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Adaptive Angle Set Method:
A new strategy for combined topology and fiber angle optimization

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Adaptive Angle Set Method
A new strategy for combined topology and fiber angle optimization

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„Wer wartet mit Besonnenheit, der wird belohnt zur rechten Zeit“

- TILL LINDEMANN
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Summary

The use of composite materials is of increasing importance over the past years. Especially unidirectional fibrous laminates are nowadays widely applied in industry. They provide mechanical advantages in terms of stiffness to weight ratios, strength and resistance against fatigue. These properties make them suitable for high-end applications as for example the aerospace industry.

Topology optimization is a mathematical technique which has recently gained importance as well. As an optimization technique with a large design freedom, it is able to design complex structures with high performance beyond human abilities. Together with the latest improvements on manufacturing techniques, the application of topology optimized structures intensifies in various fields. This research focuses on topology optimization on unidirectional fibrous laminate structures.

The problem of combined topology and fiber direction optimization is researched over the past years by a number of groups. The problem formulation where the fiber angles are directly used as design variables is highly non-convex and is likely destined to end up in a local optimum far from the global optimum.

Two other alternatives are described in literature: a discrete and continuous problem formulation. In the discrete approach, called Discrete Material Optimization (DMO), a finite number of candidate materials per element represents the different fiber orientations and penalization is applied to end up with a clear distinction between the candidate materials. The discrete formulation has the drawback that the solution is limited to the predefined candidate materials and that the number of design variables easily becomes large. Furthermore, the global optimum could never be guaranteed due to the required penalization.

The continuous approach uses lamination parameters as design variables and the optimization problem becomes convex. A shortest-distance approach is used to determine the closest realistic laminate configuration for the global optimal set of lamination parameters. Using this technique, continuous variable stiffness panels can be designed with a reasonable amount of design variables. However, the realistic laminate configuration to a set of lamination parameters is not known analytically for more complex problems. Therefore, the determination of a physically meaningful configuration may be a difficult task, and may go with a loss of performance.

Given both the pro’s and con’s of the methods from literature, there seems to be a demand for a method that can provide detailed results (continuous variable stiffness), with a reasonable amount of design variables, which also directly provides a physically realistic laminate configuration. In this research a new method called the Adaptive Angle Set Method (AASM) is proposed. AASM solves a sequence of DMO-like subproblems for fiber angle optimization, but the associated design variables are not penalized. A separate set of density variables performs the topology optimization and the combined problem is solved simultaneously.

Every subproblem in AASM is analogue to a non-penalized DMO problem with three
candidate materials for every element, representing a set of three different fiber angles. In the initial subproblem, the angle set is equal for all elements and given by \{-60^\circ, 0^\circ, 60^\circ\}, spanning the entire domain of 180° of possible fiber angles. This subproblem is solved to optimality and the subsolution is used to formulate the succeeding subproblem. Based on the subsolution of design variables, a combination of update functions estimates a new fiber angle for every element, which is defined as the middle angle of the element’s new angle set. The two other angles are valued from this middle angle plus and minus a certain offset (range) and the new subproblem is again solved to optimality. However, the range between the three candidate materials is tightened with the formulation of every new subproblem, such that the sequence of problems converges to angle sets where the three candidate materials are close to each other. This can be as close as 1° difference in the final subproblem. At the final stage, penalization is applied to create a clear distinct solution between the candidate materials, but this only causes a minimal loss of performance due to the small range in the angle set. Using this approach, the number of design variables is constant for every subproblem, namely three fiber angle design variables and one density variable per element. In the final stage, a high angle resolution is obtained with a directly known laminate configuration.

The way in which a new subproblem is formulated highly depends on the estimation of the new angle for every element. The determination of the optimal new angle using an optimization routine would be equal to solving the overall fiber angle problem, which can not be solved efficiently with a gradient based optimizer. Therefore, two heuristic update functions are introduced to estimate the new angle. The first update function makes a linear combination of the previous angle set with the corresponding optimal design vector. The second update function sets the new angle equal to the largest principal stress direction for that element.

A number of test cases showed that a mixed application of both update functions yielded the best results. The final configuration was tested on a number of compliance minimization problems, which were kept planar and single loaded during this research. For small problems, the AASM results could be compared to brute force global optima of the underlying fiber angle integer problem. Results equal or close to the global optimum were obtained. For larger problems and multiple layer laminates, AASM provided promising results as well, which were obtained faster than a comparable DMO-formulation. The promising results obtained by AASM makes the method worthwhile for further investigation on larger and more complex problems, including other objective functions, bending elements and manufacturing constrained problems.
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1 Introduction

The design process for an optimal component has been supported by computers for decades. Dozens of algorithms were developed to fulfill the always higher requirements and provide designs beyond the abilities of human creativity or calculation efforts. Nowadays, almost every part in daily human life is designed, analyzed or produced with the help of computational effort.

Topology optimization is a relatively new concept where an optimization algorithm has a large space of freedom to create an optimal product design, since the shape, size and layout are not prescribed on beforehand. Over the last years, a large amount of optimization algorithms are developed to run this process mathematically efficient.

In most cases, topology optimization is used in combination with isotropic materials. However recent developments in industry show an increasing interest in composite materials, since they can have superior mechanical properties over the classical isotropic materials. In order to combine both the advantages of topology optimization and composite materials, a couple of methods has been developed to this purpose. The combined problem has shown to be more challenging than optimization with isotropic materials.

In this Thesis, a new method for topology optimization with unidirectional fibrous laminate structures is developed. At first, a short introduction to both topology optimization and laminate mechanics is given in sections 2 and 3 respectively. The combined problem is further analyzed in section 4. Next, a brief overview of the present methods will be given in section 5. After this investigation, a new method called the Adaptive Angle Set Method is presented in section 6 and further analyzed and improved in section 7. The extension to multilayer laminates will be made in section 8. This report ends with a brief outlook to future work and recommendations in section 9 and conclusions in section 10.
2 Topology Optimization

A lot of research work has been done in the field of topology optimization. In a classic optimization problem, an optimizer tries to find the optimal shape and size of a structure, while the design space, loads and boundary conditions are predefined by the user. The difference with shape optimization is that the number of holes is not preliminary set. For every element in the structure, the task is to determine whether the element should be made out of solid material or no-material. This is a discrete integer problem, aiming for solutions with topology variables either to have value 1 (solid material) or 0 (void region). The performance of a topology optimization algorithm heavily relies on the ability of the optimizer to push the design variables to their limit values.

For isotropic materials, a well-known method is the so-called SIMP-approach (Solid Isotropic Material with Penalization) \[2\]. The described discrete problem is non-convex and generally has a large number of design variables. Therefore a 0-1 integer problem for topology optimization is computationally expensive to solve. The SIMP-method relaxes the problem to a continuous problem, meaning that the topology variable \( \rho(x) \) is allowed to vary continuously between 0 and 1. For continuous problems, design sensitivities can be calculated and the problem can be solved efficiently by gradient-based optimization techniques. Without considering micro structures, in general is aimed for a solution which has density values either 0 or 1: therefore the topology variables are penalized with a parameter \( p \). The use of a penalization scheme results in a distinct solution by driving the design variables towards 0 or 1 and penalizing intermediate values. The SIMP model is given by (2.1):

\[
C_i(x_i) = \rho(x_i)^p C_0. \tag{2.1}
\]

Here \( C_i(x_i) \) is the constitutive tensor for element \( i \) and \( C_0 \) the constitutive tensor of the used isotropic material, \( \rho(x_i) \) the topology design variable and \( x_i \) the spatial coordinate. For a penalization factor of \( p = 1 \) the method corresponds to the variable thickness sheet problem. In order to avoid singularity, the density or thickness parameter \( \rho \) is usually allowed to vary within the domain \( 0 < \rho_{\text{min}} \leq \rho \leq 1 \).

Topology optimization has been applied on several objectives and constraints \[15\]. The analyses performed are:

\[
Ku = F, \quad \text{and} \quad (K - \lambda_j K_\sigma)\Phi_j = 0, \quad j = 1, 2, ..., \tag{2.2}
\]

in order to determine displacements, compliance, failure indices and the buckling load factor. In Equation (2.2) is \( K \) the global stiffness matrix, \( u \) the global displacement vector and \( F \) the global load vector respectively. \( K_\sigma \) is the global stress-stiffness matrix for the buckling analysis and \( \lambda_j \) are the eigenvalues, which are assumed to be ordered by increasing magnitude. \( \Phi_j \) is the corresponding \( j \)th eigenmode. In the literature, topology optimization is applied for many objectives: frequently minimization of compliance or weight. However, from Equation (2.2) also optimization for buckling and strength
can be performed. Performances against local failure criteria, such as the Tsai-Wu failure criterion [21], can be investigated by introducing a load scaling factor that leads to failure. Optimization for higher buckling resistance, by maximizing the buckling factor (lowest eigenfrequency) is also performed in the literature.

In order to solve the problem with a gradient-based optimizer, the computation of sensitivities is necessary. In most cases, the design sensitivity analysis for the various objective functions are performed with use of the adjoint sensitivity method, but alternatively a finite difference method [15] can be applied to approximate the first order derivatives.

This section provided a brief overview of the possibilities in the field of topology optimization, although there exist many more or are currently in development. Since this research focuses on the interaction between topology optimization and laminate optimization, a full investigation to the possibilities of topology optimization is further omitted.
3 Composite Materials

Composite materials consist of a certain matrix material with some reinforcement material implemented. The combination of those two determines the anistropic behaviour of the composite. The fiber implementation can be done in different ways, of which three of them are illustrated in Figure 1.

Figure 1: Particulate composite, fiber reinforced composite and laminated composite.

[11]

In industry, most composite materials are laminates due to their favorable manufacturability. In a laminate, the composite is built up of several layers (plies) stacked on top of each other. The structural properties of the plies may vary. Figure 2 shows an example of a laminate, where the fiber orientation angle is different between the plies. The fiber angle is constant within a ply, but in a more general situation, the fiber orientation may also vary spatially within one ply itself. The plies are stacked on top of each other for every element, see Figure 3, the overall optimization problem runs over all the elements.

Figure 2: Laminate with constant fiber angle plies. [12]

Figure 3: Laminated elements [18]

From an optimization point of view, it is common to describe the problem in terms of objective, constraints and design variables. In the literature, composite materials have been optimized for many purposes: maximum stiffness, minimal weight, maximum buckling or failure load are the most common objective functions. When considering a
laminates, possible design variables are the spatial fiber orientation, the stacking sequence of the plies and the total laminate thickness. Another possibility is to use the material properties of the matrix and fiber material itself as design variables.

Specific constraints for laminate optimization are generally obtained from a manufacturing point of view. Since the use of these constraints may vary from problem to problem, and the implementation may differ from method to method, only a textual explanation of these constraints is given here.

First of all, laminates are often produced in so-called patches. A patch is a subdomain of the design space, representing a group of elements with the same selected constitutive properties for every layer. Thickness variation within a patch is still allowed. The use of patches makes sure that there is a layer-wise continuity over a large area (patch). It reduces the number of design variables and eases manufacturing, since it allows the usage of large predefined fiber mats.

Two constraints are described to prevent failure indirectly. An abrupt change in thickness between neighboring elements has to be prevented by limiting the maximum slope between an element and its direct neighbors. Next to that, an additional constraint limits the number of contiguous plies with the same constitutive properties. Both situations, abrupt thickness changes and too many identical stacked plies give rise to delamination and matrix cracking respectively.

The last manufacturing constraint prevents the existence of intermediate void plies and limits topology optimization to the outer layers of the laminate.

Furthermore, a lot of other constraints can be implemented: for example the requirement that a laminate has to be symmetric or balanced, see Figure 4.

![Figure 4: (a) Symmetric laminate, (b) Balanced laminate](image)

A symmetric laminate has the inverse stacking sequence around its midplane, therefore only half of the design variables can be used. This is also the case for the balanced laminate, which has the same but negative stacking sequence above and below the midplane.
Mechanics

The constitutive equation for a general laminate is given by

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa \end{bmatrix}. \tag{3.1}$$

In Equation (3.1) $N$ and $M$ are the resultant forces and bending moments, $A$ is the in-plane stiffness matrix, $D$ the bending stiffness matrix and $B$ the coupling matrix. $\varepsilon$ and $\kappa$ are the in-plane strain and curvature vectors, respectively. The laminate stiffness $ABD$ varies at any point for variable stiffness laminates and is a function of fiber angle $\theta$ and the laminate thickness $z$. Equation (3.1) becomes simpler when special laminates are used. For example, a symmetric laminate from Figure 4a decouples the in-plane and bending stiffness tensor, i.e. $B = 0$.

When the mechanics of a thin, unidirectional lamina are considered, it can be assumed that it does not carry out-of-plane loads but only in-plane loads. Hooke’s law can be reduced to two dimensions and the stress-strain relationship is given by [14]:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}. \tag{3.2}$$

In Equation (3.2) $Q_{ij}$ are the reduced stiffness coefficients and can be related to the engineering constants as:

$$Q_{11} = \frac{E_1}{1 - \nu_2\nu_1},$$

$$Q_{12} = \frac{\nu_2 E_2}{1 - \nu_2\nu_1},$$

$$Q_{22} = \frac{E_2}{1 - \nu_2\nu_1},$$

$$Q_{66} = G_{12}. \tag{3.3}$$

In Equation (3.3) $E_1$ and $E_2$ are the Young’s Moduli in the 1- and 2-direction which are the parallel and perpendicular axes to the fiber orientation respectively, $\nu$ is the Poisson’s ratio and $G_{12}$ the shear modulus.

