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Port-Hamiltonian description and analysis of the LuGre friction model

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ABSTRACT

A port-Hamiltonian formulation of the LuGre friction model is presented that can be used as a building block in the physical modelling of systems with friction. Based on the dissipation structure matrix of this port-Hamiltonian LuGre model, an alternative proof can be given for the passivity conditions that are known in the literature. As a specific example, the interconnection of a mass with the port-Hamiltonian LuGre model is presented. It is shown that the lossless-interconnection structure and dissipation structure of the port-Hamiltonian LuGre model are consistent with those of this interconnection. As an additional example, the port-Hamiltonian formulation of a quarter-car system with a LuGre-based tyre model is presented.

1. Introduction

Although friction is essential to almost every aspect of mechanical behaviour, dealing with the phenomenon remains a challenge in many engineering areas. In control systems engineering this is not different. Incorporating the complex and nonlinear behaviour of friction in a control system design has been a topic of research for decades (see [1] and the references therein). As many of the control solutions tend to be model-based, there has always been a need for a faithful but relatively simple friction model.

Starting from static models, where the friction force is described as a function of the relative velocity between the two surfaces in contact, several extensions have led to dynamical friction models that capture both the nonlinear force–velocity relation with Coulomb friction, viscous friction, and the Stribeck effect, as well as transient behaviour and stiction without a logic rule.

The Dahl model [7] is the first continuous-time model that captures stiction. However, the first continuous-time, dynamic friction model able to capture *all* the relevant friction phenomena mentioned above, is the LuGre model [5,15]. Although the model has some inaccuracy in the pre-sliding (i.e., stiction) regime and is subsequently modified in [17], it is used in numerous control applications (see, e.g., [4,11,12,20]). Further developments of the LuGre model included longitudinal and combined-slip tyre models, see [6,21,8].

In this paper, we present a port-Hamiltonian description of the LuGre model and extend the result presented in [14]. The port-Hamiltonian formalism naturally arises from network modelling of physical systems in a variety of domains (e.g. mechanical, electrical, electromechanical, hydrodynamical, and thermodynamical); see [9] for a comprehensive summary of the developments of this framework over the past decade. Exposing the relation between the energy storage, dissipation, and interconnection structure, this framework underscores the physics of the system. An attractive aspect of the port-Hamiltonian formalism is that a power-preserving interconnection between port-Hamiltonian systems results in another port-Hamiltonian (PH) system with composite energy, interconnection, and dissipation structure. Based on this principle, com-

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plex multi-domain systems can be modeled by interconnecting PH descriptions of its subsystems. Moreover, control design methodologies are available that can be directly applied to such PH model descriptions [16]. It is precisely in this context that a PH description of the LuGre model can be of great value. The PH description of the LuGre friction model that is presented in this paper can be used as a resistive element in PH descriptions of (complex) systems containing friction, which can then be used for control system design.

Since friction is, in its very nature, a dissipation phenomenon (although the pre-sliding phase should ideally be conservative), it is clear that any model describing it faithfully has to be passive. In [5], passivity of the LuGre model is proven for the mapping from relative velocity to the virtual bristle state. However, it is correctly argued in [2] that the model should be passive in the mapping from relative velocity to friction force, which is the natural power-conjugate input–output pair. In [2], necessary and sufficient conditions are derived for this physically relevant passivity property. The proof is constructed using the direct analysis of passivity in terms of time-integrals of (physical) power. The PH description of the LuGre model that we present in this paper enables us to give a short alternative proof for this passivity condition.

The remainder of this paper is organized as follows. In Section 2 we give a description of the LuGre friction model with some of its important characteristics. We also reformulate the model to render it physically more consistent. In Section 3 we present the PH description of the model, after which we derive the passivity conditions in Section 4. Finally, in Section 5, PH interconnections of a mass with Lugre friction, and a quarter-car system with LuGre friction based tyre model are presented.

