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NUMERICAL INTEGRATION OF THE MOMENTUM-INTEGRAL EQUATIONS AND APPLICATION OF THE RESULTS TO THE FLOW BETWEEN ROTATING DISKS

by

B. C. Chandrasekhara

DELFT - THE NETHERLANDS

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B.C. Chandrasekhar
ABSTRACT

The laminar boundary layers on rotating disks of finite radius contained in a stationary cylinder filled with a fluid of constant properties, with or without a uniform transverse magnetic field, has been investigated by means of von Kármán's -T method. A parabolic relation is assumed for the tangential velocity distribution which is in agreement with the model considered. Important flow quantities like, the radial mass flow in boundary layers, the axial out-flow velocity, the boundary layer thickness and the amplitude of radial velocity are computed from numerical integration. The results of the analysis indicate that the axial out-flow velocity exhibits inward or outward flowing layers for values of $\kappa > 0.5$ and the boundary layer thickness and radial mass flow also exhibit marked change in behaviour for $\kappa > 0.5$. The secondary flow in end-wall boundary layers depends on the tangential Reynolds number and can be reduced by increasing the length of the cylinder or decreasing the tangential Reynolds number.

The presence of a uniform transverse magnetic field reduces the boundary layer thickness. The axial out-flow velocity exhibits outward and inward flowing layers at low values of $\kappa$. 
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\( a_1 \)  
co-efficient of starting series for \( E^2 \), equation (41) part A

\( A \)  
velocity-profile constant, equation (32) part A

\( b \)  
induced magnetic field

\( b_1 \)  
co-efficient of starting series for \( E_1^2 \), equation (42) part A

\( B, C, D \)  
velocity-profile constants, equation (32) part A

\( \vec{B} \)  
uniform impressed magnetic field vector (0,0,B)

\( E \)  
amplitude co-efficient of radial velocity, equation (26) part A

\( \vec{E} \)  
electric field vector, \((E_r, E_\theta, E_z)\)

\( E_r \)  
electric field in radial direction

\( E_\theta \)  
electric field in azimuthal direction

\( E_z \)  
electric field in axial direction

\( f(\eta) \)  
radial velocity profile functions, equation (26) part A

\( g(\eta) \)  
tangential velocity profile function, equation (27) part A

\( I_n \)  
integral of velocity profile functions, equation (31) part A

\( \vec{J} \)  
current density vector, \((J_r, J_\theta, J_z)\)

\( J_r \)  
current density in radial direction

\( J_\theta \)  
current density in azimuthal direction

\( J_z \)  
current density in axial direction

\( \kappa \)  
ratio of angular velocities, \( \frac{2\Omega^*}{\Omega_{1}^*+\Omega_{2}^*} \)

\( L^* \)  
length of the cylinder

\( L \)  
dimensionless length of the cylinder

\( M(r) \)  
dimensionless outward radial mass flow in boundary layers, equation (49) part A

\( M_{\text{max}} \)  
maximum outward radial mass flow in boundary layer
m  Hartmann number, $\sqrt{\frac{B^2 r_1^2 \sigma}{\rho \nu^2}}$

$M_t^*$  total mass flow in the cylinder

$p^*$  static pressure

$q^*$  velocity vector $(u^*, v^*, w^*)$

$r$  dimensionless radial co-ordinate

$r_{\text{max}}$  radius at which the maximum radial mass flow in the boundary layer occurs

$r_1^*$  radius of the disks

$R$  rotational Reynolds number $\frac{\Omega_1 L^2}{\nu^*}$

$R_1^*$  suction or injection Reynolds number $\frac{v_{\text{inj}} L^2}{\nu^*}$

$R_r$  radial Reynolds number, equation (11) part A

$R_t$  tangential Reynolds number, equation (59) part A

$u^*$  dimensionless radial velocity, equation (26) part A

$v^*$  dimensionless tangential velocity, equation (27) part A

$v_{\text{inf}}^*$  dimensionless tangential velocity of outer flow

$w^*$  dimensionless axial velocity, equation (10) part A

$z^*$  dimensionless axial co-ordinate, equation (10) part A

$\delta$  dimensionless boundary layer thickness

$\eta$  boundary-layer variable $z/\delta_1$ (lower disk)

$\xi$  boundary-layer variable, $\frac{L-z}{\delta_2}$ (upper disk)

$\mu^*$  viscosity co-efficient

$\nu^*$  kinematic viscosity co-efficient

$\rho^*$  density

$\sigma^*$  electrical conductivity
$\tau_r$  dimensionless radial shear stress, equation (60) part A
$\tau_t$  dimensionless tangential shear stress, equation (60) part A
$\Omega^*_1$  angular velocity of the lower disk
$\Omega^*_2$  angular velocity of the upper disk
* asterisks refer to dimensional quantities
PART A

"THE LAMINAR BOUNDARY LAYER ON ENCLOSED ROTATING DISKS"
I INTRODUCTION AND BRIEF OUTLINE OF THE PROBLEM

A. DISKS OF INFINITE RADIUS

Many investigations have been made on the flows involving rotating and stationary disks of finite and infinite radius. Historically the problem of the steady laminar flow over an infinite rotating disk in a fluid at rest was first treated by von Kármán [1] in 1921. In von Kármán's model the moving fluid, with no pressure gradient in the radial direction to restrain it, is thrown radially outward, thereby causing an axial flow to develop over the entire disk. Von Kármán showed that for this problem the Navier-Stokes equations can be reduced with the aid of similarity arguments to a system of ordinary differential equations. He did not solve the resulting system of equations exactly, but obtained an approximate solution using his newly developed integral method. Bödewadt [2] treated the laminar flow over a stationary infinite disk with the outer flow in solid-body rotation. In this case the direction of the secondary flow in the boundary layer is radially inward rather than outward as in von Kármán's problem. Bödewadt obtained the solution to his problem by a complicated numerical procedure using power series and asymptotic series. His solution predicts that the boundary layer at a fixed radius consists of alternate layers of fluid, some moving inward along the disk others moving outward. The problem of the disk and outer flow rotating in the same direction was investigated in detail by Rogers and Lance [3] who obtained the solution by numerical integration. They found that, when the disk and the outerflow are rotating in the same direction, the solutions exhibit oscillatory character at infinity. They also observed that when the outer flow rotated faster than the disk, the oscillations are prominent in the velocity profiles.

Batchelor [4] considered a general problem of an infinite rotating disk with the outer flow also in a solid body rotation and also the problem of two rotating disks. Batchelor did not obtain any explicit solutions to his problem but, using physical arguments and general properties of ordinary differential equations, predicts the general nature of the flow. The chief characteristic of the predicted flow is that in almost all cases the main body of the fluid is also rotating and the transition of one rate of rotations to another takes place in a narrow layer. In the particular case when the two disks are rotating in opposite directions with the same angular velocity he points out that in one of the three apparently possible solutions the main body of the fluid would be in two parts with different angular velocities. Stewartson [5] also considered Batchelor's generalized problem and discussed both theoretically and experimentally the flow between two rotating co-axial disks. In the theoretical investigation he obtained the solution of the equations of motion as a power series in the tangential Reynolds number. He found that when the disks are rotating in the same direction with the same angular velocity, the
main body of the fluid also rotated with the same angular velocity for small and large values of tangential Reynolds number as well. For the case when one disk is stationary and the other is rotating, Stewardson's solutions predict that the boundary layer is formed only at the rotating disk and not at both disks as predicted by Batchelor. He found experimentally that when the disks rotate in opposite direction the main body of the fluid is almost at rest which is contrary to Batchelor's predictions. Lance and Rogers [6] investigated the case of axially asymmetric flow of a viscous fluid between two infinite rotating disks using numerical techniques. They found that for large tangential Reynolds numbers the main body of the fluid is slightly disturbed when the two disks rotate in opposite direction with same angular velocity. Pearson [7] treated the problem of time-dependent viscous flow between two rotating co-axial disks numerically. He obtained solutions for impulsively started disks and for counter rotating disks. At large Reynolds number, his numerical solutions predict that the main body of the fluid rotated at a higher angular velocity than that of either disk when the disks rotate in opposite directions. Mellor, Chapple and Stokes [8] obtained numerical solutions for the flow between a rotating and a stationary disk for arbitrary Reynolds number. For a given Reynolds number they identified three solution branches, which correspond to one, two and three flow cells in meridional plane, out of a greater number of possible solutions. They treated one-cell branch in detail and confirmed Batchelor's predictions that the main body of the fluid rotates with a constant angular velocity and boundary layers develop on both disks as the Reynolds number increases. They verified their theoretical results experimentally for small values of tangential Reynolds number, \( R < 50 \). Recently Greenspan [9] obtained numerical solutions for the flow between two rotating co-axial disks for large values of Reynolds number. His solutions predict a considerable disturbance in the main body of the fluid when the disks rotate in opposite directions with same angular velocity.

B. DISKS OF FINITE RADIUS

Schultz-Grunow [10] was the first person to consider the problem of finite-radius disk in a rotating flow. He was concerned with a disk rotating in a shallow housing. Therefore he had the two problems of the disk at rest (the endwall) in a fluid rotating as a solid body and a rotating disk in a fluid rotating with a smaller angular velocity. He used the momentum-integral method and found that, as long as the angular velocity of the outer flow is less than six times the angular velocity of the disk, a similarity solution exists for his second problem. However the similarity solution did not produce a result for his first problem. Hence he used momentum- \( T \) method which was developed by von Kármán during his investigation of turbulent boundary layer on a rotating disk. Further he assumed the boundary layer to start with zero thickness at the edge of the disk and established the initial rate of growth of this type of boundary layer. The boundary layer thickness \( \delta \), grows initially as the \( \frac{1}{2} \) power of the radial distance from the edge of the disk.
Soo [11] considered the laminar flow over an enclosed rotating disk. He showed that the friction moment coefficient of the enclosed disk is proportional to $R^{-1}$ in the laminar range and $R^{-4}$ in the turbulent range and the radial out-flow is more effective than radial in-flow for turbine-disk cooling. Daily and Nece [12] investigated the chamber dimension effects on induced flow of enclosed rotating disks. They extended their analysis to turbulent flow region as well introducing the effect of axial clearance and cylindrical wall friction as variables into momentum type analysis. Miloh and Poreh [13] discussed the resistance to rotation of free and enclosed disks. Their solution is based on Goldstein's approach for both free and enclosed smooth disks. In the Goldstein's approach the velocity is described by the universal logarithmic profile whereas in von Kármán's approach the velocity is described by polynomial profiles.

Taylor [14] considered a different type of flow which is met in a swirl atomizer. Here the model consists of an inverted cone with a hole at the apex. The fluid is directed at high pressure tangential to the cone surface at the upper end. A swirling flow is produced which exits through the hole at the apex. Taylor's investigation was directed to find whether it was possible that most of the exit-flow comes from the secondary flow induced in the boundary layer at the cone surface by the pressure gradient of the main flow. He did not calculate the radial mass flow but concluded from the fact that the boundary layer thickness at the exit occupies most of the hole at the apex of the cone that the exit-flow in a swirl atomizer comes chiefly from the boundary layer. Mack [15] treated the problem of laminar boundary layers on a disk of finite radius in a rotating flow in which the tangential velocity in the outer-flow was represented by $v_{\infty} = l/r^n$. He applied the results obtained to the flow in a vortex chamber.