However, when the fibers are placed under an angle in a lamina, i.e. the local fiber directions 1 and 2 do not longer coincide with the global $x$ and $y$ directions, the global stresses can be obtained from the local strains by:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = T^{-1} Q R T^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{3.4}$$

$T$ from Equation (3.4) is a transformation matrix according to

$$T = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}, \tag{3.5}$$
where $\theta$ is the fiber orientation angle and $R$ from (3.4) is the Reuter matrix given by

$$
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}.
$$

The $3 \times 3$ matrix $\bar{Q}$ on the right hand side of Equation (3.4) is the rotated reduced stiffness matrix, whose components can be derived from $Q_{ij}$ and $\theta$ as:

$$
\bar{Q}_{11} = Q_{11} \cos^4 \theta + Q_{22} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta,
$$

$$
\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\cos^4 \theta + \sin^4 \theta),
$$

$$
\bar{Q}_{22} = Q_{11} \sin^4 \theta + Q_{22} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta,
$$

$$
\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta,
$$

$$
\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta,
$$

$$
\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta).
$$

From the components of Equation (3.7) the element stiffness matrix is calculated for a quadrilateral element.
4 Combined problem description

In sections 2 and 3 both the advantages and opportunities of composite materials and topology optimization are discussed. The main topic of research was combining both fields and optimize both the topology as well as the fiber angle direction. A design domain for topology optimization is defined, however the material no longer has isotropic properties, but every element is treated as an unidirectional fibrous laminate. The optimizer task is to define an optimal density value as well as an optimal angle in which the fiber is oriented in the matrix material. Both the density and the fiber angle are to be determined for every element.

4.1 Optimization Problem

Stiffness is considered as objective function, which is most common in structural optimization problems. Although the research will focus on stiffness maximization, the main goal is to study the interaction between topology and fiber angle optimization, which should also hold for other objective functions like weight-minimization or resistance to failure maximization.

The stiffness maximization problem is analogue to minimizing its inverse: compliance. In this way, the overall optimization problem can be written in the negative-null form as

$$\min_{\rho, \theta} \sum_{i=1}^{N} \rho_i u_i^T k_i(\theta_i) u_i,$$

subject to: $$\sum_{i=1}^{N} \rho_i \leq V \quad 0 < \rho_i \leq 1.$$  \hspace{1cm} (4.1)

In the equation above $\rho_i$ and $\theta_i$ are the local density and fiber angle of element $i$, respectively. $k_i$ is the local stiffness matrix and $u$ is the local displacement vector. $V$ is the total volume fraction. $N$ is the total number of elements.

4.2 Design space

Using this formulation, the fiber angles $\theta_i$ are directly used as design variables. However, even without topology optimization, the pure fiber angle optimization problem is highly non-convex in this formulation. For a simple $2 \times 1$ element, cantilevered test problem, as is graphically shown in Figure 5, the design space can still be visualized in a 3d plot. Using a brute-force approach where the compliance for all possible combinations between $\theta_1$ and $\theta_2$ are calculated with a resolution of 1°; the design space is visualized in Figure 6.

From Figure 6 can be verified that the design space is highly non-convex, which causes difficulties for a gradient-based optimizer to find the global optimum. Especially when
the problems become larger and more complex due to the inclusion of topology optimization and several constraints, the global optimum will be even more difficult to find. Furthermore, it is generally expensive to calculate the derivatives from the objective function directly to the fiber angles, since the rotated local stiffness matrix is directly dependent on these angles. Therefore, most research papers on this topic make use of an alternative problem formulation where the fiber angles are not directly used as design variables. An overview of the present methods will be given in the next chapter.
5 Present Methods

This chapter summarizes the available methods for combined topology and fiber angle optimization. Aim is to describe the methods for conceptual understanding: full mathematical descriptions can be found in the corresponding references.

5.1 MMA Families

An approach for solving the original problem to find the optimal stacking sequence with the fiber angles (and optionally layer thicknesses) used as continuous variables was studied by [5]. The optimization problem is solved using a combination of MMA (Method of Moving Asymptotes [20]) algorithms. Due to the non-convex nature of the problem, the obtained solution will generally be a local one. However, it can be granted that the obtained solution is feasible. The general optimization problem can be described by:

$$\min g_0(X),$$

subject to:

$$g_j(X) \leq g_j^{\text{max}}, \quad j = 1, \ldots, m, \quad (5.1)$$

$$x_i \leq x_i \leq \bar{x}_i, \quad i = 1, \ldots, n.$$

The objective function $g_0$ can be any type of function, including weight, stiffness, strength or frequency. The vector $X = x_1, \ldots, x_n$ contains the design variables which are the fiber orientations and plies thicknesses. The design variables are allowed to vary continuously. The original problem (5.1) is replaced by $n$ approximated subproblems. The approximation is done by a first or second order Taylor series expansion. The subproblems are solved by a combination of two algorithms from the MMA-family. The use of a mixed approximation scheme has proven to be favorable over a single algorithm, in terms of computational time and number of iterations [5].

The strain energy is also a non-monotonous function with respect to the ply thickness and fiber angles. The original MMA method makes a monotonous approximation at the current design point, since only one of the moving asymptotes is active at the time. The asymptotes are fixed for all design functions $g_j$, regardless of the nature of $g_j$. Therefore, the MMA method lacks flexibility to tighten or relax the asymptotes. However, a generalization step (GMMA) can be made, where separate asymptotes are calculated for every design function. A further enhancement is made to a globally convergent version of MMA (GCMMA), where both asymptotes are active at the same time and hence a non-monotonous approximation function is created. The asymptotes are calculated based on (among others) first or, when available, second order derivative information. As a cost-effective alternative, the second order derivatives can also be approximated by the first order derivatives at the current and previous iteration point (GBMMA). The GCMMA variant is always able to find a feasible solution, however the global optimum could not be guaranteed.

Each of the different variants of MMA show their own pro’s and con’s: algorithms using second order derivative information show a good convergence near the optimum, but
are expensive and are less often able to create a convex curve between two successive
design points. Therefore, a mixed approach of the different variants is recommended.
The selection is made based on the first order derivatives at the previous and current
design point and is generally a trade off between GMMA and GBMMA/GCMMA or
even a linear approach.

5.2 Discrete Methods

DMO Method

A well-described method that combines topology optimization and composite material
optimization is the DMO-method: Discrete Material Optimization. It was first proposed
by Stegmann and Lund [19] and can be considered as an extension to the methods
developed for normal structural topology optimization. Instead of selecting a solid or
void material for an element, the constitutive tensor may now be chosen from multiple
candidate materials, as indicated in the following equation:

\[ C_e = \sum_{i=1}^{n} w_i C_i = w_1 C_1 + ... + w_n C_n, \quad 0 \leq w_i \leq 1. \]  

In Equation (5.2) the element constitutive tensor \( C_e \) is computed from a weighted sum
of the individual constitutive tensors \( C_i \) of \( n \) candidate materials. In theory, these candi-
date materials may represent any sort of materials, but for composite laminates their
only difference is the orientation of the fiber angle. For example, four candidate materials
may represent orientations of \(-45^\circ, 0^\circ, +45^\circ\) and \(90^\circ\) angles. Every candidate material is
premultiplied with a weight factor \( w_i \), which has to be in the \( 0 \leq w_i \leq 1 \) domain, since
a material cannot contribute to the solution with more than its own constitutive prop-
erties. Negative weight factors do not correspond with a physically meaningful material
and are also excluded. For every element \( e \), there are \( n \) candidate materials, meaning
that the total number of design variables becomes \( n \times N \), where \( N \) is the total number
of elements.

As for isotropic topology optimization, a clear distinction between the candidate ma-
terials has to be made for a physically meaningful solution. The desired solution has
one weight factor valued 1 and all others 0, for every element. Hence, the choice of the
weight functions is very important. Initially the same penalization scheme as for the
SIMP-method was applied to the DMO parameterization, defined as:

\[ C_e = \sum_{i=1}^{n} (x_i)^p C_i = (x_1)^p C_1 + ... + (x_n)^p C_n, \quad 0 \leq x_i \leq 1. \]  

It has to be stated that at the initial state, all weight factors should have the same value.
This prevents that the selection for one of the candidate materials is favored \emph{a priori}.
An unequal initial state generally has a great impact on the selection and may drive the
solution quickly into a local optimum.

However, the extended SIMP scheme as proposed in (5.3) does not penalize intermediate
solutions well for a larger number of candidate materials. Therefore, the penalization scheme of Equation (5.4) was introduced:

$$C^e = \sum_{i=1}^{n} (x_i)^p \prod_{j=1}^{n} (1 - (x_{j \neq i})^p) C_i, \quad 0 \leq x_i \leq 1. \quad (5.4)$$

This scheme increases the contribution of a favored design variable, while reducing the contribution of the others. This results in a more distinct choice for one candidate material.

According to Figure 3 an element can consist of multiple layers. For such multilayer problems, the interpolation method can directly be applied, but now computing the constitutive tensor $C^l$ for layer $l$ as the summation of all candidate layer materials $n^l$ over all layers $L$. The layer interpolation scheme is given by (5.5):

$$C^l = \sum_{i=1}^{n^l} (x_i)^p \prod_{j=1}^{n^l} (1 - (x_{j \neq i})^p) C_i, \quad 0 \leq x_i \leq 1. \quad (5.5)$$

For multi-layered structures, the total number of design variables per element is $n^l \times L$, which significantly increases the number of design variables. The earlier mentioned use of patches can reduce this number.

Figure 7: Design problem (left) and optimized result (right) with the DMO-method for four candidate materials. [13]

A simple example of the DMO method is presented in Figure 7. The design domain consists of a $4 \times 4$ element patched plate. The squared plate is clamped on the left edge and is loaded with a vertical force at its lower right end. There are four candidate fiber orientations $[-45^\circ, 0^\circ, +45^\circ, 90^\circ]$ and topology optimization is allowed. Figure 7 shows the obtained solution for a minimum compliance problem.
SFP Method

An extension to the DMO-method mentioned in the previous paragraph is the SFP-method: Shape Functions with Penalization [4]. The method proposes a new parameterization scheme in order to reduce the number of design variables, which reduces computational effort. Recalling the DMO-method, every candidate material requires its own design variable. In [4] a shape function is proposed that links the candidate materials to each other.

![Figure 8: SFP parameterization with a rectangular shape function](image)

For conceptual meanings, four candidate materials are considered at first. The basic idea is illustrated in Figure 8. The four candidate materials [-45°, 0°, 45°, 90°] are represented with only two design variables, R and S, by a rectangular shape function. This shape function replaces the weight function in Equation (5.2) with

\[
\begin{align*}
    w_{i}^{SF} &= \frac{1}{4}(1 \pm R)(1 \pm S), \\
    w_{i}^{SFP} &= \left(\frac{1}{4}(1 \pm R)(1 \pm S)\right)^p.
\end{align*}
\]

Using (5.6) four combinations can be made with two design variables. The bottom line of Equation (5.6) uses a penalization similar to the classic SIMP-approach in order to drive the design variables to a 0-1 solution. This method can be extended to other numbers of design variables [6], but then triangular or hexagonal shape functions are required, for example. A drawback of this method is that the shape functions become rather complex when considering larger amounts of candidate materials. The next paragraph presents a method to overcome this problem.

BCP-method

This paragraph describes the BCP-method: Bi-Coding with Penalization. In the previous paragraph, the SFP-method was mentioned, but this method however lacks the ability to handle a large amount of candidate materials, since complex shape functions are required which also have to match the shape functions used for the finite elements. In [9] the BCP-method is proposed as a parameterization scheme which is able to reduce
the number of design variables in a logarithmic manner. This is obtained by labeling every candidate material with a bi-valued code, constructed from a combination of -1 and 1 values. The weight function of (5.2) is now replaced by Equation (5.7), according to:

\[ w_{ij}^{BCP} = \left[ \frac{1}{2^m} \prod_{k=1}^{m} (1 + s_{jk} x_{ik}) \right]^p, \quad 1 \leq x_{ik} \leq 1, \quad k = 1, ..., m, \quad (5.7) \]

where

\[ m = \lceil \log_2 n \rceil. \quad (5.8) \]

The total number of candidate materials \( n \) is represented by \( m \) design variables, which is the ceiling function of \( n \) according to (5.8). In this manner, \( m \) can be used to describe the range from \([2^{m-1} + 1, 2^m]\) candidate materials. In Equation (5.7) \( s_{jk} \) is the bi-valued selection variable which generates the specific code for every material. The values of \( s_{jk} \) are calculated by:

\[
\begin{cases} 
1 & j \in [1, 2^{k-1}], \\
-1 & j \in [2^{k-1}, 2^k], \\
s_{\xi k} & j \in [2^k + 1, 2^m], \quad \text{where } \xi = 2^{\lceil \log_2 j \rceil} + 1 - j.
\end{cases} \quad (5.9)
\]

For example, the values values of \( s_{jk} \) in case \( n = 8 \) and \( m = 3 \) are listed in Table 1.