2. LuGre friction model

The LuGre friction model that is presented in [5,15] is a so-called bristle model, i.e., the dynamical part of the model describes a virtual bristle deflection. The model is described by the set of equations

$$\begin{aligned} \dot{z} &= -\frac{\sigma_0|v_r|}{g(v_r)}z + v_r, \\ F &= (\sigma_0 z + \sigma_1\dot{z} + \sigma_2 v_r)F_n, \end{aligned} \tag{1}$$

where z denotes the virtual bristle deflection, v_r the relative velocity of the surfaces in contact, and F the resulting friction force between the surfaces. The normal force between the surfaces is denoted by F_n. The function

$$g(v_r) = \mu_C + (\mu_S - \mu_C)e^{-\left|\frac{v_r}{v_S}\right|^{\alpha}},$$

parameterizes the static friction curve that is incorporated in the model, with μ_C being the Coulomb friction, μ_S the Stribeck friction, v_S the Stribeck velocity, and α a curve parameter that further tunes the Stribeck effect. The remaining terms parameterize the bristle dynamics. In the literature, σ_0 is used to denote the bristle stiffness coefficient, σ_1 the bristle damping coefficient, and σ_2 the viscous friction coefficient. These coefficients, however, are normalized by F_n and therefore do not have the appropriate units. To render the model physically more appealing, we introduce the variables

$$\begin{aligned} k_0 &:= \sigma_0 F_n, \\ d_1 &:= \sigma_1 F_n, \\ d_2 &:= \sigma_2 F_n, \\ g_0 &:= \frac{|v_r|}{F_n g(v_r)}. \end{aligned} \tag{2}$$

We now have that k_0 ([N m^{-1}]) is a proper stiffness coefficient, d_1 and d_2 ([Ns m^{-1}]) are proper damping coefficients, and g_0 ([m N^{-1} s^{-1}]) turns out to be a proper conductance coefficient. If we further eliminate $\dot{z}$ from the force output equation, we get the following formulation of the LuGre model,

$$\begin{aligned} \dot{z} &= -g_0 k_0 z + v_r, \\ F &= (1 - d_1 g_0)k_0 z + (d_1 + d_2)v_r. \end{aligned} \tag{3}$$

An important characteristic of the LuGre model is that the virtual bristle displacement z is bounded according to

$$-\frac{\mu_S}{\sigma_0} \leqslant z \leqslant \frac{\mu_S}{\sigma_0}. \tag{4}$$

The fact that the state space does not consist of the whole of $\mathbb{R}$ turns out to be crucial for the passivity analysis of the model.

Another key feature of the model is the fact that the steady-state behaviour coincides with a commonly used static friction curve parametrization [1,3], namely

$$F(v_r)|_{\dot{z}=0} = \left(\frac{1}{g_0} + d_2\right)v_r.$$

This is exactly the rationale behind the LuGre friction model, but at the same time it offers the opportunity to incorporate other friction curve parameterizations into the model by changing the conductance term g_0 in (2).

With respect to this presentation of the LuGre friction model, we like to make the following remarks:

- The LuGre model is an extension of, and therefore closely related to, the Dahl model [7]. In fact, if $g(v_r) = \mu_C$, and $d_1 = 0$, we have the original Dahl model. This means that for these values, the port-Hamiltonian model presented in Section 3 is actually a port-Hamiltonian form of the Dahl model.
- In the original work [5], the LuGre model is not assumed to be an explicit function of the normal force F_n. However, following the discussion in [8,1], and the presentation of the LuGre model in [6], the normal force dependency in (1) is used.
- Friction modelling is very much application dependent [1], and it might therefore be that in some cases the LuGre model, and subsequently its port-Hamiltonian form presented here, is not applicable.

3. Port-Hamiltonian (PH) formulation of the LuGre friction model

In this section, we give a PH description of the LuGre friction model (3). First, we recall the standard PH description of a system without direct feedthrough, as it is treated in, e.g., [18], after which we extend it to a more general form.

3.1. PH systems

A basic PH system description is given as follows [18]:

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\nabla_x H(x) + G(x)u, \\ y &= G^T(x)\nabla_x H(x), \end{aligned} \tag{5}$$

where $x \in \mathscr{X}$ is the state, and $H : \mathscr{X} \to \mathbb{R}$ the Hamiltonian, of which the gradient is denoted by $\nabla_x^T H$. The input distribution matrix is denoted by $G(x) : \mathscr{X} \to \mathbb{R}^{n\times m}$, the lossless interconnection structure matrix by $J(x) : \mathscr{X} \to \mathbb{R}^{n\times n}$, satisfying $J(x) = -J^T(x)$, and the dissipation structure matrix by $R(x) : \mathscr{X} \to \mathbb{R}^{n\times n}$, satisfying $R(x) = R^T(x)$. Although this type of PH description is suitable for describing a large class of physical systems, for the LuGre model it turns out that a more generic form is needed.

