Computational Methods and Plasticity

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LIST OF ABBREVIATIONS, INDICES AND MAIN SYMBOLS

ABBREVIATIONS

\( \mathcal{B} \) \quad \text{body (structure) level}

\( \mathcal{E} \) \quad \text{element level}

FE \quad \text{finite element}

g. \quad \text{generalized}

\text{HMH} \quad \text{Huber-Mises-Hencky}

MS \quad \text{multiple subvolume}

m. \quad \text{matrix}

NR \quad \text{Newton-Raphson}

\mathcal{P} \quad \text{point level}

\mathbb{R}^n \quad \text{n-dimensional vector space}

\mathcal{I} \quad \text{cross-section level}

TG \quad \text{Tresca-Guest}

v. \quad \text{vector}

\( (\cdot)_j = \frac{a(\cdot)}{a_\mathcal{E}_j} \)

\( (\cdot)^{\prime} = \frac{a(\cdot)}{a_\tau} \)

INDICES

Superscripts

\( c \) \quad \text{creep}

\( e \) \quad \text{elastic}

\( ep \) \quad \text{elastic-plastic}

\( j \) \quad \text{the number of the FE}

\( (i) \) \quad \text{the number of the iteration step}

\( m \) \quad \text{the step number in the incremental procedure}

\( p \) \quad \text{plastic}

\( \Theta \) \quad \text{thermal}

\( T \) \quad \text{transposed}

\( \nu \) \quad \text{viscoplastic}
Subscripts

\[ a, b \]
\[ B \]
\[ E \]
\[ i, j \]
\[ L \]

= 1.2
bifurcation
Euler's
= 1, 2, 3 or \( x, y, z \)
limit

MAIN SYMBOLS

\[ A \]
area of the cross-section
\[ A_\ell \]
area of the \( \ell \)-th part of the cross-section
\[ \alpha_{ij} \]
tensor of microstresses (position of the middle of yield surface)
\[ \varepsilon = \partial F/\partial q, \quad \varepsilon_\star = \partial F/\partial q_\star \]
gradient \( \eta \) of the yield surface
\[ B^j \]
consistency \( m \) of the \( j \)-th FE
\[ b \]
width of the rectangular cross-section
\[ \beta \]
zero-one parameter in elastoplastic constitutive equations
\[ p^e, p^{ep} \]
cross-sectional stiffness \( m \) (on the level \( \mathcal{F} \))
\[ d \]
g. displacement \( \eta \) (of nodes)
\[ \delta_{ij} \]
Kronecker's delta
\[ E \]
Young's modulus
\[ E^e, E^p, E^{ep} \]
load stiffness (modular) matrices (on the \( \mathcal{P} \) level)
\[ E^{ijkl}, E^{ijkl}, E^{ijkl}_{ep} \]
constitutive tensor components
\[ \varepsilon^{e}_{ij}, \varepsilon^{p}_{ij}, \varepsilon_{ij} \]
strain tensor components
\[ \varepsilon \in \mathbb{R}^6, \quad \varepsilon_\star \in \mathbb{R}^9 \]
strain \( \eta \)
\[ \varepsilon_0 \]
strain at the yield point at simple tension
\[ \varepsilon^0 \]
initial strain \( \eta \)
\[ \varepsilon_p \]
effective (accumulated) plastic strain
\[dc_p = \left(\frac{2}{3} \det_{i,j} \, \det_{i,j} \right)^{1/2}\]

increment of the effective plastic strain

\[F = 0\]
yield surface

\[F^i, F\]
internal force \(v\)

\[f(\delta_{ij}) = \sigma_{\text{red}}\]
reduced stress

\[\phi = 0\]
interaction surface for g. stresses

\[G = E/[2(1 + v)]\]
shear (Kirchhoff's) modulus

\[g = \frac{E}{2} \varepsilon^e + h\]
function in elastoplastic constitutive equations

\[H' = \text{d} \sigma_e/\text{d} c_p\]
plastic modulus

\[h\]
hardening function

\[h\]
height of the bar cross-section

\[I\]
momentum of inertia of a cross-section

\[I\]
unit matrix

\[J_2 = \frac{1}{2} s_{ij} s_{ij}, \quad J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}\]
invariants of the stress deviator

\[K = E/[3(1-2v)]\]
bulk modulus

\[K^i, K\]
tangent stiffness \(m\).

\[K^i, K\]
elastic stiffness \(m\).

\[K\]
locally secant stiffness \(m\).

\[k\]
hardening parameter

\[k\]
yield point at simple shear

\[\lambda, \hat{\lambda}\]
plastic parameters on \(\mathcal{P}\) and \(\mathcal{F}\) levels

\[M_{\alpha\beta}\]
moment resultants

\[\mu\]
load parameter

\[\hat{\mu}(i)\]
load parameter corresponding to a statically admissible stress field \((i)\)

\[\tilde{\mu}(i)\]
load parameter corresponding to a kinematically admissible plastic flow mechanism

\[\mu_e, \mu_p, \mu_L\]
elastic, plastic load carrying-capacities and \(N_{\alpha\beta}\)
collapse load parameters

\[\sigma\]
stress resultants

\[n\]
exponent in the stress-strain relation

\[\nu\]
Poisson's ratio
\( P^i, P \)  
\( \bar{P} \)  
\( P \)  
\( Q \)  
\( g, g_e, g_p \)  
\( R, R_a \)  
\( R^i, R \)  
\( r \)  
\( S_{ij} \)  
\( s \)  
\( \sigma_{ij} \)  
\( \sigma_i \)  
\( \sigma_o \)  
\( \sigma_e \)  
\( \sigma \in \mathbb{R}^6, \sigma_e \in \mathbb{R}^9 \)  
\( T \)  
\( T_a \)  
\( t \)  
\( t \)  
\( t \)  
\( \theta \)  
\( \tau \)  
\( u \)  
\( W \)  
\( W_p \)  
\( x \)  

g. external load v.  
reference load v. or \( m \).  
surface load v.  
g. stress v.  
g. strain v.  
radii of curvature  
residual force v.  
pseudo force v.  
deviatoric stresses  
deviatoric stress v.  
stress tension components  
principal stresses  
yield point at simple tension  
effective stress  
stress v.  
temperature parameter  
shear forces  
traction v.  
control v.  
thickness of the cross-section  
temperature  
conventional time of plasticity  
displacement v.  
work of external loads  
plastic work  
position v. of Cartesian set of coordinates
1. PRELIMINARIES

1.1. Introduction

A great number of materials, and among them metals and their alloys, exhibit plastic properties. These properties correspond to permanent (plastic) deformations, i.e. such deformations which do not disappear in spite of vanishing of the loading factors which initiated the deformation process. The theory of plasticity deals with the plastic deformation of idealized models of materials which have got two main properties:

I. The deformation process is irreversible, history-dependent, associated with plastic strains and dissipation of energy.

II. The deformation process is time-independent and rate-independent.

The first property distinguishes the theory of plasticity from the theory of elasticity and in the case if the second property does not appear the material (process) is called as viscoplastic. If the deformation process turns out to be partially reversible the material is elastoplastic and if additionally it appears to be rate sensitive the material is named as elastic-viscoplastic.

What concerns material we shall consider an idealization which is called as a model of material. A set of assumptions (properties) is defined as the physical model. Relations (equations) between quantities describing the deformation process constitute the mathematical model of material. Models of material should take into account the most important, experimentally revealed features of real materials and these models have to be consistent with principal laws of thermodynamics or, in more narrow sense, with mechanics of deformable bodies.

The complexity of models depends on many factors. A common tendency is to formulate general models for a class of materials with different properties, at wide range of their changes. But we have to consider difficulties at experiments for identification of the model parameter (functions) as well as anticipated engineering applications.

The mentioned aspects prefer rather simple, small number parameter models, specified for predicted type of loading (e.g. quasistatic, isothermal), values of deformation parameters (e.g. small strains) and the state of stresses (e.g.
uniaxial or biaxial stresses). There appears the impact problem of assessment of material models and evaluation of the range of their application.

All above mentioned remarks are of great value especially with respect to the theory of plasticity (elastoplasticity) since proposed mathematical models have to be nonlinear (or only locally linearised). The nonlinearity of models as well as history-dependence significantly complicate the analysis on the structure level. Hence the selection of a suitable model should be referred to methods of the structure analysis in order to deal with models which are numerically effective.

The theory of plasticity is over 120 year old (the first paper by Tresca was published in 1864, cf. [01] p.2). Its development has been stimulated first of all by engineering applications, related to the manufacturing processes (metal forming), then by tendency to better utilization of properties of new materials (metal alloys, concrete). The classical theory of plasticity refers to the phenomenological formulation on the base of continuum mechanics. The theory has been built up under additional assumptions which more precisely describe and refine two main assumptions of the theory of plasticity. It was originated before the Second World War (cf. [01] p. 2-4) but in fact the modern classical theory of plasticity was developed in 40-50ies, mainly due to papers by Hill, Prager, Drucker, Hodge, Ilyushin and others (cf. references in [02-06]). This theory has offered a great number of material models, theorems and heuristic type evaluations. They have been explored in various approaches and approximate methods, codes and instructions for design structures. Despite of that the analysis of elastic-plastic structures turned out to be difficult and limited to simple problems and uncomplicated structures or only to their elements.

Appearance of computers and development of numerical methods open the door to wider analysis of problems of the theory of plasticity and its engineering applications. It is evident that the Finite Element Method has been used quite early (the paper [101] by Gallagher et al. was published in 1962). On the breakthrough of 60 and 70-ties big computer codes and systems were implemented in the field of plastic analysis [14], which was successfully used to the analysis of various aerospace and naval structures, in reactor technology, in civil and mechanical engineering as well [15].
New computational possibilities enable us to go out of assumptions of the classical theory of plasticity. Large displacements and strains, influence of temperature changes and time parameter, cyclic and dynamic loadings have been recently analyzed, cf. [20,22,23,25].

A characteristic feature of computational methods for the analysis of problems, founded on the theory of plasticity, is a wide utilization of numerical methods and the software which have been developed for the analysis of elastic and geometrically nonlinear problems. The needed modifications and supplements concern the solution of constitutive equations and consideration of the time-dependence of the plastic deformation process.

The comparison of precomputer and recent approaches points out a preference of elastic-plastic models over simpler rigid, ideally plastic models and the common use of the incremental, plastic flow theory instead of the total strain, deformation theory. It is also visible that the interest in the limit state analysis decreases in favour of the analysis of the full deformation process - starting from the first yielding up to the limit state. Such an reorientation corresponds, of course, also to the common application of incremental techniques to the analysis of nonlinear problems.

Because of rapid development of computers and their software the size and number of analyzed problems are continuously increasing. The computer codes for the elastic-plastic (elasto-viscoplastic) analysis become more and more complicated and refined. In such a situation a deeper understanding of the theoretical background seems to be of great importance. This is especially of value for potential users of computer codes and systems related to the plastic analysis, to those who have not an everyday contact with the theory of plasticity. Such a knowledge is indispensable not only for the use of existing codes in a proper range of their validity but also in order to assess the codes critically and put requirements to new computer programs.

1.2. Scope of the report

There is extended literature even with restriction to computation methods for the analysis of elastic-plastic problems. That is why a certain selection has to be made according to the following suppositions:
1. Readers are familiar with foundations of the theory of deformable bodies - especially with the theory of elasticity and plasticity, with structural mechanics and the finite element method, as well as with tensorial and matrix calculus. That is why only main assumptions, relations and theorems will be remind, frequently without any derivation or proof.

2. The critical point of our consideration refers to selection of material models suitable for the computational analysis. In the course the attention will be focussed on metals and their alloys. Hence models with symmetric properties with respect to tension and compression (in [02] p.93) such models are called as isosensitive materials), homogeneous and initially isotropic will be preferred. Models developed under assumptions of the classical theory of plasticity will be more extensively discussed since just such models are commonly used in the existing computer codes. Other models will be also outlined with respect to experimental evidence and possible release of assumptions of the classical theory of plasticity.

3. There will be presented computational methods developed for the elasto-plastic analysis of the full deformation process. Classical problems of limit and shakedown analysis of structures will be merely sketched.

4. Geometrical and material nonlinear problems will be considered from the viewpoint of the displacement formulation of the finite element method.

5. Large strains, temperature dependence of material properties, cyclic loading and rate dependence will be shortly discussed as examples of an extension of the classical concepts of plasticity.

6. Specific problems of the elastic-plastic buckling analysis of structures will be pointed out.

7. A selection of extensive literature have to be made. In order to shorten the list of papers the cross-quotation will also be used, especially with respect to those positions which are rather of historical value. The bibliography is split into two parts: I. Books, conference proceedings and review papers are pointed out by two-digit numbers from the range 01-99; II. Particular papers are with numbers above 100. Such a list enable us to joint easily additional positions of literature.
1.3. Notation and levels of the analysis

Both tensors and matrices will be used. The tensors are related to the Cartesian coordinates only. The summing convention of repeated indices is used.

The absolute notation (underlined symbols) will be preferred with respect to matrices but relations for the matrix components will be also given. The one-column matrices will be shortly called as vectors. In order to save the space of the manuscript the components of vectors will be written horizontally inbetween brackets, e.g.

\[
\mathbf{a} = \{a_1 | i = 1, \ldots, n\} = \{a_1, \ldots, a_n\} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.
\]

(1.1)

If there is no misleading the enumeration of indices will be omitted, e.g. dimension of matrices (n×1) and \(i = 1, \ldots, n\) in (1.1). The indices can be put on different levels as sub or superscripts and may be related to expressions in parenthesis or brackets, e.g.

\[
(a^T \cdot b)^j = (a_i b_j)^j \quad \text{for } j = 1, \ldots, m
\]

(1.2)

\[
(a^T a)^{1/2}_i = (a_i a_i)^{1/2} = [(a_1^2 + (a_2^2 + \ldots + (a_n^2)]^{1/2}.
\]

Components of the square matrices will be put in between square brackets and diagonal matrices in not fully closed square brackets, e.g.

\[
D \mathbf{g} = |D_{11} | \{g_1 \} = |D_{11}, \ldots, D_{nn}| \{g_1, \ldots, g_n\} = \\
= D_{11} g_1 + D_{22} g_2 + \ldots + D_{nn} g_n.
\]

(1.3)

The unit matrix will be denoted as \(I\). The sign \(\equiv\) means 'is equivalent' or 'by definition'. Other symbols will be explained little by little in the text and more frequently used symbols are completed in the list at the beginning of the report.

Similarly as in [02] considerations will be carried out on different levels:
Point level $\mathcal{P}$ is the basic level, related to any or selected points of material continuum or of a model of structure. Tensorial notation and calculus are preferred on the level $\mathcal{P}$ in order to describe objects and their relations in the spaces well known from the continuum mechanics.

Cross-section level $\mathcal{J}$ corresponds to such structures as bars, plates and shells in which one dimension, e.g. thickness, is much more smaller than other dimensions. Integral quantities are called as generalized variables and they are related to each other through energy or work expressions (generalized displacement-versus $g$.loads, $g$.strains - $g$.stresses). On the level $\mathcal{J}$ both the tensors (e.g. shell equations) and matrices are used.

Element level $\mathcal{E}$ is introduced for a description of separated parts of the structure (members, substructures) or individual finite elements. In order to analyse different fields approximated functions used to be introduced (e.g. shape or basic sets of functions). Assembling of elements into a structure and internal analysis of any element is carried out through element matrices.

Body (structure) level $\mathcal{B}$ is called also as global level contrary to local, lower levels $\mathcal{P}, \mathcal{J}, \mathcal{E}$. On the level $\mathcal{B}$ algebraic relations are preferred because of the use of computers, numerical methods of algebra and matrix calculus.

Methods of the analysis has to correspond to characteristic features of the levels. In plasticity the level $\mathcal{P}$ is especially difficult for any treatment because of nonlinearity and time-type dependence of relations. That is why the transition from one to another level is not straightforward, especially with respect to transformations $\mathcal{P} \rightarrow \mathcal{J}$. In general the analysis of problems of the theory of plasticity needs more operations and additional computer memory then elastic analysis.

In order to describe more precisely the deformation process the definitions of active and passive processes are introduced against loading and unloading (cf. [02] pp.9-16). The active process is related to the increase of plastic strains on the level $\mathcal{P}$ or to the development of yielding zones on the levels $\mathcal{J}$ and correspondingly. From viewpoint of such a definition the passive process is related to the lack of increments of plastic strains or to a fixed zone of yielding as well as to the elastic behaviour of material. As a counterpart to the active and passive processes loading and unloading can be considered,
associated with the increase or decrease of a load type parameter. In what follows the names loading and unloading will be used to describe the type of process on the level $\mathcal{B}$ rather, adding name 'local' if it is associated with the other levels. It is quite possible that for a loading of structures the passive process can take place on the level $\mathcal{P}$ and vice versa.

The considerations will be conduct in different spaces, e.g. in the stress space, exertion factor space etc. The point in an appropriate space on different level of the analysis will be identified by the position vector of this space, e.g. the stress point by the vector $g$, the strain point by $\xi$, the displacement point $u(\xi)$, the load configuration point $\bar{F}$ etc.
2. MODELS ON THE POINT LEVEL $\mathcal{P}$

2.1. Assumptions of the classical theory of plasticity

The definition of the 'classical' theory of plasticity is rather an arbitrary one. On the basis of two principal assumptions I and II, mentioned in Sec. 1.1, the classical theory of plasticity is defined by:

1. **Initial yield condition**, defining limits of the elastic range for the multiaxial stress state.
2. ** Constitutive equations**, relating stresses to strains (or for their increments).
3. **Hardening rule**, which enable us to analyse subsequent active processes.

The main relations of the classical theory of plasticity are derived under following, detail assumptions:

1. The strains are small (infinitesimal) so they or their increments can be split into the elastic and plastic parts:

\[
dc_{ij} = dc_{ij}^e + dc_{ij}^p, \tag{2.1}
\]

and instead of increments $dc_{ij}$ the rates $\dot{c}_{ij}$ can be used, calculated with respect to a conventional time of plasticity (any, but monotonically increasing time of the problem under analysis):

\[
\dot{c}_{ij} \equiv \frac{dc_{ij}}{dt} = \dot{c}_{ij}^e + \dot{c}_{ij}^p. \tag{2.2}
\]

2. Material is **initially isotropic** with respect to initial yielding, without initial prestressing or prestrain, and isosensitive (symmetric properties with respect to tension and compression).

3. Material is **plastically incompressible**, i.e. the first invariant of plastic strains equals zero:

\[
dc_{11}^p + dc_{22}^p + dc_{33}^p = 0. \tag{2.3}
\]
4. No hysteresis is possible, i.e. dissipation of energy does not occur if the passive process takes place, and the elastic characteristics of material do not undergo any changes during all the deformation process.

5. The process of deformation is isothermal and temperature does not influence values of material parameters (functions).

6. Additional postulates are added in order to specify the class of admissible relations, e.g. one postulates the existence of a plastic potential or so called postulate of material stability (in Drucker's sense).

The initial assumptions have been verified experimentally by many authors in various conditions, from different points of view. These assumptions have been widely discussed and are currently accepted for metals and their alloys of plastic properties, subjected to quasi-static loads at small number of changes of their sense (small number of cycles), and small changes of temperature (around the room temperature).

There is a big number of books and review papers devoted to the theoretical background and experimental investigations. From among them only several basic monographs and conference proceedings are quoted (cf. [01-13]), as well as the papers to which individual parts of the course will be many times referred (cf. [16-25]).

2.2. Initial yield conditions

The yield condition is a scalar relation which defines the beginning of the active process (yielding of material). The initial yield condition is associated with a limit state of elastic carrying capacity of material or with a defined yielding of material. In Fig. 2.1 different definitions of such a limit state are shown with respect to the uniaxial tension, cf. [16] p.4.

Yield conditions can be considered in general from the viewpoint of so called hypothesis of failure of material, cf. [02] pp.70-72. The reduced stress is introduced in this approach:

\[ \sigma_{\text{red}} = f(\sigma_{ij}) , \] (2.4)
Fig. 2.1 Different definitions of the yield point.

which corresponds to a measure that makes it possible to compare the multiaxial and uniaxial states of stresses. If the reduced stress $\sigma_{\text{red}}$ is equal to the yield point $\sigma_o$ of the uniaxial, simple state of stresses (tension, compression, shear or torsion) then the initial yield condition can be written in the following form:

$$F(\sigma_{ij}) \equiv f(\sigma_{ij}) - \sigma_o = 0 .$$  \hspace{1cm} (2.5)

With respect to metals the Tresca-Guest (TG) and Huber-Mises-Hencky (HMH) initial yield conditions are used. The condition TG refers to the maximum shear stress:

$$\max \left[ \frac{1}{2}(\sigma_1-\sigma_2), \frac{1}{2}(\sigma_2-\sigma_3), \frac{1}{2}(\sigma_3-\sigma_1) \right] = \frac{\sigma_o}{2} .$$  \hspace{1cm} (2.6)

The condition HMH is associated with a limit value of the strain energy, or equivalently with the octaedral shear stress:

$$\tau_{\text{oct}} = \frac{1}{3}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]^{1/2} = \sqrt{\frac{2}{3}} k_o .$$  \hspace{1cm} (2.7)
As usually the yield point \( \sigma_0 \) in the course will correspond to the simple tension. Sometimes more compact formulae can be obtained for the yield point at simple shear \( k_0 \) which is associated with \( \sigma_0 \) by the relation:

\[
k_0 = \sigma_0 / \sqrt{3}. \tag{2.8}
\]

In literature the conditions (2.6) and (2.7) are related only to Tresca and von Mises respectively besides the pioneering or comparable position of other scientists (cf. [02] p.71).

The initial yield conditions (2.6) and (2.7) are formulated in the space of principal stresses. It is possible to transform them to other spaces associated with stresses, e.g. to the space of stress deviator invariances. In the case \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) the TG conditions takes the form (cf. [09] p.208):

\[
4J_2^2 - 27J_3^2 - 8\sigma_0^2 J_2 + 6\sigma_0^4 J_3 - 9\sigma_0^6 = 0, \tag{2.9}
\]

where the invariants \( J_2 \) and \( J_3 \) are defined by the formulae (cf. [09] p.71):

\[
J_2 = \frac{1}{2} s_{ij}s_{ij}, \quad J_3 = \frac{1}{3} s_{ij}s_{jk}s_{ki} \quad \text{for } i,j,k = 1,2,3. \tag{2.10}
\]

The components of the stress deviator equal:

\[
s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}, \tag{2.11}
\]

where

\[
\sigma_m = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}), \quad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{2.12}
\]

The HMF initial yield condition is of especially simple form if additionally (2.8) is taken into account:

\[
J_2 = k_0^2. \tag{2.13}
\]

It can be concluded that yield conditions are of more or less complicated form depending on the space in which they are formulated. It is worth to underline that both the TG and HMF yield conditions are independent of the first invariant of the stress tensor:

\[
I_1 \equiv 3\sigma_m = \sigma_{11} + \sigma_{22} + \sigma_{33}. \tag{2.14}
\]
It is in agreement with the assumption 3. of the classical theory of plasticity which ignores the influence of the material compressibility on the yielding of metals.

The form (2.5) of the yield conditions describes a surface in the stress space. In what follows this surface is called as the initial yield surface. In the plane stress state this surface becomes the yield locus as a result of intersection of the yield surface and the plane \( \sigma_3 = 0 \). In Fig. 2.2a the HMH yield locus is shown. The HMH yield condition, corresponding to (2.7), describes the cylinder of the radius \( k_o \) and the hydrostatic axis \( \xi \). In Fig. 2.2b the deviatoric (octaedral) plane \( D \), perpendicular to the axis \( \xi \) is shown, as well as the polar set of coordinates \((r, \theta)\) on it. The cylindrical set of variables \((\xi, r, \theta)\) corresponds to the space which can be sometimes suitable for the formulations of yielding conditions (cf. [09] Ch.5).

\[ (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_o^2 , \]  

\( (2.15) \)

or in the 6D stress space according to the engineering notation of stresses:

Fig. 2.2. a) Yield loci on the plane \( \sigma_3 = 0 \), b) Tresca-Guest hexagonal cylinder and Huber-Mises-Hencky circular cylinder in the principal stress space.

Let us remind different forms of the HMH yield condition with respect to the principal stresses
\[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2\sigma_0^2.\]  

(2.15a)

In the case of the plane stress state the above formulae become:

\[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_0^2.\]  

(2.16)

\[\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_0^2.\]  

(2.16a)

Experiments for combined loads are frequently carried out on tubular specimens subjected to axial tension or internal pressure and torsion. If two independent stresses \(\sigma_x \equiv \sigma\) and \(\tau_{xy} \equiv \tau\) are assumed to be the only nonzero stresses then relation (2.16a) takes the following simple form:

\[\sigma^2 + 3\tau^2 = \sigma_0^2.\]  

(2.17M)

For the same case the TG yield condition becomes

\[\sigma^2 + 4\tau^2 = \sigma_0^2.\]  

(2.17T)

The HMH yield locus (2.17M) approximate experimental results for metals better than the TG relation (2.17T). In Fig. 2.3, experimental results of Taylor and Quinney are shown on the background of the yield loci defined by (2.17) – cf. [01] p.22.

![Graph](image)

Fig. 2.3. HMH and TG yield loci against experimental data.
Better agreement with experiments and smoothness of the surface advocate the HMH yield condition which is recently preferred at the analysis of metallic structures. But in the limit and shakedown analysis also piecewise linear yield conditions are used (e.g. the yield surfaces combined of hyperplanes inscribing and circumscribing the HMH cylinder in the stress space, cf. [02] pp.95-100). This is caused by the application of the linear programming methods in which the yield conditions are considered as the stress constraints.

Sometimes the initial anisotropy of material should be taken into account. Such an anisotropy can be associated mainly with a manufacturing process, e.g. rolling or initial prestrain of metals. From among many of propositions (cf. [02] pp. 109-111) the Hill yield condition, [01] p. 318, is only mentioned. The condition is valid for the case if \(x, y, z\) are the principal axis of anisotropy (cf. also [03] p.77):

\[
F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2(L \tau_{yz}^2 + M \tau_{zx}^2 + N \tau_{xy}^2) = 1 . \quad (2.18)
\]

In the case of initial orthotropy the relation (2.18) becomes the HMH yield condition.

Yield conditions for various materials have to be formulated in different and sometimes more complicated forms then those for metals. In the case of soil, rock and concrete yield conditions ought to depend on all stress invariants and a good agreement with experiments can be achieved by a higher number of material parameters, cf.[09]. From such a point of view the HMH yield condition is especially simple, since it depends only on the second invariant of the stress deviator \(J_2\) and needs only one material parameter — the yield point \(\sigma_0\).

2.3. Subsequent yield surfaces, effective stress and effective strain

Each stage of the plastic deformation can be associated with a function \(f(\sigma_{ij})\) which value will display the type of process taking place. For this purpose the value of the function \(f\) is compared with the value of the functions \(\sigma_{\text{red}}\) or \(k\) which depend on the plastic strains \(\varepsilon_{ij}^p\):

\[
f(\sigma_{ij}) = k(\varepsilon_{ij}^p) . \quad (2.19)
\]
A more general form

\[ F(\sigma_{ij}, \varepsilon_{ij}^p, k) = 0, \] (2.20)

defines a subsequent yield surface in the stress space. Such a surface changes its configuration with respect to the stresses \( \sigma_{ij} \) and internal variables associated with the plastic strains \( \varepsilon_{ij}^p \) and a history-dependent scalar function \( k \), called as the hardening parameter. Evolution of the subsequent yield surfaces is controlled by the hardening law. The formulation of hardening laws is one of the main problems of the theory of plasticity.

Basic characteristics of material are usually referred to individual functions corresponding to simple load programs (tension or tension-compression, shear or torsion, proportional loading, or under control of a single parameter). In order to compare results obtained for various load programs a relation between the effective stress \( \sigma_e \) and effective strain \( \varepsilon_e \) is to be formulated. The functions \( \sigma_e \) and \( \varepsilon_e \) are usually related to the second deviators of stresses and strains correspondingly (cf. [02] p.36,45).

In the theory of plasticity the effective stress \( \sigma_e \) is associated with the simple (uniaxial) tension

\[
\sigma_e = \sigma_{\text{red}} = \sqrt{3} \ k = \sqrt{3} J_2 = \sqrt{\frac{3}{2}} \frac{s_{ij}s_{ij}}{s_{ij}} = \\
= \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} = \\
= [\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_x - \sigma_z \sigma_x + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{1/2}. \] (2.21)

The effective plastic strain \( \varepsilon_p \) can be defined in different way (cf. [102], [09] p.351). In what follows the commonly used definition of the increment at the effective plastic strain \( d\varepsilon_p \) will be explored:

\[
d\varepsilon_p = \left( \frac{2}{3} \frac{d\varepsilon_{ij}^p}{d\varepsilon_{ij}} \right)^{1/2}. \] (2.22)

In the case of simple tension test the following, basic relation should be found

\[
\sigma_e = H(\varepsilon_p) \] (2.23)
or in the incremental form

\[ d\sigma_e = H'(\sigma_e) \, dc_p , \quad (2.24) \]

where \( H'(\sigma_e) \) is the plasticity modulus

\[ H' = \frac{d\sigma_e}{dc_p} . \quad (2.25) \]

The history of the deformations process is evaluated through the length of the accumulated effective plastic strain path *):

\[ \varepsilon_p = \int_{0}^{\tau} \dot{\varepsilon}_p \, dt = \int_{0}^{\tau} \left( \frac{2}{3} \left( \epsilon_{ij}^p dc_{ij}^p \right) \right)^{1/2} = \int \frac{d\sigma_e}{H'(\sigma_e)} \quad (2.26) \]

According to the formula (2.26) the accumulated effective plastic strain is in any case the non-decreasing (monotone increasing) function of the plastic deformation process.

### 2.4. Classical hardening rules

In the simplest case the yield surface is described by the function

\[ F(\sigma_{ij}, k) = 0 , \quad (2.27) \]

which for the HMH yield condition becomes

\[ \frac{3}{2} s_{ij} s_{ij} = \sigma_e^2(\varepsilon_p) . \quad (2.28) \]

In such a case the subsequent yield surfaces correspond to the uniformly extended initially yield surface. That is why this case is called as the isotropic strain hardening.

*) In certain papers (cf. [02] p.141, [07] p.25) so called Odqvist's parameter \( I_{cp} = \sqrt{3/2} \varepsilon_p \) is used where \( \varepsilon_p \) is taken according to (2.26).
Fig. 2.4. Isotropic hardening for a) plane stresses, b) uniaxial tension-compression.

The case of plane stress is shown in Fig. 2.4a. The initial yield locus $F_0$ is extended isotropically to $F$ if the stresses change along the path $A + B$. The subsequent yield curve $F$ bounds the domain of stresses associated with the passive deformation processes, e.g. for the stress program along the path $B + C$ no plastic strains occur. The local loading process is shown also for the uniaxial tension $0 + a + b + c$ and the corresponding stress-strain curve $\sigma_1(\varepsilon_1)$ is shown in Fig. 2.4b. In the same figure the relation $\sigma_e - \varepsilon_p$ is also drawn.

The history of the process is taken into account throughout the effective plastic strain $\varepsilon_p$ which, according to the relation (2.26), is the nondecreasing function of the plastic strains $\varepsilon_{ij}^p$. This type of hardening is called as the strain hardening. In the early papers by Taylor, Quinney and Schmidt (cf.[02] p.143) instead of $d\varepsilon_p$ the increment of plastic work is introduced

$$dW_p \equiv \sigma_{ij}^p d\varepsilon_{ij} = \sigma_e d\varepsilon_p .$$

(2.29)

The hardening parameter can be related to a function of the accumulated plastic work

$$k = f(\int dW_p) .$$

(2.30)
That corresponds to so called work hardening which can be used for describing the plastic deformation process. It is worth of attention that for the HMM yield condition the work and strain hardening are equivalent to each other.

During the uniaxial tension-compression test the so called Bauschinger effect has been observed. The effect is associated with a directional anisotropy, i.e. an initial plastic deformation can influence the decrease of the yield point for unloading. In Fig. 2.5a the perfect Bauschinger effect is shown, corresponding to the relation $\sigma_b + |\sigma_c| = 2\sigma_0$.

![Fig. 2.5. Kinematic hardening: a) Uniaxial stress-strain curve for the ideal Bauschinger effect, b) Plane stress state, c) Prager's and Ziegler's kinematic hardening rules.](image)

For the multiaxial state of stresses the perfect Bauschinger effect can be easily described by the function

$$f(\sigma_{ij} - \sigma_{ij}) = \sigma_0^2. \quad (2.31)$$

It corresponds to an yield surface which undergoes only rigid translation (Fig. 2.5b). The position of the centre of the translated surface is defined by the tensor $\sigma_{ij}$. This type of hardening is called as kinematic hardening. Sometimes it is called also as anisotropic hardening since the domain of the passive processes is now nonsymmetric with respect to the origin 0. The translation depends on type and history of local loading — in Fig. 2.5b the subsequent
yield surfaces \( F \) and \( F_u \) corresponds to the two and one axial state of stresses respectively. For isotropic hardening the surfaces \( F \) and \( F_u \) coincide to each other — cf. Fig. 2.4a.

One of the most simple kinematic hardening relates the increments \( d\alpha_{ij} \) to the increments of plastic strains \( d\epsilon_{ij}^p \)

\[
d\alpha_{ij} = c \ d\epsilon_{ij}^p . \tag{2.32}
\]

If the hardening parameter \( c \) is constant then the linear Melan-Ishlinsky-Prager hardening takes place (cf. [02] p.146), commonly known as the Prager hardening rule. Kadashevich and Novozhilov [103] have interpreted \( \alpha_{ij} \) as the tensor of residual microstresses. Arutyunyan and Vakulenko [104] suggested to adopt the hardening parameter \( c \) as a function of the second deviatoric invariant

\[
c = A(s_2) . \tag{2.33}
\]

If the effective stress \( \sigma_e \) is associated with the plastic effective strain then the relation (2.33) becomes

\[
c = a(\epsilon_p) . \tag{2.34}
\]

The formulation of the appropriate hardening function \( c \) was discussed by Eisenberg and Phillips [105]. A review of more complicated relations describing kinematic hardening is given in [07] p.26-31 and [02] p.146-147, [105].

The Prager hardening rule is not consistent if simpler states of stresses are considered (cf. e.g. [09] p.355). That is why Ziegler [106] proposed the hardening rule in the following form

\[
d\alpha_{ij} = (\alpha_{ij} - \alpha_{ij}^*) \ d\mu , \quad \text{for } d\mu > 0 . \tag{2.35}
\]

The Ziegler hardening rule appears to be invariant with respect to the transition to the stress subspaces.

In Fig. 2.5c the translation of the yield locus is shown under assumptions of Ziegler's and Prager's hardening rules (additionally it is assumed in Fig. 2.5c that the 'vector' \( d\epsilon_{ij}^P = \{d\epsilon_{ij}^P\} \) is orthogonal to the yield surface \( F \). The
function $d\mu$ in (2.35) can be computed from so called consistency condition (cf. p.2.5), as it has been proposed by Ziegler [105] so it can be adopted in the following form (cf. [09] p.305)

$$d\mu = a \, d\varepsilon_p, \quad \text{for } a > 0. \quad (2.36)$$

Naghdi [107] proved that the application of both the Prager and Ziegler rules leads to the same results also in the case of the plane stresses under two following conditions: 1) the HMH condition is used, 2) the hardening parameters $c$ and $a$ are constants.

Formally it is easy to couple the isotropic and kinematic strain hardening to obtain the mixed strain hardening with the following form of the subsequent yield surface equation:

$$f(\sigma_{ij} - \sigma_{ij}) = \sigma^2_e(\varepsilon_p). \quad (2.37)$$

Eq. (2.37) describes extension and simultaneous transition of the yield surface during the deformation process. This equation can be written also in the shortened form [108]

$$f(\tilde{\sigma}_{ij}) = \tilde{\sigma}^2_e(\tilde{\varepsilon}_p). \quad (2.38)$$

where bars denote quantities associated with the relative stresses

$$\tilde{\sigma}_{ij} = \sigma_{ij} - \sigma_{ij}. \quad (2.39)$$

More extensive review of papers devoted to the mixed hardening can be found in [02] p.154-155, [07] p.28-33.

2.5. The Drucker stability postulate and plastic flow rule

Drucker [109, 110] considered the criterion of material stability, which appeared to be one of the most general postulates of the theory of plasticity. Material is stable in the Drucker's sense if the increment of the work $dW^*$ on the quasi cycle of stresses is non-negative.
\[ dw^* = (\sigma_{ij} - \sigma_{ij}^*) \, d\varepsilon_{ij}^p + d\sigma_{ij} \, d\varepsilon_{ij}^p \geq 0, \]

(2.40)

where the meaning of the terms in the inequality (2.40) can be interpreted on the example of uniaxial stress (Fig. 2.6).

Fig. 2.6. Increment of plastic work on a uniaxial stressquasi-cycle.

The Drucker postulate (2.40) may be split into two parts (so called large and small postulates):

(\sigma_{ij} - \sigma_{ij}^*) \, d\varepsilon_{ij}^p \geq 0, \quad (2.41a)

\[ d\sigma_{ij} \, d\varepsilon_{ij}^p \geq 0. \quad (2.41b)\]

Two important implications result from (2.41) - cf. [02] p.122-123. Material is stable in the Drucker's sense if the following assumptions are fullfilled:

1. The initial and subsequent yield surfaces, \( F_0 \) and \( F \), are convex.
2. The vector of increments of plastic strains \( d\varepsilon_{ij}^p \) is normal to the yield surface

\[ d\varepsilon_{ij}^p = d\lambda \, \frac{\delta F}{\delta \sigma_{ij}} \quad \text{for } d\lambda > 0. \quad (2.42)\]

The Drucker postulate and the outlined implications define a wide class of materials for which the plastic potential coincides with the yield surface function and, however, these stable materials obey the associated flow rule (2.42). Experiments carried out for metal and their alloys display that the yield surfaces are convex and increments of plastic structures are nearly normal
to these surfaces in the wide range of the deformation process (cf. [16,17,21, 24]).

In order to formulate relations between the increments of stresses and strains the assumptions listed in p. 2.1 are to be completed by the two following conditions:

i) **Continuity condition** which introduces the concept of the neutral state which separates the active and passive deformation processes.

ii) **Consistency condition** prescribes the stress point to the yield surface during the active deformation process. The consistency condition can be formally written as

\[ dF = 0 \text{ for } \sigma_{ij} \text{ fulfilling equation } F(\sigma_{ij}) = 0. \]  

(2.43)

The consistency condition can be applied to the general form of the yield surface (2.20):

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon^p_{ij}} d\varepsilon^p_{ij} + \frac{\partial F}{\partial k} dk = 0. \]  

(2.44)

The elastic response obeys the generalised Hooke's law:

\[ d\sigma_{ij} = E_{ijkl}^e d\varepsilon^e_{kl}. \]  

(2.45)

and according to (2.1) the increments of elastic strains \( d\varepsilon^e_{kl} \) can be eliminated:

\[ d\sigma_{ij} = E_{ijkl}^e (d\varepsilon_{kl} - d\varepsilon^p_{kl}). \]  

(2.45a)

After taking into account the associated flow rule (2.41) and (2.45a) the consistency equation (2.44) becomes

\[ \frac{\partial F}{\partial \sigma_{ij}} E_{ijkl}^e (d\varepsilon_{kl} - d\lambda \frac{\partial F}{\partial \sigma_{kl}}) + \frac{\partial F}{\partial \varepsilon^p_{ij}} d\varepsilon^p_{ij} + \frac{\partial F}{\partial k} \frac{\partial F}{\partial \varepsilon^p_{ij}} = 0. \]  

(2.44a)

From this Eq. (2.44a) the scalar function \( d\lambda \) can be determined:

\[ d\lambda = \frac{(\partial F/\partial \sigma_{ij}) E_{ijkl}^e d\varepsilon_{kl}}{h + (\partial F/\partial \sigma_{mn}) E_{mnpq}^e (\partial F/\partial \sigma_{pq})}. \]  

(2.46)
where \( h \) is the hardening function sometimes called as the hardening parameter (cf. e.g. [09] p.361):

\[
 h = - \frac{\partial F}{\partial \varepsilon_p^{ij}} \frac{\partial F}{\partial \sigma_i} - \frac{\partial F}{\partial \varepsilon_k^{ij}} \frac{\partial F}{\partial \sigma_j}. \tag{2.47}
\]

The relation (2.45a) can be written in the following form:

\[
 d\sigma_{ij} = (E_{ijkl}^e - E_{ijkl}^p) \, d\varepsilon_{kl} = E_{ijkl}^p \, d\varepsilon_{kl}, \tag{2.48}
\]

where the local (modular) plastic stiffness (on the \( P \) level) is defined by the formula:

\[
 E_{ijkl}^p = \beta \frac{E_{ijkl}^e}{h + (aF/\sigma_m)} \left( \frac{\partial F/\sigma_m}{\partial \varepsilon_{ik}} \right) \left( \frac{\partial F/\sigma_m}{\partial \varepsilon_{jl}} \right) \frac{E_{ijkl}^e}{h + (aF/\sigma_m)} \left( \frac{\partial F/\sigma_m}{\partial \varepsilon_{ik}} \right). \tag{2.49}
\]

The zero-one parameter \( \beta \) depends on the type of the deformation process:

a) active process

\[
 \beta = 1 \quad \text{for } F = 0 \text{ and } d\lambda > 0, \tag{2.50a}
\]

n) neutral process

\[
 \beta = 0 \quad \text{for } F = 0 \text{ and } d\lambda = 0, \tag{2.50n}
\]

p) passive process

\[
 \beta = 0 \quad \text{for } F < 0. \tag{2.45p}
\]

2.6. Classical constitutive equations

A general relationship between the ratios of the components of the strain increment and the ratios of the components of the stress deviator is known as the Lévy-Mises equations

\[
 \frac{d\varepsilon_{ij}}{\sigma_{ij}} = d\lambda. \tag{2.51}
\]
This relationship corresponds to the flow rule (2.42) in the case of the rigid plastic model of material and HMH yield condition. That is why the superscript \( p \) of Eq. (2.42) may be dropped and the components of the gradient \( \frac{\partial F}{\partial \sigma_{ij}} \) equals:

\[
\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial \sigma_{ij}} = s_{ij}.
\]

(2.52)

The Lévy-Mises equations are used extensively in cases of unrestricted plastic flow (cf. e.g. [01,03]).

