respect to the satellite because the perigee point cannot be
located. If diurnal effects were neglected (F = 0), the solutions given so far for K, e, and $\psi$ would still be valid and substantially simplified. However, for better accuracy, a method will be devised to define the perigee position at any time. The diurnal effect will then be kept in the formulation.

Going back to the beginning of Section 3.2, it is clear that an equatorial orbit represents a single Euler rotation through an angle $\alpha$ where,

$$R_{OI} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (7.6)

and

$$[\omega_r, \omega_U, \omega_h] = [0, 0, \dot{\alpha}]$$  \hspace{1cm} (7.7)

where $\alpha$ is the right ascension of the orbit. Equations (7.6) and (7.7) can be directly obtained from (3.4) and (3.5) if one sets

$$I = 0$$ \hspace{1cm} (7.8)

$$\dot{I} = 0$$ \hspace{1cm} (7.9)

$$\alpha = \Omega + U$$ \hspace{1cm} (7.10)

Equation (7.10) can be geometrically verified from Figure 3.1 when $I = 0$. From these considerations, the equations of motion (3.18) become

$$\dot{\hat{r}} - r \hat{\omega}_h^2 = f_r$$  \hspace{1cm} (7.11a)

$$2\hat{r}\omega_h + \hat{r}\dot{\omega}_h = f_U$$  \hspace{1cm} (7.11b)

$$0 = f_h$$  \hspace{1cm} (7.11c)

and

$$\dot{v} = [\dot{r} \ r \omega_h \ 0]^T$$  \hspace{1cm} (7.12)

where $(\dot{\cdot}) = d/dt$,

$$\dot{h} = [0 \ 0 \ r^2 \omega_h]^T$$ \hspace{1cm} (7.13)

$$h = r^2 \omega_h = r^2 \dot{\alpha} = r^2 \alpha/dt$$  \hspace{1cm} (7.14)

From (3.50) and (7.14), the angular displacement from the initial reference line (inertially fixed) to the current position of the satellite,
\( \dot{\theta}_h \), still satisfies the relation, \( \dot{\theta}_h = h/r^2 \) and here,

\[
\frac{d\dot{\theta}_h}{dt} = \frac{h}{r^2} = \frac{d\alpha}{dt}
\]  

(7.15)

From this we get,

\[ \dot{\theta}_h = \dot{\alpha} \]  

(7.16)

or

\[ \theta_h = \alpha + \zeta_0 \]  

(7.17)

where \( \zeta_0 \) is some arbitrary constant. Equation (7.16) is not surprising since \( \alpha \) and \( \dot{\theta}_h \) are both measured in the equatorial plane and both referenced to an inertially fixed line (\( \alpha \) to \( \xi \), \( \dot{\theta}_h \) to the i.r.l.). Consequently,

\[ \zeta_0 = 0 \]

even in the perturbed case.

From (7.17), at perigee we have

\[ \psi = \alpha_p + \zeta_0 \]  

(7.18)

and subtracting (7.18) from (7.17) yields a new equation for the true anomaly:

\[ \theta = \theta_h - \psi = \alpha - \alpha_p \]  

(7.19)

Therefore the unperturbed solution can be rewritten

\[
\begin{align*}
u &= K_0 (1 + e \cos \theta) \\
\dot{u} &= -K_0 e \sin \theta \\
K &= K_0 \\
I &= 0 \\
\theta &= \alpha - \alpha_p = \theta_h - \psi
\end{align*}
\]  

(7.20a)  

(7.20b)  

(7.20c)  

(7.20d)  

(7.20e)

where \( ('') = d/d\theta_h = d/d\alpha \)

The perturbed solution has the same form except that \( K, e \) and \( \alpha_p \) are now functions of the independent variable \( \theta_h \) (or \( \alpha \)). For \( K \) and \( e \), these
dependences on \( \theta \) are unchanged; for \( \alpha_p \), it is easily obtained from equation (7.18):

\[ \psi = \dot{\alpha}_p \]

Therefore, the variational equation for \( \alpha_p \) is exactly the same as that for \( \psi \). This is quite normal since both angles depict inertial angular displacement of the perigee. Consequently,

\[ \Delta \psi = \Delta \alpha_p \] (7.21)

The initial value for \( \alpha_p \) is easily obtained from (3.10b) evaluated at perigee. At \( I = 0 \),

\[ \cos \omega = \cos(\alpha_p - \Omega) \]

\[ \alpha_p = \omega + \Omega \] (7.22)

Now suppose that after the \( n \)th revolution, \( I \) becomes zero. Then \( \omega \) and \( \Omega \) are updated from the \( (n-1) \)st position using \( \Delta \omega \) and \( \Delta \Omega \). Then, \( \alpha_p \) is evaluated according to (7.22) and the equations for \( n \), \( \Omega \) and \( \omega \) may be forgotten. The position of the perigee is now given by \( \alpha_p \).

Now that the orientation of the orbit in the equatorial plane is determined, the bulge effect can be considered. From (3.12), with \( \delta = 0 \), we have:

\[ Q_r = \cos \alpha I_x + \sin \alpha I_y \] (7.23)

where \( \alpha = \theta + \alpha_p \) from (7.19). The bulge centre position is still given by \( b \) in (3.104), so using equation (3.103), (7.19), (7.23) and (3.104), we obtain

\[ \cos \phi = A_e \cos \theta + B_e \sin \theta \] (7.24)

where

---

1. Note that \( \alpha_p \) can be evaluated when \( I \) is very small, say \( I = 0.005 \). In this case, \( \tan(\alpha_p - \Omega) = \tan \omega \cos I = \tan \omega (1 - 3.8 \times 10^{-9}) \) and setting \( \alpha_p = \Omega + \omega \) and \( I \) to zero contributes a negligible error.
A_e = \left[ \cos^2 \frac{e}{2} \cos(\alpha_p - \lambda - L) + \sin^2 \frac{e}{2} \cos(\alpha_p - \lambda + L) \right] \quad (7.25a)\\
B_e = -[\cos^2 \frac{e}{2} \sin(\alpha_p - \lambda - L) + \sin^2 \frac{e}{2} \sin(\alpha_p - \lambda + L)] \quad (7.25b)

It is interesting to note that if we compare Eqs. (7.25) and (3.106), they are pair-wise similar if we set,

I = 0 \\
\alpha_p = \omega + \Omega \\

This conclusion is also obtained in Eqs. (7.10) and (7.22).