3.2. PH systems with feedthrough and modulation

A more general description of a PH system than (5) arises when a direct feedthrough channel is incorporated. This form is not yet widely used in the literature, but it can be found in [19,13]. If we furthermore also allow for modulations of the system matrices by external parameters, we arrive at the following PH system description:

$$\begin{aligned} \dot{x} &= [J(x,\rho) - R(x,\rho)]\nabla_x H(x,\rho) + [G(x,\rho) - P(x,\rho)]u, \\ y &= [G(x,\rho) + P(x,\rho)]^T\nabla_x H(x,\rho) + [M(x,\rho) + S(x,\rho)]u, \end{aligned} \tag{6}$$

with $\rho \in \mathbb{R}^p$ denoting the vector of external parameters. The lossless interconnection structure of the system is now determined by $J : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{n\times n}, G : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{n\times m}$, and $M : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{m\times m}$, while the dissipation structure is determined by $R : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{n\times n},\ P : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{n\times m}$, and $S : \mathscr{X} \times \mathbb{R}^p \to \mathbb{R}^{m\times m}$. The matrices $M(x,\rho)$ and $S(x,\rho)$ are respectively skew-symmetric and symmetric. In practice, the structure matrices and the Hamiltonian are often functions of the state x, but it also occurs (e.g. in power converters [18]) that they are modulated by external variables. The description above is therefore a practically relevant form. Having this generic PH form, the question now is how to put the LuGre friction model in such a framework.

3.3. PH LuGre model

The input and output of the LuGre friction model are given by v_r and F respectively, thereby constituting a natural power-conjugate input–output pair. Since we are dealing with a scalar system, the skew-symmetric interconnection terms are necessarily zero, i.e., $J_\ell = M_\ell = 0$, where we use the subscript ℓ to refer to the LuGre model. The remaining terms have to be selected sequentially. We start by selecting the feedthrough matrix $S_\ell(\rho)$. A natural choice for this matrix is

$$S_\ell(F_n) = d_1 + d_2. \tag{7}$$

Next, we choose the Hamiltonian. The most natural candidate is the elastic energy stored by the virtual bristles

$$H_\ell(z,F_n) = \frac{1}{2}k_0 z^2. \tag{8}$$

Although we have the freedom to choose other Hamiltonians, the one above has the advantage of not being modulated by the relative velocity v_r and having a clear physical meaning.

Having set both $H_\ell(x,\rho)$ and $S_\ell(\rho)$, we proceed by selecting $G_\ell(x,\rho)$ and $P_\ell(x,\rho)$ from

$$[G_\ell(x,\rho) + P_\ell(x,\rho)]^T\nabla_z H_\ell(z,F_n) = k_0(1 - d_1 g_0)z \Rightarrow [G_\ell(v_r) + P_\ell(v_r)]^T = (1 - d_1 g_0).$$

We select

$$G_\ell(v_r) = 1 - \frac{1}{2} d_1 g_0, \tag{9}$$

$$P_\ell(v_r) = -\frac{1}{2} d_1 g_0. \tag{10}$$

(The motivation for this particular form is given in Section 5.) The dissipation term $R_\ell(x,\rho)$ is now derived as

$$\dot{z} = v_r - k_0 g_0 z = -R_\ell(x,\rho) k_0 z + [G_\ell(v_r) - P_\ell(v_r)] v_r \Rightarrow v_r - k_0 g_0 z = -R_\ell(x,\rho) k_0 z + v_r \Rightarrow R_\ell(v_r, F_n) = g_0. \tag{11}$$

Hence we obtain the PH description of the LuGre friction model given by

$$\begin{aligned} \dot{z} &= -R_\ell(v_r, F_n) \nabla_z H_\ell(z, F_n) + [G_\ell(v_r) - P_\ell(v_r)] v_r, \\ F_\ell &= [G_\ell(v_r) + P_\ell(v_r)] \nabla_z H_\ell(z, F_n) + S_\ell(F_n) v_r, \end{aligned} \tag{12}$$

with $S_\ell(F_n)$, $H_\ell(z,F_n)$, $G_\ell(v_r)$, $P_\ell(v_r)$, and $F_\ell(v_r,F_n)$ as in (7)–(11), respectively.

