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Dispersion Analysis of Finite-element Schemes for a First-order Formulation of the Wave Equation

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SUMMARY

We investigated one-dimensional numerical dispersion curves and error behaviour of four finite-element schemes with polynomial basis functions: the standard elements with equidistant nodes, the Legendre-Gauss-Lobatto points, the Chebyshev-Gauss-Lobatto nodes without a weighting function and with. Mass lumping, required for efficiency reasons and enabling explicit time stepping, may adversely affect the numerical error. We show that in some cases, the accuracy can be improved by applying one iteration on the full mass matrix, preconditioned by its lumped version. For polynomials of degree one, this improves the accuracy from second to fourth order in the element size. In other cases, the improvement in accuracy is less dramatic.

Introduction

While finite-difference methods for wave propagation are popular because of their simplicity, they fail to accurately capture large material contrasts. Finite-element methods behave better if the mesh follows sharp interfaces. Mass lumping allows for explicit time stepping without the need of inverting a large sparse matrix. Here, we examine the numerical dispersion curves and error behaviour of four schemes with polynomial basis functions: the standard elements with equidistant nodes, the Legendre-Gauss-Lobatto points, the Chebyshev-Gauss-Lobatto nodes without a weighting function (Patera, 1984) and with. The analysis is carried out for the first-order similar formulation of the wave equation.

Method

Elements A first-order formulation of the acoustic wave equation is

$$\rho \frac{\partial v}{\partial t} = \frac{\partial p}{\partial x}, \quad c^{-2} \frac{\partial p}{\partial t} = \frac{\partial v}{\partial x},$$

with particle velocity $v(t,x)$ and pressure $p(x,t)$ (actually without the minus sign) as function of time t and position x. The density $\rho(x)$ and sound speed $c(x)$ will be taken as constant for the purpose of analysis. We will not consider time stepping errors and only concentrate on the spatial discretisation. Consider N elements bounded by positions $x_j = x_0 + jh_j$, $j = 0,\ldots,N$. Each element has $M+1$ nodes at relative positions $\bar{x}_k$, $k = 0,\ldots,M$, with $\bar{x}_0 = -1$ and $\bar{x}_M = 1$. Their corresponding global positions are $x_{jk} = x_j + \frac{1}{2}(\bar{x}_k + 1)jh_j$. In the periodic case, the solution on x_N is the same as on x_0. The number of degrees of freedom is $N_{\text{dof}} = MN$ on a periodic grid both for the particle velocity and pressure. For the finite-element basis functions $\psi_k(\bar{x})$, we take the Lagrange interpolating polynomials of degree M relative to the nodes, so $\psi_k(\bar{x}_l) = \delta_{kl}$, the Kronecker delta. In each element, we have a local mass matrix A and first-derivative matrix D, each with entries

$$A_{kl} = \int_{-1}^{1} w(\bar{x})\psi_k(\bar{x})\psi_l(\bar{x})\mathrm{d}\bar{x}, \quad D_{kl} = \int_{-1}^{1} w(\bar{x})\psi_k(\bar{x})\frac{\mathrm{d}}{\mathrm{d}x}\psi_l(\bar{x})\mathrm{d}\bar{x}.$$

The local lumped mass matrix, $A^{\mathrm{L}}_{kl} = \delta_{kl}\sum_{l=0}^{M} A_{kl}$ is a diagonal matrix with values proportional to quadrature weights. We consider several choice for the nodes: the standard element with equidistant nodes $x_k = k/M$, $k = 0,1,\ldots,M$ (EQUI); the Legendre-Gauss-Lobatto points (LGL) that are the zeros of $(1-\bar{x}^2)P'_M(\bar{x})$, the Chebyshev-Gauss-Lobatto points $\bar{x}_k = -\cos(\pi k/M)$ with an unweighted scalar product (CGL) and with the weighting function $w(\bar{x}) = 1/\sqrt{1-\bar{x}^2}$ (CGLw). Numerical quadrature with weights $A^{\mathrm{L}}_{kk}/\sum_{k=0}^{M} A^{\mathrm{L}}_{kk}$ is exact for polynomials up to degree $q = 1 + 2\,\mathrm{floor}(\mathrm{M}/2)$ for CGL and EQUI and degree $q = 2M-1$ for LGL and CGLw.