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A graphical representation of this situation is given in Figure 9. The eight candidate materials form a 3d-space by defining the vertices of the cube. The logarithmic reduction of the design variables per element makes the BCP-scheme very attractive when large scale problems are considered or problems with a large number of candidate materials per element.

**Combination with topology optimization**

In the original DMO-paper [19] the combination with topology optimization was made by the inclusion of void material as one of the candidate materials. However, it became clear that the introduction of a separate topology variable is favorable, since this approach
"leads to invariance with respect to the ordering of the phases when penalization is applied" [13]. This means that the ordering of the candidate materials does not influence the tendency of selecting one of the phases over the others.

In [4] the additional topology variable scales the contribution of the solid phases and is also penalized in a similar way to the SIMP-scheme. The penalization factor \( q \) may differ from \( p \) and is introduced to drive \( y_i \) towards 0 or 1.

\[
C^l = (y_l)^q \sum_{i=1}^{n} w_i^l C_i^l, \quad 0 \leq y_i \leq 1.
\] (5.10)

This latter method was also applied to the DMO-schemes and to the BCP-parameterization in [10]. With this method an additional volume constraint was introduced in the optimization problem, given by

\[
V = \sum_i y_i V_i \leq \bar{V}.
\] (5.11)

In (5.11) \( \bar{V} \) is the upper bound of the constraint can be used to control the amount of void material in the solution. \( V_i \) are the elements that contribute to the total volume, i.e. the elements filled with solid material.

5.3 Continuous Methods

Lamination Parameters

In the subsection 5.1 a continuous optimization approach was introduced by using fiber angles and ply thicknesses as design variables. However, the obtained solution is generally a local optimum. A global optimum can be reached by using Lamination Parameters as design variables. When these Lamination Parameters are used, a convex problem is created and hence a global solution can be found. Recalling Equation (3.1) from section 3, at most 12 dimensionless Lamination Parameters are able to describe the mechanical
properties of any laminate, i.e. the A, B and D-matrices. The Lamination Parameters are given by [22]:

\[
(V_1^A, V_2^A, V_3^A, V_4^A) = \frac{1}{2} \int_{-1/2}^{1/2} (\cos 2\theta, \sin 2\theta, \cos 4\theta, \sin 4\theta) d\bar{z},
\]

\[(5.12)\]

From (5.12) can be obtained that the Lamination Parameters are functions of the stacking sequence function \(\theta(\bar{z})\), which is normalized through-the-thickness by \(\bar{z} = \frac{z}{h}\), where \(h\) is the total height of the laminate. The laminate stiffness can be expressed as a linear function of the 12 Lamination Parameters:

\[
A = h(\Gamma_0 + \Gamma_1 V_1^A + \Gamma_2 V_2^A + \Gamma_3 V_3^A + \Gamma_4 V_4^A),
\]

\[
B = \frac{h^2}{4}(\Gamma_1 V_1^B + \Gamma_2 V_2^B + \Gamma_3 V_3^B + \Gamma_4 V_4^B),
\]

\[
D = \frac{h^3}{12}(\Gamma_0 + \Gamma_1 V_1^D + \Gamma_2 V_2^D + \Gamma_3 V_3^D + \Gamma_4 V_4^D).
\]

In the equations above, \(\Gamma_i\) are the material invariants. Lamination parameters are not allowed to vary totally independently, but only within a feasible region that in turn will result in feasible stacking sequences. The feasible region is a convex set of solutions for which a realistic laminate configuration exists. When only in-plane or out-of-plane stiffness is considered, the feasible domain is analytically known and given by [16]

\[
2V_1^2(1 - V_3) + 2V_2^2(1 + V_3) + V_3^2 + V_4^2 - 4V_1 V_2 V_4 \leq 1,
\]

\[V_2^2 + V_3^2 \leq 1,
\]

\[-1 \leq V_3 \leq 1,
\]

\[(5.16)\]

where the Lamination Parameters \(V_1, ..., V_4\) can either be associated with the in-plane parameters \(V_A\) or the out-of-plane parameters \(V_D\).

For a variable stiffness laminate, where the fiber orientation may spatially vary within a ply, the optimization problem is stated as a minimization of the complementary strain energy by

\[
\min_{V_i} \frac{1}{2} N_i A^{-1}(V_i) N_i, \quad \text{subject to: (5.16)},
\]

\[(5.17)\]

which is evaluated at any node \(i\) of the domain. In order to give the Lamination Parameters a smooth and continuous distribution, the design variables are associated with
the nodes and not with the elements. This problem is convex and solved with a Feasible
SQP algorithm and initialized with a feasible point in LP-space.

A feasible stacking sequence can be determined from the local problem. A good approx-
imation to determine a feasible stacking sequence close to the optimum in LP-space can
be the minimization of the least-squared distance:

\[ \min f = |V^* - \bar{V}| \]  

where \( V^* \) is the desired optimum in Lamination Parameters space. \( \bar{V} \) is the approxi-
mated combination of lamination parameters, which is in [17] calculated by curve-fitting

techniques.

5.4 Free Material Optimization

The last and most general optimization strategy is the method of Free Material Opti-
mization (FMO). The design variables are now the material properties which may vary
from point to point. The aim is to optimize not only the distribution of material but
also the material properties themselves. In the original FMO-paper [1] it was suggested
to represent the material properties as elements of the unrestricted set of positive semi-
definite constitutive tensors. The trace of the stiffness tensor is used as a measure of
resource (weight). The objective function, stiffness, is implemented by minimizing the
compliance with respect to the material properties. The compliance itself is the outcome
of a lower optimization level, namely the minimization of the potential energy. In two
dimensions, the design variables are the six defining elements of the symmetric elasticity
tensor and these variables are allowed to vary point wise throughout the structure. This
makes the problem quite complex. However, this can be reduced to only one design vari-
able (in addition to the displacement vector) by using the trace of the elasticity tensor.
The elements of the optimal tensor itself can be recovered from the optimal trace and
the related displacement.

The method was applied to optimize isotropic structures, including compliant mecha-
nisms. Using Free Material Optimization, a theoretical optimal design could be created.
However, the translation to realistic stacking sequences for orthotropic materials was
not found in the literature yet.
6 Adaptive Angle Set Method

In this section, the main focus of this research will be presented. A new method is proposed, called the Adaptive Angle Set Method (AASM) for combined topology and laminate optimization. The advantages over the present methods will be discussed first, followed by a mathematical formulation of the method. Next, the method is tested on initially a pure fiber angle optimization problem, to be continued with the combination of the topology optimization problem. The section will end with a presentation and discussion of the obtained results so far.

6.1 Motivation AASM

From the previous chapter can be concluded, that the different research groups either focus on finding a directly physically meaningful solution with a combination of algorithms or a limited number of predefined candidate angles. On the other hand, there are several trends for finding the theoretical best continuous laminate design, but facing difficulties of determining a realistic laminate configuration, especially for more complex laminates.

Given these drawbacks, there seems to be demand for a method that directly comes up with realistic laminates, while approaching continuous fiber angles. The DMO method and its derivatives mentioned in section 5.2, are theoretically able to fulfill this job. However, representing (nearly) continuous fiber angles with a discrete formulation would require an enormous amount of finite candidate materials and hence design variables. Although the SFP and BCP methods can reduce the number of design variables by clever parameterization, the number of design variables still becomes large for high angle resolutions. Furthermore, implementation of a discrete method with 1° difference between the fiber angles, has shown to have trouble in creating a clear 0-1 distinct solution, see section 7.7. This is caused by the fact that there is an increasing probability there are two or more candidate materials in the set, whose mixed performance is better than a single angle, despite penalization. This problem can partly be solved by increasing the amount of penalization, which however increases the risk of ending up in a local optimum far from the global optimum in the highly non-convex design space.

The Adaptive Angle Set Method was designed to reach an optimum close to the global optimum, while using a reasonable amount of design variables. The optimal solution directly corresponds to a realistic laminate design. In order to gain full insight of the working principles of AASM, relatively simple problems are treated. The research in this report is limited to in-plane 2D-problems. At the first stage, single layer lamina are considered, whereas in the section 8 an extension to multi-layer laminates will be given.

6.2 AASM working principle

The main task for AASM is determining the optimal fiber angle for every element. The method does not include a different treatment on topology optimization. In order to
do so, AASM is solving a sequence of convex, DMO-like subproblems. This means that there is a predefined set of finite candidate materials representing the fiber angles. However, the number of candidate materials is limited to three for every element and penalization is not applied. Every subproblem $\mathcal{P}$ is solved to optimality, which provides the global optimum since the subproblem remains convex. Based on this global sub-optimum and the initial set of candidate materials, a new set of candidate materials will be constructed for every element. An update function $f$ tries to estimate an optimal, but non-mixed angle for every element and the new set is built around this angle. This new angle set is the set of candidate materials for the next subproblem, which is again solved to optimality and the procedure is repeated. However, at every update, the distance between the three candidate materials is tightened, such that after solving a sequence of subproblems, the angle set of every element has converged to a certain set consisting of three angles close to each other. At the final stage, penalization is eventually required to determine a distinct 0-1 solution. This may cause a loss of performance, however these losses have minor influence on the objective function value, since the difference between the final set of candidate materials is small. The choice for three candidate materials per element was explicitly made, since three is the smallest number of candidates for which the suboptimal angle can be part of the set itself and the new set can be built symmetrically around it.

A mathematical representation will be given to clarify the preceding paragraph of text. First of all, a DMO-like compliance minimization subproblem $\mathcal{P}$ for fiber angle optimization is formulated,

$$\mathcal{P} : \min_{x_{i,j}} \sum_{i=1}^{N} u_i^T k_i(C_i(x_{i,j}))u_i,$$

subject to: $\sum_{j=1}^{3} x_{i,j} = 1 \quad \forall i,$

(6.1)

where $N$ is the total number of elements $i$, and $j = 3$ is the number of candidate materials. The element material tensor $C_i$ is calculated as the weighted sum of its three candidate materials $a, b, c$. The weight factors $x_{i,j}$ are the design variables such that the element material tensor becomes:

$$C_i = x_{i,1}C_{i,a}(\theta_{i,a}) + x_{i,2}C_{i,b}(\theta_{i,b}) + x_{i,3}C_{i,c}(\theta_{i,c}).$$

(6.2)

For reading reasons, the subscript $i$ in (6.1) and (6.2) is omitted in further formulations, since all equations are performed at element level, unless indicated otherwise. As mentioned, at the final stage penalization with factor $p$ is applied to end-up with a distinct integer design for the fiber angles, such that the element material tensor in the final problem is obtained by:

$$C = x_1^pC_a(\theta_a) + x_2^pC_b(\theta_b) + x_3^pC_c(\theta_c).$$

(6.3)
At the initial stage, the first subproblem $P_I$ is defined, with corresponding angle set $\Theta_I$. The set $\Theta_I$ is associated with element $i$ and consists of three candidate materials representing the fiber directions in the $\theta_a$, $\theta_b$ and $\theta_c$ direction, respectively. The angle set is equal for all elements for the first subproblem, namely $\{-60^\circ, 0^\circ, +60^\circ\}$. These three angles are not arbitrarily chosen, but they form an equal distribution between the whole range of possible fiber angles in the $[-90^\circ, 90^\circ]$ domain. In this way, the optimization procedure is not biased in a certain direction on forehand.

\[
P_I : \Theta_I = \left\{ \begin{array}{c}
\theta_a \\
\theta_b \\
\theta_c 
\end{array} \right\} \xrightarrow{\text{OPTIMIZATION}} x_{I\text{opt}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (6.4)
\]

As mentioned, subproblem $P_I$ is solved to optimality yielding the $3 \times 1$ global optimal design vector $x_{I\text{opt}}$. Due to the absence of penalization, there is no drive for the optimizer to create a 0-1 design and hence the vector $x_{I\text{opt}}$ will likely contain a mix between its entries $x_{1,2,3}$. However, ending up with a mixed optimum is an indication that the optimal angle is not included in the angle set.

Now an update function $f$ uses the information of $\Theta_I$ and $x_{I\text{opt}}$ to estimate the best single angle $\theta_{\text{new}}$ for this element, i.e.

\[
\theta_{\text{new}} = f(\Theta_I, x_{I\text{opt}}).
\quad (6.5)
\]

The new angle set $\Theta_{II}$ for the succeeding subproblem $P_{II}$ is built around the new angle $\theta_{\text{new}}$ according to:

\[
P_{II} : \Theta_{II} = \left\{ \begin{array}{c}
\theta_{\text{new}} - \gamma \cdot r_I \\
\theta_{\text{new}} \\
\theta_{\text{new}} + \gamma \cdot r_I 
\end{array} \right\}, \quad 0 < \gamma < 1. \quad (6.6)
\]

In (6.6) $r_I$ is the range of the angle set $\Theta_I$, given by the distance between the three angles:

\[
r_I = \theta_b - \theta_a = \theta_c - \theta_b = 60^\circ, \quad (6.7)
\]

whereas $\gamma$ is a reduction factor responsible for tightening the angle set at every subproblem. Every subproblem $P_p$ uses the range $r_{p-I}$ of the previous set to construct the current angle set. For now, it is further assumed that $\gamma$ is constant for every subproblem. The influence of the reduction factor is further examined in section 7.5.