If the elastic strains are taken into account the well known Prandtl-Reuss equations can be obtained as an extension of the earlier Lévy-Mises equations. In the case of HMH yield condition and isotropic hardening

\[
F = \frac{1}{2} s_{ij} s_{ij} - \frac{1}{3} \sigma^2 e = 0 ,
\]

(2.53)

the consistency condition (2.43) can be written in the following form:

\[
s_{ij} ds_{ij} - \frac{2}{3} \sigma e \frac{d\sigma}{d\epsilon_p} \frac{d\epsilon_p}{d\epsilon} = 0 .
\]

(2.54)

With (2.22), (2.25), (2.42) and taking into account the relation \( s_{ij} ds_{ij} = (2/3) \sigma e d\sigma \) the formula for the flow function \( d\lambda \) is obtained from Eq. (2.54):

\[
d\lambda = \frac{3}{2} \frac{d\sigma}{d\epsilon} = \frac{3}{2} \frac{d\epsilon}{d\epsilon} .
\]

(2.55)

This formula is well known from the literature on plasticity (cf. [01] p.39, [03], p.87) but it is not always suitable for computations. That is why the formula (2.46) is rather preferred. In the simple case of the function (2.53) the hardening function \( h \), defined by (2.47), takes the form:

\[
h = \frac{4}{9} \frac{H'}{\sigma_e}, \quad H' = \frac{E_t}{1-E_t/E} ,
\]

(2.56)

where \( E \) and \( E_t \) are the elastic and tangent moduli at uniaxial tension. The plasticity function \( d\lambda \) according to (2.46) becomes

\[
d\lambda = \frac{s_{mn} \frac{d\epsilon}{d\epsilon}}{2 \frac{\sigma^2}{3 \sigma_e} (1 + \frac{H'}{3G})} ,
\]

(2.57)
and finally the stresses are obtained from (2.48)

\[ \dot{\sigma}_{ij} = 2G \dot{e}_{ij} + 3K \dot{e}_{m} \dot{\delta}_{ij} - \beta \frac{2G \epsilon_{mn} \dot{e}_{mn}}{3 \sigma_{e} (1 + \frac{H'}{3G})} s_{ij} \]  

(2.58)

The formula (2.58) is valid for initially isotropic material with elastic moduli

\[ G = \frac{E}{2(1+v)}, \quad K = \frac{E}{3(1-2v)} \]  

(2.59)

and under assumption 1. and 2. of Sec.2.1.

In the case of elastic, ideally plastic material the formula (2.58) is also valid under assumptions that \( H' = 0 \) and \( \sigma_{e} = \sigma_{o} \).

Another set of stress-strain equations, called the Hencky-Ilyushin equations should be briefly discussed as they are applied to the analysis of special problems. These equations are formulated on the assumption of proportionality between the strain and stress deviators (cf. [02] p.74):

\[ \epsilon_{ij} = \varphi s_{ij} \]  

(2.60)

The proportionality coefficient \( \varphi \) is determined from the yield condition. In the case of HMM condition

\[ \varphi = \frac{3 \epsilon_{e}}{2 \sigma_{e}} \]  

(2.61)

where the effective total strain is defined by the following formula

\[ \epsilon_{e} = \left( \frac{2}{3} e_{ij} e_{ij} \right)^{1/2} \]  

(2.62)

In such a way the function \( \varphi \) is related to the function \( \sigma_{e}(\epsilon_{e}) \) which can be easily obtained experimentally from any simple loading test.

The deformation theory of plasticity is referred to total plastic strain contrary to the incremental formulation of the plastic flow theory. It was proved that both theories lead to the same results in the case of proportional changes of the stress deviator components (so called simple loading or straight
stress paths). The deformation theory can be used also for description of active deformation processes close to mentioned proportional type processes (cf. [111], [06] p.71-86).

The deformation theory corresponds in fact with a nonlinear elasticity and it cannot describe properly the neutral state — when the process is changed from passive into active, discontinuities of plastic strains can arise (cf. [01] p.47). Nevertheless, the Hencky-Ilyushin equation may lead to results even more close to observation then those obtained on the basis of the flow theory. Such a situation has been stated in the buckling analysis of structures and it will be discussed in appropriate part of the course.

2.7. Matrix notation on the example of mixed hardening material

In order to shorten the notation and make it more close to the computation the matrix calculus and notation are introduced also on the $\mathcal{P}$ level. In such an approach the stress and strain tensors $\sigma_{ij}$, $\varepsilon_{ij}$ are changed by appropriate vectors (one-column matrices) $\varrho_*$ and $\xi_*$. These vectors refers to the 9-dimensional vector spaces $\mathbb{R}^9$ but because of the symmetry of the tensors $\sigma_{ij}$, $\varepsilon_{ij}$ also 6-dimensional spaces $\mathbb{R}^6$ are even frequently used associated with the vectors $\varrho$ and $\xi$.

The choice of the vector dimensions should be made carefully in order to ensure the invariance of the length of vectors $\varrho$, $\xi$ and to have energy interpretation of the scalar product $\varrho^T \xi$. No problem arises if the $\mathbb{R}^9$ space is used and the following, tensor type, components are used:

$$
\varrho_* = \{ \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}, \sigma_{21}, \sigma_{32}, \sigma_{13} \}, \\
\xi_* = \{ \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{21}, \varepsilon_{32}, \varepsilon_{13} \},
$$

$$
\varrho_* = \{ s_{11}, \ldots, s_{13} \}, \quad \varepsilon_* = \{ e_{11}, \ldots, e_{13} \}, \quad (2.62)
$$

$$
\bar{a}_* = \frac{\delta F}{\delta \sigma} = \left\{ \frac{\delta F}{\delta \sigma_{11}}, \ldots, \frac{\delta F}{\delta \sigma_{13}} \right\}.
$$

Instead of the tensor-type components the engineering quantities are often applied to in the 6-dimensional space:
\[ \sigma = \{ \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \} , \]
\[ \varepsilon = \{ \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \} , \]
\[ s = \{ s_x, s_y, s_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \} , \]
\[ e = \{ e_x, e_y, e_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \} , \]
\[ a = \{ \frac{\partial F}{\partial \sigma_x} , \ldots , \frac{\partial F}{\partial \tau_{zx}} \} , \]  \hspace{1cm} (2.63)

which give the correct value of the scalar product \( \sigma^T \varepsilon \) but the vector lengths are not invariant. This drawback may be removed by another definition of three last components of the vectors. Multiplying them by \( \sqrt{2} \) the components \( \sqrt{2} \tau_{xy}, \sqrt{2} \tau_{yz}, \sqrt{2} \tau_{zx} \) are used instead of \( \tau_{xy}, \tau_{yz}, \tau_{zx} \). More details on this topic are given in [02], p.88.

In the following consideration the vectors (2.63) will be preferred as they are commonly used in the literature devoted to computational methods, as well as they are usually used in computer codes. But in places where tensorial relations are to be directly expressed in the matrix notation the matrices (2.62) with substars \(^*\) will be introduced.

The square (sometimes rectangular) matrices are of course used in order to transform the above described vectors (one-column matrix). The modular stiffness or flexibility matrices \( E_*, \zeta_* = E_*^{-1} \) replace the appropriate 4-order tensors \( E_{ijkl}, C_{ijkl} \) and the matrices \( E, \zeta \) are associated with the vectors (2.63). Of course, in the principal stress and strain spaces (3x1) and (3x3) dimension matrices are to be used.

The HNH yield condition (2.38) may be shortly written as:

\[ f(\bar{\sigma}) = \frac{3}{2} \bar{s}_s \bar{s}_s - \bar{s}_s^2 (\bar{\varepsilon}_p) , \]  \hspace{1cm} (2.64)

where overbars refer symbols to the relative stress vector

\[ \bar{\sigma}_* = \sigma_* - \sigma_* , \]  \hspace{1cm} (2.65)
which is measured from the center of the subsequent, translated and expanded yield surface.

The increment of plastic strain is split into two colinear components (this idea was suggested by Hodge [112]):

\[ d\varepsilon^P = d\varepsilon^P_{(i)} + d\varepsilon^P_{(k)} , \]  

(2.66)

where \( d\varepsilon^P_{(i)} \) and \( d\varepsilon^P_{(k)} \) are associated with the isotropic and kinematic hardening correspondingly. These two parts of plastic strain increments may be simply written as

\[ d\varepsilon^P_{(i)} = M \ d\varepsilon^P , \quad d\varepsilon^P_{(k)} = (1 - M) \ d\varepsilon^P , \]  

(2.67)

where \( M \) is the mixed-hardening parameter in the range

\[-1 < M \leq 1 . \]  

(2.68)

\( M \) is called as the parameter of mixed hardening. This parameter can also describe isotropic softening since negative values of \( M \) are admissible. Such an softening is sometimes observed in experiments. The isotropic softening is, however, not allowed to dominate over the kinematic hardening simultaneously taking place. In such a way the Drucker stability postulate can be satisfied.

The reduced plastic strain increment \( d\varepsilon^P \) is defined by the isotropic hardening share of \( d\varepsilon^P \), i.e.:

\[ d\varepsilon^P = d\varepsilon^P_{(i)} = M \ d\varepsilon^P . \]  

(2.69)

The accumulated effective plastic strain and increment of effective stress are defined as it follows:

\[ \varepsilon_p = M \int \left[ \frac{2}{3} (d\varepsilon^P)^T \ d\varepsilon^P \right]^{1/2} = M \varepsilon_p , \]

\[ \sigma_{\varepsilon_e} = \bar{H}' \ d\varepsilon^P = M \bar{H}' \ d\varepsilon_p , \]  

(2.70)

where \( \bar{H}' \) is a plasticity modulus associated with the expansion of the yield surface.
The translation of the yield surface may obey the Prager hardening rule (2.32):

\[ \dot{\sigma}_* = c \dot{\xi}^P(k) = c (1 - M) \dot{\xi}^P. \]  

(2.71)

or according to Ziegler's rule (2.35):

\[ \dot{\sigma}_* = \ddot{\sigma}_* \dot{\mu}. \]  

(2.72)

Using the relation (2.36) the function \( \dot{\mu} \) can be expressed analogically to (2.71)

\[ \dot{\mu} = \dot{\sigma}^P(k) = C (1 - M) \dot{\xi}_P. \]  

(2.73)

The consistency condition (2.44) can be written in the following form:

\[ dF \equiv \dot{\sigma}^T \dot{\sigma}_* + \frac{\partial F}{\partial \sigma} \dot{\sigma} - \frac{d(\sigma^2)}{d \sigma} \dot{\xi}_P = 0. \]  

(2.74)

The relations (2.45a) and (2.28) may be straightforward written as:

\[ \dot{\sigma}_* = E_*(\xi_*, \xi^P_*) = E_*(\dot{\xi}_* - a_\mu d\lambda) = \]

\[ = (E_0^e - E^d_0) \dot{\xi}_* = E^d_0 \dot{\xi}_*. \]  

(2.75)

After using the previously defined quantities Eq. (2.74) takes the following form:

\[ \dot{\xi}_*^T E_0^e \dot{\xi}_* - g d\lambda = 0. \]  

(2.76)

where the function \( g \) is used:

\[ g = \dot{\xi}_*^T E_0^e a_\mu + h. \]  

(2.77)

The hardening parameter \( h \) depends on the hardening rule, used to describe the increment \( \dot{\sigma}_* \). In the case of Prager's rule this parameter influences the following relation:
\[ h = c(1 - M) \frac{T}{g} \bar{\sigma}_e \bar{\sigma}_e + \frac{2}{3} M \bar{\sigma}_e \bar{H}' \left( \frac{2}{3} \frac{T}{g} \bar{\sigma}_e \bar{\sigma}_e \right)^{1/2} . \] (2.78P)

and for Ziegler's hardening rule:

\[ h = C(1 - M) \left( \frac{2}{3} \frac{T}{g} \bar{\sigma}_e \bar{\sigma}_e \right)^{1/2} \bar{\sigma}_e \bar{H}' \left( \frac{2}{3} \frac{T}{g} \bar{\sigma}_e \bar{\sigma}_e \right)^{1/2} . \] (2.78Z)

If the HMD yield condition is taken in the form (2.64) then the vector \( \bar{\sigma}_e \) equals \( \bar{\sigma}_e \) and the formulae (2.78) can be transformed to the following form

\[ h = \frac{4}{9} M \bar{H}' \bar{\sigma}_e^2 + \left\{ \begin{array}{l} \frac{2}{3} C(1 - M) \bar{\sigma}_e^2 \quad \text{Prager's h. rule} , \\ \frac{2}{3} C(1 - M) \bar{\sigma}_e^2 \bar{\sigma}_e \bar{\sigma}_e \bar{\sigma}_e \quad \text{Ziegler's h. rule} . \end{array} \right. \] (2.79)

The function \( g \) can be transformed to the following simple form:

\[ g = \frac{1}{3} G \bar{\sigma}_e^2 + h , \] (2.80)

because of the elastic stiffness matrix can be reduced to the form \( E_{\sigma}^e = 2G I \).

That is why the local plastic stiffness matrix (2.49) becomes:

\[ E_{\sigma}^p = \beta \frac{4G \bar{\sigma}_e^2 \bar{\theta}}{3 G \bar{\sigma}_e^2 + h} , \] (2.81)

where value of the control parameter \( \beta \) depends, according to (2.50), on the sign of \( d\lambda \). The function \( d\lambda \) is to be computed from (2.76):

\[ d\lambda = \frac{1}{g} \frac{\frac{T}{g} \bar{\sigma}_e \bar{\sigma}_e \bar{\sigma}_e \bar{\sigma}_e}{2G G \bar{\sigma}_e^2 + h} \] (2.82)

The constitutive relation (2.75) must be related to the simple loading test in order to identify the material parameters. Usually the tension/compression test is used to which the relative stress-tensor and corresponding reduced stress deviator can be written:

\[ \bar{\sigma} = \{ \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 \} = \{ \sigma_1 - \frac{a_1}{2}, \frac{a_1}{2}, \frac{a_1}{2} \} , \] (2.83)

\[ \bar{\sigma} = \frac{2}{3} \sigma_1 - \frac{1}{3} \sigma_1, \frac{a_1}{2} - \frac{1}{3} \sigma_1, \frac{a_1}{2} - \frac{1}{3} \sigma_1 \} . \]
The components of the residual stress vector $\bar{q}$ are related to each other owing to the assumption of the plastic incompressibility assumption (2.3). In such a way the relative effective stress $\bar{\sigma}_e$ in Eq. (2.64) becomes

$$\bar{\sigma}_e^2 = \frac{3}{2} \bar{e}^T \bar{e} = \left(\sigma_1 - \frac{3}{2} \bar{a}_1\right)^2 .$$  \tag{2.84}

Taking the square root of both sides, differentiating and making use of Eq. (2.71) for Prager's strain hardening, the following relation is obtained:

$$d\bar{\sigma}_e = d\sigma_1 - \frac{3}{2} c \left(1 - M\right) d\varepsilon^P_1 .$$  \tag{2.85}

The following relations for the uniaxial state must be hold

$$d\sigma_1 = H' \, d\varepsilon^P_1 \quad \text{and} \quad d\bar{\sigma}_e = M \, H' \, d\varepsilon^P_1 .$$  \tag{2.86}

Since $M$ is an arbitrary material constant the substitution of relations (2.86) into (2.85) enable us to deduce:

$$\bar{H}' = H', \quad c = \frac{2}{3} H' ,$$  \tag{2.87}

and finally the hardening function, defined in (2.79), becomes

$$h = \frac{4}{9} H' \, \bar{\sigma}_e^2 .$$  \tag{2.88}

In the case of Ziegler's hardening rule the comparison of (2.79) with the uniaxial case gives the relation

$$C = \frac{\bar{H}'}{\bar{\sigma}_e} .$$  \tag{2.89}

It turns out that for the $J_2$ material, Ziegler's hardening rule leads to the same hardening function $h$ in (2.88) as Prager's rule, if the appropriate coefficient $c$ and $C$ are defined by (2.87) and (2.89).

The described mixed hardening model was proposed by Axelsson and Samuelsson [108] and was comprehensibly described in [09], p.369-274. This model agrees completely with that analyzed by Tanaka [113], who at the beginning of his considerations did not split the increment plastic strain vector into transitional and expanding parts. He also had to use different definitions for
the reduced deviatoric stress vector in the space $\mathbb{R}^6$ to obtain consistent matrix relations (cf. also [114]).

Using the vectors (2.63) the following explicit modular matrices can be deduced from (2.57), (2.81) and (2.88) for triaxial stress and strain state:

$$
\mathbf{P}^e = 2G 
\begin{bmatrix}
  \frac{1-v}{1-2v} & \frac{1-v}{1-2v} & \frac{1-v}{1-2v} & \frac{1}{2} \\
  \frac{v}{1-2v} & \frac{v}{1-2v} & \frac{v}{1-2v} & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad (2.90)
$$

$$
\mathbf{P}^p = \beta \frac{2}{3} \frac{2G}{\sigma_e (1 + \frac{H'}{3G})} 
\begin{bmatrix}
  -2 & -s_x & -s_y & -s_z \\
  s_x & -s_y & s_z & s_x \\
  s_y & s_z & -s_y & s_z \\
  s_z & s_x & s_y & -s_z \\
\end{bmatrix}, \quad (2.91)
$$

These formulae were derived by Tanaka [113] and earlier by Yamada and Yoshimura [115] for the isotropic hardening.

From special cases only the plane stress state will be pointed out since such a state is considered not only in plane stress problems but also in the theory of thin plates and shells. In the plane stress state the following components vanish:

$$
d\sigma_z = d\tau_{yz} = d\tau_{zx} = 0, \quad d\gamma_{yz} = d\gamma_{zx} = 0 \quad (2.92)
$$
and the constitutive equations take the form [113]:

\[
\begin{bmatrix}
\frac{d\sigma_x}{dt} \\
\frac{d\sigma_y}{dt} \\
\frac{d\tau_{xy}}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{E}{1-v^2} \frac{S_1}{1} - \beta \frac{S_2}{S} \\
\frac{vE}{1-v^2} - \beta \frac{S_1}{S} - \beta \frac{S_2}{S} \\
- \beta \frac{S_1 S_{12}}{S} - \beta \frac{S_2 S_{12}}{S} - \frac{E}{2(1+v)} - \beta \frac{S_{12}}{S}
\end{bmatrix} \begin{bmatrix}
\frac{d\varepsilon_x}{dt} \\
\frac{d\varepsilon_y}{dt} \\
\frac{d\gamma_{xy}}{dt}
\end{bmatrix},
\]

where

\[
S = \left(\frac{4}{9}\right) \frac{H}{1-v^2} \sigma_e^2 + \bar{\sigma}_x S_1 + \bar{\sigma}_y S_2 + 2 \bar{\tau}_{xy} S_{12},
\]

\[
S_1 = \frac{E}{1-v^2} \left( \bar{\sigma}_x + v \bar{\sigma}_y \right),
\]

\[
S_2 = \frac{E}{1-v^2} \left( v \bar{\sigma}_x + \bar{\sigma}_y \right),
\]

\[
S_{12} = \frac{E}{1+v} \bar{\tau}_{xy}.
\]

2.8. Multiple Subvolume (MS) model

An interesting alternative to describe the Bauschinger effect offers the simulation of material continuum by a composite (amalgam) consisting of a number of separate components, which all undergo the same strain history. The idea was originated by Brandtsaeg in 1927 in the context of rock or concrete material, then by Duwez (1935), White (1950), refined by Besseling in 1953 [117,118] and recently extended by Zienkiewicz et al. from the University College of Swansea [118,114,119] and some research teams in the United States (ref. [18]).

The appropriate model is sometimes called as the White-Besseling model [18] but more frequently it is named as the mechanical sublayer [14] or overlay model [118,114,119]. These names are rather unfortunate, as mentioned Bushnell [22] p.71, since they implies that each component in the amalgam occupies a distinct volume or layer. It is not the case since various components are assumed to be mixed homogeneously. That is why Bushnell prefers to call this model as Multiple Subvolume (MS) Model and this name will be used in the present course parallelly with the name White-Besseling model.
In order to formulate the MS model the following assumptions are to be made:

i) All components obey the same total strain history as the element of material being modelled.

ii) Each component \( \ell \) has the same elastic properties and behaves in an elastic ideally-plastic fashion but with different yield loci.

iii) The material stress state is a superposition of the component stress states with weighting factors \( A_\ell \) referred to as components area ratios (subvolumes):

\[
\sigma = A_1 \sigma + A_2 \sigma + \ldots + A_n \sigma = \sum_{\ell=1}^{n} A_\ell \sigma.
\]

(2.95)

In the case of uniaxial stress these assumptions lead to an approximation of the stress-strain curve \( \sigma-\varepsilon \) by a piecewise linear curve as shown in Fig. 2.7a. Each

Fig. 2.7. a) Stress-strain curve for a homogenous specimen, b) Simple idealization of stress-strain curves for the components of MS model.
component is idealized by an elastic ideal-plastic model of the yield point \( \sigma_{0\ell} \) and area ratio \( A_\ell \) - Fig. 2.7b.

The governing equation for the \( \ell \) component is to be written accordingly to the perfect plasticity relation (2.75), which we rewrite in the form:

\[
\frac{d\sigma}{\rho} = \bar{E}_{\text{ep}} d\bar{\epsilon}.
\]

(2.96)

The modular matrix for the component \( \ell \) becomes:

\[
\bar{E}_{\text{ep}} = \bar{E} - \bar{E}_{\text{p}},
\]

(2.97)

where in the elasticity matrix (2.90) \( 2G = E_\perp/(1+\nu) \). The matrix \( \bar{E}_{\text{p}} \) is computed according to (2.91) in which the following values should be substituted: \( \bar{\sigma}_e = \sigma_{0\ell} \), \( H' = 0 \), \( \bar{\sigma} = \bar{\sigma}_e \).

If it is assumed that during the deformation process the values of factors \( A_\ell \) are fixed then on the base of (2.95) and (2.96) the stiffness of the MS model takes the form:

\[
\bar{E}_{\text{ep}} = \sum_{\ell=1}^{n} A_\ell \bar{E}_{\ell\text{ep}}.
\]

(2.98)

Now the identification of the component properties \( A_\ell, \sigma_{0\ell} \) should be considered. Let us assume that the input information is given by the stress-strain curve related to the simple tension test. It is easy to approximate such a curve by a piecewise linear curve in order to have values \( \{\epsilon_1, \ldots, \epsilon_n\} \), \( \{\sigma_1, \ldots, \sigma_n\} \), \( \{E_1, \ldots, E_n\} \). It is possible to use either simple or more refined approach as it has been described in [18].

The simple identification corresponds to Fig.2.7b. Each component is treated separately and do not interfere with other components of the composite (in such an approach the composite is in fact a set of mechanical sublayers). In the case of uniaxial stress each component is assumed to be also in the uniaxial stress state. Under such an assumption the parameters of the MS model equal (ref. [18], p.154-155):
\[
\sigma_{0\ell} = E_1 \varepsilon_{\ell}, \quad \mathcal{A}_\ell = \frac{E_k - E_{k+1}}{E_1} \quad \text{for } \ell = 1, \ldots, n. \tag{2.99}
\]

The outlined approach is not consistent from the mechanical point of view. The components of the composite act to each other and because of yielding even for the overall uniaxial stress state the three dimensional stress state develops in each component. In the refined identification the component stress weighting factors \(A_k\) and yield stresses \(\sigma_{0\ell}\) are obtained by subjecting the strain controlled model to the full three dimensional strain history that accompanies the uniaxial stress-strain curve \(\sigma-\varepsilon\). Such an approach was described in [18], p.156-158. The refined identification procedure will be outlined below for the notation used above.

In the case if uniaxial tension the principal overall stresses and strains are:

\[
\sigma_{11} \neq 0, \quad \sigma_{22} = \sigma_{33} = 0, \tag{2.100}
\]

\[
\varepsilon_{11} \neq 0, \quad \varepsilon_{22} = \varepsilon_{33} \neq 0.
\]

For any point of the linear segment \(k\) of the \(\sigma-\varepsilon\) curve the increments of stresses and strains equal:

\[
d\sigma_{11}^k = E_k \, d\varepsilon_{11}, \quad d\sigma_{22} = d\sigma_{33} = 0,
\]

\[
d\varepsilon_{22} = d\varepsilon_{33} = \left[-\frac{1}{2} + \left(\frac{1}{2} - \nu\right) \frac{E_k}{E_1}\right] \, d\varepsilon_{11}, \tag{2.101}
\]

where plastic incompressibility is taken into account for computing \(d\varepsilon_{22} = d\varepsilon_{33}\).

The parameter \(\sigma_{0\ell}\) is computed for the HMH yield condition

\[
\sigma_{0\ell} = \left[\frac{3}{2} \left(\frac{s_{11}}{2} + 2 \frac{s_{22}}{2}\right)^{1/2}\right]^{(\ell)} \left(\frac{s_{11}}{2} + \frac{s_{22}}{2}\right) = \sigma_{11} - \sigma_{22}, \tag{2.102}
\]

where \(\sigma_{11}, \sigma_{22}\) are defined by the elastic part of the formula (2.58):

\[
\sigma_{11} = \left[\frac{E_1}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_{11} + 2\nu \varepsilon_{22}\right]\right]^{(\ell)},
\]
\[
\sigma_{22} = \sigma_{33} = \frac{E_1}{(1+v)(1-2v)} \left[ \varepsilon_{22} + v \varepsilon_{11} \right].
\] (2.103)

After taking into account that \( \sigma_{11} = \sigma_c \) and (2.101) defines \( \sigma_{22} \) for yielding of subsequent components the following formula for \( \sigma_{ol} \) can be derived:

\[
\sigma_{ol} = \frac{E_1}{1+v} \left\{ \varepsilon_{\ell} + \frac{\ell}{3} \left[ \frac{1}{2} - \left( \frac{1}{2} - v \right) \frac{E_k}{E_1} \right] (\varepsilon_k - \varepsilon_{k-1}) \right\}. \] (2.104)

It has to be underlined that the stress increments in separate components are generally not equal zero, i.e. \( \sigma_{22}^k = \sigma_{33}^k \neq 0 \) because of yielding of subsequent components. In order to fulfill the overall conditions (2.101) it is postulated that \( \sigma_{22}^k = 0 \) at the beginning of each segment \( k \) of the piecewise linear curve \( \sigma-\varepsilon \). After joining the condition \( A_1 + \ldots + A_n = 1 \) the following set of equations is formulated in order to compute the parameters \( A_\ell \):

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
(1) & (2) & (3) & \cdots & (n) \\
\sigma_{22}^1 & \sigma_{22}^2 & \sigma_{22}^3 & \cdots & \sigma_{22}^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(1) & (2) & (3) & \cdots & (n) \\
\sigma_{22}^1 & \sigma_{22}^2 & \sigma_{22}^3 & \cdots & \sigma_{22}^n \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_n \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\] (2.105)

The MS model enables us to describe cyclic deformations corresponding to the perfect Bauschinger effect. For the uniaxial stress case this model gives the stress-strain curve shown in Fig. 2.8.

Interesting possibilities of the overlay type model (simple identification of parameters \( \sigma_{ol} \), \( A_\ell \) is performed) are pointed out by Owen, Prokash and Zienkiewicz [119]. If negative values of \( A_\ell \) are used then a strain-softening characteric of
material may be considered. They introduced also cracking overlays to simulate discontinuous behavior of brittle type materials. More extensive literature on the MS model is quoted in [244].

In 1967 Mróz [120] proposed a model which is similar to the MS model. He introduced the concept of a 'field of workhardening moduli'. The main idea of the concept lies in defining of \( n \) surfaces \( F_0, F_1, ..., F_{n-1} \) in the stress space which correspond to the initial and subsequent yield surfaces at points \( A_1, A_2, ..., A_n \) of the piecewise linear stress-strain curve in Fig. 2.7a. It is postulated that, as in classical theories of plasticity, these surfaces are geometrically similar and at the beginning of the deformation process they are formed round the stress origin. Mróz proposed to consider the rigid movement of these surfaces like it takes place in the classical kinematic hardening. During such a motion the surfaces can only make contact to each other without intersecting. This idea will be explained on an example of an approximation of the tension \( \sigma-\epsilon \) curve by 4 straight-line segment, as it is shown in Fig. 2.9a.

For simplicity let us assume that initially the yield surfaces \( F_0, ..., F_3 \) are a family of concentric circles on the \( \sigma_1-\sigma_2 \) plane as shown in Fig. 2.9b. During the uniaxial tension \( 0-A_1-A_2-B \) the point \( A_1 \) is reached on the circle \( F_0 \) and then this circle is pushing in direction of the circle \( F_2 \) to touch it in the point \( A_2 \). If the active deformation process is continued both circles are translated to make contact in subsequent yield surfaces. Fig. 2.9c presents the situation upon reaching the point \( B \). If the passive process takes place the position of
surfaces $F_i$ is fixed and the new movements starts for any subsequent yielding. Fig. 2.9d corresponds to the uniaxial stress state after local unloading and reloading along the stress path $B-C_1-C_2-D$ (continuous lines). The broken lines are used in the same Fig. 2.9 for the unloading and reloading process going along the nonproportional path $B-E-G$.

Fig. 2.9 a) The piecewise linear uniaxial stress curve, b) Initial position of the yield surfaces, c) Position of the prestressing to the point $B$, d) Yield surfaces after local unloading and reloading.

The just outlined approach was extended by Mróz in order to describe more realistically the cyclic loading. In [121] the yield functions depends on the effective plastic strain, i.e. $F_i(c_p)$. 
The Mroz model gives results very close to those obtained by means of the White-Besseling model. But these models are formulated on the base of various start points. In the MS model material is assumed to be composed of a number of elastic, ideally plastic components whereas the Mroz model deals with homogenous material of changeable hardening properties. The numerical effectiveness of both models will be discussed in next sections.

2.9. Experimental verification and other hardening models

There is quite a large experimental evidence which gives us the physical basis for verification of the proposed classical models of plasticity. Quasistatic experiments confirm quite well Drucker's stability postulate with respect to convexity of yield surfaces and normality rule in the incremental constitutive relations (cf. [16,17,21,24]). As it has been stated in p.2.2 and Fig. 2.3 the HMH condition well approximate the experimental results which define an initial yield surface of isotropic metals. Much less satisfactory is the verification of postulated hardening rules.

Experiments display that during the deformation process the yield surfaces can significantly change their shape and size. These phenomena are controlled by initial prestressing or prestrain and the shape of the stress path. Subsequent yield surfaces depend greatly on how much plastic strain defines yielding (cf. Fig. 2.1).

In Fig. 2.10, taken from papers [122,123] by Williams and Svensson, the yield points are shown for tubular aluminium specimens subjected to tension-compression and torsional loads. Results are shown only for 1% prestrain but also, for such comparatively small initial yielding, significant differences are visible for different definitions of the yield point. Significant distorsion of yield surfaces appears if the proof strain is small.

From among many other experiments (cf. [16]) results of combined tension-interval pressure tests on M-63 brass by Miastkowski and Szczepinski [124] are shown in Fig. 2.11. These results confirm those by Mair and Pugh [125] who also stated that if yielding is defined by zero or very small plastic strain then the
Fig. 2.10 Effect of the yield point definition on the initial and subsequent yield loci following 1 per cent plastic prestrain.

Fig. 2.11 Yield loci for the virgin material (---) and after prestressing along the path OA and OAB (—).
yield loci are anisotropic. If yielding is defined by large plastic strain offsets (0.1 - 0.5%) then the yield loci are usually found to expand nearly isotropically. Results from [124] display also that for nonproportional prestressing path the yield loci move around, usually in direction of prestress. In Fig. 2.11 also direction of strain increments are pointed out.

In order to describe more exactly a subsequent yield locus the beginning of yielding should be found not only for simple tension but also for the secondary yielding which develops as a continuation of the local unloading process (cyclic type process). In Fig. 2.12a, drawn after [126], the primary and secondary yield points are shown. The points 1 and 1' corresponds to the proportional limit, points 2 and 2' to the 20μ offset strains, points 3 and 3' to the back extrapolation points.

![Diagram](image)

Fig. 2.12 a) Uniaxial stress-strain curve, b) Yield loci corresponding to proportional limits, c) Yield loci for 20μ offset strains, d) Yield loci for back extrapolation points.

In Fig. 2.12a,b,c,d the initial and subsequent yield loci are shown on the tension-shear plane, corresponding to the appropriate definitions of the primary and secondary yield points.

Depending on the definition of the yield point the Bauschinger effect can be significant (Fig. 2.12b) or the deformation of the yield locus is close rather
to an isotropic extension (Fig. 2.12d). Existence of so called cross effect (change in yield strength in direction 90 deg. from the loading direction) depends also on the definition of yield point. In Fig. 2.12c positive cross effect is displayed, corresponding to an increase of the subsequent yield locus width.

In the outlined, classical law hardening the subsequent yield surfaces are usually assumed to be congruent to each other. As it has been shown in Figs. 2.10-12 the experimental evidence does not confirm such an assumption, especially for nonproportional prestressing and advanced yielding. That is why several authors tried to formulate more general yield surfaces. Baltov and Sawczuk [127] proposed a more general hardening rule

\[
\sigma_{ij} \sigma_{ij} - 2 \sigma_{ij} \sigma_{ij} + A \sigma_{ij} \sigma_{kl} (\sigma_{ij} - \sigma_{ij}) (\sigma_{kl} - \sigma_{kl}) - \frac{2}{3} \sigma_e^2 = 0 ,
\]

(2.106)

where the tensor \( \sigma_{ij} \) is determined according to Prager's rule (2.32). Eq. (2.106) describes a symmetric, second order surface which can undergo transition, non-symmetric extension and rotation.

The Baltov-Sawczuk yield surface is a special case of the family of anisotropic surfaces given in the following, general form *) [126]:

\[
\frac{1}{2} N_{ijkl} \sigma_{ij} \sigma_{kl} - \frac{1}{3} \sigma_e^2 = 0 .
\]

(2.107)

where \( N_{ijkl} \) is a fourth-order tensor. Axelsson [126] extended the mixed-hardening rule, described in Sec. 2.7, in order to take into account also distortion of the yield surface. He proposed the following tensor

\[
N_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + A \epsilon_{ij}^p \epsilon_{kl}^p.
\]

(2.108)

In order to satisfy the consistency condition the increments of the microstress tensor is to be computed according to the formula:

*) This form results from a generalization of the second invariant of the stress deviator in order to describe anisotropic properties of material either initial or implied by the deformation process (cf. [126,25]). The Hill yield criterion (2.18) is a special case in the frame of the outlined formulation.
\[ \begin{align*}
\sigma_{ij} &= c \sigma_{ij}^P + A \sigma_{ij}^P \sigma_{kl}^P s_{kl}^P .
\end{align*} \] (2.109)

Substitution of (2.108) and (2.109) leads to the following formula for the local plastic stiffness:
\[ \begin{align*}
E_{ijk\ell} &= \beta \frac{2G}{g} t_{ij} t_{k\ell} .
\end{align*} \] (2.110)

where \( t_{ij} \) is the distortional stress tensor and \( g \) is a generalization of (2.80):
\[ \begin{align*}
t_{ij} &= N_{ijmn} \tilde{s}_{mn} , \\
g &= \left[ \frac{4}{3} G + \frac{4}{9} H'(1-M) \right] \tilde{t}^2 - (A R + \frac{4}{9} M H') \tilde{c}_e^2 .
\end{align*} \] (2.111)

The Axelsson hardening rule depends on three parameter \( H' \), \( M \) and \( A \). The parameter \( H' \) corresponds to the slope of the strain-stress uniaxial tension curve, the parameter \( M \epsilon(-1,1) \) describes kinematic-isotropic mixed strain-hardening. The distortional parameter \( A \) is observed to take values from zero to over \( 10^5 \) (cf. [126,127]).

The Baltov-Sawczuk or comparatively simple, three parameter Axelsson's rule cannot fully described the deformation of the subsequent yield surfaces. That is why several others, more complicated relations were proposed (cf.[02] p.160-161). As an example only the Phillips and Weng [128] proposition is sketched. They postulated that in the stress space the yield surface undergoes rigid body motion and deformation, described by 3 scalar function \( \lambda, \xi, \eta \). These functions depend not only on \( \sigma_{ij}^P \) and \( \sigma_p \) but also on the prestressing tensor \( K_{ij} \). The values of the functions \( \lambda, \xi, \eta \) are used as factors for the increment of prestressing \( \Delta K \) as it is shown in Fig. 2.13a.

Phillips stated in his experiments that practically it is impossible to eliminate completely time parameter from tests and consideration [21,26]. In fact instead of one stress-strain curve there are a family of such curves depending on time of test realization. In Fig. 2.13b two of such curves are shown for times \( t \) and \( t+t_{\Delta t} \). If during the experiment under strain control a prestress is applied in the time \( t \) and the yield locus \( F_A \) traced for the time
t + \Delta t then the prestressing point A is situated 'before' the locus as shown in Fig. 2.13a.

Fig. 2.13 a) Yield surfaces for the Phillips-Weng hardening rule, b) Stress-strain curves for different types of tension test realization.

Conceptions similar to those by Phillips can lead to excellent agreement of theoretical and experimental results. In Fig. 2.14, taken from [128], a comparison of four hardening laws is shown. The yield surface by Phillips and Weng approximates so good experimental results that it can be treated as the basis for an evaluation of the theoretical predictions from experiments. But on

Fig. 2.14 Comparison of 4 hardening laws for proportional loading OP where (a) Initial yield locus, (b) First subsequent yield loci.
the other side the Phillips-Weng type models need a great number of tests, made for various prestressing situations in order to identify model parameters (functions).

In the theory of plasticity a great discussion has taken place on the existence of corners in the yield surfaces. The formation of a corner in the stress point can be deduced from the elastic-plastic behaviour of polycrystals. On the base of such considerations Batdorf and Budiansky [129, 130] proposed their slip theory. Many different formulations related to the concept of slip have been proposed in the past thirty years (cf.[16] p.10, [05] p.35-38, [02] p.189-192). Such an approach is referred to the statistical background and localization of plastic strains - cf.[07] p.103-124.

The existence of corners is of great importance for the analysis of yielding. In the corner the plastic strain increment is not uniquely defined - it must only fall within a fan of normals to the segment of the yield surface, as shown in Fig. 2.15a. Budiansky [111] has indicated that a corner can enable us to extend

![Graphs showing yield loci](image)

Fig. 2.15 a) Yield locus according to the Batdorf-Budiansky slip theory, b) Subsequent yield locus after cyclic loading to point A until plastic strain was exhausted, [16] p.16.

the range of applicability of deformation (total strain) theories. Sewell [131] has proposed an associated flow rule for yield surfaces with corners and has demonstrated that buckling loads computed for such relations can be close to
those predicted by the total strain theory. The existence of a corner during the deformation process may explain necking in biaxial tension and instability in plastic sheet stretching - cf.[16], p.15. Hence, the question of a corner is certainly more just of academic interest.

Hecker reviewed in [16] about 90 papers, devoted especially for the biaxial stress state devoted especially for the biaxial stress state. He concluded that most experimentally determined yield loci are smooth. Experiments indicate that an actual corner can not be conducted properly because of many difficulties. First of all different experimental techniques are not sufficiently sensitive to distinguish between a corner and a rounded vertex - - cf. Fig. 2.15b. Unloading, which is required to probe further for the yield locus, erases the corner, if such a corner exists. As it has been mentioned, it is practically impossible to eliminate the time parameter. Viscous properties of material may be responsible for unsuccessful results of many experiments in this field, as Phillips [26] recently concluded.

Quite another problem is related to the formulation of yield surface as combined of hyperplanes in the stress space, i.e. with ribs and corners on them. Such a yield surface was originated by Tresca in 1864 and then the concept of such surfaces was developed by many authors, especially by Hodge - cf. literature listed in [28], and many others, cf. [02] p.161-163. In Fig. 2.16, taken from [25], some piecewise linear hardening surfaces are shown. Such surfaces can be

![Fig. 2.16. Piecewise linear hardening: a) Independent yield surfaces, b) Interdependent yield surfaces, c) Special case of b).](image-url)
considered as a certain approximation to the smooth yield surfaces what implies that corners are obtained as a formal result of such a mathematical treatment and they are not related to the hypothetic physical phenomenon.