Mass matrix and defect correction With the local mass and first-derivative matrices, we can assemble the global mass matrix $\mathbf{M}$ and derivative matrix $\mathbf{D}$. A leap-frog time discretisation with time step Δt is

$$\frac{1}{\Delta t}\mathbf{M}_{\mathbf{v}}(\mathbf{v}^{n+1} - \mathbf{v}^{n}) = \mathbf{D}\mathbf{p}^{n+1/2}, \quad \frac{1}{\Delta t}\mathbf{M}_{\mathbf{p}}(\mathbf{p}^{n+3/2} - \mathbf{p}^{n+1/2}) = \mathbf{D}\mathbf{v}^{n+1}.$$

Here, the material properties are absorbed into the mass matrices and the superscript n denotes the solution at time $t^n = t_0 + n\Delta t$. The time-stepping stability limit for a leap-frog scheme is given by the CFL number $2/\sqrt{\rho(L^{(2)})}$, with $L^{(2)} = \mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1}\mathbf{D}$ and where $\rho(\cdot)$ now denotes the spectral radius. For time stepping, we want to avoid the cost of inverting the full mass matrix and replace it by its lumped version. Depending on the choice of nodes, this may or may not harm the spatial accuracy. Formally, the lumped version should be exact for numerical quadrature of polynomials up to a degree of at least $2M-2$. If its accuracy is less, we can iterate with the lumped mass matrix as preconditioner. This approach resembles defect correction (Stetter, 1978), which has the following convenient property. Consider two operators $\mathbf{L}_1$ and $\mathbf{L}_2$ where $\mathbf{L}_k$ has an order of accuracy p_k ($k = 1,2$) and $p_1 > p_2$. We can try to solve $\mathbf{L}_1\mathbf{u} = \mathbf{f}$ with the iterative scheme $\mathbf{u}^{-1} = 0$, $\mathbf{u}^{j+1} = \mathbf{u}^{j} + \mathbf{L}_2^{-1}(\mathbf{f} - \mathbf{L}_1\mathbf{u}^{j})$, where $j = 0,1,\ldots$ denotes the iteration count, not the time step. Convergence is obtained if the operator $\mathbf{G} = \mathbf{I} - \mathbf{L}_2^{-1}\mathbf{L}_1$ has a spectral radius $\rho(\mathbf{G}) < 1$.

Table 1 *Leading error terms in the dispersion curves for a polynomial basis of degree M and various sets of nodes, using the full or lumped mass matrix or lumped with one iteration based on* $\mathbf{G}$*. Its spectral radius* $\rho(\mathbf{G})$ *is given, as well as the CFL number without and with mass lumping.*