As can be seen in Equation (6.6), the estimated new angle is the center angle of the new angle set. In this way, there is always some space for the optimizer to step away from this estimated angle by successively selecting the lower or upper bound of the set. This provides some margin for errors in the estimation by the update function. However, this margin becomes smaller with the definition of every new subproblem.

Since the subproblems are convex, the suboptima are independent on the initial values of the design variables. Therefore it is unnecessary to update the design vector to an equal distribution or fully to $\theta_{\text{new}}$, although the number of iterations may change whether the
design vector is ‘reset’ or not.

Using this update formulation, a series of subproblems is solved sequentially until the range is sufficiently small to approach a continuous variable stiffness panel. In practice, this is depending on the accuracy of the manufacturing machines or user criteria, but here the process is terminated when the range has decreased to a certain value $r_{\text{min}}$.

It should be noticed that although a sequence of convex problems is solved, it can not be guaranteed that the final optimum of the Adaptive Angle Set Method is equivalent to the global optimum of the underlying non-convex problem mentioned in section 4 or the global optimum of the DMO-problem with $r_{\text{min}}$ separated candidate materials. The complete process is graphically summarized in the flowchart of Figure 10.

Figure 10: AASM flowchart
6.3 Update functions

From section 6.2 can be concluded that the update function \( f \) plays a crucial role in the process, since it defines the new subproblem. Ideally, the update function finds the angle \( \theta_{\text{new}} \) which is exactly the best representation of the previous optimum with the use of a sub-optimization routine. However, solving such a sub-optimization problem is equal to solving the overall fiber angle problem, which could not be done efficiently. Therefore, two different heuristic update functions are assigned, whose performances will be examined.

**Linear Combination**

The first update function constructs the new angle by a weighted sum of the angles from the previous set multiplied with their associated design variables. A linear combination as update function expresses Equation (6.5) as:

\[
\theta_{\text{new}} = (x_1^{\text{opt}})^T \Theta = x_1 \theta_a + x_2 \theta_b + x_3 \theta_c.
\]

The main idea behind the linear combination is that if a mixture between two or more candidate materials is obtained, the optimal angle is situated somewhere in between them by ratio of their design variables. This approximation becomes increasingly correct for tighter angle sets.

**Principal Stress Direction**

A second idea for an update function is to use the mechanics of the composite material: its favorable mechanical properties such as stiffness are parallel to the fiber direction. When optimality is reached, the largest principal stress angle is calculated for every element and assigned as \( \theta_{\text{new}} \) for that element. For planar problems the element principal stress angles and values are obtained by calculating the eigenvalues and eigenvectors of the \( 2 \times 2 \) Cauchy stress tensor. The element stresses are obtained from the local displacement vector \( \mathbf{u} \) and the strain-displacement matrix \( \mathbf{B} \), yielding the strains by

\[
\varepsilon = \mathbf{B} \mathbf{u}
\]

and hence the local stresses by:

\[
\mathbf{\sigma} = (x_1 \mathbf{Q}_a + x_2 \mathbf{Q}_b + x_3 \mathbf{Q}_c) \varepsilon.
\]
dimension $3 \times 4 \times N$ where $N$ is the total number of elements. From the nodal stresses $\sigma$ the average stress per element is calculated from which the $2 \times 2$ Cauchy stress tensor is set up by:

$$\sigma_{\text{Cauchy}} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix},$$

(6.11)

Now by calculating the eigenvalues $\omega_{1,2}$ and eigenvectors $V = [V_1 \ V_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ from $\sigma_{\text{Cauchy}}$, the principal stress directions are calculated by

$$\varphi_I = \tan^{-1} \left( \frac{v_{21}}{v_{11}} \right),$$

$$\varphi_{II} = \tan^{-1} \left( \frac{v_{22}}{v_{12}} \right),$$

(6.12)

with corresponding principal stresses

$$\sigma_{I,II} = \frac{1}{2} (\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}}.$$  

(6.13)

Either $\varphi_I$ or $\varphi_{II}$ is selected as the principal stress direction, based on the largest value of $\omega_{1,2}$.

### 6.4 Fiber Angle Optimization

Since the Adaptive Angle Set Method is focused on fiber optimization, a pure fiber angle optimization problem is treated firstly. This means that no topology changes are allowed and all elements must remain a density of 1 (fully solid). For now, only single layer lamina are considered. Both update functions are tested on a compliance minimization subproblem, defined by:

$$\min_{x_{i,j}} \ c = \sum_{i=1}^{N} u_i^T k_i(x_{i,j}) u_i \quad 0 \leq x_{i,j} \leq 1$$

subject to:

$$\sum_{j=1}^{3} x_{i,j} - 1 \leq 0 \ \forall \ i$$

(6.14)

In problem (6.14) $i = 1 \ldots N$ is the element number and $j = 1 \ldots 3$ is the index corresponding to the three candidate materials. $u_i$ and $k_i$ are the element displacement vector and stiffness matrix respectively. The optimization problem has $N$ constraints, stating that the sum of the three design variables per element must be equal or smaller than 1. In a pure fiber angle optimization problem, this constraint is stronger and should be implemented as an equality constraint. Due to the inability of MMA to deal with equality constraints, the constraint is implemented as indicated in (6.14). However, since a compliance minimization problem is considered, there will not be any drive for
the optimizer to drive this constraint smaller than 1 and the equivalent equality constraint is always satisfied. The large number of (equality) constraints, although linear, is disadvantageous for the efficiency of MMA. However, since a convex problem (see next paragraph) is considered here with linear constraints, a possible improvement can be the use of constraint aggregation as proposed in [8].

The sensitivities were obtained by using an adjoint formulation, according to

\[
\frac{\partial c_i}{\partial x_{i,j}} = -\rho_i u_i^T \frac{\partial k_i(x_{i,j})}{\partial x_{i,j}} u_i. \tag{6.15}
\]

Convexity of problem

When recalling the non-convex design space from Figure 6, it can be shown that the subproblems of (6.14) are convex. Since AASM uses more than two design variables to solve the \(2 \times 1\) element problem from Figure 5, the AASM design space can only be visualized by two (even further) reduced problems. The first reduced problem only optimizes the second element, whereas the first element has a fixed angle at \(-78^\circ\), which is known as the optimal value for element 1 by brute force calculations, but may also be arbitrarily chosen. The second element uses the AASM-parameterization of equation (6.2), where \(x_3 = 1 - x_1 - x_2\), which automatically satisfies the constraint in the lower line of 6.14. Summarizing, both element stiffness matrices are calculated according to:

\[
\begin{align*}
    k_1 &= k_{-78} \text{ FIXED AT } -78^\circ, \\
    k_2 &= x_{2,1} Q_a + x_{2,2} Q_b + (1 - x_{2,1} - x_{2,2}) Q_c. \tag{6.16}
\end{align*}
\]

The design space can now be plotted by calculating all possible combinations of \(x_1, x_2 = [0, 1]\) with a 0.01 step size, yielding the plot of figure 11.

Figure 11: Design space for a 2x1 element problem, \(\theta_1\) is fixed.
Assuming a smooth function behavior, Figure 11 clearly shows a convex design space. The blue triangle in the lower left corner indicates non-feasible designs, where the sum of $x_1$ and $x_2$ is greater than 1. The matrices $Q_a$, $Q_b$ and $Q_c$ represent an arbitrarily chosen angle set. Changing the set only results in heightening or lowering the corners of the feasible triangular space.

The second reduced subproblem can be formulated by optimizing both elements, but now only two candidate materials are in the angle set. The element stiffness matrices are now calculated by:

\[
\begin{align*}
k_1 &= x_{1,1}Q_a + (1 - x_{1,1})Q_b, \\
k_2 &= x_{2,1}Q_a + (1 - x_{2,2})Q_b.
\end{align*}
\]  
(6.17)

Although not indicated in problem (6.17), the angle set $\{Q_a, Q_b\}$ may be different for the two elements. The design space of problem (6.17) is plotted in Figure 12.

Figure 12: Design space for a 2x1 element problem, 2 candidate materials

In Figure 12 the entire design space is feasible, and again the design space is convex. Now, for larger problems either the number of elements or the number of candidate materials is increased, which should however not affect the convexity. The design space of the full AASM $2 \times 1$ elements problem is 4-dimensional, since at least 4 design variables are required to describe the objective function. Although not visualized, the convexity of this 4-dimensional problem can also be shown using a convex hull approach. A convex hull of a set of points is the smallest convex set which contains all the points of the set. The set of points can be considered as convex if the set is equal to its convex hull, meaning that all points are part of the convex hull, which is a convex set by definition.

If the $2 \times 1$ element problem is reconsidered once again, now using the full AASM parameterization with 3 candidate materials per element, the design space can be described
by 4 design variables, by combining (6.16) and (6.17). A smooth function behavior is assumed such that the design space can be explored with a limited number of interpolation points. When a step size of 0.04 is used to interpolate between the domain of the four design vectors $0 \leq x_{i,j} \leq 1$, the total number of combinations that satisfies the constraints is 105625. The convex hull is calculated by the union of the Delaunay Triangulation of the finite set of points and contains also 105625 unique vertices. This means that every point in the set is part of the convex hull and the set can be considered as convex. Some additional explanation on this technique can be found in Appendix A.

Results

Now AASM is applied to several test problems to measure its performance. The reduction factor $\gamma$ mentioned in section 6.2 is kept constant at 0.7. The angle set is reduced until a resolution of $1^\circ$ is reached. Given the initial range $r_1 = 60^\circ$, means that a sequence of

$$p = \left\lceil \log_{60^\circ} \frac{1^\circ}{60^\circ} \right\rceil = 12$$

(6.18)

subproblems has to be solved to reach the desired resolution. In Equation (6.18) the ceiling function is used to solve an integer number of problems. A subproblem is considered as converged when the change of objective function and constraint functions $g$ is less than 0.001, i.e. $\max(|\Delta c & \Delta g|) < 0.001$.

At first, the problem of Figure 5 is treated, but now parameterized with AASM, which is illustrated in Figure 13. Secondly, the $2 \times 2$ element problem from Figure 14 is treated. These two small problems are chosen, since the limited number of design variables makes it possible to calculate the global optimum by a brute force approach and hence compare the performance of AASM to a global reference.

![Figure 13: 2 $\times$ 1 elements cantilever problem, parameterized with AASM](image)

![Figure 14: 2 $\times$ 2 elements cantilever problem, parameterized with AASM](image)

The orthotropic material properties for the fibrous laminate are listed in Table 2 and are the same as used by Bruyneel in [4].
Table 2: Material Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{11}$</td>
<td>$146.86 \cdot 10^9$ Pa</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>$10.62 \cdot 10^9$ Pa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.33</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>$5.45 \cdot 10^9$ Pa</td>
</tr>
</tbody>
</table>

Figure 15 shows the optimized results for the linear update function in 15a and the principal stress angle based update function in Figure 15b respectively.

The global optimum of the underlying problem from Figure 5 was calculated using a brute force approach and yields the optimal angles as $\Theta_{\text{opt}} = \{-78^\circ, -55^\circ\}$ respectively. From Figure 15 can be concluded that the linear update function is exactly able to find the global optimum, whereas the $\varphi$-based method finds an optimum very close by, which objective function value is only 0.2% higher than the global optimum.

For the problem shown in Figure 14, both AASM update functions are providing the results showed in Figure 16.

The global optimum of the underlying fiber angle problem was calculated at $\Theta_{\text{opt}} = \{18^\circ, -72^\circ, -28^\circ, -44^\circ\}$ with a corresponding compliance of $c = 4.25$. In this case, none of the AASM update functions is able to find the global optimum, however the obtained compliances are less than 1% higher. So both AASM optima are situated very closely to the global optimum.
Given the promising results from the previous subsection, the obvious next step is to examine AASM in combination with topology optimization. In the simplest implementation, only one global volume constraint has to be added to the optimization problem (6.14), which can be rewritten as:

$$\min_{x_{i,j}} \quad c = \sum_{i=1}^{N} u_i^T k_i(x_{i,j}) u_i \quad 0 < x_{i,j} \leq 1,$$

subject to:

$$\frac{\sum_{j=1}^{3} x_{i,j}}{V} - 1 \leq 0 \quad \forall \ i,$$

where $V$ is the total allowable volume, given as the product of the volume fraction $V$ and the total number of elements $N$, i.e.:

$$V = V \cdot N \quad 0 < V \leq 1.$$

For such a compliance minimization problem, the optimizer is forced to void elements due to the volume constraint. For other objective functions, like for example weight minimization with stiffness or stress constraints, this volume constraint becomes redundant.