Phillips in his experiments (cf. [21, 26]) stated significant motion of yield surfaces, even far from the stress origin - cf. Fig. 2.17. He pointed out also a possibility of defining a bounding type surface (loading surface) which extends isotropically and can be tangent to subsequent yield loci.

![Diagram of yield surfaces](image)

Fig. 2.17 Subsequent yield surfaces and loading surfaces, ref. 42 in [16].

The concept of two-surface plasticity theories was mentioned in [29, 30] in order to describe a transition from hardening to ideally plastic behaviour. A more original idea was proposed by Phillips and Sierakowski [132]. They referred to well known, experimentally observed phenomenon that unloading and reloading paths do not correspond to each other - in Fig. 2.18a paths A-B and B-C respectively. The close to real behaviour is an approximation shown in Fig. 2.18b in which the point D separates purely elastic and plastic reloading (plastic drop, shown in Fig. 2.12a). Phillips and Sierakowski have distinguished two surfaces. The yielding surface \( F_0 \) encloses purely elastic region and corresponds to the point F of the uniaxial stress-strain curve. The stress point for the multiaxial state lies on a loading surface \( G \) which initially corresponds to the point A and can be called as the outer loading surface \( G_A \) in Fig. 2.18c. During unloading intermediate loading surfaces are developing until the loading and yield surfaces make a contact (point d in Fig. 2.18c) and the
Fig. 2.18 a) Experimental strain-stress curve, b) An approximation to the experimental σ-ε curve, c) Yield surfaces $F_i$ and loading surfaces $G_i$ in the stress space.

movement of the loading surface is stopped for purely elastic unloading. During reloading the loading and yield surfaces are changing for the active deformation process - cf. the surfaces $G_C$ and $F_H$ in Fig. 2.18c. Justusson and Phillips [133] considered material stability in the Drucker sense with respect to double surfaces, Eisenberg and Phillips [134] developed the theory under restriction of isotropic hardening to the outer loading and yield surfaces.

Slightly another approach was developed by Dafalias and Popov [135,136]. In their theory the outer loading surface was called as the bounding surface which always encloses the yield surface. During the active process the stress point lies on the yield surface to which the changeable plastic modulus $E^P$ is prescribed, related to the slope of the DC curve of uniaxial stress state (Fig. 2.18b). The deformation and transition of both surfaces are controlled by the value of plastic work during the most recent loading and the parameter $\delta$, defined by the relative position of the bounding and yield surfaces - cf. Fig. 2.19. The approach by Dafalias and Popov is not tied to any hardening rule and, as it was concluded in [23], it appears to be sufficiently general and sound to warrant farther examination.
Fig. 2.19. Yield and bounding surfaces.

Quite similar concept was developed by Krieg [137] who called the boundary surface as limit surface. In the 'metaelastic' zone between the yield and limit surfaces he introduced relations which enable us to describe a variety of situations including reloading. Both surfaces can vary according to a combined kinematic-isotropic hardening. The motion of the yield surface is identical to that assumed by Mróz. For the general multiaxial case the theory requires the retention of 3 vectors and 3 scalars — a small increase over 2 vectors required for kinematic hardening alone.

It is worth to mention about models without a yield surface. Valanis [138,139] proposed to introduce an intrinsic time parameter, independent of external physical time. The intrinsic time is chosen to be a monotonically increasing function of the deformation process. That is why the theory is known as the endochronic theory of plasticity. The stress response is associated with the history of an appropriate deformation. The endochronic theory has been developed by Bazant and his coworkers for modelling the concrete, cf. [31].

From a physical point of view the endochronic theory has many advantages. Phenomena such as noncoincident of yield and loading points and cyclic hardening can be considered. On the other hand, the absence of a yield surface does not facilitate the computational effort. Against the endochronic theories is also
the very difficult identification of material parameters, needed in proposal models.

In this section only isothermal and rate-independent models have been discussed. We will outline more general models in next chapters, where a later release of assumptions of the classical theory of plasticity is considered.

2.10. Comparison of simple hardening rules

The elastic-plastic analysis is much more time consuming than the elastic analysis. That is the most significant reason that more general models have not yet been incorporated into general-purpose programs. Another reason is the material parameter identification which becomes very difficult in the case of multiple parameter models.

The analysis on the $\mathcal{P}$ level has to be associated with the analysis on higher levels. The deformation process in any individual point of the structure is controlled by the loading process on the level $\mathcal{S}$. Because of nonlinearity and history-dependence of the elastic-plastic equations it is practically impossible to establish analytic relations between the processes on different levels of the analysis. Such relations are displaying during the computation. It can occur that simple load programs on the $\mathcal{S}$ level correspond to very complicated deformation processes on the $\mathcal{P}$ level.

In such a situation a comparison of the capability of simple models and their relation to experiments is of great importance. In the literature, devoted to applications of the theory of plasticity to the analysis of structures, there are few appropriate papers. The published results point out on certain limitations and possible fields of application of the simple elastic-plastic models of material. That is why some of this results will be presented below.

In [140] by Hunsaker, Vaughan and Stricklin a comparison of the capability of four hardening rules was made. In the paper only these modes were considered which require only a uniaxial tension test, namely isotropic hardening, Prager-Ziegler kinematic hardening, the Mróz model in noncombined hardening form (the surfaces of constant work-hardening moduli translate rigidly), and the MS,
Besseling-White model. Axial cyclic loading and proportional and nonproportional biaxial loading were considered.

In Fig. 2.20 results, computed for reverse loading [140] are compared with the experimentally evaluated curves by Christman et al. (ref.32 in [140]) and Wells [141].

The theoretical results from [140] are shown in Fig. 2.21 against the experimental curves obtained by Marin and Hsu [142] for tubular steel specimens under tension and internal pressure.

The most important conclusion from the above presented results is that the isotropic strain hardening rule can be superior even for a nonproportional loading path (Fig. 2.20). Poor results obtained with kinematic hardening are due to unimodulus approximation of the uniaxial yielding. For the uniaxial reverse loading the MS and Mróz models give the same results.

![Graph](image)

**Fig. 2.20** Stress-strain relations for reverse compression-tension uniaxial tests for a) Alpha titanium, ref.32 in [140], b) Udimet 700, [141].
Fig. 2.21 Stress-strain relations for biaxial test.

In [140] computer storage requirements have been considered with regard to the four applied models. The kinematic strain hardening needs seven arrays to store data for the plane stress state against four arrays in the case of the isotropic hardening. The number of arrays N needed for the MS and Mroz models depends on number of components (sublayers) n and equals $N = 3 \times n$ and $N = 3+3 \times n$ respectively.

Because of comparatively low computational time and storage requirements the isotropic hardening is recommended in [140] for materials which do not strain-harden appreciably. When dealing with high strain-hardening materials the MS model should be preferred.

Interesting results were published by Owen et al. [119] and complemented by Axelsson and Samuelsson [108]. Thick aluminium ring was subjected to reverse loading and unloading under two diametrically applied point loads as shown in Fig. 2.22a. The load-displacement curves were computed using plane finite elements and the bilinear strain-stress approximation to the experimental
uniaxial tension curve. The mixed strain-hardening parameter $M = 0.20$ leads to excellent approximation of the experimental curve, especially in the case if 'non-linear' hardening is considered (the dropped line in the $\sigma$-$\epsilon$ diagram in Fig. 2.22a). From the viewpoint of the structural response the mixed strain-hardening gives better results then those due to the MS model, cf. Fig. 2.22b.

The motion and deformation of the subsequent yield surfaces strongly depends on the definition of the yield point as it has been emphasized in previous Sec. 2.9. In order to check it Axelsson [126] computed a cantilever beam using plane finite elements. The proportional and back-substitution limits have been used to define the yield point on the approximated uniaxial stress-strain curves - cf. Fig. 2.23a. The computation displays that the deflection of the beam associated
with the approximation B is almost 50% larger than that related to the strain-stress curve A. This is caused by the more developed yielding zones for the case B then for A as it is shown in Fig. 2.23b. In [126] the law of distortional-kinematic hardening was assumed for the approximation A and mixed hardening for B.

![Graphs showing stress-strain and load-deflection relations](image)

**Fig. 2.23** a) Approximations of the uniaxial stress-strain curve, b) Load-deflection relation of a cantilever beam.

The two above discussed examples point out that the structural response may be more or less sensitive to the assumed strain hardening rule and definition of the yield point. For cyclic loading associated with small permanent prestrain the kinematic hardening can give good approximation to experimental results (cf. paper [143] by Armen et al.) but for advanced prestrain the Dafalias and Popov, two-surface type model should be used [136].
3. MODELS ON THE CROSS-SECTION LEVEL $S$

3.1. Generalized variables

All real problems of mechanics are four dimensional. That means that a position of any point (particle) of the deformable body is defined by three geometrical coordinates and the time parameter with respect to an immovable set of coordinates. In statics or in quasistatic problems the physical time does not influence the body response and only for computational reasons a time type parameter can be introduced. Such a parameter is called as conventional time $\tau$ and can be identified with any, monotonically increasing parameter of the problem under consideration. A time type parameter is selected rather on the level $S$ of the analysis, e.g. the load or a selected displacement parameter.

On the base of experimental observations and theoretical evaluations of the behaviour of deformable structures it is possible to introduce additional relations which enable us to decrease the number of independent geometrical variables. These additional relations are called as geometrical hypotheses. From physical point of view the hypotheses are associated with internal constraints which force in a prescribed manner the movement of individual points of the structure. From mathematical viewpoint the hypotheses correspond to additional equations which enable us either to separate or to eliminate certain (geometrically dependent) variables. Similarly are introduced stress hypotheses which enable us to decrease the number of independent components of the stress tensor.

The application of the geometrical and stress hypotheses opens the door to transition from 3D to 2D and 1D problems in such a sense that instead of three independent geometrical variables only two or one variables become independent. Such a transition corresponds to well known reduction of plate and shell structures to a reference surface (usually to the middle surface) and bar structures to reference lines (axis of members). With every point of the reference geometrical object the cross-section is associated.

The definition of the cross-section in shell structures is understood in generalized sense. In Fig. 3.1 the shell cross-section is shown as two perpendicular cross-sections of the thickness $h$ and unit width measured along the shell middle surface.
Fig. 3.1 Cross-section of a shell structure.

With respect to the cross-section stress and moment resultants are introduced called also as sectional forces or generalized stresses. In the case of bar structures the stress vector components

$$ Q = \{N, T_y, T_z, M_x, M_y, M_z\} $$

are defined by the following formulae

Fig. 3.2 a) Generalized stresses for a bar, b) G. stresses and g. deformations for the bar plane bending.
\[ N = \iint \sigma_x \, dA, \quad \tau_y = \iiint \tau_{xy} \, dA, \quad \tau_z = \iiint \tau_{xz} \, dA, \]
\[ M_x = \iiint (\tau_{xz} - \tau_{xy}) \, dA, \quad M_y = \iiint \sigma_z \, dA, \quad M_z = -\iiint \sigma_y \, dA. \] (3.2)

These definitions correspond to the notation and sign convention with respect to the local (cross-sectional) set of coordinates as shown in Fig. 3.2a.

In the special case of the plane bending three non-zero g. stresses are: \( N, T = T_z, M = M_y \). In Fig. 3.2b also g. deformations corresponding to these g. stresses are shown. If the hypothesis of non-deformable cross-section is applied then the displacements of any point P are the functions of two variables \( x \) and \( z \):

\[ u(x,z) = u_0(x) + \beta(x)z, \quad w(x,z) = w_0(x). \] (3.3)

Small strains are calculated according to Cauchy's formulae:

\[ \varepsilon_x = \frac{\partial u}{\partial x} = \frac{du_0}{dx} + \frac{dw_0}{dx}, \quad \gamma_z = \gamma \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \beta + \frac{dw_0}{dx}, \] (3.4)

where \( \varepsilon, \kappa \) and \( \gamma \) are called as the generalized strains:

\[ \varepsilon = \frac{du_0}{dx}, \quad \kappa = \frac{dw_0}{dx}, \quad \gamma = \beta + \frac{dw_0}{dx}. \] (3.5)

The g. stresses and strains has to be consistent with respect to the strain energy or to the virtual work of internal forces:

\[ \delta W_i = \left[ \left( \iint \sigma \, T \xi \, dA \right) \delta x = \int (N\delta \varepsilon + T\delta \gamma + M\delta \kappa) dx = \right. \]
\[ = \int \Phi^T \delta q \, dx, \] (3.6)

where for the plane bending the g. stress and g. strain vectors are:
Q = \{N, T, M\}, g = \{\varepsilon, \gamma, \kappa\}.

(3.7)

The hypothesis of non-deformable cross-section leads to well known Timoshenko's beam theory. The appropriate generalized quantities can be deduced in the theory of thin plates under assumption of straight segment (non-deformable generalized cross-section). This assumption is associated with the Reissner-Mindlin plate theory. In this theory the displacements along the axis x and y equals:

\begin{equation}
\begin{align*}
u(x, y, z) &= u_0(x, y) + \beta_x(x, y)z, \\
\gamma(x, y, z) &= v_0(x, y) + \beta_y(x, y)z, \\
\tau(x, y, z) &= w_0(x, y).
\end{align*}
\end{equation}

(3.8)

The strains are linear functions of z:

\begin{equation}
\begin{align*}
\varepsilon_x &= \varepsilon_0^x + \kappa_x z, \\
\varepsilon_y &= \varepsilon_0^y + \kappa_y z, \\
\gamma_{xy} &= \gamma_{0xy} + \chi_{xy} z, \\
\varepsilon_z &= 0, \\
\gamma_x &= \beta_x + \frac{\partial w_0}{\partial x}, \\
\gamma_y &= \beta_y + \frac{\partial w_0}{\partial y}.
\end{align*}
\end{equation}

(3.9)

The corresponding g. strain vector is:

\begin{equation}
g = \{\varepsilon_0^x, \varepsilon_0^y, \gamma_{0xy}, \kappa_x, \kappa_y, \chi_{xy}, \gamma_x, \gamma_y\} = 
\begin{bmatrix}
\frac{\partial u_0}{\partial x}, \\
\frac{\partial u_0}{\partial y}, \\
\frac{\partial u_0}{\partial z}, \\
\frac{\partial v_0}{\partial x}, \\
\frac{\partial v_0}{\partial y}, \\
\frac{\partial v_0}{\partial z}, \\
\frac{\partial w_0}{\partial x}, \\
\frac{\partial w_0}{\partial y}
\end{bmatrix}.
\end{equation}

(3.10)

The g. stress vector

\begin{equation}
Q = \{N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}, T_x, T_y\}.
\end{equation}

(3.11)

has the following components (plate stress and moment resultants)

\begin{equation}
\begin{align*}
N_{\alpha\beta} &= \int \sigma_{\alpha\beta} dz, & M_{\alpha\beta} &= \int \sigma_{\alpha\beta} z dz, & T_\alpha &= \int \tau_\alpha dz,
\end{align*}
\end{equation}

(3.12)
where \( \alpha = x, \beta = y \). In Fig. 3.3 the sign convention is shown. The convention corresponds to the positive value of stresses in the first quadrants of the planes \((x,z)\) and \((x,y)\). It does not agree with the convention for the bar moments \( M_x, M_y, M_z \) in Fig. 3.2a, where all \( g \) stresses are referred to the set of cross-section coordinates \((x,y,z)\).

![Diagram showing stress and moment conventions](image)

Fig. 3.3 The convention for the plate \( g \) stresses (plate stress and moment resultants).

In the general case of thin shell analysis the \( g \) stress and \( g \) strain vectors are of the same form (3.10) and (3.11) but the appropriate formulae for their components are more complicated. These components can be consistent if the principle of virtual work is explored for their derivation (cf. paper [144] by Sanders).

The number of components of the vectors of \( g \) stresses and \( g \) strains \( Q \) and \( g \) depends on the number of interval constraints which are used in the frame of geometrical hypotheses. If besides the non-deformability of the cross-section its orthogonality to the deformed reference surface (reference line) is kept then the shear strains disappear, i.e. \( \gamma_{xz} = 0 \). In such a case the shear forces \( T_g \) are non-active since they do not give the work on the \( g \) strains \( \gamma_g \). That is why sometimes the shear forces are called as passive forces or reactions - their values can be computed from the equilibrium equations only.
The nondeformable and normal cross-section corresponds to the Kirchhoff-Love hypothesis (KL) in the theory of plates and shells and to the Bernoulli hypothesis (B) for bar structures. According to this hypothesis the vector of g. stresses and g. strains has the following components *):

a) Plane bar structures

\[ \mathbf{Q} = \{N, M\}, \quad \mathbf{g} = \{\epsilon, \kappa\} , \quad (3.13a) \]

b) Plates and shells

\[ \mathbf{Q} = \{N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}\} , \]

\[ \mathbf{g} = \{\epsilon_x, \epsilon_y, \gamma_{xy}, \kappa_x, \kappa_y, \chi_{xy}\} . \quad (3.13b) \]

*) The relations for thin shells can be associated with the simplified theories, e.g. Love's or Sanders' equations for which the membrane force and moment tensors are symmetric, i.e. \( N_{xy} = N_{yx}, M_{xy} = M_{yx} \).

3.2. Relations on the level \( \mathcal{P} \)

The relations between g. stresses and g. strains may be obtained after the substitution of the appropriate relations at the \( \mathcal{P} \) level for the functions in the integrals (3.2) or (3.12). The incremental form of the constitutive relations, which correspond to the matrix Eq. (2.75) **) can be written in general form:

\[ \mathbf{d} \mathbf{Q} = \mathbf{D}^{ep} \mathbf{d} \mathbf{g} , \quad (3.14) \]

where components of the cross-section stiffness matrix \( \mathbf{D}^{ep} \) depend not only on material, but also on the structure and adopted geometrical hypotheses.

In the case of KL hypothesis the increments of active g. stresses \( \mathbf{d} \mathbf{Q} = \{dN, dM\} \) are associated with the increments of g. strains \( \mathbf{d} \mathbf{g} = \{d\epsilon, d\kappa\} \) by the following equation:

**) The star * is omitted for the definitions (2.63) of the stress and strain vectors in the 6-dimension spaces.
\[
\begin{bmatrix}
\frac{dN}{dq} \\
\frac{dM}{dg}
\end{bmatrix} =
\begin{bmatrix}
\frac{h}{2} & \frac{h}{2} \\
\frac{-h}{2} & \frac{-h}{2}
\end{bmatrix}
\begin{bmatrix}
\int E_{ep}^{dz} \\
\int E_{ep}^{zdz}
\end{bmatrix} 
\begin{bmatrix}
\frac{d\varepsilon_{0}}{d\varepsilon} \\
\frac{d\varepsilon}{d\varepsilon}
\end{bmatrix},
\]
\[(3.15)\]

where the modular matrix \(E_{ep}^{\ast}\) (on the \(\mathcal{P}\) level) corresponds to the plane stress state, cf. Eq. (2.93).

Quite similar relations may be obtained for the uniaxial state of stresses, which is assumed to be in a slender bar subjected to plane bending (Bernoulli's hypothesis is applied):

\[
\begin{bmatrix}
\frac{dN}{dq} \\
\frac{dM}{dg}
\end{bmatrix} =
\begin{bmatrix}
\int \int E_{ep}^{dz A} \\
\int \int E_{ep}^{zdz A}
\end{bmatrix} 
\begin{bmatrix}
\frac{d\varepsilon}{d\varepsilon} \\
\frac{d\varepsilon}{d\varepsilon}
\end{bmatrix},
\]
\[(3.15a)\]

where \(E_{ep}^{\ast}\) is the actual tangent modulus of the uniaxial stress-strain curve.

Coefficients of the stiffness matrix \(D_{ep}^{\ast}\) are defined by the integrals taken along the plate thickness \(h\) or over the cross-section area \(A\). The computation of the integrals

\[
D_{ij}^{k} = \int_{-h/2}^{h/2} E_{ij}^{zp}(z)^{k}dz \quad \text{or} \quad D^{k} = \int_{A} E_{ij}^{zp}(z)^{k}dz \quad \text{for} \quad k = 0,1,2.
\]
\[(3.16)\]

is the main problem of the transition from the level \(\mathcal{P}\) to level \(\mathcal{S}\), i.e. \(\mathcal{P} \rightarrow \mathcal{S}\).

The computation of the integrals (3.15) turns out to be easy if the passive (elastic) process takes place in all points of the cross section. In this case the coefficient \(\beta=0\) in (2.93) and \(E_{ij}^{ep} = E_{ij}^{e}\) or \(E_{ij}^{p} = E\). If the plane \((x,y)\) is the middle plane of the plate or the central (and principle for the plane bending) plane of the bar then for homogeneous material \(D_{ij}^{1} = 0\) and \(D^{1}\) equals zero.
This corresponds to the separation of the membrane and bending states. Finally the elastic cross-section stiffness matrices equal:

\[
P^e = \begin{bmatrix} D_M & 0 \\ 0 & D_B \end{bmatrix} \quad \text{or} \quad P^e = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix},
\]

where the following notation is used:

\[
D_M = \frac{Eh}{1-v^2}, \quad D_B = \frac{Eh^3}{12(1-v^2)}, \quad D = \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}.
\]

The computation of the integrals (3.16) becomes difficult if different type of the deformation process takes place in the points of the cross-section. The shape of the bar cross-section and its structure (uniform, sandwich or multilayer), as well as material properties should be considered. Even in the simplest cases the analytical formulae, if they can be derived, appear to be very complicated. This is shown below for the rectangular cross-section of the bar made of elastic, perfectly plastic material.

In Fig. 3.4 the development of the active deformation process is shown - from one-side yielding (Fig. 3.4c) to the limit state of full, two-side yielding (Fig. 3.4e).

Fig. 3.4 a) Rectangular cross-section, b) Distribution of elastic strain and stresses, c) One-side yielding, d) Two-side yielding, e) Limit state.
For the case of two-side yielding the stress distribution obeys the following relations:

\[
\sigma_x = \begin{cases} 
-\sigma_0 & \text{for } -h/2 \leq z \leq z_- , \\
E(c + \kappa z) & \text{for } z_- \leq z \leq z_+ , \\
\sigma_0 & \text{for } z_+ \leq z \leq h/2 .
\end{cases}
\] (3.19)

After integration and some mathematics the following formulae can be derived (cf. [02] p.302):

\[
z_\pm = -\frac{h}{2} \left[ n \mp \sqrt{3(1-n^2-|m|)} \right] ,
\]

\[
c = -\frac{\sigma_0}{E} \frac{z_+ + z_-}{z_+ - z_-} = \frac{\sigma_0}{E} \frac{n}{\sqrt{3(1-n^2-|m|)}} ,
\] (3.20)

\[
\kappa = \frac{\sigma_0}{E} \frac{2}{z_+ - z_-} = \frac{2\sigma_0}{E h} \frac{\text{sign } m}{\sqrt{3(1-n^2-|m|)}} ,
\]

where \(n = N/(\sigma_0 bh)\) and \(m = 4M/(\sigma_0 bh^2)\) are the non-dimensional axial force and bending moment respectively.

In Fig. 3.5 moment-curvature relation is shown for the pure bending, i.e. for the case \(n = 0\). The spreading of the yielding leads to the nonlinear relation, according to Eq. (3.20).

Fig. 3.5 Moment-curvature relation for the pure bending of a rectangular, elastic-perfectly plastic bar.
In the case of combined action of the bending moment $m$ and axial force $n$ the bending and tension states are not separated after a yielding, as it takes place in elastic range – cf. (3.17).2. The relation for one-side yielding are more complicated then these for two-side yielding.

Because of mentioned difficulties the only general approach to the analysis on the level $\mathcal{Y}$ can be supported on numerical methods. The most evident approach to the computation of the integrals (3.16) depends on the approximation by a weighted algebraic sum:

$$I = \int \int f(z) \, dA = \sum_{j=1}^{J} A_j f(z_j),$$  \hspace{0.5cm} (3.21)

where the number of integration points $J$, their distribution and values of the parameters $A_j$ influences the accuracy of the approximation (3.21).

In the above approach the analysis on the $\mathcal{P}$ level is carried out at every integration point $j$. Appropriate information associated with the integration points has to be stored and modified in order to follow the deformation process.

Different approximation formulae of the type (3.21) have been used in the elastic-plastic analysis. Some of them are more suitable for the analysis of bar structures and others are useful rather for plates and shells (cf.[02]).

3.3. Multilayer and multipoint substitutive cross-sections

The simplest approximation is associated with the division of the cross-section into finite number $J$ of layers parallel to the reference plane. This approach, introduced by Popov et al. [145], has been explored by many authors and used in various computer codes.

A more deep discussion of the multilayer model was presented by Balmer and Doltinis [146]. In the case of a monosymmetric bar cross-section the reference points $j$ are usually placed in the middle of the individual layer. If the homogeneous stress distribution is assumed in the layer (cf. Fig. 3.6b) then the axial force $N$ and the bending moment $M$ equal:

$$N = \sum_{j=1}^{J} \sigma_j A_j, \quad M = \sum_{j=1}^{J} \sigma_j z_j A_j,$$  \hspace{0.5cm} (3.22)
where the average area equals \( A_j = h_j b_j \) for \( b_j = b(z_j) \).

![Diagram](image)

Fig. 3.6 Multilayer cross-section for a monosymmetric bar.

According to the multilayer approximation the integrals (3.16) are equal:

\[
D^k = \sum_{j=1}^{J} E_j^p(z_j)^k A_j \quad \text{for } k = 0, 1, 2.
\] (3.23)

The accuracy of the approximation (3.23) depends on the number of layers \( J \) and their thickness \( h_j \) (or coordinates \( z_j \)). The appropriate choice of these parameters leads to so called multipoint substitutive cross-sections.

An interesting approach to the optimal choice of a multipoint substitutive cross-section was suggested by Orkisz [147]. He assumed that the uniform and multipoint cross-sections should be equivalent to each other with respect to the subsequent moments of the area of both cross-sections. This approach will be explained on the example of the 4-point equivalent cross-section.

Let us assume that two points are below and two of them are above the central axis \( y \) - cf. Fig. 3.7a. The moments are to be computed for the areas \( A_I \) and \( A_{II} \) respectively. For the area \( A_I \) the following set of equations may be formulated:

\[
A_1 + A_2 = \int \int_{A_I} dA = A_I,
\]

\[
A_1z_1 + A_2z_2 = \int \int_{A_I} zdA = S_I,
\] (3.24)

\[
A_1z_1^2 + A_2z_2^2 = \int \int_{A_I} z^2dA = I_I.
\]
Fig. 3.7 a) Uniform cross-section, b) Multipoint substitutive cross-section, c) Values of the function $f$ in the integration points $j$, d) Trapezoidal approximation for the constant distances $\Delta z = h/(J-1)$.

If this set of equations is completed by the relation $z_1 = h_I$ then one obtains (cf. [02] p.317):

$$A_1 = \frac{A_{II} - S_I}{I_I - 2S_{II}h_I + A_I h_I^2}, \quad z_1 = h_I,$$

$$A_2 = \frac{(S_I - A_I h_I)^2}{I_I - 2S_{II}h_I + A_I h_I^2}, \quad z_2 = \frac{I_I - S_{II}h_I}{S_I - A_I h_I}.$$

(3.25)

The areas $A_3$, $A_4$ and coordinate $z_3$ are computed in the same way for the coordinate $z_4 = h_{II}$, the area $A_{III}$, the static moment $S_{III}$ and the moment of inertia $I_{III}$ about $y$ for the lower part II of the cross-section - cf. Fig. 3.7a.

The proposed 4-point cross-section has been called as the equivalent 4-point cross-section since it has got the same elastic and plastic load carrying capacity with respect to tension and pure bending as the uniform cross-section has. In the case of equivalence of the moments of higher order, i.e. for $k > 2$ in (3.24) it is possible to construct the equivalent cross-section for a larger number $J > 4$ of concentrated areas - cf. [147].
In many papers devoted to the plastic analysis of structures the so called ideal sandwich cross-section has been applied. Such a model obeys the KL or B hypotheses and the membrane stress assumption in the faces - cf. Fig. 3.8. The ideal sandwich c-s can be considered as an approximation to the standard uniform I cross-section.

![Diagram](image)

Fig. 3.8 Ideal sandwich cross-section, distribution of the normal stress and strain.

If the ideal sandwich c-s is treated as 2-point substitution cross-section then two conditions of equivalency to the uniform rectangular cross-section \( b \times h \) can be formulated. The condition of the same load carrying capacity for tension \( 2bt = bh \) gives

\[
t = \frac{h}{2} \quad \text{or} \quad A_1 = A_2 = \frac{bh}{2}, \tag{3.26a}
\]

and from the second condition of equivalency for pure bending

\[
H = \begin{cases} 
    h/3 & \text{elastic carrying capacity} , \\
    h/2 & \text{plastic c.c.} , \\
    h/3 & \text{elastic stiffness} . 
\end{cases} \tag{3.26b}
\]

The c-s stiffness matrix for the ideal sandwich bar structure becomes

\[
\mathbf{D}^{ep} = \begin{bmatrix}
    \frac{1}{2} (E^{ep} + E^{ep})bt & \frac{1}{2} (E^{ep} - E^{ep})btH \\
    \frac{1}{2} (E^{ep} - E^{ep})btH & \frac{1}{4} (E^{ep} + E^{ep})btH^2
\end{bmatrix}, \tag{3.27a}
\]

and for the plates and shells
The number of stress components in faces equal the number of the active resultant forces in the ideal sandwich cross-section. That makes easy transition $P \rightarrow \mathcal{P}$ and $\mathcal{P} \rightarrow \mathcal{F}$ on the basis of evident linear relations.

Different possibilities to formulate approximations of the type (3.21) are offered by the quadrature formulae (cf. [34]). In these formulae the weight coefficients $A_j$ depend on the approximation curve. For the case of the piecewise linear approximation (trapezoidal rule) and constant distance between integral points $\Delta z = h/(J-1)$ the coefficients $A_1 = A_j = 0.5$, $A_2 = \ldots = A_{J-1} = 1$ and the integrals (3.16) are approximated according to (3.20):

$$\mathcal{P}_k = \frac{\Delta z}{2} \left( f_1 + 2f_2 + \ldots + 2f_{J-1} + f_J \right).$$

(3.28)

where $f_j = E^{ep}(z_j)^{k} b_j$. For the odd number of points $J$ the piecewise parabolic approximation can be applied that leads to the Simpson formula:

$$\mathcal{P}_k = \frac{\Delta z}{3} \left[ f_1 + 4(f_2 + f_4 + \ldots + f_{J-1}) + 2(f_3 + f_5 + \ldots + f_{J-2}) \right].$$

(3.29)

Both the trapezoidal and Simpson formulae are special cases of the Newton-Cotes closed formulae - cf. [32] p.118-119. Their accuracy depends on the number of the integral stations $J$ as well as on the function $f(z)$. For smooth functions the number $J$ can be decreased if more accurate quadrature formulae are applied, e.g. the Gaussian quadratures - cf. [32] p.87-100 (in [32] evaluation of errors associated with the above mentioned quadratures can be found).

In the case of rectangular cross-section the multilayer equivalent cross-section gives the same approximation as appropriate numerical quadratures. For $J = 5$ and $z_1 = -z_5 = h/2$ the following parameters can be computed: $A_1 = A_5 = bh/12$, $A_2 = A_4 = bh/3$, $A_3 = bh/6$, $z_2 = -z_4 = h/4$, $z_3 = 0$ which fully agree with the Simpson formula. For a larger number $J$ of concentrated areas the equivalent multipoint cross-section corresponds to the Lobatto formula, and if the constraints $z_1 = -z_J = h/2$ are rejected then the Gauss coefficients are obtained. In Tab. 3.1 the coordinates $\xi_j = 2z_j/h$ and coefficients $W_j = 2A_j/(bh)$ are shown for the Legendre-Gauss and Lobatto quadrature formulae (cf. [34] p.89,107).
Table 3.1 The abscissas $\xi_j = 2z_j/h$ and weights $W_j = 2A_j/(bh)$ for the Legendre-Gauss and Lobatto quadrature formulae.

The models discussed above for the bar cross-section can be directly applied to the analysis of plates and shells if the unit value of the width is adopted, i.e. for $b = 1$.

In every integration point $j$ the stress analysis on the level $\theta$ has to be carried out. That is why the computational effort is proportional to the number of these points J.

The number of integration points J has to be fixed at the beginning of the computations because of the history-dependent deformation process. Evaluation of the value of $J$ is not simple because of many factors which influence the accuracy of the elastic-plastic analysis. Material properties, type of structures and boundary conditions, distribution of loads, development of yielding, local unloading and reloading zones can significantly influence the computational errors.

That is why a rigorous theoretical treatment is not available for general situations and numerical experience is limited to special cases. The number of layers in the multilayer model ranges from 40 in the paper [145] by Popov et al., devoted to plate bending to 10 layers in the case of elastic-plastic...
analysis of shells, as it was reported by Bäcklund and Wennerström [148]. The
last result was supported by Balmer and Doltsinis [146] with respect to the flat
plate triangular element TRIB3. Parish [149] advocates for 16 layers in the
isoparametric quadrilateral element QUAD4.

Marcal and Pilgrim [150] and Marcal [151] used 11 integration points in the
trapezoidal formula for the analysis of elastic-plastic shells of revolution.
Ramm [152] obtained satisfactory results using 7 points in Simpson's rule for
the large displacement analysis of general shells.

The Gaussian quadratures were used by Crisfield [153] in the large-deflection
elastic-plastic buckling analysis of thin plates and Corneau [154] in bending of
thick plates and shells. Crisfield used 3 and 5 points in the Legendre-Gauss
formula. A more deep analysis by Corneau indicates that nearly the same
satisfactory results can be obtained for 5-6 points in the Legendre-Gauss or
Lobatto formulae instead of 10 layers of equal thickness.

The Lobatto quadrature can be recommended because it gives information about
yielding of bounding surfaces of plates and shells (external fibers of bars).
That is why this formula was applied by Nguyen-Cao-Duong and Waszczyszyn to the
large deflection of circular panels and arches [155] and Cichon to plane bending
of frames [156]. In [155] it was stated that instead of 5 points in the Lobatto
quadrature only the 3-point equivalent cross-section is enough to predict the
limit external pressure of a circular panel.

Numerical experience resulting from many computations can be concluded as it has
been expressed by Corneau [154] p.211:

i) The multilayer model underestimates inelastic displacements.

ii) The trapezoidal formula overestimates them.

iii) Gaussian rules require smaller number of integration stations.

iv) The pure bending state is more sensitive to the choice of the quadrature
    rule then mixed bending-membrane states.

3.4. Plastic interaction surfaces and associated flow rule on the level $\varepsilon$

Let us assume that the material is elastic, perfectly plastic and consider the
rectangular cross-section under action of the axial force $N$ and bending moment
M. In the limit state of full yielding (Fig. 3.3e) the condition \( z_+ = z_- \) is fulfilled and from (3.20) the following two relations can be deduced:

\[
\begin{align*}
 n^2 + m &= 1 \quad \text{for} \quad m \geq 0 , \\
 n^2 - m &= 1 \quad \text{for} \quad m \leq 0 .
\end{align*}
\]  

(3.30)

The relations (3.30) are associated with curves on the plane \((n, m)\) - cf. Fig. 3.9a which determine the limit state of full yielding of the rectangular cross-section.

![Diagram](image)

**Fig. 3.9 Interaction plastic curves for the rectangular cross-section a) Exact plastic interaction curve and strain rate vectors b) Different approximations to the exact interaction curve.**

These curves can be treated as a special case of the plastic interaction surface

\[
\Phi(Q) = 0 .
\]  

(3.31)

which corresponds to the limit state of full yielding of the structural cross-section.

In the case of rigid, perfectly plastic material the surface \( \Phi \) bounds the domain of passive deformation processes for which the rate of g. strain vector is equal to zero, i.e. \( \dot{g} = 0 \). For any active deformation process the vector \( \dot{g} \neq 0 \) and the vector of g. stress \( Q \) is associated with the plastic interaction surface also by
the consistency condition \( \dot{\varphi} = 0 \). In such a way the following relations are fulfilled for the active (a) and passive (p) processes respectively:

\[(a) \quad \dot{\varphi}(Q) = 0, \quad \dot{\xi} = 0, \quad \dot{\varphi} \neq 0, \quad (3.32a) \]

\[(p) \quad \dot{\varphi}(Q) < 0, \quad \dot{\xi} < 0, \quad \dot{\varphi} = 0. \quad (3.32p) \]

In the case of convex surface \( \varphi \) the condition of neutral change of stresses

\[ \dot{\sigma}_{ij} \dot{\varepsilon}_{ij} = 0 \]

leads to the relation

\[ \dot{Q}_i \dot{q}_i = 0, \quad \dot{q}_i = 0. \quad (3.33) \]

The differentiation of (3.32a) gives the following relation

\[ \frac{d\varphi}{dt} = \frac{3\varphi}{3Q_1} \dot{q}_1 = 0. \quad (3.34) \]

Comparing (3.33) and (3.34) the associated flow rule on the level \( \mathcal{Y} \) can be deduced in the following form

\[ \dot{q}_1 = \lambda \frac{3\varphi}{3Q_1}, \quad \lambda > 0. \quad (3.35) \]

The relations on the level \( \mathcal{Y} \) fully correspond to the appropriate relations on the level \( \varphi \). That is why the plastic interaction curves \( \varphi \) are called also as yield surfaces. The function (3.31) can be interpreted as the plastic potential from the viewpoint of the associated flow rule (3.35) and the parameter \( \lambda \) corresponds to the scalar function \( d\lambda \) in (2.42).

Returning to the example of rectangular cross-section the generalized variables and the plastic interaction curves (p.i.c) are:

\[ Q = (n, m), \quad \dot{g} = (\dot{\xi}, \dot{h}) \quad (3.36) \]

\[ \Phi_1 = n^2 + m - 1 = 0, \quad \Phi_2 = n^2 - m - 1 = 0. \]

The p.i.c. is singular in the points A, C for \( n = 1 \) - cf. Fig. 3.9a. In these points the vector \( \dot{g} \) is to be determined by the formula
\[
\dot{q}_i = \dot{l}_k \frac{a_k}{\dot{a}_i},
\]

which places the rate vector \(\dot{q}\) in the cone formulated by the vectors orthogonal to the surfaces \(a_k\).

With the active process the dissipation function \(D\) can be also associated

\[
D = cQ_i \dot{q}_i > 0,
\]

where \(c > 0\) is a constant.

The exact p.i.c. can be approximated by different linear relations as it has been shown in Fig. 3.9b. These approximation may be supported on the multipoint equivalent cross-section model. For instance the ideal sandwich cross-section with the condition \((3.26b)_2\) gives the straight lines A-B, B-C, C-D, D-A.

The considered case of the bar cross-section under action of the axial force and bending moment correspond to the transition \(\mathcal{P}_1 + \mathcal{S}_2\) according to Zyczkowski's classification [02]. It means that only one stress is active on the level \(\mathcal{P}\) versus two g. stresses on the level \(\mathcal{S}\).

Formulation of the p.i. surfaces for the more complicated cases \(\mathcal{P} \to \mathcal{S}_5\) may be very difficult and can need additional assumptions. In the case of combined action of bending with tension of shear, i.e. for \(\mathcal{P}_2 + \mathcal{S}_3\) (Timoshenko's beam), a distribution of shear stress is to be assumed (cf. [02] p.403) to obtain an approximate plastic interaction surface.

In the case of multiaxial stresses, i.e. for \(r > 1\) in \(\mathcal{P}\), the p.i. surfaces depend on the yield condition and constitutive relations on the level \(\mathcal{P}\), on assumed geometrical and stress hypothesis as well as on the mode of cross-section. The problem will be illustrated on the example of pure bending of plates under KL hypotheses, i.e. for the case \(\mathcal{P}_2 + \mathcal{S}_3\).