M	nodes	full	lumped	1 iteration	$\rho(\mathbf{G})$	CFL (full)	CFL (lumped)
1	LGL	$-\frac{1}{180}\xi^4$	$-\frac{1}{6}\xi^2$	$-\frac{1}{30}\xi^4$	2/3	$2/\sqrt{3}=1.155$	2
2		$\frac{1}{270}\xi^4$	$-\frac{4}{270}\xi^4$	$-\frac{4}{945}\xi^4$	3/5	$\sqrt{2}/3=0.471$	$2/3=0.667$
3		$-\frac{81}{39200}\xi^8$	$-\frac{27}{2800}\xi^6$	$-\frac{6}{2800}\xi^6$	4/7	0.278	0.365
4		$\frac{128}{496125}\xi^8$	$-\frac{1024}{496125}\xi^8$	$-\frac{4096}{6449625}\xi^8$	5/9	0.188	0.239
5		$\frac{-9765625}{19179224064}\xi^{12}$	$\frac{-78125}{67060224}\xi^{10}$	$-\frac{15625}{50295168}\xi^{10}$	6/11	0.138	0.171
3	CGL	see LGL	$-\frac{333}{10240}\xi^4$	$-\frac{21}{1460}\xi^2$	3/5	see LGL	0.311
4			$\frac{8}{1395}\xi^8$	$-\frac{1042}{35397}\xi^4$	5/7		0.198
5			$-\frac{231125}{134217728)}\xi^4$	$\frac{5115}{4502764}\xi^2$	0.966		0.132
1	CGLw	$-\frac{1}{24}\xi^2$	$-\frac{1}{6}\xi^2$	$-\frac{1}{24}\xi^2$	1/2	1.414	2
2	CGLw	$\frac{1}{30}\xi^2$	$-\frac{2}{135}\xi^4$	$\frac{1}{48}\xi^2$	1/2	0.426	2/3
3	CGLw	$\frac{9}{1280}\xi^4$	$-\frac{9}{320}\xi^4$	$-\frac{9}{5120}\xi^4$	1/2	0.213	0.354
4	CGLw	$-\frac{1}{405}\xi^4$	$-\frac{32}{4725}\xi^6$	$-\frac{1}{530}\xi^4$	1/2	0.132	0.224
5	CGLw	$-\frac{625}{344064}\xi^6$	$\frac{625}{258048}\xi^6$	$\frac{-625}{1032192}\xi^6$	1/2	0.0909	0.155
3	EQUI	see LGL	$-\frac{42}{295}\xi^2$	$-\frac{42}{295}\xi^2$	0.651	see LGL	0.369
4			$\frac{40}{1137}\xi^4$	$\frac{56825}{157068}\xi^4$	(1.72)		0.184
5			$\frac{-92807}{312500}\xi^4$	$\frac{33740850}{26406233}\xi^2$	(1.96)		0.125

In a finite-difference context, the order of accuracy of $\mathbf{u}^j$ is $\min(p_2,(j+1)p_1)$, which suggests that a few iterations will often suffice to get a sufficiently accurate though not necessarily fully converged result. In our case, we can take the lumped mass matrix for $\mathbf{L}_2=\mathbf{M}^{\mathrm{L}}$ and the full mass matrix as $\mathbf{L}_1=\mathbf{M}$. However, for degree $M>1$, the eigenvalues and eigenvectors are mixed up in a non-trivial way (Mulder, 1999) and the property that the accuracy increases by an order p_1 per iteration may be lost.

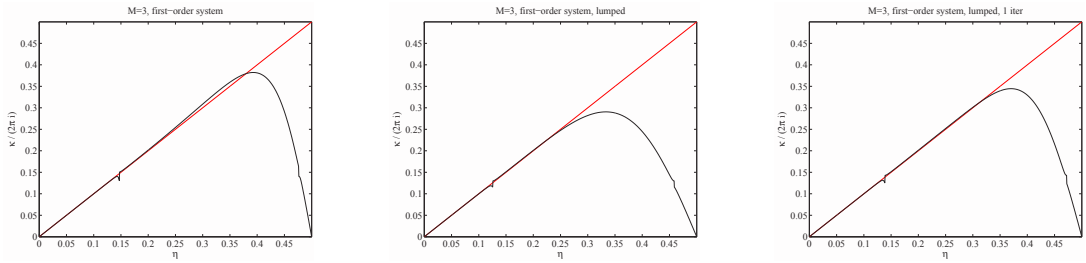
Dispersion The numerical dispersion of the finite-element scheme can be analyzed by considering the eigenvalues of the first-order operator $\mathbf{M}^{-1}\mathbf{D}$ or $(\mathbf{M}^{\mathrm{L}})^{-1}\mathbf{D}$ when discretized on a sufficiently fine periodic mesh with constant material properties and a constant element size h. Alternatively, we can use the fact that the elements are translation-invariant if all is constant and perform a Fourier transform on the solution. We then have to take the M degrees of freedom inside an element as a vector and do a transform on each component over the N elements. This results in a small $M\times M$ matrix in the Fourier domain. However, we can go one step further and also involve the M individual components. These are aliased but still can be considered separately by looking at the eigenvalues of the $M\times M$ block and unwrapping the result (Mulder, 1999). This produces a discrete approximation $\mathrm{i}\kappa$ to the exact operator $\mathrm{i}\xi$, where $\xi=k(x_N-x_0)/(NM)=kh/M\in[-\pi,\pi]$ is scaled version of the wavenumber k. The relative dispersion error can than be characterized by $\kappa/\xi-1$. Note that the error in the dispersion curve does not tell the full story, because errors in the eigenvectors also play a rôle.

