INSTITUTE

FOR

AEROSPACE STUDIES

UNIVERSITY OF TORONTO

THEORY OF WIND TURBINES WITH CONTRAROTATION

by

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July, 1977

UTIAS Report No. 218
CN ISSN 0082-5255

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Submitted September, 1976

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UTIAS Report No. 218
CN ISSN 0082-5255

Abstract

In view of the current interest in unconventional energy sources, research on the design of wind turbines of high efficiency done by the author some years ago has been reviewed and prepared for publication. The underlying theory is contained in a series of papers on ducted fans (Refs. 1-6).

Emphasis has been placed on a ducted contrarotating system of high efficiency capable of a wide range of operating conditions.

Contents

<u>Notation</u>

Note: The subscripts 1 and 2 applied to many symbols refer to rotor 1 and rotor 2, respectively. Numbers in brackets refer to equations in the text.

a	Radius of boss fairing (hub) (Fig. 2)
b	Radius of each rotor tip (Fig. 2)
c_1, c_2	Chord of the rotor blade element at radius r
C_{D_1}, C_{D_2}	Two-dimensional drag coefficient of the blade element (68)
C_{F_1}, C_{F_2}	Coefficient of axial force acting on the rotor (51, 52)
C_{L_1}, C_{L_2}	Two-dimensional lift coefficient of the blade element (67)
C_{Q_1}, C_{Q_2}	Coefficient of torque for a rotor (61, 62)
C_{X_1}, C_{X_2}	Coefficient of force on the rotor blade element acting in the plane of rotation (69, 70)
dD_1, dD_2	Drag on the blade element of a rotor (Fig. 3)
E	Total energy per unit volume of flow extracted from the incident wind by the two rotors (2)
E_1, E_2	Input of energy from unit volume of flow to one rotor (Fig. 1) (2, 7, 8)
F_1, F_2	Force acting on a rotor parallel to the axis
dF_1, dF_2	Axial force on the blade element of a rotor (16, 17, 34, 35, 53, 54)
H	Total energy extracted by the wind turbine from unit volume of flow (Fig. 1) (2, 3)
H_D	Energy loss per unit volume of flow in the duct (Fig. 1) (2)
k	Coefficient of total energy input per unit volume of flow to both rotors (26, 47)
k_1, k_2	Coefficient of energy input per unit volume of flow to one rotor (7, 8)

k_D	Coefficient of energy loss per unit volume of flow in the duct (46)
dL_1, dL_2	Lift on the blade element of a rotor (Fig. 3)
N_1, N_2	Number of blades of a rotor
p_∞	Pressure in the undisturbed, incident wind (Fig. 1) (1)
p_w	Pressure in the settled wake far downstream (Fig. 1) (1)
p_1, p_2, p_3	Pressures in front of rotor 1, between rotors at radius r and behind rotor 2, respectively (Fig. 2) (4)
Q_1, Q_2	Torque developed by a rotor
dQ_1, dQ_2	Torque developed by the blade element of a rotor (18, 34, 35)
r	Radius of the elementary annulus from the axis (Fig. 2)
dr	Radial width of the elementary annulus (Fig. 2)
R	Radius ratio (12)
R_a	Radius ratio of the hub (a/b)
Re_1, Re_2	Reynolds number of the blade element (71)
S_1, S_2	Rotor solidity (72)
u	Axial velocity through the rotors (Figs. 2, 3)
u_∞	Velocity in the undisturbed, incident wind (Fig. 1) (1)
u_w	Velocity in the settled wake far downstream (Fig. 1) (1)
w_1, w_2	Resultant velocity of flow relative to the blade element (Fig. 3)
dX_1, dX_2	Force on the blade element acting in the plane of rotation (30, 31)
dY_1, dY_2	Force on the blade element acting parallel to the axis (30, 31)

α_1, α_2	Angle of incidence of the chord of the blade element to the resultant flow (Fig. 3)
β	The constant ϵR with respect to radius between the rotors (48)
γ_1, γ_2	Two-dimensional lift/drag ratio for a rotor blade element (65, 66)
ϵ	Ratio of one half of the circumferential induced velocity between rotors at radius r to the axial velocity (13)
η	Overall blade element efficiency for both rotors (25)
η_1, η_2	Blade element efficiency for one rotor (19, 22)
θ_1, θ_2	Angle of the blade section chord to the plane of rotation (77)
λ	Value when $\lambda_1 = \lambda_2$
λ_1, λ_2	Ratio of the circumferential speed of the blade element of a rotor at radius r to the axial velocity (20, 23)
Λ_1, Λ_2	Value of λ_1, λ_2 at rotor tip (21, 24)
μ	Coefficient of viscosity
ρ	Density
φ_1, φ_2	Angle made by the resultant velocity of flow at the blade element with the plane of rotation (Fig. 3) (32, 33)
ω	Angular velocity of rotation of the flow at radius r between the rotors measured in a plane perpendicular to the axis (Fig. 3)
Ω_1, Ω_2	Angular velocity of a rotor (20, 23)

1. INTRODUCTION

The aerodynamic theory of contrarotating wind turbines (ducted windmills with contrarotation) presented here was initiated many years ago (1940's) when the author was working on ducted fans as a wartime project (Refs. 1-5). This theory was developed at that time as a natural extension of the ducted fan research, using the same aerodynamic fundamentals, but was left in abeyance when other priorities emerged. In view of the current interest in unconventional energy sources, the writer decided to review and publish the theory as a retirement project.

The possibility of converting wind energy to man's use will always be attractive since the winds are an inexhaustible source of energy which is available on many sites and free for the taking. Combined with an accessibility to water, which so often occurs in Canada, wind energy can be made available in a self-contained system that requires no other energy input. With increasing demands for energy, diversification of sources may well become established policy and the wind as a potential source will receive more serious consideration.

In the following analysis the basic aerodynamic theory is presented, followed by a suggested design procedure. The possibility of highly efficient designs based on the principle of contrarotation is emphasized.

2. BASIC REQUIREMENT

The basic requirement for a wind energy converter is the extraction of the maximum energy from the undistributed, incident airflow of a given cross section with a minimum loss of energy in the process. The overall system is outlined diagrammatically in Fig. 1. We assume uniformly constant pressure and velocity in the undisturbed, incident wind (p_∞, u_∞) and in the settled wake far downstream (p_w, u_w), the velocity in these initial and final regions being parallel to the axis of symmetry (Ref. 6). We also assume that no significant compressibility is associated with the flow, i.e. that the density (ρ) is everywhere constant and the same.

Under these circumstances the conservation of energy, applied to each unit volume of flow throughout the process, requires that (Ref. 7)

$$p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p_w + \frac{1}{2}\rho u_w{}^2 + H \qquad (1)$$

where

$$H = E + H_D = E_1 + E_2 + H_D \qquad (2)$$

In these expressions H is the total energy extracted by the wind turbine from unit volume of flow, composed of inputs to the rotors of E_1, E_2, respectively, and an energy loss H_D arising from inlet and outlet pressure conversion in the duct and viscous action on the cylindrical walls, boss fairing and support components.

Far downstream from the wind turbine the wake pressure returns to the atmospheric value ($p_w = p_\infty$) and hence (1) becomes

$$H = \frac{1}{2}\rho u_\infty^2 - \frac{1}{2}\rho u_w^2 \qquad (3)$$

Thus the energy available for conversion is always less than the kinetic energy in the incident wind. For maximum energy extraction the kinetic energy in the wake must be small compared with that in the undisturbed wind and the energy losses in the rotors and the duct must be minimal to ensure that an optimal proportion of H is available for useful work. A high efficiency of energy conversion by the rotors and an aerodynamically "clean" duct are essential to good design.

