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A theoretical model for suspended sediment transport in river bends

December 1989

A.M. Talmon
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**Note:** The changes include corrections for the equations and expressions involving \( \mathbf{u}_c \), \( \mathbf{u}_s \), \( \mathbf{v}_c \), and \( \mathbf{v}_s \) with respect to the coordinate axes and bases.
A THEORETICAL MODEL FOR
SUSPENDED SEDIMENT TRANSPORT
IN RIVER BENDS

by

A.M. Talmon

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ABSTRACT

A two dimensional depth-averaged model for the concentration field of suspended sediment in river bend flow is formulated. Transport of suspended sediment in horizontal and vertical directions is modelled. Convection by the main and secondary flow and turbulent diffusion are incorporated. The model is capable of computing the exchange of sediment with the bed-load layer adjacent to the bed.

The model is based on the three dimensional convection-diffusion equation formulated in a cylindrical bed following coordinate system. Non-orthogonality of the coordinate system is included. The concentration field is modelled by a first order asymptotic solution of the convection-diffusion equation. The zeroth order contribution to the solution is given by equilibrium shape concentration verticals, the first order contribution is due to the convection by the flow. The result is formulated in depth-averaged variables.

The time and length scales of the model are computed. These scales are significantly affected by the choice of reference level, the choice of boundary condition at this level, the suspension parameter and the Chezy coefficient.

The model will have to be calibrated and verified by measurements and numerical computations.
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Appendix A Scales of the depth-averaged convection-diffusion equation
SYMBOLS

\( a \)  
water depth  \([\text{L}]\)

\( a \)  
acceleration of sediment  \([\text{L/T}^2]\)

\( A \)  
length scale for water depth  \([\text{L}]\)

\( a_b \)  
constant in the transport formule for bed-load  \([-\text{]}\)

\( a_s \)  
constant in the transport formule for suspended-load  \([-\text{]}\)

\( \{ b_{ij} \} \)  
perturbation profiles, which contribute to the first order perturbation profile \( c_1 \)  \([-\text{]}\)

\( b_b \)  
exponent in the transport formule for bed-load  \([-\text{]}\)

\( b_s \)  
exponent in the transport formule for suspended-load  \([-\text{]}\)

\( B \)  
constant of the \( r_c \) shape function  \([-\text{]}\)

\( c \)  
concentration  \([-\text{]}\)

\( \dot{c}_h \)  
depth-averaged concentration, (over \( h \))  \([-\text{]}\)

\( C \)  
Chézy coefficient  \([-\text{]}\)

\( c_e \)  
the equilibrium concentration  \([-\text{]}\)

\( \dot{c}_e \)  
the depth-averaged equilibrium concentration  \([-\text{]}\)

\( \dot{c}_h \)  
the depth-averaged concentration  \([-\text{]}\)

\( c_0 \)  
zeroth order contribution to the concentration  \([-\text{]}\)

\( c_j \)  
first order contribution to the concentration  \([-\text{]}\)

\( c_{j+1} \)  
local concentration, approximated to \( j \)th order  \([-\text{]}\)

\( d \)  
sediment diameter  \([\text{L}]\)

\( D \)  
the diffusion parameter  \([-\text{]}\)

\( E \)  
sediment flux between the bed- and the suspen.-load region  \([\text{L/T}]\)

\( f_* \)  
friction factor  \([-\text{]}\)

\( g \)  
acceleration due to gravity  \([\text{L/T}^2]\)

\( g_z \)  
\( \frac{\partial r_c}{\partial Z} \)  \([-\text{]}\)

\( g_{ij} \)  
the metric tensor  \([\text{L}^2]\)

\( g_{ij}^{-1} \)  
the inverse of the metric tensor  \([\text{L}^{-2}]\)

\( J = \sqrt{g} \)  
the jacobian  \([\text{L}^3]\)

\( h \)  
depth of suspended-load region  \([\text{L}]\)

\( h_i \)  
normalizing factor of the \( \tilde{x}_i \) coordinate system  \([\text{1/L}]\)

\( k_s \)  
roughness height  \([\text{L}]\)

\( L \)  
length scale in streamwise direction  \([\text{L}]\)

\( L_z \)  
length scale in vertical direction over which the concentration changes  \([\text{L}]\)
\[ L_{hs} \quad L_{hn} \quad L_{us} \quad L_{un} \quad L_{vun} \]
\[ n \quad \text{coordinate in transversal direction} \quad [L] \]
\[ \text{OP}_{\xi}[] \quad \text{the Galappatti operator as function of the } \xi \text{-coordinate} \quad [-] \]
\[ r_c \quad \text{shape function for zeroth order concentration vertical} \quad [L] \]
\[ r_s \quad \text{radius of curvature of the flow} \quad [L] \]
\[ r_u \quad \text{shape functions of the velocity profiles} \quad [-] \]
\[ r_v \quad \text{shape function of the turbulent eddy viscosity} \quad [-] \]
\[ R_u \quad \text{integral of } r_u \text{ function (} R_u = \int r_u d\xi \text{)} \quad [-] \]
\[ R_v \quad \text{integral of } r_v \text{ function (} R_v = \int r_v d\xi \text{)} \quad [-] \]
\[ s \quad \text{water surface level} \quad [L] \]
\[ s_s \quad \text{coordinate in streamwise direction} \quad [L] \]
\[ \text{a scalar} \quad [-] \]
\[ S_{\text{bed } s} \quad \text{bed-load transport rate in } s \text{-direction (per unit width)} \quad \frac{L^2}{T} \]
\[ S_{\text{bed } n} \quad \text{bed-load transport rate in } n \text{-direction (per unit width)} \quad \frac{L^2}{T} \]
\[ S_{\text{susp } s} \quad \text{suspended-load transport rate in } s \text{-dir. (per unit width)} \quad \frac{L^2}{T} \]
\[ S_{\text{susp } n} \quad \text{suspended-load transport rate in } n \text{-dir. (per unit width)} \quad \frac{L^2}{T} \]
\[ t \quad \text{time} \quad [T] \]
\[ T_{ct} \quad \text{time scale} \quad [T] \]
\[ T_{ht} \quad \text{time scales of the depth-averaged c-d equation} \quad [T] \]
\[ u \quad \text{velocity in streamwise direction} \quad \frac{L}{T} \]
\[ u_r \quad \text{velocity in radial direction} \quad \frac{L}{T} \]
\[ \ddot{u} \quad \text{depth-averaged flow component in } \theta \text{-direction} \quad \frac{L}{T} \]
\[ u_{sec} \quad \text{secondary flow component} \quad [L/T] \]
\[ u_{sr} \quad \text{velocity component of the sediment relative to the flow} \quad [L/T] \]
\[ u_* \quad \text{friction velocity} \quad [L/T] \]
\[ U \quad \text{velocity scale in streamwise direction} \quad [L/T] \]
\[ U_* \quad \text{velocity scale for friction velocity} \quad [L/T] \]
\[ v \quad \text{velocity in normal direction} \quad [L/T] \]
\[ \dot{v} \quad \text{depth-averaged flow component in } r \text{-direction} \quad [L/T] \]
\[ V \quad \text{velocity scale in lateral direction} \quad [L/T] \]
\[ w \quad \text{velocity in vertical direction} \quad [L/T] \]
\( w_s \)  
fall velocity of sediment [L/T]

\( W \)  
length scale in lateral direction [L]

\( \hat{x}^i \)  
coordinates of the rectangular cartesian coordinate system [L]

\( \hat{x}^i \)  
coordinates of the transformed coordinate system [-]