For this formulation, it can immediately be seen that there is an additional difficulty with the linear update function as defined in (6.8). According to the volume constraint, the sum of the design variables $x_{i,j}$ does not longer add up to unity for every element. An element for which the density is smaller than unity gets an updated angle $\theta_{\text{new}}$ which goes towards $0^\circ$ if the element density becomes smaller. Even though a (partly) voided element contributes less to the total stiffness, a $0^\circ$ angle is likely a poor representation.
of the true optimum. Therefore, the new angle $\theta_{\text{new}}$ is scaled with the element density which is the sum of the three design variables according to:

$$\theta_{\text{new}} = \frac{(x_1^{\text{opt}})^T \Theta_1}{\sum_{j=1}^{3} x_{i,j}}$$

(6.21)

Results

The scaled linear combination and the principal angle based update function are tested on the $2 \times 2$ cantilever problem from Figure 14. The volume fraction $\mathcal{V} = \frac{3}{4}$ i.e. one element has to be voided. The results for both update strategies are shown in Figure 17.

\[\Theta = \{14^\circ \ -73^\circ \ -30^\circ \ -42^\circ\}, \ c = 7.68\]  
(a) Linear update function

\[\Theta = \{13^\circ \ -72^\circ \ -31^\circ \ -42^\circ\}, \ c = 7.67\]  
(b) $\varphi$-based update function

Figure 17: $2 \times 2$ elements cantilevered beam results, $\mathcal{V} = \frac{3}{4}$

The brute force global optimum was calculated by manually voiding the upper right element, which for this problem is obviously the element that has to be void in a clear black-white topology design. The optimal angle set for the remaining three elements turns out to be $\Theta^{\text{opt}} = \{-15^\circ \ -56^\circ \ -45^\circ \ -\}$ with corresponding compliance $c = 8.46$.

Topology fixing phenomenon

Although both update methods have reached an optimum with an even lower compliance, it can immediately be seen that the results do not have a clear black-white topology design, but contain a mixture between elements 1 and 4 respectively. This is indicated by the lighter gray fill of those elements with half-densities, whereas elements 2 and 3 are fully solid (dark gray fill). Since the design variables are both optimizing the fiber angles as well as the topology, the absence of penalization ‘allows’ the optimizer to create grey elements. So in all subproblems, the angle set is optimized for a gray topology design.
Only at the final, penalized stage according to Equation (6.3), gray designs become unattractive. However, at the final stage the angle set range has decreased so far, that the optimizer is unable to create a black-white topology design with the current angle set, since a good performing black-white topology design would likely require a different angle set. This theory is supported when the performance of the nearest black-white design is computed for the angle sets given in Figures 17a and 17b respectively. The corresponding compliances were \( c = 12.61 \) and \( c = 14.11 \), respectively, which is a lot higher than the compliances from the gray designs. It is assumable that despite the penalization, the black-white design is even more unattractive and that the fiber angles have more or less fixed the topology at the final stage. This fixing phenomenon was further endorsed by changing the material properties from Table 2 to a more isotropic material, i.e. \( E_{22} \rightarrow E_{11} \). By doing so, the fiber angles only have a minor influence on the performance. For an almost isotropic material, the method is perfectly able to create a black-white topology design, since gray designs are now more unattractive for the optimizer than the non-matching angle set. However, since the design space is flattened, the termination criteria had to be extended with an inequality stating that the maximum absolute change of the design variables must be smaller than 0.001. This additional criterion was also adopted in further problems to exclude possible disturbances.

### 6.6 Continuous penalization

The most obvious choice of creating a black-white topology design for stronger anisotropic materials is the application of penalization at all subproblems.

![Figure 18: 2x2 elements cantilevered beam results, \( V = \frac{3}{4} \), \( p = 3 \)](image)

\[ \Theta = \{10^\circ, -72^\circ, -32^\circ, -41^\circ\}, \quad c = 7.66 \]

(a) Linear update function

\[ \Theta = \{10^\circ, -70^\circ, -32^\circ, -41^\circ\}, \quad c = 7.68 \]

(b) \( \varphi \)-based update function

The results are visualized in Figure 18 and are still showing light-gray elements. Analysis showed that despite the penalization the (mixed) topology is again fixed from the first subproblem. Furthermore, the continuous penalization causes the subproblems to...
be non-convex and the update becomes based on a local optimum. This also means that there is influence on the final result on how the start vector for the new subproblem is defined. On one hand it can be argued that the design vector should be reset to an equal distribution scaled with the overall volume fraction, such that no material is favored at the initial state. On the other hand it can be argued that the new angle $\theta_{new}$ should be a good approximation of the previous optimum, and favoring $\theta_{new}$ should provide a good starting point for the next subproblem. A drawback of this is however, when $\theta_{new}$ is a poor representation, the optimizer may still converge to a local optimum with a high fraction of $\theta_{new}$. More of less the same arguments hold for the topology: the total volume can be redistributed equally over all elements to provide more freedom of changing the topology with the new angle set. However, it may be harder for the optimizer to create a black-white topology design with a tight angle set if the topology information is lost and entirely gray after every subproblem. Furthermore, keeping the element density after sub-optimization may provide a better starting point for the next sub-optimization. Sometimes, it may also be desirable to start with a rather poor design vector to escape local optima. It was decided to initialize every subproblem with an equally distributed design vector, but preserving the topology information by scaling with the density $\rho$, i.e. $x_{start} = \begin{bmatrix} \frac{\rho}{3} & \frac{\rho}{3} & \frac{\rho}{3} \end{bmatrix}^T$. In this way, the risk of converging to a possible bad update angle is reduced. It should be noted that even though the suboptima are now forced to become distinct, the sequence of problems provides some space to compensate for this penalization effect.

### 6.7 Isotropic Pre-Optimization

In order to reduce the effect of the fixing phenomenon, a pure topology problem under gradually increasing penalization (common SIMP-method) with an isotropic material is solved first. Here after the new angle set is constructed by one of the two update functions and the sequence of combined problems is solved using AASM. In this way, the topology is not directly affected by the orthotropic material from the start. The results on the same $2 \times 2$ elements problem for this strategy, called isotropic pre-optimization, are shown in Figures 19a and 19b respectively.

For both update functions, a black-white topology design is now obtained. For all tested cases, isotropic pre-optimization yielded results with less presence of gray elements. The drawback of solving an isotropic subproblem is however, that there is an improved chance of just optimizing the fiber angles for the isotropic topology from the pre-optimization, rather than simultaneously optimizing both the topology as the fiber angles. For the $2 \times 2$ problem it is known that the optimal isotropic topology is equal to the orthotropic topology, but for complex problems this may certainly not hold in general. It should be noted that there is still freedom to change the topology, but initiating the sequence of problems with the isotropic topology together with the earlier mentioned fixing problem, definitely makes this approach dependent on how well the isotropic topology is representing the real optimal topology.
\[ \Theta = \{-9^\circ, -47^\circ, -44^\circ, -\\}, \quad c = 8.96 \]

(a) Linear update function

\[ \Theta = \{-18^\circ, -62^\circ, -46^\circ, -\\}, \quad c = 8.91 \]

(b) \(\varphi\)-based update function

Figure 19: 2 \times 2 elements cantilevered beam, isotropic pre-optimization

6.8 Separate Topology Variable

Despite the improved results of isotropic pre-optimization, there is also the disturbing effect by the simultaneous topology optimization of the fiber angles, since the same design variables are used to optimize both problems. The fiber angle subproblems are forced to become penalized and hence end up with distinct solutions, which is unwanted since a mixed suboptimal design vector provides useful information for the update function. For this reason, a separate density variable \(\rho\) for the element topology is introduced and the element material tensor from equation (6.3) is now determined by:

\[ C = \rho^q \left( x_1^p C_a(\theta_a) + x_2^p C_b(\theta_b) + x_3^p C_c(\theta_c) \right). \] (6.22)

Using the formulation of Equation (6.22) the density variable \(\rho\) can be penalized independently with a penalization factor \(q\), whereas the fiber angle design variables \(x_{1,2,3}\) can remain unpenalized \((p = 1)\) until the final subproblem. The subproblems are again non-convex due to the multiplication of design variables. This was also indicated by the number of unique points of the convex hull, which is much smaller than the number of points of the non-linear design space. Another drawback of this method is the need of an additional design variable per element, which slows down the iteration process. Since the topology design is treated separately, the fixing phenomenon is absent and hence isotropic pre-optimization becomes redundant. The benefits of the separate density variable become particularly visible in cases where there is a clear difference between the optimal isotropic and orthotropic topology. At first, the common 2 \times 2 cantilever problem with 75% volume fraction is presented in Figure 20. Next a larger 8 \times 8 problem is shown in Figure 23, which is a left hand sided clamped beam under a vertical load of 10 kN downwards, acting on the right hand edge at \(y = 1\). For this problem, a volume constraint stating that of \(11/16\) of the total volume must remain solid was implemented.
From Figure 20b can be obtained that the $\varphi$-based update function almost finds the same optimum as the $\varphi$-based update function in combination with isotropic pre-optimization in Figure 19b. This can easily be explained by the fact that in this case the optimal isotropic topology is equal to the optimal orthotropic topology, which was found by both strategies. The principal stress directions used for the update coincide and more or less the same optimum is found. For the linear update function however, the compliance is significantly increased. The nature of this decline is discussed in section 7.1.

Figure 21: $8 \times 8$ elements Middle point loaded cantilevered beam
It is interesting to mention that the topology after isotropic pre-optimization is the same as the final topology in Figure 21a, which clearly differs from the topology from Figure 25b, which was optimized using a separate density variable. When both compliances are compared, there is a significant improvement of 13%. A drawback of the method is that the multiplication with the additional design variable significantly increases the iteration time. For a simple $2 \times 2$ problem, the iteration time was already increased with roughly 750% whereas the depicted $8 \times 8$ problem required around 3500% more iteration time compared to the isotropic pre-optimization case. Despite this drawback, the promising results due to a better handling of the topology optimization, make the formulation of equation (6.22) recommended for AASM.
7 Analyses on functionality

Although section 6 showed promising results, the method will be subjected to a closer look in the following subsections. At first, both update functions were checked on their performance, other parameters such as the reduction factor and optimizer will be discussed afterwards. When the result of Figure 20 is compared to the brute force optimum of the $2 \times 2$ element problem from Figure 14, both update functions were still unable to reach the global optimum. A closer look at the history of both updates provides insight on the imperfections of the update functions.

7.1 Linear update function

The linear update function suffers from the weakness that it poorly updates in situations where the suboptimal vector corresponds to high fractions of the upper and lower bound of the angle set and rather a low fraction of the center angle. Such a suboptimal vector will be introduced as a ‘split’-solution and may particularly occur when the fiber angle design variables are not penalized. It is attractive to assign an element under shear loading with both the minimal and maximal angle from the set, since the high stiffness property of the composite material in the fiber direction can not be represented with a single angle in this case. An illustration to this example is the problem of Figure 22, where a $2 \times 2$-element problem is considered, clamped on the left and vertically loaded with equal forces pointing in the upward and downward direction. This problem was optimized while considering only fiber angles, i.e. all elements must remain solid.

![Figure 22: 2 × 2 elements design problem, double vertically loaded](image)

For the situation in Figure 22, both the linear and the principal stress based update functions were used to solve the problem, yielding to the results in Figure 23a and 23b respectively. For the problem in Figure 22 the global optimal angle set $\Theta_{BF}$ with corresponding compliance $c$ was also calculated using a brute force approach:

$$\Theta_{BF} = \{-80^\circ, -89^\circ, 80^\circ, 89^\circ\}, \quad c = 1.77.$$  \hspace{1cm} (7.1)
\[ \Theta = \{31^\circ, 54^\circ, -31^\circ, -54^\circ\}, \ c = 8.74 \]

(a) Linear update function

\[ \Theta = \{-79^\circ, -89^\circ, 79^\circ, 89^\circ\}, \ c = 1.77 \]

(b) \( \varphi \)-based update function

Figure 23: Double loaded 2 \( \times \) 2 elements problem results

From Figure 23a can immediately be seen that the angle set of the linear update function deviates heavily from the global optimal angle set. The corresponding compliance is almost 5 times higher, whereas the principal stress direction update function in Figure 23b is almost able to find the global optimum with roughly the same objective function value. These results are in contrast with the earlier treated pure fiber angle optimization problems from section 6.4, where the linear update function ended up with results very close to the global optimum. An investigation to the history of the updates between the subproblems provides insight on how the - poorly performing - angle set from Figure 23a is formed.