4. Passivity analysis of the LuGre model via dissipation structure

As stated in the Introduction, passivity of the LuGre model in the $v_r \mapsto F$ mapping is crucial. In this section, we show that exploiting the physical structure of the PH description (12), the passivity conditions for the LuGre friction model naturally follow by a direct analyses of the dissipation structure of the system. Our results coincide with the passivity condition derived in [2], with the modest exception that our results hold for any constant F_n. This implies that in the passivity analysis below, we assume that ρ remains constant. The definitions of passivity used in this paper are based on the seminal works [18,22]. Details on passivity of port-Hamiltonian systems with direct feedthrough can be found in [9].

4.1. Passivity of PH systems

A particularly appealing feature of PH systems of the form (6) is that, because of skew-symmetry of $J(x)$, the energy flow of the system satisfies (for sake of brevity we omit the arguments)

$$\dot{H} = u^T y - \begin{pmatrix} \nabla_x^T H & u^T \end{pmatrix} \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} \nabla_x H \\ u \end{pmatrix}, \tag{13}$$

expressing that the power associated to the energy stored by the system equals the power supplied to it minus the power that is dissipated. Furthermore, if $H(x)$ is bounded from below and if the dissipated power is such that

$$\begin{pmatrix} \nabla_x^T H & u^T \end{pmatrix} \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} \nabla_x H \\ u \end{pmatrix} \geqslant 0, \tag{14}$$

for all $x \in \mathscr{X}$ and admissible inputs $u : [t_0, t_1] \to \mathbb{R}^m$, the system satisfies the power balance inequality $\dot{H} \leqslant u^T y$. Integrating the latter from time t_0 to t_1 yields the inequality

$$H(x(t_1)) - H(x(t_0)) \leqslant \int_{t_0}^{t_1} u^T(t) y(t) dt, \tag{15}$$

which states that the system cannot store more energy than it receives from the environment. In other words, the system is passive with respect to the supply rate $u^T y$ and storage function the Hamiltonian $H(x)$. In the special case that (14) is identically zero, we have that $H = u^T y$, which implies that the system is lossless.

4.2. Passivity conditions for the LuGre friction model

Although it is often stated that (14) is satisfied if

$$\begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \succeq 0, \tag{16}$$

this is only true when the latter is not a function of the states, otherwise the product in (14) cannot be treated as a pure quadratic form. Furthermore, one also has to take into account the nature of the state space. If the state space is a bounded subset $\mathscr{X} \subset \mathbb{R}^n$, the inequality has to be satisfied only on the domain of interest.

Taking into account the fact that the state space of the LuGre friction model is bounded according to (4) and using the result of the previous section, the passivity condition (14) for the LuGre friction model (12) is given by

$$\begin{pmatrix} k_0 z & v_r \end{pmatrix} \begin{pmatrix} g_0 & -\frac{1}{2} d_1 g_0 \\ -\frac{1}{2} d_1 g_0 & d_1 + d_2 \end{pmatrix} \begin{pmatrix} k_0 z \\ v_r \end{pmatrix} \geqslant 0,$$

$$\forall z \in \left[-\frac{\mu_S}{\sigma_0}, \frac{\mu_S}{\sigma_0}\right], \quad v_r \in \mathbb{R}, \tag{17}$$

which, under the assumption that F_n is constant, implies that (for details, see the Appendix)

$$d_1 \leqslant d_2 \frac{\mu_C}{\mu_S - \mu_C}. \tag{18}$$

Hence, under this condition, together with the fact that the Hamiltonian (8) is positive semi-definite (and hence bounded from below) for all z and $k_0 \geqslant 0$, the LuGre friction model is passive.

Not surprisingly, the inequality (18) precisely coincides with the passivity condition derived in [2]. The difference, however, is that our result directly follows from the energetic and dissipative structure of the PH description of the LuGre friction model, whereas the result in [2] is derived from the positivity requirement of the supplied power.