\( x_b \)  
bed level [L]

\( x_r \)  
reference level [L]

\( Z \)  
shape parameter of the zeroth order concentration profile [-]

\( z \)  
coordinate in vertical direction, relative to the bed [-]

\( z_r \)  
reference level, relative to the bed [-]

\( \alpha_s \alpha_n \alpha_d \)  
integrals used in the depth-averaged c-d equation [-]

\( \beta_s \beta_n \)  
and the depth-averaged transport equations [-]

\( \gamma_s \gamma_n \)  
small parameter [-]

\( \delta_s \delta_n \)  
[-]

\( \epsilon_s \epsilon_n \)  
[-]

\( \epsilon \)  
[\( \nu \sqrt{\nu_t} \)] [-]

\( \kappa \)  
Von Karman constant [-]

\( \tau_b \)  
bond shear-stress [ML^2/T]

\( \nu \)  
viscosity [L^2/T]

\( \nu_t \)  
eddy viscosity [L^2/T]

\( \nu \chi \)  
turbulent diffusion coefficient of mass [L^2/T]

\( \Theta \)  
coordinates of the cylindrical, bed following coordinate [-]

\( \zeta \)  
system [-]

\( \Omega \chi \)  
mixing scale for diffusivity of mass [L^2/T]

\( \Gamma \)  
porosity of the bed [-]

\( \Gamma_{ij}^k \)  
the Christoffel symbol [-]

\( A \)  
a first order tensor [-]

\( \mathbf{i} \)  
unit vector [-]

\( u \)  
velocity vector of the flow [L/T]

\( u_s \)  
velocity vector of sediment relative to the flow [L/T]

\( \lambda_\theta \)  
unit vector in \( \theta \)-direction [-]

\( \lambda_r \)  
unit vector in \( r \)-direction [-]

\( \eta_1 \)  
contravariant basis vector [1/L]

\( r_1 \)  
covariant basis vector [L]

superscript ' : normalized variable (eq. 4,4)
1. INTRODUCTION

The project at hand is directed towards the computation of river bend morphology in case of rivers transporting a significant part of the bed material in suspension.

Important effects of suspended sediment on the morphology of river bends have been predicted with a strongly simplified analytical model (Olesen, 1987). These effects are to be attributed to the different physical processes of bed-load transport and suspended-load transport in curved flows. The morphology of a river bend with suspended sediment transport is intended to be computed with the aid of a 2-D depth-averaged model.

In this report the theoretical derivation of a depth-averaged suspended-load model is reported. The model is based on the method of asymptotic solution of the convection-diffusion equation which was originally developed by Galappatti (1983). The present theoretical model incorporates the sediment fluxes due to turbulent diffusion and the 3-D sediment fluxes by convection. Wang (1989) has developed a model of the same type which, however, is suited for estuary flow. Due to the morphology of river bends some complicating circumstances are encountered. The convection-diffusion equation has to be formulated in a 3-D bed-following cylindrical coordinate system. This introduces additional terms due to curvature and to non-orthogonality of the coordinate system. It is unknown whether these additional terms will significantly contribute to the solution.

To investigate the physics of the problem and to calibrate the model use will be made of laboratory measurements. Under these conditions, however, the effects of non-orthogonality and curvature are expected to play a larger role than in full scale conditions because of different geometries. Consequently the additional terms are included in the derivation of the theoretical model.

In chapter 2 the model is described briefly. In chapter 3 the transformation of the convection-diffusion to a bed-following cylindrical coordinate system is reported. In chapter 4 the equation is simplified by investigation of the magnitude of the individual terms. In chapter 5 the depth-averaged
convection-diffusion equation, in the bed-following cylindrical coordinate system, is formulated by applying the Galappatti method. In chapter 6 the depth-integrated transport equations are investigated. In chapter 7 the model is discussed. In chapter 8 the conclusions are presented.
2. **BRIEF OUTLINE OF THE MODEL**

To compute the morphology of river bends in case of suspended-load transport, the concentration field, suspended-load transport and sediment exchange with the bed-load layer will have to be computed.

A model for the concentration field is to be incorporated in the morphological model of Olesen (1987). At present (1989) the model computes the morphology of river bends in case of bed-load transport only, no bank erosion is incorporated. The model consists of a depth-averaged flow model and a bed-load transport model. Bed level changes are computed by use of a bed-load sediment balance. When the overall boundary conditions applied to the model are constant, the model will converge to a stationary state. The time dependent development of a river bend, without bank erosion, can be computed when time dependent boundary conditions are applied.

A suspended-sediment model will be incorporated. The concentration field has to be computed at every time-step. In one time-step, first the flow field will be computed, next the concentration field, next the transport rates of suspended-load and bed-load, and finally a new bed topography will be computed.

In conditions of non-curved flow, uniform water depth, stationary flow and stationary concentration field the vertical upward sediment flux due to turbulence is balanced by the downward flux due to gravity. Under these conditions the convection-diffusion equation reads:

\[
\frac{\partial c}{\partial t} + \frac{\partial}{\partial s} (\nu \frac{\partial c}{\partial s}) + \frac{\partial}{\partial n} (\nu \frac{\partial c}{\partial n}) + \frac{\partial}{\partial z} (\nu \frac{\partial c}{\partial z}) = 0
\]  
\( (2.1) \)

In case of a non-constant water depth and curved flow the convection-diffusion equation becomes more complicated. Sediment fluxes due to convection, by the main and the secondary flow, will occur. Also sediment fluxes, in horizontal directions, due to turbulent exchange will be present. In an orthogonal cartesian coordinate system the convection-diffusion reads (the eddy diffusion coefficient is assumed isotropic):

\[
\frac{\partial c}{\partial t} + \frac{\partial c}{\partial s} + \frac{\partial c}{\partial n} + \frac{\partial c}{\partial z} = \frac{\partial}{\partial s} (\nu \frac{\partial c}{\partial s}) + \frac{\partial}{\partial n} (\nu \frac{\partial c}{\partial n}) + \frac{\partial}{\partial z} (\nu \frac{\partial c}{\partial z})
\]  
\( (2.2) \)

in which: \( s = \) the coordinate in streamwise direction
n = the coordinate in transverse direction
z = the coordinate in vertical direction

The orthogonal cartesian coordinate system is not practical to model the curved flow of varying depth. In case of estuary flow, however, such a coordinate system can be applied in the horizontal (Wang 1989). For river bend flow a cylindrical bed-following coordinate system is used. Expressed in this coordinate system the convection-diffusion equation becomes more complicated. This is due to the coordinate transformation which introduces additional terms. The formulation of the model in a bed-following cylindrical coordinate system is the subject of chapters 3 and 4.

The method of asymptotic solution of the concentration field yields a depth-averaged formulation of the convection-diffusion equation. This method is briefly described as follows:

The 3-D concentration field is a result of sediment transport fluxes. These fluxes are caused by convection of sediment by the flow field, by movement of sediment relative to the flow and by turbulent diffusion. Except for abrupt changing conditions, the vertical component of movement of sediment relative to the flow, i.e. the fall velocity \( w_s \) and the upward sediment flux by turbulent diffusion will dominate the process.

The concentration field is modelled in a region bounded by the water surface and a reference level near the bed, fig 2.1. The concentration field is modelled to the first order.

The zeroth order solution \( (c_0) \) is based on a local balance of vertical fluxes due to the fall velocity and the upward turbulent diffusion. This means that, to the zeroth order, the whole concentration field is modelled by vertical concentration profiles which have a shape that is identical with the shape under equilibrium conditions. At this condition concentration gradients in horizontal directions are absent. In non-homogeneous flow the local zeroth order solution is determined by the local ratio of fall velocity \( w_s \) and the shear velocity \( u_\theta \).