Rasmussen [16] investigated the flow between two finite disks contained in a rotating cylinder. His model consisted of the flow between two disks of finite radius rotating with angular velocity $\Omega$ and $\Omega(1+\epsilon)$ respectively. The disks are contained in a cylinder of equal radius which rotated with angular velocity $\Omega(1+\epsilon)$. The difference $\epsilon$ between the angular velocities of the disks is taken to be small so that the governing equations of motion may be linearized as shown by Proudman [17]. Even the linearized equations in his case pose mathematical difficulties and hence he constructs approximate solutions which are valid for small values of Eckmann number and postulates that the flow at low Eckmann numbers has a boundary layer on the cylinder.

C. OUTLINE OF THE PROBLEM

It is evident from the above brief survey of literature that no single theory is unique and explains all the observed phenomena associated with flows involving rotating disks of finite or infinite radius. In any special case of flow a compromise has to be made between the existing theories and the experimental observations. Hence the flows associated with rotating disks are still attracting the attention of workers in the
field to develop refined theories, if possible.
However, the present investigation is devoted to the study of flow be-
tween two rotating disks contained in a stationary cylinder of equal
radius as the disks. This problem is different from Rasmussen's problem
in which the cylinder also rotates in the same direction as the disks.
The type of flow situation considered in the present analysis is en-
countered in industrial applications like centrifugal machinery.
Attention is confined to the solution of end-wall boundary layers and
the total mass flow in the cylinder. To obtain the solution for the end-
wall boundary layers von Kármán's - T method is adopted.
Von Kármán's - T method is briefly described below.
To study the problem of steady laminar flow over an infinite rotating
disk in a fluid at rest von Kármán developed his integral method. He
took as the two unknown functions in his integral method the boundary
layer thickness and a quantity related to the slope of the radial velo-
city at the disk surface. The boundary conditions he imposed on the ve-
locity profiles included the conditions of compatibility known as radial
and tangential compatibility conditions. These are boundary layer equa-
tions evaluated at the disk surface. To extend his analysis to turbulent
region von Kármán developed a second method which avoids the compat-
bility conditions altogether, because there are no compatibility conditions
for turbulent boundary layer. This method is referred to as von Kármán's
- T method. In this method the second unknown function is an amplitude
co-efficient for radial velocity. When this method is applied momentum
equations reduce to a system of first order differential equations in
two unknowns. Then the system of equations is solved numerically. Though
von Kármán developed his T method for the study of turbulent flow region,
Schultz-Grunow and Taylor applied this method successfully to the study
of laminar boundary layers in their respective problems.
In the present analysis the system of first order differential equations
is solved by Runge-Kutta-Gill method. From the numerical results impor-
tant flow quantities like outward radial mass flow, axial out-flow ve-
locity and fraction of total mass flow in the end-wall boundary layers
are calculated and presented in graphs. The aspect ratio of the cylinder
for which the total mass flow would be in the end-wall boundary layers
is also evaluated. As mentioned earlier the present analysis is restrict-
ed to the solutions of the end-wall boundary layers.
II DERIVATION OF EQUATIONS

A. FORMULATION OF PROBLEM

The formulation of the problem is based on the following assumptions.
1) The flow in the boundary layer is incompressible
2) The flow in the boundary layer is laminar
3) The radial velocity in the outer flow is zero
4) The outer flow tangential velocity distribution is parabolic
5) The flow is axi-symmetric.

![Physical model diagram]

The physical model consists of a cylinder of length \(L^Z\) and radius \(r^Z\) in which two disks of same radius are situated at \(z^Z = 0\) and \(z^Z = L^Z\). The two disks rotate with angular velocities \(\Omega_1^Z\) and \(\Omega_2^Z\) respectively in the same direction \((\Omega_1^Z > \Omega_2^Z)\). The cylinder itself is at rest. The region between the disks is occupied by a fluid of constant properties. The boundary-layer equations will be written in a cylindrical co-ordinate system where \(z\) is the axial co-ordinate and \(r\) is the radial co-ordinate. (Fig. 1a).

The governing equations of motion are:

\[
\begin{align*}
\frac{\partial u} {\partial r} + \frac{w} {r} \frac{\partial u} {\partial z} - \frac{v^2}{r} &= - \frac{1}{\rho} \frac{\partial p} {\partial r} + \nu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right] \\
\frac{u}{r} \frac{\partial u} {\partial r} + w \frac{\partial v} {\partial r} + \frac{u v}{r} &= \nu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right] \\
\frac{u}{r} \frac{\partial w} {\partial r} + w \frac{\partial w} {\partial z} &= - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right]
\end{align*}
\]

(1) (2) (3)

and the continuity equation is:
\[
\frac{\partial}{\partial r^*} (r^* u^*) + \frac{\partial}{\partial z^*} (r^* w^*) = 0
\]  

(4)

Where \( u^* \) is the radial velocity, \( v^* \) the tangential velocity, \( w^* \) the axial velocity, \( p^* \) the pressure, \( \rho^* \) the density, and \( \nu^* \) the kinematic viscosity coefficient. The asterisks refer to dimensional quantities. Equations (1), (2) and (3) are transformed into boundary layer equations in the usual manner. The derivatives in the \( z^* \) direction are considered to be an order of magnitude larger than those in the \( r^* \) direction, and \( w^* \) to be an order of magnitude smaller than \( u^* \) and \( v^* \). The resulting boundary-layer equations are:

\[
\frac{u^*}{r^*} \frac{\partial u^*}{\partial r^*} + v^* \frac{\partial u^*}{\partial z^*} - \frac{v^*}{r^*} \frac{\partial v^*}{\partial z^*} = \frac{1}{r^*} \frac{\partial p^*}{\partial r^*} + \nu^* \frac{\partial^2 u^*}{\partial z^*^2}
\]  

(5)

\[
\frac{u^*}{r^*} \frac{\partial v^*}{\partial r^*} + v^* \frac{\partial v^*}{\partial z^*} + \frac{u^* v^*}{r^*} = \nu^* \frac{\partial^2 v^*}{\partial z^*^2}
\]  

(6)

\[
0 = \frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*}
\]  

(7)

and the continuity equation is unchanged. As usual the pressure is found to be constant through the boundary layer and therefore is a function only of \( r^* \) as given by the outer flow. The equations of motion governing the outer flow are:

\[
\frac{1}{\rho^*} \frac{d}{dr^*} \left( r^* v_\infty^* \right) = \frac{v_\infty^*}{r^*}
\]  

(8)

\[
0 = \frac{d}{dr^*} \left[ r^* \frac{d}{dr^*} \left( \frac{v_\infty^*}{r^*} \right) \right]
\]  

(9)

Where \( v_\infty^* \) is the tangential velocity in the outer flow. Equation (8) follows from equation (1) and the assumption that as \( z^* \to \infty, u^* = 0 \). Equation (9) follows from equation (2) when \( v_\infty \) is independent of \( z^* \). Equation (9) is exactly satisfied for a solid body outer flow, \( v_\infty \sim r^* \), and for a free vortex outer flow, \( v_\infty \sim 1/r^* \). But it is not zero in general for the type of tangential outer flow of the form \( v_\infty \sim (r^* - a)^2 \), assumed in the present analysis. This situation explains the fact that \( v_\infty^* \sim (r^* - a)^2 \), \( u^*_\infty = 0 \) is an approximation, not a solution of Navier-Stokes equations.

B. DERIVATION OF MOMENTUM-INTEGRAL EQUATIONS.

In the present model it is assumed that both the disks are rotating in
the same direction but with different angular velocities \( \Omega_1^* \) and \( \Omega_2^* \). Hence the boundary layer equations are solved separately. The same analysis is valid when the disks rotate with the same angular velocity (say \( \Omega \)).

Equations (5), (6) and (7) written in dimensionless form, separately for the two disks, introducing the following dimensionless variables:

\[
\begin{align*}
  u &= \frac{u^*}{v_1^*}, & v &= \frac{v^*}{v_1^*}, & w &= \frac{w^*}{\sqrt{v^* \Omega_1^*}} \\
  r &= \frac{r^*}{r_1^*}, & z &= z^* \sqrt{\frac{\Omega_1^*}{v^*}} \quad \text{lower disk} \\
  u &= \frac{u^*}{v_2^*}, & v &= \frac{v^*}{v_2^*}, & w &= \frac{w^*}{\sqrt{v^* \Omega_2^*}} \\
  r &= \frac{r^*}{r_1^*}, & z &= z^* \sqrt{\frac{\Omega_2^*}{v^*}} \quad \text{upper disk}
\end{align*}
\]

where \( v_1^* \) is the tangential velocity of the lower disk at \( r_1^* \)
\( v_2^* \) is the tangential velocity of the upper disk at \( r_1^* \)
\( \Omega_1^* = \frac{v_1^*}{r_1^*} \) is the angular velocity of the lower disk
\( \Omega_2^* = \frac{v_2^*}{r_1^*} \) is the angular velocity of the upper disk
\( r_1^* \) is the radius of the disks

The boundary layer equations in dimensionless form for both disks are:

\[
\begin{align*}
  u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{v_\infty^*}{r} + \frac{\partial^2 u}{\partial z^2} & (11) \\
  u \frac{\partial v}{\partial r} + w \frac{\partial u}{\partial z} + uv &= \frac{\partial^2 v}{\partial z^2} & (12) \\
  \frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (rw) &= 0 & (13)
\end{align*}
\]

In equation (11) the radial pressure gradient is replaced by the outer flow tangential velocity using equation (8).
To apply the momentum-integral procedure of von Kármán and Pohlhausen
the momentum equations (11) and (12) are rewritten in the form:

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru^2) + \frac{\partial}{\partial z} (uw) - \frac{v^2}{r} = - \frac{v_\infty^2}{r} + \frac{\partial^2 u}{\partial z^2} \tag{14}
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 uv) + \frac{\partial}{\partial z} (vw) = \frac{\partial v}{\partial z} \tag{15}
\]

The above two equations are now integrated from \( z = 0 \) to \( z = \delta_1 \), for the lower disk, and from \( z = L \) to \( z = L - \delta_2 \), for the upper disk, where \( \delta_1 \) and \( \delta_2 \) functions of \( r \) to be determined, are the dimensionless boundary-layer thickness of the lower and the upper disk respectively. Both the radial and tangential velocities are assumed to reach their outer flow value at the same value of \( \delta_1 \) and \( \delta_2 \).

At \( z = \delta_1 \), and \( z = L - \delta_2 \), \( u = 0 \), \( \frac{\partial u}{\partial z} = 0 \), and \( \frac{\partial v}{\partial z} = 0 \).

At \( z = 0 \) and \( z = L \), \( u = 0 \) and \( w = 0 \).