Let us assume the HNNH yield condition. The function \(F\) for the plane stresses, corresponding to \((2.16a)\), may be written in the following form:

\[
F = 3\sigma_{\alpha\beta} \sigma_{\alpha\beta} - \sigma_{\alpha\alpha} \sigma_{\beta\beta} - 2\sigma_0^2 = 0,
\]

(3.39)
where summing is to be made for subscripts \( \alpha, \beta = x, y \). The material is assumed to be rigid ideally-plastic and incompressible \((\dot{\epsilon}_{33} = -\dot{\epsilon}_{yy})\). The associated flow rule gives the following relations:

\[
\dot{\epsilon}_{\alpha\beta} = \dot{\lambda}(3\sigma_{\alpha\beta} - \sigma_{yy}\delta_{\alpha\beta}) \rightarrow \sigma_{\alpha\beta} = \frac{1}{3\dot{\lambda}}(\dot{\epsilon}_{\alpha\beta} + \dot{\epsilon}_{yy}\delta_{\alpha\beta}) .
\]  
(3.40)

According to the KL hypothesis the strains for pure bending equal:

\[
\dot{\epsilon}_{\alpha\beta} = z\dot{k}_{\alpha\beta} .
\]  
(3.41)

Substitution of (3.41) and (3.40) into the yield condition (3.39) leads to the value of \( \dot{\lambda} \):

\[
\dot{\lambda} = \frac{z}{\sigma_o \sqrt{6}} \left( \kappa_{\alpha\beta} \dot{\kappa}_{\alpha\beta} + \kappa_{\alpha\alpha} \dot{\kappa}_{\beta\beta} \right)^{1/2} .
\]  
(3.42)

Coming back to (3.40) the stresses are expressed in the form independent of the variable \( z \):

\[
\sigma_{\alpha\beta} = \sqrt{\frac{2}{3}} \frac{\sigma_o}{(\kappa_{\alpha\beta} \dot{\kappa}_{\alpha\beta} + \kappa_{\alpha\alpha} \dot{\kappa}_{\beta\beta})^{1/2}} (\dot{\epsilon}_{\alpha\beta} + \dot{\epsilon}_{yy}\delta_{\alpha\beta}) .
\]  
(3.43)

The moment resultants equal

\[
M_{\alpha\beta} = 2\sigma_{\alpha\beta} \int_0^{h/2} zdz = \frac{h^2}{4} \sigma_{\alpha\beta} .
\]  
(3.44)

and the yield condition (3.39) can be easily transformed to the form

\[
m_x^2 + m_y^2 - m_{xy} m_y + 3m_{xy}^2 = 1 .
\]  
(3.45)

where dimensionless moment resultants are used: \( m_{\alpha\beta} = 4M_{\alpha\beta}/(\sigma_o h^2) \) and \( m_x = m_{xx}, m_y = m_{yy} \).

Eq. (3.45), associated with the HHH yield condition formulates an ellipsoid in the 3D space of moment resultants \((m_x, m_y, m_{xy})\) as it is shown in Fig. 3.10.
The flow mechanism is determined by the rates of g. strains according to the flow rule (3.35):

\[
\begin{align*}
\dot{\varepsilon}_x &= \dot{\varepsilon}_{xx} = \dot{\lambda}(2m_x - m_y), \\
\dot{\varepsilon}_y &= \dot{\varepsilon}_{yy} = \dot{\lambda}(2m_y - m_x), \\
\dot{\varepsilon}_{xy} &= \dot{\varepsilon}_{yx} + \dot{\varepsilon}_{yx} = 6\dot{\lambda} m_{xy}.
\end{align*}
\] (3.46)

In case of the TG yield condition the p.i. surface is defined by relations which can be deduced from (2.6) and (3.44):

\[
\max (|m_1|, |m_2|, |m_1 - m_2|) = \pm 1, \tag{3.47}
\]

where the principal moments \( m_1, m_2 \) are related to moments \( m_{\alpha\beta} \) by relations:

\[
\begin{pmatrix}
m_1 \\ n_2
\end{pmatrix} = \frac{1}{2} (m_x - m_y) \pm \frac{1}{2} \sqrt{[(m_x + m_y)^2 + 4m_{xy}^2]}^{1/2}. \tag{3.48}
\]

Fig. 3.10 Plastic interaction surfaces for the pure bending state of thin plates and shells.

From relations (3.47) the following equations are derived (cf. [157]):

\[
m_x + m_y = m_{xy} + m_{xy}^2 = 1,
\]
\[ -m_x - m_y - m_x m_y + m_{xy}^2 = 1 , \]  
\[ m_x^2 + m_y^2 - 2m_x m_y + 4m_{xy}^2 = 1 , \]  
which define two elliptic cones and an elliptic cylinder in the space \((m_x, m_y, m_{xy})\) — cf. Fig. 3.10.

In the general case of the bending-membrane state of plates and shells under the KL hypothesis the transition \(P_3 \rightarrow \mathcal{G}_6\) is much more difficult. The pioneering work was made by Ilyushin [04] who used the HMM yield condition and the deformation theory of plasticity. He defined the following dimensionless quadratic g. stress intensities:

\[
\begin{align*}
P_n &= n_x^2 + n_y^2 - n_x n_y + 3n_{xy}^2 , \\
P_m &= m_x^2 + m_y^2 - m_x m_y + 3m_{xy}^2 , \\
P_{mn} &= m_x n_x + m_y n_y - \frac{1}{2} (m_x m_y + m_y m_x) + 3m_{xy} n_{xy} .
\end{align*}
\]  

(3.50)

These intensities were related by means of complex parametric relationships of the form

\[
P_n = f_1(\alpha, \beta) , \quad P_m = f_2(\alpha, \beta) , \quad P_{mn} = f_3(\alpha, \beta) ,
\]  

(3.51)

where the two parameters \(\alpha\) and \(\beta\) are:

\[
\alpha = \frac{e_2}{e_1} , \quad \beta = \frac{e_0}{e_1} .
\]  

(3.52)

The equivalent strains on the top surface, on the bottom surface, and minimum value, \(e_1, e_2, e_0\) respectively, are used in (3.52).

Unfortunately, Eqs. (3.50) are not in a form suitable for computation. That is why a number of attempts have been made to approximate these equations by elimination the parameters \(\alpha\) and \(\beta\). Ilyushin proposed the approximation:

\[
P_n + P_m + \frac{1}{\sqrt{3}} |P_{nm}| = 1 .
\]  

(3.53)
Robinson [158] has shown that this is the best of the various linear approximations in the space $(P_n, P_m, P_{nm})$.

A more accurate approximation was given by Ivanov [159]:

$$p_n + \frac{1}{2} \left[ P_m + \left( P_m^2 + 4P_m^{2_{mn}} \right)^{1/2} \right] - \frac{1}{4} \frac{P_n P_m - P_{nm}^2}{P_n^2} = 0.48 \left( P_n^2 + P_m^2 \right) = 1.$$  \hspace{1cm} (3.54)

Interesting approximations to Ilyushin's and Ivanov's p.i. surfaces were proposed by M.A. Crisfield [153,160,161].

Similar results were obtained by Sawczuk and Rychlewski [162] on the base of the plastic flow theory equations. They considered the HMH and TG yield conditions and different types of cross-sections. In case of the ideal sandwich cross-sections the TG yield conditions leads to linear relations between g. stresses (hyperplanes in the $R^6$ space).

In special cases the p.i. surfaces can be described by simpler relations as it has been shown for the pure bending state. P.i. surfaces for plates and shells of revolution, related to the transition $P_2 \rightarrow Y_4$ are discussed in [36]. For cylindrical shells the problem can be reduced to $P_2 + Y_3$ or $P_2 + Y_2$ if the influence of $m_0$ or $m_0$ and $n_x$ is neglected — cf. [34a].

The literature devoted to p.i. surfaces for plates and shells is enormous — cf. list of books and review papers in [02] p.413. This was caused by the analytic approach to the limit analysis. Introduction of computers and development of numerical methods enable us to eliminate a number of simplifications and use more complicated and accurate relations.

Plastic interaction surfaces can be used as plastic potential functions in the approximate elastic-plastic analysis on the level $P$. For purpose of such an analysis let us assume that the strain rate vector $\dot{g}$ is composed of elastic and plastic parts:

$$\dot{g} = \dot{g}^e + \dot{g}^p.$$  \hspace{1cm} (3.55)

where the elastic strain rate $\dot{g}^e$ and plastic strain rate $\dot{g}^p$ equal
\[ \dot{\mathbf{e}} = \mathbf{D}^e \mathbf{q}, \quad \dot{\mathbf{p}} = \lambda \frac{\partial \Phi}{\partial \mathbf{q}}. \] (3.56)

The derivation procedure is quite similar to that on the level \( \Phi \). Making use of the consistency condition \( \dot{\mathbf{q}} = 0 \) for the active process (3.32a) the plastic parameter \( \nu \) can be calculated and the constitutive relations are of the form (3.14) with respect to the stress and strain rates

\[ \dot{\mathbf{q}} = \mathbf{D}^{ep} \dot{\mathbf{g}}. \] (3.57)

Eqs. (3.56-57) had been explored by Crisfield [153,163] and by Crisfield and Puthli [163] who applied these equations to the large deflection analysis of plate and shell structures. Crisfield named the approach associated with the cross-section potential function \( \Phi \) as the 'area' approach versus the 'volume' approach related to the multilayer model of the plate cross-section [153]. Similar approach has been earlier used in the elastic-plastic analysis of bar structures. From among many papers that one [165] by Argyris et al. can be quoted, devoted to the elastic-plastic analysis of space frames with \( \gamma_6 \) (plane frames are treated as the special cases \( \gamma_2 \) or \( \gamma_3 \)).

The main advantage of the 'area' approach is a significant decrease of computational time and storage requirements because of elimination of the integration along the thickness (area) or the cross-section. Disadvantages are associated with all those problems which accompany the formulation of plastic interaction surfaces. The surfaces are related in fact to the elastic, perfectly plastic model of material. Another, serious deficiency corresponds to the disregarding of the spread of yielding along the cross-section. That, of course, can be a source of errors which are difficult for estimating 'a priori'.

The mentioned deficiencies of the 'area' model and increasing poverty of computers and their software cause that in great majority of computer codes the 'volume' type models are used. Despite of that it is worth of attention that for specific problems the 'area' approach can be succesfully applied to, as it was shown in papers by Crisfield, Algyris et al.
4. MODELS ON THE ELEMENT AND BODY (STRUCTURE) LEVELS $E$ AND $B$

4.1. Kinematic and static equations, principles of virtual work

In Ch. 2 the constitutive (physical or material) equations have been discussed. In order to analyze boundary value problems (BVP) of the elastoplasticity the constitutive equations have to be completed by kinematic (geometrical) and equilibrium equations as well as by appropriate boundary conditions. In case of incremental plasticity initial values should be added, consistent with the above mentioned set of equations. All the equations will be recapitulated below with respect to the 3D continuum.

Let us consider a body $B$ of volume $V$, bounded by the surface $S$. The position of every point $P$ of the body is determined by the Cartesian set of coordinates $(x,y,z)$. Let us assume that in the initial configuration $V$, $S$ none of external load factors act and that under such an action a new (actual) configuration $V'$, $S'$ is reached - cf. Fig. 4.1.

![Fig. 4.1. Initial and actual configuration of the body $B$.](image)

The displacement vector $u = \overrightarrow{PP'}$ has 3 components

$$u = \{u,v,w\}, \quad (4.1)$$

and each of them is assumed to be a continuous differentiable function of the coordinates $x,y,z$:

$$u = f_1(x,y,z), \quad v = f_2(x,y,z), \quad w = f_3(x,y,z). \quad (4.2)$$
The infinitesimal strain vector $\varepsilon$

\[ \varepsilon = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\} \]  \hspace{1cm} (4.3)

is referred to the displacement vector $u$ by the Cauchy strain-displacement relation, called as kinematic equation:

\[ \varepsilon = L u. \]  \hspace{1cm} (4.4)

where the consistency matrix of differential operators is

\[ L = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix} \] \hspace{1cm} (4.5)

On a part of the surface $S_u$ the displacements are given

\[ u = u_0 \quad \text{on} \quad S_u. \] \hspace{1cm} (4.6)

The relation (4.6) formulates so called kinematic boundary conditions (k.b.c.). In many structures (especially in civil engineering) the k.b.c. are formulated in such a way to eliminate the rigid body motion, e.g. $u_0 = 0$ on $S_u$ in Fig. 4.1.

The displacements and strains are called as to be kinematically admissible $u = u$, $\varepsilon = \varepsilon$ if they fulfill the relations (4.4) and (4.6).

Let us assume that the body is subjected to the action of the volume forces $g$ and tractions $t$:

\[ g = \{g_x, g_y, g_z\} \quad \text{in} \quad V, \quad t = \{t_x, t_y, t_z\} \quad \text{on} \quad S_t, \] \hspace{1cm} (4.7)

which cause a stress state, defined by the stress vector $\sigma$:

\[ \sigma = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}. \] \hspace{1cm} (4.8)
The volume forces $\mathbf{g}$ and the stresses $\sigma$ are related by the **equilibrium equations**

$$ \mathbf{g} = L^* \sigma ,$$  \hspace{1cm} (4.9)

where

$$ L^* = -L^T .$$  \hspace{1cm} (4.10)

At the surface $S$ the equilibrium of tractions $\mathbf{t}$ and stresses $\sigma$ is defined by the following relation

$$ \mathbf{t} = L^*_\sigma \sigma ,$$  \hspace{1cm} (4.11)

where the algebraic matrix $L^*_\sigma$ corresponds to the matrix $L^T$ through the substitution of differential operators by the components of the unit vector $\mathbf{v}$ (directional cosines):

$$ L^*_\sigma = \begin{bmatrix} v_x & 0 & 0 & v_y & 0 & v_z \\ 0 & v_y & 0 & v_x & v_z & 0 \\ 0 & 0 & v_z & 0 & v_y & v_x \end{bmatrix} .$$  \hspace{1cm} (4.12)

**Static boundary conditions** (s.b.c.) correspond to given value of tractions

$$ \mathbf{t} = \mathbf{t}_0 \text{ on } S_t .$$  \hspace{1cm} (4.13)

The loads and stresses are called as to be **statically admissible** $\mathbf{g} = \mathbf{g}$, $\mathbf{t} = \mathbf{t}$, $\sigma = \sigma$ if they fulfill the equilibrium equations (4.9) and (4.11), as well as the static boundary conditions (4.13).

All outlined equations are linear since they are supported on assumptions of small displacements and infinitesimal strains. That is why any equilibrium state can be referred to the initial configuration $V,S$ instead to the actual one $V',S'$. More precise formulation has to be done in the analysis of large displacement (strain) problems as it will be shown in next chapters.

The **principles of virtual work** (PVW) serve as a general tool to formulate models consistent with the kinematic and equilibrium equations on different levels of
the analysis. In Ch. 3 the generalized variables on the level \( \mathcal{G} \) have been chosen in such a way.

Two versions of PVW can be used. The first one, called also as the principle of virtual displacements uses any of possible kinematically admissible fields of consistent displacements and strains, i.e. \( \delta \mathbf{u} = \mathbf{\tilde{u}} \), \( \delta \mathbf{e} = \mathbf{\tilde{e}} \), in order to formulate the following PVW:

\[
\int_V \sigma^T \mathbf{\tilde{\varepsilon}} \, dV = \int_V \mathbf{g}^T \mathbf{\tilde{u}} \, dV + \int_S \mathbf{\mathbf{l}}^T \mathbf{\tilde{u}} \, dS \quad (4.14a)
\]

The other version, called as the complementary PVW is associated with the statically admissible fields of loads and stresses. For \( \delta \mathbf{g} = \mathbf{\tilde{g}} \), \( \delta \mathbf{t} = \mathbf{\tilde{t}} \), \( \delta \mathbf{\sigma} = \mathbf{\tilde{\sigma}} \) the following PVW can be formulated:

\[
\int_V \varepsilon^T \mathbf{\mathbf{\sigma}}^T \mathbf{\tilde{\varepsilon}} \, dV = \int_V \mathbf{u}^T \mathbf{\tilde{g}} \, dV + \int_S \mathbf{\mathbf{u}}^T \mathbf{\tilde{t}} \, dS \quad (4.14b)
\]

If the fields are statically and kinematically admissible they correspond to exact (real) solutions and the balance of VW gives the following relation

\[
\int_V \sigma^T \mathbf{L} \mathbf{u} \, dV = \int_V \mathbf{u}^T \mathbf{L}^* \sigma \, dV + \text{(boundary terms)} \quad (4.15)
\]

The linear operators \( \mathbf{L} \) and \( \mathbf{L}^* \) which fulfill the condition of the type (4.15) are called as self-adjoint. In case of algebraic matrices the self-adjointing depends on transposition of the matrices, whereas for the operator matrices the change of sign should be additionally taken into account as it is shown in (4.10).

The kinematic and static pair of variables, \( \mathbf{u}, \mathbf{e} \) and \( \mathbf{g}, \mathbf{\sigma} \) respectively, are related by dual relations and are associated with the reciprocal correspondence of the strain and stress states, as it is shown in Fig. 4.2.

The outlined relations, formulated for the 3D continuum, are valid on the level \( \mathcal{P} \). Transition to the generalised variables on the level \( \mathcal{G} \) is straightforward. The internal virtual work (3.6) defines the g. stress and strain vectors \( \mathbf{g}, \mathbf{e} \) and the external VW the g. displacements and loads \( \mathbf{w}, \mathbf{p} \) as well as. The g. variables
on the level $\mathcal{Y}$ are to be consistent with those on the level $\mathcal{P}$ in the sense of fulfilling the PWV:

$$
\mathcal{P}^T \mathcal{w} = \int_A \mathcal{g}^T \mathcal{u} \, dA + \int_C \mathcal{e}^T \mathcal{u} \, dC,
$$

$$
\mathcal{Q}^T \mathcal{q} = \int_A \mathcal{e}^T \mathcal{e} \, dA.
$$

(4.16)

The $g$ variables are related by dual relations, quite similar to those on the level $\mathcal{P}$. In Fig. 4.2 these relations are expressed by means of the consistency operator matrix $\mathcal{C}$.

![Diagram of dual relations for kinematic and static variables.]

Fig 4.2. Dual relations for kinematic and static variables.

All outlined relations have been presented in the matrix form. Such a formulation makes them more applicable for computation. An exact derivation of all mentioned equations may be found in many books and papers, in which the analogy on different level of analysis for the matrix formulation was pointed out, cf. e.g. the book [36] by Przemieniecki, papers [166,167] by Besseling.

The presented formulation has been taken from the book [37] by Borkowski as an introduction to the application of linear programming methods in the rigid-plastic and elastic-plastic analysis of structures. This problem will be considered in the next chapter.

4.2. Discrete models of structures

In general the models of structures can be divided into two groups: (C) continuous models, (D) discrete models. Differential and integral operators are
used in the (C) models and results are chosen in the form of functions (analytical solutions). Algebraic operators (matrices) are applied to the analysis of (D) models which give numerical solutions. The computational methods of analysis are adjusted to (D) models.

Usually a transition $C \rightarrow D$ is performed. The transition is carried out in various ways. In the finite difference method (FDM) derivatives are approximated by finite differences; in the finite elements the transition $C \rightarrow D$ is made on the level $\mathcal{E}$. In both methods the variables are lumped in the nodes related to the level $\mathcal{B}$.

It is possible to build up also mixed models $C/D$ which lead to semi-analytical solutions. The application of such models (e.g. finite strip models or conic type FE models) is rather restricted to a small class of problems - mainly in the field of linearly elastic analysis.

In Fig. 4.3 the transition $C \rightarrow D$ is shown for various models of the type (D) on an example of a plate in the plane stress state.

Discrete models can be associated with an approximation of the continuous fields by appropriate sets of admissible functions. In majority of (D) models the displacement vector is approximated

$$ u(x) = X(x)g \quad \text{in} \quad V, $$

where $X(x)$ is the matrix of basic functions and $g$ is the vector of generalized DOF.

The overall, Ritz type approximation (4.16) can be applied to rather restricted cases in which the distribution of the displacements $u$ is smooth (regular) the FE approach is widely explored. The FE approach is widely explored in order to extend the analysis on more complex cases of various shapes of bodies and their boundary conditions, variable properties of material as well as on different configuration and changes of loads. In this approach the body is divided into a finite number of elements (cf. Fig. 4.3b) and the approximation is restricted to the subvolume $V_j$ (subarea $A_j$):

$$ u^j = N(x)d^j \quad \text{in} \quad V^j. $$

(4.18)
where the shape functions $N(x)$ can be used instead of $X(x)$ and the displacement vector $d^j$ is rather introduced instead of the vector $g$ of generalized DOF. The FEM is extensively used in the elastoplastic analysis — that is why it will be discussed in next chapters of the present course.

Fig. 4.3. Various models for the analysis of a plate. a) Continuous model, b) Plane FE model, c) Boundary FE model, d) Mixed FE model, e) Regular FD mesh, f) Irregular FD model.

In recent years the boundary FE method (BEM) was applied to the analysis of various field and structural problems. BEM is supported on the Trettiez type approximations (cf. [38,39]):

$$u(s) = N(s)a \quad \text{on } \Gamma, \quad (4.19)$$

where the basic functions depend on the boundary variables $s$, and the displacements $a$ are related to the boundary nodes (cf. Fig. 4.3c). The difficult problem of formulation of the admissible set of functions $N(s)$ causes that BEM is rather rarely used in plasticity.
Both approaches FEM and BEM can be successfully joint (cf. e.g. [169]) if appropriate consistency conditions are fulfilled along the common boundary $\Gamma_1$ - cf. Fig. 4.3d.

The variational version of FDM, close to FEM was developed by Bushnell et al. [170] and Bushnell [171,172] who applied this version to the inelastic, large deflection analysis of axisymmetric shells.

An interesting extension of FDM has been recently done with respect to irregular finite difference meshes — cf. Fig. 4.3f. Orkisz [173], Liszka and Orkisz [174] adopted this approach also in the analysis of plasticity field problems.

The spread of yielding zones during the deformation process is a source of significant complications in the computational process. That is why approaches, earlier developed for the limit analysis and dynamics of structures have been applied. They correspond to models with lumped parameters. The models refer first of all to the analysis of bar structures.

In Fig. 4.4 a simple plane frame is shown under vertical and horizontal concentrated forces $V$ and $H$. The continuous structure can be divided into deformable FE - cf. Fig. 4.4b, or rigid FE can be connected in deformable joints - cf. Fig. 4.4c.

Fig. 4.4. a) Continuous plane frame, b) Division into deformable FE, c) Introduction of deformable joints.

In case b) the level $E$ is considered in individual FE in order to prepare data for the analysis on the level $B$. In case c) the level $E$ is omitted and the analysis can be carried out directly on the level $B$. 
The transition $E \rightarrow B$ has been developed in structural mechanics — especially in the frame of displacement (stiffness) method, cf. [38]. On a similar background the displacement FEM is supported where the level $E$ is useful to automatize the computational process. Introduction of the level $E$ enable us to generalize the analysis, since after specification of FE the analysis on the level $B$ is in fact independent of the type of structures.

The above mentioned advantages of the intermediate level $E$ cause that this level is introduced also to the analysis of rigid-plastic models of structures. The joints are then associated with so-called plastic hinges in which discontinuities of g. strains are associated with the plastic flow of material. In Fig. 4.5b a rigid-plastic, plane beam FE is shown under assumption of the Bernoulli hypothesis (active g. stresses are the axial force $N$ and bending moment $M$).

![Diagram](image)

Fig. 4.5. a) Deformable FE, b) Rigid-plastic FE, c) Elastic-plastic FE with lumped plastic properties.

The continuous FE, shown in Fig. 4.5a, are commonly used in the elastic analysis. For elastic-plastic material the spread of yielding zones inside the element has to be taken into account. This disadvantage can be overcome if the area type cross-sections are considered in selected points of the element, as it is shown in Fig. 4.5c. The elastic-plastic FE with lumped plastic properties leads of course to better approximation in comparison with the rigid-plastic FE.

In case of FE with lumped material properties only a part of g. variables, shown in Fig. 4.4a,b, is independent. The outlined FE were formulated by Borkowski [37,175], Boni and Kleiber [176], Kleiber and Sosnowski [177]. These elements
were used by Argyris et al. [165], Borkowski and Saran [178] in the analysis of large displacements of framed structures.

Formulation of models with lumped properties for surface structures is also possible. Ang and Lopez [179] used the ideal sandwich cross-sections as deformable nodes joining rigid bars and torsional elements for the analysis of elastic-plastic plates. A plate FE with discontinuities along the element edges was suggested by Hodge and McMahou [180]. Extensive literature on FE with lumped plastic properties can be found in [40, 41].

Continuous FE, supported on an approximation of the displacement field, are the most commonly used FE in elastoplasticity. They offer a wide spectrum of various possibilities and enable us to take into account various material properties, described in Ch. 2. That is the main reason why this type of FE will be discussed in the present report.
5. LIMIT AND SHAKEDOWN ANALYSIS OF STRUCTURES BY LINEAR PROGRAMMING

5.1. Fundamentals of the limit analysis *

Let us consider the simple loading case called as proportional loading. That means that all external loads are proportional to one load parameter \( \mu \)

\[
p = \mu \overline{p},
\]

where \( \overline{p} \) is the reference load vector which is kept constant during the deformation process.

At a certain value \( \mu_0 \) the first yielding occurs in a certain cross-section of the considered structure. The corresponding load \( p_0 = \mu_0 \overline{p} \) is called as the elastic load carrying capacity of the structure.

An increase of the load parameter is associated with a development of yielding zones in the structure. This behaviour is called as constrained plastic flow, because the deformations of the structure are still controlled by elastic strains. Above certain value of the load parameter the spread of yielding or the localization of yielded zones may cause that the remaining elastic material does not contribute effectively to sustaining the load. Such a behaviour is called as unconstrained plastic flow.

The unconstrained plastic flow may be associated with a real collapse of the structure. This is the case which can take place for bar structures and shells under certain load configuration \( \overline{p} \). In bar structures localization of plastic zones can be modelled as plastic hinges. At certain number of plastic hinges the bar structure can change into a mechanism.

In case of surface structures either collapse may occur or deformations become so large that the structure is no longer usable.

The outlined behaviour of elastic-plastic structures can be illustrated by means of the equilibrium path which relates a representative displacement \( d \) to the load parameter \( \mu \). In Fig. 5.1a the equilibrium path 1 corresponds to that considered behaviour of an elastic-plastic structure for which collapse occurs at the value \( \mu_L \) of the load parameter.

*) In order to simplify considerations the analysis is performed with respect to the generalized variables.
Fig. 5.1 a) Stable and unstable equilibrium paths for large displacement behavior of elastic-plastic and rigid-plastic structures, b) Equilibrium path for small displacement formulation and a structure with lumped plastic properties.

The equilibrium path 2 corresponds to stable behavior of a plate-type structure. The collapse does not take place but because of plastic flow a significant increasing of displacements can be stated.

The equilibrium paths 1 and 2 are associated with the elastic-plastic model of material and nonlinear, large displacement - strain relations. If the model of material is changed into the rigid-plastic one then the equilibrium paths 1' and 2' can be calculated. These paths start from the plastic load carrying capacity parameters $\mu_p$ which are referred to initial plastic flow mechanisms. In case of perfectly elastic-plastic or rigid-plastic models of materials an increase or decrease of the load parameter along the equilibrium paths are associated with so called geometrical hardening (softening) of the structure under consideration. In order to analyse structural hardening/softening nonlinear (large displacement) strain-displacement kinematic relations have to be taken into account.

Elastic, perfectly plastic structures are more flexible than those structures made of rigid, perfectly plastic material. That is the reason why the equilibrium paths 1' and 2' bound from above the equilibrium paths 1 and 2, computed for elastic, perfectly plastic structures. The post-yield behaviour
(equilibrium paths B-C of B-A) can be traced combining the rigid-plastic model of material and large displacement equations.

In the classical approach to the limit analysis of structures we are interested in computation of the value $\mu_p$ of the plastic load c.c. parameter and with associated plastic flow mechanisms. These quantities correspond to the beginning of the plastic flow of rigid-plastic structures. That is why linear, small displacement - strain relations can be used in the analysis. Since in such a formulation the equilibrium is referred to an initial, undeformed configuration of a structure the solutions corresponding to the limit state (the value of $\mu_p$ and plastic flow mechanisms) are the same, both for the elastic, perfectly plastic and rigid, perfectly plastic models of material - cf. Fig. 5.1b.

In case of application of the elastic, perfectly plastic model of material the development of yielding in the structure can be followed. The analysis may be considerably simplified if the model of structure with lumped plastic properties is used. In Fig. 5.1b subsequent points 1,...,4 on the equilibrium path are associated with the creation of plastic hinges (discontinuity lines in surface structures) in these points up to changing the structure into a mechanism. In the example considered in Fig. 5.1a the creation of the 4th plastic hinge causes that the structure becomes a mechanism and this last hinge provokes unrestricted plastic flow, i.e. the displacements increase without any restriction. Of course, such considerations are possible only in the frame of small displacement assumptions.

Another approach depends on assumption of the rigid, perfectly plastic model of material for the structure with lumped plastic properties. In the approach a sequence of plastic flow mechanisms (p.f.m.) is considered. Every p.f. mechanism is created by an appropriate number of plastic hinges from among all possible hinges.

The above sketched approach is presented on the example of a portal frame similar to that from Fig. 4.4. The bending moment is assumed to be the only g. stress in the rigid, ideally plastic nodes 1,...,4. In order to simplify the computation let us assume that the plastic limit moment is of the value $M_0 = \overline{W}G$ in the columns and $2Mo$ in the beam. In Fig. 5.2b,c the so called basic p.f. mechanisms are shown which are called as sway and beam p.f. mechanisms respectively. The relations for the rotation rates $\dot{\theta}_i$ as well as the value of the load
parameter are shown — the lowest value \( \mu_c = 8M_p/(3\bar{P}l) \) has been computed for the combined p.f. mechanisms, shown in Fig. 5.2d.

\[ H = \mu \bar{P} \]
\[ V = 2\mu \bar{P} \]

\[ \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \bar{u}/l \]
\[ \mu_c = 8M_p/(3\bar{P}l) \]

Fig. 5.2. a) Data for a portal, plane frame, b) Sway plastic flow mechanism, c) Beam p.f. mechanism, d) Combined p.f. mechanism.

The first approach, depending on the development of subsequent yield hinges, is called as the static approach. In this approach statically admissible fields of g. variables are used. That means that according to the definition, introduced in p. 4.1., the equilibrium equations and static boundary conditions are fulfilled in the static approach, as well as the g. stress fulfill the yield conditions in all points of the structure. With respect to the static approach the following statical theorem of limit analysis is valid:

Any load parameter \( \bar{p}^{(i)} \) corresponding to a statically admissible stress field (a statically admissible g. stress vector) cannot exceed the actual collapse load parameter \( \bar{\mu}_p \) (plastic load c.c.):

\[ \bar{p}^{(i)} \leq \bar{\mu}_p. \]  \hspace{1cm} (5.2)
The second approach depends on the analysis of different p.f. mechanisms and it is called as kinematic approach. With respect to this approach the following kinematical theorem of limit analysis can be proved:

Any load parameter \( \hat{\mu}_i \) corresponding to a kinematically admissible plastic flow mechanism by the power equation is not less than the limit load parameter \( \mu_p \):

\[
\hat{\mu}^{(i)} \geq \mu_p .
\]  

(5.3)

The power equation corresponds to the principle of virtual work (4.13) in which the virtual strains and displacements are replaced by the rates of g. strains and g. displacements which are assumed to be kinematically admissible:

\[
\int_\Omega Q^T \dot{g} \, d\Omega = \mu \left( \int_\Omega \tilde{\varepsilon}^T \dot{\varepsilon} \, d\Omega + \int_{\partial \Omega} \tilde{\tau} \dot{\gamma} \, d\Omega \right) .
\]  

(5.4)

The name 'power' is used to be explored in the limit analysis since the product \( Q^T \dot{g} \) corresponds to the product of a force and velocity in physics i.e. to \( \mathbf{F} \dot{\mathbf{V}} \). The left hand side of Eq. (5.4.) can be called as the dissipation function \( D \) on the level \( \mathcal{B} \), correspondingly to (3.38) on the level \( \mathcal{F} \).

The static and kinematic theorems are called also as the lower and upper-bound theorems of the limit analysis. They are used for evaluation of the limit load parameter:

\[
\hat{\mu}^{(i)} \leq \mu_p \leq \hat{\mu}^{(j)} .
\]  

(5.5)

A solution corresponding to \( \mu_p \) fulfill the requirements of static and kinematic admissibility and is called as exact or complete solution for the limit analysis.

The limit analysis problems were formulated firstly for beam-type steel structures by Kazincky in 1914 and Kist in 1917 but a real progress was made by Gvozdyev at the end of 30-ies. The complete formulation of the limit analysis theorems were given at the beginning of 50-ies due to papers by Greenberg, Prager, Drucker and Hill — cf. [42] p. 405-407. A deeper discussion of the outlined problems can be found in many excellent books, cf. e.g. [08] by Save
and Massonnet or [42] by Martin as well as in many review papers, cf. e.g. Massonnet [20].

The limit analysis can be treated much more wide as in fact it is related to the evaluation of the load carrying capacity of structures. From such a point of view we have to return to Galileo. According to Prager [44], Galileo considered in his 'Discorsi', which were published in Leyden in 1638, the failure of a cantilever beam. Stability of masonry vaults and domes' Coulomb's analysis of retaining walls, metal forming can be pointed out as next examples of applications of the limit analysis.

5.2. Shakedown and inadaptation of structures

The theory of limit analysis considers the collapse of structures under proportionally increasing loads. However, the loads applied to engineering structures vary often independently, changing their magnitude, sense and direction. The loads are mostly repetitive in nature and vary within given limits, but neither the sequence of loading nor its frequency being prescribed.

The theory of shakedown deals with determination of the permissible range of variation of loads independently acting on elastic-plastic structures, in order to avoid some kinds of plastic collapse. The elastic-plastic response to variable repeated actions can be defined as an adaptation (called commonly as structural shakedown) or an inadaptation.

The inadaptation of a structure is associated with structural crisis caused by incremental collapse or by low-cycle fatigue. The incremental collapse can occur because of accumulation of plastic strains which lead to such an increment of displacements that the structure becomes unserviceable. This kind of inadaptation is related to loading programs which induce the plastic strain increments of the same sign in such a number of yielding zones (hinges) that unconstrained plastic flow of the structure will start.

If the strain increments change sign they can tend to cancel each other and the total deformation may remain small. This is so called alternating plasticity. After a number of load cycles local material failure can be provoked due to low-cycle fatigue.
In order to explain more comprehensively the shakedown phenomenon let us introduce the load parameter vector

\[ \mathbf{y} = (\mu_i) \in \mathcal{L} \quad \text{for } i = 1, \ldots, M, \]  

(5.6)

which traces the loading path (or defines the loading program) in the space of the independent load parameters \( \mu_i \). Any closed loading path which is repeated in time corresponds to a loading cycle. Usually the loading programs are bounded and can vary in a loading domain. In Fig. 5.3a, typical loading paths are shown in the loading domain \( \mathcal{L} \) bounded by the piece-wise linear boundary \( \partial \mathcal{L} \).

Fig. 5.3. a) Loading paths in the loading domain \( \mathcal{L} \), b) Permissible loading domain \( \mathcal{L}_s \) for the shakedown of a structure.

In the theory of shakedown the loading program can be usually identified with the components of the vector \( \mathbf{y}(t) \) which vary in time \( t \) in a generally unknown way, but within a prescribed range of variation, i.e. \( \mathbf{e}_1^- \leq \mu_i \leq \mathbf{e}_1^+ \). Thus, any loading path related to the shakedown phenomenon is to be contained in the permissible loading domain \( \mathcal{L}_s \) - cf. Fig. 5.3b.

Several criteria of shakedown have been formulated. The simplest criterion is of intuitive nature. It refers shakedown to a stabilization of plastic deformations. That means that after some plastic deformation in the initial load changes (cycles) the structural response becomes eventually elastic. Such a criterion prevents the incremental collapse.
As an example of the shakedown or inadaptation phenomena the experimental results, taken from König [45] p. 6, are shown in Fig. 5.4. For the load amplitude $P = 225$ N the shakedown occurs after the first loading cycle. For $P = 325$ N the inadaptation is associated with an increase of the permanent deflection $\delta$ at every subsequent loading cycle.

![Graph showing permanent deflection $\delta$ versus number of cycles for different load amplitudes $P$.](image)

Fig. 5.4. Permanent deflection $\delta$ versus the number of loading cycles for different load amplitudes $P$.

The classical theory of shakedown of elastic-plastic structures can be regarded as a generalization of the plastic limit analysis. That is why similar assumptions are adopted for the SD analysis. From among all assumptions, listed by König and Maier [46], the main assumptions are:

1. Material is elastic, perfectly plastic, stable in Drucker’s sense, of properties time- and temperature-independent as well as cycle-insensitive.
2. Strains and displacements are small, i.e. the geometrically linear (first order) theory is valid.
3. Loads are assumed as varying slowly so structural response is quasi-static.
4. The assessment of the safety factor $s$ with respect to inadaptation is the primary or even the exclusive scope of the shakedown analysis.

Similarly as in the limit analysis both the statistical and kinematical theorems were formulated. The static theorem is called also as Melan's theorem, since he proved it in 1936-38, cf. [42] p. 669. This theorem is supported on introduction
of two fields: i) fictitious elastic stresses $\sigma^e_{ij}(x,t)$, ii) self-equilibrated residual stresses $\rho_{ij}(x)$.

The fictitious elastic stress vector $\sigma^e(x,t)$ is associated in every time $t$ with an elastic response of the body $S$ to external actions. That means that the kinematical equilibrium and boundary condition equations are fulfilled but yield conditions can be violated. The self-equilibrated stress vector $\rho(x)$ is time-independent and fulfill homogeneous equilibrium equation (4.9) and homogeneous boundary conditions (4.11) on $S_t'$ in Fig. 4.1:

$$\begin{align*}
L^* \rho &= 0 \text{ in } V, \\
L_{o}^* \rho &= 0 \text{ on } S'_t.
\end{align*}$$

(5.7)

The Melan theorem states that the structure shakes down to the prescribed loading range if there exists a time independent residual stress field $\rho(x)$ which, superimposed on the fictitious elastic stress field $\sigma^e(x,t)$ does not violate the yield condition in any point of the structure:

$$f \left[ \sigma^e(x,t) + \rho(x) \right] \leq \sigma_0(x).$$

(5.8)

The form (5.8) of the static theorem is not related to any restriction on plastic strains or on the plastic work before the structure reaches the shakedown state. That is why more precise criteria of shakedown has been formulated. From among those, mentioned in [45] p. 30-31, the bounding of the average plastic work

$$w_p = \frac{1}{V} \int_t \int_0^T \sigma^T \xi P \, dV \, dt < w_o,$$

(5.9)

can be applied to evaluate whether the structure shakes down.

The Melan theorem (5.8) can be modified to the following form:

$$f(s[\sigma^e(x,t) + \rho(x)]) \leq \sigma_0(x) \quad \text{for } s > 1,$$

(5.10)

which can be worded as follows:
The body shakes down to the prescribed loading range if there exists any
time-independent residual stress field \( g(x) \) and a positive multiplier \( s > 1 \)
such that the relation (5.10) is fulfilled.

As it was proved by Konieczny and König (cf. ref. 14 and 28 in [47]) the
relation (5.10) leads to the evaluation of the mean value of the plastic work:

\[
\frac{1}{V} \int_0^V \mathbf{q}^T \mathbf{e}^p \, dV \, dt \leq \frac{s}{2(s-1)} \int_0^V \mathbf{q}^T (\mathbf{P}^e)^{-1} \mathbf{q} \, dV.
\]  

(5.11)

Formulation of an appropriate static theorem with respect to generalized
variables is not so evident as it takes place in the limit analysis. Whereas on
the point level \( \mathcal{P} \) material can be either elastic or plastic, a part of the
cross-section can be elastic and another part goes yielded. Thus, although a
yield surface \( \Phi(Q) = 0 \) is not reached on the level \( \mathcal{P} \) plastic strains can
develop. The same conclusion is valid if zones of passive processes occur
instead of elastic regions.

In order to be assured that nowhere in the cross-section yielding occurs it is
convenient to introduce the elastic locus \( \Phi^e(Q) = 0 \). This locus bounds in the
space of g. stresses the domain of such a stress variation that no plastic
strains occur at any point of the cross-section.

Residual stresses in a cross-section can be split into two parts, namely \( \rho_{ij} =
\rho_{ij}^o + \rho_{ij}^p \). The stresses \( \rho_{ij}^p \) are self-equilibrated in every cross section. That
means that the stresses \( \rho_{ij}^p \) do not produce the residual stress resultants at any
point of the body of coordinates \( x \), i.e. \( Q(q(x)) = \mathbf{0} \). The part \( \rho_{ij}^o \) gives the
residual stress resultants \( Q^o = Q(q^o) \) which satisfy the homogeneous equilibrium
equations for the generalized stresses

\[
\mathbf{e}^T Q^o = \mathbf{0}.
\]  

(5.12)

The Melan theorem in terms of generalized stresses can be worded as follows
[47]:

The structure shakes down for the prescribed loading range \( \mathcal{L} \) if there
exists a time-independent field of g. residual stress \( Q^o(x) \) and a field of
elastic loci \( \Phi^e(x) \) such that \( Q^o(x) \) superimposed on the field of fictitious
elastic g. stress \( Q^e(x,t) \) remains within the elastic locus
\( \mathbf{\dot{\varepsilon}}^e(x,t) + \mathbf{\dot{\varepsilon}}^o(x) \subseteq \mathbf{\dot{\varepsilon}}^e(x) \) \hspace{1cm} (5.13)

The kinematic theorem proved by Koiter (1956) (cf. [42] p.670), is associated with inadaptation of structures. He introduced so called admissible plastic strain rate cycle. This corresponds to any \( \mathbf{\dot{\varepsilon}}^p \) whose integral over some time interval \( T \)

\[
\Delta \mathbf{\dot{\varepsilon}}^p = \int_0^T \mathbf{\dot{\varepsilon}}^p(t) \, dt ,
\]

(5.14)
gives a compatible, i.e. stressless, plastic strain field associated with a kinematically admissible field of displacement velocities (rates) \( \mathbf{\dot{v}} = \mathbf{\dot{\varepsilon}}^p \). Once the plastic strain rates \( \mathbf{\dot{\varepsilon}}^p \) are known, the associated flow rule enables us to calculate the dissipation function \( D(\mathbf{\dot{\varepsilon}}^p) = \mathbf{Q}^T \mathbf{\dot{\varepsilon}}^p \).