The first order contribution to the solution \( (c_1) \) is mainly due to convection by the flow. The first order contribution \( (c_1) \) is modelled as a perturbation on the zeroth order solution, and is assumed not to contribute to the depth-averaged concentration. The first order contribution to the
concentration is computed with the aid of the mathematical expressions which Galappatti (1983) derived.

When a boundary condition is applied at reference level, the depth-averaged concentration field will be computed by the model. The boundary condition will be either a concentration condition, by which $c_0 + c_1$ is modelled at the reference level, or a gradient condition, by which $\frac{\partial (c_0 + c_1)}{\partial z}$ is modelled.

Once the depth-averaged concentration field is known the depth-integrated suspended sediment transport and the local sediment exchange with the bed can be computed.

The model is characterized by:

- The model computes the depth-averaged concentration field and depth-integrated suspended sediment transport.

- The model is suited to be used in rivers of nearly constant width and non-horizontal bed-topography.

- The model applies to a 3-D concentration field in an area bounded by the water surface and by a reference level close to the bed. This reference height can be chosen arbitrarily.

- The 3-D concentration field is split into a 2-D concentration field in the horizontal direction and a number of vertical profiles, derived using an asymptotic solution technique for the convection-diffusion equation.

- Two types of bed boundary conditions can be applied:
  - Locally the concentration at reference height is prescribed.
  - Locally the concentration gradient at reference height is prescribed.

These conditions are modelled as if the concentration profiles were in equilibrium (zeroth order solution).
3. TRANSFORMATION OF THE CONVECTION-DIFFUSION EQUATION TO A CYLINDRICAL BED FOLLOWING COORDINATE SYSTEM

3.1 Introduction

A cylindrical bed following coordinate system will have to be used to formulate the convection-diffusion equation in river bend flow. To this purpose the convection-diffusion equation, formulated in a rectangular cartesian coordinate system, is transformed to a cylindrical bed following system. This is achieved with the aid of tensor analysis.

3.2 The convection-diffusion equation

The convection-diffusion equation in general form reads:

\[ \frac{\partial c}{\partial t} + \nabla \cdot uc + \nabla \cdot u_s c - \nabla \cdot \chi c = 0 \]  \hspace{1cm} (3.1)

In which: \( \nabla \cdot uc \) = divergence of sediment flux due to convection by the flow
\( \nabla \cdot \chi c \) = divergence of sediment flux due to turbulent diffusion
\( \nabla \cdot u_s c \) = divergence of sediment flux due to convection of sediment relative to the flow

with: \( c \) = concentration of sediment
\( u \) = velocity vector of the flow
\( \chi \) = turbulent diffusion coefficient of mass (isotropic)
\( u_s \) = velocity vector of the sediment relative to the flow

The convection-diffusion equation can be represented in any coordinate system, provided that the formulation of the diffusion equation is invariant with eq.(3.1)

In a rectangular (=orthogonal) cartesian coordinate system the expressions involving the \( \nabla \) operator are equal to:

\[ \nabla s = \sum_{i} \frac{\partial s}{\partial x^i} \hat{i}_i + \sum_{i} \frac{\partial s}{\partial x^i} \hat{i}_i \frac{\partial}{\partial x^i}, \quad \nabla A = \sum_{i} \frac{\partial A}{\partial x^i} \hat{i}_i + \sum_{i} \frac{\partial A}{\partial x^i} \hat{i}_i \frac{\partial}{\partial x^i} \]  \hspace{1cm} (3.2)

with: \( x^i \) = coordinates
\( \hat{i}_i \) = unit vector of the coordinate system
\( s \) = a scalar
A = a first order tensor (= vector)
\(A^1\) = the component of a vector perpendicular with a surface of constant \(x^i\).

In coordinate systems different from the rectangular cartesian system the expressions involving to \(V\) operator are modified due to non-cartesian coordinates and non-orthogonality of the coordinate system. With the aid of tensor analysis the transformation of the convection-diffusion equation to a cylindrical and bed topography following coordinate system is accomplished. This coordinate system is non-cartesian and non-orthogonal.

3.3 Characteristics of the coordinate transformation

To comply with the geometry of a river bend a transformation of coordinates is employed. The transformed coordinate system is cylindrical and follows the bed topography. The coordinates are independent. The transformation is given by, see fig. 3.1:

\[
x^1 = r \cos \theta \\
x^2 = r \sin \theta \\
x^3 = h \xi + x^3_r, \quad h = s - x^3_r, \quad x^3 = h \xi + s - h
\]

\[
\hat{x}^1 = \theta = \arctg(x^2/x^1) \\
\hat{x}^2 = r/a_0 = ((x^1)^2 + (x^2)^2)^{1/2}/a_0 \\
\hat{x}^3 = \zeta = (x^3 - x^3_r)/(s - x^3_r) = (x^3 - x^3_r)/h
\]  

(3.3)

In which: \(x^1, x^2, x^3\) = a rectangular cartesian coordinate system, surfaces of \(x^3\)-constant are parallel to the water surface \(\hat{x}^1, \hat{x}^2, \hat{x}^3 = \theta, r/a_0, \zeta\) = a cylindrical bed following coordinate system

\(s\) = top of the model region (the water surface), \(s\)-constant
\(x^3_r\) = bottom of the model region, the reference level close to the bed
\(a_0\) = overall mean water depth
\(h\) = local depth of the model region
The \( \zeta \) coordinate has a value of \( \zeta = 1 \) at the top, and \( \zeta = 0 \) at the bottom of the model, and is a linear function of the position in the vertical, see also fig. 2.1.

In the tensor analysis (Dutton, 1976, chapter 5), two types of basis-vectors are discerned, see fig. 3.2:
- covariant basis-vectors which are tangent to the coordinate-axes;
  \[
  \hat{\tau}^i = \frac{\partial x^i}{\partial x}\tag{3.4}
  \]
- contravariant basis-vectors which are normal to surfaces of constant \( \xi^i \); \( \eta^i = \frac{\partial x^i}{\partial x^j} \)
  \[
  \hat{\eta}^i = \left( \begin{array}{c}
  \cos \theta \\
  \sin \theta
  \end{array} \right) \frac{\partial x}{\partial r} = \left( \begin{array}{c}
  \cos \theta \\
  \sin \theta
  \end{array} \right) \frac{\partial x}{\partial \zeta} \quad \hat{\eta} = \left( \begin{array}{c}
  \frac{\partial x}{\partial \eta} \\
  \frac{\partial x}{\partial \zeta}
  \end{array} \right) \frac{\partial x}{\partial h}
  \]
with: \( \hat{\tau} \) the unit vector of the original system.
In case the transformed coordinate system is orthogonal \( \hat{\tau} \) and \( \hat{\eta} \) become identical in direction.
For the transformation to the cylindrical bed following coordinate system the covariant and contravariant basis-vectors are:

\[
\hat{\tau}_1 = \frac{\partial x}{\partial \theta} = \begin{bmatrix} -r \sin \theta \\ r \cos \theta \\ (\zeta - 1) \frac{\partial h}{\partial \theta} \end{bmatrix} \quad \hat{\tau}_2 = a_0 \frac{\partial x}{\partial r} = a_0 \begin{bmatrix} \cos \theta \\ \sin \theta \\ (\zeta - 1) \frac{\partial h}{\partial r} \end{bmatrix} \quad \hat{\tau}_3 = \frac{\partial x}{\partial \zeta} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} \tag{3.6}
\]