3. FLOW CONDITIONS

The theoretical considerations that will form the basis of a method for the design of a contrarotating wind turbine are subject to certain flow conditions:

(a) The flow in front of rotor 1 and behind rotor 2 is directed parallel to the axis of symmetry (Fig. 2) and the velocity and pressure are constant for all radii in these planes (i.e. u, p_1, p_3 are constant with respect to r). We note further that u is constant and the same throughout the flow in the wind turbine from considerations of flow continuity.

(b) The vortex theory of aerofoils applies. The velocity relative to the blade element (w_1, w_2) is the resultant of the axial velocity (u), the geometrical velocity of rotation ($\Omega_1 r$, $\Omega_2 r$) and the circumferential component of induced velocity ($1/2\ \omega r$) for rotor 1 and rotor 2, respectively. The aerodynamic action of the blade element is, therefore, the same as that for two-dimensional flow if the latter is referred to the resultant velocity (w_1, w_2). It is noted that continuity of flow does not permit an axial component of induced velocity.

(c) Rotor 2 will be designed to remove all the slipstream rotation introduced by rotor 1. To facilitate this, the design will be such that there is no radial component of flow between the rotors. At any radius (r) the streamlines are confined to the surface of a cylinder which is coaxial with the walls and the boss fairing (Fig. 2). This is required so that the circumferential velocity (ωr), induced at radius r by rotor 1, can be removed at the same radius by rotor 2.

According to these flow conditions the energy equation for unit volume of flow in the annulus between r and $r + dr$ (i.e. r, dr) may be written

$$p_1 + \frac{1}{2}\rho u^2 = p_2 + \frac{1}{2}\rho u^2 + \frac{1}{2}\rho\omega^2 r^2 + E_1 = p_3 + \frac{1}{2}\rho u^2 + E_1 + E_2 \qquad (4)$$

from which we deduce that the energy inputs to the rotors are, respectively,

$$E_1 = (p_1 - p_2) - \frac{1}{2} \rho\omega^2 r^2 \tag{5}$$

$$E_2 = (p_2 - p_3) + \frac{1}{2} \rho\omega^2 r^2 \tag{6}$$

It is useful to introduce the input coefficients k_1, k_2 for rotor 1 and rotor 2, respectively, as follows:

$$E_1 = k_1 \cdot \frac{1}{2} \rho u^2 \tag{7}$$

$$E_2 = k_2 \cdot \frac{1}{2} \rho u^2 \tag{8}$$

The condition for no radial flow between the rotors limits the permissible radial variation of the slipstream rotation induced by rotor 1. Thus, if the radial pressure gradient behind rotor 1 must be that which supports a rotating flow only, without convergence or divergence, then

$$\frac{\partial p_2}{\partial r} = \rho\omega^2 r \tag{9}$$

Now the differentiation of (4) with respect to r, noting that u and p_1 are constant with respect to r, yields the result

$$\frac{\partial p_2}{\partial r} + \frac{1}{2} \rho \frac{\partial}{\partial r} (\omega^2 r^2) + \frac{1}{2} \rho u^2 \frac{\partial k_1}{\partial r} = 0 \tag{10}$$

or, from (9),

$$\omega^2 r + \omega r \frac{\partial}{\partial r} (\omega r) + \frac{1}{2} u^2 \frac{\partial k_1}{\partial r} = 0 \tag{11}$$

If we introduce the convenient dimensionless notation

$$R = \frac{r}{b} \tag{12}$$

where b is the diameter of the rotors, and

$$\epsilon = \frac{\frac{1}{2} \omega r}{u} \tag{13}$$

then (11) becomes

$$\frac{\epsilon^2}{R} + \epsilon \frac{\partial \epsilon}{\partial R} + \frac{1}{8} \frac{\partial k_1}{\partial R} = 0 \tag{14}$$

If the design is such that the energy input per unit volume of flow (E_1) is constant and the same over the whole face of rotor 1, then $\partial k_1/\partial r = 0$, and the condition for no radial flow between rotors will be met if

$$\epsilon R = \text{constant} \tag{15}$$

This relation defines an "irrotational" or zero vorticity flow relative to the fluid element behind rotor 1.

4. FLOW IN THE ELEMENTARY ANNULUS

We now consider the aerodynamic characteristics of the flow in the annulus between r and r + dr (Fig. 2), including the elements of force on each rotor, acting parallel to the axis of symmetry, the elements of torque developed by the rotors and the elementary efficiencies of the energy conversion process for the rotors separately and in combination.

The elements of force acting on rotor 1 and rotor 2, respectively, in the direction of u in the annulus r,dr are

$$dF_1 = (p_1 - p_2) \cdot 2\pi r dr \tag{16}$$

$$dF_2 = (p_2 - p_3) \cdot 2\pi r dr \tag{17}$$

arising from the reduction in pressure across each rotor.

The magnitudes of the elements of torque generated by the airstream in the annulus r,dr for rotor 1 and rotor 2, respectively, are

$$dQ_1 = dQ_2 = \rho u \cdot 2\pi r dr \cdot \omega r \cdot r \tag{18}$$

determined from the rate of change of angular momentum in the annulus for each rotor. It should be noted that dQ_1 and dQ_2 have the same magnitude but act in opposite directions.

The input of energy in unit time to rotor 1 in the annulus r,dr is $E_1 \cdot 2\pi r dr \cdot u$ and the output in unit time is $\Omega_1 dQ_1$ where Ω_1 is the angular velocity of rotor 1. Then the efficiency of the energy conversion for rotor 1 in the annulus r,dr is (see (7))

$$\eta_1 = \frac{4\lambda_1 \epsilon}{k_1} \tag{19}$$

where we have written

$$\lambda_1 = \frac{\Omega_1 r}{u} = \Lambda_1 R \tag{20}$$

$$\Lambda_1 = \frac{\Omega_1 b}{u} \tag{21}$$

Similarly for rotor 2 the efficiency in the annulus r,dr is (see (8))

$$\eta_2 = \frac{4\lambda_2 \epsilon}{k_2} \tag{22}$$

where

$$\lambda_2 = \frac{\Omega_2 r}{u} = \Lambda_2 R \tag{23}$$

$$\Lambda_2 = \frac{\Omega_2 b}{u} \tag{24}$$

Then the combined efficiency for the two rotors in the annulus r,dr is

$$\eta = \frac{\Omega_1 dQ_1 + \Omega_2 dQ_2}{(k_1 + k_2) \cdot \frac{1}{2}\rho u^2 \cdot 2\pi r dr \cdot u} = \frac{4\epsilon(\lambda_1 + \lambda_2)}{k} \tag{25}$$

where

$$k = k_1 + k_2 \tag{26}$$

The rotor efficiency in the annulus r, dr can also be expressed in a form more specifically related to the characteristics of the rotor blade element. Thus the input of energy to rotor 1 in the annulus r,dr may be written (see (5)),

$$E_1 \cdot 2\pi r dr \cdot u = \left[(p_1 - p_2) - \frac{1}{2}\rho\omega^2 r^2 \right] \cdot 2\pi\, r dr \cdot u$$

$$= u\, dF_1 - \frac{1}{2}\omega dQ_1 \tag{27}$$

The corresponding output of energy is $\Omega_1 dQ_1$ and hence we can write for the efficiency of rotor 1 in the elementary annulus r,dr

$$\eta_1 = \frac{\Omega_1 dQ_1}{u dF_1 - \frac{1}{2}\omega dQ_1} \tag{28}$$

Similarly from (6) the efficiency of rotor 2 in r,dr becomes

$$\eta_2 = \frac{\Omega_2 dQ_2}{u dF_2 + \frac{1}{2}\omega dQ_2} \tag{29}$$

where dQ_2 has the same magnitude as dQ_1.