For readability reasons, only the update for the second element (lower right, see Figure 22) is displayed here, however, the same argumentation holds for the first element. Due to symmetry, the third and fourth element are updated analogue to the first and second element respectively, only with mirrored design vectors.
The update history for the 1st, 2nd, 3rd and 5th subproblem is given by

\[ P_1: \Theta_{I,2} = \{-60^\circ, 0^\circ, 60^\circ\}, \quad x_{I,2}^{\text{opt}} = \begin{bmatrix} 0.46 \\ 0.1 \\ 0.44 \end{bmatrix} \rightarrow \theta_{\text{new}} = -1^\circ \]

\[ P_{II}: \Theta_{II,2} = \{-43^\circ, -1^\circ, 41^\circ\}, \quad x_{II,2}^{\text{opt}} = \begin{bmatrix} 0.18 \\ 0.41 \\ 0.41 \end{bmatrix} \rightarrow \theta_{\text{new}} = -1^\circ \]

\[ P_{III}: \Theta_{III,2} = \{-30^\circ, -1^\circ, 28^\circ\}, \quad x_{III,2}^{\text{opt}} = \begin{bmatrix} 0.38 \\ 0.62 \\ 0.01 \end{bmatrix} \rightarrow \theta_{\text{new}} = 6^\circ \]

\[ P_V: \Theta_{V,2} = \{13^\circ, 23^\circ, 33^\circ\}, \quad x_{V,2}^{\text{opt}} = \begin{bmatrix} 0.01 \\ 0.0 \end{bmatrix} \rightarrow \theta_{\text{new}} = 33^\circ. \]

respectively (decimal values are rounded towards nearest integer angles, for readability reasons). From the first two subproblems \( P_1 \) and \( P_{II} \) can be seen that the (global) suboptimal vectors \( x_{I,2}^{\text{opt}} \) and \( x_{II,2}^{\text{opt}} \) are showing a ‘split’ solution and that the update calculates a new angle \( \theta_{\text{new}} \) around the center angle. From subproblem \( P_{III} \) the right hand angle from the set becomes slightly favored, and from subproblem \( P_V \) till the last subproblem the suboptimal design vector fully assigns to the right hand angle. From this point, the update function tries to reach the global value of 91° (equivalent to -89° from Equation (7.1)). However, due to reduction of the distance in the angle set and the inability of the update function to ‘escape’ the set within one subproblem, the global value could not be reached anymore and the final solution sticks with an angle of 54°. The principal stress based method however can create a new angle \( \theta_{\text{new}} \) outside the current angle set: although the same design vector \( x_{I,2}^{\text{opt}} \) is (obviously) obtained after the first subproblem, the principal stress based method updates the new angle for the second element immediately as \( \theta_{\text{new}} = -84^\circ \) and hence is less sensitive for split suboptima.

The problems with the linear update function in combination with split solutions can (partly) be circumvented by using a larger reduction factor. In this way there is more space available for the update function to escape from the current angle set by repeatedly assigning the entire design vector to one of the bounds. However, the larger number of subproblems that has to be solved to reach a sufficient high angle resolution is disadvantageous for the efficiency of the method. If the reduction factor \( \gamma \) was increased from 0.7 to 0.9, a total number of 40 instead of 12 subproblems had to be solved in order to reach the same final angle distance of 1°, requiring a three times longer solving time. However, the final angle set after 40 subproblems yields \( \Theta = \{-71^\circ, -92^\circ, 71^\circ, 92^\circ\} \) with a corresponding compliance \( c = 1.79 \). If the reduction factor was further increased to \( \gamma = 0.95 \) and 80 subproblems were solved, there was also enough space for the first element angle to reach a value at almost the global optimal angle of 80°. This small example shows that a lot of subproblems may be required to overcome the problem of split suboptima and the correct value of the reduction factor is problem dependent. Especially for larger problems, just adopting a large reduction factor while solving a large
number of subproblems will be rather expensive.

Secondly, the occurrence of split solutions can be reduced by applying penalization, forcing the intermediate solutions towards a 0-1 optimum. For the problem from Figure 22 the linear update function with \( p = 3 \) finds an angle set of \( \Theta = \{-53^\circ, 91^\circ, 53^\circ, -91^\circ\} \), with a compliance \( c = 1.87 \). Although there is a significant improvement in angle set and compliance, the benefits of penalization are limited. First of all, split solutions can not always be avoided if the penalized mixture still performs better than the distinct solution. Secondly, the penalized suboptimal design vector may be fully assigned to the center angle, i.e. \( x_{\text{opt}} = [0 \ 1 \ 0] \) which still results in the same \( \theta_{\text{new}} \). Lastly, the design vectors may jump from fully left-bound \( ([1 \ 0 \ 0]) \) to fully right-bound \( ([0 \ 0 \ 1]) \) from subproblem to subproblem, resulting that after a number of subproblems, the center angle is still close to the original one. All three occasions were observed in the 2 x 2 problem from Figure 22 with penalization. Therefore, penalization does not provide a robust solution to this problem, beside the drawback of additional non-convexity.

The formation of split solutions is particularly unfavorable in the first subproblems, where the distance between the angles is large and selection of the center angle as \( \theta_{\text{new}} \) means a huge difference from one of the bounds.

7.2 Principal stress based update function

A shortcoming of the principal stress based update function could also be observed, when analyzing the results of Figure 19b. The second fiber angle of -62° is a rather large deviation from the earlier determined global optimal angle of -56°. The origin of this deviation was again found when the update history for this element was examined. In (7.3) the 8th and 9th subproblem are displayed successively.

\[
\mathcal{P}_{\text{XIII}} : \Theta_{\text{XIII},2} = \{-66^\circ, -63^\circ, -60^\circ\}, \quad x_{\text{opt}}^{\text{XIII},2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \theta_{\text{new}} = -63^\circ
\]

\[
\mathcal{P}_{\text{IV}} : \Theta_{\text{IV},2} = \{-65^\circ, -63^\circ, -61^\circ\}, \quad x_{\text{opt}}^{\text{IV},2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \theta_{\text{new}} = -63^\circ
\]

The updated angle from subproblem \( \mathcal{P}_{\text{XIII}} \) is -63°, although the suboptimal design vector \( x_{\text{opt}}^{\text{XIII},2} \) clearly indicates that -63° is not the optimal value for this element, otherwise the optimizer would have selected -63° directly from the angle set of \( \mathcal{P}_{\text{XIII}} \). However the principal stress direction apparently is -63° and is thus selected as \( \theta_{\text{new}} \), although the design vector indicates that the optimal angle is -60° or even higher. In the next problem however, the set is reduced and only values lower than -60° are part of the set. The design vector \( x_{\text{opt}}^{\text{IV},2} \) again indicates that the optimal angle is on the right hand side of the angle set, but \( \theta_{\text{new}} \) is again calculated at -63°. Due the reduction of the set, the method is unable to reach the greater optimal value of -56°. In this way, the principal
stress based update function can force an element angle towards the principal stress angle, although this angle may differ from the real optimal angle. The principal stress based update function is particularly sensitive for this error in the later stages, where the angle set distance is already small and the real optimal angle is outside the entire angle set.

7.3 Combined update functions

Considering the drawbacks of both methods, it has been mentioned that the linear update function is sensitive for errors in the early stage, whereas the principal stress direction update function fails to determine the real optimal angle at the later stage. In order to solve this problem both algorithms were used. An additional statement was added to the algorithm which identifies whether one of the update functions fails according to the situations described in the previous subsections. Since both update functions are failing in another event, there is always one update function that can handle the situation correctly. When none of the update functions fails, the principal stress based update function is used, since this update function generally requires the smallest number of iterations to converge to a suboptimum, according to the numbers of Table 3.

<table>
<thead>
<tr>
<th>Problem size</th>
<th>Volume fraction</th>
<th>Linear Update iterations</th>
<th>ϕ-Based Update iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 × 2</td>
<td>1</td>
<td>499</td>
<td>249</td>
</tr>
<tr>
<td>2 × 2</td>
<td>3/4</td>
<td>384</td>
<td>258</td>
</tr>
<tr>
<td>4 × 4</td>
<td>1</td>
<td>1534</td>
<td>340</td>
</tr>
<tr>
<td>4 × 4</td>
<td>11/16</td>
<td>814</td>
<td>334</td>
</tr>
<tr>
<td>8 × 8</td>
<td>11/16</td>
<td>1466</td>
<td>899</td>
</tr>
</tbody>
</table>

The three occasions for which the linear update function is used are identified with three if-statements: for every element the pseudo-code (7.4) is evaluated whether the problem described in section 7.2 occurs.

\[
\text{if } \begin{cases} 
(x^{opt} = [x_1 > 0.95 \ x_2 \ x_3]^T \text{ and } \varphi > \theta_b) \\
(x^{opt} = [x_1 \ x_2 \ x_3 > 0.95]^T \text{ and } \varphi < \theta_b) \\
x^{opt} = [x_1 \ x_2 > 0.95 \ x_3]^T
\end{cases}
\]

USE LINEAR UPDATE FUNCTION

else

USE ϕ-BASED UPDATE FUNCTION

end

In code (7.4) \( \varphi \) is the dominant principal stress angle \( \varphi_1 \) or \( \varphi_{II} \) as described in section 6.3. Clearly the code (7.4) only works for pure fiber angle optimization problems or with
the separate density variable in combination with topology optimization, since the fiber design variables have to sum up to 1.

Results

A number of test problems were treated using the strategy as introduced in the previous subsection. These test problems include the $2 \times 2$ and $8 \times 8$ elements cantilever beam, the opposite vertically loaded $2 \times 2$ elements beam problem from Figure 22 and the $8 \times 8$ cantilever loaded on the middle right end. The implementation settings can be found in Appendix B.

\[ \Theta = \{-15^\circ, -54^\circ, -45^\circ, -\}\}, \ c = 8.57 \]

(a) $2 \times 2$ element problem, cantilever beam

\[ \Theta = \{-78^\circ, -89^\circ, 78^\circ, 89^\circ\}, \ c = 1.77 \]

(b) $2 \times 2$ elements, opposite loaded beam

From the results in Figures 24a and 24b, it can be seen that the combined update function strategy produces better or equally results than the same problems treated with a single update function (Figure 20b and Figure 23b respectively). The results for the $8 \times 8$ element problems are given in Figures 25a and 25b respectively. The results of Figure 25 show that the combined update function algorithm reaches optima better or equal as good as the best performing single update algorithm for almost all cases, as overviewed in Table 4. The only exception is the $8 \times 8$ cantilever problem, where merely use of the linear update function yields a better compliance. This is further discussed in subsection 7.5. The single update data of column 2 and 3 in Table 4 are obtained using the formulation of equation (6.22).
(a) $8 \times 8$ element problem, cantilever beam

(b) $8 \times 8$ elements, middle loaded cantilever beam

Figure 25: $8 \times 8$ elements test problems, combined update functions

<table>
<thead>
<tr>
<th>Problem</th>
<th>Volume fraction</th>
<th>Linear Update compliance</th>
<th>$\varphi$-Based Update compliance</th>
<th>Combi Update compliance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 2$ Cantilever</td>
<td>$3/4$</td>
<td>9.71</td>
<td>8.89</td>
<td>8.57</td>
</tr>
<tr>
<td>$2 \times 2$ Opposite load</td>
<td>1</td>
<td>8.74</td>
<td>1.77</td>
<td>1.77</td>
</tr>
<tr>
<td>$8 \times 8$ Cantilever</td>
<td>$11/16$</td>
<td>4.75</td>
<td>4.86</td>
<td>4.86</td>
</tr>
<tr>
<td>$8 \times 8$ Mid. cantilever</td>
<td>$11/16$</td>
<td>0.79</td>
<td>0.91</td>
<td>0.79</td>
</tr>
</tbody>
</table>

7.4 Iteration history

The iteration history for the objective and constraints is monitored for the combined update function strategy. In Figure 26a the iteration history for the $2 \times 2$ element problem with 75% volume fraction from Figure 24a was displayed. Figure 26b displays the development of the angle values from the update function (in red) and the history of the densities (in blue), after the first update. Figure 26a shows that the combination of update functions is converging towards one single angle, while only slightly increasing the compliance. The kinks corresponds with the events where the angle set is updated and a new subproblem is formulated. For every subproblem, a smooth convergence is observed. The peak around 1000 iterations has its origin at the point where the end penalization of $p = 3$ on the fiber angle design variables is applied.

In Figure 26b, there are three lines at $y_2 = 1$, only they are completely overlapping. For this particular example it can be seen that the optimal topology is found directly after the first subproblem, since the densities remain at the same value for all subproblems. The red lines are indicating the estimated optimal angle $\theta_{new}$ after update. Here some changing over the sequence of problems is obtained, which underlines that just defining
the principal stress direction after the first subproblem not automatically equals the true optimal angle for the problem.

7.5 Reduction Factor influence

Another parameter of influence in the process is the reduction factor $\gamma$, which has not been investigated yet. As mentioned in Equation (6.18) the reduction factor was equal to 0.7 for all obtained results. A motivation for this choice and the influence of this parameter will be given by examining various test problems with different reduction factors. The results, summarized in Table 5, are obtained by the strategy of combined update functions with a separate topology variable, as described in the subsection 7.3. From Table 5 can be obtained, that there only is a minor influence of the reduction factor on the final result. Only when the reduction factor becomes very small ($\gamma = 0.3$) and only 4 subproblems have to be solved to reach a 1° angle resolution, the reduction factor significantly lowers the performance. Naturally spoken a larger reduction factor is preferable for the performance, since more space is provided to escape from poorly chosen angle sets. This was also mentioned in section 7.1, however the analyzed problem here was already solved by using a combination of update functions. Table 5 also shows that a smaller reduction factor not always leads to a smaller amount of total iterations or evaluation time. This is probably caused by the fact that it takes more iterations for the optimizer to converge when the new angle set is much tighter than the previous one, which is the case for small reduction factors.