\[
\hat{\eta}_1 = \begin{bmatrix} 1 \\ \frac{1}{r} \cos \theta \\ 0 \end{bmatrix} \quad \hat{\eta}_2 = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \quad \hat{\eta}_3 = \begin{bmatrix} -\frac{\zeta - 1}{r} \frac{\partial h}{\partial \theta} \\ -\frac{\zeta - 1}{h} \frac{\partial h}{\partial r} \\ \frac{1}{h} \frac{\partial x}{\partial \zeta} \end{bmatrix} \tag{3.7}
\]
in which use has been made of:

\[
\frac{\partial x^3}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial x^3}{\partial \theta} - (\zeta - 1) \frac{\partial h}{\partial \theta} \frac{\partial x^3}{\partial r}, \quad \frac{\partial x^3}{\partial r} = \frac{\partial x}{\partial h} \frac{\partial x^3}{\partial r} = (\zeta - 1) \frac{\partial h}{\partial r} \frac{\partial x^3}{\partial \zeta} - h
\]

\[
\begin{aligned}
\frac{\partial \theta}{\partial x} &= \frac{\partial \theta}{\partial x} - \frac{\partial \theta}{\partial \zeta} = \frac{1}{1 + \tan \theta} - \frac{1}{r} \tan \theta = -\frac{\sin \theta}{r} \\
\frac{\partial \zeta}{\partial x} &= \frac{x^3 - x - x^3}{1 + \tan \theta} = \frac{\zeta - 1}{h} \\
\frac{\partial \zeta}{\partial x^3} &= -\frac{\partial \zeta}{\partial x} - \frac{\partial \zeta}{\partial \theta} \frac{\partial x^3}{\partial x} = -\frac{\zeta - 1}{h} \frac{\partial h}{\partial x^3}
\end{aligned}
\]
\[ \frac{\partial \xi}{\partial x^1} = \frac{\partial \xi}{\partial x^2} = \frac{\partial \xi}{\partial x^3} = \frac{\partial h}{\partial x^r} = \frac{\partial h}{\partial x^s} = \frac{1}{h}, \text{ because } s = \text{constant} \]

The \( \partial h/\partial \theta \) and \( \partial h/\partial r \) terms are related to the \( \partial h/\partial x \) and \( \partial h/\partial y \) terms by:

\[
\frac{\partial h}{\partial x^1} = \cos \theta \frac{\partial h}{\partial r} - \sin \theta \frac{\partial h}{\partial \theta}, \quad \frac{\partial h}{\partial x^2} = \sin \theta \frac{\partial h}{\partial r} + \cos \theta \frac{\partial h}{\partial \theta} \quad (3.8)
\]

The metric tensor of the transformation is defined by:

\[
\hat{g}_{ij} = \hat{g}_i \cdot \hat{g}_j \quad (3.9)
\]

The inverse of the metric tensor is given by:

\[
\hat{g}^{ij} = \eta^i \cdot \eta^j \quad (3.10)
\]

\[
\hat{g}_{ij} = \begin{bmatrix}
\frac{r^2}{h} + (\xi - 1)^2 \frac{\partial h}{\partial r} \frac{\partial h}{\partial \theta} & a_0 (\xi - 1)^2 \frac{\partial h}{\partial r} \frac{\partial h}{\partial \theta} & (\xi - 1) a_0 h \frac{\partial h}{\partial \theta} \\
0 & 1 + (\xi - 1)^2 \left( \frac{\partial h}{\partial r} \right)^2 & a_0 (\xi - 1)^2 h \frac{\partial h}{\partial r} \\
0 & 0 & a_0 h \frac{\partial h}{\partial r}
\end{bmatrix} \quad (3.11)
\]

\[
\hat{g}^{ij} = \begin{bmatrix}
\frac{1}{r^2} & 0 & \frac{\xi - 1}{rh} \frac{\partial h}{\partial \theta} \\
0 & \frac{1}{a_0} & -\frac{\xi - 1}{h} \frac{\partial h}{\partial r} \\
\frac{\xi - 1}{rh} \frac{\partial h}{\partial \theta} & 0 & \frac{1}{a_0} \frac{\xi - 1}{h} \frac{\partial h}{\partial r} + \frac{(\xi - 1)^2}{h^2} \left[ (\frac{\partial h}{\partial r})^2 + (\frac{\partial h}{\partial \theta})^2 \right] + \frac{1}{h^2}
\end{bmatrix} \quad (3.12)
\]

The determinant (-Jacobi) of the transformation is:

\[
J = \frac{\frac{\partial (x^1, x^2, x^3)}{\partial (\xi^1, \xi^2, \xi^3)}}{\frac{\partial (x^1, x^2, x^3)}{\partial (\xi^1, \xi^2, \xi^3)}} = \sqrt{\hat{g}} \quad (3.13)
\]

\[
\sqrt{\hat{g}} = \begin{bmatrix}
-r \sin \theta & a_0 \cos \theta & 0 \\
-r \cos \theta & a_0 \sin \theta & 0 \\
(\xi - 1) \frac{\partial h}{\partial \theta} & a_0 (\xi - 1) \frac{\partial h}{\partial r} & h
\end{bmatrix} \quad (3.14)
\]

Unit vectors of the transformed coordinate system are:

\[
\hat{f}_i = h_i \hat{v}_i \quad \text{with: } h_i^{-1} = |\hat{v}_i| = |\hat{\eta}^i| = \sqrt{(\hat{\eta}^i \cdot \hat{\eta}^i)} \quad (3.15)
\]

for \( i = 1, 2, 3 \)
\[ \frac{1}{h_1} = \sqrt{\left(\hat{\eta}^1 \cdot \hat{\eta}^1\right)} = \frac{1}{r}, \quad \frac{1}{h_2} = \frac{1}{a_0}, \quad \frac{1}{h_3} = \left(\frac{\mathbf{\tau} \cdot \mathbf{\tau}}{h}\right)^2 + \left(\frac{\mathbf{\phi}}{\partial \mathbf{\tau}}\right)^2 + \left(\frac{\mathbf{\phi}}{\partial \mathbf{\theta}}\right)^2 + \left(\frac{1}{h}\right)^2 \]  

(3.16)

The transformation of \( \partial s/\partial x^k \), the differentiation of a scalar, is given by:

\[ \frac{\partial s}{\partial x^k} = \frac{\partial s}{\partial x^1} \frac{\partial x^1}{\partial x^k} + \frac{\partial s}{\partial x^2} \frac{\partial x^2}{\partial x^k} + \frac{\partial s}{\partial x^3} \frac{\partial x^3}{\partial x^k} \]  

(3.17)

Transformation of \( \partial A/\partial x^k \), the differentiation of a vector, involves the Christoffel symbol \( \Gamma^k_{ij} \). In transformed coordinates the differentiation of the contravariant components of a vector \( A^i \) reads:

\[ A^i_{;k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{kj} A^j \]  

(3.18)

with:

\[ \Gamma^i_{kj} = \frac{\partial A^i}{\partial x^1} \frac{\partial x^1}{\partial x^k} + \frac{\partial A^i}{\partial x^2} \frac{\partial x^2}{\partial x^k} + \frac{\partial A^i}{\partial x^3} \frac{\partial x^3}{\partial x^k} \]  

(3.19)