The integrated equations for the lower disk are:

\[
\frac{d}{dr} \left( r \int_0^{\delta_1} u^2 dz \right) + \int_0^{\delta_1} (v_\infty^2 - v^2) dz = - r \left( \frac{\partial u}{\partial z} \right)_{z=0} \tag{16}
\]

\[
\frac{d}{dr} \left( r^2 \int_0^{\delta_1} uv dz \right) + r^2 v_\infty w_\infty = - r^2 \left( \frac{\partial v}{\partial z} \right)_{z=0} \tag{17}
\]

Similarly the integrated equations for the upper disk are:

\[
\frac{d}{dr} \left( r \int_L^{L-\delta_2} u^2 dz \right) + \int_L^{L-\delta_2} (v_\infty^2 - v^2) dz = - r \left( \frac{\partial u}{\partial z} \right)_{z=L} \tag{18}
\]

\[
\frac{d}{dr} \left( r^2 \int_L^{L-\delta_2} uv dz \right) + r^2 v_\infty w_\infty = - r^2 \left( \frac{\partial v}{\partial z} \right)_{z=L} \tag{19}
\]

Where the order of integration and differentiation has been interchanged in the first term of each equation. This operation is allowed here because both integrands are zero at the upper limit of integration. The axial velocity at the edge of the boundary layers, \( w_\infty \), which will be referred to as axial outflow velocity, is determined from the integration of the continuity equation to be:
\[ w_\infty = - \frac{1}{r} \frac{d}{dr} (r \int_0^\delta_1 u \, dz) \quad \text{lower disk} \] (20)

\[ w_\infty = - \frac{1}{r} \frac{d}{dr} (r \int_0^{L-\delta_2} u \, dz) \quad \text{upper disk} \] (21)

With further change of variables
\[ \eta = \frac{z}{\delta_1} \quad \text{and} \quad \xi = \frac{L - z}{\delta_2} \]

The above two sets of momentum equations become:

**Lower disk**

\[ \frac{d}{dr} \left( \delta_1 r \int_0^1 u^2 \, d\eta \right) + \delta_1 \int_0^1 (v_\infty^2 - v^2) \, d\eta = - \frac{r}{\delta_1} \left( \frac{\partial u}{\partial \eta} \right)_{\eta=0} \] (22)

\[ \frac{d}{dr} \left( \delta_1 r^2 \int_0^1 uv \, d\eta \right) - rv_\infty \frac{d}{dr} \left( \delta_1 r \int_0^1 u \, d\eta \right) = - \frac{r^2}{\delta_1} \left( \frac{\partial v}{\partial \eta} \right)_{\eta=0} \] (23)

**Upper disk**

\[ \frac{d}{dr} \left( \delta_2 r \int_0^1 u^2 \, d\xi \right) + \delta_2 \int_0^1 (v_\infty^2 - v^2) \, d\xi = - \frac{r}{\delta_2} \left( \frac{\partial u}{\partial \xi} \right)_{\xi=0} \] (24)

\[ \frac{d}{dr} \left( \delta_2 r^2 \int_0^1 uv \, d\xi \right) - rv_\infty \frac{d}{dr} \left( \delta_2 r \int_0^1 u \, d\xi \right) = - \frac{r^2}{\delta_2} \left( \frac{\partial v}{\partial \xi} \right)_{\xi=0} \] (25)

The form of the above two sets of equations are the same except for the variables \( \eta \) and \( \xi \). Hence in what follows the analysis is confined to one set and the same analysis is valid for the other set of equations. Considering the first set of equations, (22) and (23), we find that the velocity components \( u \) and \( v \) in these two equations are functions of both \( r \) and \( \eta \). To proceed further, it is necessary to choose a method of representing \( u \) and \( v \) in which the variables are separated. We know already that \( \delta_1 \) provides one unknown function of \( r \), hence the representation chosen must introduce one and only one additional function of \( r \). The simplest method known is the one developed by von Kármán for the study of turbulent boundary layer on a rotating disk. The same method is adopted here. According to this method the velocity components are expressed as:

\[ u(r, \eta) = v_\infty(r) E(r) f(\eta) \] (26)
\( v(r, \eta) = v_\infty(r) \ g(\eta) \) \hfill (27)

Where \( E(r) \) is the additional unknown function of \( r \) and can be referred to as the amplitude co-efficient of radial velocity. With this form of the radial velocity only homogeneous boundary conditions can be satisfied. The two functions \( f(\eta) \) and \( g(\eta) \) in equations (26) and (27) are the velocity-profile functions, and, since they are independent of the radius, the velocity profiles are assumed similar in shape for all \( r \). When equations (26) and (27) are substituted into equations (22) and (23), the equations become:

\[
\frac{d}{dr} \left( r v_\infty^2 \frac{\delta_1}{E} \right) f^2 d\eta + v_\infty^2 \frac{\delta_1}{E} \int_0^1 (1-g^2) d\eta = - rv_\infty f'(0) \frac{E}{\delta_1} \tag{28} 
\]

\[
\frac{d}{dr} \left( r v_\infty^2 \frac{\delta_1}{E} \right) f g d\eta - rv_\infty \frac{d}{dr} \left( r v_\infty \frac{\delta_1}{E} \right) \int_0^1 g d\eta = - r^2 v_\infty g'(0) \frac{1}{\delta_1} \tag{29} 
\]

where the primes refer to differentiation with respect to \( \eta \).

Equation (29) is rewritten in the form:

\[
rv_\infty \frac{\delta_1}{E} \frac{d}{dr} \left( r v_\infty \right) \left( \int_0^1 f d\eta - 2 \int_0^1 fg d\eta \right) 
\]

\[
+ r^2 v_\infty \frac{d}{dr} \left( \frac{\delta_1}{E} \right) \left( \int_0^1 f d\eta - \int_0^1 fg d\eta \right) = r^2 v_\infty g'(0) \frac{1}{\delta_1} \tag{30} 
\]

The following integrals of equations (28) and (30):

\[
I_1 = \int_0^1 f \ d\eta \\
I_2 = \int_0^1 f^2 \ d\eta \\
I_3 = \int_0^1 (1-g^2) \ d\eta \\
I_4 = \int_0^1 fg \ d\eta 
\]

are used to define four constants:

\[
A = \frac{I_3}{I_2}, \\
B = \frac{I_1 - 2I_4}{I_1 - I_4}, \\
C = \frac{f'(0)}{I_2} \\
D = \frac{g'(0)}{I_1 - I_4} 
\]

The constants \( A, B, C \) and \( D \) will be referred to as velocity profile constants. When equations (28) and (30) are expressed in terms of these
new constants, they take the form:

\[ \frac{d}{dr} \left( r v_\infty \delta_1 E^2 \right) + A v_\infty E \delta_1 = - C v_\infty \frac{E}{\delta_1} \]  \hspace{1cm} (33)

\[ \frac{d}{dr} (\delta_1 E) + B \left( \frac{1}{rv_\infty} \frac{d}{dr} (rv_\infty) \right) \delta_1 E = \frac{D}{v_\infty} \frac{1}{\delta_1^2} \]  \hspace{1cm} (34)

The above equations are rearranged in the form:

\[ \frac{d}{dr} \frac{d\delta_1^2}{dr} + \frac{E^2}{\delta_1} \frac{d\delta_1}{dr} + E^2 \frac{d}{dr} \left( \log rv_\infty \right) + \frac{A}{r} = - \frac{C}{v_\infty} \frac{E}{\delta_1^2} \]  \hspace{1cm} (35)

and

\[ \frac{1}{2} \frac{dE^2}{dr} + \frac{E^2}{\delta_1} \frac{d\delta_1}{dr} + BE^2 \frac{d}{dr} \left( \log rv_\infty \right) = \frac{D}{v_\infty} \frac{E}{\delta_1^2} \]  \hspace{1cm} (36)

From the above equations two separate equations for \( E^2 \) and \( \delta_1^4 \) are obtained. They are of the form:

\[ \frac{dB^2}{dr} = - 2E^2 \left[ \frac{d}{dr} \left( \log rv_\infty \right) + B \frac{d}{dr} \left( \log rv_\infty \right) \right] - \frac{2A}{r} - \frac{2(C+D)}{v_\infty} \frac{E}{\delta_1^2} \]  \hspace{1cm} (37)

\[ \frac{d\delta_1^4}{dr} = 4\delta_1^4 \left[ \frac{d}{dr} \left( \log rv_\infty \right) - 2B \frac{d}{dr} \left( \log rv_\infty \right) \right] + \]  \hspace{1cm} (38)

\[ + \frac{4A}{r} \left( \frac{\delta_1^2}{E} \right)^2 + \frac{4(C+2D)}{v_\infty} \frac{\delta_1^2}{E} \]

To obtain the starting behavior for \( E \) and \( \delta_1 \) it is necessary to prescribe the initial conditions for the equations (37) and (38). Since we are considering the situation where the cylinder is stationary it is evident that the radial velocity should satisfy the no-slip boundary conditions at the edge of the disk. Hence we can take \( E \) to be zero at the edge of the disk. For \( \delta_1 \) it is assumed that the boundary layer starts with zero thickness at the intersection of the end and side wall, so that \( \delta_1(1) = 0 \). Therefore the initial conditions are \( E(1) = 0 \) and \( \delta_1(1) = 0 \).

The starting behavior of \( E \) and \( \delta_1 \), is determined from equations (37) and (38) and the initial conditions. It is seen that if a solution exists, that is, finite derivatives of \( E^2 \) and \( \delta_1^4 \), \( E^2/\delta_1^2 \) and \( \delta_1^2/E \) must have
the same behaviour at \( r = 1 \). As a consequence \( E/\delta_1^2 \) is a constant and \( \delta_1^2 \sim E \). The differential equations then immediately establish that:

\[
E \sim (1-r)^4, \quad \delta_1 \sim (1-r)^4
\]

for \( r \approx 1 \). This initial behaviour of \( E \) and \( \delta_1 \) was found by Schultz-Grunow and Taylor for their problems.

Though the equations (37) and (38) are suitable for numerical integration, the problem of finding starting values is considerably simplified if the dependent variable \( \delta_1^4 \) is replaced by \( E/\delta_1^2 \).

The equation for \( E \delta_1^2 \) is found to be:

\[
\frac{d}{dr} (E\delta_1^2) = E\delta_1^2 \left[ \frac{d}{dr} \left( \log \left( r v_\infty^2 \right) \right) - 3B \frac{d}{dr} \left( \log \left( r v_\infty^2 \right) \right) \right] + \]

\[
+ \frac{A}{r} \frac{E\delta_1^2}{E^2} + \frac{C+3D}{v_\infty}
\]

Hence the two equations to be solved numerically are equations (37) and (40).

The starting behavior of \( E^2 \) and \( E\delta_1^2 \), from the equation (39), is given by:

\[
E^2 = a_1 (1-r) + \ldots
\]

\[
E\delta_1^2 = b_1 (1-r) + \ldots
\]

The co-efficients \( a_1 \) and \( b_1 \) are determined from the differential equations (37) and (40) and from the form of \( v_\infty \).

In the present analysis, since the cylinder is stationary, the tangential velocity at the edge of the boundary layer (i.e. \( v_\infty \)) should satisfy the no-slip condition at the edge of the disks and should follow a parabolic distribution. In view of the above fact, the tangential velocity at the edge of the boundary layer is assumed to be of the form:

\[
v_\infty = (r - ar^2)
\]

where \( a \) is a constant which takes values, \( 0 < a < 1 \).

The tangential velocity distribution is illustrated in fig. 2 for different values of \( a \) (0.9, 0.95 and 0.99). The equations (32) and (40) become singular for \( a = 1 \) hence the case \( a = 1 \) is avoided in the analysis.