5. BLADE ELEMENT THEORY

The efficient transfer of energy in the annulus is accomplished by designing each rotor with blade elements of appropriate shape, attitude to the resultant flow, and size. The flow and force diagrams based on the vortex theory of aerofoils applied to rotor 1 and rotor 2 are shown in Fig. 3.

Resolving the lift and drag (dL_1, dD_1; dL_2, dD_2) on the blade element in the annulus r,dr in directions parallel and perpendicular to u, we have for rotor 1 and rotor 2, respectively,

$$\begin{aligned} dX_1 &= dL_1 \sin\varphi_1 - dD_1 \cos\varphi_1 \\ dY_1 &= dL_1 \cos\varphi_1 + dD_1 \sin\varphi_1 \end{aligned} \tag{30}$$

and

$$\begin{aligned} dX_2 &= dL_2 \sin\varphi_2 - dD_2 \cos\varphi_2 \\ dY_2 &= dL_2 \cos\varphi_2 + dD_2 \sin\varphi_2 \end{aligned} \tag{31}$$

where, according to the velocity diagrams in Fig. 3,

$$\tan\varphi_1 = \frac{u}{\Omega_1 r + \frac{1}{2}\omega r} = \frac{1}{\lambda_1 + \epsilon} \tag{32}$$

and

$$\tan\varphi_2 = \frac{u}{\Omega_2 r - \frac{1}{2}\omega r} = \frac{1}{\lambda_2 - \epsilon} \tag{33}$$

The axial force and torque on the blade elements in r,dr of rotor 1 and rotor 2 are, respectively,

$$dF_1 = N_1 dY_1, \quad dQ_1 = N_1 r dX_1 \tag{34}$$

and

$$dF_2 = N_2 dY_2, \quad dQ_2 = N_2 r dX_2 \tag{35}$$

where N_1, N_2 are the numbers of blades for rotor 1 and rotor 2, respectively.

Substituting in (28) and (29), we have

$$\eta_1 = \frac{\lambda_1}{\frac{dY_1}{dX_1} - \epsilon} \tag{36}$$

$$\eta_2 = \frac{\lambda_2}{\frac{dY_2}{dX_2} + \epsilon} \tag{37}$$

From (30) and (31)

$$\frac{dY_1}{dX_1} = \frac{\gamma_1 + \tan\varphi_1}{\gamma_1 \tan\varphi_1 - 1} \tag{38}$$

$$\frac{dY_2}{dX_2} = \frac{\gamma_2 + \tan\varphi_2}{\gamma_2 \tan\varphi_2 - 1} \tag{39}$$

where we write

$$\gamma_1 = \frac{dL_1}{dD_1} \tag{40}$$

$$\gamma_2 = \frac{dL_2}{dD_2} \tag{41}$$

These are the aerodynamic lift/drag ratios for the blade elements. Then the above blade element efficiencies become

$$\eta_1 = 1 - \frac{(\lambda_1 + \epsilon)^2 + 1}{\gamma_1\lambda_1 + \epsilon(\lambda_1 + \epsilon) + 1} \tag{42}$$

and

$$\eta_2 = 1 - \frac{(\lambda_2 - \epsilon)^2 + 1}{\gamma_2\lambda_2 - \epsilon(\lambda_2 - \epsilon) + 1} \tag{43}$$

for rotor 1 and rotor 2, respectively.

An examination of these expressions for η_1 and η_2 shows that a high efficiency corresponds to large values for γ_1 and γ_2 provided λ_1, λ_2 and ϵ are of order 1 or less. Aerodynamic information on various blade sections (see Ref. 1 for example) shows that lift/drag ratios in excess of 50 are possible. The variations of η_1 and η_2 over a range of λ_1, λ_2 for $\gamma_1 = \gamma_2 = 50$ and given values of ϵ are shown in Fig. 4. A significant fact indicated by these curves is that for the high values of γ_1, γ_2 selected and a relatively wide range of ϵ, the maximum blade element efficiencies (η_1, η_2) correspond to values for both λ_1 and λ_2 approximately between 1 and 2. We note also that η_1 decreases as ϵ increases for given λ_1, λ_2 and for η_2 the reverse is the case. Also, since rotor 2 recovers the rotational energy lost by rotor 1, then rotor 2 operates at a higher efficiency.

The choice of λ_1 and λ_2 is an important question for the designer. Various factors other than aerodynamic requirements may be involved. In this investigation, which involves aerodynamic theory only, the emphasis is on combinations of λ_1 and λ_2 that will ensure the highest overall blade element efficiency. To this end values of η have been determined for various combinations of λ_1, λ_2 at two values of ϵ (0.2 (small) and 1.0 (large), Tables I, II). By equating the two expressions for the blade element efficiency for rotor 1 (see (19) and (42)) we obtain the following relation for k_1,

$$k_1 = 4\epsilon \left[\lambda_1 + \frac{(\lambda_1 + \epsilon)^2 + 1}{\gamma_1 - (\lambda_1 + \epsilon)} \right] \tag{44}$$

Similarly, by equating (22) and (43), we have

$$k_2 = 4\epsilon \left[\lambda_2 + \frac{(\lambda_2 - \epsilon)^2 + 1}{\gamma_2 - (\lambda_2 \quad \epsilon)} \right] \tag{45}$$

We can now evaluate the overall blade element efficiency for the two rotors combined for various choices of λ_1 and λ_2 (see (25)). The results are given in Tables I and II.

We conclude from Table I, corresponding to a small value of ϵ and large value of γ_1, γ_2 that overall blade element efficiencies of about 95% are possible for many combinations of λ_1 and λ_2 and values of k up to about 3. These results show that, so long as ϵ is small and γ_1, γ_2 large, the values of λ_1 and λ_2 can be selected according to other requirements as well as the aerodynamic and still maintain a high efficiency. For best results λ_1 and λ_2 should be in the neighbourhood of 1 - 1.5. At larger values of ϵ and k overall blade element efficiencies of over 90% are still possible (Table II). Best efficiencies occur under these conditions for $\lambda_2 > \lambda_1$.

The designer may find it convenient to choose the same values for λ_1 and λ_2 along the radius. The overall blade element efficiency (η) for $\lambda_1 = \lambda_2$ is plotted in Fig. 4. We note that high values for η correspond to low values for ϵ with λ_1, λ_2 above 0.5 and below 2.0.

6. ENERGY INPUT TO THE ROTORS

Returning to (1) and (2), we can write

$$k\left(\frac{u}{u_\infty}\right)^2 + k_D\left(\frac{u}{u_\infty}\right)^2 = 1 - \left(\frac{u_w}{u_\infty}\right)^2 \tag{46}$$

where

$$k = \frac{H}{\frac{1}{2}\rho u^2} = k_1 + k_2 \tag{47}$$

is the coefficient of total energy input to the rotors. The basic purpose of design is to make the right hand side of (46) as close to 1 as possible by maximizing the left hand side such that the energy input to the rotors is very much greater than the energy loss in the duct ($k \gg k_D$). The magnitude of the total duct loss coefficient (k_D) can be kept small compared with the energy input coefficient (k) since a large value of k is possible without serious loss of efficiency for a contrarotating system (see Tables I, II).

Duct losses in aerodynamic systems similar to that considered here are discussed in Ref. 6. Losses arise mainly during pressure recovery at the inlet ($u < u_\infty$) and from viscous action around the boss fairing and obstructions such as supports (Fig. 2). Loss due to skin friction is comparatively small and can be neglected. This subject needs further study as it relates to wind turbines, but information presently available suggests that k_D is of order 0.1. By comparison the value of k might be placed at 2. The selection of k is also a matter for further experimental investigation.