The non-zero Christoffel symbols of the transformation are:

\[ \Gamma^1_{11} = -\frac{1}{a_0} \]
\[ \Gamma^1_{12} = \Gamma^1_{21} = a_0 / r \]
\[ \Gamma^1_{13} = \left(\mathbf{\tau} \cdot \mathbf{\tau}\right) \frac{\partial \mathbf{\phi}}{\partial \mathbf{\tau}} \]
\[ \Gamma^1_{21} = \left(\mathbf{\tau} \cdot \mathbf{\phi}\right) \frac{\partial \mathbf{\tau}}{\partial \mathbf{\tau}} + \left(\mathbf{\tau} \cdot \mathbf{\theta}\right) \frac{\partial \mathbf{\phi}}{\partial \mathbf{\theta}} \]
\[ \Gamma^1_{22} = a_0 \left(\mathbf{\tau} \cdot \mathbf{\phi}\right) \frac{\partial \mathbf{\phi}}{\partial \mathbf{\tau}} \]
\[ \Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{a_0} \frac{\partial \mathbf{\phi}}{\partial \mathbf{\theta}} \]
\[ \Gamma^1_{33} = \frac{1}{a_0} \frac{\partial \mathbf{\phi}}{\partial \mathbf{\theta}} \]  

(3.20)

The contravariant velocity components (velocity components perpendicular to surfaces of \( \hat{\tau}^i \) is constant) are:

\[ \hat{u}^i = \hat{\eta}^i \cdot u \]
\[ u^1 = -\frac{1}{r} \sin \theta \quad u^1 + \frac{1}{r} \cos \theta \]  

(3.21)
\[ \hat{u}^2 = \frac{1}{a_0} \cos \theta u^1 + \frac{1}{a_0} \sin \theta u^2 \]  
\[ \hat{u}^3 = \frac{r-1}{h} \frac{\partial h}{\partial x^1} u^1 - \frac{r-1}{h} \frac{\partial h}{\partial x^2} u^2 + \frac{1}{h} u^3 \]  
with: \( u^i \) the velocity components in the rectangular cartesian coordinate system \([\text{m/s}]\) 
\( \hat{u}^i \) the contravariant velocity components in the transformed coordinate system \([\text{l/s}]\)

3.4 Convection due to the flow in transformed coordinates

The convective term of the convection diffusion equation is written in transformed coordinates. The convective term \( \nabla \cdot \mathbf{u} \cdot c \) is the divergence of the vector \( \mathbf{u} \cdot c \), and represents the differentiation of a vector. In tensor notation: \( \nabla \cdot \mathbf{u} \cdot c = \nabla \cdot \mathbf{A} = A^i_{;i} \), with: \( A^i_{;i} = \frac{\partial A^i}{\partial x^i} \)

In both coordinate systems \( \nabla \cdot \mathbf{A} \) has to be invariant. This amounts to: 
\( \nabla \cdot \mathbf{A} = A^i_{;i} = \hat{A}^i_{;i} \), According to Dutton (1976, p.142): 
\[ \hat{A}^i_{;i} = \frac{1}{h} \frac{\partial}{\partial \hat{x}^i} (\hat{h} \hat{A}^i) \]

The convective term of the convection-diffusion equation reads:

\[ \nabla \cdot \mathbf{u} \cdot c = \frac{1}{\hat{h}} \frac{\partial}{\partial \hat{x}^i} \left( \hat{h} \hat{u}^i \right) = \frac{\partial}{\partial \hat{x}^i} \left( \frac{\partial}{\partial \hat{x}^i} \hat{u}^i \right) + \hat{u}^i \frac{\partial c}{\partial \hat{x}^i} \]

\[ \frac{\partial}{\partial \hat{x}^i} \left( \hat{h} \hat{u}^i \right) - \hat{u}^1 \frac{\partial c}{\partial \theta} + \hat{u}^2 \frac{\partial c}{\partial r} + \hat{u}^3 \frac{\partial c}{\partial \hat{r}} \]

In which: \( \frac{1}{\hat{h}} \frac{\partial}{\partial \hat{x}^i} \left( \hat{h} \hat{u}^i \right) = \nabla \cdot \mathbf{u} = 0 \), the continuity equation.

The contravariant velocity components will be expressed in the physical components \( u_\theta \) and \( u_r \) which are tangent to the \( \theta \) and \( r \) coordinate-axes of the transformed coordinate system.

The vertical velocity component is calculated by considering the continuity equation and applying the Leibniz rule. The velocity component at the water surface is equal to: \( u^3_s \). Integration of the continuity equation from \( x^3 \) to the water surface \( s \) gives:
\[ u_3^3 - u_3^3 = \int_x^s \left( \frac{\partial u_1^1}{\partial x_1} + \frac{\partial u_2^2}{\partial x_2} \right) dx^3 \]  

(3.23)

The velocity at the water level is equal to:

\[ u_3^3 = \frac{\partial s}{\partial t} + u_1^1 |_{s} \frac{\partial s}{\partial x_1} + u_2^2 |_{s} \frac{\partial s}{\partial x_2} \]  

(3.24)

According to the Leibniz rule the following relationship holds:

\[ \int_x^s \frac{\partial u_1^1}{\partial x_1} dz = \frac{\partial s}{\partial x_1} \int_x^s u_1^1 dx^3 - u_1^1 |_{s} \frac{\partial s}{\partial x_1} \]  

(3.25)

These equations yield the vertical velocity component in the orthogonal cartesian coordinate system:

\[ u_3^3 = \frac{\partial s}{\partial t} + \frac{\partial}{\partial x_1} \left( \int_x^1 u_1^1 dx^3 \right) + \frac{\partial}{\partial x_2} \left( \int_x^1 u_2^2 dx^3 \right) - \nabla \cdot A, \text{ for } \frac{\partial s}{\partial t} = 0 \]  

(3.26)

in which: \( \nabla \cdot A = A_{i}^{i} = A_{i}^{i} ; i = 1, 2 \)

with: \( A = \int_x^1 u \, dx^3 - h \int_x^1 \hat{u}^i d\hat{x}^3 \) and \( A_{i}^{i} = h \int_x^1 \hat{u}^i d\hat{x}^3 \)

The vertical component expressed in variables of the transformed coordinate system is:

\[ u_3^3 = A_{i}^{i} ; i = \frac{\partial A_{i}^{i}}{\partial x^i} + \Gamma_{ij}^{i} A_{j}^{j} \]

\[ = \frac{\partial}{\partial x^i} \left( h \int_x^1 \hat{u}^i dx^3 \right) + \frac{\partial}{\partial x^j} \left( h \int_x^2 \hat{u}^2 dx^3 \right) + \frac{\partial}{\partial x^k} \left( h \int_x^3 \hat{u}^3 dx^3 \right) \]  

(3.27)

The contravariant \( \hat{u}^3 \) velocity component of the transformed coordinate system is: \( \hat{u}^3 = \hat{\eta}^3 \cdot u \)

\[ \hat{u}^3 = \frac{\hat{\eta}^1}{\hat{\eta}^3} \frac{\partial u_1^1}{\partial x} \frac{\partial u_1^1}{\partial \hat{x}} + \frac{\hat{\eta}^2}{\hat{\eta}^3} \frac{\partial u_1^1}{\partial \hat{x}} + \frac{\hat{\eta}^3}{\hat{\eta}^3} \frac{\partial u_3^3}{\partial \hat{\eta}^3} + \frac{\hat{\eta}^1}{\hat{\eta}^3} \frac{\partial u_3^3}{\partial \hat{\eta}^1} \]  

(3.28)

substitution of \( u_3^3 \) yields:
\[ \ddot{u}_3 = - \frac{r-1}{h} \frac{\partial h}{\partial \theta} \dot{u}_1 + a_0 + \frac{r-1}{h} \frac{\partial h}{\partial r} \dot{u}_2 + \]
\[ + \frac{1}{h} \frac{\partial}{\partial r} \left( h \int_{0}^{1} u_1 d\xi \right) + \frac{1}{h} \frac{\partial}{\partial \theta} \left( h \int_{0}^{1} u_2 d\xi \right) + \frac{a_0}{r} \int_{0}^{1} u_2 d\xi \]  

(3.29)

The physical horizontal components of the velocity in \( \theta \) and \( r \) directions are calculated.