Results

We compared the various spatial discretisations in terms of their dispersion curves, obtained by Fourier analysis, as well by set of numerical experiments. As an example, Figure 1 shows dispersion curves for polynomials of degree $M=3$ on Legendre-Gauss-Lobatto points (LGL). For reference, Figure 1(a)

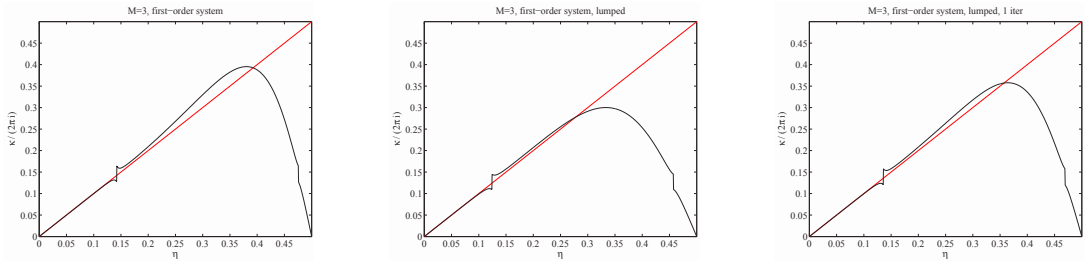
depicts the result without mass lumping. With lumping, the deviation from the exact dispersion curve, the straight line, increases, but not so much at the smaller values of $\eta = \xi/(2\pi)$. With one iteration of $\mathbf{G} = \mathbf{I} - (\mathbf{M}^{\mathrm{L}})^{-1}\mathbf{M}$, the result is improved. For the smaller wavenumbers, we have analytically determined the asymptotic error behaviour. by taking the leading term in the series expansion of $\kappa/\xi - 1$ for the eigenvalue that is valid at small ξ. The results are listed in Table 1 for various cases. For degree $M = 1$ and $M = 2$, the standard element (EQUI), the Legendre-Gauss-Lobatto points (LGL) and the unweighted Chebyshev-Gauss-Lobatto (CGL) points all provide the same results. The same is true when the full mass matrix is used. Then, the choice of nodes does not matter. The exception is the weighted scheme with Chebyshev-Gauss-Lobatto nodes (CGLw), where the weighting functions changes the outcome. Note that for the latter, the error analysis did *not* involve a weighted norm. Figure 2 show dispersion curves for degree $M = 3$.

Interestingly, the LGL scheme without mass lumping has a fourth-order error instead of the usual second-order. In the finite-difference world, Lele (1992) found the same behaviour. Without lumping and just a single iteration, this fourth-order behaviour is recovered, albeit with a larger error constant and not necessarily on a finite-difference grid with constant mesh spacing.



(a) (b) (c)

Figure 1 *Dispersion curves for Legendre-Gauss-Lobatto points and $M = 3$ without (a) and with (b) mass lumping and (c) after one iteration.*