Finally, the complete solution when I = 0 and e \neq 0 is,

\Delta K_{2\pi} = \Delta K_{D,2\pi} \\
\Delta e_{2\pi} = \Delta e_{D,2\pi} \\
\Delta \alpha_{p,2\pi} = \Delta \psi_{D,2\pi} + 2 C^2 K^2 J_\pi \\
\Delta I_{2\pi} = 0 \\
T = \left( \frac{2\pi + \Delta \alpha_{p,2\pi}}{2\pi} \right) T_0 

where \Delta K_{D,2\pi}, \Delta e_{D,2\pi} and \Delta \psi_{D,2\pi} are given in (4.8), (4.12) and (4.14), respectively, with I = 0 and the following changes in the density coefficients:
A becomes A_e (7.25a) and B becomes B_e (7.25b) and all \delta_i are zero except \delta_0 = 1.

This follows from equations (3.115) and (3.109) with I = 0, so that the equations for the \bar{g}_i and \bar{h}_i are simplified as,

\bar{g}_i = a_i \\
\bar{h}_i = b_i \\

Finally, the initial value for \alpha_p is

\alpha_p = \omega + \Omega \\

where \omega and \Omega are evaluated when I becomes zero. For practical purposes, this can be evaluated when I is small, say I = 0.005, with negligible error.
7.2 Circular Orbits \((e = 0)\)

In this case, the following relations are true:

\[
\begin{align*}
\omega_h &= \dot{\alpha} \quad (7.28a) \\
\theta_h &= \dot{\alpha} \quad (7.28b) \\
I &= e = 0 \quad (7.28c) \\
h, \Omega, \omega &= \text{undefined} \quad (7.28d)
\end{align*}
\]

The unperturbed solution becomes:

\[
\begin{align*}
\ddot{u} &= \kappa \quad (7.29a) \\
\dot{u} &= 0 \quad (7.29b) \\
\kappa &= \kappa_0 \quad (7.29c) \\
I &= 0 \quad (7.29d)
\end{align*}
\]

where \(\dot{\alpha}_0\) is some arbitrary constant. The perturbed solution is similar except that \(\kappa_0\) is now a function of the independent variable \(\theta_h\) or, in the present case, of \(\alpha\) because of \((7.28b)\). The development is thus similar to that given in Section 6 except that now, the angular displacement of the satellite is referred to \(I_x\) instead of to the line of nodes, \(N\). The variational equations are, from \((6.3)\):

\[
\begin{align*}
\ddot{\kappa} &= 2\bar{B} \rho \quad (7.30a) \\
\dot{I} &= 0 \quad (7.30b)
\end{align*}
\]

since \(\ddot{\kappa}_h = 0\).

Note that

\[
(\dot{\cdot}) \equiv \frac{d}{d\theta_h} = \frac{d}{d\alpha}
\]

The variational equations \((7.30)\) can now be integrated with respect to \(\alpha\). However, the limits of the integration must first be determined. Two possible situations arise.

(i) \(e = 0\) before \(I = 0\)

From Section 6, when \(e = 0\), the integration is carried out from \(\omega_0\) to \(\omega_0 + 2\pi\), \(\omega_0\) being the angle from the line of nodes to the lower limit of integration. Consequently, from Section 7(a),

\[
\alpha_0 = \omega_0 + \Omega
\]

when \(I\) tends to zero, where \(\alpha_0\) represents the required lower limit. The angle \(\Omega\) is obtained from the last increment \(\Delta\Omega_{2\pi}\) before \(I = 0\).

(ii) \(I = 0\) before \(e = 0\)

From Section 7(a), when \(I = 0\), the integration is carried out from
perigee to perigee (with respect to \( \theta = \alpha - \alpha_p \)). Therefore, as \( e \) approaches zero, the last integration (before one sets \( e \) to zero) finishes at the perigee, located at an angle \( \alpha_p \) from \( \Gamma_x \). Consequently, the original displacement \( \alpha_0 \) from \( \Gamma_x \) will be given by the last \( \alpha_p \):

\[
\alpha_0 = \alpha_p
\]  

(7.33)

It is now shown that performing the integration on \( \mathbf{K} \) (7.30a) will give the same result as obtained earlier in Eqs. (6.14). From (6.4), the density is given by:

\[
\overline{\rho} = \left[ \mathbf{P} \mathbf{D} + P\mathbf{e} \right] \left[ 1 + F_p \cos \phi \right]
\]  

(7.34)

From Eqs. (7.23), (3.104), (3.103) and (3.11), one gets,

\[
\cos \phi = A_{ec} \cos \alpha + B_{ec} \sin \alpha
\]  

(7.35)

\[
A_{ec} = \cos^2(\varepsilon/2) \cos(-L - \lambda) + \sin^2(\varepsilon/2) \cos(L - \lambda)
\]  

(7.36a)

\[
B_{ec} = -[\cos^2(\varepsilon/2) \sin(-L - \lambda) + \sin^2(\varepsilon/2) \sin(L - \lambda)]
\]  

(7.36b)

Therefore, (7.32) is rewritten as,

\[
\overline{\rho} = \mathbf{D}' \left[ 1 + F_p A_{ec} \cos \alpha + F_p B_{ec} \sin \alpha \right]
\]  

(7.37)

where \( \mathbf{D}' \) is given by (6.11). Consider now \( \alpha \) to be a dummy variable to be integrated from \( \alpha_0 \) to \( \alpha_0 + 2\pi \). Then, \( \overline{\rho} \) has the same form as (6.6) with \( I = 0 \) (\( c = 0 \)), \( A_c = A_{ec} \) and \( B_c = B_{ec} \). Therefore, Eqs. (6.14) (with \( I = 0 \)) give the required solution to (7.30a):

\[
\alpha_1 K = 2B \mathbf{D}' \left[ \pi - 2A_{ec} F_p \sin \alpha_0 + 2B_{ec} F_p \cos \alpha_0 \right]
\]  

(7.38a)

\[
\alpha_2 K = 2B \mathbf{D}' \left[ \pi + 2A_{ec} F_p \sin \alpha_0 - 2B_{ec} F_p \cos \alpha_0 \right]
\]  

(7.38b)

where \( \alpha_0 \) is given by (7.32) or (7.33). Since the integration is carried out with an inertial angle, the period becomes

\[
T = \left( T_{01} + T_{02} \right)/2
\]  

(7.39)

where \( T_0 \) is successively evaluated at \( \pi \) and \( 2\pi \). Note that from equations (6.5) and (7.36), we have,
This makes sense because in Section 6, the bulge-spacecraft angle was expressed in terms of its displacement from the line of nodes while it is here referenced to \( I_x \), as if \( \Omega \) became zero in the first case. Similarly, when we go from \( I = 0 \) to \( I = e = 0 \), the reference goes from the perigee \((\alpha_p)\) to \( I_x \) so that

\begin{align}
(A_c)_{\Omega=0} &= A_{ec} \\
I=0
\end{align}

\begin{align}
(B_c)_{\Omega=0} &= B_{ec} \\
I=0
\end{align}

It is easy to see that Eqs. (7.38) are the same as (6.14) with \( c \) and \( \Omega \) set to zero. It can also be shown that (7.27a) reduces to

\[
\bar{K}_{2\pi} = 4B_{\pi\pi}D_p \bar{K} = 4B_{\pi\pi}D'
\]

when \( e = 0 \), which is exactly \( \Delta_1K + \Delta_2K \) when \( D' \), \( A_{ec} \) and \( B_{ec} \) are not updated at \( \pi \) in (7.38).