When the above form of \( v_\infty \) is substituted into equations (37) and (40) the following relations for the co-efficients \( a_1 \) and \( b_1 \) are obtained in terms of the profile constants \( A, B, C \) and \( D \).

\[
a_1 = \frac{4AD}{3C + 5D}
\]
\[ b_1 = -\frac{40}{3} \cdot D \quad \text{for} \quad v_\infty = (r - 0.9 r^2) \quad (44) \]

\[ a_1 = \frac{4AD}{3C + 5D} \]

\[ b_1 = -\frac{80}{3} \cdot D \quad \text{for} \quad v_\infty = (r - 0.95 r^2) \quad (45) \]

and

\[ a_1 = \frac{4AD}{3C + 5D} \]

\[ b_1 = -\frac{400}{3} \cdot D \quad \text{for} \quad v_\infty = (r - 0.99 r^2) \quad (46) \]

In order to have a numerical solution for \( E^2 \) and \( E \delta_1^2 \) the velocity profile functions \( f(n) \) and \( g(n) \) must be so chosen that both \( a_1 \) and \( b_1 \) are positive otherwise the roots of either \( E^2 \) or \( E \delta_1^2 \) become imaginary.

C. EXPRESSIONS FOR THE IMPORTANT FLOW QUANTITIES.

When the solutions for \( E \) and \( \delta_1 \) are obtained it is easy to calculate the other quantities of interest. Of these quantities the outward radial mass flow is of primary importance because the magnitude of outward radial mass flow will give us an idea of how much total mass flow appears as secondary flow in the end wall boundary layers. The outward radial mass flow, \( M^* \), is given by:

\[ M^*(r^*) = 2\pi r^* \int_0^{\delta^*} \rho^* u^* dz \quad (47) \]

When the relation (26) is substituted into equation (47), the dimensionless mass flow, \( M \), defined by:

\[ M = \frac{M^*}{2\pi \rho^* \delta^* (\delta^*)^{\frac{1}{2}}} \quad (48) \]

is found to be:

\[ M(r) = -rv_\infty \delta EI_1 \quad (49) \]

Where \( I_1 \) is given by equation (31)

The axial outflow velocity, \( v_\infty^* \) can be obtained in two ways, by either integrating the continuity equation or by considering the mass-flow balance of the boundary layer as a whole. However, the latter method is
followed in this report. The total mass flow into an annular region of width \(dr^\text{r}^\text{r}\) must be zero. Therefore:

\[
(M^\text{r}^\text{r} + dM^\text{r}^\text{r}) - M^\text{r}^\text{r} - 2\pi r^\text{r} \rho^\text{r} \omega^\text{r} \omega^\text{r} dr^\text{r} = 0
\]  
(50)

and

\[
\omega^\text{r} (r^\text{r}) = \frac{1}{2\pi r^\text{r}} \frac{dM^\text{r}^\text{r}}{dr^\text{r}}
\]  
(51)

In dimensionless form:

\[
\omega^\infty (r) = \frac{1}{r} \frac{dM}{dr}
\]  
(52)

Using the relation (49) and the equations (37) and (40), the relation for axial out-flow velocity, in terms of the boundary-layer thickness, \(\delta_1\) and radial amplitude co-efficient \(E\), is found to be:

\[
\omega^\infty (r) = \left[ \frac{D}{\delta_1} + (1 - B) \frac{\delta_1 E}{r} \frac{d}{dr} (rv^\infty) \right] \frac{I_1}{I_1}
\]  
(53)

The radial shear stress at the lower disk is:

\[
\tau_{r^\text{r}} = \mu^\text{r} \left( \frac{\partial u^\text{r}}{\partial z} \right)_{z=0}
\]  
(54)

and the tangential shear stress at the lower disk is:

\[
\tau_{t^\text{r}} = \mu^\text{r} \left( \frac{\partial v^\text{r}}{\partial z} \right)_{z=0}
\]  
(55)

Defining the dimensionless shear stress as:

\[
\tau = \frac{\tau^\text{r}}{\rho^\text{r} u^\text{r}\delta_1^{3/2} v^\text{r}^{1/2}}
\]  
(56)

and with the use of equations (26) and (27), the dimensionless radial and tangential shear stresses at the surfaces of the lower and upper disks are expressed as:

\[
\tau_{r^\text{r}} (r) = \frac{\partial u^\text{r}}{\partial z}_{z=0} = f'(0) \frac{E v^\infty}{\delta_1}
\]  
(57)

and

\[
\tau_{t^\text{r}} (r) = \frac{\partial v^\text{r}}{\partial z}_{z=0} = g'(0) \frac{v^\infty}{\delta_1}
\]  
(58)
\[ \tau_r(r) = \left( \frac{\partial u}{\partial z} \right)_{z=L} = f'(0) \frac{E_{\infty}}{\delta_2} \]  

(57a)

and

\[ \tau_t(r) = (\frac{\partial v}{\partial z})_{z=L} = g'(0) \frac{v_{\infty}}{\delta_2} \]

(58a)

To calculate the fraction of total mass flow in end wall-boundary layers, the tangential Reynolds numbers \( R_t \) and \( R_{t'} \) are specified. The tangential Reynolds numbers referred to lower and upper disks are defined as,

\[ R_{t_1} = \frac{\rho \cdot v_1 \cdot r_1}{\mu}, \quad R_{t_2} = \frac{\rho \cdot v_2 \cdot r_2}{\mu} \]

(59)

In terms of \( R_{t_1} \) and \( R_{t_2} \), the dimensionless quantities which have previously been defined take the form:

\[ w = \frac{R_t^*}{v_1^*} \left( \frac{R_{t_1}}{r_1} \right)^{\frac{1}{2}}, \quad z = \frac{z^*}{r_1} \left( \frac{R_{t_1}}{r_1} \right)^{\frac{1}{2}} \]

lower disk

\[ M = \frac{M^*}{2\pi \rho \cdot v_1^* \cdot r_1 \cdot v_1^*} \left( \frac{R_{t_1}}{r_1} \right)^{\frac{1}{2}} \]

\[ \tau_r = \frac{\tau_r^*}{\rho \cdot v_1^* \cdot r_1 \cdot v_2^*} \left( \frac{R_{t_1}}{r_1} \right)^{\frac{1}{2}}, \quad \tau_t = \frac{\tau_t^*}{\rho \cdot v_1^* \cdot r_1 \cdot v_2^*} \left( \frac{R_{t_1}}{r_1} \right)^{\frac{1}{2}} \]

\[ w = \frac{w^*}{v_2^*} \left( \frac{R_{t_2}}{r_2} \right)^{\frac{1}{2}}, \quad z = \frac{z^*}{r_1} \left( \frac{R_{t_2}}{r_2} \right)^{\frac{1}{2}} \]

upper disk

\[ \tau_r = \frac{\tau_r^*}{\rho \cdot v_2^*} \left( \frac{R_{t_2}}{r_2} \right)^{\frac{1}{2}}, \quad \tau_t = \frac{\tau_t^*}{\rho \cdot v_2^* \cdot r_1 \cdot r_2 \cdot v_2^*} \left( \frac{R_{t_2}}{r_2} \right)^{\frac{1}{2}} \]  

(60a)
III POLYNOMIAL VELOCITY PROFILES

The usual procedure in applications of momentum-integral method is to represent the velocity profile functions \( f(\eta) \) and \( g(\eta) \) by polynomials. The deciding factor in the selection of polynomials is the boundary conditions on the velocity components \( u \) and \( v \). In the present case the fluid in the outer flow rotates in the same direction as the disks but with an angular velocity between the angular velocities of the disks. If \( \beta^* \) represents the angular velocity of the outer flow, then \( \Omega_2^* \leq \beta^* \leq \Omega_1^* \). To take this factor into account a dimensionless quantity \( \kappa \), is defined such that

\[
\kappa = \frac{2\beta^*}{\Omega_1^* + \Omega_2^*}
\]

The value of \( \kappa \) lies between 0 and 1.

The boundary conditions on \( u \) and \( v \) are:

\[
\begin{align*}
  u &= 0 & v &= v_\infty & \text{at } z = 0 & \text{lower disk} & (61) \\
  u &= 0 & v &= \kappa v_\infty & \text{at } z = \delta_1 \\
  u &= 0 & v &= v_\infty & \text{at } z = L \\
  u &= 0 & v &= \kappa v_\infty & \text{at } L = \delta_2 & \text{upper disk} & (62)
\end{align*}
\]

These are the minimum boundary conditions the polynomial profiles should satisfy. However, in order to be consistent with the derivation of the integral-momentum equations, the velocity components \( u \) and \( v \) should satisfy the conditions \( u' = 0 \) and \( v' = 0 \) at \( z = 0 \). But it is still possible to use the equations as derived for polynomials that do not satisfy these conditions. The above boundary conditions expressed in terms of \( f(\eta) \) and \( g(\eta) \) become:

\[
\begin{align*}
  f(0) &= 0 & g(0) &= 1 & \text{at } \eta = 0 & \text{lower disk} & (63) \\
  f(1) &= 0 & g(1) &= \kappa & \text{at } \eta = 1 \\
  f(0) &= 0 & g(0) &= 1 & \text{at } \xi = 0 \\
  f(1) &= 0 & g(1) &= \kappa & \text{at } \xi = 1 & \text{upper disk} & (64)
\end{align*}
\]

Conditions given by equation (64) are the same as equation (63) except for the change in variable. For further discussion only equation (63) will be mentioned.

In this report four different sets of profile functions, \( f(\eta) \), and \( g(\eta) \) are considered.
Set 1

\[ f(\eta) = \eta - \eta^2 \]

and

\[ g(\eta) = 1 - 10\eta (1-\kappa) + 9\eta^2 (1-\kappa) \quad (65) \]

In this set, \( f(\eta) \) and \( g(\eta) \) satisfy only the minimum boundary conditions given by equation (63).

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>25.950</td>
<td>1.130</td>
<td>30.000</td>
<td>-26.0869</td>
</tr>
<tr>
<td>0.67</td>
<td>14.829</td>
<td>0.6825</td>
<td>30.000</td>
<td>-26.0869</td>
</tr>
<tr>
<td>0.75</td>
<td>21.480</td>
<td>0.2808</td>
<td>30.000</td>
<td>-26.0869</td>
</tr>
</tbody>
</table>

Table 1. Velocity-profile constants referred to set 1.

Set 2

\[ f(\eta) = 5(\eta - \eta^2) \]

and

\[ g(\eta) = 1 - 3\eta (1-\kappa) + 3\eta^2 (1-\kappa) - \eta^3 (1-\kappa) \quad (66) \]

In this set the profile function \( g(\eta) \) satisfies additional condition \( g'(1) = 0 \)

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.802</td>
<td>-0.0833</td>
<td>6</td>
<td>-4.5</td>
</tr>
</tbody>
</table>

Table 2. Velocity-profile constants referred to set 2.

Set 3

\[ f(\eta) = 5(\eta - 3\eta^3 + 2\eta^4) \]

and

\[ g(\eta) = 1 - 3\eta (1-\kappa) + 3\eta^2 (1-\kappa) - \eta^3 (1-\kappa) \quad (67) \]

In this set the profile functions \( f(\eta) \), and \( g(\eta) \), satisfy the conditions \( f'(1) = 0 \), and \( g'(1) = 0 \) required by the momentum integral formulation in addition to satisfying the boundary conditions given by equation (63).