The unit vectors in these directions are:

\[ \lambda_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \quad \lambda_r = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \]  

(3.30)

The physical horizontal velocity components are:

\[ u_\theta = u_{(\lambda_\theta)} - \lambda_\theta \cdot u = \left( \lambda_\theta \cdot \hat{r}_i \right) \dot{u}_i = r \dot{u}_1, \quad u_r = u_{(\lambda_r)} = a_0 \dot{u}_2 \]  

(3.31)

In which use has been made of the fundamental relation: \( u = \hat{r}_i \dot{u}_i \)

Expressed in these physical components \( \ddot{u}_3 \) becomes:

\[ \ddot{u}_3 = - \frac{r-1}{h} \frac{\partial h}{\partial \theta} u_\theta + \frac{r-1}{h} \frac{\partial h}{\partial r} u_r + \frac{1}{h} \frac{\partial}{\partial r} \left( h \int_{0}^{1} u_\theta d\xi \right) + \]
\[ + \frac{1}{h} \frac{\partial}{\partial \theta} \left( h \int_{0}^{1} u_r d\xi \right) + \frac{1}{r} \int_{0}^{1} u_r d\xi \]  

(3.32)

Finally the convective term \( \nabla \cdot uc \) expressed in the cylindrical bed following coordinate system becomes:

\[ \nabla \cdot uc = u \cdot \nabla c + u_\theta \frac{\partial c}{\partial \theta} + u_r \frac{\partial c}{\partial r} + \dot{u}_3 \frac{\partial c}{\partial \zeta} = \]
\[ = u_\theta \frac{\partial c}{\partial \theta} + u_r \frac{\partial c}{\partial r} + (- (\zeta-1) \frac{\partial h}{\partial \theta} u_\theta \cdot (\zeta-1) \frac{\partial h}{\partial r} u_r + \frac{\partial}{\partial r} \left( h \int_{0}^{1} u_\theta d\xi \right) + \]
\[ + \frac{\partial}{\partial \theta} \left( h \int_{0}^{1} u_r d\xi \right) + h \int_{0}^{1} u_r d\xi \frac{\partial c}{h \partial \zeta} + \]  

(3.33)

The \(- (\zeta-1) \frac{\partial h}{\partial \theta} u_\theta \frac{\partial c}{\partial h \partial \zeta} \) and \(- (\zeta-1) \frac{\partial h}{\partial r} u_r \frac{\partial c}{h \partial \zeta} \) terms originate from the transformation of the vertical coordinate. The \( h \int_{0}^{1} u_r d\xi \frac{\partial c}{h \partial \zeta} \) term originates from the transformation to the cylindrical coordinate system.
3.5 Convection due to sediment moving relative to the flow

The density of sediment is larger than the density of water. Consequently the sediment will not follow the same trajectories as the flow. The sediment velocity relative to the flow is denoted by: \( u_s \)
This velocity component will also induce additional fluxes. The flux due to the sediment moving relative to the flow is equal to: \( u_s c \).
The \( u_s \) vector in an orthogonal cartesian coordinate system is equal to:

\[
\begin{pmatrix}
  u_{s1} \\
  u_{s2} \\
  u_{s3}
\end{pmatrix},\quad \text{in non-curved flow: } u_s = \begin{pmatrix}
  0 \\
  0 \\
  -w_s
\end{pmatrix}
\]  
\tag{3.34}

In which \( w_s \) = fall velocity of sediment.

In the transformed coordinate system the contravariant velocity components are:

\[
\begin{align*}
  \hat{u}_s^1 &= 0 \\
  \hat{u}_s^2 &= u_{sr}/a_0 \\
  \hat{u}_s^3 &= -\frac{c-1}{h} \frac{\partial h}{\partial r} u_{sr} - \frac{1}{h} w_s
\end{align*}
\]  
\tag{3.35}

In which it is assumed that the velocity of the sediment in \( \theta \)-direction is equal to \( u_\theta \). The \( u_{sr} \) component is due to the higher density of the sediment. The centripetal force acting on the sediment, due to \( \partial p/\partial r \), is insufficient to yield the same trajectory as the flow.

The force, in \( r \)-direction, acting on the sediment is equal to:

\[
F = (\rho_s - \rho) \frac{u_{sr}^2}{r^2} \frac{1}{6} \pi \, d^3
\]  
\tag{3.36}

This force is balanced by a drag force. The suspension consist of small particles with characteristic diameters of 0.1 to 0.3 mm. If the sphere Reynolds number is small the Stokes drag formula applies:
\[ F = 3\pi \nu \rho d u_s \] 

(3.37)

In a quiescent fluid the sediment particles attain a fall velocity \( w_s \) which is due to the acceleration by gravity. The Reynolds number \( \text{Re} = w_s d / \nu \) under these conditions is of the order one. In that case the fall velocity can be computed by the Stokes formula for the terminal velocity:

\[ w_s = \frac{1}{18} \Delta \frac{d^2}{\nu} \] 

(3.38)

in which \( \Delta \) is the relative density of the sediment.

The velocity of the sediment relative to the flow, in \( r \)-direction, is calculated by virtue of the eq. (3.37) and eq. (3.36). Equation (3.38) is substituted:

\[ u_{s2} = \frac{F}{3\pi \nu \rho d} - \frac{1}{g} \frac{u_r}{r} \frac{\partial h}{\partial r} |_{w_s} \] 

(3.39)

In the transformed coordinate system the expression for \( \nabla \cdot u_s c \) is:

\[ \nabla \cdot u_s c = u_{s \rho} \frac{\partial c}{\partial \rho} + \left( - (\zeta - 1) \frac{\partial h}{\partial r} \right) \frac{u_r}{r} \frac{\partial c}{\partial h} |_{w_s} \] 

(3.40)

3.6 The turbulent diffusion term in transformed coordinates

The turbulent diffusion term \( \nabla \cdot \nu \nabla c \) is written in transformed coordinates. This term accounts for the divergence of \( \nu \nabla c \). \( \nu \) and \( c \) are functions of the coordinates \( \chi^i \). The turbulent diffusion term is written in orthogonal cartesian coordinates as:

\[ \nabla \cdot \nu \nabla c = \nabla \cdot A = \frac{\partial A^i}{\partial x^i}, \text{ with } A = \nu \frac{\partial c}{\partial x^i} \] 

(3.41)

In the transformed coordinate system this is equal to:

\[ \nabla \cdot \nu \nabla c = \nabla \cdot A = \hat{\nabla} \cdot A = \hat{\nabla} \cdot \left( \frac{\partial A^i}{\partial \hat{x^i}} \right) \] 

(3.42)

In which:

\[ A^i = \hat{\nabla} \cdot A = \hat{\nabla} \cdot \left( \frac{\partial c}{\partial x^i} \right) = \hat{\nabla} \cdot \left( \frac{\partial c}{\partial \hat{x^i}} \right) = \hat{\nabla} \cdot \left( \frac{\partial c}{\partial \hat{x}^k} \right) \]
This yields:

\[ v \cdot \nu \cdot v_c = \frac{1}{g} g^{ik} \frac{\partial}{\partial x^i} (g \nu \frac{\partial c}{\partial x^k}) , \text{ because } g^{ij}_{,k} = 0 , \text{ (Dutton, 1976, p. 141)} \]

This is equivalent with:

\[ v \cdot \nu \cdot v_c = g^{ik} \frac{\partial}{\partial x^i} (\nu \frac{\partial c}{\partial x^k}) + \frac{1}{g} g^{ik} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} \]

(3.43)

The last term accounts for the spatial variation of unit volumes in the transformed coordinate system.