In conclusion, the equations to be used are as follows:

(i) For \( I \neq 0, e \neq 0 \), use (5.17) and (5.18).

(ii) For \( I \neq 0, e = 0 \), use (6.19) and (6.20), with \( \omega_0 \) given by the last perigee position \( (e_0 \neq 0) \) or zero \( (e_0 = 0) \).

(iii) For \( I = 0, e \neq 0 \), use (7.27), with initial \( \alpha_p \) given by \( \alpha_p = \omega + \Omega \) when the initial inclination is nonzero.

(iv) For \( e = 0 \), then \( I = 0 \), use (6.19) and (6.20) which naturally reduce to equations (7.38) and (7.39) when we set \( c = 0 \) and \( \Omega = 0 \) in \( A_c \) and \( B_c \) and change \( \omega_0 \) by \( \alpha_0 = \omega_0 + \Omega \) (7.32).

(v) For \( I = 0 \), then \( e = 0 \), use (7.27), which reduce to (the sum of) (7.38a) and (7.38b) with \( \alpha_0 \) replaced by

\[
\alpha_p = (\alpha_p)_{I=0} + \sum B_{2\pi} = (\omega + \Omega)_{I=0} + \sum B_{2\pi}
\]
where \((\omega + \Omega)_{1=0}^{p}\) denotes the value of \(\alpha\) just before the orbit becomes equatorial. This is used as an initial value to which are added the extra terms \(\Delta\psi_{2\pi}\), for the (now equatorial) orbits as \(e \to 0\).

8. DISCUSSION

Closed-form expressions have been presented for the per-orbit change in the orbital elements of a near-Earth satellite under the combined effects of Earth's oblateness, atmosphere flattening and rotation, and diurnal, 11-year cyclic, semi-annual, and random variations in density. The random uncertainty in \(C_{DA}\) was also treated in the formulation. These expressions can then be efficiently integrated on a computer as difference equations with the orbital period \(\tau\) as the unit time giving accurate and low-cost lifetime predictions and mean trajectory estimation.

Due to the static density model used, this theory is accurate when the perigee height lies between 150 and 500 km. When it falls below 150 km, re-entry is imminent and the sum of the computed orbital periods gives a good estimate of the lifetime. Note that when the eccentricity is small \((e < 0.001)\) the equations for circular orbits can be used, avoiding large oscillations in \(\psi\) (proportional to \(1/e\)). Similarly, when \(I\) is small \((< 0.005)\), the equations for equatorial orbits can be used.

In this section, some general comments about the perturbation models and the method used will be given followed by a list of the contributions of this thesis. Some suggestions for future work will close this discussion.

8.1 Perturbation Models

The gravitational perturbation model used in this work includes the dominant effect i.e. the oblateness of the Earth \((J_2)\). Although it is the most important factor, the higher harmonics \((J_n, n \geq 3)\) can induce additional dispersion to the trajectory described by the solutions given here. Therefore, this must be kept in mind in interpreting the results. Note that higher harmonics could be included by simply adding the relevant terms to the existing solutions.

1. In the case of circular orbits, the smallest time interval is \(\tau/2\).
The static part of the density model is quite accurate. It is based on a theoretical foundation, and an empirical fit to the United States Standard Atmosphere 1976 results in small discrepancies, considering the large unpredictable density variations that may occur at any time. The altitude dependence of the scale-height was considered and a scale-height correction factor introduced to provide better accuracy to the perigee-approximated expressions. Flattening effects were also investigated.

A simple dynamic density model was derived from the available data and formulas. It relies on empirical considerations alone, although some formulas are based on physical evidence (e.g. sinusoidal density variations with the solar cycle and the Sun's position). The main limitations of this model are the inaccuracy in the relation between the solar flux $F$ and the time, the assumption that the bulge factor is independent of the 11-year cycle, and the approximation of random density fluctuations by straight lines of variable length and slope.

Finally, the uncertainty in $C_D$ and $A$ due to the tumbling of an uncontrolled satellite was considered in terms of an interval within which the product $C_D A$ has a uniformly distributed probability with a mean given by $2.2A_{AV}$ where $A_{AV}$ is given by King-Hele's averaging technique. The influence of this random distribution, as well as that of the random density variations, can be analyzed in a Monte-Carlo simulation.

8.2 Method

The method used is similar to that of King-Hele in [30]. However, since the static density model was slightly different here, the analytical integration over one orbit was carried out in terms of the true anomaly, while King-Hele expressed his equations in terms of the eccentric anomaly. Some perturbation models were taken from King-Hele (flattening, day-to-night, scale-height variation and correction factor) and new ones were added (bulge factor variations, other dynamic density variations and Earth's oblateness). Since the "slow" motion is to be solved here by computer integration, unlike King-Hele's work, all the dispersion parameters can be combined in a "unified" theory instead of considering their individual effects. Due to its semi-analytical nature, this present work bears more resemblance to that of Santora [45] although a more elaborate density model is used here (in particular, the static part is different and random density variations are included). In other words,
the methods used in the present work are basically the ones used by other theorists, but with modifications and additions incorporated.

8.3 Advantages and Limitations

Due to the possibility of including realistic perturbation models while keeping the computational costs low, semi-analytical methods are very desirable in the theory of lifetime prediction. This is why this approach is taken in the present work. Let us now return to the three criteria expressed in Section 1.2, and assess this work in terms of its complexity (Step 1), its cost of operation (Step 2) and the accuracy of its results (Step 3).