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.142</td>
<td>0.6841</td>
<td>6.666</td>
<td>-5.262</td>
</tr>
<tr>
<td>0.25</td>
<td>1.017</td>
<td>0.2457</td>
<td>6.666</td>
<td>-5.262</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8193</td>
<td>-0.193</td>
<td>6.666</td>
<td>-5.262</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7856</td>
<td>-0.6244</td>
<td>6.666</td>
<td>-5.262</td>
</tr>
</tbody>
</table>

Table 3. Velocity-profile constants referred to set 3.

The set 3 satisfies all the necessary conditions required by momentum-integral formulation and hence in the discussion of numerical results it is given more attention.
Set 4
\[ f(\eta) = 2(\eta - \eta^3) \]
and
\[ g(\eta) = 1 - \frac{3}{2} \eta (1-\kappa) + \frac{1}{4} \eta^3 (1-\kappa) \]  \hspace{1cm} (68)

In this set \( f(\eta) \) satisfies the minimum conditions given by equation (63), whereas, \( g(\eta) \) satisfies additional conditions \( g'(1) = 0 \) and \( g''(0) = 0 \) together with the conditions given by equation (63). The condition \( g''(0) = 0 \), is the tangential compatibility condition which is obtained by evaluating equation (12) at \( z = 0 \). However, the corresponding condition from equation (11) cannot be satisfied in the T-method as discussed earlier in the outline of the problem.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>2.089</td>
<td>-0.08331</td>
<td>6.562</td>
<td>-4.375</td>
</tr>
</tbody>
</table>

Table 4. Velocity-profile constants referred to set 4.

In this report only polynomial velocity profiles have been used. However, there is no restriction to use only polynomial functions. In fact any functions which satisfy the boundary conditions and resemble the actual velocity profiles are acceptable.

All the four sets of polynomial functions discussed above are also valid for the calculation of boundary layer on the upper disk.
IV RESULTS OF NUMERICAL INTEGRATIONS

A. REMARKS ON THE NUMERICAL INTEGRATIONS OF THE MOMENTUM-INTEGRAL EQUATIONS

The numerical problem is to obtain solutions of equations (37) and (40) from \( r = 1 \) to \( r \) near zero for the assumed velocity profiles and the different values of \( \kappa \) with the initial conditions \( E^2 = 0 \) and \( E\delta_1^2 = 0 \) at \( r = 1 \). Though such a computation appears to be straightforward the presence of the \( E^2/E\delta_1^2 \) term in equation (37) and the \( E\delta_1^2/E^2 \) term in the equation (40) introduces a complication. Both of these terms are of the form 0/0 at \( r = 1 \). Hence it is necessary to specify the initial derivatives which are the co-efficients of the first terms of the two series expansions given by equations (41) and (42) about \( r = 1 \). To continue with numerical integrations the values of \( E^2 \) and \( E\delta_1^2 \) were found from equations (41) and (42) at \( r = 0.995 \) and the derivatives at that point computed from the differential equations. With the starting values obtained in this manner, the modified Runge-Kutta-Gill method was used to continue the integrations inward to \( r = 0.01 \). A fixed step size of 0.005 was used from \( r = 0.995 \) to \( r = 0.940 \) and then the step size was increased to 0.01 at \( r = 0.940 \). The same step size was maintained till the end of the integration. For a given error bound the numerical results obtained by Runge-Kutta-Gill method did not exhibit any divergence in values when the step size was decreased. The results obtained with step size of 0.005 and 0.01 agreed up to fourth significant figure. However, the numerical integration exhibited initial oscillation which damped out rapidly.

Solutions were obtained for all the four sets of polynomial profiles listed earlier. For the set 3 detailed investigations were made of the behaviour of the solutions as functions of \( \kappa \), for values of \( \kappa \) between 0 and 0.5. Before the numerical integrations were started the programme used was checked for reliability of solutions feeding Taylor’s equations. The results obtained were quite in agreement with those of Taylor’s. The programme used is given in the appendix.

B. RESULTS OBTAINED WITH POLYNOMIAL VELOCITY PROFILES

The results of numerical integrations of equations (37) and (40) are presented in figs. 3a-12a.

Figures 3a and 4a represent the behaviour of the boundary layer thickness \( \delta_1 \) for \( v_\infty = r - 0.95 r^2 \) and \( v_\infty = r - 0.99 r^2 \) respectively. It is observed that for \( \kappa = 0 \) corresponding to free disk case (fig. 3a) boundary layer thickness \( \delta_1 \) is more or less constant between \( r = 0.45 \) and 0.85 and has finite value at \( r = 0 \). For other values of \( \kappa \), the boundary layer thickness \( \delta_1 \) attains a maximum and then falls off to zero. Also \( \delta_1 \) exhibits a point of inflexion for \( \kappa = 0.4 \) and 0.5. Similar trend is also observed for \( v_\infty = (r - 0.99 r^2) \) (fig. 4a).

The important flow quantities are the axial out-flow velocity and the radial mass flow. The solution for axial out-flow velocity is given in equation (53). Equation (53) reveals that the solution obtained by von
Kármán's-T method exhibits a singularity at $r = 1$ due to the initial condition $\delta_1 = 0$ at $r = 1$. From physical considerations, for the model considered in this report, $w_\infty$ should be zero at $r = 1$, as $w_\infty$ has to satisfy the no-slip condition. Hence it may be considered that the solution obtained for $w_\infty$ is valid for points other then $r = 1$. The axial out-flow velocity is presented in figs. 5a and 6a. It is interesting to observe that for values of $\kappa$ up to 0.5 the axial out-flow velocity $w_\infty$ is inwards at $r = 1$ and outward at $r = 0$ as expected. However, for $\kappa = 0.5$ the axial flow velocity is inwards at $r = 1$ and again near $r = 0$ exhibiting a type of cellular flow. The same trend is observed for $v_\infty = (r - 0.99 r^2)$ and $\kappa = 0.5$ (fig. 6a). In fig. 10a the axial out-flow velocity $w_\infty$ related to the set 4 of the polynomial functions is presented. Even in that case also $w_\infty$ exhibits the similar trend as mentioned above for $v_\infty = (r - 0.99 r^2)$ and $\kappa = 0.3$. Hence it can be surmised that for a given velocity profile function there is a critical value of $\kappa$ above which the axial out-flow velocity exhibits a cellular type of flow.

In other words, if the flow is as represented by the assumed velocity profiles then we can expect inward and outward flowing layers in the axial out-flow velocity $w_\infty$ above a critical value of $\kappa$. The only evidence that can be adduced for the existence of this type of axial flow pattern is from experimental observation. However, to the author's knowledge there are no experimental observations on the flows involving the model considered in this report.

In fig. 11a the radial mass flow is presented for $v_\infty = (r - 0.95 r^2)$ and $\kappa = 0.25$ and $\kappa = 0.5$. It is observed that for $\kappa = 0.25$ the radial mass flow attains a maximum value at $r = 0.55$ and then falls off to zero. For $\kappa = 0.5$ the radial mass flow attains a maximum at $r = 0.655$ and falls to a minimum at $r = 0.23$ and then increases in magnitude near $r = 0$. The behaviour of $M$ with the radius for $\kappa = 0.5$ confirms the prediction that flow conditions in the boundary layer as well as the outer flow are different for values of $\kappa < 0.5$.

C. FRACTION OF TOTAL MASS FLOW IN END-WALL BOUNDARY LAYERS

The dimensional maximum outward radial mass flow in the boundary layer can be written from equation (60) as:

$$M_{\text{max}}^* = 2\pi u^*_1 r^*_1 (R t^*_1)^{1/2} M_{\text{max}}^*(k)$$

(69)

It may be recalled that $R t^*_1 = v^*_1 r^*_1 / \nu^*$ is the tangential Reynolds number at $r = 1$ and $M_{\text{max}}^*(k)$ is the dimensionless mass flow (shown in fig. 11a) which is a function of $k$.

The total mass flow $M_t^*$ can be written as:

$$M_t^* = 2\pi u^*_1 r^*_1 L^*$$

(70)

where $L^*$ is the length of the cylinder. $u^*_1$ appearing in equation (70) is a fictitious radial velocity which is used to define a second Reynolds number:
\[ \text{Re}_1 = \frac{\rho \frac{u_1}{r_1}}{\mu} \] (71)

which is called the radial Reynolds number. Equation (70) is rewritten, using equation (71) in the form:

\[ M_{t}^{x} = 2\pi \mu \frac{L^{x} \text{Re}_1}{r_1} \] (72)

The ratio of radial mass flow into end-wall boundary layers to the total mass flow through the cylinder is from equations (69) and (72):

\[ \frac{2M_{\text{max}}^{x}}{M_{t}^{x}} = 2 \frac{r_1^{x}}{L^{x}} \left( \text{Re}_1 \right)^{1/2} \frac{M_{\text{max}}}{\text{Re}_1} \] (73)

The quantity \( L^{x}/r_1^{x} \) is the aspect ratio of the cylinder.

Figure 12a has been prepared from equation (73) for \( 10^3 < \text{Re}_1 < 10^5 \), \( 50 < \text{Re}_1 < 1000 \) and \( k = 0.5 \). This figure in which:

\[ \frac{(L^{x})}{(r_1^{x})} \left( \frac{2M_{\text{max}}^{x}}{M_{t}^{x}} \right) \]

is plotted against \( \text{Re}_1 \) for two different values of \( \text{Re}_1 \) may be considered as giving, for a fixed \( \text{Re}_1 \) and \( \text{Re}_1 \) the aspect ratio of the cylinder for which the total mass flow would be entirely in the end-wall boundary layers, provided the tangential velocity distribution remains unchanged. Fig. 12a exhibits that the aspect ratio of the cylinder, for which the total mass flow would be entirely in the boundary layers, goes on decreasing with increasing values of \( \text{Re}_1 \). Hence to reduce the secondary flow in the end wall boundary layers it is necessary to increase \( \text{Re}_1 \), in other words the total mass flow or to decrease the tangential Reynolds number \( \text{Re}_1 \). The same effect can also be achieved by increasing the length of the cylinder for a fixed radius.
REFERENCES


APPENDIX

Programme used for the numerical integration.