If (46) is solved for u_w/u_∞, then we find that for $k + k_D = 2.1$ the value of u/u_∞ must be less than 0.69 (corresponding to zero velocity in the wake).

7. SLIPSTREAM ROTATION BETWEEN ROTORS AND OVERALL EFFICIENCIES

We have seen in Section 3 that, if k_1 is chosen to be constant and the same for all values of r for rotor 1, then

$$\epsilon R = \beta \tag{48}$$

where β is constant. Then the blade element efficiency for rotor 1 becomes (see (19))

$$\eta_1 = \frac{4\lambda_1 \epsilon}{k_1} = \frac{4\Lambda_1 \beta}{k_1} \tag{49}$$

Thus the blade element efficiency η_1 is constant and the same for all radial distances for rotor 1 and therefore η_1 becomes the overall efficiency for rotor 1.

The combined blade element efficiency for the two rotors may be written (see (25))

$$\eta = \frac{4\epsilon(\lambda_1 + \lambda_2)}{k} = \frac{4\beta}{k}(\Lambda_1 + \Lambda_2) \tag{50}$$

Hence the product $k\eta$ is constant and, since η will be kept close to 1 for all r, little variation of k would be expected and we can take k = constant and η (also constant) now becomes the efficiency for the total energy conversion.

According to (26), k_2 is also constant and from (22) η_2 is constant and becomes the efficiency for rotor 2.

8. OVERALL FORCE AND TORQUE

We define the coefficients of force acting on rotor 1 and rotor 2, respectively, as follows:

$$C_{F_1} = \frac{F_1}{\frac{1}{2}\rho u^2 \cdot \pi b^2} \tag{51}$$

$$C_{F_2} = \frac{F_2}{\frac{1}{2}\rho u^2 \cdot \pi b^2} \tag{52}$$

Substituting for (5) and (6) in (16) and (17), respectively, then

$$dF_1 = (E_1 + \frac{1}{2}\rho\omega^2 r^2) \cdot 2\pi r dr \tag{53}$$

and

$$dF_2 = (E_2 - \frac{1}{2}\rho\omega^2 r^2) \cdot 2\pi r dr \tag{54}$$

Introducing the force coefficients defined above,

$$dC_{F_1} = 2R(k_1 + 4\epsilon^2)dR \tag{55}$$

and

$$dC_{F_2} = 2R(k_2 - 4\epsilon^2)dR \tag{56}$$

These expressions can be integrated readily with the help of (48). Thus

$$C_{F_1} = 2\int_{R_a}^{1} R\left(k_1 + \frac{4\beta^2}{R^2}\right) dR \tag{57}$$

$$C_{F_2} = 2\int_{R_a}^{1} R\left(k_2 - \frac{4\beta^2}{R^2}\right) dR \tag{58}$$

where the integration is taken over the range from the radius of the boss ($r = a$) to the blade tip ($r = b$). Then integration gives

$$C_{F_1} = k_1(1 - {R_a}^2) - 8\beta^2 \log R_a \tag{59}$$

and

$$C_{F_2} = k_2(1 - {R_a}^2) + 8\beta^2 \log R_a \tag{60}$$

We also define the coefficient of torque developed for each rotor as follows:

$$C_{Q_1} = \frac{Q_1}{\frac{1}{2}\rho u^2 \cdot \pi b^3} \tag{61}$$

$$C_{Q_2} = \frac{Q_2}{\frac{1}{2}\rho u^2 \cdot \pi b^3} \tag{62}$$

where C_{Q_1} and C_{Q_2} are equal in magnitude but opposite in direction. Then from (18)

$$dC_{Q_1} = dC_{Q_2} = 8R^2 \epsilon \, dR \tag{63}$$

and upon integration with the help of (48),

$$C_{Q_1} = C_{Q_2} = 8\beta \int_{R_a}^{1} R dR = 4\beta(1 - R_a{}^2) \tag{64}$$

It should be noted that η, η_1, η_2, C_{F_1}, C_{F_2}, C_{Q_1}, C_{Q_2} can all be determined without detailing the geometrical shape of the rotor blade so long as the requirement for large (but permissible) values of γ_1, γ_2 (high η_1, η_2) is met.

9. EFFICIENCY AND THE LIFT/DRAG RATIO

It will be noted that the condition for pure rotating flow in the transverse planes between the rotors has led us to a design method based on constant blade element efficiency along the radius for both rotors and for the combination. The choices of η_1, η_2 and η are governed by the possible values of γ_1, γ_2 that are available for known aerofoil sections (see Ref. 1). For given values of η_1 and η_2, γ_1 and γ_2 can be found from (42) and (43) expressed in the form

$$\gamma_1 = \frac{1}{\lambda_1}\left[\frac{(\lambda_1 + \epsilon)^2 + 1}{1 - \eta_1} - \epsilon(\lambda_1 + \epsilon) - 1\right] \tag{65}$$

$$\gamma_2 = \frac{1}{\lambda_2}\left[\frac{(\lambda_2 - \epsilon)^2 + 1}{1 - \eta_2} + \epsilon(\lambda_2 - \epsilon) - 1\right] \tag{66}$$

As an illustration of the restriction which γ_1, γ_2 place on η_1, η_2, Fig. 5 has been prepared for a design in which $1 \leq \lambda_1$, $\lambda_2 \leq 2$ and $\epsilon\lambda_1 = 0.2$. It is evident from Fig. 5 that $\gamma_1 > \gamma_2$ for the same blade element efficiency and for both rotors the lift/drag ratio is greater at the tip than it is at the hub. The range of variation of γ_2 from hub to tip for a given efficiency is greater than the corresponding range for γ_1.

10. OTHER BLADE SECTION PROPERTIES

When γ_1, γ_2 have been calculated, the designer must choose an appropriate aerofoil shape with known two-dimensional aerodynamic properties as the blade section at the appropriate radius r. The two-dimensional aerodynamic characteristics of an aerofoil, determined experimentally, are available from many sources. It was convenient for the writer to obtain his information from Ref. 1, but many other references can be used. The required information includes the coordinates of the aerofoil shape and tables or plots of the lift/drag ratio (γ_1, γ_2), the lift coefficient (C_{L_1}, C_{L_2}) and the drag coefficient (C_{D_1}, C_{D_2}) versus the angle of incidence (α_1, α_2), where

$$C_{L_1} = \frac{dL_1}{\frac{1}{2}\rho w_1^2 \cdot c_1 dr}, \qquad C_{L_2} = \frac{dL_2}{\frac{1}{2}\rho w_2^2 \cdot c_2 dr} \qquad (67)$$

$$C_{D_1} = \frac{dD_1}{\frac{1}{2}\rho w_1^2 \cdot c_1 dr}, \qquad C_{D_2} = \frac{dD_2}{\frac{1}{2}\rho w_2^2 \cdot c_2 dr} \qquad (68)$$

Then

$$C_{X_1} = \frac{dX_1}{\frac{1}{2}\rho w_1^2 \cdot c_1 dr} = C_{L_1}\sin\varphi_1 - C_{D_1}\cos\varphi_1 \qquad (69)$$

$$C_{X_2} = \frac{dX_2}{\frac{1}{2}\rho w_2^2 \cdot c_2 dr} = C_{L_2}\sin\varphi_2 - C_{D_2}\cos\varphi_2 \qquad (70)$$

The variations of γ_1, γ_2; C_{L_1}, C_{L_2}; C_{D_1}, C_{D_2} with α_1, α_2 will be different for various Reynolds numbers (Re_1, Re_2) where

$$Re_1 = \frac{\rho c_1 w_1}{\mu}, \qquad Re_2 = \frac{\rho c_2 w_2}{\mu} \qquad (71)$$

(see Ref. 1) and μ is the coefficient of viscosity for an average atmospheric temperature. Before appropriate values of C_{L_1}, C_{D_1}, α_1 and C_{L_2}, C_{D_2}, α_2 can be selected from available information consistent with the calculated values of γ_1 and γ_2, respectively, an estimate of the Reynolds number is required.