The formulation and implementation of the theory were time consuming but straightforward. As stated above, the cost of operation should be less than for a purely numerical integration of the equations of motion, although more than for a purely analytical approach. In the latter, however, one loses accuracy (simplistic perturbation models). Finally, since no computer integration is included in this work, it is relatively difficult to estimate how accurate the per-orbit expressions are. However, some general inferences can be made. Since the perturbation model for the density variations is relatively sophisticated, one would expect the effects of the dominant perturbation (air drag) to be accurately depicted by the solutions, subject to the limitations given in Section 8.1. With regard to gravitational perturbations, one may expect the theory to be less accurate for very low and very high altitude, and for very eccentric orbits where geopotential harmonics and luni-solar perturbations can introduce important trajectory dispersions. Also, by the very nature of the method used, slight deviations of the satellite about its mean path are not taken into account here. This is not always important but when the geopotential field or the luni-solar attraction become relatively more significant, the density profiles actually experienced by the satellite differ slightly from those assumed in the equations and the resulting discrepancies may add up. In summary, lifetime predictions are probably quite accurate while trajectory estimation may be a little less exact. Obviously, this accuracy decreases as the particular orbit and the specific properties of the satellite depart from the various assumptions made throughout this work.
The three existing problems in the theory of lifetime prediction, depicted in Section 1.2, are shown to be partially solved in this thesis. First, a relatively realistic perturbation model has been used, but the costs are low since they are split up into Steps 1 and 2. Consequently, the "analytical costs" are low since the equations are not integrated for the entire lifetime but for one orbit only, and the "numerical costs" are also low since the smallest step size is one (or one half) orbit. A slight improvement was also made in the atmospheric density model. However, the third problem is still acutely present: the theory lacks a precise model for $F = F(t)$, and it does not consider "how" random the sporadic variations in the density are nor what probability distribution function best expresses them.

Finally, it should be noted that the effects of the perturbations can be easily visualized from the solutions even before the "slow" motion is numerically integrated.

8.4 Contributions

The main contributions of this thesis are:

(i) to provide a relatively accurate and low cost semi-analytical method of predicting the lifetime of near-Earth satellites, combining the advantages of the analytical methods and the efficiency of the numerical methods.

(ii) to provide equations for analyzing the influence of random density and (CDA) variations in a Monte Carlo simulation. The probability of re-entry within a given time span can then be obtained.

(iii) to provide basic equations which can be used to determine the individual or combined effects of the following orbital perturbations (and their associated dispersion parameters):
   - Earth's oblateness ($J_2$)
   - scale-height variation with altitude ($\alpha, m, \beta_f$)
   - atmosphere flattening ($c$)
   - atmosphere rotation ($\omega_A$)
   - 11-year solar cycle density variations ($D_{sc}$)
   - semi-annual density variations ($D_{sa}$)
- diurnal density variations (F)
- altitude dependence of the bulge factor \( \delta_f \)
- random density variations (and 27-day recurrence of high peaks) (R)
- random variations in \( (CDA) \) due to tumbling and random orientation of the spin axis \( (\mu) \).

(iv) to provide a semi-empirical, g-and scale-height-varying static density model accurate to within 17% in the range \([150-500 \text{ km}]\) and 12% in \([500-900 \text{ km}]\).

(v) to provide an empirical formula relating the density variation to that of the exospheric temperature during the course of a solar cycle. It is accurate to within 30% and, below 350 km, to within 18%.

8.5 Suggestions for Future Work

In view of the assumptions made throughout this work, it is expected that the following improvements will increase the reliability of its results:

(i) include luni-solar perturbations and hence extend the range of permissible eccentricity,

(ii) consider the variation of the bulge factor during the solar cycle,

(iii) develop a method of predicting the solar flux at any future time,

(iv) carry out a detailed analysis of the random fluctuations in the density,

(v) include flattening effects in the bulge factor and the scale-u correction factor \( \beta_f \),

(vi) develop more accurate (but tractable) formulas expressing the dynamic behavior of the density.
<table>
<thead>
<tr>
<th>Reference</th>
<th>Authors</th>
<th>Title and Details</th>
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<tbody>
<tr>
<td>No.</td>
<td>Author(s)</td>
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</tbody>
</table>
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| Scott, D. W.  
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| Ching, B. K.  
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| 63. King-Hele, D. G.  
| 66. Jacchia, L. G.  


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<th>No.</th>
<th>Author</th>
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<tr>
<td>87</td>
<td>Hughes, P. C.</td>
<td>&quot;Attitude Dynamics of Spacecraft&quot;</td>
<td>To be published.</td>
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</tbody>
</table>
Appendix A

Orbital Perturbations

Basic orbital mechanics tell us that a point-mass satellite interacting in vacuo with the inverse-square gravitational field of a much more massive primary describes a conic section. For bounded (negative energy) orbits, this path is elliptic (or, in particular, circular) and inertially fixed. This situation is very idealistic since the presence of other physical bodies and fields perturb this trajectory and the masses involved are, in general, asymmetric. When these disturbances are relatively small compared to the main attractive field, the resulting trajectory approximates an ellipse. Periodic and secular variations are super-imposed on the basic Keplerian motion.

These variations can be divided into two groups depending on whether they originate from the environment (environmental perturbations) or whether they are caused by the particular properties of the satellite under consideration (specific perturbations). It is recognized that the latter alters the importance of the former and vice versa. Table A-1 gives the main perturbations in these two groups.

<table>
<thead>
<tr>
<th>TABLE A-1</th>
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<tbody>
<tr>
<td>(a)</td>
</tr>
<tr>
<td>Environmental Perturbations</td>
</tr>
<tr>
<td>i) Solar radiation pressure</td>
</tr>
<tr>
<td>ii) Earth's magnetic field</td>
</tr>
<tr>
<td>iii) Presence of charged particles</td>
</tr>
<tr>
<td>iv) Metoroid Impacts</td>
</tr>
<tr>
<td>v) Presence of an atmosphere</td>
</tr>
<tr>
<td>vi) Luni-solar attractions</td>
</tr>
<tr>
<td>vii) Non-symmetric mass distribution of the Earth</td>
</tr>
<tr>
<td>(b)</td>
</tr>
<tr>
<td>Specific Perturbations</td>
</tr>
<tr>
<td>i) Finite satellite dimensions, non-symmetrical shape and mass distribution</td>
</tr>
<tr>
<td>ii) Large area-to-mass ratio, very large size</td>
</tr>
</tbody>
</table>

1. This discussion follows closely that given in [1].
iii) Changes in mass
iv) Ablation and break-up upon re-entry

These perturbations are now briefly discussed keeping in mind the scope of this thesis. From Table A-1, it is clear that the environmental perturbations are caused by the physical bodies and fields surrounding the satellite. They are usually functions of spatial position.

Solar radiation pressure is a dominant perturbation at high altitudes but for near-Earth satellites, it is negligibly small in view of the importance of drag at these altitudes. Exceptions to this remark are the so-called "balloon-type" satellites (large area-to-mass ratio) for which drag and solar pressure are often of the same order. For the average satellite, it is neglected. Similarly, the influences of the Earth's magnetic field, atmospheric charged particles and small meteoroid impacts are considered negligible compared to air drag.