The Koiter theorem was proved for the variables on the level \( \mathbf{\rho} \), but it has an identical form for generalized variables and can be formulated as follows [47]:

The structure does not shake down over any loading path from the prescribed domain \( \mathcal{L}_S \) if for an admissible plastic strain rate cycle the following inequality takes place

\[
\int_0^T \mathbf{\dot{p}}^T \cdot \mathbf{\dot{w}} \, dA \, dt > \int_0^T \mathbf{Q}^T \mathbf{\dot{\varepsilon}}^p \, dA \, dt .
\]

(5.15)

With respect to the definition of shakedown related to (5.9), the kinematic theorem can be worded as follows [47]:

The structure will shake down to a prescribed load range if there exists a number \( s > 1 \) such that

\[
s \int_0^T \mathbf{\dot{p}}^T \cdot \mathbf{\dot{w}} \, dA \, dt < \int_0^T \mathbf{Q}^T \mathbf{\dot{\varepsilon}}^p \, dA \, dt .
\]

(5.16)

Both the static and kinematic theorems reduce the shakedown analysis to the searching either of a safety factor \( s \) or an admissible loading domain (range of variation of load parameters) \( \mathcal{L}_S \).
In case of application of the static theorem the essential problem is to find the 'best' self-equilibrated stress field among all the stresses satisfying internal equilibrium requirements under zero external loads. The problem is not trivial and many approaches has been proposed (cf. [47,46,45]) in order to formulate appropriate fields which can lead to good approximations to the SD problems from below, i.e. from the safe side from engineering point of view.

In Fig. 5.5 two distributions of the uniaxial stress $\sigma$ is shown which corresponds to self-equilibrated stress states on the level $\sigma'$ (i.e. $Q(\sigma) = 0$) of the rectangular bar subjected to bending and tension/compression. Depending on the value of the parameters $\alpha$ and $\beta$ two families of elastic loci $\theta^e$ are shown in Figs. 5.5a, b (cf. [45] p. 82-83). According to the assumptions in the Melan

![Diagram](image)

Fig. 5.5. Distributions of residual stresses in the rectangular c-s of the limit axial force $N_o = bh\sigma_o$ and limit bending moment $M_o = bh^2\sigma_o/4$ and two families of corresponding elastic loci.
theorem the residual-stress distribution can be arbitrary but self-equilibrated, in particular it can have discontinuities—cf. Fig. 5.5b. The residual stresses can be interpreted as a residuum associated with initial yielding after external unloading or as a remain of technological processes.

As an example of computation of the SD domain the portal frame from Fig. 5.6. is considered, cf. [45] p. 113. Referring to data accepted for the limit analysis—cf. Fig. 5.2d, the SD analysis has been carried out either for the uniform, rectangular cross-section or for the ideal, sandwich c-s under the assumption that both the cross-sections have the same limit plastic moment $M_O$ (influence of the normal force $N_O$ an yielding has been neglected). For the concentrated forces $H = \mu_1 H_O$, $V = \mu_2 V_O$ and the range of variation $-1 \leq \mu_1 \leq 1$, $0 \leq \mu_2 \leq 1$ the SD domains are shown in Fig. 5.6. These domains are bounded by the loci defined by the pairs of the load amplitudes $(H_O, V_O)$.

![Diagram](image)

**Fig. 5.6.** SD domains for a rectangular c-s (a) and ideal-sandwich c-s (b) versus the limit locus (c) for the plastic load carrying capacity (point L corresponds to the plastic l.c.c. for the example in Fig. 5.2.)

The SD for the rectangular, uniform c-s is smaller than that for the ideal-sandwich c-s because of differences in elastic loci on the level $H$. Both the domains a and b are bounded by the limit analysis locus c which is independent of the elastic locus if the same limit plastic moments are assumed.
for both the cross-sections. For the frame with the sandwich c-s the incremental collapse is dominant in the SD analysis, whereas low cycle fatigue (inadaptation to alternating) is more stringent for uniform cross-sections except for small values of the horizontal load H.

Problems of the SD analysis were considered in many papers and books. From among them only the review papers by Sawczuk [47], König and Maier [46] as well as the books by Martin [42] and König [45] have been quoted. In this literature (especially in [45]) extensive references are listed.

The SD analysis has been much more difficult than the limit analysis. A real progress has taken place since mathematical programming methods started to have been applied via computer codes. Fulfilling assumptions of the classical SD analysis enable us to explore the linear programming which causes that algorithms for the SD and limit analysis are quite similar.

5.3. Linear programming and appropriate formulation of the limit and SD problems

In the linear programming (LP) both the objective function \( f \) and constraints are linear with respect to the state vector \( x \). The LP problem can be formulated as follows (cf. [49]):

\[
\begin{align*}
\text{minimize} & \quad f = -c^T x \\
\text{subject to} & \quad A x \leq b, \quad x \geq 0
\end{align*}
\]

or in the shorter form

\[
\min \left\{ -c^T x \mid A x \leq b, \quad x \geq 0 \right\}.
\]

The vector \( x^* \), satisfying the constraints (5.17), is called as a feasible solution, and if it satisfies also the condition \( \min f \) it is called the optimal solution of LP.

In structural mechanics and also in the elastoplasticity basic equations are to be considered and it is rather inefficient to change them into inequalities in order to decrease the number of constraints. That is why the LP problem is modified into the form (cf. [37] p. 206):
maximize \[ f = c_1^T x_1 + c_2^T x_2 \]
subject to \[ \begin{align*} A_{11} x_1 + A_{12} x_2 &= b_1, \\
A_{21} x_1 + A_{22} x_2 &\leq b_2, \\
& x_2 \geq 0. \end{align*} \] (5.18)

The formulation (5.18) is called as the LP primal problem. It is completed by the following dual LP problem:

minimize \[ f' = b_1^T y_1 + b_2^T y_2 \]
subject to \[ \begin{align*} A_{11}^T y_1 + A_{21}^T y_2 &= c_1, \\
A_{12}^T y_1 + A_{22}^T y_2 &\leq c_2, \\
& y_2 \geq 0. \end{align*} \] (5.19)

The formulation (5.19) is dual with respect to the primal formulation (5.18) if the complementary slackness conditions are fulfilled

\[ \begin{align*} c^T x &= b^T y, \\
y_2^T (A_{21} x_1 - A_{22} x_2 - b_2) &= 0, \end{align*} \] (5.20)

where the matrices \( A, x, y \) are split into submatrices:

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \] (5.21)

Let us reformulate the basic relations on the level \( \mathcal{B} \) for the discrete model of a structure with lumped parameters. The kinematic and equilibrium equations, corresponding to (4.4), (4.9) or to those in Fig. 4.2., can be written in the following form:

\[ \dot{g} = \mathcal{B} \dot{\mathcal{G}}, \] (5.22)

\[ \mathcal{B}^T \mathcal{Q} = \mathcal{P} \mathcal{U}. \]
where: $\mathbf{B}$ - kinematic consistency matrix, $\mathbf{P}$ - matrix of reference loads, $\mathbf{y}$ - vector of load parameters. The displacement, load, strain and stress vectors $\mathbf{y}$, $\mathbf{P}$, $\mathbf{g}$, $\mathbf{Q}$ respectively, are related to all nodes where plastic properties of the structure are lumped.

Plastic interaction surfaces are approximated by piece-wise linearized relations

$$ \mathbf{q} = \mathbf{N}^T \mathbf{Q} - \mathbf{k} = 0 $$

(5.24)

where $\mathbf{N} = \partial \mathbf{q} / \partial \mathbf{Q}$ is the matrix of gradients to the surfaces $\mathbf{q}$. The physical relations are written for elastic-plastic materials

$$ \mathbf{g} = \mathbf{g}^e + \mathbf{g}^p, \quad \dot{\mathbf{g}}^e = (\mathbf{d}^e)^{-1} \mathbf{Q}, \quad \dot{\mathbf{g}}^p = \mathbf{N} \dot{\mathbf{A}}, $$

(5.25)

$$ \dot{\mathbf{A}} \succ 0, \quad \mathbf{F} \preceq 0, \quad \mathbf{A}^T \mathbf{F} = 0. $$

The complete solution of the limit analysis, corresponding to the plastic load carrying capacity parameter $\mu_p$, can be achieved assuming the rigid, perfectly plastic model of material. That is why the elastic part of the g. strain rates $\dot{\mathbf{g}}^e$ can be neglected and in what follows the notation $\dot{\mathbf{g}} = \dot{\mathbf{g}}^p$ is used.

The static theorem of the limit analysis can be formulated in the form suitable for LP:

$$ \max \{ \mu \mid \mathbf{B}^T \mathbf{Q} - \mu \mathbf{P} = 0, \mathbf{N}^T \mathbf{Q} \preceq \mathbf{k} \} $$

(5.26)

The matrices corresponding to the LP primal formulation are:

$$ \mathbf{A} = \begin{bmatrix} -\mathbf{P}^T & \mathbf{P} \\ \mathbf{N}^T & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{0} \\ \mu \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{k} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}. $$

(5.27a)

In order to formulate the dual LP problem the matrices are completed by the vector:

$$ \mathbf{y} = \begin{bmatrix} \mathbf{\dot{d}} \\ \mathbf{\dot{A}} \end{bmatrix}. $$

(5.27b)
which corresponds to the kinematic part of solution. According to (5.19) the following dual LP problem is:

\[
\min \left( k^T \dot{\lambda} \mid B \dot{\lambda} - N \dot{\lambda} = 0, \, \bar{p}^T \dot{\lambda} = 1, \, \dot{\lambda} \geq 0 \right). \tag{5.28}
\]

The LP problem (5.28) can be identified as the kinematic theorem since the power equation (5.4) may be written in the form:

\[
\mu \bar{p}^T \dot{\lambda} = q^T N \dot{\lambda}. \tag{5.29}
\]

If the relations \( \bar{p}^T \dot{\lambda} = 1 \) and \( (N^T q)^T = k^T \) are taken into account then the LP problem (5.28) becomes the kinematic theorem of the limit analysis:

\[
\min \left( \mu \mid B \dot{\lambda} - N \dot{\lambda} = 0, \, \dot{\lambda} \geq 0 \right). \tag{5.30}
\]

The LP dual formulation (5.26, 30) gives the solution which corresponds to the feasible solution \( x^*, \, y^* \). That means that the value \( \mu^* \) of the load parameter is the plastic load c.c. factor \( \mu_p \) since the solutions \( q^*, \, \dot{q}^*, \, \dot{\lambda}^* \) are statically and kinematically admissible. The uniqueness of solution is guaranteed only for the extreme minimal value of the parameter \( \mu^* = \mu_p \). Both the g. stress vector \( q^* \) as well as the plastic flow mechanism associated with \( \dot{q}^* \) and \( \dot{\lambda}^* \) can be non-unique because of rigid-plastic model of the structure. The unique distribution of g. stresses can be obtained if elastic properties of material are taken into account but such an approach leads to the nonlinear (at least quadratic) programming.

Evaluation of the safety factor \( s \) against inadaptation of the structure is the main goal of the SD analysis. The g. loads are defined in the form

\[
\bar{p} = \bar{p} \mu, \tag{5.31}
\]

where the load parameter vector \( \mu \) is associated with the reference load matrix \( \bar{p} \) which determines a configuration of loads applied to the structure. In the SD analysis by means of LP the domain of load feasibility is assumed to be convex and bounded by a set of hyperplanes (cf. Fig. 5.3a):

\[
L \mu \leq \eta \in \mathcal{L}. \tag{5.32}
\]
The g. fictitious elastic stresses $\mathbf{Q}^e$ are obtained as the solution of Eqs (5.23, 24) and the elastic relation (5.25)$_2$. Of course, a unique solution can be obtained after assuming appropriate supporting (boundary) conditions. With respect to the loads (5.31) the fictitious, elastic solution can be written as:

$$
\mathbf{Q}^e = \mathbf{M} \mathbf{u} . 
$$

(5.33a)

The vector of g. residual stresses $\mathbf{Q}^o$ is any solution of the homogeneous equilibrium equation $\mathbf{B}^T \mathbf{Q}^o = 0$ which corresponds to Eq. (5.23). The solution can be written in the form:

$$
\mathbf{Q}^o = \mathbf{H} \mathbf{N} . 
$$

(5.33b)

The vector $\mathbf{N}$ and matrix $\mathbf{H}$ can be easy interpreted in bar structures from the force method point of view. $\mathbf{N}$ is the vector of redundants and the matrix $\mathbf{H}$ is a transformation matrix in an isostatic complementary system if subjected to unit redundants — cf. [45] p.119. Appropriate interpretation in FEM was given in [181].

The application of LP in the SD analysis starts with solving of the following, sometimes trivial, LP subproblem:

$$
\max \{ S_i = N_i^T \mathbf{M} \mathbf{u} \mid L \mathbf{u} \leq b \} 
$$

(5.34)

where: $N_i$ - vector of the $i$-th plastic mode in the plastic interaction relation (5.24). The parameters $S_i$ can be interpreted as the limit values of variations of elastic fictitious stresses (5.33).

The static constraints, corresponding to those (5.8) on the level $\mathcal{P}$, and to (5.13) on the level $\mathcal{Y}$, are in the form:

$$
N^T (\mathbf{Q}^e + \mathbf{Q}^o) \leq 0 .
$$

(5.35)

After taking into account the mentioned relations and the constraints related to the range of variation

$$
S \leq \mathbf{u} \leq b \in \mathcal{U}
$$

(5.36)
the static constraints (5.35) can be transformed into the following form (cf. [45] p.119, [181] p.253):

\[ N^T H X + sS \leq k , \]

(5.37)

where \( S \) is the vector of components (5.34).

Now the static theorem of the SD analysis can be formulated as the following primal LP problem:

\[ \max \left\{ s \mid N^T H X + sS \leq k , s \geq 0 \right\} \]

(5.38)

In order to write the relation (5.38) in the form consistent with (5.18) the vector of redundants \( X \) is defined by two non-negative vectors \( \bar{X}^+ \) and \( \bar{X}^- \):

\[ X = \bar{X}^+ - \bar{X}^- \] where \( \bar{X}^+ \geq 0 , \ \bar{X}^- \geq 0 . \]

(5.39)

Finally the static SD theorem is equivalent to the following primal LP problem (cf. [181]):

\[ \max \left\{ [1, 0, 0] \begin{bmatrix} s \\ \bar{X}^+ \\ \bar{X}^- \end{bmatrix} \right\} \begin{bmatrix} [s, N^T H, - N^T H] \begin{bmatrix} s \\ \bar{X}^+ \\ \bar{X}^- \end{bmatrix} \leq k , \begin{bmatrix} s \\ \bar{X}^+ \\ \bar{X}^- \end{bmatrix} \geq 0 \right\} \]

(5.40)

The dual LP problem, associated with (5.40) takes the form:

\[ \min \left\{ \begin{bmatrix} T \\ \Lambda \end{bmatrix} \begin{bmatrix} S^T \\ H^T N \end{bmatrix} \right\} \begin{bmatrix} 1 \\ \Lambda \end{bmatrix} \geq 0 , \ \Lambda \geq 0 \]

(5.41)

The dual LP problem (5.41) can be identified as the kinematic SD theorem, corresponding to (5.16), on the base of following considerations. First of all let us remark that the first constraint in (5.41) is the equality \( S^T \Lambda = \frac{c}{2} \). In case \( c > 1 \) the vector \( \Lambda/c \) is feasible but not optimal since it leads to a smaller value of the objective function than for \( c = 1 \). The second and third inequalities in (5.41) are equivalent to the equation \( N^T H = 0 \). In such a way the dual LP problem (5.41) can be reformulated to the form:
\[
\min \left\{ k^T \dot{A} : \begin{array}{c}
S^T \dot{A} = 1 , \\
H^T \dot{A} = 0 , \\
\dot{A} \geq 0 \end{array} \right\} . \tag{5.42}
\]

The scalar product equals the safety factor \(s\) according to the following manipulations:

\[
k^T \dot{A} = s (N^E)^T \dot{A} = s S^T \dot{A} = s . \tag{5.43}
\]

The relation \(k^T \dot{A} = q^T q^p = s S^T \dot{A}\) corresponds to the power equation according to the relation (5.16). The equation \(H^T \dot{A} = 0\) can be written as:

\[
H^T q^p = 0 \tag{5.44}
\]

and interpreted as the compatibility condition of the plastic strain rates, which has been assumed in the Koiter theorem (5.15).

Thus, the dual LP problem (5.42) can be worded as the kinematic theorem in which the minimal value of the safety factor is searched for kinematically admissible distributions of plastic strain rates.

Application of LP to the SD analysis leads to the unique, optimal solution for the safety factor \(s\). The solutions related to \(X\) and \(\dot{A}\) are non-unique since they correspond to plastic flow mechanisms which are associated with the variation of loads.

That is why a great attention has been devoted to evaluation of permanent displacements associated which the shakedown phenomena. Various bounding techniques need at least quadratic programming methods. These problems have been widely discussed in literature (cf. e.q. [48,46,45]) but they are out of scope in this report.

5.4. Some remarks on application of mathematical programming in the plastic analysis of structures

The presented formulations fully correspond to the SIMPLEX algorithm supported on the dual formulation (5.18,19). The algorithm is implemented as a standard subroutine in computer codes, practically accessible for all computers. Computational effectiveness strongly depends on the size of problems under
consideration. That is why the problem of reduction of variables has been widely considered in the literature — cf. [45,48]. For instance the active constraints are only considered in the multistage iteration procedure [182]. Another effective approach was suggested by Zavellani-Rossi [184] who proposed to formulate the linearized plastic interaction surface by means of a linear combination of the vectors which are associated with the corners of the surface.

Only the simplest formulations, related to LP analysis has been presented. Another field of application is related to the use of LP in the optimum design of plastic structures. Assuming various cost functions it is possible to compute an optimal distribution of components of the vector \( k \) which defines the plastic interaction curves in (5.24). The optimum design procedure can lead either to a choice of dimensions of the cross-sections or to an appropriate distribution of plastic properties (yield points) of material. The problems of optimum design were extensively considered by Čyraš [50] and his coworkers [51]. A review of papers devoted to these problems can be found in [42,45].

Linear objective functions and constraints limits significantly applications of LP in the analysis of structures. The elastoplastic analysis requires at least a quadrature objective function which corresponds to an energy-type functional. Methods of quadratic programming (QP) are especially effective if such an objective function is convex.

Other problems which are in the frame of the convex analysis [52] have been recently developed. The hardening of material, postyield analysis, incremental procedures are considered on the basis of different methods and modifications of QP. These problems have been extensively discussed and reviewed — cf. [45-52].
6. STATIC, FINITE-ELEMENT ANALYSIS OF ELASTIC-PLASTIC STRUCTURES

6.1. General remarks

In recent years the finite element method has emerged as the most powerful
general method of structural analysis and has provided engineers with a tool of
very wide applicability. Enormous number of papers and conference proceedings,
as well as an increasing number of books devoted to the theory and applications
of FEM (cf. Noor [53]) reflect a rapid development in this field during recent
years. The application of FEM to the elastic-plastic analysis of structures
threw the door open to general analysis of the deformation process — from the
beginning of loading through limit states to advanced post-yield response of
structures.

The fundamentals of FEM are taught in all engineering courses so, according to
the assumptions in Sec. 1.2 of this report, they will not be discussed in
detail. Only problems associated with elastic-plastic models of material will be
considered and special attention will be called to differences between the
elastic and elastoplastic FE analysis.

Displacement version of FEM is only presented since advantages of such an
approach has dominated the majority of algorithms and computer codes for
elastoplastic analysis. The incremental formulation is presented as a background
for the analysis of the deformation process in a wide range of load changes on
the base of compatible, incremental constitutive relations.

The derivation of the FE equations is shortly repeated under assumptions of the
geometrically linear theory. The attention is focussed on such formulations
which correspond to two main methods of the elastoplastic analysis, namely to
the tangent stiffness method and the pseudo-load method.

Various methods for the nonlinear FE analysis on the level $S$ are shortly
recapitulated. These methods have been developed for the elastic, nonlinear
analysis but they can also be applied to the analysis of elastic-plastic
problems with minor modifications.

The most important problem of the elastoplastic analysis is the numerical
integration (solution) of constitutive equations for a finite increment (step)
of the plasticity time $\Delta \tau$. Appropriate algorithms on the $\Phi$ level are briefly discussed as the background of subroutines for the analysis by means of multilayer-type models of the cross-section.

Yielding of material can cause large displacements of structures and therefore assumptions of the small displacement theory are not valid. That is why the material and geometrically nonlinear formulation is presented with special emphasize put on the small strain, large displacement theory. This theory has a wide range of applications to various, static and stability problems of engineering structures.

The principle of virtual work for kinematically admissible displacements is adopted as the theoretical background for unified formulation of all, above mentioned problems.

References are limited to basic positions of literature and they are devoted especially to computational methods of the elastoplastic analysis.

In this chapter many details concerning FEM are omitted since they are well described in literature devoted to the elastic analysis, e.g. the approximation of a displacement field in FE, prediction of the incremental step, definition of norms for evaluating the errors in iteration procedures etc. are not discussed in detail.

6.2. Displacement FE for incremental elastoplasticity

Let us focus our attention on the geometrically linear theory. That means that nonlinearity of the elastic-plastic analysis is only due to material properties. The formulation of the FE relations is quite similar to that as in elasticity. That is why only a short recapitulation of the derivation of the FE matrices is given below. All relations are referred to small but finite increments, which can be associated with the increment of conventional time of plasticity $\Delta \tau$. For instance the constitutive equation (3.14) takes the form

$$
AQ = D^{ep} \Delta q \quad \text{for} \quad \Delta \tau_m = \tau_{m+1} - \tau_m, \quad (6.1)
$$

where the c.-s. stiffness matrix $D^{ep}$ is computed at the time $\tau_m$ and is independent of the increments $AQ$ and $\Delta q$. 

Let us assume an approximation of the displacement increments $\Delta w$ in the finite element $j$):

$$\Delta w(\xi) = X(\xi) \Delta q \quad \text{in } \Omega^j$$

(6.2)

where $\xi = \{\xi^a\}$ are the local FE coordinates, $X(\xi)$ - set of basic functions. Instead of the generalized DOF $q$ the generalized node displacements $d$ can be used

$$d^j = M^j q.$$  

(6.3)

In case of the nonsingular matrix $M^j$ the approximation (6.2) takes the form:

$$\Delta w^j(\xi) = X(\xi) (M^j)^{-1} \Delta d^j = N^j(\xi) \Delta d^j,$$

(6.4)

where $N(\xi) \in C^{(n)}$ are called as shape functions.

In many formulations it is possible to omit the transformation (6.3) and to achieve (6.4) in another way. The shape function $N(\xi)$ should fulfill three criteria of compatibility:

1. Rigid body motion does not produce strains in the element.
2. The homogeneous strain state can be reproduced.
3. Kinematical consistency inside FE and continuity conditions along the boundary $\partial \Omega^j$ are fulfilled.

That is why the shape functions should be of the $C^{(n)}$ class of continuity, where the order $n$ depends on the order of differential operators in the kinematic consistency matrix $C$ in Fig. 4.2.

FE in which the above mentioned three criteria are fulfilled are called as compatible finite elements. These elements guarantee that a sequence of FE solutions will converge to the analytical solution which could be achieved for a continuous model if dimensions of FE monotonically decrease - cf. e.g. Zienkiewicz [54].

*) The domain $\Omega^j$ of the finite element $j$ can be specified as the volume $V^j$, area $A^j$ or the length $L^j$ depending on the model of a structure and applied generalized variables. The same refers to the boundary $\partial \Omega^j$ which will be denoted as $S^j$, $T^j$ or the edge points $i$ correspondingly.
The approximation (6.4) is substituted into the kinematic incremental equation:

$$\Delta \mathbf{q}^j = \mathcal{C} \Delta \mathbf{w}^j = \mathcal{C} N^j \Delta \mathbf{d}^j = B^j \Delta \mathbf{d}^j,$$  \hspace{1cm} (6.5)

where $B^j$ is the consistency matrix of the $j$-th FE.

If the shape functions $N^j$ are kinematically admissible then for any time $\tau_{m+1} = \tau_m + \Delta \tau_m$ the principle of VW can be written in the following form:

$$\int_{\Omega^j} (G^T \mathbf{+} \Delta \mathbf{G}^T) \delta \mathbf{g} \ d\Omega^j =$$

$$= \int_{\Omega^j} (P^T \mathbf{+} \Delta P^T) \delta \mathbf{w} \ d\Omega^j + \int_{\alpha \Omega^j} (\mathbf{t}^T \mathbf{+} \Delta \mathbf{t}^T) \delta \mathbf{u} \ dS + (G^j + \Delta G^j)^T \delta \mathbf{g}^j \hspace{1cm} (6.6)$$

where $G^j = \{G_i^j\}^j$ and $g^j = \{g_i^j\}^j$ are the reactions and displacements in the nodes $i$ of the $j$-th FE.

After substitution of (6.5) and (6.1) into the left hand side of Eq. (6.6) and rearrangement we have:

$$(\delta \mathbf{d}^j)^T (K^j \mathbf{d}^j - \Delta \mathbf{P}^j + P^j - F^j) = (\delta \mathbf{d}^j)^T (\Delta \mathbf{G}^j + G^j),$$  \hspace{1cm} (6.7)

where the following integrals come into play:

$$K^j = \int_{\Omega^j} B^T \mathcal{P}^e B \ d\Omega^j \hspace{1cm} \text{— tangent stiffness matrix},$$

$$\Delta \mathbf{P}^j = \int_{\Omega^j} N^T \mathcal{P} \ d\Omega^j + \int_{\alpha \Omega^j} N^T \mathbf{t} \ dS \hspace{1cm} \text{— vector of external load increments},$$

$$P^j = \int_{\Omega^j} N^T \mathcal{P} \ d\Omega^j + \int_{\alpha \Omega^j} N^T \mathbf{t} \ dS \hspace{1cm} \text{— vector of external loads},$$

$$F^j = \int_{\Omega^j} B^T \mathbf{q} \ d\Omega^j \hspace{1cm} \text{— vector of internal forces}.\hspace{1cm} (6.8)$$
If components of the vector $\Delta \mathbf{d}^j$ are linearly independent then the following, FE incremental equation can be deduced from (6.7) for an individual finite element $j$:

$$K^j \Delta \mathbf{d}^j - \Delta \mathbf{P}^j - \mathbf{R}^j = \Delta \mathbf{G}^j + \mathbf{G}^j,$$  \hspace{1cm} (6.9)

where the vector of residual FE forces appears

$$\mathbf{R}^j = \mathbf{P}^j - \mathbf{F}^j.$$  \hspace{1cm} (6.10)

Assembling of finite elements into a structure, i.e. the transition depends on summing the reactions in the nodes $i$ from the finite elements which come together at these nodes

$$G_i = \sum_j G_i^j, \quad \Delta G_i = \sum_j \Delta G_i^j, \quad \text{for } i = 1, \ldots, N,$$  \hspace{1cm} (6.11)

where $N$ is the number of DOF of the structure. The assembling (6.11) requires, of course, the transformation from the local FE coordinates to a node or global system of coordinates

$$G_i^j = (T_i^j)^T G_i^j.$$  \hspace{1cm} (6.12)

The fulfilling of the quilibrium equations *)

$$G_i + \Delta G_i = 0 \quad \text{for } i = 1, \ldots, N$$  \hspace{1cm} (6.13)

leads to the following FE incremental equations of the structure (on the level $\mathcal{E}$):

$$K \Delta \mathbf{d} = \Delta \mathbf{P} + \mathbf{R}.$$  \hspace{1cm} (6.14)

The residual (unbalanced) force vector $\mathbf{R}$ equals zero if the structure is in equilibrium. However, this vector is taken into account in (6.14) since this equation will be used in subsequent iterations.

*) In case of the application of concentrated external forces $\mathbf{S}_i + \Delta \mathbf{S}_i$ in the node $i$ Eq. (6.13) should be changed by $\Delta G_i + G_i = \mathbf{S}_i + \Delta \mathbf{S}_i$. 
The stiffness matrix $K$ depends on elastic-plastic properties of material throughout the local stiffness matrix $D_{EP}$ in the integral (6.8)\textsubscript{1}. During the active plastic deformation process the components of this matrix have to be updated in elements which undergo the yielding. This approach is known in the literature as the tangent stiffness method (tangent modulus method).

The updating of the stiffness matrix is time consuming and that is why another approach has been developed. An alternative approach depends on treating the plastic effects as corrections to the elastic behaviour of structures. This can be achieved by transformation of Eq. (6.14) into the following form

$$K_0 \Delta d = \Delta \Phi + R + \Delta \Phi^P,$$  \hspace{1cm} (6.15)

where $K_0$ is the elastic stiffness matrix and $\Delta \Phi^P$ is the plastic pseudo-load vector.

In order to derive Eq. (6.15) let us return to the constitutive relation (6.1) in FE and for the sake of brevity let us omit for a while the superscript $j$ as the FE number. According to (2.45) the relation (6.1) can be written as

$$\Delta \Phi = D^e \Delta \Phi_e = D^e (\Delta \Phi - \Delta \Phi^P).$$  \hspace{1cm} (6.16)

Let us consider the scalar product $\Delta \Phi^T \Delta \Phi$ in the integral of the left hand side in (6.6). Substitution of (6.16) leads to the relation

$$(\Delta \Phi^T \Delta \Phi^e (\Delta \Phi - \Delta \Phi^P) = (\Delta \Phi^T (B^T D^e B \Delta d - B^T D^e \Delta \Phi^P).$$  \hspace{1cm} (6.17)

In such a way one can arrive at the following quantities:

$$K_0^j \int \mathcal{D}^e B \mathcal{D} \, d\Omega \quad \text{--- elastic stiffness matrix}, \hspace{1cm} (6.18)$$

$$(\Delta \Phi^P)^j \int \mathcal{D}^e \Delta \Phi^P \, d\Omega \quad \text{--- 'initial strain' plastic pseudo-load vector}. \hspace{1cm} (6.19a)$$

It is possible to formulate the vector $(\Delta \Phi^P)^j$ in another form. If the plastic g. strain increment vector $\Delta \Phi^P$ is expressed in the form
\[ \Delta \mathbf{d}^P = [(\mathbf{D}^e)^{-1} - (\mathbf{D}^e)^p] \Delta \mathbf{Q} \equiv \mathbf{c}^P \Delta \mathbf{Q} . \]

then the formula (6.19a) takes the form

\[ (\Delta \mathbf{P}^e)^j = \int_{\Omega^j} \mathbf{B}^T \mathbf{D}^e \mathbf{c}^P \Delta \mathbf{Q} \, d\Omega - \text{'initial stress' plastic pseudo-load vector.} \quad (6.19b) \]

The approach in which the FE equations (6.15) are solved by means of the plastic pseudoload vector of the form (6.19a) is called as the initial strain approach versus the initial stress approach if (6.19b) is used.

### 6.3. Methods of analysis of FE incremental equations

The incremental equation (6.14) can be written in the form

\[ K \Delta \mathbf{d} = \Delta \mathbf{r} , \quad (6.20) \]

where the increment of pseudo-forces is:

\[ \Delta \mathbf{r} = \Delta \mathbf{P} + \mathbf{R} = (\mathbf{P} + \Delta \mathbf{P}) - \mathbf{P} . \quad (6.21) \]

Eq. (6.20) can be solved either by means of direct (simple) incremental procedure or in an iteration way.

The solution process is controlled by the time-type parameter \( \tau \), which in the linear elastic analysis is identified with the load parameter, i.e. \( \tau = \mu \). Such a control parameter can also be applied to the analysis of nonlinear problems under assumption that the solution is unique with respect to \( \mu \). That means that the solution process under load control can be continued below the limit point on the equilibrium path. In Fig. 6.1, equilibrium paths are shown as the diagrams \( \mu - d \) where \( d \) is an representative displacement of the considered structure.

An application of the direct incremental procedure can give only an approximated solution which drifts from the exact one depending on the length of the step, i.e. on the load parameter increment \( \Delta \mu \) - cf. Fig. 6.1a.

The interaction procedure depends on realization of the approach which is known as the predictor-corrector technique. If the iteration starts from the equi-
librium state associated with the point \( m \) on the equilibrium path then the residual force vector equals zero, i.e. \( \mathbf{R}^{(0)} = 0 \) for \( i = 1 \). For the next iteration steps \( i > 1 \) the residual forces appear, \( \mathbf{R}^{(i-1)} \neq 0 \) at zero value of the load increment \( \Delta \mathbf{F}^{(i-1)} = 0 \). This approach corresponds to the sequence of solutions of the equation:

\[
\mathbf{m}_K^{(i-1)} \Delta \mathbf{d}^{(i)} = \Delta \mathbf{m}^{(i-1)},
\]

or after missing the superscript \( m \) and solution of the corresponding set of linear equations:

\[
\Delta \mathbf{d}^{(i)} = (\mathbf{K}^{(i-1)})^{-1} \Delta \mathbf{F}^{(i-1)}.
\]

That means that the solution for subsequent iteration steps depends on the vector of the residual forces:

\[
\begin{align*}
\Delta \mathbf{F}^{(0)} = \Delta \mathbf{F} = \Delta \mathbf{u} \mathbf{R} & \quad \text{— predictor,} \\
\Delta \mathbf{F}^{(i-1)} = \mathbf{R}^{(i-1)} & \quad \text{— corrector for } i > 1.
\end{align*}
\]

**Fig. 6.1.** a) Direct incremental procedure, b) Pseudo-load method (om NR), c) Classical Newton-Raphson method (NR), d) Modified NR method (m NR).
The direct incremental approach fully corresponds to the Euler method which can be used to numerical integration of ordinary differential equations with given initial condition (initial value problem) - cf. [32]. As an initial state the state of elastic carrying capacity can be adopted - cf. Fig. 6.1a. The approach enable us to evaluate the exact solution which is associated with disappearing of the residual forces $R(\mu) = 0$ for every point of the equilibrium path. In case of elastic-plastic structures application of the direct incremental approach leads to an approximate solution which does fulfill neither the global (overall) equilibrium equations on the levels $\mathcal{E}$ and $\mathcal{B}$ nor the yield conditions on the $\mathcal{P}$ level at individual points of the analyzed structure.

Despite of such fundamental defects the direct incremental approach is sometimes used in order to evaluate the exact solution without any iteration. That is why the computations can be easy repeated for different, decreasing increments of the load parameter, e.g. $\Delta \mu^m$, $\Delta \mu^m / 2$ etc. The approach was included into some computer codes, especially into the older ones (cf. table 2 in [14]).

Iteration methods have been used since the very beginning of application of FEM to the elastoplastic analysis. The Newtonian methods are extensively explored. The well known Newton-Raphson (NR) method is used in the classical form if the tangent stiffness approach is adopted (cf. Fig. 6.1c). In this method the tangent stiffness matrix $m^T_k(1)$ is updated at every iteration step $i$. If the stiffness matrix $m^T_k(0)$ is not updated (the matrix is modified at the end of iteration at the previous step) the modified Newton-Raphson method (mNR) is realized - cf. Fig. 6.1d. The initial mNR method (omNR) can be obtained as a special case of the mNR method. In omNR the stiffness matrix is associated with the initial state (points 0 or 1 in Fig. 6.1b) which corresponds to the elastic matrix $m^T_k(i-1) = o^T_k(0) = l^T_k(0) = K_0$.

In all Newtonian methods the exact solution can be achieved if no approximations are made at the computation of the residual force vector $m^T_R(i-1)$. The exact, residual force vector guarantees that for every iteration a contact with the equilibrium path is maintained.

The plastic pseudo-load approach can be identified as the omNR method for which the increment of pseudo-forces vector (6.21) is completed by the increment of plastic pseudo-loads according to (6.19)

$$\Delta F = (P + \Delta P) - F + \Delta F^P.$$ (6.25)

Both the plastic pseudo-load and stiffness approaches were originated earlier by Ilyushin [04] and Birger [184] in USSR, then by Mendelson et al. [185,186] and
Marcal [187]. With respect to FEM the initial strain approach was used by Gallagher et al. [101] in 1962, then by Argyris et al. [188]. The initial stress approach was proposed by Zienkiewicz et al. [189]. The tangent stiffness approach was used early in papers by Pope [190], Marcal and King [191].

A comparison of the plastic pseudo-load and tangent stiffness approaches has been made in many papers. Both approaches give satisfactory results for small yielding as stated by Marcal [192]. For more advanced yielding the iteration process for pseudo-loads can converge slowly or even can converge to a wrong solution as it was proved by DeDonato [193]. Henkel (ref. 45 in [55]) concluded that the initial stress solution diverges when the instantaneous slope is twice the initial, linear slope. The initial stress approach, proposed by Zienkiewicz et al. [189], gives more rapid convergence then the initial strain approach, permits large load increments and establishes a lower bound solution. The tangent stiffness method enable us to extend significantly the step size in comparison with that used in the plastic pseudo-load method. Mc Namara and Marcal [194] show that for the same accuracy the step can be eight times larger in the tangent stiffness method.

Even without a deeper analysis it is evident that for the same elastic-plastic structures the fastest iteration procedure corresponds to the NR method. An disadvantage of the application of NR is related to the updating of the stiffness matrix at every iteration step. This may cause, despite of a small number of iterations, that the total computational effort can be bigger then for the modified Newtonian methods. That is why the mNR is often used for the analysis of elastic-plastic structures. Effectiveness of mNR can be increased if the updating is performed some times per one step increment of load parameter or if the number of iteration exceeds the desired number.

Updating of the stiffness matrix and residual force vector requires going back to the individual finite elements j. After modification of the stresses on the $\mathcal{P}$ level the transition $\mathcal{P} \to \mathcal{E} \to \mathcal{E}$ is repeated in the loops over cross-section layers and numerical integration points in FE. Then assembling of $\mathbf{K}^j$ and $\mathbf{F}^j$ is made in order to obtain the global matrices $\mathbf{K}$ and $\mathbf{F}$. 

Besides the data updating inside the finite elements the tangent direction to the equilibrium path defines the vector of displacement increments $\Delta \mathbf{u}^{(1)}$ in the NR method. Another approach is explored in the Quasi-Newton methods (QN). The
main idea of QN lies in updating the stiffness matrix using information from the previous iteration step. In the QN method, the direction of searching the new solution is locally secant (cf. Fig. 6.2), according to the condition:
\[ d^T_h = g^T g, \]  
(6.26)

where the following, short notation is used: \( d^i \), \( d = d^{(i-1)} \), \( g = \Delta r^{(i-1)} \), \( h = \Delta r^{(i-2)} - g \). A new approximation for the increment of the displacement vector equals
\[ d^i = \hat{K}^{-1} g = (\hat{K}^{(i-1)})^{-1} \Delta r^{(i-1)}, \]
(6.27)

where the locally secant stiffness matrix \( \hat{K} \) depends on the vectors \( h \) and \( d \) associated with the previous iteration steps \( i-1 \) and \( i-2 \).

In literature devoted to mathematical programming there are various formulae devoted to computation of the locally secant matrix \( \hat{K} \). From among them only the Broyden-Fletcher-Goldfarb-Shanno (BFGS) formula is presented (cf. [56]):
\[ K_{\text{BFGS}} = K + \frac{h h^T}{d^T h} - \frac{K d d^T K}{d^T K d}, \]  
(6.28a)

and the formula for the inverse matrix:
\[ K_{\text{BFGS}}^{-1} = (I - \frac{d d^T}{d^T h}) \hat{K}^{-1} (I - \frac{h d^T}{d^T h}) + \frac{d d^T}{d^T h}, \]  
(6.28b)
where: \( \hat{K}^{\text{BFGS}} \equiv \hat{K}^{(i-1)} \), \( K \equiv \hat{K}^{(i-2)} \). The BFGS method has been introduced to FEM due to the algorithm by Matties and Strang [195] which depends on the multiplication of appropriate vectors \( a \) and \( b \) in the following formula:

\[
\Delta d^{(i)} = [I + ab^T]^{-1} [\hat{K}^{(i-1)}]^{-1} [I + ba^T] \Delta R^{(i-1)}.
\]  
(6.29)

The BFGS method is explored in many computer codes, e.g. the Matties-Strang algorithm has been incorporated in the ADINA system [196] for the analysis of nonlinear problems.

Other methods suitable for the analysis of nonlinear equations are reviewed in many papers; an extensive list of references is given in [57,60].

The load controlled computational process becomes badly convergent and even divergent in the vicinity of limit points on the equilibrium path. One of possibilities to pass over the limit point is using a selected displacement as the control parameter. From among algorithms which are incorporated into computer codes only that by Batoz and Dhatt [197] is presented.