The selection of these aerodynamic characteristics for the blade section will be valid so long as two-dimensional data applies. Thus, if multiplane interference occurs between adjacent blades of the rotors due to close proximity, the actual values of the coefficients will be different and some allowance for this form of interference may be necessary by appropriate adjustments of the coefficients (see Fig. 2 in Ref. 4). To assess possible multiplane interference effects, the solidities S_1 and S_2 for rotor 1 and rotor 2, respectively, should be determined where

$$S_1 = \frac{N_1 c_1}{2\pi r}, \qquad S_2 = \frac{N_2 c_2}{2\pi r} \tag{72}$$

From (34), (61) and (69) we can write

$$d\left(C_{Q_1} \cdot \frac{1}{2}\rho u^2 \cdot \pi b^3 \right) = N_1 r \cdot C_{X_1} \cdot \frac{1}{2}\rho w_1{}^2 \cdot c_1 dr \tag{73}$$

which becomes

$$\frac{dC_{Q_1}}{dR} = \frac{2S_1 R^2 C_{X_1}}{\sin^2\varphi_1} \tag{74}$$

Equating (74) with (63), we have for rotor 1

$$S_1 = \frac{4\epsilon \sin^2\varphi_1}{C_{X_1}} \tag{75}$$

Similarly for rotor 2 we obtain

$$S_2 = \frac{4\epsilon \sin^2\varphi_2}{C_{X_2}} \tag{76}$$

Difficulties with regard to the use of two-dimensional aerodynamic information can develop also if the tip speed exceeds about half the speed of sound and compressibility effects occur. Then the assumption of constant density (ρ) is no longer valid. This limitation on tip speed will not likely be a problem in contrarotating wind turbines since, as we have already seen, for good design the ranges of $\Omega_1 r/u$, $\Omega_2 r/u$ are both approximately $0.5 < \lambda < 2$ and u ($< u_\infty$) should be well below the speed of sound (see Fig. 2 in Ref. 4).

The final property of the blade section, required to complete the geometrical shape, is the angle of the blade section to the plane of rotation. From Fig. 3,

$$\theta_1 = \varphi_1 - \alpha_1, \qquad \theta_2 = \varphi_2 - \alpha_2 \tag{77}$$

11. SUGGESTED PROCEDURE FOR AERODYNAMIC DESIGN

In this section a procedure is suggested for the aerodynamic design of a contrarotating wind turbine system. The calculations are shown in dimensionless form so that the design applies to any prescribed output of power. The various steps are described as follows:

(1) The value of k is selected in accordance with the discussion in Section 6 in which the purpose is to optimize the energy input to the rotors and minimize duct losses. We choose k = 2.

(2) As suggested by Table I and Fig. 4 we select $\lambda_1 = \lambda_2 = \lambda$ as consistent with high efficiency. We note that $\Lambda_1 = \Lambda_2 = \Lambda$ and $\lambda = \Lambda R$.

(3) From the data presented in Fig. 4 and Tables I and II we note that for the same λ we expect η_2 to be greater than η_1 and that $\eta_1 = 0.94$, $\eta_2 = 0.96$ are possible efficiencies. With $\lambda = \lambda_1 = \lambda_2$, then

$$\eta_1 = \frac{4\lambda\epsilon}{k_1}, \quad \eta_2 = \frac{4\lambda\epsilon}{k_2}, \quad \eta = \frac{8\lambda\epsilon}{k} \tag{78}$$

and therefore

$$\eta = 2\left(\frac{\eta_1\eta_2}{\eta_1 + \eta_2}\right) \tag{79}$$

According to the above selection of η_1, η_2 we find that $\eta = 0.95$.

(4) The condition for rotating flow only in the planes between the rotors (Section 3) now takes the form

$$\epsilon\lambda = \frac{k\eta}{8} = 0.2375 \tag{80}$$

(5) The coefficients of energy input to rotor 1 and rotor 2 are, therefore,

$$k_1 = \frac{4\lambda\epsilon}{\eta_1} = 1.01 \tag{81}$$

$$k_2 = \frac{4\lambda\epsilon}{\eta_2} = 0.99 \tag{82}$$

respectively. Then $k = k_1 + k_2 = 2.00$.

(6) The selection of the blade element Reynolds number relates to the scale of the turbine system and must be estimated accordingly. In the example Re has been chosen arbitrarily to be 0.3×10^6, a value consistent with a turbine system of moderate scale.

(7) The determination of the actual geometry of the rotor shapes should begin with a calculation of γ_1 and γ_2 to ensure that the choices of η_1, η_2 and η are consistent with possible values of the lift/drag ratios for the estimated Reynolds number. The remaining details are shown in Tables III and IV.

It is important to note that the dimensionless design procedure recommended here is based on calculations for a range of values of λ. This procedure permits the appropriate choice of the range of R after the calculations have been completed.

Tables III and IV show that the tip value of R (R = 1) has been chosen to correspond to $\lambda = 2.0$. Then $\Lambda = \lambda/R = 2$. This choice of R at the tip was considered feasible since no excessive values of γ_1, γ_2 were encountered up to $\lambda = 2$. In other designs this may not happen and it might be necessary to choose Λ so that R = 1 corresponds to a lower value of λ (e.g. $\lambda = 1.8$).

The choice of λ at the hub also needs to be studied. For example, good structural strength would require c_1/c_{b_1}, c_2/c_{b_2} (or RS_1/S_{b_1}, RS_2/S_{b_2}) to increase along the blade from tip to root. In the design example presented here this occurs for rotor 1 down to $\lambda = 1.0$. In these circumstances it may be advisable to exclude the stations for $\lambda < 1.0$ and choose $R_a = 0.5$ (see Tables III, IV).

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TABLE I

COMBINED BLADE ELEMENT EFFICIENCIES FOR $\epsilon = 0.2$, $\gamma_1 = \gamma_2 = 50$

λ_1	λ_2	k_1	k_2	$\lambda_1+\lambda_2$	k_1+k_2	η
0.5	0.5	0.424	0.418	1.0	0.842	0.950
	1.0		0.827	1.5	1.251	0.959
	1.5		1.244	2.0	1.668	0.959
	2.0		1.670	2.5	2.094	0.955
1.0	0.5	0.840	0.418	1.5	1.258	0.954
	1.0		0.827	2.0	1.667	0.960
	1.5		1.244	2.5	2.085	0.960
	2.0		1.670	3.0	2.510	0.956
1.5	0.5	1.264	0.418	2.0	1.682	0.951
	1.0		0.827	2.5	2.091	0.956
	1.5		1.244	3.0	2.508	0.957
	2.0		1.670	3.5	2.934	0.954
2.0	0.5	1.698	0.418	2.5	2.116	0.945
	1.0		0.827	3.0	2.525	0.950
	1.5		1.244	3.5	2.942	0.952
	2.0		1.670	4.0	3.368	0.950

TABLE II

COMBINED BLADE ELEMENT EFFICIENCIES FOR $\epsilon = 1.0$, $\gamma_1 = \gamma_2 = 50$