start
  /* rungku * lost een stelsel van n le-orde diff. vgl op */;
  IF "first THEN GO TO start2;
  first, text = 0;
  ON ATTENTION GO TO start1;
  PUT LIST ("Wilt U tekst en uitleg");
  text=antwr;
  IF "text THEN GO TO start1;
  PUT LIST ("Dit programma lost een stelsel van n le-orde differen-");
  PUT LIST ("tiaalvergelijkingen op over het interval (a,b). Er wordt");
  PUT LIST ("een Runge Kutta-methode gebruikt waarbij de stapgrootte");
  PUT LIST ("in het programma zelf wordt bepaald n.a.v. een door U op");
  PUT LIST (" te geven nauwkeurigheid eps. U moet zorgen voor een");
  PUT LIST ("procedure fct(X,Y,DERY) waarin het stelsel beschreven");
  PUT LIST ("wordt en wel zo dat het array DERY de n afgeleiden bevat");
  PUT LIST ("in het punt (X,Y(1),Y(2),....,Y(n)).");
  PUT LIST (";

start1:  ON ATTENTION SYSTEM;
  PUT LIST ("Hebt u de procedure fct(X,Y,DERY) al gedefinieerd?");
  IF antwr THEN GO TO start2;
  PUT LIST ("Doe dat dan nu in statements met nrs 200 en hoger.");
  STOP ;

outp:    PROCEDURE;
  PUT LIST (";
  PUT LIST ('x=',x);
  DO i=1 TO n;
    PUT LIST ('y(','i',')=',y(i));
  END ;
  END outp;
  DECLARE A(4),B(4),C(4);
  DECLARE y(n) CONTROLLED ,dery(n) CONTROLLED , aux(n,8) CONTROLLED ,
  ,xoutp (m) CONTROLLED;
start2:  IF text THEN GO TO start3:
       PUT LIST ('Geef nu achtereenvolgens a,b (intervalgrenzen), n')
       PUT LIST ('(aantal vergelijkingen), stap (eerste stapgrootte)')
       PUT LIST ('eps (nauwkeurigheid) en rmax (max aantal interval-')
       PUT LIST ('halveringen).');
start3:  GET LIST (a,b,n,stap,eps,rmax);
       hp=b-a;
       IF MOD(hp,stap)<.5 THEN m=FLOOR(hp/stap); ELSE m=CEIL(hp/stap);
       IF alloca(y) THEN FREE y,der,y,aux,xoutp;
       ALLOCATE y,der,y,aux,xoutp;
       GET LIST (y);
       dery=1/n;
       DO i=1 TO m;
       xoutp(i)=a+i*stap;
       END;
L1:     DO i=1 TO n;
       aux(i,8)=.0666666666666667*dery(i);
       END L1;
       x=a;
       h=stap;
       CALL fct(x,y,dery);
       test=h*(b-x);
       IF test<0 THEN GO TO L38;
       IF test=0 THEN GO TO L37;
       A(1),C(1),C(4)=.5;
       A(2),C(2)=.29289321881345;
       A(3),C(3)=1.7071067811865;
       A(4)=.1666666666666666;
       B(1),B(4)=2;
       B(2),B(3)=1;
L3:     DO i=1 TO n;
       aux(i,1)=y(i);
       aux(i,2)=dery(i);
aux(i,3),aux(i,6)=0;
END L3;
irec, istep, iend=0;
ihlf=-1;
k=1;
.L4:
test=(x+h-b)xh;
IF test<0 THEN h=b-x;
IF ABS (x-xoutp(k))>.2E-10 THEN GO TO ON;
CALL outp;
k=k+1;
ON: itest=0;
L9: istep=istep+1;
j=1;
L10
aj=A(j);
bj=B(j);
cj=C(j);
L11: DO i=1 TO n;
    rl=hmdery(i);
    r2=aj*bj*aux(i,6);
    y(i)=y(i)+r2;
    r2=r2+r2+r2;
    aux(i,6)=aux(i,6)+r2-cj*rl;
END L11;
IF j>=4 THEN GO TO L15;
    j=j+1;
IF j>=3 THEN x=x+.5xh;
CALL fct(x,y,dery);
GO TO L10;
L15 IF itest>0 THEN GO TO L20;
L17 DO i=1 TO n;
aux(i,4)=y(i);
END L17;
itest=1;
istep=istep+istep-2;
L18:  ihlf=ihlf+1;
    x=x-h;
    h=.5wh;
L19:  DO i=1 TO n;
    y(i)=aux(i,1);
    dery(i)=aux(i,2);
    aux(i,6)=aux(i,3);
END L19;
    GO TO L9;
L20:  imod=trunc(istep/2);
    IF istep-imod-imod=0 THEN GO TO L23;
    CALL fct(x,y,dery);
L22:  DO i=1 TO n;
    aux(i,5)=y(i);
    aux(i,7)=dery(i);
END L22;
    GO TO L9;
L23:  delt=0;
L24:  DO i=1 TO n;
    delt=delt+aux(i,8)*wabs(aux(i,4)-y(i));
END L24;
    ep=eps*wh/stap;
    IF delt<=ep THEN GO TO L28;
    IF ihlf>=rmax THEN GO TO L36;
L27:  DO i=1 TO n;
    aux(i,4)=aux(i,5);
END L27;
    istep=istep+istep-4;
L28:  CALL fct(x,y,dery);
L29:  DO i=1 TO n;
    aux(i,1)=y(i);
aux(i,2)=dery(i);
aux(i,3)=aux(i,6);
y(i)=aux(i,5);
dery(i)=aux(i,7);
END L29;

L31: DO i=1 TO n;
y(i)=aux(i,1);
dery(i)=aux(i,2);
END L31;
IF ABS(x-b)<.2E-10 THEN GO TO L39;
irec=ihlf;
ihlf=ihlf-1;
istep=trunc(istep/2);
h=h+h;
IF ihlf<0 THEN GO TO L4;
imod=trunc(istep/2);
IF istep-imod-imod=0 THEN GO TO L4;
IF delt>.02xep THEN GO TO L4;
ihlf=ihlf-1;
istep=trunc(istep/2);
h=h+h;
GO TO L4;

L36: PUT LIST ('Het aantal intervalhalveringen is groter dan ',r);
CALL fct(x,y,dery);
GO TO L39;

L37: PUT LIST ('a=b');
GO TO L39;

L38: PUT LIST ('a>b');
L39: CALL outp;
PUT EDIT ('**EINDE**') (SKIP(2),A);
antwrd: PROCEDURE;
DECLARE answer CHAR(3), ANSWER CHAR(3);
opnieuw READ INTO (answer);
ANSWER=upcase(answer);
IF ANSWER='JA' OR ANSWER='YES' OR ANSWER='1' THEN RETURN (1);
IF ANSWER='NEE' OR ANSWER='NO' OR ANSWER='O' THEN RETURN (0);
PUT LIST ('''answer,''' begrijp ik niet. Opnieuw.'');
GO TO opnieuw;
END antwr;

_XEQ
Wilt U tekst en uitleg?

_NEE
Hebt U de procedure fct(X,Y,DERY) al gedefinieerd?

_NEE
Doe dat dan nu in statements met nrs 200 en hoger.

_XEQ 19.XEQ "STOP".

fct: procedure (r,z,dz)
q1=(3.-4.95xr)/(r-.99xr*2)
q2=(2.-2.97xr)/(r-.99xr*2)
dz(1)=-2.xz(1)x(q1+.193xz2)-1.7826/r+2.808xz(1)/z(2)/(r-.99xr*2)
dz(2)=z(2)x(q1+.579xz2)+.8913xz(2)/(rz(1))-9.120/(r-.99xr*2)
end fct
PART B

"THE MAGNETOHYDRODYNAMIC FLOW BETWEEN ENCLOSED ROTATING DISKS"
I INTRODUCTION

The magneto-hydrodynamic flow over a single disk has been investigated by several authors. Rizvi [1] has examined the problem of the steady rotation of a disk in a weak magnetic field. He assumes the disk to be a perfect conductor which yields an indeterminate electric field. Sparrow and Cess [2] investigated the case of an insulating rotating disk in a conducting fluid whose angular velocity is zero at large distances from the disk. The solution of the problem was obtained by numerical integration. Similar problem was solved by Katutani [3] by joining the two expansions. King and Lewellen [4] examined the behaviour of a rotating fluid over a stationary infinite disk with or without an axial magnetic field. They obtained the solution by numerical integration for power-law variation in external tangential velocity (i.e. \( v_0 r^n \) where \(-1 < n < 1\).

The magneto-hydrodynamic flow between a rotating and a stationary disk has been analysed by Srivatsava and Sharma [5]. Their solution is valid for small values of \( R \) and arbitrary values of Hartmann number \( m \). They find that both the shearing stress and force of suction at the stationary disk decrease with an increase in \( m \) which is qualitatively similar to the effect of elastic and cross viscous forces. Stephenson [6] treated the case of conducting flow between a rotating and a stationary disk including the effect of radial electric field which was omitted by Srivatsava and Sharma. He obtained asymptotic solutions for \( R < m^2 \) on the basis of similarity and numerical solutions for arbitrary \( R \) and \( m \). His experimental results show a disposition in favour of the theoretical similarity flows. Chandrasekhar and Rudraiah [7-8] extended Stephenson's problem to the case of porous disks with the application of uniform injection or suction at both the disks. They obtained asymptotic solutions valid for \( R_1 < m^2 \) and compared their results with those of Stephenson's. They found that the effect of injection or suction predominates over the effect of rotation and suppresses the inward or outward flowing layers in the radial velocity making it uni-directional.

The present analysis is confined to the study of conducting flow, between two rotating disks enclosed in a stationary cylinder in the presence of a transverse uniform magnetic field. The solution for the end-wall boundary layers is obtained using von Kármán's -T method and numerical integration. The results of numerical integration are presented in graphs and discussed.
II A. FORMULATION OF THE PROBLEM

A uniform incompressible fluid of density $\rho^\mu$, viscosity $\nu^\mu$ and electrical conductivity $\sigma^\mu$ is contained between two non-conducting disks of radius $r^1_1$ situated at $z^\mu = 0$ and $z^\mu = L^\mu$ in a stationary co-axial cylinder of length $L^\mu$ and radius $r^1$. The disks have thickness $t^a_1$ and $t^b_1$ and electrical conductivity $\sigma^a_1(z^\mu)$ and $\sigma^b_1(z^\mu)$. The disks make contact with the fluid between but are electrically isolated from the environment. They rotate with angular velocities $\Omega^a_1$ and $\Omega^b_2$ ($\Omega^a_1 > \Omega^b_2$) in the same direction. A uniform magnetic field $B$ is applied in the axial direction. (Fig.1b).

The equations of motion governing the steady axi-symmetric flow are:

\[ \rho^\mu (\vec{q} \cdot \vec{\nabla}) \vec{q} = - \vec{\nabla} P + \nu^\mu \nabla^2 \vec{q} + \vec{J} \times \vec{B} \]  
(1)

\[ \nabla \cdot \vec{q} = 0 \]  
(2)

\[ \nabla \times \vec{B} = \mu_m \sigma^\mu (\vec{E} + \vec{q} \times \vec{B}) \]  
(3)

\[ \nabla \cdot \vec{B} = 0 \]  
(4)
\[ \nabla \times \vec{E} = 0 \quad \text{(5)} \]
\[ \vec{J} = \sigma \vec{E} + \nabla \times \vec{B} \quad \text{(6)} \]
\[ \nabla \cdot \vec{J} = 0 \quad \text{(7)} \]

Equation (5) and the assumption that the flow is axi-symmetric leads to:
\[ \vec{E}_\theta = 0 \quad \text{everywhere} \quad \text{(8)} \]

Further, if the induced magnetic field \( b \ll B \) then \( \vec{B} = \hat{z}B \). Under the above assumptions it is found that:
\[ J_\theta = - \sigma \nabla \cdot \vec{B} \quad \text{(9)} \]
\[ J_r = \sigma (E_r + \nabla \times \vec{B}) \quad \text{(10)} \]

Stephenson [6] from similarity considerations found that \( E_r \), the induced radial electric field is a function of \( r \) only and can be expressed as:
\[ E_r = - \chi B \Omega_1 r \quad \text{(11)} \]

The quantity \( \chi \) gives the strength of the induced electric field \( E_r \).