Given these results, a reduction factor of 0.7 was chosen as the standard for all test problems treated in this research, since a relatively large reduction factor should theoretically provide more space to escape bad intermediate solutions and is therefore ‘more safe’. The total evaluation time is only a little bit higher for most of the cases than for a reduction factor of 0.5. For practical use, a reduction factor of 0.5 should be satisfying,
Table 5: Influence of reduction factor

<table>
<thead>
<tr>
<th>Problem</th>
<th>Volume Fraction</th>
<th>$\gamma$</th>
<th>Compliance</th>
<th>Iterations</th>
<th>Evaluation Time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 2$ Cantilever</td>
<td>$\frac{3}{4}$</td>
<td>0.3</td>
<td>8.82</td>
<td>880</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>8.57</td>
<td>979</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7</td>
<td>8.57</td>
<td>1137</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9</td>
<td>8.56</td>
<td>2472</td>
<td>89</td>
</tr>
<tr>
<td>$2 \times 2$ Opposite Load</td>
<td>1</td>
<td>0.3</td>
<td>1.90</td>
<td>1100</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>1.77</td>
<td>497</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7</td>
<td>1.77</td>
<td>741</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9</td>
<td>1.77</td>
<td>999</td>
<td>43</td>
</tr>
<tr>
<td>$4 \times 4$ Cantilever</td>
<td>$\frac{11}{16}$</td>
<td>0.3</td>
<td>5.17</td>
<td>963</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>5.00</td>
<td>1019</td>
<td>93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7</td>
<td>5.00</td>
<td>958</td>
<td>89</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9</td>
<td>5.00</td>
<td>1450</td>
<td>140</td>
</tr>
<tr>
<td>$8 \times 8$ Cantilever</td>
<td>$\frac{11}{16}$</td>
<td>0.3</td>
<td>4.93</td>
<td>2012</td>
<td>1606</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>4.84</td>
<td>2095</td>
<td>2176</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7</td>
<td>4.85</td>
<td>2112</td>
<td>2048</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9</td>
<td>4.73</td>
<td>2832</td>
<td>1990</td>
</tr>
</tbody>
</table>

since there is hardly any loss in performance but still some savings on evaluation time. Only for the $8 \times 8$ problem, the larger reduction factor of $\gamma = 0.9$ provides a significantly better optimum than the smaller reduction factors. This optimum is even better than the best performing update strategy for this problem, see the third row of Table 4. Surprisingly, the evaluation time for $\gamma = 0.9$ is even lower than for $\gamma = 0.5$ or $\gamma = 0.7$, although the number of iterations is still higher. At last should be noted that the evaluation time may rely on some intervening variables, such as small differences between student computers, network business and software versions. Therefore, the number of iterations is a more reliable indication for the time demand. Based on this and earlier arguments, $\gamma = 0.7$ is kept as the reduction factor for the test problems in this research.

### 7.6 Optimizer

All presented results in this report were obtained by using the Method of Moving Asymptotes (MMA) as optimizer, which is a widely used optimizer for topology optimization problems. It is a powerful algorithm to handle problems with a large amount of design variables with relatively few constraints. However, the multi-material like implementation of the subproblems, requires a large number of local constraints to prevent the selection of more than one candidate material per element. These constraints are required when three or more candidate materials are implemented per element and are disadvantageous for the efficiency of the MMA-algorithm. Although this research was not focused on choosing the best optimizer for AASM, a small comparison with Sequen-
tial Quadratic Programming (SQP) will be made in this section. SQP is theoretically better able to handle the large amount of local constraints and also provides the possibility to implement equality constraints directly.

Table 6 shows the results of several test problems treated with MMA and SQP respectively. For all test problems the combined update formulation with a separate topology variable has been used. The SQP algorithm was terminated when the tolerances were smaller than 1e-6.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Volume Fraction</th>
<th>Algorithm</th>
<th>Compliance</th>
<th>Iter</th>
<th>Eval. Time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 1 Cantilever</td>
<td>1</td>
<td>MMA</td>
<td>2.09</td>
<td>538</td>
<td>20</td>
</tr>
<tr>
<td>2 x 1</td>
<td>SQP</td>
<td>2.09</td>
<td>81</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2 x 2 Cantilever</td>
<td>3/4</td>
<td>MMA</td>
<td>8.57</td>
<td>1137</td>
<td>42</td>
</tr>
<tr>
<td>2 x 2</td>
<td>SQP</td>
<td>8.71</td>
<td>962</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2 x 2 Opposite Load</td>
<td>1</td>
<td>MMA</td>
<td>1.77</td>
<td>741</td>
<td>30</td>
</tr>
<tr>
<td>2 x 2</td>
<td>SQP</td>
<td>1.77</td>
<td>138</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4 x 4 Cantilever</td>
<td>11/16</td>
<td>MMA</td>
<td>5.00</td>
<td>958</td>
<td>89</td>
</tr>
<tr>
<td>4 x 4</td>
<td>SQP</td>
<td>5.00</td>
<td>364</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>8 x 8 Cantilever</td>
<td>11/16</td>
<td>MMA</td>
<td>4.85</td>
<td>2112</td>
<td>2048</td>
</tr>
<tr>
<td>8 x 8</td>
<td>SQP</td>
<td>5.31</td>
<td>1030</td>
<td>316</td>
<td></td>
</tr>
<tr>
<td>8 x 8 Mid. Cantilever</td>
<td>11/16</td>
<td>MMA</td>
<td>0.79</td>
<td>2012</td>
<td>797</td>
</tr>
<tr>
<td>8 x 8 Mid.</td>
<td>SQP</td>
<td>0.88</td>
<td>336</td>
<td>129</td>
<td></td>
</tr>
</tbody>
</table>

From Table 6 can be concluded that SQP requires a smaller number of iterations than MMA does and also converges within a shorter timespan. However, the obtained results were equal or worse in performance than MMA. For pure fiber angle problems, a significant difference in objective function could not be obtained. However SQP performs less for the combined topology problems. Especially for the 8 x 8 problems, differences in the final topologies could be seen between both optimization algorithms. Since this research is focused on reaching the best performance of AASM, MMA was used for all test problems in this research, since it always provided the better results and is generally known as a more tranquil algorithm.

7.7 Comparison with DMO

In this subsection, the Adaptive Angle Set Method will be compared to the popular Discrete Material Optimization method, as described in section 5.2. A comparison between both methods will be made both on performance as well on time demand. In order to reach a 1° angle resolution, the DMO method requires 180 different candidate materials representing all possible angle directions and hence 180 design variables per element. Although different parameterization schemes like SFP and BCP were developed to reduce the number of design variables, the design problems always become larger when a higher angle resolution is demanded, whereas AASM solves problems with the same
(small) amount of design variables, independent on the demanded angle resolution.

The results of AASM were compared to the results of a simple DMO-implementation with 180 candidate materials, i.e. the material tensor $C_i$ for an element $i$ is calculated by

$$C_i = \sum_{j=1}^{180} x_{i,j}^p C_j, \quad 0 < x_{\text{min}} \leq x_{i,j} \leq 1,$$  \hspace{1cm} (7.5)

where the penalization factor $p$ is increased from 1 to 3 during optimization.

The results are shown in Table 7.

Table 7: Results DMO

<table>
<thead>
<tr>
<th>Problem</th>
<th>Volume Fraction</th>
<th>Compliance</th>
<th>Iterations</th>
<th>Evaluation Time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 1 Cantilever</td>
<td>1</td>
<td>2.36</td>
<td>126</td>
<td>192</td>
</tr>
<tr>
<td>2 x 2 Cantilever</td>
<td>1</td>
<td>12.95</td>
<td>91</td>
<td>484</td>
</tr>
<tr>
<td>2 x 2 Cantilever</td>
<td>3/4</td>
<td>19.47</td>
<td>113</td>
<td>711</td>
</tr>
<tr>
<td>2 x 2 Opposite load</td>
<td>1</td>
<td>4.19</td>
<td>127</td>
<td>811</td>
</tr>
<tr>
<td>4 x 4 Cantilever</td>
<td>11/16</td>
<td>5.30</td>
<td>141</td>
<td>24107</td>
</tr>
</tbody>
</table>

When Table 7 is compared to the MMA results of Table 6, it can easily be seen that the compliances resulting from AASM are always lower than the results obtained with DMO. Beside, it can be seen that the total evaluation time is much longer for the DMO-parameterization, despite using a comparable amount of iterations. This can easily be explained by the fact that the sequence of problems solved by AASM requires a lot of iterations, but the evaluation time is a lot shorter, due to the use of relatively few design variables.

The results from Table 7 contain some mixtures between angles and densities, which probably originates from the implementation of DMO in its simplest formulation. The original DMO paper [19] proposed a different parameterization to create a better distinct solution. However, it was stated that this mainly purposes an improvement of the physical meaningfulness of the solution and less of the objective function. Although the compliances in Table 7 are partly based on mixtures, their values are still higher than the AASM results.
8 Multi-layer Laminates

In this section a brief extension will be made to investigate the performance of the Adaptive Angle Set Method on multi-layer laminates. Only a small amount of test problems will be treated, just to investigate the treatment of the multiple layers. As for the single layer lamina, all test problems are limited to in-plane cases, meaning that the bending matrix $D$ and the coupling matrix $B$ from the ABD-formula (3.1) of section 3 are not involved. The extensional stiffness matrix $A_i$ for every element is calculated by:

$$A_i = h_m \sum_{m=1}^{M} \bar{Q}_{m,i}. \quad (8.1)$$

In Equation (8.1) $h_m$ is the thickness of layer $m$, which is assumed to be equal for all elements within one layer. $\bar{Q}_{m,i}$ is the rotated stiffness matrix associated with the layer $m$ and element $i$, whereas $M$ is the total number of layers.

Since the $B$ and $D$ matrices are zero for in-plane cases, the stacking sequence of the layers becomes arbitrary. The main goal is to investigate whether AASM is able to treat multi-layer laminates correctly and end up with distinct solutions with a high angle resolution. The additional layers provide a larger design freedom. In theory, due to the enlarged design freedom, the multi-layer solutions should be able to represent solutions which are physically impossible to create with a single layer lamina, such as mixtures between angles and half-densities. Therefore, the expectation is that the multi-layer solutions have a better performance, i.e. a lower compliance than the equivalent single-layer test cases. In order to compare both situations, the total sum of layer heights is always equal to the height of the single layer case.

8.1 Fiber Angle Optimization

The amount of design variables required for a multiple layer optimization problem is increased proportional to the total amount of layers. The same relation holds for the amount of local constraints, meaning that multiple layer optimization problem can easily grow large and complex. However, the industry standard of mostly using balanced and symmetric laminates slows down this increase in design variables a bit. For the problems treated in this research, the amount of layers is limited to $m = 2$, just to keep a clear overview of the interaction between both layers.

Analogous to the single layer case, the first problem is the $2 \times 1$ element cantilever beam problem as illustrated in Figure 13. Both elements are now parameterized with a double layer of half the height, according to the sketch of Figure 27.

The result is shown in Figure 28a, where the upper row of set $\Theta$ contains angles of the top layer of element 1 and 2 respectively, whereas the lower row contains the element angles in the bottom layer.

Figure 28a was obtained by AASM as described in the previous section. Immediately can be seen that the angle set is the same for both layers, which are exactly equal to the (global optimal) angles from the single layer case in Figure 13 and hence yielding
the same objective value of 2.09. However for the multiple layer case, this is not the global optimal angle set. The optimizer is unable to optimize both layers independently, despite resulting in a rather poor compliance. The suspicion is that this problem is caused by the fact that here a purely in-plane problem is treated and the optimizer is unable to decide which layer should be placed on top, since this is irrelevant for in-plane problems. A more detailed research should be performed to investigate whether this is the real origin of this problem. This could for example include the consideration of bending problems and see whether the problem is solved or not.

For in-plane cases, this problem was solved by introducing a tiny offset in the initial sets for both layers, such that both layers could be optimized independently. The initial sets for the elements in both layers were set to:

$$\Theta^m_{p=1} = \left\{ \begin{array}{c}
\Theta_1 \\
\Theta_2 \end{array} \right\} = \left\{ \begin{array}{c}
-59^\circ \\
-61^\circ
\end{array}; \begin{array}{c}
1^\circ \\
-1^\circ
\end{array}; \begin{array}{c}
61^\circ \\
59^\circ
\end{array} \right\}. \quad (8.2)$$
While using the initial sets from Equation (8.2), the result of Figure 28b was obtained. This solution shows different angle sets for both layers and a compliance that is decreased by 32%, as a result of the additional freedom by the extra layer.

For the $2 \times 1$ element case, the global optimal angle set could again be determined by a brute force approach, which turned out to be:

$$
\Theta_{BF} = \begin{bmatrix}
-72^\circ & -44^\circ \\
25^\circ & -44^\circ
\end{bmatrix}, \quad c = 1.42
$$

(8.3)

where $c$ is the corresponding compliance value. When this result is compared to the AASM result of Figure 28b, it can be obtained that the AASM angle set has values close to the global optimal values from (8.3), with a corresponding compliance which is only 0.11% higher.