4.3. Modified bristle damping

The passivity condition (18) is often in contradiction with the friction parameter values that are identified in experiments. Besides that, in [15] it is argued that a constant σ_1 leads to inconsistent behaviour in the transition from slip to stick. In order to resolve these two issues, a relative velocity-dependent bristle damping

$$\hat{\sigma}_1 = \sigma_1 e^{-\left(\frac{v_r}{v_d}\right)^2} \tag{19}$$

is proposed, with v_d being the bristle damping parameter [15]. For a general relative velocity-dependent $\hat{\sigma}_1$, the condition for passivity is derived in [15] reads

$$0 \leqslant \hat{\sigma}_1 \leqslant 4\frac{|v_r|}{g(v_r)}. \tag{20}$$

The proof for this condition is given in terms of direct integrals of power. Again, we can use the dissipation structure of the PH description of the LuGre model, and in particular condition (17), to conclude the following.

Assuming that we have a relative velocity-dependent bristle damping $d_1^* = \hat{\sigma}_1 F_n$, a sufficient condition for (17) is given by

$$\begin{pmatrix} g_0 & -\frac{1}{2}d_1^* g_0 \\ -\frac{1}{2}d_1^* g_0 & d_1^* + d_2 \end{pmatrix} \succcurlyeq 0, \tag{21}$$

which, in turn, by taking the Schur complement, is satisfied if and only if the following three inequalities are satisfied:

$$\begin{aligned} & g_0 \geqslant 0, \\ & d_1^* + d_2 \geqslant 0, \\ & \frac{1}{4}(d_1^*)^2 g_0 - (d_1^* + d_2) \leqslant 0. \end{aligned}$$

First note that both g_0 and d_2 are positive. Due to positivity of d_2, a sufficient condition for the last inequality to hold is

$$\frac{1}{4}(d_1^*)^2 g_0 - d_1^* \leqslant 0,$$

which, due to positivity of g_0, is satisfied if and only if

$$0 \leqslant d_1^* \leqslant \frac{4}{g_0}. \tag{22}$$

This inequality is equivalent to the passivity condition (20).

4.4. Discussion

Concerning the passivity analysis, we can state the following remarks:

- The analysis above shows that a zero bristle damping d_1, renders the model passive. Some authors indeed found that the influence of this parameter is negligibly small to be observable on their test setup (which is specifically the case for tyre friction applications discussed in Section 5.3.1). The need of introducing such a term is therefore questionable.
- Although the passivity discussion here is centered around the energy dissipative nature that one would expect from a friction model, one can argue that for control purposes, this type of passivity is not easily exploited. In many cases, v_r is not available for measurement, and an estimate of v_r has to be used. Such an estimation procedure might introduce delays that hamper the exploitation of passivity. For a further discussion on this issue, we refer to [10].

5. PH description of systems with friction

In this section, we present a PH description of a mass subject to friction and a quarter-car with LuGre-based longitudinal tyre force model. These interconnections show how the PH description of the LuGre model can be used as a building block for physical modelling of systems in a PH framework. Moreover, due to the specific nature of these interconnection examples, the way in which the dissipation structure of the PH LuGre friction and the LuGre tyre force building blocks carry over to the interconnected systems, becomes particularly transparent.

5.1. Interconnection of a PH system with the PH LuGre model

Let us first introduce a standard (negative) feedback interconnection of a basic PH system (without feedthrough) with the PH Lugre model. If we denote the state, the input, and the output of the LuGre model by $x_\ell = z$, $u_\ell = v_r$, and $y_\ell = F_\ell$, respectively, and further omit the arguments of the matrices in (12), the PH LuGre model is described by

$$\Sigma_\ell : \begin{cases} \dot{x}_\ell = -R_\ell \nabla_{x_\ell} H_\ell + [G_\ell - P_\ell] u_\ell, \\ y_\ell = [G_\ell + P_\ell]^T \nabla_{x_\ell} H_\ell + S_\ell u_\ell, \end{cases} \tag{23}$$

with R_ℓ, H_ℓ, G_ℓ, P_ℓ and S_ℓ given in (11), (8)–(10) and (7) respectively. The basic general PH system that we interconnect the PH LuGre model with, is described by

$$\Sigma_p : \begin{cases} \dot{x}_p = [J_p - R_p] \nabla_{x_p} H_p + G_p u_p, \\ y_p = G_p^T \nabla_{x_p} H_p. \end{cases} \tag{24}$$