Atmospheric drag is thus the dominant orbital perturbation regulating the expected lifetime of a satellite. The satellite loses its energy through friction and as a result, its orbit slowly contracts, ending its life upon re-entry. Luni-solar attraction and the asymmetry of the geopotential field are conservative forces and cannot, by themselves, lead to orbit contraction. However, they modulate the effects of air drag and in some cases, they can be determinant factors in lifetime predictions. The details of this modulation are now briefly described.

It is well known that the importance of drag is directly proportional to the density of the medium (air) and the square of the speed of the body relative to the medium:

\[ a_D \sim \rho v_R^2 \]

\( a_D \) = drag deceleration
\( \rho \) = air density
\( v_R \) = velocity of the satellite with respect to air

It so happens that the density, \( \rho \), and the satellite velocity, \( v \), (hence \( v_R \)) are maximized at the same point on the orbit: the perigee. Therefore, the perigee location (altitude and angular distance from some
the critical region where the effects of drag are greatest. Since the luni-solar attractions and the asymmetry of the
geopotential field directly affect the perigee distance and its argument, it is clear that they alter the influence of air drag on the decay of
the orbit.

From this discussion on environmental perturbations (Table A-1), one can see that the theory of lifetime prediction (for the majority of
satellites) can be relatively accurate by including the last three of these
perturbations in its formulation. Note that for low-eccentricity, near-Earth orbits, luni-solar attraction is negligibly small.

Specific perturbations are related to the particular physical
properties of the spacecraft considered. The finite dimensions and the
non-symmetrical shape and mass distribution of a satellite alter the
influence of atmospheric drag, the geopotential field and solar pressure.
Generation of lift, gravity-gradient and solar torques induce attitudinal
motion which causes trajectory dispersion due to the altitude/orbital
coupling exerted by the aerodynamic force. In particular, large area-
to-mass satellites and large extent spacecraft are very subject to these
perturbations. Under these circumstances, not only the orbital motion but
also the attitude motion of the satellite must be considered in the formulation
of the theory. But, as will be obvious later, this refinement is rarely
worth the complications it brings. Finally, a variable mass modifies the
importance of the drag deceleration,

\[ a_D \sim \frac{1}{m} \]

and lifetime estimates are affected accordingly. In particular, oblation
and breakup, which usually occur upon re-entry, involve a large variation
in several parameters \((C_D, m, A)\).

The consideration of specific perturbations brings about two major
drawbacks: the theory must become particular by nature, and more complex due
to the consideration of the attitude motion. However, by restricting the
range of applicability of the theory, most of these perturbations can be
neglected. This thesis, like most of the existing theories, thus considers
"compact" satellites (small area-to-mass), of small extent, undergoing
orbital decay only (Stage A). Therefore solar forces and gravity-gradients
are excluded as well as oblation and breakup, which are assumed to take
place in Stage B only (re-entry). It is also tacitly assumed that no change in the total mass of the satellite is occurring. This simplifies the problem greatly but the finite and non-symmetrical shape of the body can still cause important trajectory dispersions, especially if aerodynamic lift is present. In this case, the assumption of uncontrollability, which is discussed in some detail in Appendix C, makes this last specific perturbation negligible as well.

Consequently, the theory can thus be derived in a more general manner with no consideration of the attitude motion of the satellite. Actually, in view of the large uncertainties in atmospheric density, this last consideration is probably unjustified anyway.

On a more detailed basis, the classes of orbital perturbations can be broken down into the specific parameters that must be realistically modelled. The "dispersion parameters" can be defined [1] as "any parameter, variable or constant, which affects the dynamic behaviour of a satellite, its trajectory and lifetime". Table A-2 gives the important dispersion parameters relevant to our problem.

**TABLE A-2**

<table>
<thead>
<tr>
<th>Dispersion Parameters</th>
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<tbody>
<tr>
<td>i) Relative to the atmospheric drag</td>
</tr>
<tr>
<td>- altitude dependence of the density, scale-height variations</td>
</tr>
<tr>
<td>- atmosphere flattening</td>
</tr>
<tr>
<td>- diurnal bulge</td>
</tr>
<tr>
<td>- 11-year solar cycle and semi-annual density variations</td>
</tr>
<tr>
<td>- sporadic density fluctuations</td>
</tr>
<tr>
<td>- atmospheric rotation</td>
</tr>
<tr>
<td>ii) Relative to the potential field</td>
</tr>
<tr>
<td>- asymmetry of the Earth's mass</td>
</tr>
<tr>
<td>- luni-solar attraction</td>
</tr>
<tr>
<td>iii) Relative to the satellite</td>
</tr>
<tr>
<td>- variations in the projected area, A</td>
</tr>
<tr>
<td>- uncertainty in the value of $C_D$ and its variation with the attitude motion</td>
</tr>
<tr>
<td>- generation of lift</td>
</tr>
</tbody>
</table>
Table A-2 excludes the perturbations given in Table A-1 that were judged negligible in the theory of satellites in an atmosphere.

Dispersion parameters related to atmospheric drag are briefly discussed in Appendix B while Appendix C treats dispersion parameters related to satellite properties. Models for gravitational perturbations are usually straightforward and taken directly from the literature.
Appendix B

Model Atmospheres

The choice of a model atmosphere that accurately describes the present and future variations of atmospheric properties (especially density) as a function of position and time is a critical stage of any theory. The basic techniques available to obtain these models are here briefly reviewed.

First, empirical techniques use the method of curve fitting. It closely follows the following lines.

(i) obtain numerical values of the density as a function of position and time from satellite observations and/or rocket explorations.
(ii) using a mathematical expression with undetermined (matching) coefficients, obtain numerical values for these coefficients to get the best fit to the data.

This technique is particularly welcome when the available density formulas are not suitable to analytical integration. These formulas, in closed form or differential form, are obtained from the second technique, the theoretical method.

Theoretical techniques rely on the closed-form or numerical solution of the equations of state of the atmospheric gas. The steps are:

(i) obtain the state equations relating the properties of the atmosphere
(ii) solve these equations
(iii) obtain the density as a function of the other variables in the case of closed-form solutions, and numerical tables for the computational solutions.

The closed-form solutions resulting from the theoretical method are usually inaccurate for some range of the independent variables and the numerical solutions are not desirable in an analytical or semi-analytical approach to lifetime predictions.

From these remarks, it can be seen that while empirical techniques are not based on theoretical grounds, the theoretical techniques are usually intractable or too inaccurate in an analytical solution of our problem. However, a combination of the two techniques seems to be adequate.

1. More detailed analysis can be found in [85] and [1].
Semi-empirical techniques follow the pattern,

(i) obtain an approximate closed-form expression of the density

\[ \rho = \rho(t, a_1, a_2, \ldots, a_n) \]

from theoretical methods where the \( a_i \)'s are coefficients obtained from the derivation.