For the $2 \times 2$ cantilever problem, a similar gain in objective function could be noticed with a decrease of 34% in compliance compared to the single layer case from Figure 16.

### 8.2 Combined Fiber Angle and Topology Optimization

The Adaptive Angle Set Method for multiple layers was also examined on combined fiber angle and topology optimization. Here the use of more than one layer is also favorable to represent physically impossible half-densities. The volume constraint is now constraining both layers. Again the $2 \times 2$ element cantilever subproblem is considered, with a volume constraint of $\frac{3}{4}$. The result of this problem is given in Figure 29a. As a comparison, the same problem with single layer is shown in Figure 29b which was optimized used the separate density variable from Equation (6.22). However, the penalization factor $q$ for the density variables $\rho$ was kept at 1. This intentionally creates mixed density values, which could be represented better by the multiple layer case.

$$
\Theta = \begin{bmatrix}
-11^\circ & -67^\circ & -34^\circ & -22^\circ \\
-31^\circ & -40^\circ
\end{bmatrix}, \quad c = 3.49
$$

(a) $2 \times 2$ multilayer, volume constraint

$$
\rho = \begin{bmatrix} 0.59 & 1 & 0.94 & 0.47 \end{bmatrix}, \quad c = 5.03
$$

(b) $2 \times 2$ single layer, $q = 1$

Figure 29: $2 \times 2$ element multi layer comparison
Figure 29b clearly shows that without penalization on the density, it is favorable to create a mixture between (roughly) element 1 and 4. The compliance of 5.03 for this situation is way better than the penalized single layer compliance of 8.57 from Figure 24a. Figure 29a shows nicely that this desired mixture in density is applied by splitting this over the 2 layers, while creating a perfectly physical meaningful solution. The allowance of a different angle in each layer for every element further improves the compliance, such that a total decrease of 31% could be reached. The same behavior could be obtained for a larger $4 \times 4$ problem, as illustrated in Figure 30, where an improvement of 16% in objective function value was reached: $c = 4.22$ for the multi-layer case and $c = 5.00$ for the single layer case (from Table 5).
9 Extensions and Recommendations

In the preceding section, the working principle of AASM is pointed out and discussed. The optimization problems treated in sections 6 and 8 are kept rather small and general. A full investigation is considered to be beyond the scope of this project and hence left for future work. This section describes a number of possibilities which are considered as a worthwhile extension to AASM, in order to better approximate the behavior of real life laminates. The recommendations are divided into extensions on the method itself and manufacturing constraints.

9.1 Extensions to AASM

Degrees of Freedom

At first place, an elaborated research can include out-of-plane bending and coupling of membrane and bending stiffness, such that the stacking sequence of the laminate is no longer redundant. For that case, the rotated stiffness matrix $\bar{Q}$ from Equation (8.1) is also used to calculate the bending stiffness matrix $D$ and coupling matrix $B$ from Equation (3.1). This requires the implementation of finite elements which support both membrane and bending deformations, such as shell elements. Since industrial laminates are mostly used layer by layer to construct thin structures, a further extension to solid elements is probably not necessary. Note that a flat shell element has 5 degrees of freedom per node, whereas the implemented 4-node quadrilateral elements has only 2 DoF per node. Therefore, a significant increase in evaluation time is expected.

Other optimization problems

A large amount of extensions can be made to different optimization problems. As far as this Thesis went, only compliance minimization problems were considered, with constraints only to guarantee physical meaningfulness and a volume constraint to trigger topology optimization. Given the different applications of composite materials, different objective functions are likely worthwhile to be considered. Common in the aerospace industry is weight minimization with constraints on stiffness and/or failure modes like buckling loads or stresses. For unidirectional fibrous composites, a common failure criterion is the Tsai-Wu failure criterion from [21]. The implementation of buckling and/or stress constraints in the optimization procedure is a topic of research on itself.

Multiple Loads

All problems treated in this report were restricted to single load cases, however in reality a component or panel may be applied under various loads. A multiple load case introduces another complexity for AASM, since it is impossible to determine one overall principal stress direction per element. Possibilities to tackle this problem may include the use of a weighted sum of the different load cases or optimize one or more layers set wise for a specific load case. For both situations, the problem should be well-constrained.
in order to meet the combined load case requirements. Other strategies may be the optimization of different load cases separately and find an optimal combination afterwards, which causes a minimal loss of performance.

Layer Thicknesses

In section 8 the layer thickness of both layers was kept constant and equal as half the thickness of the single layer lamina, in order to make comparisons. However, the layer thickness can also be used as an additional design variable, which can even better represent gray topology elements. Furthermore, a design variable for layer thickness is worthwhile for bending problems where the total height of the laminate plays a crucial role. This becomes even more important when more than two layers are considered.

9.2 Manufacturing Constraints

An important aspect in the process of designing a real laminate is the inclusion of manufacturing constraints. Manufacturing constraints may differ depending on the application: commercial companies are seeking for a product of the highest economical value: here lowest production costs, maintenance or user friendliness and durability are keywords. This contrasts with companies demanding the highest possible performance, like research institutes or high end applications such as spacecraft or top sports. Given the end application, different machinery is purchased which is generally a trade-off between costs/speed and performance.

Fiber Placement

The Adaptive Angle Set Method is able to approximate variable stiffness panels, however there are restrictions on how well this mathematically obtained stiffness panel can be manufactured in real life. A fiber has to be placed in a continuous way through the layer, meaning that the discrete solution with sudden changes in fiber angle between the elements is not possible. In literature, different methods are investigated to achieve an optimal fiber placement throughout the elements, of which the modified Fast Marching Method from [7] is a nice example. Next to the continuous way of fiber placement, there is also a limitation on the maximum curvature a fiber can make between adjacent elements. This originates both from the resolution of the machinery, such as Tow Placement Fiber Machines and as well limitations on the (elastic) flexibility of the fibers themselves. A constraint can be implemented stating that the maximum difference in fiber angle between adjacent element should be restricted. Such a constraint is likely to become active on elements where there are high gradients in maximum principal stress directions. For in-plane multilayer problems, such a constraint may also favor a fiber angle to be placed in a certain layer to achieve a more smooth behavior in angle transition.
Topology Restrictions

In many industrial applications the formation of holes in the middle of the structure is undesired. Especially for multiple layer laminates, it is not uncommon to restrict topology changes exclusively to the outer layers for manufacturing reasons. Beside, large variations of the thickness between two adjacent elements can give rise to crack growing and failure in the form of delamination and can be constrained as well.

Stacking Sequence

Industrial laminates are mostly symmetric and/or balanced. For a symmetric laminate the coupling effect from the $B$-matrix from equation (3.1) is canceled out. Other classes of laminates are cross-plied laminates, angle ply laminates and anti-symmetric laminates [3]. These restrictions generally limit the number of design variables and simplify the problem. However, the design freedom becomes restricted, which possibly causes a loss of performance. Obviously, the (maximum) total number of applicable layers should be restricted with a constraint too.

Patches

In industry, continuous variable stiffness panels are not widely used yet, due to their complexity and related with that, their required production time. It is common to assign a single fiber angle to a larger area of elements, a so called patch. The use of patches decreases the number of design variables and eases the manufacturing process. For AASM this means that the update function has to estimate the optimal angle for an entire patch. Therefore the patches should be chosen carefully, since the update becomes less accurate if the stress field is highly changing within one patch. It becomes more challenging to determine a linear combination or principal stress direction which is representative for the entire patch.

A more extreme case of patches is that there is only one patch per layer and that the fiber angle may only vary between the layers. AASM can be used in the same way, but the three candidate materials are now assigned for the entire layer instead of an element in the layer. When patches of entire layers are assigned with the same fiber angle, AASM is especially a useful method, since the fiber angles are calculated with a high resolution. Since larger patches have a greater influence on the total performance, it is particularly favorable to assign such a patch with exactly the right angle. Therefore, the selection of patches should be performed carefully: a large uniformity makes an area suitable for a patch. However, a large area with the same properties can again be sensitive for crack growing and failure of the laminate, which further complexes the choice of a suitable patch.

9.3 Recommendations

The previous two subsections provide a large number of possibilities and potentials to design a laminate which is directly useful for production without the need of extensive
post-processing. This research was however focused on the working principle of AASM and showing its potential on small, but manageable problems. Many of the extensions or manufacturing constraints are already developed for general topology optimization with composite materials and are not directly affecting AASM. Especially the multiple load case and the implementation of shell elements would be a useful next step in the development of AASM, since a close interaction with the update functions is to be expected. Most of the manufacturing constraints should be implemented relatively easily and as history learns, it will be a matter of time before technical solutions are evolved to overcome manufacturing difficulties and justify their economic deployment.
10 Conclusions

In this Thesis, a new method for combined topology and fiber angle optimization for unidirectional fibrous laminates was proposed. This method, called the Adapative Angle Set Method (AASM) is able to reach solutions with a high angle resolution, by solving a sequence of multi-material like optimization subproblems. The element material tensor is calculated by a weighted sum of a set of three candidate materials representing three different fiber directions, pre-multiplied with a density variable for topology optimization. The solution of every subproblem is used by a combination of two update functions to estimate one new angle per element, which represents this subsolution. Penalization on the fiber angle design variables is not applied until the last subproblem, to avoid the risk of updating from a local optimum far from the global solution. A new angle set is formulated around the newly estimated angle and with every new subproblem the distance between the three angles is tightened. In this way, the method converges to a final solution which has a high angle resolution and is directly physically meaningful, while only using 4 design variables per element.

AASM has been tested on several 2-dimensional planar subproblems and angle sets close to the global optimal values were obtained for small problems. The limited number of design variables makes AASM able to reach these solutions within a shorter total evaluation time than a comparable problem with DMO parameterization. For larger problems and multiple layer laminates, the results are considered as promising. Considering the high potential of AASM, extensions to more complex problems, including shell elements, multiple load cases and manufacturing constraints are strongly recommended for further investigation. The inclusion of these extensions will provide a closer approach of the real life laminate behavior and may lead to designs which can be applied for modern day applications without the need of extensive post-processing.
11 Appendix

A: Convex Hull Approximations

In section 6 convex hulls were briefly mentioned as a method to prove convexity of the problem. A convex hull of a set of points in Euclidean space contains the smallest convex set that contains these set of points. A two dimensional example of a convex hull of finite set is given in Figure 31.

A brute force approach was used to research the design space of the problem. The main idea behind this approach is to determine whether the convex hull contains the set of all possible combinations, as obtained by the brute force algorithm. If this is the case, the set equals the convex hull - which is a convex set by definition - and proves that the set is convex.

The convex hull \( \mathbf{k} \) of a set of points is calculated in MATLAB using the command \texttt{convhull} or \texttt{convhulln}, depending on the dimensions. \texttt{convhulln} returns a three-columned matrix where every row contains the indices of the facets of the convex hull. Since one point of the convex hull can be a vertex of more than one facets, the total number of unique indices is calculated by \texttt{size(unique(k))}, in order to determine how many points are part of the convex hull. For the non-penalized pure fiber angle problem, the total number of unique points of the convex hull equals the total number of possible combinations, meaning that the design space is convex.

On the other hand, for penalized problems and problems with multiplication of design
variables, the convex hull of the set of combination contains less unique points, which may indicate that the set is non-convex. However, the non-convexity could be observed by obtaining different (sub)optimal solutions for different start vectors, indicating the existence of local optima and hence non-convexity.

B: Implementation parameters

Appendix B provides an overview of the implementation parameters, for reproducibility. All scripts were implemented in MATLAB versions R2012b or R2013b. The design variables \( x_{i,j} \) were allowed to vary between 0.0001 and 1. A subproblem was considered as converged by a threefold termination criterion, stating limitations to the maximum absolute change in design variables, objective and sum of constraint values between two succeeding iterations respectively:

\[
\begin{align*}
\max(\text{abs}(x_{\text{iter}} - x_{\text{iter-1}})) &< 0.001, \\
\text{abs}(c_{\text{iter}} - c_{\text{iter-1}}) &< 0.001, \\
\max(\text{abs}(g_{\text{iter}} - g_{\text{iter-1}})) &< 0.001.
\end{align*}
\] (11.1)

In equation (11.1) \( x \) and \( g \) are the vectors with design variables and constraint function values. \( c \) is the objective function, which was the compliance for all test problems evaluated at the iteration number \( \text{iter} \). The layer thickness of a single layer laminate was set to 0.01, whereas the layer thickness for the multi-layer laminates was 0.005, such that the total height was the same for all test cases.

All time measurements were performed on a Dell Optiplex 790 desktop, equipped with Windows 7 and an Intel i5 2.93 GHz 8 MB Dual Core processor. None of the algorithms were optimized for speed.

The Method of Moving Asymptotes optimizer was implemented using the standard recommended parameters, according to:

\[
\begin{align*}
a_0 &= 1, \\
a &= 0, \\
c &= 1000, \\
d &= 0.
\end{align*}
\] (11.2)

Some experiments with move limits were performed, but did not lead to improvements in performance and was therefore not implemented for all test problems. The Sequential Quadratic Programming algorithm was implemented using the standard MATLAB command \texttt{fmincon}, while setting the optimization settings to ‘SQP’. Furthermore, all options were kept at their default values.
References


