Why can’t we do $\int e^{-x^2} \, dx$?

Non impeditus ab ulla scientia

K. P. Hart

Faculty EEMCS
TU Delft

Delft, 10 November, 2006: 16:00–17:00
Outline

1 Integration in finite terms

2 Formalizing the question
   - Differential fields
   - Elementary extensions
   - The abstract formulation

3 Applications
   - Liouville’s criterion
   - \( \int e^{-z^2} \, dz \) at last
   - Further examples

4 Sources

5 Technicalities

K. P. Hart

Why can’t we do \( \int e^{-x^2} \, dx \)?
What does ‘do $\int e^{-x^2} \, dx$’ mean?

To ‘do’ an (indefinite) integral $\int f(x) \, dx$, means to find a formula, $F(x)$, however nasty, such that $F' = f$.

- What is a formula?
- Can we formalize that?
- How do we then prove that $\int e^{-x^2} \, dx$ cannot be done?
We recognise a formula when we see one. E.g., Maple’s answer to $\int e^{-x^2} \, dx$ does not count, because

$$\frac{1}{2} \sqrt{\pi} \text{erf}(x)$$

is simply an abbreviation for ‘a primitive function of $e^{-x^2}$’, (see Maple’s help facility).
A formula is an expression built up from elementary functions using only

- addition, multiplication, . . .
- other algebra: roots ’n such
- composition of functions

Elementary functions: \( e^x \), \( \sin x \), \( x \), \( \log x \), . . .
Yes.

- Start with $\mathbb{C}(z)$ the \textit{field} of (complex) rational functions and add, one at a time,
- algebraic elements
- logarithms
- exponentials
How do we then prove that $\int e^{-x^2} \, dx$ cannot be done?

We do not look at all functions that we get in this way and check that their derivatives are not $e^{-x^2}$.

We do establish an *algebraic* condition for a function to have a primitive function that is expressible in terms of elementary functions, as described above.

We then show that $e^{-x^2}$ does not satisfy this condition.
A differential field is a field $F$ with a *derivation*, that is, a map $D : F \to F$ that satisfies

- $D(a + b) = D(a) + D(b)$
- $D(ab) = D(a)b + aD(b)$
Main example(s)

The rational (meromorphic) functions on (some domain in) \( \mathbb{C} \), with \( D(f) = f' \) (of course).
We write \( a' = D(a) \) in any differential field.
Easy properties

Exercises

- $(a^n)' = na^{n-1}a'$
- $(a/b)' = (a'b - ab')/b^2$ (Hint: $f = a/b$ solve $(bf)' = a'$ for $f'$)
- $1' = 0$ (Hint: $1' = (1^2)'$)
- The ‘constants’, i.e., the $c \in F$ with $c' = 0$ form a subfield
Exponentials and logarithms

- $a$ is an exponential of $b$ if $a' = b'a$
- $b$ is a logarithm of $a$ if $b' = a'/a$
- so: $a$ is an exponential of $b$ iff $b$ is a logarithm of $a$.
- ‘logarithmic derivative’:

$$\frac{(a^m b^n)'}{a^m b^n} = ma' + nb'$$

Much of Calculus is actually Algebra . . .
Definition

A *simple* elementary extension of a differential field $F$ is a field extension $F(t)$ where $t$ is

- algebraic over $F$,
- an exponential of some $b \in F$, or
- a logarithm of some $a \in F$

$G$ is an elementary extension of $F$ is $G = F(t_1, t_2, \ldots, t_N)$, where each time $F_i(t_{i+1})$ is a simple elementary extension of $F_i = F(t_1, \ldots, t_i)$. 
Elementary integrals

We say that $a \in F$ has an elementary integral if there is an elementary extension $G$ of $F$ with an element $t$ such that $t' = a$. The Question: characterize (of give necessary conditions for) this.
Theorem (Rosenlicht)

Let $F$ be a differential field of characteristic zero and $a \in F$. If $a$ has an elementary integral in some extension with the same field of constants then there are $v \in F$, constants $c_1, \ldots, c_n \in F$ and elements $u_1, \ldots u_n \in F$ such that

$$a = v' + c_1 \frac{u_1'}{u_1} + \cdots + c_n \frac{u_n'}{u_n}.$$ 

The converse is also true: $a = (v + c_1 \log u_1 + \cdots + c_n \log u_n)'$. 

Why can’t we do $\int e^{-x^2} \, dx$?
Comment on the constants

Consider \( \frac{1}{1 + x^2} \in \mathbb{R}(x) \)

- an elementary integral is

\[
\frac{1}{2i} \ln \left( \frac{x - i}{x + i} \right),
\]

using a larger field of constants: \( \mathbb{C} \)

- there are no \( v, u_i \) and \( c_i \) in \( \mathbb{R}(x) \) as in Rosenlicht’s theorem.
When can we do \( \int f(z)e^{g(z)} \, dz \)?

Let \( f \) and \( g \) be rational functions over \( \mathbb{C} \), with \( f \) nonzero and \( g \) non-constant.