(ii) use this mathematical expression in an empirical curve fitting of density data obtained from observation. The \( n \) coefficients \( a_i \) are then used as matching coefficients and a minimum of \( n \) fitting points (where the model is exactly accurate) result. Some of the matching coefficients could possibly be expressed as a function of altitude, for better fit.

Many of the available standard atmospheres combine theoretical considerations and observational data and are thus semi-empirical (for instance, the USA 1976). In this thesis also, a semi-empirical method will be used and the source of the density data will be the United States Standard Atmosphere 1976 (for the static part of the model).

Atmospheric density models are very often grouped into the classes: static and dynamic. A static density model considers the variation of density with altitude only. It gives a mean density, averaged over all the time-dependent density fluctuations. The USSR 1976 is a static model. A dynamic model is basically a static model to which are added the time-varying density fluctuations induced by the Sun and the Earth's magnetic field.

Most static models assume that atmosphere is (1) a perfect gas and (2) a homogeneously mixed under hydrostatic equilibrium (i.e. the equilibrium between gravitational and pressure forces). An elementary derivation (see [1]) shows that the hydrostatic equilibrium equation takes the form:

\[ \frac{d\rho}{\rho} = -\frac{g}{k} \, dh \]  \hspace{1cm} (B-1)

where \( g \) is the local acceleration due to gravity and \( k = P/\rho \) (\( P \) = pressure). The parameter \( k \) relates the pressure of the atmospheric gas (equal to the sum of all the partial pressures of the constituents) and its density. It is given by
where \( T \) is the gas temperature and
\[
M = \sum_{i} M_i x_i \tag{B-3}
\]
where \( M \) is the molecular weight of the air, \( M_i \) and \( x_i \) being the molecular weight and the volumetric fraction of constituent \( i \), respectively. This parameter \( k \) is thus a variable which is related to the speed of sound in the atmosphere, as shown in [1].

It must be noted that the hydrostatic equilibrium equation (B-1) is only valid when assumptions (1) and (2) are valid. It is therefore accurate in altitudes below approximately 85-100 km only because above this point, photo-dissociation and diffusive separation of the constituents render the assumption of homogeneity inapplicable. The molecular weight starts then to decrease with altitude.

However, Eq. (B-1) can be solved and the resulting expression used in a semi-empirical technique where some of the matching coefficients are varied to take into account the non-applicability of the basic assumptions. For instance, taking \( k \) and \( g \) constant in Eq. (B-1) gives the well-known exponential density model:
\[
\rho = \rho_0 \exp\left[\frac{(r_0 - r)}{H}\right] \tag{B-4}
\]
where \( H \), the scale height, is given by
\[
H = \frac{k}{g_0} = \frac{RT}{Mg} \tag{B-5}
\]
Since the temperature, the molecular weight and, to a lesser degree, the acceleration due to gravity are variable with altitude, the scale-height can be expressed as a linear function of altitude, as in [30] and fitted to observational data. Some other theoretical static models consider some simple altitude dependence of \( T, M \) and \( g \) so that Eq. (B-1) can still be integrated [1].

Finally, some other models use the multi-fit or layer-technique

1. In this method, the atmosphere is divided into \( n \) layers with respect to the altitude and the mathematical expression is fitted to each interval, resulting in \( n \) sets of matching coefficients, each one relevant to its associated layer.
or a purely empirical expression such as the power-law model,

\[ \rho = \rho_0 \left( \frac{h}{h_0} \right)^n \]  

(B-6)

where \( \rho_0 \) and \( n \) can be used as matching coefficients.

Static models are usually expressed as a function of altitude while the orbital theory gives the satellite position as a function of its radial distance from the Earth's centre. Therefore a change of variable from \( h \) to \( r \) is required for the static models. Obviously, the relation is,

\[ h = r - r_E \]  

(B-7)

where \( r_E \) is the Earth's radius. It is recognized that \( r_E \) varies with longitude and latitude but, for simplicity, many models take \( r_E \) to be a constant, equal to its mean value at the equator \( (r_{Ee}) \). The resulting models are thus termed "spherical", "symmetrical" or "equatorial" density models since they assume the density to be constant on a spherical surface.

A better approximation considers the main divergence of the Earth from sphericity: its flattening. Then \( r_E \) is approximately given by [30],

\[ r_E = r_{Ee}(1 - f \sin^2 \delta) \]  

(B-8)

where

\[ f = \text{Earth's flattening} \]
\[ \delta = \text{geocentric latitude} \]

Static models that include the change of variable (B-7) with \( r_E \) given by (B-8) are referred to as "flattened" or "oblate" static density models.

In a more general approach, the flattened spheroid of reference need not to be that represented by the Earth's surface and a constant-altitude reference surface could be defined, with negligible error [30], as:

\[ \sigma = \sigma_e(1 - f \sin^2 \delta) \]  

(B-9)

where

\[ \sigma_e = \text{radius of the reference spheroid at the equatorial plane} \]
\[ \delta = \text{geocentric latitude of the spatial point considered} \]

1. This is equivalent to a change of reference frame from the Earth's surface to the Earth's centre.
2. This tacitly assumes the static relation \( \rho = \rho(h) \) to be the same at all latitudes and longitudes.
Equation (B-9) reduces to Eq. (B-8) at the Earth's surface. If the reference surface is required to pass through a given point \((r_r, \delta_r)\), we write,

\[
\sigma_r = \sigma_{er}(1 - f \sin^2 \delta)
\]  

(B-10)

where it is easy to find that

\[
\sigma_{er} = r_r/(1 - f \sin^2 \delta_r)
\]  

(B-11)

The change of variable can then be

\[
h^* = r - \sigma_r
\]  

(B-12)

where \(h^*\) represents the altitude of the satellite with respect to the spheroid of reference, \(\sigma_r\).

The dynamic density variations of the atmosphere are mainly caused by the influence of the Sun. Two agents are usually recognized: solar radiation and the solar wind, and they will now be briefly discussed.