λ_1	λ_2	k_1	k_2	$\lambda_1+\lambda_2$	k_1+k_2	η
0.5	0.5	2.268	2.099	1.0	4.367	0.916
	1.0		4.080	1.5	6.348	0.945
	1.5		6.101	2.0	8.369	0.956
	2.0		8.163	2.5	10.431	0.959
1.0	0.5	4.417	2.099	1.5	6.516	0.921
	1.0		4.080	2.0	8.497	0.942
	1.5		6.101	2.5	10.518	0.951
	2.0		8.163	3.0	12.580	0.954
1.5	0.5	6.611	2.090	2.0	8.710	0.918
	1.0		4.080	2.5	10.691	0.935
	1.5		6.101	3.0	12.712	0.944
	2.0		8.163	3.5	14.774	0.948
2.0	0.5	8.851	2.099	2.5	10.950	0.913
	1.0		4.080	3.0	12.931	0.928
	1.5		6.101	3.5	14.952	0.936
	2.0		8.163	4.0	17.014	0.940

TABLE III

ROTOR 1

$k = 2$, $k_1 = 1.01$, $\eta = 0.95$, $\eta_1 = 0.94$, $\lambda_1 = \lambda$, $\epsilon\lambda = 0.2375$, $Re \doteq 0.3 \times 10^6$, Section E (Ref. 1)

λ	ϵ	γ_1	C_{L_1}	C_{D_1}	α_1°	φ_1°	θ_1°	C_{X_1}	S_1	c_1/c_{b_1}	R
0.6	0.3958	53.0	0.790	0.0149	3.60	45.12	41.52	0.549	1.447	1.316	0.3
0.8	0.2969	44.2	0.625	0.0141	1.75	42.35	40.60	0.411	1.313	1.592	0.4
1.0	0.2375	40.9	0.565	0.0138	1.20	38.95	37.75	0.345	1.090	1.652	0.5
1.2	0.1979	40.0	0.552	0.0138	1.15	35.58	34.43	0.309	0.867	1.578	0.6
1.4	0.1696	40.3	0.558	0.0138	1.20	32.50	31.30	0.288	0.680	1.443	0.7
1.6	0.1484	41.5	0.580	0.0140	1.30	29.77	28.47	0.276	0.531	1.287	0.8
1.8	0.1319	43.1	0.605	0.0140	1.35	27.37	26.02	0.266	0.420	1.145	0.9
2.0	0.1188	45.1	0.645	0.0143	1.90	25.27	23.37	0.262	0.330	1.000	1.0

$C_{F_1} = 0.8357$, $C_{Q_1} = 0.3563$, $\beta = 0.1188$

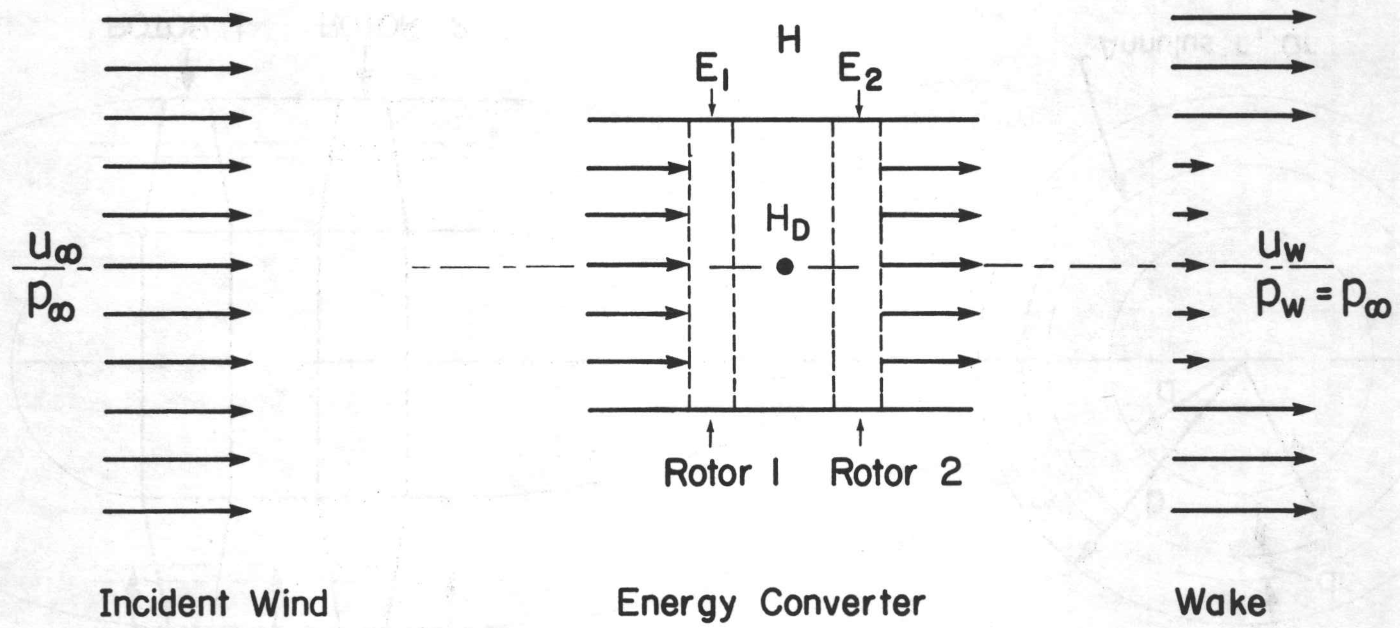
TABLE IV

ROTOR 2

$k = 2$, $k_2 = 0.99$, $\eta = 0.95$, $\eta_2 = 0.96$, $\lambda_2 = \lambda$, $\epsilon\lambda = 0.2375$, $Re \doteq 0.3 \times 10^6$, Section E (Ref. 1)

λ	ϵ	γ_2	C_{L_2}	C_{D_2}	α_2°	φ_2°	θ_2°	C_{X_2}	S_2	c_2/c_{b_2}	R
0.6	0.3958	41.9	0.585	0.0140	1.40	78.47	77.07	0.570	2.665	3.255	0.3
0.8	0.2969	38.1	0.522	0.0137	0.85	63.30	62.45	0.460	2.060	3.354	0.4
1.0	0.2375	38.7	0.540	0.0139	0.93	52.68	51.75	0.421	1.427	2.905	0.5
1.2	0.1979	41.1	0.575	0.0140	1.33	44.93	43.60	0.396	0.997	2.435	0.6
1.4	0.1696	44.3	0.628	0.0142	1.85	39.10	37.25	0.385	0.701	1.997	0.7
1.6	0.1484	48.1	0.690	0.0144	2.40	34.57	32.17	0.380	0.503	1.640	0.8
1.8	0.1319	52.1	0.782	0.0150	3.40	30.95	27.55	0.389	0.359	1.314	0.9
2.0	0.1188	56.4	0.939	0.0167	5.00	20.00	15.00	0.426	0.246	1.000	1.0

$C_{F_2} = 0.6643$, $C_{Q_2} = 0.3563$, $\beta = 0.1188$



H
E_1
E_2
H_D
u_∞
p_∞
u_w
$p_w = p_\infty$
Rotor 1
Rotor 2
Incident Wind
Energy Converter
Wake