The contributions to the first term on the right hand side are:

(i=1; in \( \theta \)-direction, i=2; in \( r \)-direction, i=3; in \( \phi \)-direction)

\[ i=1; \quad g^{1k} \frac{\partial}{\partial x^i} (\nu \frac{\partial c}{\partial x^k}) = \frac{1}{r^2} \frac{\partial}{\partial \theta} (\nu \frac{\partial c}{\partial \theta}) - \frac{1}{r^2} \frac{\partial}{\partial \theta} (\nu \frac{\partial c}{\partial \phi}) \]

(3.44)

\[ i=2; \quad g^{2k} \frac{\partial}{\partial x^i} (\nu \frac{\partial c}{\partial x^k}) = \frac{1}{h} \frac{\partial}{\partial r} (\nu \frac{\partial c}{\partial r}) - \frac{1}{h} \frac{\partial}{\partial r} (\nu \frac{\partial c}{\partial \phi}) \]

The products of more than two derivatives are due to the non-orthogonality of the coordinate system.

Contributions to the second term on the right hand side are:

\[ i=1; \quad \frac{1}{g} g^{1k} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} = \frac{1}{r^2} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^i} - \frac{1}{r^2} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^i} \]

(3.45)

\[ i=2; \quad \frac{1}{g} g^{2k} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} = \frac{1}{h} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} + \frac{1}{h} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^i} \]

i=3; \quad \frac{1}{g} g^{3k} \nu \frac{\partial c}{\partial x^k} \frac{\partial g}{\partial x^i} = 0
In these expressions, the products of more than two derivatives are also due to the non-orthogonality of the coordinate system.

Neglect of products of more than two derivatives finally gives:

\[ \nabla \cdot \mathbf{v}_c = \frac{1}{r^2} \frac{\partial}{\partial \theta} (\mathbf{v}_r \times \partial \phi) + \frac{\partial}{\partial r} (\mathbf{v}_r \times \partial r) + \frac{1}{h^2} \frac{\partial}{\partial \zeta} (\mathbf{v}_r \times \partial \zeta) + \]

\[ + \frac{1}{r^2} \frac{\partial}{\partial \theta} \mathbf{v}_r \frac{\partial h}{\partial \theta} + \frac{1}{h} \frac{\partial}{\partial \phi} \mathbf{v}_r \frac{\partial h}{\partial \phi} + \frac{1}{h} \frac{\partial}{\partial \zeta} \mathbf{v}_r \left( \frac{r-1}{\phi} \frac{\partial h}{\partial \phi} \right) \frac{\partial \phi}{\partial \zeta} \]

(3.46)

The last 4 terms originate from the determinant of the transformation, which depends on the coordinates. The determinant relates to the transformation of volume: \( dV = dx^1 dx^2 dx^3 = g^1 dx^1 dx^2 dx^3 \). A transformation to cylindrical yields a Jacobian which is a function of \( r \); \( g^1 = f(r) \). In that case the \( \frac{1}{r} \frac{\partial}{\partial r} \mathbf{v}_r \) term appears. If only the transformation of the vertical is performed, terms appear comparable with the \( \frac{1}{r^2} \frac{\partial}{\partial \zeta} \mathbf{v}_r \frac{\partial h}{\partial \zeta} \) and \( \frac{1}{h} \frac{\partial}{\partial \phi} \mathbf{v}_r \frac{\partial h}{\partial \phi} \) terms, but with differentials to \( x^1 \) and \( x^2 \). The \( \frac{(r-1)}{r} \frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial \zeta} \) term is due to the combination of both transformations.
3.7 The convection-diffusion equation in transformed coordinates

The convection diffusion equation in the cylindrical bed following coordinate system now reads (neglecting products of more than two derivatives):

\[
0 - \frac{\partial c}{\partial t} + \nabla \cdot uc + \nabla \cdot u_c s - \nabla \cdot \nabla c -

= \frac{\partial c}{\partial t} + u_\Theta \frac{\partial c}{\partial \Theta} + u_r \frac{\partial c}{\partial r} + (-\zeta \frac{\partial h}{\partial \Theta} u_\Theta - \zeta \frac{\partial h}{\partial r} u_r +

\frac{\partial}{\partial \Theta} (h \int_\zeta^1 u_\Theta \, dz) + \frac{\partial}{\partial r} (h \int_\zeta^1 u_r \, dz) + \frac{h}{r} \int_\zeta^1 u_r \, dz \right) \frac{\partial c}{\partial \Sigma} +

+ u_{sr} \frac{\partial c}{\partial r} + \left(-\zeta \frac{\partial h}{\partial r} u_{sr} - \Sigma \right) \frac{\partial c}{\partial \Sigma} +

\frac{1}{r^2} \frac{\partial}{\partial \Theta} \left( \nu \frac{\partial c}{\partial \Theta} - \frac{\partial}{\partial r} \left( \nu \frac{\partial c}{\partial r} \right) \right) - \frac{1}{h^2} \frac{\partial}{\partial \Sigma} \left( \nu \frac{\partial c}{\partial \Sigma} \right) +

\frac{1}{r^2} \frac{\partial}{\partial \Theta} \left( \nu \frac{\partial c}{\partial \Theta} - \frac{\partial}{\partial r} \left( \nu \frac{\partial c}{\partial r} \right) \right) - \frac{1}{h^2} \frac{\partial}{\partial \Sigma} \left( \nu \frac{\partial c}{\partial \Sigma} \right) +

\frac{1}{r^2} \nu \frac{\partial c}{\partial h} \frac{\partial h}{\partial r} - \frac{1}{h} \nu \frac{\partial c}{\partial h} \frac{\partial h}{\partial r} + \frac{1}{r} \nu \frac{\partial c}{\partial h} \frac{\partial h}{\partial r} + \zeta \frac{\partial c}{\partial h} \frac{\partial h}{\partial r} + \zeta \frac{\partial c}{\partial h} \frac{\partial h}{\partial r}

(3.47)

\]

with: \( u_r = \bar{u} r + \bar{v} r \)
4. SIMPLIFICATION OF THE TRANSFORMED CONVECTION-DIFFUSION EQUATION

4.1 Introduction

The magnitude of terms of the convection-diffusion equation is investigated. To this purpose a scale analysis is applied. Implications of the mathematical investigations by Wang (1989) are incorporated. The resulting convection-diffusion equation is formulated in terms of depth averaged velocity components.

4.2 Origin of terms

The origin of terms of the transformed convection-diffusion equation is recapitulated.