Ultraviolet rays represent the main source of solar energy incident on the atmosphere, in which they are absorbed near the 200 km region. The density of this flux cannot be directly measured from the Earth's surface but it is closely correlated to that of the radio waves 10 cm - 20 cm in wavelength. Since the latter are not absorbed by the atmosphere, they provide a useful index expressing the amount of heat absorbed in the upper atmosphere. The 10.7 cm wavelength is commonly used and the resulting solar activity index is denoted \(F_{10.7}\). An increase in \(F_{10.7}\) is usually correlated to an increase in the density. The solar flux varies somewhat unpredictably with the appearance and disappearance of sunspots, solar flares and other disturbances on the solar disc. These fluctuations are also modulated [30] by the periodic reappearance of the active regions of the Sun corresponding to its 27-day rotation (with respect to the Earth) on its axis. In this random behaviour of the solar activity can be recognized the slowly varying and more predictable average flux, \(F_{10.7}^{\text{av}}\). When \(F_{10.7}\) is averaged over several solar rotations, the resulting mean varies with the 11-year sunspot cycle [49]. Therefore, while \(F_{10.7}\) is related to the sporadic activity of some active regions on the solar disc, \(F_{10.7}^{\text{av}}\) corresponds to the average activity of the solar disc taken as a whole. Tables of \(F_{10.7}^{\text{av}}\) between 1958 and 1976 are given in [49] where \(F_{10.7}^{\text{av}}\) is a weighted mean of \(F_{10.7}\) given by

1. Note that \(\sigma_r - r_E\) varies slightly with \(\delta\) but this variation is negligibly small (\(\approx 1\ km\) at 300 km, [30]).
where
\[
F_{10.7} = \frac{\sum w F_{10.7}}{\sum w}
\]  
(B-13)

and where \( t_0 \) is the time at which we want \( F_{10.7} \) and \( \tau \) is three solar rotations.

There are two possible mechanisms for the solar wind to transmit its energy to the upper atmosphere [30]. First the ionized particles interact with the earth's magnetosphere and the magneto-hydrodynamic waves so produced dissipate their energy in heat, thus increasing the density at a given height. Second, these particles may directly collide with the air particles and hence produce heat as well. Since the solar wind interacts with the Earth's magnetic field and thus perturbs it, the recording of the geomagnetic planetary index, \( A_p \), represents a good measure of the heat released by the solar wind, and consequently of the probable increase in air density.

The main dynamic variations of the density that are of importance in a general theory of lifetime predictions are:

(i) 11-year solar cycle variations
(ii) diurnal bulge
(iii) semi-annual variations
(iv) random variations

The 11-year cycle density variations have already been briefly described: the average flux of the Sun, \( F_{10.7} \), increases the exospheric temperature, \( T_\infty \), of the atmosphere and this is directly related to an increase in density. The problem in modelling this dynamic variation lies in the fact that while \( F_{10.7} \) is easily measurable, it is hardly predictable.

The diurnal bulge is caused by the constant heating of the atmosphere on the sunlit side of the Earth. The importance of this hump varies with altitude and during the course of the 11-year solar cycle since it is related to the activity of the Sun. The bulge centre lags some 25 to 30° behind the sub-solar point due to the atmosphere rotation and it is now recognized that the bulge does not have a definite symmetry about its
centreline. However, a very acceptable first approximation is of the form:

$$\rho = \rho_0 [1 + F_B(h,t)S(\phi)] \quad (B-15)$$

where $\rho_0$ is the density with no diurnal effects considered. In Eq. (B-15), $F_B(h,t)$ represents the "bulge factor" or the altitude- and time-varying amplitude of the bulge, while $S(\phi)$ usually represents a zero-mean periodic function of the satellite-bulge centre angular displacement, $\phi$. In the simple case, Cook and King-Hele [31] give

$$\rho = \rho_0 [1 + F \cos \phi] \quad (B-16)$$

where the bulge factor, $F$, is taken as a constant.

The semi-annual density variations are thought to be more dependent on the orientation of the magnetosphere to the solar wind than to the actual activity of the Sun [64]. Consequently, this effect can be considered uncoupled from the 11-year solar cycle. Empirical formulas giving the semi-annual density variations are investigated by Jacchia in [49].

The last dynamic density variation considered here is induced by sporadic solar eruptions and the appearance of active solar regions on the solar disc. These perturbations, whose magnitude can be appreciated by the recording of $F_{10.7}$ and $A_p$, are random in nature. They can be quite large sometimes, up to a factor of 6 at 600 km altitude [48]. They render any deterministic approach subject to substantial uncertainties when lifetime estimates are considered.

Finally, the atmosphere is rotating at, or slightly above, the Earth's angular rate. This rotation is usually expressed in terms of days, $A$

$$A = \frac{\omega_A}{\omega_E} \quad (B-17)$$

where $\omega_A$ and $\omega_E$ represent the angular rate of the atmosphere and the Earth respectively. Tables of $A$ vs. altitude are given and discussed in [56-61].
Appendix C

Satellite Properties

It is recognized that the importance of environmental perturbations is directly linked to important satellite properties: shape, mass distribution, dimensions and constituent material. We then speak of the specific perturbations associated with the given satellite. Solar, gravity-gradients, and aerodynamic torques govern the attitudinal motion of the satellite which in turn alters the magnitude and the direction of these torques. This variable orientation of the satellite of general shape can lead to significant trajectory dispersion due to the aerodynamic force.

Fortunately, when the satellites undergoing orbital decay are small and relatively dense, many of the specific perturbations become negligible compared to the drag perturbations, as pointed out in Appendix A. However, the satellite still has finite dimensions (not a point mass) and, in general, its shape is not symmetric. Consequently, even if solar and gravity gradient torques are negligibly small, aerodynamic lift is likely to perturb the orbital motion.

The aerodynamic influence on a body moving through a fluid can be resolved into three components,

i) a force parallel to the flow (drag)
ii) a force perpendicular to the flow (lift)
iii) a torque about the centre of mass.

While the direction of the drag is unique, that of the lift force can be anywhere in a plane perpendicular to the relative velocity vector, depending on the geometry and the orientation of the satellite. Consideration of this force is consequently a very complex task but it is nevertheless necessary in trajectory prediction when a particular orientation is preferred as in attitude controlled spacecraft. However, in the case of uncontrolled satellites the aerodynamic torque will normally be destabilizing [30]. The spacecraft will then tumble about its axis of maximum inertia (the only stable mode since atmospheric friction is present). This spin axis is approximately fixed in space, apart from a probable precession due to small
environmental torques, and the spin rate is slowly decaying, mainly because of aerodynamic damping. Under this condition of uncontrolled tumbling, King-Hele [30] showed that for nearly spherical and cylindrical satellites (1/d > 1), the resultant lift acting on them is practically zero. It is slightly more significant for disc-shaped satellites (1/d << 1) but as the lift-to-drag ratio is usually small for satellites (< 0.1), the combination of these two factors renders the effect of lift generation negligible. Therefore, the assumption of uncontrollability is equivalent to assuming a point-mass satellite subject to drag. This assumption is usually realistic since we are concerned with satellites which will eventually re-enter the atmosphere and hence they are, in general, uncontrolled.