Using a standard negative (power-preserving) feedback interconnection

$$\begin{pmatrix} u_p \\ y_p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_\ell \\ y_\ell \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \tag{25}$$

with external input u, we get the PH system

$$\Sigma : \begin{cases} \dot{x} = [J - R] \nabla_x H + Gu, \\ y = G^T \nabla_x H, \end{cases}$$

with $x = (x_p \quad x_\ell)^T$ and $G = (G_p \quad 0)^T$. The total energy (Hamiltonian) is given by

$$H = H_p + H_\ell,$$

and the lossless interconnection and dissipation structure matrices by

$$J = \begin{pmatrix} J_p & -G_p G_\ell^T \\ G_\ell G_p^T & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} R_p + G_p S_\ell G_p^T & G_p P_\ell^T \\ P_\ell G_p^T & R_\ell \end{pmatrix}.$$

The description above shows which terms of both systems (23) and (24) contribute to the lossless interconnection structure of the closed-loop system and what terms contribute to its dissipation structure.

5.2. Feedback interconnection of the PH LuGre friction model with a mass

Having described a general negative feedback interconnection of a PH system with the PH LuGre model, it is now straightforward to substitute the PH description of a mass m into this feedback loop, see Fig. 1. The PH description of the mass is given by

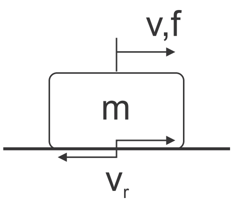

Fig. 1. Mass with friction model with mass m, forward velocity v, and relative velocity v_r.

$$\Sigma_m : \begin{cases} \dot{x}_m = G_m u_m, \\ y_m = G_m \nabla_{x_m} H_m, \end{cases} \tag{26}$$

with $x_m = p_m$ being the momentum of the mass, input $u_m = F$ the total force acting upon the mass, and output y_m the velocity of the mass. Furthermore, the Hamiltonian consists of the kinetic energy of the mass

$$H_m(p_m) = \frac{p_m^2}{2m}.$$

Of all the other terms in the PH description of the mass, only $G_m = 1$ is non-zero. Using again the power-preserving interconnection (25)

$$\begin{pmatrix} u_m \\ y_m \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_\ell \\ y_\ell \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_F,$$

where the external input u_F is an external force, the closed-loop system is given by

$$\dot{x} = \begin{bmatrix} \dot{p}_m \\ \dot{x}_l \end{bmatrix} = \left[\begin{pmatrix} 0 & -1 + \frac{1}{2} d_1 g_0 \\ 1 - \frac{1}{2} d_1 g_0 & 0 \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} d_1 + d_2 & -\frac{1}{2} d_1 g_0 \\ -\frac{1}{2} d_1 g_0 & g_0 \end{pmatrix} \right] \begin{pmatrix} \frac{p_m}{m} \\ k_0 x_l \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_F,$$

$$y = v = [1 \quad 0] \begin{bmatrix} \frac{p_m}{m} \\ k_0 x_l \end{bmatrix}, \tag{27}$$

with Hamiltonian

$$H(p_m, x_l) = \frac{p_m^2}{2m} + \frac{k_0 x_l^2}{2}, \tag{28}$$

representing the sum of the kinetic energy of the mass and the elastic energy stored by the virtual bristles. Since the mass is a lossless system, the dissipation structure of the mass-friction interconnection is determined solely by the friction part. On the other hand, it can be shown that the chosen Hamiltonian (28) admits only one unique realization of both the lossless interconnection structure matrix and the dissipation structure matrix. The specific form of both $G_\ell(\cdot)$ and $P_\ell(\cdot)$, proposed in Section 3, are chosen such that the PH LuGre description is consistent with the PH description of the mass-friction system.