Fig. 1

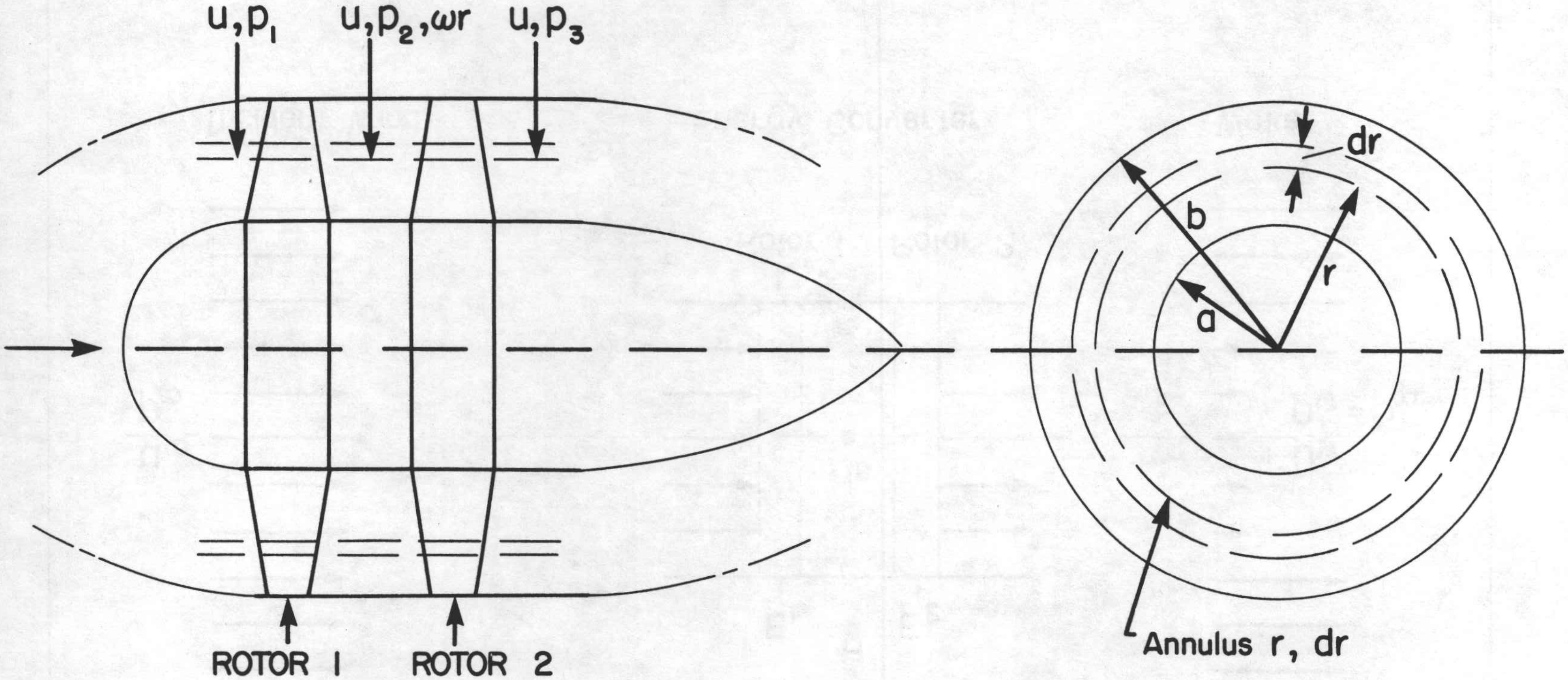
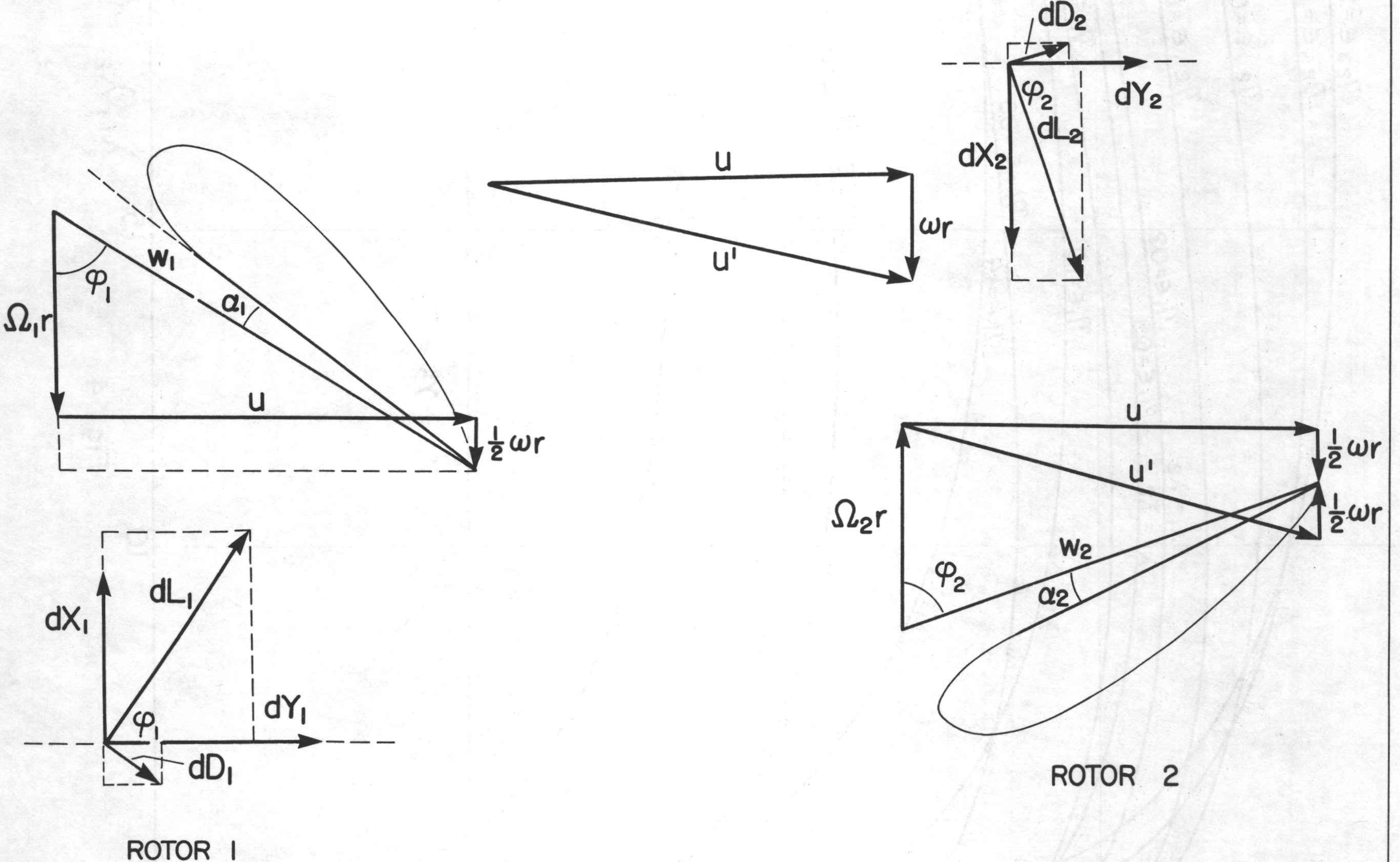