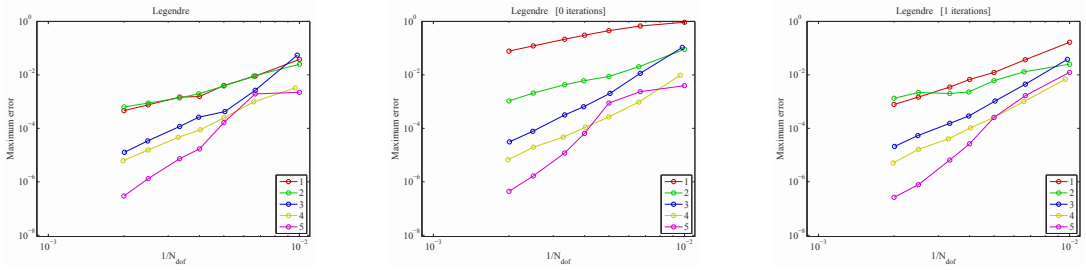


(a) (b) (c)

Figure 2 *Dispersion curves for degree $M = 3$ and CGLw without (a) and with (b) mass lumping and (c) after one iteration.*

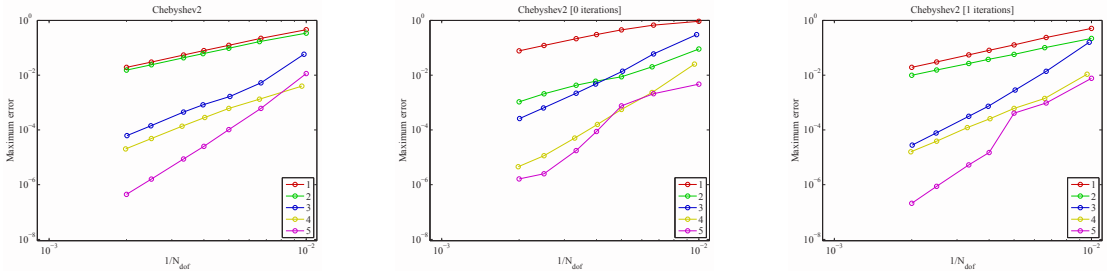
In addition to the above dispersion-curve analysis, we have performed a set of numerical experiments in the form of a Ricker pulse travelling around once on a periodic domain. The reason is that the error is not only described by the dispersion curves, which measure the error in the eigenvalues, but also by the error in the eigenvectors for degrees $M > 1$.

The pulse, the second-derivative of a Gaussian, was centred at 0.74 and had a standard deviation of 0.0375 on the interval of length 1. The simulation ran at at a fraction of 10^{-3} times the maximum time step dictated by the CFL condition to avoid the imprint of the time stepping error. Since finite-difference methods are known to suffer from abrupt changes in the grid spacing, we used two difference spacing h_L and h_R and set $h_j = h_L$ for $j = 0, \frac{1}{2}N - 1$, $h_j = h_R$ for $j = \frac{1}{2}N, N - 1$, N chosen even and $h_L = 0.8h_R$. Figure 3 plots the maximum errors in $u(t,x)$ for a varying number of degrees of freedom without and with mass lumping and with one extra iteration for polynomial degrees $M = 1$ to 5.



(a) (b) (c)

Figure 3 *Maximum error in u as function of the inverse number of degree of freedom,* $1/N_{\mathrm{dof}}$*, for the Legendre-Gauss-Lobatto nodes (LGL) with the full mass matrix (a), its lumped version (b), or with one iteration (c).*



(a) (b) (c)

Figure 4 *Maximum error in u as function of the inverse number of degree of freedom,* $1/N_{\mathrm{dof}}$*, for the Chebyshev-Gauss-Lobatto nodes with weighting (CGLw) with the full mass matrix (a), its lumped version (b), or with one iteration (c).*

Conclusions

We have compared four finite-element schemes with polynomial basis functions for the first-order formulation of the acoustic wave equation, using Legendre-Gauss-Lobatto nodes, Chebyshev-Gauss-Lobatto without and with weighting function or the standard element. Mass lumping, desired for numerical efficiency since it allows for explicit time stepping, tends to decrease the spatial accuracy. The remaining accuracy is best for the Legendre-Gauss-Lobatto nodes. In some cases, the accuracy can be improved by applying one iteration on the full mass matrix, preconditioned by its lumped version. For polynomials of degree one, this improves the accuracy from second to fourth order in the element size. In other cases, the improvement in accuracy is less dramatic.

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