5.3. Interconnection of a PH LuGre friction based longitudinal tyre model with a quarter-car

A second example of how a PH description of the LuGre friction model can be used in modelling nonlinear systems, we present a PH description of a quarter-car model with a LuGre-based longitudinal tyre force model. First we introduce the longitudinal tyre force model as it is presented in [6].

5.3.1. LuGre friction based longitudinal tyre force model

Based on the LuGre friction model, several tyre models have been developed that describe either the longitudinal [6], or the combined longitudinal and lateral tyre–road interaction [21,8]. These models are generally derived by setting up the partial differential equations (PDE) describing the LuGre bristle dynamics for the tyre–road contact patch. Under the assumption of some normal force distribution, these PDEs are then averaged, which produces a lumped, single-bristle LuGre model with an extra conductivity term capturing the 'convective losses' effect of the bristles moving in and out of the contact patch due to the rolling motion.

The longitudinal tyre–road interaction model in [6] is used in this section and is, in the notation of Section 2, given by the following equations:

$$\dot{z} = v_r - k_0 g_t z,$$

$$F = -k_0(1 - d_1 g_t) z - (d_1 + d_2) v_r, \tag{29}$$

with k_0, d_1, and d_2 as described in (2) and the tyre conductivity g_t being the sum of the original LuGre bristle conductivity g_0 (2) and a new term g_1, that is,

$$g_{\mathrm{t}} := g_0 + g_1,$$
$$g_0 := \frac{|v_r|}{g(v_r)F_n},$$
$$g_1 := \frac{\kappa|r\omega|}{\sigma_0 F_n}.$$

The extra conductivity term g_1 captures the rolling motion of the tyre, with ω being the rotational velocity of the tyre/wheel, r the effective radius of the tyre, and κ a term representing the averaging of the friction PDEs over some assumed normal force distribution. We furthermore have that the relative velocity v_r is related to the forward velocity of the wheel v, and its rotational velocity ω according to

$$v_r = v - r\omega.$$

5.3.2. Quarter-car model with longitudinal tyre force model

Fig. 2 shows a quarter-car with car body mass m_b and wheel mass m_w, tyre/wheel moment of inertia I, forward velocity v, and wheel rotational velocity ω. The PH description of the combined car body and wheel mass $m = m_b + m_w$ is given by (26), while the PH description of the rolling tyre is given by

$$\Sigma_I : \begin{cases} \dot{x}_I = G_I u_I, \\ y_I = G_I \nabla_{x_I} H_I x_I, \end{cases}$$

with $x_I = p_I$ the momentum of the rolling wheel, input $u_I = T$ the total torque acting upon the wheel, and output y_I the rotational velocity. The input distribution matrix is given by $G_I = 1$. The Hamiltonian consists of the kinetic energy of the wheel

$$H_I(p_I) = \frac{p_I^2}{2I}.$$

The last component is the PH description of longitudinal tyre force model, given by

$$\Sigma_{\mathrm{t}} : \begin{cases} \dot{x}_{\mathrm{t}} = -R_{\mathrm{t}} \nabla_{x_{\mathrm{t}}} H_{\mathrm{t}} + [G_{\mathrm{t}} - P_{\mathrm{t}}] u_{\mathrm{t}}, \\ y_{\mathrm{t}} = [G_{\mathrm{t}} + P_{\mathrm{t}}]^T \nabla_{x_{\mathrm{t}}} H_{\mathrm{t}} + S_{\mathrm{t}} u_{\mathrm{t}}. \end{cases}$$

Using the longitudinal tyre force model introduced in the previous section, the PH description of the total system is now given by

$$\dot{x} = [J - R]\nabla_x H(x) + Gu,$$

with

$$x = \begin{pmatrix} p_m \\ p_I \\ x_t \end{pmatrix}.$$

Hamiltonian

$$H(x) = \frac{p_m^2}{2m} + \frac{p_I^2}{2I} + \frac{k_0 x_t^2}{2},$$

and lossless-interconnection and dissipation structure matrices

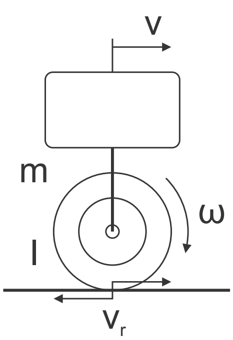

Fig. 2. Quarter-car model with mass m, moment of inertia I, forward velocity v, wheel rotational velocity ω and relative velocity v_r.