The main idea of the algorithm lies in double solution of the FE equilibrium equation:

\[
K^{(i-1)} \cdot [d^P | \Delta d^R(i)] = [\bar{P} | R^{(i-1)}], 
\]  
(6.30)

and computation of the load parameter increment from the equation corresponds to the selected displacement \( d_k \):

\[
\Delta d_k^{(i)} = \Delta \mu^{(i)} d_k^{P(i)} + \Delta d_k^{R(i)}.
\]  
(6.31)

For a fixed value of \( \Delta d_k \), i.e. \( \Delta d_k^{(1)} = \Delta d_k \) and \( \Delta d_k^{(i)} = 0 \) for \( i > 1 \), the following formula is evident if we take into account that \( \Delta d_k^{R(1)} = 0 \):

\[
\Delta \mu^{(i)} = \begin{cases} 
\frac{\Delta d_k}{d_k} & \text{predictor for } i = 1, \\
\frac{\Delta d_k^{R(i)}}{d_k^{P(i)}} & \text{corrector for } i > 1.
\end{cases}
\]  
(6.32)
The predictor-corrector approach can be extended on various control parameters if instead of Eq. (6.31) a more general, so called the constraint equation is introduced:

$$\varphi (\Delta \tilde{x}, \Delta \mu; \Delta \tau) = 0 .$$  \hspace{1cm} (6.33)

In a special case of linear constraint equation:

$$t_\mu^T \Delta d + t_\mu \Delta \mu = \Delta \tau ,$$  \hspace{1cm} (6.34)

the control vector \( \underline{z} = [t, t_\mu] \in \mathbb{R}^{N+1} \) can be introduced in the load-displacement space \( \mathbb{R}^{N+1} \), where \( N \) is the number of DOF of the considered structure. After solution of Eq. (6.30) with respect to \( \Delta \tilde{d}^P \) and \( \Delta \tilde{d}^R \) the increment \( \Delta \mu \) can be computed from (6.34):

$$\Delta \mu (i) = \frac{\alpha \Delta \tau - (1-\alpha) t_\mu^T \Delta \tilde{d}^R(i)}{t_\mu + t_\mu^T d^P(i)} ,$$  \hspace{1cm} (6.35)

where the parameter \( \alpha \) switches the computational process from prediction into correction of solution, i.e. \( \alpha = 1 \) for \( i = 1 \) and \( \alpha = 0 \) for \( i > 1 \).

Specification of the control vector leads to various procedures:

a) load control (Fig. 6.3a)

$$t = 0 , \quad t_\mu = 1 , \quad \Delta \tau = \Delta \mu_m ,$$  \hspace{1cm} (6.36a)

b) displacement control (Fig. 6.3b)

$$t_k = 1 , \quad t_\mu = t_\ell = 0 \quad \text{for} \quad \ell \neq k , \quad \Delta \tau = \Delta d_k ,$$  \hspace{1cm} (6.36b)

c) arc-length control (Riks-Wempner method) - Fig. 6.3c

$$\frac{t}{|\tilde{z}|} = \frac{\Delta \mu}{|\tilde{z}|} , \quad \Delta \tau = \Delta \mu_m ,$$  \hspace{1cm} (6.36c)

where \( |\tilde{z}| = (\Delta \tilde{d}^T \Delta \tilde{d} + \Delta \mu^2)^{1/2} \).
Fig. 6.3. a) Load control, b) Displacement control, c) Arc-length control.

In the original Riks-Wempner method the control vector $\tilde{f}$ is assumed to be not only unit but also close to the vector tangent to the equilibrium path at the point $m$ (e.g. parallel to the direction $m$, $m-1$ or to the vector $m^{-1}(4d, 4\mu)^{(n)}$ for the last iteration step $i=n$ at the point $m-1$) and is fixed during the iteration. That corresponds to the hyperplane constraint (Fig. 6.4a). In Fig. 6.4 subsequent solutions (points 1,2,..., $m+1$) are obtained by means of the mNR method but, of course, any other method can be used for solving the FE incremental equations.

Fig. 6.4. a) Riks-Wempner method, b) A modification of the RW method, c) Crisfield's method.

The control vector can be updated for subsequent iterations. That corresponds to the piece-wise linear constraint (6.34). Such an approach is close to the
Crisfield method in which the constraint corresponds to the sphere surface of radius \( \rho_m = \Delta \tau_m \).

The above presented methods have been extensively discussed in many papers listed in [58, 59]. Besides the outlined methods also other constraint equations are possible, e.g. Fried's orthogonal trajectory method [198] or energetic-type constraint by Bathe and Dvorkin [199].

In the presented approach the FE equations (6.20) and the constraint equation (6.33) are solved separately. It may lead to a slowly convergent iteration procedure of \( \det | \tilde{K}^{(1)} | \rightarrow 0 \). That is why Waszczyszyn [59] proposed to analyse the extended set of equations

\[
\tilde{K} \Delta \tilde{d} = \tilde{R}.
\]

which corresponds to Eqs. (6.20) and (6.34). The structure of the matrices is shown in Fig. 6.5. If an arc-type control parameter is used then the determinant of the extended matrix \( \tilde{K} \) is not equal zero at limit points, i.e. \( \det | \tilde{K} | \neq 0 \) at \( \det | K | = 0 \).

![Fig. 6.5. Structure of matrices in the extended Eq. (6.37).](image)

The matrix \( \tilde{K} \) is non-symmetric because of the row \( z^T \) and column \( -P \) but it is easy to modify solvers for a linear set of equations of a symmetric matrix at a small increase of the number of operations.

The analysis of Eq. (6.37) was called in [59] as the solving in the load-configuration space \( \mathbb{R}^{N+1} \) since both the displacement and load parameter...
increments are treated as virtually equivalent. The control vector can be easy changed (even during the iteration process) and the predictor-corrector technique is realized throughout the zero-one factor $\alpha$ at $\Delta t$.

All above discussed methods correspond to a continuation method which depends of the transition from one configuration of the structure into the neighbor configuration — cf. Fig. 6.6a. The advantage of such a procedure depends on storing data only from the previous iteration step. But on the other hand the procedure is in fact history-independent between the point $m$ and $m+1$ on the equilibrium path and at large incremental steps can lead to a drift from the exact elastic-plastic solution.

![Fig. 6.6. Two possible iteration procedures.](image)

The procedure sketched in Fig. 6.6b is better fitted into the elastoplastic analysis. Subsequent iterations are referred to the same reference configuration $m\Omega$ in the sense of computation of residual forces. The procedure b) enable us to consider better passive processes on the $P$ level, which can be perturbed by unbalanced forces in the procedure a). Also correction of stresses for fulfilling yield conditions is more correct for the procedure b).

All above mentioned remarks point out that for the analysis of an elastic-plastic structure the iteration procedure b) should be preferred in spite of simpler subroutines for the procedure a) — cf. [61] p. 265-267.
6.4. Numerical integration of constitutive equations

The discrete continuation method (step-by-step procedure) needs the coming down to the level $P$ for all integration points, both in the FE domain $\Omega^j$ and along the thickness of the structure. The transition $B\rightarrow \xi \rightarrow P$ has to be made for every iteration step in order to compute the g. stress vector $g$ which is needed for computation of the vector of internal forces $F^j$, according to (6.8)\(_4\). In the case of the $m$NR or NR methods the same has to be made with respect to $F^j(Q)$ and $K^j$ for every increment $m$ or iteration step $i$ correspondingly. The computation of the actual value of the stress vector $g(\tau)$ on the level $P$ is the main problem of elastoplastic analysis.

In the incremental analysis infinitesimal values of variables (full differentials) are to be changed by their finite increments. In case of the stress increment vector $\Delta g$ corresponding to the increment of time parameter $\Delta \tau$ the following formula is valid:

$$
\Delta g = \int_{\tau}^{\tau+\Delta \tau} \dot{g} \, d\tau = \int_{\xi}^{\xi+\Delta \xi} \frac{\partial g}{\partial \xi} \, d\xi = \int_{\xi}^{\xi+\Delta \xi} E^{ep} \, d\xi.
$$

(6.38)

The formula (6.38) can be approximated by relation $\Delta g = E^{ep}(a \Delta \tau) \Delta \xi$, where the coefficient $0 < a < 1$. In case of the passive process the local stiffness matrix $E^E$ is independent of time, but for active processes the integration (6.38) has to be carried out in order to fulfill yield condition $F(g + \Delta g) = 0$.

Because of mathematical difficulties only numerical integration can be practically used. This problem will be shortly presented below for the infinitesimal relations, discussed in Sec. 2. All relations are rewritten in the matrix form, according to Sec. 2.7 *):

$$
\xi = \dot{\xi} + \dot{\xi} = (E^e)^{-1} \frac{d\xi}{dt} + \frac{d\lambda}{dt}, \quad \lambda = \frac{1}{g} b^T \xi.
$$

(6.39)

$$
\Delta \xi = E^{ep} \xi = (E^e - E^P) \xi = (E^e - \beta \frac{b}{g} b^T) \xi.
$$

*) The relations (6.39) are formulated in the $F^6$ spaces, i.e. the vectors (2.63) are used and the matrices $E^e, E^P$ should be consistent with them.
where
\[ a = \frac{\delta F}{\delta \sigma}, \quad b = F^e a, \quad g = h^T E a, \]
\[ F(q, \varepsilon^p, k(\varepsilon_p)) = 0, \quad d\varepsilon^p_p = \left( \frac{2}{3} (d\varepsilon_p^p)^T d\varepsilon_p^p \right)^{1/2}. \] (6.40)

Let us consider a stress state just before the yielding, associated with the stress vector \( \sigma_A^{(1)} = \sigma_A \) in Fig. 6.7a. Let us assume that the increment of stresses are known, i.e. the vector \( \Delta \varepsilon \) has been obtained as a result of the transition \( B \rightarrow P \). The first step of computation depends on the elastic prediction:
\[ \Delta \sigma_A = E^e \Delta \varepsilon, \quad \sigma_B = \sigma_A + \Delta \sigma_A. \] (6.41)

Fig. 6.7. a) Crossing of the yield surface and approximate computation of the stress increment \( \Delta \sigma_C \), b) Subincremental procedure, c) Consistent approach (elastic prediction, orthogonal return, consistent modular matrix).
For simplicity of considerations let us assume that the material is elastic, perfectly plastic (in Fig. 6.7 the yield surface is immovable). Value of the function $F$, corresponding to points $A$ and $B$ in Fig. 6.7a, are:

$$F_A = F(\sigma_A, k) < 0 ,$$

$$F_B = F(\sigma_B, k) > 0 .$$  \hspace{1cm} (6.42)

The vector $\Delta \sigma_A$ is multiplied by a value of $0 \leq \gamma \leq 1$ in order to reach the surface $F$ in the point $C$

$$F_C (\sigma_A + \gamma \Delta \sigma_A, k) = 0 .$$  \hspace{1cm} (6.43)

The first approximation to $\gamma$ can be computed from the 'regula falsi':

$$\gamma_1 = - \frac{F_A}{F_B - F_A} .$$  \hspace{1cm} (6.44)

but because of nonlinearity of $F$ (as it takes place for the HMH yield condition) the next correction is recommended from the approximation (cf. [114]):

$$F_1 + a_1^T \Delta \sigma_A \Delta \gamma_1 = 0$$  \hspace{1cm} (6.45)

where: $F_1 = F(\sigma_A + \gamma_1 \Delta \sigma_A, k), a_1 = a(\sigma_A + \gamma_1 \Delta \sigma_A)$. In such a way the parameter $\gamma$ equals

$$\gamma = \gamma_1 + \Delta \gamma_1, \quad \Delta \gamma_1 = - \frac{F_1}{a_1^T \Delta \sigma_A} .$$  \hspace{1cm} (6.46)

In case of the HMH yield condition and elastic, perfectly plastic material

$$\frac{3}{2} s^T s - \sigma_o^2 = 0 ,$$  \hspace{1cm} (6.47)

the exact value of the parameter $\gamma$ can be computed from the quadratic equation:

$$a\gamma^2 + b\gamma + c = 0 ,$$  \hspace{1cm} (6.48)
where the coefficients are:

\[ a = \frac{3}{2} \Delta s_A^T \Delta s_A', \quad b = 3 \Delta s_A^T s_A, \quad c = \frac{3}{2} s_A^T s_A - s_0^2. \]  

(6.49)

After computation of the parameter \( \gamma \) the increment of strains can be split into two parts which leads to the following relation — cf. Fig. 6.7a:

\[ \sigma_C = \sigma_A + \gamma \Delta \sigma_A, \quad \Delta \varepsilon_C = (1-\gamma)\Delta \varepsilon, \quad \Delta \sigma_C = \frac{\Delta \varepsilon_C}{\varepsilon^p}(\sigma_C)\Delta \varepsilon_C, \]

(6.50)

\[ \sigma_D = \sigma_C + \Delta \sigma_C. \]

The solution \( \sigma_D \) does not fulfill the yield condition. That is why a correction \( \Delta \sigma_D \) should be introduced. The correction can be made by the factor \( r \) of radial return (scaling factor)

\[ \sigma_D' = r \sigma_D, \]

(6.51)

which gives the fulfilling of the yield condition

\[ F(\sigma_D', k) = 0 \]

(6.52)

and the equation similar to (6.48) — cf. [62], p. 252.

Another possibility is similar to that which gave the approximation \( \Delta Y_1 \) from Eq. (6.45). The correction is assumed to be in the direction of the normal to the yield surface:

\[ \Delta \sigma_D = \alpha \frac{\varepsilon_D}{\sigma_D}, \]

(6.53)

where the coefficient \( \alpha \) is computed from the consistency condition of the form

\[ F_D + dF = F_D + \frac{\varepsilon_D}{\sigma_D} \Delta \sigma_D = F_D + \alpha \frac{\varepsilon_D}{\sigma_D} \sigma_D = 0. \]

(6.54)

In such a way the correction vector becomes (cf. [114]):

\[ \Delta \sigma_D = - \frac{F_D}{\frac{\varepsilon_D}{\sigma_D} \sigma_D}. \]

(6.55)
The error of approximation to the vector \( \Delta \xi \) can be diminished by the subincremental approach. The idea of this approach depends on division of the strain increment \( \Delta \xi^C \) into \( n \) equal parts

\[
\Delta \xi_n = \frac{1}{n} \Delta \xi^C ,
\]

(6.56)

and compute the subincrements \( \Delta \xi^D \) into the direction tangent to the yield surface in the subincremental points \( D^D \). Finally the stress increment equals:

\[
\Delta \sigma^C = \sum_{\ell=1}^{n} \Delta \sigma^D \ell = \sum_{\ell=1}^{n} F^{ep} (\sigma^D) \Delta \xi_n .
\]

(6.57)

In Fig. 6.7b the value \( \Delta \sigma^C \) is shown for \( n=3 \). It is evident that even for small number of increments the differents between vectors \( \overline{CD} \) and \( \overline{CD^T} \) can be significant.

The subincremental approach is commonly accepted in computer codes. One-step approach can lead to divergent solution on the \( B \) level, as remarked by Bushnell [200]. On the other side the subincrements extend the computational time on the \( P \) level. That can drastically influence the increase of the total time of computations because the interval loops are carried out over all points (sections of numerical integration) of the discrete model of the considered structure.

That is why the number of subincrements \( n \) should be minimized. Nayak and Zienkiewicz [114] proposed to take \( n \) as the next highest integer of the ratio

\[
R = \frac{F_A}{\sigma_e (\varepsilon_p^C)} ,
\]

(6.58)

where \( \sigma_e (\varepsilon_p^C) \) is the effective stress in the point \( C \) of Fig. 6.7a and \( \alpha \) is a small number. In [114] the value of this parameter is recommended to be \( \alpha=0.05 - 0.10 \). Bushnell [203] suggested to use the ratio

\[
R = \frac{\Delta \bar{\varepsilon}}{0.0002} = \frac{1000}{\sqrt{3}} (\Delta \xi^C \Delta \xi^C)^{1/2} .
\]

(6.59)

In [09] p. 406 the number \( n = 10 - 16 \) is recommended.
More precise evaluation of the number of subincrements was given by Kriegs [201] and Schreyer et al. [202].

In papers [201, 202] the error analysis was made for different numerical approaches which depend on the elastic of tangent prediction - radial correction procedures in the deviatoric space $\bar{e}^e \in \mathbb{R}^9$.

On the basis of computation of the coefficient $\gamma$ for crossing of the yield surface and the subincremental technique with correction to the subsequent, actual yield surface appropriate algorithms have been carried out. The algorithm in [114] was worked out for material with isotropic strain hardening. Detail description of this algorithm with the FORTRAN subroutine has been given in [62] p. 249-257. In [09] p. 406 and [61] p. 282-283 the algorithms similar to that from [114] have been modified for arbitrary strain hardening.

In the mentioned algorithms the return to the actual yield surface is recommended to be performed at every subincremental step. If the number of subincrements is small the return can be made only at the final point $D_n$.

As the final result of the subincremental loop the stress vector $\sigma_D = \sigma_C + \Delta \sigma_C^p$ is obtained as the input data for computation of the interval force vector $F^j$ and then the residual forces $R = P - F$ -- cf. subroutine RESIDU in [62] p. 253-257.

Return mapping, analyzed in [201] was developed in recent papers by some authors — cf. Simo and Taylor [240], Ortiz and Simo [241], Ramm and Matzenmiller [242], Mitchell and Owen [243]. Contribution of elastic prediction, orthogonal return and consistent modular matrix leads to effective algorithms of numerical integration of elastoplastic constitutive equations.

Using elastic prediction (6.41) the stress $\sigma_F$ is computed as shown in Fig. 6.7c. Then the stress $\sigma_D$ on the yield surface is written in the following form

$$\sigma_D = \sigma_A + E^e (\Delta \epsilon - \Delta \epsilon_F^p) = \sigma_B - \Delta \lambda_D E^e \sigma_D$$

(6.60)

where

$$\sigma_D = \frac{\partial F}{\partial \sigma} \bigg|_D = P \sigma_D.$$  

(6.61)

From (6.60) the following relation results
After substitution of (6.62) into the yield condition a nonlinear equation is obtained

\[ F(\Delta \lambda_D) = 0. \quad (6.63) \]

Different numerical methods can be explored to solve Eq. (6.63) with respect to the increment of yielding parameter \( \Delta \lambda_D \) — cf. [242,243].

The relation (6.62) corresponds to the orthogonal return on the actual yield surface without computation of the crossing point C — see Fig. 6.7c.

After the parameter \( \Delta \lambda_D \) is known, the consistent modular matrix \( E_D^{ep} \) can be formulated to

\[ dq_D = E_D^{ep} \, d\varepsilon_D. \quad (6.64) \]

The strain vector \( \varepsilon_D \)

\[ \varepsilon_D = \varepsilon_A + \Delta \varepsilon_D + \Delta \varepsilon_D^P = \varepsilon_A + H (q_A - q_A) + \Delta \lambda_D \, q_D \]

can be differentiated:

\[ d\varepsilon_D = d\varepsilon_A + H (dq_A - dq_A) + d\Delta \lambda q_D + \Delta \lambda \frac{\partial q_D}{\partial q_D} \, dq_D \]

where \( H = (E_D^{ep})^{-1} \). For stationary values of \( \varepsilon_A \), \( q_A \) their differentials disappear, i.e. \( d\varepsilon_A = 0 \) and \( dq_A = 0 \), and the following relation can be easily obtained:

\[ dq_D = E_D \, (d\varepsilon_D - d\Delta \lambda q_D), \quad (6.64) \]

where

\[ E_D = [(E_D^{ep})^{-1} + \Delta \lambda_D P]^{-1} \quad (6.65) \]

and \( \partial q_D / \partial q_D = P \) is taken into account. Parameter \( d\Delta \lambda \) is computed from the consistency condition
\[ \text{d}F = 0 \rightarrow \text{d}A = \frac{a_D^T E_D}{g} \text{d}e_D . \]

In case of the HMH yield condition and isotropic strain hardening the parameter \( g \) is:
\[ \hat{\gamma} = \frac{T}{A_D} \hat{E}_D a_D + \hat{h} \quad \hat{h} = \left[ \frac{2}{3} \frac{G_e}{H'} \right] \frac{H'}{1 - \frac{2}{3} H' A_D} \]

Substitution of (6.66) into (6.64) gives the consistent modular matrix
\[ \hat{E}_D^{\text{ep}} = \hat{E}_D - \frac{\hat{h}}{g} \left( \hat{E} \circ \hat{a} \circ \hat{E} \right)_D . \] (6.68)

In the above sketched derivation all the quantities are computed at the actual point \( D \), at the end of interval \( \Delta t \) in (6.38). This corresponds to an implicit scheme of numerical integration to find the finite stress increment \( \Delta \gamma_D \). The matrix (6.68) is consistent with the NR method of solution in the sense of the quadratic rate of convergence of iteration procedure — cf. [240].

The use of the elastic predictor (6.41), the orthogonal return (6.62) and the consistent modular matrix (6.68) creates the base for much more effective algorithms than those supported on the subincremental approach — cf. [242,243].

Local unloading can significantly influences the convergence of the iteration on the \( \mathcal{B} \) level. As it has been mentioned in the previous section 6.3 the strategy b) from Fig. 6.6 is preferable in the elastoplastic analysis. If this strategy is accepted then the yield surfaces \( \Phi \) associated with the configuration \( \Psi \) should be taken as the starting surfaces for computing the stress vector \( \sigma_c \) in the subincremental procedure for every iteration step (i) on the \( \mathcal{B} \) level.

6.5. FE incremental formulations for large elastic-plastic deformations

The classical theory of plasticity has been developed under the assumption of geometrical linearity. According to that assumption strains and displacements are small -- all equilibrium equations and boundary conditions are referred to the initial, undeformed configuration. On such a base linear kinematic and equilibrium equations (4.4) and (4.9) can be derived. But there are many problems which cannot be analyzed in the frame of classical formulation. Large strains and displacements have to be taken into account in the analysis of plastic metal forming. Large displacement (rotations) - small strain relations
establish the background for formulation of the structural stability equations. Similar equations ought to be used for advanced yielding which can (especially in shell structures) influence the development of large deformations. That is why very early geometrical and material nonlinearities were joined in the FE analysis (cf. Table 1 in [22] p. 57 where Bushnell points out Felippa’s D.Th. from 1966, as the first paper in this field).

Large deformations require accurate consideration of the motion of the considered structure (deformable body $\mathcal{B}$). This demands refined definitions of strain and stress measures consistent with a reference configuration. In incremental formulation a relation between different configurations have to be derived in order to obtain the consistent description of the body motion.

A great number of papers was devoted to the formulation of large deformation relations (cf. references in [61,63,204,205]). In what follows only a sketch of main ideas will be presented in this section on the base of the book by Washizu [63] and papers [204,205] by Bathe et al. and Hibbitt et al. [207].

Let us consider three configurations, namely: i) the initial configuration $^0\Omega$ related to the time $\tau = 0$, ii) the current configuration $^\tau\Omega$, iii) the incremental configuration $^{\tau+\Delta\tau}\Omega$. The movement of a body will be referred to the immovable Cartesian system of coordinates $(x_1, x_2, x_3)$ and the position vector $^x\mathbf{r}$ is associated with the point $^\tau\mathbf{P} = \mathbf{P}(^x\mathbf{r})$ at the time $\tau$. Of course, it is possible to introduce also a system of material-type, curvilinear coordinates $[^x\mathbf{t}]$ - cf. Fig. 6.8.

The left superscripts refer to the configuration of $\mathcal{B}$ for considered time while the left subscripts the reference configuration. In the index notation the right hand side subscripts correspond to the Cartesian coordinate axis. The comma denotes differentiation with respect to the coordinate following, e.g.

$$
^0u_{i,j} = \frac{\partial^\tau \Delta\tau u_i}{\partial x_j}, \quad ^\tau\Delta\tau x_{m,n} = \frac{\partial^\tau \Delta\tau x_n}{\partial x_m}.
$$

(6.69)

Increments are considered between the configurations $^\tau\Omega$ and $^{\tau+\Delta\tau}\Omega$. The left hand side indices can be omitted if it does not cause a misleading, e.g.

$$
\Delta u_i = ^{\tau+\Delta\tau}u_i - ^\tau u_i \quad \text{but} \quad \Delta \mathbf{p}_j = ^{\tau+\Delta\tau}p_j - ^\tau p_j.
$$

(6.70)
If the initial configuration $^0\Omega$ is assumed to be the reference configuration then the 2nd Piola-Kirchhoff stress tensor $S_{ij}$ is used, related to the Cauchy stress tensor $\sigma_{kl}$ according to the formula:

$$
\frac{\tau^{+\Delta\tau S}}{\rho} = \frac{\sigma}{\tau^{+\Delta\tau}} \frac{o_{i}}{\tau^{+\Delta\tau}} \frac{\tau^{+\Delta\tau}}{\tau^{+\Delta\tau}^{i,k}} \frac{\sigma}{\tau^{+\Delta\tau}} \frac{o_{j}}{\tau^{+\Delta\tau}} \frac{\tau^{+\Delta\tau}}{\tau^{+\Delta\tau}^{j,l}}
$$

(6.71)

where $\rho$ is the material density. The Green-Lagrange strain tensor $E_{ij}$ *):

$$
\frac{\tau^{+\Delta\tau}}{o} E_{ij} = \frac{1}{2} \left( \frac{\tau^{+\Delta\tau}}{o} u_{i,j} + \frac{\tau^{+\Delta\tau}}{o} u_{j,i} + \frac{\tau^{+\Delta\tau}}{o} u_{k,i} + \frac{\tau^{+\Delta\tau}}{o} u_{k,j} \right)
$$

(6.72)

is consistent with the 2nd PL stress tensor in the sense of the principle of VW. The increment of the GL strain tensor can be split into two parts:

* The GL strain tensor is denoted as $E$ and the local stiffness tensor as $D_{ijkl}$ instead of $E_{ijkl}$ as it was been used in Ch.2.
\[ \Delta E_{ij} = \tau^{\Delta T}_{ij} \sigma_{ij} - \tau_{ij} = \]
\[ = \frac{1}{2} \left( \Delta u_{i,j} + \Delta u_{j,i} + \tau_{u_k,i} \Delta u_{k,j} + \tau_{u_k,j} \Delta u_{k,i} \right) + \]
\[ + \frac{1}{2} \Delta u_{k,i} \Delta u_{k,j} = \Delta e_{ij} + \Delta \eta_{ij}, \]
(6.73)

where: \( \Delta e_{ij} \) - increments linear with respect to \( \Delta u_{i,j} \) and \( \Delta \eta_{ij} \) - non-linear strain increments.

The increments of 2nd PL stress tensor
\[ \Delta S_{ij} = \tau^{\Delta T}_{ij} \sigma_{ij} - \tau_{ij}, \]
(6.74)

are related to the increments of GL strain tensor:
\[ \Delta S_{ij} = \tau^{D}_{ijkl} \Delta E_{kl}, \]
(6.75)

where \( \tau^{D}_{ijkl} \) is the locally secant stiffness tensor.

As an approximation to (6.75) the following relation can be assumed
\[ \Delta S_{ij} = \tau^{D}_{ijkl} \Delta e_{kl}, \]
(6.75a)

where now the constitutive tensor \( \tau^{D}_{ijkl} \) can be interpreted as the locally tangent stiffness tensor.

Increments of displacement and displacement-dependent loads equal:
\[ \Delta u_{i} = \tau^{\Delta T}_{ui} - \tau_{ui}, \]
\[ \Delta g_{i} = \tau^{\Delta T}_{gi} - \tau_{gi} = \tau^{T}_{i;j} \Delta u_{j} + \tau^{T}_{i} \Delta \mu_{r}, \]
\[ \Delta t_{i} = \tau^{\Delta T}_{ti} - \tau_{ti} = \tau^{T}_{i;j} \Delta u_{j} + \tau^{T}_{i} \Delta \mu_{r}. \]
(6.76)

The increments of internal forces \( \Delta g_{i}(u_{j}, \mu_{r}) \) and tractions \( \Delta t_{i}(u_{j}, \mu_{r}) \) are expressed approximately as linear functions of the displacement increments \( \Delta u_{j} \) and load parameter increments \( \Delta \mu_{r} \) where \( r = 1, \ldots, M \). This can be obtained due to
expansion of the load functions into Fourier series and retaining only linear parts, where derivatives are denoted as \( \frac{\tau_{o} e_{i;j}}{o_{i;j}} = \frac{\partial_{o} e_{i;j}}{\partial_{o} u_{j}} \), etc.

Loads and their behaviour should be precisely defined, e.g. dead or live load, related to initial or to actual configuration etc. This influences the form of derivatives in (6.76), cf. e.g. [204, 206, 207].

If all variables, associated with current and incremental configurations, \( \tau_{o} \) and \( \tau_{k} \) respectively, are referred to the initial configuration \( \tau_{o} \) then appropriate formulation is called as the total Lagrangian (TL) formulation. For the TL formulation the principle of virtual work takes the form:

\[
\int_{0V}^{\tau_{k} + \Delta \tau_{o}} \delta \delta E_{ij} \, d\delta V = \int_{0V}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta V + \int_{0S}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta S. \tag{6.77}
\]

After substitution of relations (6.73, 74, 75a) and rearranging (6.77) becomes:

\[
\int_{0V}^{\tau_{k} + \Delta \tau_{o}} \delta \delta E_{ij} \, d\delta V + \int_{0S}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta S - \int_{0V}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta V - \int_{0S}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta S - \int_{0V}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta V - \int_{0S}^{\tau_{k} + \Delta \tau_{o}} \delta \delta u_{ij} \, d\delta S = 0. \tag{6.78}
\]

Let us assume the FE approximation, similar to that in (6.4) and come back to the matrix notation using the superscript \( j \) as the FE number:

\[
\Delta u^{j}(\xi) = N^{j}(\xi) \Delta d^{j}. \tag{6.79}
\]

The incremental kinematic relations (6.73) can be written in the following, nonlinear form:

\[
\Delta \delta^{j} = \left[ \frac{\tau_{o} B_{o}}{o_{o}} + \frac{\tau_{o} B_{1}}{o_{o}} \Delta \delta^{j} + \frac{\tau_{o} B_{2}}{o_{o}} \Delta \delta^{j} \right] \Delta d^{j} = \frac{\tau_{o} (B_{L} + B_{N}) \Delta d^{j}}{o_{o}}. \tag{6.80}
\]
where the matrices $\tau_{B}^{j}$ and $\tau_{O}^{B}$ gives linear and nonlinear terms with respect to the node displacement increments $\Delta d^{j}$. After substitution of (6.80) into (6.75a) the constitutive equation becomes:

$$\Delta \omega^{j} = \tau_{O}^{D} \Delta \omega^{j}.$$  (6.81)

After substitution of (6.79-81) into (6.78) and manipulations as in Sec. 6.2 the FE incremental equation on the $\mathcal{B}$ level takes the form:

$$\tau_{O}(K^{j} + K_{u}^{j} + K_{s}^{j} - K_{p}^{j}) \Delta d^{j} - \Delta \alpha^{j} - \tau_{O}^{p} + \tau_{O}^{f} = \Delta \alpha^{j} + \tau_{O}^{G} + \tau_{O}^{S}.$$  (6.82)

where the following matrices and vectors are used:

$$\tau_{O}^{k} = \int_{\Omega} \tau_{O}^{B} \tau_{O}^{D} \tau_{O}^{B} d\Omega$$  - stiffness matrix of small displacements,

$$\tau_{O}^{u} = \int_{\Omega} \left[ \tau_{O}^{B} \tau_{O}^{D} \tau_{O}^{B} + \tau_{O}^{D} \tau_{O}^{B} + \tau_{O}^{B} \tau_{O}^{D} \tau_{O}^{B} \right] d\Omega$$  - initial displacement matrix,

$$\tau_{O}^{p} = \int_{\Omega} \tau_{O}^{N} \tau_{O}^{T} \tau_{O}^{N} d\Omega$$  - initial stress matrix,

$$\tau_{O}^{f} = \int_{\Omega} \tau_{O}^{G} \tau_{O}^{T} \tau_{O}^{G} d\Omega$$  - initial load matrix,

$$\Delta \alpha^{j} = \int_{\Omega} \tau_{O}^{T} \Delta \alpha^{j} d\Omega + \int_{\Omega} \tau_{O}^{T} \tau_{O}^{T} \tau_{O}^{T} \tau_{O}^{T} d\Omega$$  - vector of external load increments,

$$\tau_{O}^{p} = \int_{\Omega} \tau_{O}^{G} \tau_{O}^{T} \tau_{O}^{G} d\Omega + \int_{\Omega} \tau_{O}^{T} \tau_{O}^{T} \tau_{O}^{T} \tau_{O}^{T} dS$$  - vector of external loads,

$$\tau_{O}^{f} = \int_{\Omega} \tau_{O}^{S} \tau_{O}^{T} \tau_{O}^{S} d\Omega$$  - vector of internal forces.

In the integral of the matrix $\tau_{k}^{j}$ the stress matrix $\tau_{s}^{j}$ is used versus the stress vector $\tau_{s}^{j}$ in the integral of the vector $\tau_{p}^{j}$. In $\tau_{k}^{j}$ the matrices $\tau_{p}^{j} = \tau_{p}^{T} = \tau_{T}^{i} [t_{i};j]$ are used as well.
After assembling of FE into a discrete structure (transition $E \rightarrow B$) the following incremental equations for the TL formulation are obtained:

$$
\tau_0 (K + K_U + K_Q - K_p) \Delta d = \tau_0 P \Delta u + \tau_0 R ,
$$

(6.84)

where: $\tau_0 P$ — matrix of reference loads, $\tau_0 R = \tau_0 P - \tau_0 F$ — vector of residual forces.

In the TL formulation all function and matrices are related to the initial configuration $\Omega^0$ but it is possible to assume the current $c. \tau \Omega$ as a reference configuration. For such an approach the updated 2nd PL stress tensor has to be used:

$$
\tau^* S_{ij} = \tau_0 S_{ij} + \Delta \tau S_{ij} ,
$$

(6.85)

as well as the updated GL strain tensor

$$
\tau^* E_{ij} = \frac{1}{2} (\Delta \tau u_{i,j} + \Delta \tau u_{j,i}) + \frac{1}{2} \Delta \tau u_{k,i} \Delta \tau u_{k,j} = \\
= \Delta \tau \epsilon_{ij} + \Delta \tau \eta_{ij} .
$$

(6.86)

The constitutive equation is also associated with the current configuration:

$$
\Delta \tau S_{ij} = \tau_{ijrs} \tau^* E_{ij} = \tau_{ijrs} \Delta \tau \epsilon_{rs} .
$$

(6.87)

The quantities (6.86-87) are used in the frame of updated Lagrangian (UL) formulation. The principle of VW takes the form:

$$
\int V \tau^* S_{ij} \delta \tau^* E_{ij} \ dV = \int V \tau^* \delta \tau_{i} \delta \tau u_{i} \ dV + \int S \tau_{i} \delta \tau u_{i} \ dS .
$$

(6.88)

The appropriate FE incremental equations can be written in the form:

$$
\tau (K + K_Q - K_p) \Delta d = \tau P \Delta u + \tau R ,
$$

(6.89)

where the matrices are quite similar to those for the TL formulation except the transformation which refers them to the configuration at time $\tau$. 
Eq. (6.89) seems to be simpler than that for the TL formulation (6.84) because of lack of the initial displacement matrix $K_u$. But on the other side all variables and also their increments have to be updated for every time increment $\Delta t$. That causes such additional number of arithmetic operations that computational effort can be comparable if either the TL or TU formulations are used.

Geometrical nonlinearities influences the FE incremental equations in the same way as in the elastic analysis. These nonlinearities are introduced through the matrices of initial displacements (only in TL), initial stresses and loads, $K_u$, $K_0$ and $K_p$ respectively, as well as transformation to the reference configuration. In both formulations large deformations have to be considered also with respect to constitutive equations.

Incremental models of elastic-plastic materials should correspond to the TL or UL formulation. Using the TL formulation the 2nd PK and GL tensors must be applied to define the stress and strain history in the constitutive tensor $\tau^D_{ijkl}$. In the UL formulation the tensor $\tau^D_{ijkl}$ is defined by the history of the Cauchy stresses and the accumulation of the instantaneous plastic strain increments - cf. [205].

The constitutive Eq. (6.87) may be more appealing than the TL material Eq. (6.75) since physical components are used to define the material functions. Additionally the increments $\Delta \epsilon_{ij}$ in the approximated Eq. (6.87) are just as in small displacement analysis. Having calculated $\Delta S_{ij}$ the updated tensor $\tau^{\Delta \tau}S_{ij}$ can be computed from (6.85) and transformed into the Cauchy stress tensor

$$
\tau^{\Delta \tau}_{ij} = \frac{\tau^{\Delta \tau}_{ik}}{\tau_{j}} \tau^{\Delta \tau}_{k,i} \tau^{\Delta \tau}_{ij} \tau^{\Delta \tau}_{k,l,j},
$$

in order to come to the next time step $\tau^{\Delta \tau} \rightarrow \tau$.

Both the TL and UL are equivalent to each other if appropriate transformations are made. With respect to the constitutive tensors the following relation exist

$$
\tau^{D}_{mnpq} = \frac{O_i}{O_{\rho}} \tau^{X}_{m,i} \tau^{X}_{n,j} \tau^{D}_{ijkl} \tau^{X}_{p,k} \tau^{X}_{q,l},
$$

The relation (6.91) is of importance since experimental evidence in elastoplasticity is associated with the current configuration.
The idea of Lagrangian formulation can be associated with different reference configurations (cf. [61]). In case of quasi-static problems and the UL formulation the current configuration can be associated either with $m_Q^{(1)}$ or with $m_Q$ depending on the iteration procedures shown in Fig. 6.6.

Another possibility is to describe the deformation process on the $\mathcal{R}$ level using a stress rate defined with respect to the current moving coordinates within the time interval $(\tau, \tau+\Delta \tau)$. The stress rate is invariant with respect to rigid body rotations if the Jaumann stress rate is used (cf. [204]):

$$\dot{\sigma}_{ij} = \frac{D}{D\tau} \sigma_{ij} - \sigma_{ip} \dot{\omega}_{pj} - \sigma_{jp} \dot{\omega}_{pi}, \quad (6.92)$$

where the components of spin tensor are

$$\dot{\omega}_{pj} = \frac{1}{2} \frac{D}{D\tau} (\Delta u_{i,p} - \Delta u_{p,i,j}) \quad (6.93)$$

and $D/D\tau$ denotes the time derivative operator with $\tau_{x_i}$ kept constant. The constitutive relation takes the form

$$\dot{\sigma}_{ij} = \tau_{ij} D_{ijkl} \frac{D}{D\tau} \Delta \varepsilon_{kl} \quad (6.94)$$

Proper formulation of constitutive equations for large strains requires deeper considerations on the background of continuous mechanics in order to obtain consistent relations. A great number of papers devoted to this problem is listed in [02] p.196-199.

For many problems of structural mechanics satisfactory results are obtained under assumptions of small strains but large displacements (rotations). In such an approach the integration can be performed for the initial configuration and without change of material density:

$$\tau_{ij} = 0_{ij}, \quad \varepsilon_{ij} = 0_{ij}, \quad \rho = 0_{\rho}, \quad (6.95)$$

If the motion of any material line element is characterized as a rigid body rotation and a pure deformation
\[
\frac{\alpha^{O}_{x_{i}}}{\alpha^{T}_{x_{j}}} = (\delta_{ik}a_{kl})^{T}k_{j}, \quad \frac{\alpha^{T}_{x_{i}}}{\alpha^{O}_{x_{i}}} = T_{jk}(\delta_{kl}a_{kl}) + O(a^{2}) \tag{6.96}
\]

then for small strains \(a_{ik} \ll 1\) and all formulae can be significantly simplified (cf. [206]). In the small strain - large displacement theory the constitutive matrix \(T_{ijkl}^{D}\) can be defined as that at the small strain theory but if referring to the configuration \(O\Omega\) then the relation (6.91) becomes:

\[
T_{D}^{O} \cdot mnpq = T_{ijkl}^{D} T_{mi} T_{nj}^{T} T_{pk} T_{qk}^{T} \tag{6.97}
\]

If the local updated system of coordinates is used then the constitutive relation of small strains can be used:

\[
\Delta S_{ij}^{O} = T_{ijkl}^{D} \Delta E_{ij} \tag{6.98}
\]

where \(d^{O}_{x_{j}} = T_{ij}d^{O}_{x_{j}}\) defines a coordinate transformation.

Contrary to the pointed out transition from large to small strains the transition in opposite direction, sometimes provoked by experiments and existing software, ought to be performed very carefully, cf. Nemat-Nasser [208].

The geometrical nonlinearity causes that the tangent stiffness matrix in Eqs. (6.84) and (6.89) has to be extended by addition of the matrices \(K_{O}\) and \(K_{U}\) in comparison with the small displacement Eq. (6.14). In case of displacement-dependent loads the matrix \(K_{D}\) should be also introduced. The additional effort is associated with transformation of variables from the current to the reference configurations. Despite these main differences all methods discussed in Sec. 6.3 can be applied to the analysis of material and geometrically nonlinear problems.