Integrating equation (11) between \( z^* = -t_a \) and \( z^* = t_b + L^* \) and putting:
\[ \int_{-t_a}^{t_b} j_{r} d z^* = 0 \quad \text{(12)} \]

An expression for \( \chi \) is obtained and it is written in the form:
\[ \chi = \int_{-t_a}^{t_b} \sigma(z^*) v^* d z^*/r \Omega_1 \int_{-t_a}^{t_b} \sigma(z^*) d z^* \quad \text{(13)} \]

Letting:
\[ s = \int_{-t_a}^{t_b} \sigma(z^*) d z^* \quad \text{(14)} \]

and introducing the dimensionless parameters of conductivity:
\[ S_1 = \frac{1}{s} \int_{-t_a}^{0} \sigma(z^*) d z^* \quad \text{lower disk} \quad \text{(15)} \]
\[ S_j = \frac{\sigma L}{s} \quad \text{fluid} \quad (16) \]
\[ S_2 = \frac{1}{s} \int_L^{*} t(z^*) \, dz^* \quad \text{upper disk} \quad (17) \]

It can be shown from equation (13) that,
\[ \chi = \frac{S_1 \Omega_1^* + S_2 \omega_{av} + S_2 \Omega_2^*}{\Omega_1^*} \quad (18) \]

Where \( \omega_{av} \) is the average angular velocity of the fluid.

For non-conducting disks considered in this report, \( \sigma_{a}^* \) and \( \sigma_b^* \) are zero and hence \( S_1 \) and \( S_2 \) are both zero. Therefore the expression for \( \chi \) reduces to:
\[ \chi = \frac{\omega_{av}}{\Omega_1^*} \quad (19) \]

\( \chi \) depends on the average velocities of the disks and fluid. Substituting equation (19) into equation (10), the relation for \( J_r \) takes the form:
\[ J_r = \sigma^* (v^* B - \chi \Omega_1^* Br^*) \quad (20) \]

B. DERIVATION OF MOMENTUM-INTEGRAL EQUATIONS

The governing equations of motion, equation (1), written in scalar form become:
\[ u^* \frac{\partial u^*}{\partial r^*} + v^* \frac{\partial u^*}{\partial z^*} - \frac{v^*}{r^*} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial r^*} + \bar{v}^* \]
\[ \left[ \frac{\partial^2 u^*}{\partial r^*^2} + \frac{\partial}{\partial r^*} \left( \frac{u^*}{r^*} \right) + \frac{\partial^2 u^*}{\partial z^*^2} \right] + \frac{J_B}{\rho^*} \quad (21) \]
\[ u^* \frac{\partial v^*}{\partial r^*} + v^* \frac{\partial v^*}{\partial z^*} + \frac{u^* v^*}{r^*} = v^* \left[ \frac{\partial^2 v^*}{\partial r^*^2} + \frac{\partial}{\partial r^*} \left( \frac{v^*}{r^*} \right) + \frac{\partial^2 v^*}{\partial z^*^2} \right] - \frac{J_B}{\rho^*} \quad (22) \]
\[ u^* \frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*} + v^* \left[ \frac{\partial^2 w^*}{\partial r^*^2} + \frac{1}{r^*} \frac{\partial w^*}{\partial r^*} + \frac{\partial^2 w^*}{\partial z^*^2} \right] \quad (23) \]
Substituting for $J_r$ and $J_\theta$ from equations (9) and (20) into equations (21) and (22), the radial and azimuthal momentum equations take the form:

\[
\begin{align*}
\frac{u}{r} \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v}{r^2} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \left( \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] - \frac{\sigma}{\rho} B^2 u \\
\frac{v}{r} \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{u w}{r} &= \nu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2} \left( \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial z^2} \right] - \frac{\sigma}{\rho} B^2 v \\
- \frac{\sigma}{\rho} B^2 v + B^2 \Omega^2 r^2 \chi &= \frac{\sigma}{\rho}
\end{align*}
\]  

Equations (24), (25) and (23), under the boundary layer approximation reduce to:

\[
\begin{align*}
\frac{u}{r} \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v}{r^2} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \left( \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] - \frac{B^2 u}{\rho} \\
\frac{v}{r} \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{u w}{r} &= \nu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2} \left( \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial z^2} \right] - \frac{B^2 v}{\rho} + \chi B^2 \Omega^2 r \frac{\sigma}{\rho} \\
0 &= -\frac{1}{\rho} \frac{\partial p}{\partial z}
\end{align*}
\]  

The continuity equation (2) takes the form:

\[
\frac{\partial}{\partial r} (r u) + \frac{\partial}{\partial z} (r w) = 0
\]  

Introducing the dimensionless variables defined in equation (10) of part A, equations (26) and (27) take the form:
\[ u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{v_\infty^2}{r} + \frac{\partial^2 u}{\partial z^2} - \frac{m^2}{Rt_1} u \] \tag{30}

\[ u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + uv = \frac{\partial^2 v}{\partial z^2} - \frac{m^2}{Rt_1} v + \frac{m^2}{Rt_1} \chi r \] \tag{31}

Where \( m \) is the Hartmann number \( m = 2 r_1 x_0^H / \rho^H \nu^H \)

\[ Rt_1 = \frac{\delta_{x} r}{{\nu}^H} \] is the tangential Reynolds number.

The continuity equation is expressed as:

\[ \frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (rw) = 0 \] \tag{32}

Following the same procedure as in part A, the integrated momentum equations for the lower and the upper disk are obtained. They are:

**Lower disk**

\[ \frac{d}{dr} (\delta_1 r \int_0^1 u^2 d\eta) + \delta_1 \int_0^1 (v\omega - v^2) d\eta = -\frac{r}{\delta_1} \frac{\partial u}{\partial \eta} \bigg|_{\eta=0} - \frac{m^2}{Rt_1} \delta_1 r \int_0^1 u d\eta \] \tag{33}

\[ \frac{d}{dr} (\delta_1 r \int_0^1 uvd\eta) - rv_\infty \frac{d}{dr} (\delta_1 r \int_0^1 u d\eta) = -\frac{r^2}{\delta_1} \frac{\partial v}{\partial \eta} \bigg|_{\eta=0} - \frac{m^2}{Rt_1} \delta_1 r \int_0^1 v d\eta + \frac{m^2}{Rt_1} \delta_1 r^3 \int_0^1 \chi d\eta \] \tag{34}

**Upper disk**

\[ \frac{d}{dr} (\delta_2 r \int_0^1 u^2 d\xi) + \delta_2 \int_0^1 (v\omega - v^2) d\xi = -\frac{r}{\delta_2} \frac{\partial v}{\partial \xi} \bigg|_{\xi=0} - \frac{m^2}{Rt_1} \delta_2 r \int_0^1 u d\xi \] \tag{35}
\[
\frac{d}{dr} \left( \delta_2 r^2 \int_0^1 uv \, d\xi \right) - \nu v_{\infty} \frac{d}{dr} \left( \delta_2 r \int_0^1 ud\xi \right) = - \frac{r^2}{\delta_2} \frac{\partial v}{\partial \xi} \bigg|_{\xi=0} - \\
- \frac{m^2}{R \tau_1} \delta_2 r^2 \int_0^1 v d\xi + \frac{m^2}{R \tau_1} \delta_2 r^3 \int_0^1 \chi d\xi
\]

where the variables \( \eta_1 = \frac{z}{\delta_1} \) and \( \xi = \frac{1-z}{\delta_2} \).

For further discussion only one set of equations (33) and (34) is considered. Making use of the relations (26) and (27) of part A, equations (33) and (34) become:

\[
\frac{d}{dr} \left( r v_{\infty}^2 \delta_1 E^2 \right) \int_0^1 f^2 \, d\eta + \nu v_{\infty} \frac{d}{dr} \left( \delta_1 g \int_0^1 \left( 1-g^2 \right) \, d\eta \right) = \\
= - r v_{\infty} f'(0) \frac{E}{\delta_1} - \frac{m^2}{R \tau_1} \delta_1 r E v_{\infty} \int_0^1 f \, d\eta
\]

\[
\frac{d}{dr} \left( r^2 v_{\infty} \delta_1 E \right) \int_0^1 f g \, d\eta - r v_{\infty} \frac{d}{dr} \left( r v_{\infty} \delta_1 E \right) \int_0^1 f \, d\eta = \\
= - r^2 v_{\infty} \frac{g'(0)}{\delta_1} - \frac{m^2}{R \tau_1} \delta_1 r^2 v_{\infty} \int_0^1 g \, d\eta + \frac{m^2}{R \tau_1} \delta_1 r^3 \int_0^1 \chi \, d\eta
\]

In the last term of equation (38), \( \chi \) represents the term due to the induced radial electric field. \( \chi \) depends on the average angular velocities of the fluid in the entire region between the disks and the average angular velocity of the disks. In the present analysis attention is directed to obtain the solution for the end-wall boundary layers and if the effect of induced electric field is to be included in the discussion, the value of \( \chi \) is to be evaluated in the boundary layers. However, it is found that if \( \chi \) is evaluated in the boundary layer, the term due to \( \chi \) and the term due to the magnetic field cancel each other making the second integrated equation of momentum independent of the effect of the magnetic field.

From equation (19) \( \chi = \frac{\omega_{av}}{\Omega_1} \).
\[ \omega_{av} = \frac{v_{av}}{r} = \frac{\Omega_1 r_{\infty}}{r} \frac{1}{\delta_1} \int_0^{\delta_1} g \, dz \]

\[ = \frac{\Omega_1 r_1 v_{\infty}}{rr_1} \int_0^{\delta_1} g \, d\eta \quad (39) \]

Hence \( \chi = \frac{v_{\infty}}{r} \int_0^{\delta_1} g \, d\eta \quad (40) \)

If equation (40) is substituted into equation (38), it is found that
the last two terms cancel each other. Hence to study the effect of
magnetic field on the boundary layer the term due to induced electric
field is neglected in the boundary layer for further discussion.
Neglecting the last term in equation (38) and rearranging, equation (38)
becomes:

\[ rv_{\infty} \delta_1 E \frac{d}{dr} (rv_{\infty}) (\int_0^{\delta_1} f d\eta - 2 \int_0^{\delta_1} f g d\eta) + r^2 v_{\infty} \frac{d}{dr} (\delta_1 E) \]

\[ (\int_0^{\delta_1} f d\eta - \int_0^{\delta_1} f g d\eta) = r^2 v_{\infty} \frac{g'(0)}{\delta_1} + \frac{m^2}{\delta_1} r^2 v_{\infty} \int_0^{\delta_1} g d\eta \quad (41) \]

The five integrals:

\[ I_1 = \int_0^{\delta_1} f \, d\eta \quad I_2 = \int_0^{\delta_1} f^2 \, d\eta \]

\[ I_3 = \int_0^{\delta_1} (1 - g^2) \, d\eta \quad I_4 = \int_0^{\delta_1} f \, d\eta \]

\[ I_5 = \int_0^{\delta_1} g \, d\eta \]

are used to define six constants A, B, C, D, G and H and these are
referred to as velocity profile constants.