Having ruled out a requirement for attitude motion equations, the difficulties introduced by specific perturbations are not altogether solved. The direction of the drag was said to be well defined but its magnitude, directly proportional to the drag coefficient (C_D) and the projected area (A) of the satellite, is not easily obtained. In fact, an uncontrolled satellite of general shape will present different cross-sectional areas to the flow in its tumbling motion and the leading portion of the satellite will be at different angle of attack to the flow so that C_D will also be variable. Therefore, these two parameters must be considered in some detail.

Since the satellite is assumed to be tumbling it is meaningful to compute an averaged value for A over one revolution which would then represent the effective projected area of the satellite. This leads to the averaging technique for evaluating A. This technique is taken from [30]. It basically assumes that, if the spin axis is unknown, it has an equal probability of taking any direction, the two extremes being parallel to, and perpendicular to, the flow. In each case, the average projected area over one revolution of the satellite about its spin axis can be easily computed, giving A_∥ and A_⊥. Then, from the basic assumption, A_∥ and A_⊥ are equally likely over one orbit so that the average or effective projected area of the satellite, A_{AV}, can be obtained as,

---
1. These represent the most common shapes of satellites.
Obviously, if some knowledge of the preferred orientation of the spin axis exists, a weighted mean would replace Eq. (C-1). This method has the advantage of being very general and can be applied to any satellite shape. In particular, King-Hele [30] obtained the following results:

(i) Rod-shaped Satellites \((1/d > 2)\)

\[
A_{AV} = \frac{1}{2} \left( A_{∥} + A_{⊥} \right)
\]  

(C-1)

(ii) Disc-shaped Satellites \((1/d \leq 1/2)\)

\[
A_{AV} = \pi d^2 + \frac{1d}{2}
\]  

(C-2)

For nearly spherical and spherical satellites, no moment of inertia is substantially larger than the others so that any faces of the satellite can be directed towards the flow with equal probability. In this case, a relatively accurate value for \(A_{AV}\) is [33],

\[
A_{AV} = \frac{\text{Total Surface Area}}{4}
\]  

(C-4)

The maximum possible error introduced by \(A_{AV}\) is of the order of 15% in (C-2) and 22% in (C-3) [30]. For the nearly spherical satellites, the error in (C-4) is of the same order as that for cylindrical satellites and decreases as the shape becomes more and more spherical. Therefore, it must be kept in mind that the use of \(A_{AV}\) in the motion equations leads to some degree of inaccuracy in the results, depending on satellite shape. All in all, the use of \(A_{AV}\) represents a relatively accurate and efficient way of avoiding the specific perturbation introduced by the varying projected area.

Proper evaluation of the drag coefficient is even more complex. Many assumptions and suppositions must be resorted to due to the incomplete knowledge of the basic mechanism of the surface interaction between a satellite and a free molecular flow. Many parameters and formulas have been derived and the reader is referred to [77, 78, 79, 84 and 1] for a detailed discussion of them. It is just noted here that Cook [79] gave
a value of $C_D = 2.2 \pm 15\%$ for a satellite in the altitude range [140-400 km] during low solar activity or [140-600 km] during high solar activity. Above these intervals, the drag coefficient increases with altitude since the molecular weight of the atmosphere decreases (leading to smaller molecular speed ratio). Below these intervals, in the transition region, the re-emitted particles collide with the incident particles so that $C_D$ diminishes. Therefore, using a value of $C_D = 2.2$ in the orbital theory may represent a 15% error or more if the satellite is outside the intervals just given. This also must be kept in mind in interpreting the results.
Fig. 2.1 Static Density Model SM1 Compared with the USSA 1976 (Low Attitude).
Fig. 2.2 Percent Error of SM1 (Low Altitude) with respect to the USSA 1976.
Fig. 2.3 Effect of the Perigee Approximation of the Scale-\(u\) for Different Reference Altitudes.
$V_r$ at 150 Km
$S_p = S_p (U_p$ at 150 Km)

Fig. 2.4 Divergence of SM1-s(PA) from SM1.
Fig. 2.5 Models SM3 and SM1-s(PA) when \( u_f < u_r < u_p \) \( (h_p = 150 \text{ km}) \).
Fig. 2.6 Models SM3 and SM1-s(PA) when $u_f < u_r < u_p$ ($h_p = 300$ km).
Fig. 2.7 Model SM3 ($h_p = 200$ km).

- $U_p$ at 200 Km
- $U_r$ at 150 Km
- $\beta_f$ at 350 Km

(e = 0.023 if $U_f = U_1$)
Fig. 2.8 Model SM3 ($h_p = 300$ km).
Fig. 2.9 Model SM3 ($h_p = 500$ km).

$U_p$ at 500 Km
$U_r$ at 150 Km
$\beta_f$ at 650 Km
($e = 0.022$ if $U_f = U_\perp$)
Fig. 2.10 Static Density Model SM1 Compared with the USSA 1976 (High Altitude).
Fig. 2.11 Percent Error of SMI (High Altitude) with respect to the USSA 1976.

\[ \alpha = 0.8957 \]
\[ m = 0.07589 \]
Fig. 2.12 Density as a Function of Altitude and Exospheric Temperature (Taken from [49]).
Fig. 2.13 Accuracy of Equations (2.68)-(2.70) when compared with Jacchia [49].
\[ F = 135 + 55 \cos \left( \frac{2\pi (t - t_o)}{T} \right) \]

\[ t_o = 36,500 \]

\[ T = 4,200 \]

Fig. 2.14 Mean Solar Flux from 1958 to 1974 (Taken from [49]) and Equation (2.72).
$S_{10.7}$: solar flux

$\rho_{STD}$: density adjusted to a fixed flux ($S_{10.7} = 130 \times 10^{-22} \text{Wm}^{-2} \text{Hz}^{-1}$)

$\rho_0$: density at a height of 300 km (before correction to $S_{10.7}$)

Fig. 2.15 Random Variations in Density (Taken from [86]).
Fig. 3.1 Frames $F_0$ and $F_I$.

Fig. 3.2 Frames $F_s$ and $F_I$. 
Fig. 3.4a  Initial Conditions at $\theta_h \neq \psi_o$.

Fig. 3.4b  Initial Conditions at $\theta_h = \psi_o = 0$. 

(in the orbital plane)
After a brief review of the field, the pertinent elements of the lifetime prediction problem are assembled. A quite complete model of the atmosphere is constructed which includes all the important dynamic effects, and which possesses several novel features. The equations of motion for the satellite are then presented together with a discussion of methods for their solution. The equations are first solved for the drag-only dispersions in a stationary atmosphere, and then, based on this foundation, more 'unified' solutions are defined due to the combined effects of air drag and Earth oblateness. Both circular and elliptical orbits are considered.