\( fe^g \) belongs to the field \( F = C(z, t) \), where \( t = e^g \) (and \( t' = gt \)).

\( F \) is a transcendental extension of \( C(z) \).

If \( fe^g \) has an elementary integral then in \( F \) we must have

\[
ft = v' + c_1 \frac{u'_1}{u_1} + \cdots + c_n \frac{u'_n}{u_n}
\]

with \( c_1, \ldots, c_n \in \mathbb{C} \) and \( v, u_1, \ldots, u_n \in C(z, t) \).
The criterion

Using *algebraic* considerations one can then get the following criterion.

**Theorem (Liouville)**

The function $f e^g$ has an elementary integral iff there is a rational function $q \in \mathbb{C}(z)$ such that

$$f = q' + q g'$$

the integral then is $qe^g$ (of course).
In this case \( f(z) = 1 \) and \( g(z) = -z^2 \).
Is there a \( q \) such that \( 1 = q'(z) - 2zq(z) \)?
Assume \( q \) has a pole \( \beta \) and look at principal part of Laurent series

\[
\sum_{i=1}^{m} \frac{\alpha_i}{(z - \beta)^i}
\]

Its contribution to the right-hand side should be zero.
We get, at the pole $\beta$:

$$0 = \sum_{i=1}^{m} \left( -\frac{i\alpha_i}{(z-\beta)^{i+1}} - \frac{2z\alpha_i}{(z-\beta)^i} \right)$$

Successively: $\alpha_1 = 0, \ldots, \alpha_m = 0$. So, $q$ is a polynomial, but $1 = q'(z) - 2zq(z)$ will give a mismatch of degrees.
Here \( f(z) = 1/z \) and \( g(z) = z \), so we need \( q(z) \) such that

\[
\frac{1}{z} = q'(z) + q(z)
\]

Again, via partial fractions: no such \( q \) exists.

\[
\int e^z \, dz = \int \frac{e^u}{u} \, du = \int \frac{1}{\ln v} \, dv
\]

(substitutions: \( u = e^z \) and \( u = \ln v \))
In the complex case this is just \( \int \frac{e^z - e^{-z}}{z} \, dz \).

Let \( t = e^z \) and work in \( \mathbb{C}(z, t) \); adapt the proof of the main theorem to reduce this to \( \frac{1}{z} = q'(z) + q(z) \) with \( q \in \mathbb{C}(z) \), still impossible.
These slides at: fa.its.tudelft.nl/~hart

J. Liouville.
*Mémoire sur les transcendents elliptiques considérées comme functions de leur amplitudes*, Journal d’École Royale Polytechnique (1834)

M. Rosenlicht,
Lemma

Let $F$ be a differential field, $F(t)$ a differential extension with the same constants, with $t$ transcendental over $F$ and such that $t' \in F$. Let $f(t) \in F[t]$ be a polynomial of positive degree. Then $f(t)'$ is a polynomial in $F[t]$ of the same degree as $f(t)$ or one less, depending on whether the leading coefficient of $f(t)$ is not, or is, a constant.
Lemma

Let $F$ be a differential field, $F(t)$ a differential extension with the same constants, with $t$ transcendental over $F$ and such that $t'/t \in F$. Let $f(t) \in F[t]$ be a polynomial of positive degree.

- for nonzero $a \in F$ and nonzero $n \in \mathbb{Z}$ we have $(at^n)' = ht^n$ for some nonzero $h \in F$;
- if $f(t) \in F[t]$ then $f(t)'$ is of the same degree as $f(t)$ and $f(t)'$ is a multiple of $f(t)$ iff $f(t)$ is a monomial $(at^n)$. 

K. P. Hart
Write \( F = \mathbb{C}(z) \) and \( t = e^z \).
If \( \int \frac{\sin z}{z} \, dz \) were elementary then

\[
\frac{t^2 - 1}{tz} = v' + c_1 \frac{u'_1}{u_1} + \cdots + c_n \frac{u'_n}{u_n}
\]

with \( c_1, \ldots, c_n \in \mathbb{C} \) and \( v, u_1, \ldots, u_n \in F(t) \).
By logarithmic differentiation: the \( u_i \)'s not in \( F \) are monic and irreducible in \( F[t] \).
If \( \int \frac{\sin z}{z} \, dz \) were elementary then

\[
\frac{t^2 - 1}{tz} = v' + c_1 \frac{u'_1}{u_1} + \cdots + c_n \frac{u'_n}{u_n}
\]

with \( c_1, \ldots, c_n \in \mathbb{C} \) and \( v, u_1, \ldots, u_n \in F(t) \).

By the lemma just one \( u_i \) is not in \( F \) and this \( u_i \) is \( t \).

So \( c_1 \frac{u'_1}{u_1} + \cdots + c_n \frac{u'_n}{u_n} \) is in \( F \).
Finally, in

$$\frac{t^2 - 1}{tz} = v' + c_1 \frac{u'_1}{u_1} + \cdots + c_n \frac{u'_n}{u_n}$$

we must have $v = \sum b_j t^j$ and from this: $\frac{1}{z} = b'_1 + b_1$. 