Fig. 2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φ2
dL2
dX2
u
ωr
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w1
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α1
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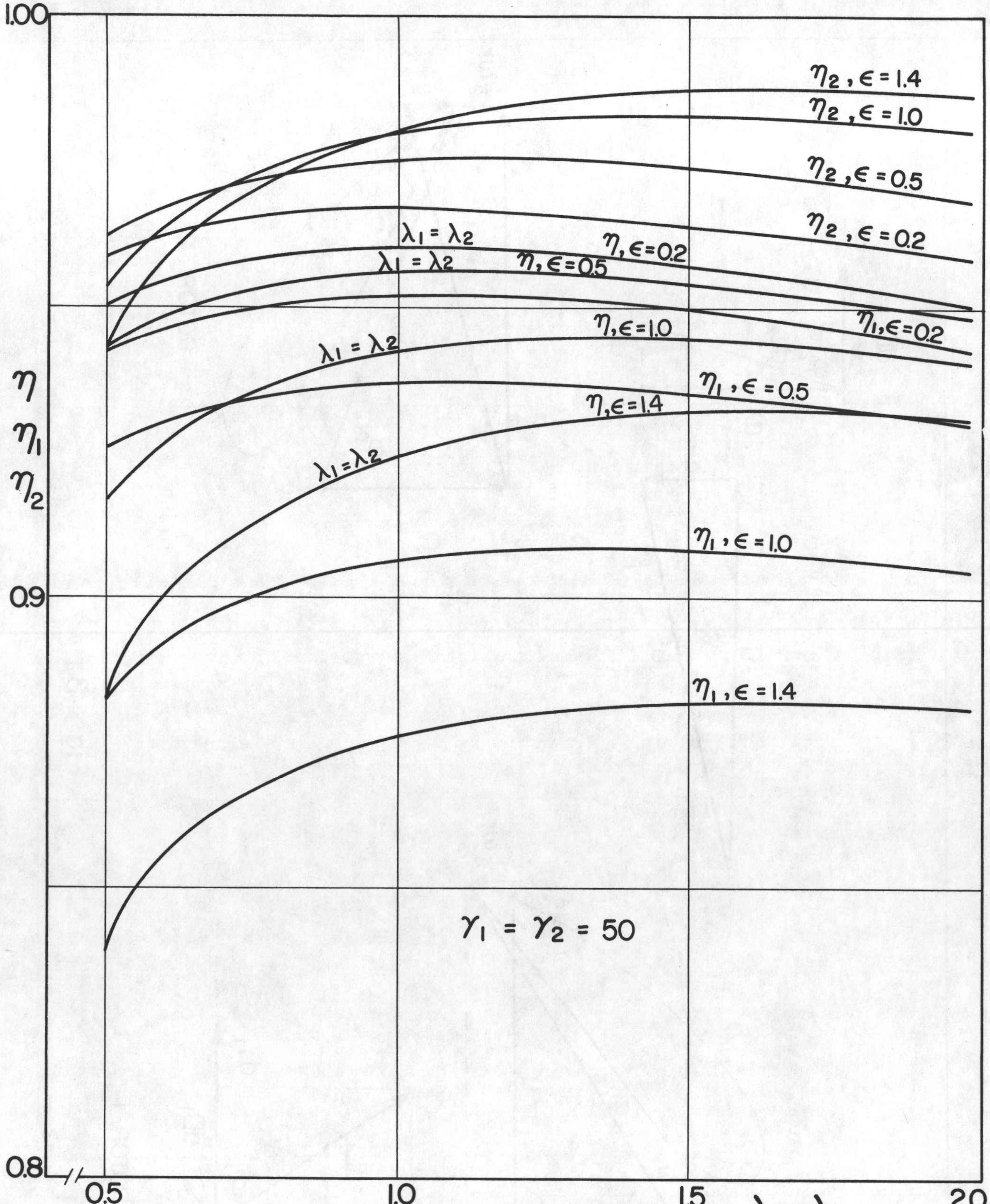
u
½ωr
u'
½ωr
Ω2r
w2
φ2
α2
dX1
dL1
dY1
φ1
dD1
ROTOR 1
ROTOR 2

Fig. 3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0.8
η
η_1
η_2
0.5
1.0
1.5
2.0
λ_1, λ_2
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$\eta_2, \epsilon = 1.0$
$\eta_2, \epsilon = 0.5$
$\eta_2, \epsilon = 0.2$
$\lambda_1 = \lambda_2$
$\lambda_1 = \lambda_2$
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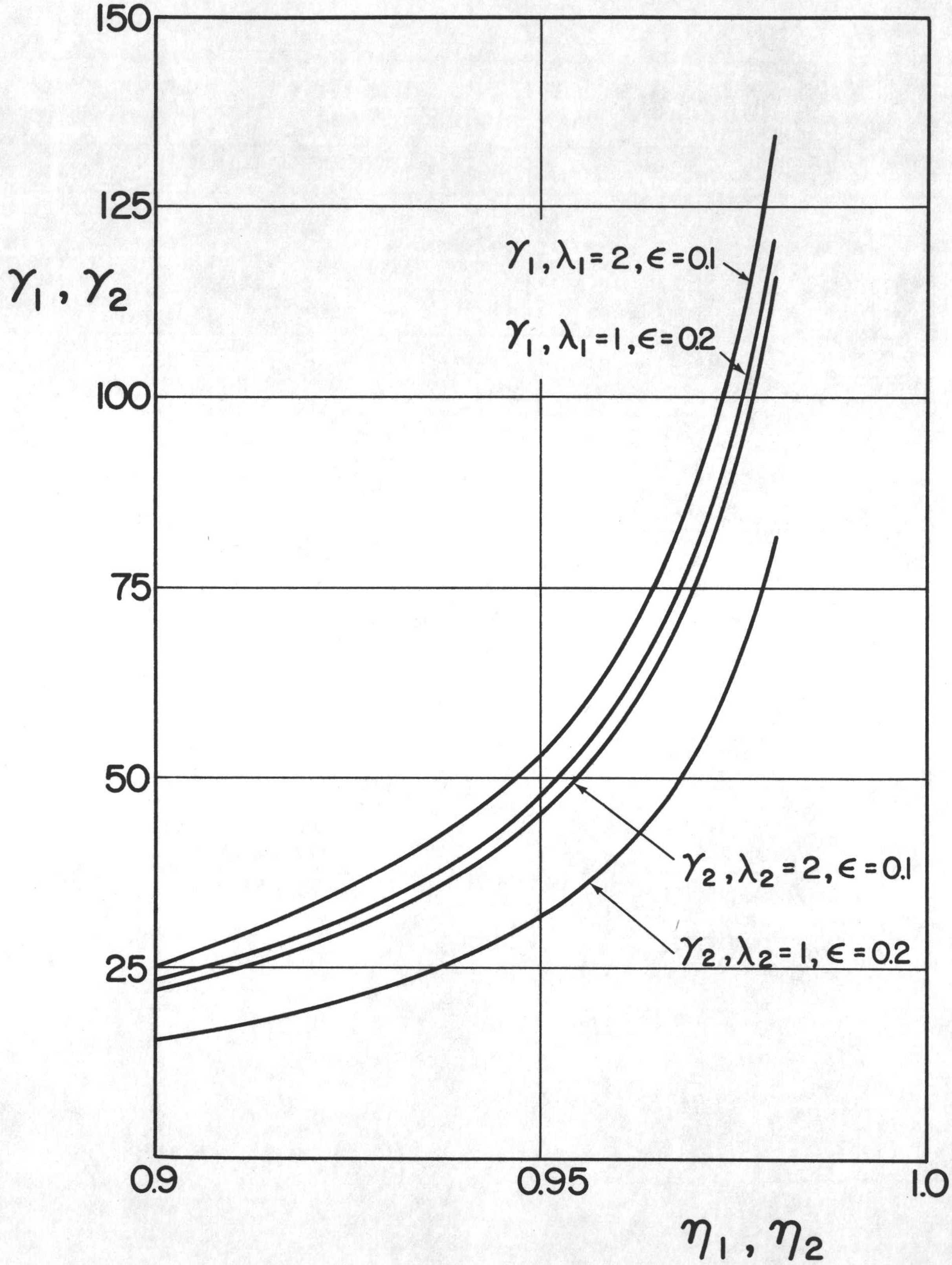
$\eta_1, \epsilon = 1.4$
$\gamma_1 = \gamma_2 = 50$

Fig. 4


150
125
100
75
50
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γ_1 , γ_2
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$\gamma_2 , \lambda_2 = 2, \epsilon = 0.1$
$\gamma_2 , \lambda_2 = 1, \epsilon = 0.2$
0.9
0.95
1.0
η_1 , η_2

Fig. 5