\[ A = \frac{I_3}{I_2} \quad B = \frac{I_1 - 2I_4}{I_1 - I_4} \quad C = \frac{f'(0)}{I_2} \quad (43) \]

\[ D = \frac{g'(0)}{I_1 - I_4} \quad G = \frac{I_1}{I_2} \quad H = \frac{I_5}{I_1 - I_5} \]
In terms of profile constants equations (37) and (41) become:

\[
\frac{d}{dr} \left( rv_\infty^2 \delta_1 E^2 \right) + Av_\infty^2 \delta_1 = - Crv_\infty \frac{E}{\delta_1} - \frac{m^2}{R_t} \left( \delta_1 rv_\infty E^2 G \right) \tag{44}
\]

\[
\frac{d}{dr} (\delta_1 E) + B \left[ \frac{1}{rv_\infty} \frac{d}{dr} (rv_\infty) \right] \delta_1 E = \frac{D}{v_\infty} \delta_1 + \frac{m^2}{R_t} \frac{\delta_1^H}{v_\infty} \tag{45}
\]

Following the method in part A, we finally arrive at the two equations to be solved numerically. They are:

\[
\frac{dE^2}{dr^2} = - 2E^2 \left[ \frac{d}{dr} (\log rv_\infty^2) + \frac{2A}{r} \right] - 2 \left( \frac{C+D}{v_\infty} \right) \frac{E^2}{E_\delta^2} - \frac{2m^2}{R_t} \frac{E^2}{E_\delta^2} \tag{46}
\]

\[
\frac{d}{dr} (E\delta_1^2) = E\delta_1^2 \left[ \frac{d}{dr} (\log rv_\infty^2) - 3B \frac{d}{dr} (\log rv_\infty) \right] + \frac{A}{r} \frac{E^2 \delta_1^2}{E} \tag{47}
\]

The starting behaviour of \( E^2 \) and \( E\delta_1^2 \) is as given by equations (41) and (42) of part A. From which the constants \( a_1 \) and \( b_1 \) are determined using the differential equations (46) and (47). It is found that:

\[
a_1 = \frac{4AD}{3C+5D} \quad b_1 = \frac{40}{3} D \quad \text{for} \quad v_\infty = (r-0.95r^2) \tag{48}
\]

\[
a_1 = \frac{4AD}{3C+5D} \quad b_1 = - \frac{400}{3} D \quad \text{for} \quad v_\infty = (r-0.99r^2) \tag{49}
\]

It may be observed from equations (48) and (49), that the starting behaviour of \( E^2 \) and \( E\delta_1^2 \) remains the same even in the presence of a transverse magnetic field.
C. EXPRESSIONS FOR IMPORTANT FLOW QUANTITIES

The dimensionless radial mass flow in the presence of a magnetic field is found to be:

$$ M(r) = r v_\infty \delta_1 E I_1 $$ (50)

The axial outflow velocity is obtained as before by considering the mass flow balance of the boundary layer as a whole. The expression for the axial outflow velocity is written as:

$$ w_\infty (r) = \left( \frac{D}{\delta_1} + (1-B) \delta_1 \frac{E}{r} \frac{d}{dr} (rv_\infty) + \frac{m^2}{R t_1} H \delta_1 \right) I_1 $$ (51)

The radial shear stress at the surface of the lower disk is:

$$ \tau_r (r) = (\frac{\partial u}{\partial z})_{z=0} = f'(0) \frac{E v_\infty}{\delta_1} $$ (52)

The tangential shear stress at the surface of the lower disk is:

$$ \tau_t (r) = (\frac{\partial v}{\partial z})_{z=0} = g'(0) \frac{v_\infty}{\delta_1} $$ (53)
III POLYNOMIAL VELOCITY PROFILES AND RESULTS OF NUMERICAL INTEGRATIONS

To solve equations (46) and (47) the profile constants $A$, $B$, $C$, $D$, $G$ and $H$ are to be specified. These constants depend on the profile functions assumed. In this analysis three sets of profiles are used. The profiles satisfy the following boundary conditions.

\[
\begin{align*}
  f(0) &= 0 & g(0) &= 1 & \text{at } \eta = 0 & \text{lower disk} \\
  f(1) &= 0 & g(1) &= \kappa & \text{at } \eta = 1 \\
  f(0) &= 0 & g(0) &= 1 & \text{at } \xi = 0 & \text{upper disk} \\
  f(1) &= 0 & g(1) &= \kappa & \text{at } \xi = 1
\end{align*}
\]

The last three sets of polynomial functions of part A are used for the conducting flow case.

\[f(\eta) = 5(\eta^2 - 3\eta^3 + 2\eta^4)\]

\[g(\eta) = 1 - 3\eta(1 - \kappa) + 3\eta^2(1 - \kappa) - \eta^3(1 - \kappa)\]

As discussed earlier this set satisfies all the boundary conditions and is consistent with the derivation of momentum-integral equations which requires the conditions that $f'(1) = 0$ and $g'(1) = 0$ at $\eta = 1$. The constants $A$, $B$, $C$, $D$, $G$ and $H$ are evaluated for different values of $\kappa$, the dimensionless angular velocity and the values are tabulated as shown in table 1.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.142</td>
<td>0.6841</td>
<td>6.666</td>
<td>-5.262</td>
<td>1.0</td>
<td>0.4385</td>
</tr>
<tr>
<td>0.25</td>
<td>1.017</td>
<td>0.2457</td>
<td>6.666</td>
<td>-5.262</td>
<td>1.0</td>
<td>1.023</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8193</td>
<td>-0.193</td>
<td>6.666</td>
<td>-5.262</td>
<td>1.0</td>
<td>1.608</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7856</td>
<td>-0.6244</td>
<td>6.666</td>
<td>-5.262</td>
<td>1.0</td>
<td>2.194</td>
</tr>
</tbody>
</table>

Table 1. Velocity-profile constants referred to set 1.

\[
\begin{align*}
  f(\eta) &= 5(\eta^2 - \eta^4) \\
  g(\eta) &= 1 - 3\eta(1 - \kappa) + 3\eta^2(1 - \kappa) - \eta^3(1 - \kappa)
\end{align*}
\]

\[
\begin{align*}
  \kappa &\quad A &\quad B &\quad C &\quad D &\quad G &\quad H \\
  0.4 &\quad 0.802 &\quad -0.0833 &\quad 6.0 &\quad -4.5 &\quad 1.0 &\quad 1.375
\end{align*}
\]

Table 2. Velocity-profile constants referred to set 2 for $\kappa = 0.4$. 

Set 3

\[ f(\eta) = 2(\eta - \eta^3) \]
\[ g(\eta) = 1 - \frac{3}{2} \eta(1-\kappa) + \frac{1}{4} \eta^3(1-\kappa) \]

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>2.089</td>
<td>-0.0833</td>
<td>6.562</td>
<td>-4.375</td>
<td>1.641</td>
<td>2.339</td>
</tr>
</tbody>
</table>

Table 3. Velocity-profile constants referred to set 3 for \( \kappa = 0.3 \)

RESULTS OF NUMERICAL INTEGRATION

The results of numerical integrations are presented in figs.2b-8b. In figures 2b, 3b and 4b the behaviour of the boundary layer thickness \( \delta_l \) with respect to the radius of the disk is presented. It is found that the boundary layer thickness \( \delta_l \) is reduced compared to the boundary layer thickness in the absence of a magnetic field and it is indeed the characteristic effect of a magnetic field in conducting flows. In fig.5b the axial out-flow velocity is presented. The axial out-flow velocity exhibits the inward and outward flowing layers for \( \kappa = 0.4 \) where as, in the absence of a magnetic field this effect was observed at \( \kappa = 0.5 \). Similar trend is also observed in fig.6b. Hence it may be concluded that the presence of a magnetic field induces the formation of inward and outward flowing fluid layers in the axial velocity even for small values of \( \kappa \). In fig.7b the radial amplitude co-efficient \( E \) is represented. It is found that \( E \) approaches infinity as \( r \to 0 \), which is similar to the behaviour of the flow without a magnetic field. Fig.8b shows the behaviour of radial mass flow \( M \) with \( r \). It is found that the mass flow in the boundary layer decreases with increase in value of \( \kappa \) and the radius at which the maximum flow occurs shifts towards the edge of the disk.

The above graphs are prepared for \( R_{c1} = 1000 \) and \( m^2 = 100 \). Similar trend is observed for other values of \( R_{c1} \) and \( m \) but with a change in magnitude. The two important effects of the presence of a transverse magnetic field are:

1) To decrease the boundary layer thickness of the end-wall boundary layers.
2) To induce the formation of inward and outward flowing layers of fluid in the outer-flow axial velocity for small values of \( \kappa \).
IV CONCLUSIONS AND REMARKS ON FURTHER WORK

The momentum-integral method developed by von Kármán for the study of turbulent boundary layer is used here to obtain solutions of the laminar boundary layer equations for the flow, between two rotating disks contained in a stationary cylinder, with or without a transverse magnetic field. The method adopted avoids the radial compatibility condition and the boundary layer velocity profiles assumed are independent of the radius. The tangential velocity of the outerflow is represented by a parabolic distribution which is in confirmity with the physical model. Attention is directed to obtain solutions of the end-wall boundary layers and the distribution with radius of the outward radial mass flow induced in the boundary layers by the external pressure gradient. The following main conclusions are drawn from the numerical integrations of the momentum-integral equations.

1) In the absence of a transverse magnetic field the axial out-flow velocity exhibits inward and outward flowing layers for values of \( \kappa > 0.5 \). So also the boundary layer thickness and radial outward mass flow also exhibit marked change in behaviour for values of \( \kappa > 0.5 \). Suggesting that if the angular velocity of the disks is such that the main body of the fluid attains a critical angular velocity represented by \( \kappa \) then the flow pattern in the boundary layer as well as in the outer flow undergoes a complete change and the outer flow axial velocity breaks up into cells.

2) The secondary flow in the boundary layer depends on the tangential Reynolds number and the secondary flow can be reduced by either decreasing the tangential Reynolds number or increasing the length of the cylinder.

3) Similar behaviour is observed even in the presence of a transverse magnetic field but at lower values of \( \kappa \).

4) The transverse magnetic field reduces the boundary layer thickness considerably.

REMARKS ON FURTHER WORK

The present analysis is confined to end-wall boundary layers and the effect of rotation of the main body of fluid on the flow in general. However, the complete analysis of the problem should in fact reconcile the existence of side-wall boundary layers, end-wall boundary layers and the smooth merging of these two sets of boundary layers at the edge of the disks. The present author intends to undertake, as further work, the above problem.
REFERENCES


FIG. 2a: TANGENTIAL VELOCITY DISTRIBUTION.

LEGEND:
- $V_{\infty} = (r - 0.90 r^2)$
- $V_{\infty} = (r - 0.95 r^2)$
- $V_{\infty} = (r - 0.99 r^2)$
FIG. 11(a): DISTRIBUTION WITH RADIUS OF INWARD RADIAL MASS FLOW.
FIG. 12a: RATIO OF RADIAL MASSFLOW IN END-WALL BOUNDARY-LAYERS TO TOTAL MASSFLOW IN THE CYLINDER.
Fig. 7a: Distribution of amplitude of radial velocity with radius.

Fig. 7b: Distribution of outflow axial velocity with radius.

Set 1

$V_0 = (r - 0.95)^2$

$K = 0$

$K = 0.4$

$R_1 = 100$

$M_2 = 1000$
FIG. 8b: DISTRIBUTION OF INWARD RADIAL MASSFLOW WITH RADIUS.

LEGEND:

\[ V_\infty = (r - 0.95r^2) \]

- \[ K = 0 \]
- \[ K = 0.4 \]

\[ R_1 = 1000 \]
\[ M^2 = 100 \]