If the small strain assumption is accepted then in the frame of large displacement formulation only subroutines related to the numerical integration of elastic-plastic constitutive equations have to be incorporated into computer codes implemented for the elastic, nonlinear analysis, cf. e.g. programs by Owen and Figueiras in [64].
7. THERMO-ELASTOPLASTIC, CREEP AND VISCOPLASTIC ANALYSIS

7.1. General remarks

The development of structures which operate at elevated temperatures (e.g. turbines, jet propulsion systems) or are subjected to large thermal gradients (e.g. steel structures in fire conditions, welding and other fabrication processes) needs a temperature-dependent analysis. Elevated temperatures and their changes create not only thermal stresses but also influence upon mechanical properties of materials and cause development of strains and displacements in time. That is why time-dependent behaviour of structures should also be considered. This phenomenon, related to changes of strains under fixed loads or relaxation of stresses for established deformation are commonly called as creep of structures if the relation between stresses and strains is time-dependent. In the mechanical analysis the total strain can be considered as combined of elastic, thermal, plastic and creep parts:

$$\varepsilon = \varepsilon^e + \varepsilon^t + \varepsilon^p + \varepsilon^c.$$  (7.1)

Elastic and thermal strains are reversible and depend on the instantaneous stress and temperature levels. Irreversible strains are associated with plastic and viscous properties of materials and their complete determination requires that stress and temperature histories must be known.

For many practical problems mechanical or thermal loadings are of short duration. Hence, creep strains are not significant and adequate description of appropriate behaviour of structures can be obtained by the use of non-isothermal elastoplasticity.

For structures exposed to long duration loadings, especially at elevated temperature, creep strains have to be considered since they influence not only a decrease of stiffness but also a redistribution of stresses in structures. That leads to the need of the non-isothermal, elastic-plastic-creep analysis. Viscous and elastic-plastic properties of materials ought to be considered simultaneously for transient loadings with multiple complex cycles.

In metals the dependance of creep rate upon stress is quite nonlinear and most of the creep deformation is irreversible. As a consequence linear theories of
viscoelasticity (hereditary theories) do not apply to metals. Experiments point out that plastic (time-independent) and creep (time-dependent) properties are coupled and they cannot be treated separately, especially for time dependent loadings. That leads into splitting of the total strain into three parts

\[ \varepsilon = \varepsilon^e + \varepsilon^{vp} + \varepsilon^o, \quad (7.2) \]

where \( \varepsilon^e, \varepsilon^{vp} \) stand for elastic and viscoplastic strains respectively. The strain \( \varepsilon^o \) can be treated as an initial strain of autogenous character or due to temperature changes.

In the analysis of structures both approaches, supported on the (7.1) and (7.2) strain splitting are considered. The first approach corresponds to the elastic-plastic-creep analysis, the second one is related to the elastic-viscoplastic analysis. In the both approaches isothermal and non-isothermal deformation processes can be considered associated with thermal (initial) strains and with or without changes of material properties.

On purpose of computational methods two incremental formulations are used. The first formulation explores the constitutive relation in the following form:

\[ \dot{\sigma} = \mathbf{E}^e \dot{\varepsilon} - \dot{\varepsilon}^o, \quad (7.3) \]

where into the initial stress rate \( \dot{\varepsilon}^o \) all inelastic effects are put. The relation (7.3) leads to the initial stress approach, as it has been called in Sec. 6.2. Another possibility is related to the formulation

\[ \dot{\sigma} = \mathbf{E}^* \dot{\varepsilon} - \dot{\varepsilon}^*, \quad (7.4) \]

where all strain-independent effects are included in the initial stress rate \( \dot{\varepsilon}^* \). The material matrix \( \mathbf{E}^* \) can be interpreted as the tangent matrix with respect to the \((\sigma + \dot{\sigma}^*, \varepsilon)\) vector relation. On the base of the relation (7.4) the tangent stiffness approach can be formulated.

The splitting of total strain (or its increment) and the constitutive relations (7.3) and (7.4) are valid, in principle, for small strains and they can be easy introduced to large displacement formulations.
The main difference between time-independent and time-dependent analysis lies in the treatment of rates of variables. In time-independent problems the rates can be related to an artificial, time-type parameter, e.g. plasticity time $\tau$ or temperature parameter $T$. In time-dependent problem the rates correspond to velocities and the physical time $t$ must be considered as the basic independent variable. That means that in the time-dependent analysis initial-boundary-value problems are considered which, of course, influences the formulation of computational methods.

In this chapter the non-isothermal elastoplasticity relations are given under assumptions of small strains and uncoupled thermal and mechanical fields. The uncoupling of these fields enables us to solve independently the heat equation in order to obtain a temperature distribution for the time $t$:

$$\theta(x,t,T) = T \bar{\theta}(x,t),$$  \hfill (7.5)

where $T$ can be treated as the temperature parameter (load-type parameter). For non-isothermal elasto-plastic problems the asymptotic solution

$$\theta(x,T) = T \bar{\theta}(x),$$  \hfill (7.5a)

can be obtained applying FEM to the solution of the Poisson-type equation for the steady heat transfer - cf. [54]. Solutions (7.5) or (7.5a) can be obtained on the base of the same FE mesh as it is applied to the analysis of mechanical fields in the structure. In what follows we assume that temperature distribution is known.

Creep constitutive models are discussed under assumption of small strains. The attention is focussed on the strain-hardening law and associated creep flow. The viscoplastic Perzyna's model is related to the HMH yield condition and associated flow rule.

Modification of the incremental FE equations, corresponding to the constitutive relations (7.3) and (7.4) are shown from the viewpoint of geometrically nonlinear formulations and elastic-viscoplastic model of material.

New problems, related to numerical integration of time-dependent equations are pointed out. As an interesting approach to elastoplasticity a stationary state

can be considered as an elastic-viscoplastic, asymptotic-type solution.

7.2. Basic relations for thermo-elastoplasticity

Experiments at elevated temperature display that plastic properties of material can undergo significant changes. In Fig. 7.1, initial and subsequent yield loci are shown according to Phillips and Tang [209]. The experiments were made on thin-walled tubes of commercially pure aluminium under combined tension and torsion. The presented results correspond to different specimens but scattering of the results for the initial yield loci were minor, so only one yield locus for the specimen S-5 is shown.

![Graph showing yield loci for tubular aluminium specimens](image)

**Fig. 7.1. Yield loci for tubular aluminium specimens**

The yield loci were obtained for the proportional limit definition of yielding. That caused significant change when the yield loci pass from the initial states into subsequent ones - cf. Sec.2.9. The initial yield loci, with the base at 70°F being nearly the HMM ellipse, and significant kinematic-type strain hardening manifest.
Temperature can be treated as a special type of generalized load which influences not only the strain and stress fields but above certain level it changes also material properties. That is why in the analysis of structures, exploited at deviated temperatures (especially above 1/3 of the melting point), special attention should be paid to that effect which can significantly decrease the load carrying capacity and stiffness of structures. In case of metals and their alloys especially the yield surfaces undergo a considerable contraction if the temperature increases, but in what follows changes of other material parameters will be also taken into account.

In Fig. 7.2., similar to that from paper [210] by Sharifi and Yates, a tendency of changing of plastic parameters for metal-type material is shown for the increasing temperature $\theta$.

Fig. 7.2 Dependence on temperature of the material parameters $E$, $E_T$, $H'$, $\zeta$ and $\phi$.

In this section basic relations of thermo-elastoplasticity are derived. Only assumption 5. is dismissed from those listed in Sec. 2.1. That means that a non-dimensional deformation process is considered and material properties are assumed to be temperature-dependent.
Let us assume the yield condition as a certain generalization of the relation (2.20):

\[ F (\sigma_{ij}, \varepsilon_{ij}^p, k, \theta) = 0 , \]  

(7.6)

and the continuity condition of neutral state (at fixed values of the plastic parameter k and plastic strains \( \varepsilon_{ij}^p \)):

\[ \varphi = \frac{\partial F}{\partial \sigma_{kl}} \, d\sigma_{kl} + \frac{\partial F}{\partial \theta} \, d\theta = 0 . \]  

(7.7)

The associated plastic flow rule satisfying the condition (7.7) can be written in the following form:

\[ d\varepsilon_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}} \left( \frac{\partial F}{\partial \sigma_{kl}} \, d\sigma_{kl} + \frac{\partial F}{\partial \theta} \, d\theta \right) . \]  

(7.8)

The consistency condition takes the form

\[ dF = \frac{\partial F}{\partial \sigma_{kl}} \, d\sigma_{kl} + \frac{\partial F}{\partial \theta} \, d\theta + \frac{\partial F}{\partial \varepsilon_{ij}^p} \, d\varepsilon_{ij}^p + \frac{\partial F}{\partial k} \, dk + \frac{\partial F}{\partial \varepsilon_{ij}^p} \, d\varepsilon_{ij}^p = 0 . \]  

(7.9)

After substitution of (7.8) into (7.9) the consistency condition becomes:

\[ \left( \frac{\partial F}{\partial \sigma_{kl}} \, d\sigma_{kl} + \frac{\partial F}{\partial \theta} \, d\theta \right) [1 + \lambda \frac{\partial F}{\partial \sigma_{ij}} \left( \frac{\partial F}{\partial \varepsilon_{ij}^p} + \frac{\partial F}{\partial k} \frac{\partial \varepsilon_{ij}^p}{\partial k} \right)] = 0 . \]  

(7.10)

For a plastically active process the expression in brackets is positive. Thus, the plastic parameter \( \lambda \) equals:

\[ \lambda = - \left[ \frac{\partial F}{\partial \sigma_{ij}} \left( \frac{\partial F}{\partial \varepsilon_{ij}^p} + \frac{\partial F}{\partial k} \frac{\partial \varepsilon_{ij}^p}{\partial k} \right) \right]^{-1} \approx \frac{1}{h} , \]  

(7.11)

where h corresponds to the hardening parameter (2.47). Substitution of (7.11) to (7.8) leads to a relation for the plastic strain increments:

\[ d\varepsilon_{ij}^p = \frac{1}{h} \left( a_{kl} \, d\sigma_{kl} + F, \theta \, d\theta \right) a_{ij} , \]  

(7.12)

where for the sake of brevity the following notation is used:
\[ a_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, \quad \frac{\partial F}{\partial \theta} = F', \theta. \]  

(7.13)

Total strain increment consists of the elastic, plastic and thermal strains:

\[ d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} + d\varepsilon_{ij}^{\theta}. \]  

(7.14)

For temperature-dependent elastic properties the elastic strain increments are given by:

\[ d\varepsilon_{ij}^{e} = H_{ijkl} \, d\sigma_{kl} + dH_{ijkl} \, \sigma_{kl}, \]  

(7.15)

where \( H_{ijkl} = [E_{ijkl}^{e}]^{-1} \). From (7.7), (7.14) and (7.15) appropriate relation for \( d\varepsilon_{ij} - d\varepsilon_{ij}^{\theta} \) can be calculated, and after multiplication by \( E_{ijrs}^{e} \) we come to the following relation:

\[ E_{ijrs}^{e} \left( d\varepsilon_{ij} - d\varepsilon_{ij}^{\theta} \right) = d\sigma_{rs} + E_{ijrs}^{e} \, dH_{ijkl} \, \sigma_{kl} + \]
\[ + \frac{1}{k} E_{ijrs}^{e} a_{ij} (a_{kl} d\sigma_{kl} + F', \theta, d\theta) \]  

(7.16)

After some mathematics (cf. Zudans et al. [211]) the following constitutive relation can be computed from (7.15):

\[ d\sigma_{ij} = E_{ijkl}^{ep} \left( d\varepsilon_{kl} - d\varepsilon_{kl}^{\theta} - d\varepsilon_{kl}^{p} \right), \]  

(7.17)

where the stiffness matrix \( E_{ijkl}^{ep} \) and the vector of plastic non-isothermal strain increments \( d\varepsilon_{ijkl}^{p} \) equal:

\[ E_{ijkl}^{ep} = E_{ijkl}^{e} - E_{ijkl}^{p}, \]  

\[ d\varepsilon_{ijkl}^{p} = dH_{klmn} \, \sigma_{mn} + \frac{1}{h} a_{kl} \, F', \theta, d\theta, \]  

(7.18)

where: \( E_{ijkl}^{p} \) - local plastic stiffness matrix given in (2.49).

The relation (7.17) can be easily transformed into the form (7.3) or (7.4) depending on the need of computational methods.

Let us consider the \( J_2 \) material (generalized HMH yield condition) with isotropic strain-hardening:
\[
F = \frac{1}{2} s_{ij} s_{ij} - \frac{1}{3} \sigma_e^2 (\epsilon_p, \theta) = 0 .
\]

(7.19)

In such a case the following relations occur:

\[
a_{ij} = s_{ij}, \quad E_{ijkl} a_{kl} = 2G s_{ij}, \quad dH_{ijkl} a_{kl} = \frac{1}{2} d\left(\frac{1}{G}\right) s_{ij},
\]

\[
k = \frac{4}{9} H' \sigma_e^2, \quad F_{ij} = -\frac{2}{3} \sigma_e H' \theta = \frac{2}{3} \sigma_e \xi .
\]

(7.20)

where functions \( \xi = \frac{\partial \sigma_e}{\partial \theta} \) and \( \phi = \frac{\partial \epsilon_p}{\partial \theta} \) are interpreted in Fig. 7.2c, d. The increment of plastic strains (7.7) takes the form

\[
d_{ij}^p = \frac{9 s_{ij}}{4 H' \sigma_e^2} (s_{kl} d \sigma_{kl} + \frac{2}{3} \sigma_e \xi d \theta) .
\]

(7.21a)

or after substitution of (7.11):

\[
d_{ij}^p = \frac{3 s_{ij}}{2 \sigma_e (1 + \frac{H'}{3G})} d \sigma_{kl} (d \sigma_{kl} - d \sigma_{kl} - \frac{1}{2} d\left(\frac{1}{G}\right) \sigma_{kl})
\]

\[+ \frac{3 s_{ij}}{2 \sigma_e (3G + H')} \xi d \theta .
\]

(7.21b)

From (7.21a) it can be deduced that non-isothermal plasticity material must have a strain hardening \( H' > 0 \). For the perfectly plastic material only isothermal plasticity can be considered. This problem was deeper analyzed from thermodynamical point of view by Drucker [110].

Constitutive relations for kinematic strain-hardening can be formulated after a temperature dependence of the tensor of microstress increment is defined, \( d \sigma_{ij}(\theta) \) - cf. [210,211].

The condition of neutral state (7.7) is used to define the plastically active and passive processes:

a) **active process**

\[
F = 0 \quad \text{and} \quad \phi = a_{kl} d \sigma_{kl} + F_{ij} d \theta > 0 ,
\]

(7.22a)

b) **neutral process**
\[ F = 0 \] and \[ \varphi = 0 , \]  
(7.22n)

p) passive process

\[ F \leq 0 \] and \[ \varphi < 0 . \]  
(7.22p)

The conditions (7.22) are to be used instead (2.50) since the plasticity function \( \lambda \) was used in (7.8) instead of \( d\lambda \), as it was in (2.42).

In computations for non-isothermal elastoplasticity the parameter \( \beta = 1 \) should be taken in (2.49) for active processes. If either neutral or passive processes take place then \( d\varepsilon^P_{ij} = 0 \) and instead of (7.17) the constitutive relation becomes:

\[ d\sigma_{ij} = E_{ijkl}^e (d\varepsilon^e_{kl} - d\varepsilon^\theta_{kl}) - A_{ijkl} \sigma_{kl} , \]  
(7.23)

where the thermal strains \( d\varepsilon^\theta_{kl} \) and the tensor of thermal stiffness degradation \( A_{ijkl} \) are - cf. [211]:

\[ d\varepsilon^\theta_{kl} = \alpha \varepsilon_{kl} d\theta , \]  
(7.24)

\[ dA_{ijkl} = E_{ijmn}^e dH_{mnkl} = G \delta_{ik} \delta_{jl} d(\varepsilon^e_j) - \frac{\delta_{ij} \delta_{kl}}{(1-2v)(1+v)} dv . \]

The relations (7.24) are given for isotropic material with three characteristics: the Kirchhoff modulus \( G \), Poisson's ratio \( v \) and coefficient of thermal extension \( \alpha \).

Both definitions of the type of plastic deformation processes and the formulae (7.12), (7.17), (7.23) ought to be taken into account in order to apply the algorithms which have been discussed in Sec. 6.4. If other relations are formulated for the thermo-elastic-plastic constitutive relations special attention should be paid to crossing of the yield surface, cf. e.g. algorithms by Cyr and Tefer [212] and Zudans et al. [211].

Though several simple thermoelastic problems were solved at the 40-ies the first general formulation was given by Prager [213] in 1958. He assumed the
consistency condition (7.2) and derived equations for rigid plastic, work-hardening materials. Generalization of the theory on the thermo-elastoplasticity was made by Naghdi [107].

In [02] p.193-195 extensive literature is quoted, devoted to formulation of constitutive equations on the more substantial background. The internal state variables theory is especially perspective from thermomechanical viewpoint.

7.3. Creep constitutive models for metals

As in the theory of plasticity, there exists a big number of models which have been proposed for describing creep behaviour of materials. Unlike the plasticity models, the various creep models differ significantly in mathematical form and physical basis - cf. [06,214]. With respect to metal and their alloys the phenomenological creep theory is widely used and that is why appropriate models have been used in computer codes, cf. eg. [66]. In this section only these models are discussed on the basis of review papers by Boresi and Sibetton [67] and Nickell [66].

The phenomenological creep theory is similar to the incremental theory of plasticity. In order to model the creep behaviour three relationships are used:
1. Uniaxial creep law, obtained usually for simple tension test.
2. Flow rule as a generalization of the creep law to multiaxial stress and strain states.
3. Hardening rule generalizes the flow rule relations to time-varying stress levels.

The creep law should describe two main phenomena, namely: i) changes of strains in time in a specimen under fixed value of uniaxial stress, ii) stress-strain relations for fixed time levels. In Fig. 7.3. appropriate diagrams are shown for a specimen under uniaxial tension.

The creep-time curves (Fig. 7.3a.) display 3 stages of creep behaviour. From among them the most important is the longest in time, secondary (steady-state) creep, when the creep rate \( \dot{\varepsilon}_c \) remains nearly constant. A great number of relations has been proposed to approximate experimental data - cf. [67] p. 448-451. The Norton relation for the secondary creep is reminded:

\[
\dot{\varepsilon}_c = B \sigma^\delta ,
\]  

(7.25)
where the parameter $\delta = 2.5-7$ shows how highly non-linear dependence on stress level takes place in metal. The parameters $B$, $\delta$ depend on temperature.

In order to include the effects of elastic deformation, the instantaneous plastic strain and primary stage creep, Odqvist (cf. ref. 4 in [67]) proposed a generalization of Norton's law

$$\dot{\epsilon} = \frac{\partial c}{\partial \sigma} |_{t=0} \dot{\sigma} + \dot{\epsilon}_{sc} (\sigma),$$

(7.26)

where $\dot{\epsilon}_{sc}$ is the steady-state creep rate. This approximation starts at $\epsilon_0$ at $t = 0$ and gives a straight line asymptotic to the steady state curve of creep (Fig. 7.3a).

Uniaxial creep equations can be formulated in the following form (cf. [215]):

$$\epsilon_c (\sigma, t, \theta) = f(\sigma, \theta) \left[ 1 - e^{-r(\sigma, \theta)t} \right] + g(\sigma, \theta)t = \epsilon_{pc} + \epsilon_{sc},$$

(7.27)

where $f, r, g$ are functions of stress $\sigma$ and temperature $\theta$. The relation (7.27) is combined of two parts. The first component describes a primary or transient creep $\epsilon_{pc} = f[-\exp(-rt)]$. The second component corresponds to steady-state creep $\epsilon_{sc} = gt$.

More generally it is natural to assume the existence of an equation of state which relates the creep rate to the applied stress $\sigma$, temperature $\theta$ and a number
of structural parameters $s_i$ for $i = 1, \ldots, m$ which characterize the creep process

$$F (\dot{\varepsilon}, \sigma, \theta, s_i) = 0.$$  \hfill (7.28)

Simpler cases corresponds to $m=1$ and are called as time-hardening rule if $s_1 = t$

$$F (\dot{\varepsilon}, \sigma, t, \theta) = 0.$$  \hfill (7.28a)

and strain-hardening rule for $s_1 = \varepsilon_C$

$$F (\dot{\varepsilon}, \sigma, \varepsilon_C, \theta) = 0.$$  \hfill (7.28b)

The time-hardening (aging) theory is an attempt to use an analogy with viscous flow and it can be, therefore, effective when secondary creep dominates, e.g. at very high temperatures. According to (7.28a) the creep rate at any time and stress level is assumed to be independent of the current value of creep strain. If the stress is changed from the level $\sigma_1$ to $\sigma_2$ at time $t_1$, the creep rate is determined at the point B - cf. Fig. 7.4a.

Fig. 7.4. a) Models of creep hardening, b) Auxiliary strain-hardening rule.
The strain-hardening theory is an attempt to use analogy with work-hardening plasticity theories. According to (7.28b) the creep strain rate at any time and stress level depend upon the total creep strain. If the stress level is changed from $\sigma_1$ to $\sigma_2$ the creep rate at time $t_1$ is determined at point $C$.

In order to take into account local unloading a so-called auxiliary strain-hardening rule can be formulated. It depends on reflection of the $\sigma$ curve for reverse value of stress — cf. Fig. 7.4b taken from [66].

As a generalization of the uniaxial creep law the flow rule has been proposed in accordance with similar terminology in time-independent plasticity. For the isotropic strain-hardening rule the following flow rule is accepted in the form slightly modified in comparison to that originally proposed by Odqvist [68], Ch.5. but similar to relations (2.51) and to (2.55):

$$
\dot{\epsilon}_{ij}^c = \frac{3}{2} \frac{\epsilon_{ij}^c (\sigma_e, \epsilon_c, \theta)}{\sigma_c} \epsilon_{ij}^c.
$$

(7.29)

where: $\sigma_e$ is effective stress defined in (2.21) and $\epsilon_c$ is effective creep strain similarly to (2.22):

$$
\sigma_e^2 = \frac{3}{2} \epsilon_{ij}^c \epsilon_{ij}^c, \quad \epsilon_c^2 = \frac{3}{2} \epsilon_{ij}^c \epsilon_{ij}^c.
$$

(7.30)

The relation (7.29) is valid under following assumptions — cf. [68] p. 21:

1. Strains are small and creep strains can be separated from total strains, according to (7.1).
2. Material is homogeneous and isotropic.
3. With respect to creep strains material is assumed to be incompressible.
4. There exists a potential flow — in case of (7.29) the strain rates are coaxial with components of the stress deviator.
5. Material obeys the isotropic strain-hardening.

The listed assumptions fully corresponds to those which are adopted in Sec.2.1 for the classical theory of plasticity with respect to plastic strain rates $\dot{\epsilon}_{ij}^p$. The assumption 4. corresponds to the $J_2$ material with HMH yield condition as the creep potential. In such case the relation (7.29) can be called an associated flow rule.
The strain hardening rule, used in (7.29), can be formulated in a different way. The simplest approach was proposed by Øqvist [68] p. 21 as a generalization of Norton's law (7.25) in the following form:

\[ \dot{\varepsilon}_c = \frac{\sigma_c}{\sigma_c^\delta}, \]  
(7.31)

where \( \sigma_c \) and \( \delta \) are material constants. Following Crussard (ref. 16 in [203,200]) Bushnell has derived the strain-hardening rule from the uniaxial creep law of the form:

\[ \varepsilon_c = A \sigma_e^n t_e^m, \]

where effective time \( t_e^{(i)} \) is defined for iteration steps \( (i) \) by the formula:

\[ t_e^{(i)} = \left[ \frac{\varepsilon_c^{(i)}}{A(\sigma_e^{(i)})^n} \right]^{1/m}. \]  
(7.32)

A more general approach to strain-hardening depends on elimination of time from a time dependent creep law. In order to explain this approach let us consider the relation (7.27). Two equations:

\[ \varepsilon_c = f.(1-e^{-\alpha t}) + gt, \quad \dot{\varepsilon}_c = f.(re^{-\alpha t}) + g, \]  
(7.33)

can be numerically solved with respect to \( \dot{\varepsilon}_c \) and \( t \) for given values of \( \varepsilon_c, \sigma, \theta \) - cf. [211] p. 426. Alternative approach, related to prediction of strain-hardening from transient creep strain, was also proposed in [211].

7.1. Viscoplastic model of material

Following Phillips [21,26] it has been stated in Sec.2.9 that phenomena of creep and plasticity cannot be treated separately as only the combined effects are measured. The coupling is evident for quasi-static loads at elevated temperatures and for the rate-dependent problems, e.g. at the propagation of waves in inelastic media. To deal with observed facts the viscoplastic strain rates \( \dot{\varepsilon}^{vp}_{ij} \) are preferred to be considered instead of the sum \( \dot{\varepsilon}^{p}_{ij} + \dot{\varepsilon}^{c}_{ij} \).
In order to describe the transition from the elastic into a viscoplastic state, the yield condition can be used as it takes place in the theory of plasticity. This assumption was used by Malvern [215] in the uniaxial analysis of wave propagation. From among the formulations in which a generalization of Malvern's rate-dependent relation on multiaxial stress states was undertaken (cf. references in [216,217,69]), commonly accepted is the relation given by Perzyna [217, 69]:

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \langle \Phi(\bar{F}) \rangle \frac{\partial Q}{\partial \sigma_{ij}}
\]  

(7.34)

where: \(\gamma(\theta)\) - fluidity coefficient, \(\Phi(\bar{F})\) - scalar function of \(\bar{F}\), \(Q(\sigma_{ij}; t)\) - viscoplastic potential. The relative yield condition \(\bar{F}\) and the switch-on/switch-off operator \(\langle \cdot \rangle\) are defined as follows:

\[
\bar{F} = F/\sigma_e = [f(\sigma_{ij}) - \sigma_e(x, \theta)]/\sigma_e(x, \theta)
\]  

(7.35)

\[
\langle \Phi(\bar{F}) \rangle = \begin{cases} 
\Phi(\bar{F}) & \text{if } \bar{F} > 0, \\
0 & \text{if } \bar{F} \leq 0.
\end{cases}
\]  

(7.36)

In what follows the special cases of (7.35) and (7.36) are only considered. According to Perzyna's proposition the isotropic strain-hardening associated with the MMH yield condition and the power-type function can be used - cf. [69] p. 283:

\[
\bar{F} = \sqrt{3} J_2/\sigma_e(\kappa) - 1, \quad \text{(7.35a)}
\]

\[
\Phi(\bar{F}) = F^{\delta}, \quad \text{(7.36a)}
\]

where \(\delta(\theta)\) is to be experimentally established, and \(\kappa(\theta)\) is a strain-hardening parameter, related to the viscoplastic strains \(\varepsilon_{ij}^{vp}\).

If the potential \(Q = F\) then the relation (7.34) becomes the **viscoplastic associated flow**:

\[
\dot{\varepsilon}_{ij}^{vp} = \frac{\gamma}{k} \langle \Phi(\bar{F}) \rangle s_{ij}
\]  

(7.37)

where \(\gamma = \sqrt{3} \gamma/2\) and \(k = \sigma_e/\sqrt{3}\). In the case of an active viscoplastic deformation process the relation (7.37) takes the form.
\[ \varepsilon^{VP}_{ij} = \gamma \#(\tilde{F}) \frac{S_{ij}}{k} \quad \text{for} \quad \sqrt{J_2} > k. \]  
(7.37a)

Squaring both sides of Eq. (7.37a) the following relation can be calculated:

\[ \tilde{I}_{ij} = \gamma \#(\tilde{F}) \quad \text{where} \quad \tilde{I}_{ij} = \frac{1}{2} \varepsilon^{VP}_{ij} \varepsilon^{VP}_{ij} \]  
(7.38)

From (7.38) and for \( \tilde{F} = \sqrt{J_2}/k - 1 \) the value \( r = \sqrt{2J_2} \) can be computed:

\[ r = \sqrt{2J_2} = k \sqrt{2} \left[ 1 + \#^{-1}(\tilde{I}_{ij} / \gamma) \right]. \]  
(7.39)

In Fig. 7.5a the deviotoric plane (cf. 2.2b) is shown with circles which correspond to the HMM yield condition. The circle of the radius \( r \) is related to so called dynamic yield surface and \( \sqrt{2} k \) corresponds to the static yield surface which is considered in the rate-independent plasticity. For the active process the rate \( \tilde{F} \) can be either positive or negative if \( \tilde{F} > 0 \). The passive process is controlled by \( \tilde{F} \leq 0 \), according to the definition (7.36) of the switching function \( \#(\tilde{F}) \).

Fig. 7.5. a) Yield loci on the deviatoric plane for the HMM yield condition.

b) Dynamic and static stress-strain curves for uniaxial tension.

In Fig. 7.5b the static and dynamic stress-strain are shown for the uniaxial tension test. Besides the family curves \( \dot{\varepsilon} = \text{const.} \), a more realistic dynamic curve is shown for the case \( \dot{\varepsilon}(\varepsilon) \).
Elastic-plastic and creep problems can be considered as the limit cases of the constitutive relation (7.34). An elastic-plastic state occurs if stationary conditions are reached at which no further change of strains occurs and $\bar{F} \leq 0$ - cf. [216,218].

Another possibility is to increase the parameter $\bar{r}$. In the case $\bar{r} \rightarrow \infty$ the radius $r \rightarrow k\sqrt{2}$ and it is independent of strain rates. In such a case the multiplier in (7.37) has to be calculated according to the transition $\gamma <\Phi(\bar{F})>/k + \lambda$.

If the static yield surface tends to zero then the viscoplastic relation (7.34) degenerates to that which describes the creep strain rate. In the case of relation (7.35a) the function $\bar{F} = \sigma_e/\sigma_c$ should be applied, where the reference value $\sigma_c$ is used to render $\bar{F}$ non-dimensional. For the form (7.36a) of the function $\Phi(\bar{F})$ the Norton law of the form (7.31) is obtained, and for $\gamma = \sqrt{3}/2$ the relation (7.37) becomes (7.29).

7.5. FE approach to the analysis of thermo-elasto-plastic, creep and viscoplastic problems

During the development of FEM and its application to the analysis of thermo-elastoplastic, creep and then viscoplastic problems some algorithms have been carried out. All these algorithms can be treated as a combination of the initial strain and tangent stiffness methods. Early the initial strain method was preferred (cf. Greenbaum and Rubinstein [379]). The disadvantage of this approach is the small time steps that are required in order to ensure proper satisfaction of the creep law and achieve convergence of an iteration procedure. That is why variable stiffness has been considered, updated to a known stress and temperature level (cf. Sharifi and Yates [210]). An improving of the iteration convergence can be achieved by a subincremental technique with respect to the inelastic part of strain increments (cf. Bushnell [171,200,203]). More advanced algorithms to the analysis of thermo-elastoplastic and creep problems have been given by Levy [245], Snyder and Bathe [246]. The algorithms can be applied to the mixed isotropic-kinematic strain-hardening material, cf. Allen and Haisler [247]. In the case of absence of the creep strains (rate-dependent terms) an artificial time-type parameter can be used and formulations and
methods similar to those from Ch.6 can be effectively applied (cf. Ueda and Yamakawa [220], Hibbitt and Marcal [221]).

In the analysis of time-dependent problems the crucial point is related to the computation of the strain increment. This problem will be discussed with respect to the rate of strains of the form (7.2) and the UL formulation but the transitions to other formulations can be easily made.

Let us focus our attention on computation of an increment of the viscoplastic strain:

\[
\Delta e^{\text{VP}} = \int_{t}^{t+\Delta t} \dot{e}^{\text{VP}} dt .
\]  

(7.40)

For the given time interval \(\Delta t_m = t_{m+1} - t_m\) the relation (7.40) is assumed to be approximated by the following formula:

\[
\Delta e^{\text{VP}}_m = [(1-\beta) \frac{m^* e^{\text{VP}}}{\Delta t} + \beta \frac{m^{*1} e^{\text{VP}}}{\Delta t}] \Delta t , \quad \beta \in [0,1]
\]  

(7.41)

The exact form of this relationship depends on the selection of the time stepping parameter \(\beta\). The case \(\beta = 0\) corresponds to the Euler or 'fully explicit' scheme (forward difference method). On the other hand \(\beta = 1\) leads to the 'fully implicit' (or backward difference) scheme. The case \(\beta = 1/2\) is known as the Crank-Nicolson, trapezoidal rule in the context of linear equations (cf. [70]).

To define \(m^{*1} e^{\text{VP}}\) a limited Taylor series expansion can be used:

\[
m^{*1} e^{\text{VP}} = \frac{m^* e^{\text{VP}}}{\Delta t} + m H \Delta \sigma + \left( \frac{m \dot{e}^{\text{VP}}}{\dot{\sigma}} \right) \Delta \sigma .
\]  

(7.42)

where

\[
m_H = m \left( \frac{\dot{e}^{\text{VP}}}{\dot{\sigma}} \right) = m_H (m).
\]  

(7.43)

Thus, (7.41) can be written as
\[
\dot{\varepsilon}_{\text{VP}} = m_{\text{VP}} \dot{\varepsilon} + \frac{m_{\text{G}}}{m_{\text{H}}} \Delta_m \sigma + \frac{m_{\text{H}}}{m_{\text{H}}} \Delta_m \varepsilon \quad (7.44)
\]

where

\[
m_{\text{G}} = \beta \Delta_m t_m \quad H_m = \beta \Delta_m \tau \left( \frac{\beta \phi}{\phi_e} \right) \Delta_m \sigma_e \quad (7.45)
\]

The matrix \(H\) and the increment of hardening-type coefficient \(\Delta_m \varepsilon\) are computed for the constitutive relationship (7.34) which can be rewritten in the matrix form

\[
\dot{\varepsilon}_{\text{VP}} = \tau \langle \Phi \rangle \dot{a} \quad (7.46)
\]

where the vector \(a = \Phi Q / \Phi_e\) is introduced. Use of (7.46) and (7.43) gives

\[
H = \tau \left( \frac{\beta \dot{a}}{\phi_e} + \frac{\beta \phi}{\phi_e} a \right) a^T \quad (7.47)
\]

where the symbols \(\langle \rangle\) and the superscript \(m\) are dropped for convenience. Restricting the discussion to the associated flow rule and HMH yield condition the vector \(a\) becomes

\[
a = \frac{\sqrt{3}}{2k} \left( s_x, s_y, s_z, 2 \tau_{xy}, 2 \tau_{yz}, 2 \tau_{zx} \right) \quad (7.48)
\]

and the matrix \(H\) can be expressed in the form (cf. [62] p.280):

\[
H = c_1 H_1 + c_2 H_2 \quad (7.49)
\]

where

\[
c_1 = \frac{\beta \tau}{2 \sigma_e} \langle \Phi \rangle, \quad c_2 = \frac{\beta \tau}{4 \sigma_e^2} \langle \frac{\Phi}{dF} - \frac{\Phi}{\sigma_e} \rangle \quad (7.50)
\]

\[
H_1 = \begin{bmatrix}
2 & -1/3 & -1/3 & 0 & 0 & 0 \\
-1/3 & 2 & -1/3 & 0 & 0 & 0 \\
-1/3 & -1/3 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
s_x^2 & s_x s_y & s_x s_z & 2s_x \tau_{xy} & 2s_x \tau_{yz} & 2s_x \tau_{zx} \\
s_y^2 & s_y s_x & s_y s_z & 2s_y \tau_{xy} & 2s_y \tau_{yz} & 2s_y \tau_{zx} \\
s_z^2 & s_z s_x & s_z s_y & 2s_z \tau_{xy} & 2s_z \tau_{yz} & 2s_z \tau_{zx} \\
2s_x \tau_{xy} & 2s_y \tau_{xy} & 2s_z \tau_{xy} & 4s_y \tau_{xy} & 4s_y \tau_{yz} & 4s_y \tau_{zx} \\
2s_x \tau_{yz} & 2s_y \tau_{yz} & 2s_z \tau_{yz} & 4s_y \tau_{yz} & 4s_y \tau_{zx} & 4s_y \tau_{zx} \\
2s_x \tau_{zx} & 2s_y \tau_{zx} & 2s_z \tau_{zx} & 4s_y \tau_{zx} & 4s_y \tau_{zx} & 4s_y \tau_{zx}
\end{bmatrix}
\]
Using the incremental form for elastic constitutive relation and FE approximation:

\[
\Delta \sigma_m^e = F^e \Delta m^e = F^e (\Delta m^e - \Delta m^{\nu P} - \Delta m^{\nu O}),
\]

(7.51)

\[
\Delta m^e = m_B \Delta m^d.
\]

the stress increment vector becomes

\[
\Delta \sigma_m^e = m_E^* (m_B \Delta m^d - m^{\nu P} \Delta m^t - m\Delta m^h - \Delta m^{\nu O}),
\]

(7.52)

where the local stiffness matrix takes the form:

\[
m_E^* = (I + m_E m_Q)^{-1} F^e = [(m_E)^{-1} + m_Q]^{-1}.
\]

(7.53)

Specification of the matrix \(m_B\) depends on geometrical formulation (total or updated Lagrangian) and solution technique (initial strain versus tangent stiffness approach). In the case of UL formulation and tangent stiffness method the principle of VW leads to the following FE incremental equation:

\[
\Delta m^e = (K + K_{\sigma} - K_p) \Delta m^d = \Delta m^r.
\]

(7.54)

where the FE matrices on the level \(\mathcal{E}\) takes the form:

\[
m_{K_j} = \int_{Q_j} m_B (B_L^T E^* B_L) \, d\Omega,
\]

(7.55)

\[
m_{K_{\sigma} j} = \int_{Q_j} [mB_N (\sigma - \dot{\sigma}^*) B_N] \, d\Omega.
\]

In the vector of pseudo-forces

\[
\Delta m^r = m(\Delta P + \tilde{P}) - m_{F^*},
\]

(7.56)

the modified vector of internal forces is included

\[
m_{F^* j} = \int_{Q_j} [mB_L (\sigma - \dot{\sigma}^*)] \, d\Omega.
\]

(7.57)
where according to (7.52):

$$\Delta_m \sigma^* = m \sigma^* \left( \frac{m^*}{m} \Delta_m t + m \Delta_m h + m \Delta_m O \right).$$

(7.58)

Similarly to (6.83) in the matrix $K^j_\sigma$ the stress matrix $\sigma - \Delta \sigma^*$ is used instead of the vector $(\sigma - \Delta \sigma^*)$ which is in the integral (7.57).

The above presented formulation is in fact close to the initial stress approach. In the case of explicit scheme $\beta = 0$ and according to (7.45) and (7.53) we obtain $F^* = F^E$ and $\Delta \sigma^* = \Delta \sigma^E = \frac{\sigma^E}{\varepsilon^P} (\varepsilon^P \Delta t + \Delta \varepsilon^o)$. 

The incremental equation (7.54) can be solved applying a series of equilibrium iteration cycles to each time step until the pseudo-forces $\Delta F^m$ become negligibly small. A computationally economic alternative is to avoid the iteration process but to add the residual forces to the pseudo-forces to be applied for the next time step. This can be made by accumulation of stresses

$$m^{+1} \sigma = m \sigma + \Delta_m \sigma$$

(7.59)

and computation of residual forces in each finite element

$$m^{+1} R^j = m^{+1} F^j - \int_{\Omega^j} m^{+1} \left( \frac{\sigma^T}{\varepsilon^P} \right) d\Omega.$$ 

(7.60)

As it has been mentioned in Sec.7.5 the visco-plastic analysis can be applied to obtain solutions for elasto-plastic problems. The idea lies in fixing of the load level and continuation of computations in time until the stationary level is reached. This can be checked by computation of the rate of visco-plastic strain

$$m^{+1} \varepsilon^P = \gamma \left( m^{+1} \phi \right) m$$

(7.61)

and halting the marching process if the criterion

$$\max_{\ell} \left| \left| m^{+1} \varepsilon^P \right| \right| < \text{ERL}$$

(7.62)

is fulfilled with respect to all integration FE points $\ell$, where ERL is a small value.
It was proved by Zienkiewicz and Cormeau [218] that for a so called critical time step the implicit scheme for a viscoplastic model is more efficient than an appropriate initial stress approach in the elastoplastic analysis. The efficiency increases if the time steps are extended due to implicit schemes. That is why the selection of the time step length is of great importance in the elasto-viscoplastic analysis.

Application of various time integration schemes requires examination of their stability, accuracy and convergence characteristics. From among a number of papers devoted to stability analysis of viscoplastic, or more generally, inelastic rate solution procedures only papers by Cormeau [222], Owen and Danjanic [70] and Hughes and Taylor [223] are mentioned. In [222] stability of the implicit scheme has been discussed and in [70,223] implicit schemes have been considered.

If the time stepping parameter $\beta \geq 1/2$ then the implicit scheme is unconditionally stable - cf. [223]. This implies that the time marching scheme is stable but does not guarantee the accuracy of solution at any stage. That is why also for $\beta \geq 1/2$ step increments must be limited in order to achieve a valid solution.

For $\beta < 1/2$ the implicit scheme is only conditionally stable and the length of the time increment has to be less than some critical value, i.e. $\Delta m t < \Delta t_{cr}$.