$$J(p_v,p_\omega,F_n)=\begin{pmatrix}0 & 0 & -1+\frac{1}{2}d_1g_t\\ 0 & 0 & (1-\frac{1}{2}d_1g_t)r\\ 1-\frac{1}{2}d_1g_t & (-1+\frac{1}{2}d_1g_t)r & 0\end{pmatrix},$$

$$R(p_v,p_\omega,F_n)=\begin{pmatrix}(d_1+d_2) & -(d_1+d_2)r & -\frac{1}{2}d_1g_t\\ -(d_1+d_2)r & (d_1+d_2)r^2 & \frac{1}{2}d_1g_tr\\ -\frac{1}{2}d_1g_t & \frac{1}{2}d_1g_tr & g_t\end{pmatrix}.$$

The actuation input $u=\tau$ is the traction or brake torque, for which the input distribution matrix is given by

$$G=\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

6. Concluding remarks

In this paper we have presented a port-Hamiltonian (PH) description of the LuGre friction model. Apart from establishing a closer connection between the mathematics and the physical background of the parameters in the model, a PH description of the LuGre model also enables us to assess the passivity conditions, as originally derived in [2] from an input–output perspective, in a straightforward and natural manner using the dissipation structure. Moreover, we showed that the passivity conditions can be given in case of any constant normal force between the surfaces in contact.

As a specific example of the use of the PH Lugre model, we presented an interconnection with a mass. It was shown that the lossless interconnection structure and dissipation structure of the port-Hamiltonian LuGre model are consistent with those of the interconnection, which are in turn uniquely determined by the choice of the Hamiltonian. A further example was presented in the form of a quarter-car with longitudinal tyre force model.

Appendix A

Writing out (17) results in

$$\begin{aligned}&k_0^2g_0z^2-k_0d_1g_0zv_r+(d_1+d_2)v_r^2\geqslant 0,\\ &\forall z\in\left[-\frac{\mu_S}{\sigma_0},\frac{\mu_S}{\sigma_0}\right],\quad v_r\in\mathbb{R}.\end{aligned}\tag{30}$$

First of all we note that the left hand term of the inequality is equal to zero for $v_r=0$, so for this case the inequality is trivially satisfied. Furthermore, we have that the inequality in (30) is equivalent to

$$k_0^2z^2\frac{g_0}{v_r^2}-k_0d_1z\frac{g_0}{v_r}+(d_1+d_2)\geqslant 0.\tag{31}$$

for all $v_r\in\mathbb{R}\setminus\{0\}$. The first two terms on the left-hand side have their infimum for $|v_r|\to\infty$, producing the strongest conditions on the friction parameters. Indeed, both g_0/v_r and $1/v_r$ are odd and their product is non-negative for all $v_r\in\mathbb{R}$ with

$$\lim_{|v_r|\to\infty}\left(k_0^2z^2\frac{g_0}{v_r^2}\right)=0.$$

With z^2 also being non-negative, this results in

$$\inf_{\substack{v_r\in\mathbb{R}\\ -\frac{\mu_S}{\sigma_0}\leqslant z\leqslant\frac{\mu_S}{\sigma_0}}} k_0^2z^2\frac{g_0}{v_r^2}=0.$$

For the second term in (31) the infimum is given by

$$\inf_{\substack{v_r\in\mathbb{R}\\ -\frac{\mu_S}{\sigma_0}\leqslant z\leqslant\frac{\mu_S}{\sigma_0}}}\left(-k_0d_1z\frac{g_0}{v_r}\right)=\lim_{v_r\to\infty}\left(-k_0d_1z\frac{g_0}{v_r}\right)\Bigg|_{z=\frac{\mu_S}{\sigma_0}}=\lim_{v_r\to-\infty}\left(-k_0d_1z\frac{g_0}{v_r}\right)\Bigg|_{z=-\frac{\mu_S}{\sigma_0}}=-d_1\mu_S\frac{1}{\mu_C}.$$

This results in the passivity condition

$$-d_1\mu_S\frac{1}{\mu_C}+(d_1+d_2)\geqslant 0,$$

which is equivalent to (18).

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