Cormeau [222] proved that for the viscoplastic constitutive relation with the HMH yield condition and $\Phi = \Phi$ and for the explicit scheme the following evaluation can be used:

$$\Delta t \leq \frac{4(1+v) \sigma}{3 \gamma E}.$$  \hspace{1cm} (7.63)

A practical evaluation of time steps is given in [62] p. 277. For the step $m$ the time increment should fulfill the inequality

$$\Delta m t \leq \Delta t \cdot \min \left[ \frac{\varepsilon_{ii}}{\varepsilon_{ii}^{*}}, \frac{\varepsilon_{ii}^{*}}{\varepsilon_{ii}^{*}} \right]^{1/2}$$  \hspace{1cm} (7.64)

where $\ell$ corresponds to the numbers of integral points in the finite elements of $\varepsilon$ and $\Delta t$ is the time increment parameter. The value of this parameter must be specified by the user - for the explicit marching scheme a parameter in the
range $0.01 < \tau < 0.15$ is recommended. For implicit schemes $\tau < 10$ gives stable results though the accuracy deteriorates.

Another practical suggestion is to change the time increment according to the evaluation [70]:

$$\Delta_{m+1} \leq k \Delta_m$$

(7.65)

where $1.0 \leq k \leq 2.0$.

The elasto-viscoplastic analysis by means of FEM was originated by Zienkiewicz and Cormeau [218] and was early applied to the analysis of geometrically nonlinear problems [216,224,154]. In paper [224] Kanchi et al. pointed out that the outlined approach can be effectively applied to the analysis of continuous damage problems as well as to creep buckling. The viscoplastic approach can be effectively used for softening behaviour of materials or, more generally, for nonassociated flow rules [218,224].

In Sec.2.8 the multiple subvolume model has been discussed with respect to the elastoplastic analysis. The MS model was proposed by Besseling [117,237] also to the creep analysis. Pande et al. [238], Owen et al. [62] p.304-315, Meijers and Rode [239] discussed the use of MS model from the viewpoint of elasto-viscoplastic analysis (cf. also [244]).
8. BUCKLING OF ELASTIC-PLASTIC STRUCTURES

8.1. Elastic versus plastic buckling of structures

The term 'buckling' is used in the static analysis of structures in order to define two phenomena which may occur during the increase of loading - cf. Fig. 8.1: i) collapse at the maximum value of the load parameter (limit point L on the equilibrium path), ii) bifurcation buckling (point B at intersection of equilibrium paths). If the load parameter increases from its zero value then equilibrium states are associated with the fundamental path up to a critical point is reached and in the case of bifurcation point the secondary path occurs. Equilibrium states can be stable or unstable depending on possible increase or decrease of the loading parameter (in Fig. 8.1 and further unstable equilibrium paths are marked by broken lines).

Fig. 8.1. Equilibrium paths for perfect and imperfect structures.

The theory of structural buckling has been mostly developed for elastic structures and first of all for conservative (history-independent) problems. The buckling analysis is usually restricted to one-parameter loads. Under such assumptions above mentioned definitions of equilibrium paths and critical points are useful in what follows.

Bifurcation critical points are associated with so called perfect structures for which buckling modes and corresponding to them buckling mode displacements \( \delta_p \) can be calculated. These displacements can be inactive in the pre-buckling state...
(Fig. 8.1b) but they are useful describing a post-buckling behaviour. Bifurcation points can be single (if only one secondary path exists) or multiple. Bifurcation points on the fundamental path are called as primary b.p. while those on secondary equilibrium paths are named secondary b.p. – cf. [71,72].

From engineering point of view we are interested in the basic primary point of the lowest value of load parameter. But in case if such a bifurcation point $B_1$ is of stable character (cf. Fig. 8.1) a basic secondary point $B_2$ may be of importance. That is because of the relation of perfect structures to imperfect structures.

Perfect structures are considered as an useful idealisation of real, engineering structures. Solutions obtained for 'perfect' models evaluate from above solutions for models with idealized imperfections (curves 2 versus 3 in Fig. 8.1) and can give answer with respect to the imperfection sensitivity of structures.

In the case of unstable primary b.p. the asymptotic analysis of the initial post-buckling state in Koiter's sense can give a satisfying evaluation for the imperfection sensitivity. If on a stable secondary path the unstable b.p. occurs then the asymptotic analysis should be performed in the vicinity of this point. Such a situation can take place on so called coupled buckling modes of structures – cf. [73], for which imperfections cause creation of limit b.p. – cf. equilibrium paths 5, 6 and 7 in Fig. 8.1b.

Application of approaches developed for the buckling analysis of elastic structures to the plastic buckling should be made very carefully. This is because of history-dependence of plastic problems and specific features of material models.

Plastic buckling of structures involves a complex interaction between material and geometrical non-linearities. First studies concerning the elastic-plastic columns were originated a hundred years ago (cf. [74] p. 96-97) but it was not until 1947 that Shanley [225] explained the significance of the tangent modulus, increasing load at the critical bifurcation point, at which the straight column configuration looses its uniqueness, but not its stability.
A general theory of bifurcation and uniqueness in elastic-plastic solids was developed in subsequent papers by Hill (cf. references in [74,80]). This theory now forms the theoretical background for the buckling analysis of elastic-plastic structures.

Post-buckling analysis beyond the elastic range is considerably complicated by the necessity of consideration of the scatter of regions with local unloadings. Papers by Hutchinson, Needleman and Tvergaard tried to develop the asymptotic analysis for perfect and imperfect elastic-plastic structures. They give an important basis for understanding phenomena associated with plastic buckling. However, for structures with more complex geometry, various types of imperfections and/or more realistic constitutive relations we have to apply almost entirely numerical methods.

In this chapter main features of plastic buckling and post-buckling behaviour of elastic-plastic structures are shown on the example of a Shanley-type discrete structure.

A generalization of considerations on buckling of continuous structures is shortly discussed. Main ideas of the Hill theory are presented as well as the difference between predictions obtained by different elastic-plastic models of material are pointed out.

Some problems and examples of the buckling analysis of elastic-plastic structures are considered on the base of numerical methods.

No attempt is made here to give a complete review of the field of plastic buckling. The readers interested in more deep and complete insight into the problems only sketched in this chapter are referred to a number of review papers, e.g. Sewell [74], Hutchinson [75], Tvergaard [76,80], Needleman and Tvergaard [77], Budiansky and Hutchinson [78], Bushnell [22,79].

8.2. Shanley-type model and buckling analysis of elastic-plastic columns

Basic problems associated with buckling of structures can be comprehensively discussed on examples of simple discrete models. In Fig. 8.2 a model of rigid column with springs of nonlinear characteristics is shown. The model was used by many authors (cf. references in [226]) and among them by Shanley (without
spring 3) in his famous paper [225] devoted to solution of the crucial problem on elastic-plastic buckling of structures. The following considerations are supported on papers by Hutchinson [226,75] but the notation is slightly changed to be similar to that used in this report.

![Diagram](image)

Fig. 8.2 a) Discrete model, b) Displacements and forces, c) Characteristic of springs.

The basis of the rigid column can move vertically and rotate so the model has 2 DOF corresponding to components of the g. displacement vector $\mathbf{d} = (u, \varphi)$. The springs 1, 2 have an elastic-plastic characteristic as it is shown in Fig. 8.2c. The nonlinear elastic spring 3 introduces into the model stabilizing or destabilizing effects depending on the sense of the force $K$. In the case shown in Fig. 8.2b the action of the spring 3 will cause unstable effects since the force $K$ will influence the increase of displacements.

The analysis is carried out under assumption of moderate rotations, i.e. $\sin \varphi = \varphi, \cos \varphi = 1$. The equilibrium equations

$$Q_1 + Q_2 = P,$$

$$(Q_2 - Q_1)l + PL(\varphi + \varphi_0) + LK = 0,$$

(8.1)

can also be written in the incremental form:

$$dQ_1 + dQ_2 = dP,$$

$$l(dQ_1 - dQ_2) + LP d\varphi = L(\varphi + \varphi_0) dP + LdK,$$

(8.2)
where \( \varphi_0 \) denotes initial imperfection. Geometrical relations are evident from Fig. 8.2b:

\[
dq_1 = du + \ell d\varphi, \quad dq_2 = du - \ell d\varphi.
\] (8.3)

Stress-strain relations are in the simple form:

\[
dQ_i = D_i dq_i \quad \text{for } i = 1, 2.
\] (8.4)

After simple manipulations Eqs. (8.1-4) can be reduced to the set of incremental equations:

\[
(D_1 + D_2)du + \ell(D_1 - D_2)d\varphi = dP,
\] (8.5)

\[
\ell(D_1 - D_2)du + [\ell'(D_1 + D_2) - LP]d\varphi = L(\varphi + \varphi_0)dP + LdK.
\]

Eqs (8.5) corresponds to the displacement method with the stiffness matrix

\[
K = \begin{bmatrix}
D_1 + D_2 & \ell(D_1 - D_2) \\
\ell(D_1 - D_2) & \ell'(D_1 + D_2) - LP
\end{bmatrix}.
\] (8.6)

The term \(-LP\) in the component \(K_{22}\) corresponds to the geometrical matrix (initial stress matrix) which has to be taken into account in the buckling analysis. The term \(dK\) can be written in the form

\[
dK = (2k_1\varphi + 3k_2\varphi^2)d\varphi,
\] (8.7)

which is associated with the initial displacement matrix.

The necessary condition of a critical state corresponds to the zero value of the stability determinant:

\[
det | K(P_{cr}) | = 0.
\] (8.8)

Let us assume linear elastic properties of the strings 1 and 2. In such a case \(D_1 = D_2 = D_e\) and from (8.8) one obtains:
\[ P_E = \frac{2 D e^2}{L} . \] (8.9)

From Eqs. (8.2) the slope of the equilibrium curve can be computed:

\[ \frac{dP}{d\phi} = \frac{P_E - P}{\phi + \phi_0} - (2k_1 + 3k_2\phi) \frac{\phi}{\phi + \phi_0} . \] (8.10)

The maximum external load, corresponding to the limit points on the equilibrium paths for the structure with imperfections, is given by the asymptotic formulae. In the case \( k_1 > 0, k_2 = 0 \) the asymmetric bifurcation point occurs for the perfect model and maximum load for \( \phi_0 = 0 \) equals:

\[ \frac{P_L}{P_E} = 1 - \frac{2Lk_1}{k^2D} \phi_0^{1/2} + \ldots \quad \text{for} \quad k_1 > 0, k_2 = 0 . \] (8.11a)

Symmetric bifurcation point occurs for \( k_1 = 0, k_2 \neq 0 \) and the asymptotic formula is of the following form:

\[ \frac{P_L}{P_E} = 1 - \frac{3}{2} \frac{2Lk_2}{k^2D} \phi_0^{2/3} + \ldots \quad \text{for} \quad k_1 = 0, k_2 \neq 0 . \] (8.11s)

Well known relations, shown in Fig. 8.3, can be obtained from Eqs. (8.1-8) specifying only values of the parameters \( \phi_0, k_1 \) and \( k_2 \).

Fig. 8.3. Elastic post-bifurcation behaviour and imperfection sensitivity.
Let us assume that buckling takes place beyond the elastic range and also that at the moment of buckling in both springs the tangent stiffness equals $D_1 = D_2 = D_t$. In this case from Eq. (8.8) the Shanley tangent-modulus load is obtained:

$$P_s = \frac{2 D_t \ell^2}{L}. \quad (8.12)$$

Let us assume additionally that the neutral state takes place in spring 2, i.e. $dQ_2 = 0$. Under such an assumption from Eqs. (8.2) and (8.3) the slope $dP/d\varphi$ can be computed in the bifurcation point $P_s$:

$$\frac{dP}{d\varphi} \bigg|_{P_s} = 2D_t \ell. \quad (8.13)$$

In such a way we came to the main statement of Shanley's increasing load approach:

After buckling under tangent-modulus critical load the external load parameter increases at the beginning of the post-buckling state.

Another possibility is associated with the assumption that at buckling the passive process occurs in spring 2, i.e. $D_1 = D_t$, $D_2 = D_e$. From the buckling condition (8.8) so called reduced-modulus buckling load of von Karman can be computed:

$$P_K = \frac{4 D_t D_e \ell^2}{(D_e + D_t)L} = \frac{2P_s}{1 + D_t/D_e}. \quad (8.14)$$

It is easy to examine the Eqs. (8.5) that for Karman's buckling load $P_K$ the slope $dP/d\varphi = 0$. That is why the Karman approach of the reduced-modulus load is also called as constant buckling load approach.

In Fig. 8.3a. it is pointed out that the perfect model can buckle for any bifurcation load $P_B$ from the range $P_s \leq P_B \leq P_K$ under increasing load. The relation between external load $P$ and rotation $\varphi$ in the vicinity of $P_B$ can be calculated from Eqs. (8.2) under assumption $\varphi = d\varphi - \text{cf.}[75]$:

$$P = \frac{P_B + c(P_K \varphi - K)}{1 + c\varphi} = P_B + c(P_K - P_B)\varphi - [c^2(P_K - P_B) + ck_1]\varphi^2 + \ldots , \quad (8.15)$$

where
\[ c = \frac{L}{\ell} \left( \frac{D_e + D_t}{D_e - D_t} \right) . \] (8.16)

Fig. 8.4. Behaviour of the model from Fig. 8.1 in the plastic range, a) Shanley model, \( k_1 = k_2 = 0 \), b) Model with a destabilizing geometrical non-linearity \( k_1 > 0 \), \( k_2 = 0 \).

Maximum load \( P_L \) for \( P_B = P_S \) and \( k_2 = 0 \) can be computed from the equation:

\[ (P_K - P_L)^2 - 4 \frac{k_1}{c} (P_L - P_S) = 0 . \] (8.17)

It can be proved that \( P_L > P_S \) but the value \( \Delta P = P_L - P_S \) is rather small - cf. Fig. 8.4b. In case of \( k_1 > 0 \) the maximum load approaches the Karman load \( P_L^\infty + P_K \) and the equilibrium path starting from \( P_S \) asymptotically approaches the line \( P_K = \text{const.} \) - cf. Fig. 8.4a.

Two-segment characteristics of the springs 1, 2 - cf. Fig. 8.2c, can cause that for certain geometrical parameters (ratios \( \ell/L \)) a value of the bifurcation load cannot be computed. It is shown in Fig. 8.5a where the first approximation to the g. stress starts from the elastic range, i.e. \( Q_i^{(1)} = Q_E \) and for the perfect model \( Q_i^{(1)} = D_t \ell^2/L \). If \( Q_i^{(1)} > Q_p \) then the value \( Q_S = D_t \ell^2/L \equiv Q_p^{(2)} \) and in the case \( Q_i^{(2)} < Q_p \) the iteration process is not convergent. The only possibility to have a unique value of the tangent-modulus load \( Q_S \) is to assume a smooth stress-strain curve - cf. Fig. 8.5b, for which a function \( D(Q) \) exists. In such a case the buckling load can be computed from the equation \( Q_S = D(Q_S) \ell^2/L \).
The discrete Shanley-type model has lumped plastic properties at the basis of the rigid column. It can be considered as a rough model of a continuous, cantilever-type column with an deformable, ideal sandwich cross-section of the damped basis. If parameters of a perfect, axially compressed column are such that before buckling the yielding occurs in all points of the column then for the post-buckling state the zone of local unloading develops - Fig. 8.6. The plastic passive processes started from the neutral state in the point A at which $d_{\varepsilon_p} = 0$ and $c = 0$ are associated with the tangent-modulus buckling load

$$P_S = \frac{\pi^2 E_T I}{4L^2},$$

where $E_T = E^p$ is local tangent stiffness and $I$ - inertia momentum of the column cross-section.

Fig. 8.6. a) Continuous column, b) Equilibrium paths of perfect and imperfect columns with bifurcation buckling at $\sigma_S > \sigma_0$, c) Equilibrium paths at $\sigma_E < \sigma_0$. 

Fig. 8.5. Two-segment and smooth g. stress-strain curves.
In case of a slender perfect column the buckling can occur in the elastic range, i.e. for the uniform stress state \( \sigma = \sigma_E < \sigma_0 \). The buckling load is defined by Euler's formula

\[
P_E = \frac{E \pi^2}{4L^2} \text{ for } \sigma_E = \frac{E \pi^2 r^2}{4L^2} < \sigma_0.
\] (8.19)

where \( r \) is the radius of gyration of the cross-section and \( \sigma_0 \) is the yield point. From the elastic analysis it is well-known that buckling of elastic columns occur at the stable critical point (cf. Fig. 8.3c) but because of yielding the limit points \( L \) are reached on the equilibrium paths - cf. Fig. 8.6c. In the considered case of the column shown in Fig. 8.6a the first yielding starts at the point \( C \) of the column basis.

An extensive discussion on the buckling of columns was presented by Sewell [74] p. 88-123. It is worth to remind that the analogy between formulae (8.18) and (8.19) leads to an exact evaluation of the tangent-modulus buckling load only for a uniform stress state. The change \( E_T \rightarrow E \) in case of nonuniform stresses is difficult because of needed average value of the tangent stiffness modulus \( E_T \).

8.3. General analysis of bifurcation buckling in the plastic range

In his excellent paper [74] Sewell gave an extensive survey of more than 600 papers. His bibliography covers practically all the field of elastic-plastic buckling analysis of structures until 1970. The main conclusion from this paper is that the bifurcation buckling analysis involving plasticity was applied almost entirely to simple structures with uniform prestress. The analysis of more complicated problems require computer methods which make use of earlier experimental and analytical results with a theoretical background for testing and interpretation of the numerical solution. Hill's general theory originated in [227], deals with basic problems of uniqueness and bifurcation in elastic-plastic solids at small strains. The theory generalizes Shanley's approach on multi-axial, nonuniform stresses. In subsequent papers by Hill, Sewell, Hutchinson, Tvergaard et al. (cf. references in [22, 74-80]) the theory has been extended on materials with non-smooth yield surface, finite strain plasticity and shallow shells analysis. Only main ideas of this theory are discussed below in order to enable us to understand basic problems of the plastic bifurcation theory.
Let us start with models of a material with a general smooth yield surface for which the constitutive tensor of the instantaneous moduli is of the form

$$E_{ijkl} = E_{ijkl}^e - \beta g^{-1} a_{ij} a_{kl}.$$  \hspace{1cm} (8.20)

where $\beta$ is the switching, zero-one parameter defined in (2.50), the scalar function $g$ can be deduced from (2.49), $a_{ij} = \delta F/\delta \sigma_{ij}$ corresponds to the outward unit normal to the yield surface $F$.

In the case of initially isotropic material with the HMH yield surface ($J_2$ material) and the isotropic strain-hardening the following constitutive tensors can be deduced from (2.58):

$$E_{ijkl}^e = \frac{E}{1+v} \left[ \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} \right],$$

$$E_{ijkl}^p = \beta g^{-1} s_{ij} s_{kl} \quad \text{where} \quad g^{-1} = \frac{1}{\sigma^2} \frac{E}{1+v} \frac{E/E_T - 1}{E/E_T - (1-2\nu)/3}.$$  \hspace{1cm} (8.21)

In $g^{-1}$ the elastic and tangent moduli $E = E^e$, $E^p$, $E_T$ are used instead of the moduli $G = E/(1+v)$ and $H = E/(E/E_T - 1)$ in (2.58). These moduli are referred to the uniaxial tension test.

The TL formulation and small strains are assumed. For convenience left side indices are omitted in (6.65) and the Green-Lagrange rate strain tensor takes the form:

$$\dot{e}_{ij} = \frac{1}{2} (\ddot{u}_{i,j} + \ddot{u}_{j,i} + u^o_{k,j} \dot{u}_{k,i} + u^o_{k,i} \dot{u}_{k,j}).$$  \hspace{1cm} (8.22)

Rates ($\dot{\cdot}$) are used instead of increments $\Delta(\cdot)$ for the sake of brevity and $(\cdot)^0$ denotes the fundamental state of equilibrium.

The usual constructions of uniqueness proof leads to the following equation - cf. [75] p. 88:

$$H = \int_V (\ddot{\sigma}_{ij} \ddot{e}_{ij} + \sigma^o_{ij} \ddot{u}_{k,i} \ddot{u}_{k,j}) dV = 0.$$  \hspace{1cm} (8.23)

where the difference between two kinematically admissible solutions is denoted by $(\cdot) = (\cdot)^a - (\cdot)^b$. 
Hill introduced an elastic comparable solid such that $E_{ijkl}^C$ are equal the elastic-plastic moduli $E_{ijkl}$ where the stress is currently on the yield surface and $E_{ijkl}^E$ elsewhere. The quadratic functional

$$ \Phi (\mu, \tilde{u}) = \int_V (E_{ijkl}^C \tilde{e}_{ij} \tilde{e}_{ij} + c_{ij}^{\circ} \tilde{u}_k, i \tilde{u}_k, j) dV, $$

(8.24)

is formulated for the elastic comparable body. It can be proved (cf. [75] p. 91) that the integrand of $H$ is nowhere smaller than that of $\Phi$, and thus

$$ H \geq \Phi. $$

(8.25)

The requirement $\Phi > 0$ for any admissible non-vanishing displacement fields is a sufficient condition for uniqueness, as it was shown by Hill [227].

Bifurcation condition requires the simultaneous vanishing of $H$ and $\Phi$ or wording it more precisely: for the lowest eigenvalue $\mu_c$ of $F$ both solutions $\dot{u}_1^a$ and $\dot{u}_1^b$ are possible if and only if these both solutions have the property that no elastic unloading occurs, i.e. if $a_{ij}^a e_{ij}^a > 0$ and $a_{ij}^b e_{ij}^b > 0$.

If the fundamental solution (')$^0$ is identified as (')$^a$ then the other distinct solution can be written as

$$ \dot{u}_1^b = \dot{u}_1^0 + \xi (1), $$

(8.26)

where $(1)$ is the normalized eigenmode of $F$ associated with $\mu_c$ and $\xi > 0$ is an amplitude. The variation of the load parameter $\mu$ immediately after bifurcation is

$$ \mu = \mu_c + \mu_1 \xi + \ldots, $$

(8.27)

where $\mu_1$ must be chosen sufficiently large so that the plastic loading condition $a_{ij}^b e_{ij}^b > 0$ is fulfilled everywhere. Generally $\mu_1 > 0$ is required and thus bifurcation takes place under increasing load in the Shanley sense as it was proved by Hill.

The elastic-plastic models of material, discussed so far have been formulated under assumption of a smooth yield surface. It has been pointed out in Sec.2.9. that a vertex on the yield surface can be considered. Such a singularity may
have a rather strong effect on bifurcation prediction and opens the door to more
general considerations. Assuming existance of a vertex it is possible to obtain
reasonably good agreement of bifurcation loads computed by means of incremental
and total strain theories.

A generalization of uniqueness and stability theorems on plastic models based on
yield surfaces with vertices was comprehensively discussed by Hutchinson [75] p.
91-93. In papers [77,80] the application of so called J₂ corner theory, proposed
by Christoffersen and Hutchinson [228] to the bifurcation analysis is reviewed.

This theory corresponds to the J₂ deformation theory for which the incremental
consstitutive relation is
\[ \varepsilon_{ij}^* = \varepsilon_{ij}^e + \varepsilon_{ij}^p = H_{ijkl} \delta_{kl}^* + C_{ijkl} \dot{\sigma}_{kl}^* . \] (8.28)

where the elastic and plastic compliances are used:
\[ H_{ijkl} = (E_{ijkl}^e)^{-1} = \frac{1}{2G} \tilde{I}_{ijkl}^e + \frac{1 - 2\nu}{3E} \delta_{ij} \delta_{kl}^* . \] (8.29)
\[ C_{ijkl} = g_1 \tilde{I}_{ijkl}^e + \frac{9}{4} g_2 \frac{s_{ij} s_{kl}}{\sigma_e^2} . \]

Here, the special identity tensor \( \tilde{I}_{ijkl}^e \) and functions \( g_1 \) and \( g_2 \) are used:
\[ \tilde{I}_{ijkl}^e = \frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) = \frac{1}{3} \delta_{ij} \delta_{kl}^* . \] (8.30)
\[ g_1 = \frac{3}{2} \left( \frac{1}{E_S} - \frac{1}{E} \right), \quad g_2 = \frac{1}{E_T} - \frac{1}{E_S} . \]

where \( E \) is Young's modulus, \( E_S \) and \( E_T \) - secant and tangent moduli of the
uniaxial tension curve.

An angular measure \( \psi \) is introduced in the J₂ corner theory, defined as
\[ \cos \psi = \left( 1 + \frac{g_1}{g_2} \frac{s_{ij} s_{ij}}{\sigma_e^2} \right)^{1/2} . \] (8.31)
The angle $\psi$ is measured from the cone axis in the stress-rate space, where the axis is parallel to $s_{ij}$ - cf. Fig. 8.7a. The angle $\psi$ is coupled with the angle $\beta$ which was introduced in the stress space by Budiansky [111]:

$$\tan \beta = \sqrt{a} \tan \psi, \quad \text{where} \quad a = (E/E_t - 1)/(E/E_S - 1). \quad (8.32)$$

Fig. 8.7. a) Vertex in the stress-rate space, b) Yield surface with a vertex.

Assuming that there exists the stress-rate potential the constitutive relation is formulated as

$$\dot{e}_{ij} = H_{ijkl} \dot{\sigma}_{kl} = f(\psi) C_{ijkl} \dot{\sigma}_{kl} = M_{ijkl}(\psi) \dot{\sigma}_{kl}, \quad (8.33)$$

where the compliance function $f(\psi)$ is used:

$$f(\psi) = \begin{cases} 
1 & \text{for} \quad \psi \leq \psi_o , \\
0 & \text{for} \quad \psi_o \leq \psi \leq n . 
\end{cases} \quad (8.34)$$

The angle $\psi_o$ define a cone of total loading and $\psi_c$ describes the domain of elastic unloading. In the transition range $\psi_o \leq \psi \leq \psi_c$ the nonlinear function $\psi$ is chosen such that the function $f(\psi)$ varies continuously in the range $[1,0]$ - cf. [228].

The inverse tensor $E_{ijkl} = [M_{ijkl}(\psi)]^{-1}$ can be used to compute $\dot{\sigma}_{ij} = E_{ijkl} \dot{e}_{ij}$ and formulate functionals $H$ and $\Phi$. According to Sewell [131] a comparison solid, defined by the total loading moduli, satisfies the inequality (8.25).

For many problems of the elastic-plastic buckling the fundamental solution is associated with proportional or nearly proportional loading everywhere, so that means $\psi \leq \psi_o$. In such a case the solution obtained from the $J_2$ corner theory
fully corresponds to that which is obtained from the deformation theory. In a case of a fundamental solution with $\phi_o \leq \phi \leq \phi_c$ in some material points an alternative comparison solid may be defined by the instantaneous moduli associated with the fundamental solution. Such an approach leads to an upper bound for $\mu_c$ - cf. ref. 15 in [80].

As the limit of a thoroughly nonlinear vertex description the transition $\mu_1 \to = \text{ for } \phi_o = 0$ occurs in (8.27). This corresponds to so called smooth bifurcation point when the two incremental solution $(\cdot)^a$ and $(\cdot)^b$ differ for the comparison solid but not for the elastic-plastic body - cf. [80] p. 152.

8.4. Flow theory versus deformation theory in the elastic-plastic buckling analysis

Comparison of experimental results and those based on simple theories of plasticity revealed that better predictions for the bifurcation buckling come from the deformation theory. This has been treated as a paradox since this theory is not consistent comparing with incremental theories.

Perhaps the best example for illustrating a discrepancy between the buckling predictions of the simplest flow and deformation theories is the cruciform column under axial compression (Fig. 8.8a). If the column is not too long it undergoes torsional buckling and the critical stress according to any flow theory with smooth yield surface is equal to that in the elastic range

$$\sigma_B^f = G \left( \frac{E}{G} \right)^2. \tag{8.35a}$$

The buckling stress obtained on the base of the $J_2$ deformation theory is - cf. [75] p. 100:

$$\sigma_B^d = \frac{G}{1 + 2G} \left( \frac{E}{G} \right)^2. \tag{8.35b}$$

Experimental results, performed on specimens of 2024-T4 aluminium was taken from Gerard and Becker - cf. [75] p. 101. The results are in good agreement with the prediction (8.35b) according to the deformation theory. The flow theory completely breaks down.
Results obtained by many authors (cf. [74,81]) have confirmed the statements known since 40-ties: i) the flow theory gives buckling loads greater than the deformation theory, ii) results of experiments are close to predictions by the deformation theory.

![Diagram](image)

**Fig. 8.8.** Buckling loads for various structures, a) Comparison of tests and theory for torsional buckling of a compressed cruciform column, b) Clamped circular plate under radial pressure, c) Spherical shell under external pressure.

The discrepancies between two theories in various structures can not be so large as in the case of a cruciform column. In Fig. 8.8b bifurcation stresses are plotted for a circular clamped plate under radial pressure. Computation was
performed by Needleman [229] for the power-type strain-stress relation in the plastic range:

$$
\varepsilon / \varepsilon_0 = \begin{cases} 
\sigma / \sigma_0 & \text{for } \sigma \leq \sigma_0 , \\
1 + [(\sigma / \sigma_0)^n - 1]/n & \text{for } \sigma > \sigma_0 .
\end{cases}
$$

(8.36)

The elastic buckling stress is defined by the formula

$$
\sigma_E = - k^2 E(t/R)^2/12(1-v^2) ,
$$

(8.37)

where the coefficient equals $k = 3.832$ for a clamped plate. For $\alpha < 2.6$ predictions by the two theories are practically the same but for $\alpha = 6.0$ the flow theory gives $\sigma_B^f$ about 50% higher than the deformation theory - cf. Fig. 8.8b.

Next example in Fig. 8.8c was taken from Hutchinson [226]. A spherical shell under external pressure is not to much sensitive for the choice of a theory of plasticity. At $\alpha = 7.0$ the flow theory leads to the buckling stress $\sigma_B^f \approx 1.13 \sigma_B^d$.

Many authors have tried to explain the paradox: why the intrinsic consistent flow theory gives worse results than the deformation theory from the viewpoint of agreement with experiments? The first more interesting observation was made by Onat and Drucker [230] who stated that the flow theory is very sensitive to imperfections. This was confirmed by Hutchinson and Budiansky [231]. Their computations proved that even small imperfections drastically influence decrease of collapse loads. This is visible in Fig. 8.9a,b where results of computation taken from [226] are shown. The axisymmetric imperfection is taken proportional to the bifurcation mode of the perfect sphere and the amplitude \( \tilde{\delta}\) is the inward initial deflection of the pole of the sphere (the increment of deflection \( \Delta \delta \) pole is measured from the imperfect middle surface at the pole of the shell).

In Fig. 8.9c,d results obtained for the cruciform column from Fig. 8.8a are shown. The geometrical imperfection corresponds to the initial deviation of the flanges. Needleman and Tvergaard [77] used the stress-strain relation (8.36) at different $n$. They assumed $\sigma_0 / E = 0.002, \nu = 0.3$ and $P_B^d / P_E = 0.5$. For higher $n$ (more flat $\sigma - \varepsilon$ curve) the upper bound, given by the flow theory, is quite close to the curve predicted by the deformation theory. Results obtained due to
the $J_2$ corner theory are closer to those from deformation theory than predicted by the flow theory. Especially good results have been obtained for the angle $\phi = \phi_n/2$, where $\phi_n = \phi_c - \pi/2$, but also $\phi = 0$ leads to a satisfactory evaluation.

8.5. Post-buckling and imperfection analysis

The analysis of the Shanley-type model and then the discussion in Sec.8.3. point out that initial post-bifurcation, plastic analysis requires to take into account.

![Diagram](image)

**Fig. 8.9. Load parameter-displacement curves and imperfection sensitivity:**

a,b) Spherical shell under external pressure, c,d) Cruciform column.
account an increase of the load parameter. That is why the term $\mu_1 \xi$ must be kept in (8.27) and the solution is predicted in the form

$$\mu = \mu_c + \mu_1 \xi + \mu_2 \xi^{1+\beta} + \ldots,$$

where $\beta < 1$.

A great effort has been made in order to extend the asymptotic analysis beyond the elastic range. Papers by Hutchinson, Needleman, Tvergaard and their coworkers, cf. references in [75-80], discovered serious difficulties. They can be caused by the shape of equilibrium path and features of material models.

As it has been shown at the analysis of post-bifurcation behaviour the load parameter initially increases and then tends toward the collapse value. An effective asymptotic approach should base on the expansion of variables around a critical point, but in the elastoplastic analysis the collapse-type points are not known 'a priori' and extrapolation can lead to serious inaccuracies.

The next problem is related to local unloading. Passive plastic deformation processes start to develop just after the bifurcation. Zones of local unloading are deformation-dependent and introduce significant non-regularity into analytical calculations. It was examined in earlier papers (cf. [81] p. 414-421) that equations of plastic stability can be significantly reduced if local unloading is omitted. In such a way we come into models of hyperelastic materials (if an appropriate potential exists) or, more commonly, a hypoelastic model of material is used.

Results of the hypoelastic model of material to the analysis of imperfect structures are shown in Fig. 8.10a,b,c taken from [80]. An axially compressed cylindrical panel with longitudinal stiffeners is considered. Only local buckling modes are analysed and no mode interaction is taken into account. Asymptotic approach leads to solutions quite close to numerical results which have been obtained by the finite strip method - cf. Tvergaard [232].

In case of a U-shape column (cf. Fig. 8.10d,e) asymptotic approach gives good predictions for stronger strain-hardening ($n = 4$). For a more flat $\sigma-\epsilon$ curve, e.g. for $n = 10$, the differences between analytical and numerical results are significant — for $\xi = -0.8$ the differences are over 30% (Fig. 8.10d). Similar
Fig. 8.10 Comparison of numerical results and asymptotic, hyperelastic predictions for imperfection-sensitivity of elastic-plastic cylindrical panels and columns: a) Panel with $\theta = 0.5$, $\sigma_0/E = 0.002$, b,c) Panels with $\theta = 0.75$, $\sigma_0/E = 0.0028$, d,e) U-shape columns, f) Triangular cross-section.

Results have been obtained for a column of triangular cross-section (Fig. 8.10f).

Results shown in Fig. 8.10 are completed by those in Fig. 8.11a where equilibrium paths are shown for the columns of U-shape cross-section - cf. Tvergaard and Needleman [233]. The perfect columns have such parameters that the bifurcation takes place either in the elastic or plastic range. The equilibrium paths have been computed also for initial imperfections in the shape of the buckling mode with the ratio $\xi$ between the amplitude and the radius of gyration. Quite similar results have been obtained for the eccentric loading. The equilibrium path for a perfect column and suppressed local unloading is more smooth than the path if unloading is taken into account.
Fig. 8.11 Equilibrium paths for: a) U-shape columns when bifurcation takes place in the elastic region (---), in plastic region (---) and when unloading is suppressed (----). b) Axially compressed eccentrically stiffened panels.

In Fig. 8.11b results of numerical computations are shown for eccentrically stiffened elastic-plastic panels with $\sigma_0/E = 0.001$, $v = 0.3$, $a/b = 4$, $e/b = 0.1$, $\alpha = (1 + A_s/bk) = 0.65$. Imperfections are related to wide column buckling and local buckling, $\xi_w$ and $\xi_1$ respectively - cf. Tvergaard and Needleman [234,235].

All presented results have been obtained for the isotropic hardening. Quite similar results are for the kinematic strain-hardening up to the point where after local unloading the secondary yielding occurs. Such a case is shown in Fig. 8.12, taken from [236] by Waszczyszyn et al. Large deflections of annular plates under radial compression are shown. The plates have an ideal sandwich cross-section made of material of bilinear characteristics $\sigma_e = \sigma_0 + A_p$ for isotropic s.-h. and $C = 2A/3$ for kinematic s.-h. In case of strong strain-hardening ($A = 0.5$ or $C = 0.333$) a snap-through can occur for a simply supported external contour of the plate. For the elastic, perfectly plastic material ($A = 0$, $C = 0$) the plate will collapse for the small deflection $w_L/H = 0.05$. The collapse load for clamped plates occurs at the deflection $w_o/H = 0.4$. It is visible that for advanced deflections the equilibrium paths which start from the Karman bifurcation points asymptotically approach paths related to the Shanley increasing load bifurcation.
Fig. 8.12. Post-bifurcation deflections of annular plates under radial uniform compression.

The computation of post-buckling states may be difficult since the vectors tangent to the fundamental and secondary equilibrium paths can be close to each other - cf. Fig. 8.13. Usually the basic eigenvector $\mathbf{a}$ is used to predict the first approximation $\mathbf{z}_B^{(1)}$. In order to trace the secondary path the stiffness matrix has to be updated according to the formula - cf. [59]:

$$K_B = K \left( \mathbf{\tilde{z}}_B + \beta \mathbf{\tilde{z}}_B \Delta \tau_B \right).$$  \hspace{1cm} (8.39)

where $\Delta \tau_B$ is an increment of the control parameter (cf. Sec. 6.3) and $\beta < 1$.

Fig. 8.13 Prediction of secondary equilibrium path in the bifurcation point $B$. 
Fig. 8.14 Large deflections of plane portal frames: a) Data for computations, b, c) Equilibrium paths for the frames I and II, d) Imperfection sensitivity.
Deflections of portal plane frames are shown in Fig. 8.14 taken from [83] p. 113-121 by Cichoń. Two portal frames I and II are different only with respect to the length of the columns, i.e. $H = 120r$, $80r$ where $r$ is the radius of gyroscoy of the I-shape cross-section - cf. data in Fig. 8.13a. Twelve FE have been used with 5 Lobatto's integration points along the cross-section thickness (the elements are described in [156]). The perfect frame I buckles in the elastic range (point $B_1$ on the fundamental path 1). The secondary bifurcation points $B_2$ on path 2 is associated with the first yielding and the postbifurcation path 3 - cf. 8.14b.

The bifurcation point $B$ for frame II occurs in the plastic range (Fig. 8.14c). According to the Shanley theory of increasing load the limit point $L$ is reached on path 3 and then the snap-back point $S$ is obtained with respect to the deflection $w$ in the middle of the beam.

As it is shown in Fig. 8.14d frame II is more sensitive to the 'imperfection' related to the application of the horizontal force $P_3$.

The presented examples of numerical approach to the elastic-plastic analysis give only a rough picture of possibilities in this field of interest. Many other applications of FDM and FEM to the buckling analysis can be found in appropriate papers and books, cf. e.g. [22, 79, 53].
9. PROBLEMS NOT DISCUSSED

In Sec. 1.2 the scope of the course has been defined but it is worth to mention other topics which have not been discussed in detail. This concerns especially new directions which are under development. The best information in this field can be found in proceedings of last conferences devoted to plasticity, its computational methods and engineering applications - cf. [84,85,86].

A great deal of attention is paid to formulation of new models which can describe more exactly inelastic properties of materials in extreme exploitation conditions (low and elevated temperatures, cyclic loadings) in a wide range of deformation. That is why in phenomenological plasticity a dependance on all stress invariants is considered and non-associated flow rules are used. This concerns especially non-metallic materials, as soil, rock and concrete. In order to explain experimentally observed behaviour of such materials plasticity has to be combined with continuous damage, fracture and failure phenomena.

Mathematical modelling of finite elastoplasticity is considered in order to describe very large deformations and instability at tension. Continuous effort is directed toward models supported on microscopic properties of materials. Computational methods are used also to the identification of material parameters on the base of new experiments — cf. e.g. [86].

In the course thermal loads in shakedown analysis have not been discussed. Variable thermal stresses superimposed on stresses from other external factors can lead to large deformation and failure of structures — cf. [87].

Dynamics of inelastic structures has been completely omitted in the course. Appropriate problems are related to permanent displacement under impact loads, transient analysis and wave propagation — cf. [88].

Also optimum design of structures has not been considered. In this field interesting results have been obtained mainly due to mathematical programming on the base of convex analysis — cf. [48].

There are many papers devoted to implementation of various methods and algorithms into particular computer codes. Such a review can be found in papers [14,22,55] and it has not been discussed in the course.
In the course the attention has been focused on new algorithms or modification of computational methods which have been formulated in the frame of elastic analysis. Many problems which are transferred from elasticity into the elastic-plastic analysis have been completely omitted despite the fact that such transitions are not quite evident. This concerns, for instance, the problems of approximation of displacement fields and numerical integration in FEM.
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Abbreviations used in references

AIAAJ = AIAA Journal
AMec = Acta Mechanica
CMAME = Computer Methods in Applied Mechanics and Engineering
C&S = Computers and Structures
FEAD = Finite Elements in Engineering and Design
IJN Mech = International Journal of Non-Linear Mechanics
IJNME = International Journal for Numerical Methods in Engineering
IJMS = International Journal of Mechanical Sciences
IJPlast = International Journal of Plasticity
IJSS = International Journal of Solids and Structures
JAM = Journal of Applied Mechanics, Transactions of ASME
JEMD = Journal of Engineering Mechanics Division, Transactions of ASCE
JMES = Journal of Mechanical Engineering Science
JMPHS = Journal of Mechanics and Physics of Solids
JPVT = Journal of Pressure Vessel Technology, Transactions of ASME
JRAS = Journal of the Royal Aeronautical Society
JSA = Journal of Strain Analysis
MK = Mechanika i Komputer
NED = Nuclear Engineering and Design
QAM = Quarterly of Applied Mathematics
TRRL = Transport and Road Research Laboratory, Crownthorne, Berkshire, U.K.