The transport terms due to convection through surfaces of \( \theta \)-constant and \( r \)-constant are:

\[
\frac{\partial c}{\partial \theta} \quad \text{and} \quad (u_r + u_{sr}) \frac{\partial c}{\partial r}
\]  

The transport terms due to convection through surfaces of \( \zeta \)-constant are:

\[
(-1) \frac{\partial h}{\partial \theta} u_{\theta} - (\zeta - 1) \frac{\partial h}{\partial r} (u_r + u_{sr}) + \frac{\partial}{\partial \theta} (h \int_{\zeta}^{\zeta+1} u_{\theta} \, d\zeta) + \frac{\partial}{\partial r} (h \int_{\zeta}^{\zeta+1} u_r \, d\zeta) + \frac{h}{h \delta_r} \frac{\partial c}{\partial \zeta}
\]  

The \(- (\zeta - 1) \frac{\partial h}{\partial \theta} u_{\theta} \frac{\partial c}{\partial h} \) and \(- (\zeta - 1) \frac{\partial h}{\partial r} (u_r + u_{sr}) \frac{\partial c}{\partial h} \) terms are a consequence of the non-orthogonality of the \( \theta, r, \zeta \) coordinate system. They represent the transport, by convection, through the surfaces of \( \zeta \)-constant by components of the horizontal velocities \( u_{\theta}, u_r \) and \( u_{sr} \).

The remaining terms represent the transport, by convection, through the surfaces of \( \zeta \)-constant by the vertical velocity component. This component depends on the horizontal velocity components, and is also affected by the non-constant water depth and the cylindrical coordinate system.
The \( \frac{h}{r} \int_{\theta}^{1} u \frac{d\theta}{\partial r} \frac{\partial c}{\partial \theta} \) term is a consequence of the cylindrical coordinate system.

The terms representing turbulent diffusion are:

\[
\frac{1}{r^2} \frac{\partial}{\partial \theta} (\nu \frac{\partial c}{\partial \theta}) - \frac{\partial}{\partial r} (\nu \frac{\partial c}{\partial r}) - \frac{1}{h^2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} (\nu \frac{\partial c}{\partial \xi}) + \]

\[
- \frac{1}{r^2} \frac{1}{h} \frac{\partial c}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{1}{r} \frac{\partial c}{\partial r} \frac{\partial h}{\partial r} - \frac{1}{r} \frac{\partial c}{\partial r} \frac{h}{\partial r} + \frac{c-1}{r} \frac{\partial h}{\partial r} \frac{\partial c}{\partial \xi} \frac{1}{\partial \xi} \quad (4.3)
\]

The \(- \frac{1}{r^2} \frac{\partial}{\partial \xi} (\nu \frac{\partial c}{\partial \xi})\) term represents the gradient of transport, by diffusion, through surfaces of \( \xi \)-constant.

The terms \(- \frac{1}{r^2} \frac{\partial}{\partial \theta} (\nu \frac{\partial c}{\partial \theta})\) and \(- \frac{1}{r^2} \frac{1}{h} \frac{\partial c}{\partial \theta} \frac{\partial h}{\partial \theta}\) represent the gradients of transport, by diffusion, through surfaces of \( \theta \)-constant.

The terms \(- \frac{\partial}{\partial r} (\nu \frac{\partial c}{\partial r})\), \(- \frac{1}{h} \frac{\partial c}{\partial r} \frac{\partial h}{\partial r}\), \(- \frac{1}{r} \frac{\partial c}{\partial r} \frac{h}{\partial r}\), and \(\frac{c-1}{r} \frac{\partial h}{\partial r} \frac{\partial c}{\partial \xi} \frac{1}{\partial \xi}\) represent the gradients of transport, by diffusion, through surfaces of \( r \)-constant.

Compared with the case of an orthogonal cartesian coordinate system additional terms have emerged. Terms due to non-orthogonality are not incorporated in the equation because these yield products of more than two differentials. The terms due to the spatial variation of the dimensions of a unit volume in the \( \theta, r, \xi \) system are:

\[
- \frac{1}{r^2} \frac{1}{h} \frac{\partial c}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{1}{h} \frac{\partial c}{\partial r} \frac{\partial h}{\partial r} - \frac{1}{h} \frac{\partial c}{\partial r} \frac{h}{\partial r} \frac{c-1}{r} \frac{\partial h}{\partial r} \frac{\partial c}{\partial \xi} \frac{1}{\partial \xi} \]

4.3 Estimation of the magnitude of terms

The magnitude of terms of the convection diffusion equation is investigated. A scale analysis is employed. The objective of the analysis is to estimate the order of magnitude of the terms in eq. (3.47). The variables are normalized by a suitable scale to obtain normalized variables which are of the order one; \( O(1) \).
\( z' = z / L_z \) with: \( L_z \) = length scale in vertical direction over which the concentration changes \( t' = t / T \) \( T \) = time scale \( u' = u_\theta / U \) \( U \) = velocity scale in streamwise direction \( v' = u_y / V \) \( V \) = velocity scale in transverse direction \( x' = x / L \) \( L \) = length scale in streamwise direction \( y' = y / W \) \( W \) = length scale in transverse direction \( a' = a / A \) \( A \) = length scale for water depth \( u'_* = u_*/U_\ast \) \( U_\ast \) = velocity scale for friction velocity \( v'_* = v_*/\Omega \) \( \Omega \) = mixing scale for diffusion of mass

The derivative of a variable \( f \) equals: \( \frac{\partial f}{\partial x} = \frac{E}{L} \frac{\partial f'}{\partial x'} \), in which: \( \frac{\partial f'}{\partial x'} = \text{O}(1) \)

In the present application of the analysis the following remarks are made:

- The vertical extent of the analysis is confined to: \( x_r^3 < z < s \)
- The region close to the bed, where \( u_\theta \) approaches zero is excluded, consequently \( u' = \text{O}(1) \).
- At about mid water depth the secondary flow component equals zero, consequently, at this level \( v' < \text{O}(1) \)
- The length scales in streamwise and transverse directions on which concentration, flow velocity and flow depth vary are \( L \) and \( W \) respectively.
- Due to large concentration gradients in vertical direction, the vertical length scale of variation of concentration will be smaller than the depth of flow: \( L_z < A \)

Substitution of eq.\((4.4)\) in eq.\((3.47)\) and subsequent multiplication with \( L_z / (w, c) \) yields an normalized equation eq.\((4.5)\)

\[
\frac{\partial c}{\partial z'} + D \frac{\partial}{\partial z'} \left( \nu' \frac{\partial c}{\partial z'} \right) =
\]

\[
- \frac{L}{A} \left( \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x'} + S v' \frac{\partial c}{\partial y'} \right) +
\]

\[
+ \varepsilon \frac{\partial c}{\partial z} \left( 1 - (\gamma - 1) u \frac{\partial a'}{\partial x'} - (\gamma - 1) S v' \frac{\partial a'}{\partial y'} \right) +
\]
\[
\frac{\partial}{\partial x'}(a' \int_\zeta^1 u'd\zeta) + \frac{\partial}{\partial y'}(a' \int_\zeta^1 v'd\zeta) + \int_\zeta^1 w'd\zeta + D \left( \frac{L_z^2}{L_z^2} \frac{\partial}{\partial x'}(\nu' \frac{\partial c}{\partial x'}) - \frac{L_z^2}{W^2} \frac{\partial}{\partial y'}(\nu' \frac{\partial c}{\partial y'}) \right) + \\
- \frac{L_z^2}{L_z^2} \nu' \frac{\partial c}{\partial x'} \frac{\partial a}{\partial x'} - \frac{L_z^2}{W^2} \nu' \frac{\partial c}{\partial y'} \frac{\partial a}{\partial y'} + \frac{L_z^2}{R} \nu' \frac{\partial c}{\partial y'} + \frac{L_z^2}{R} \zeta (\zeta - 1) \frac{\partial a'}{\partial x'} + \frac{\partial c}{\partial z'}) \tag{4.5}
\]

In which \( \epsilon = \frac{A_{\nu}}{L_z W} \) - a small parameter

\( D = \frac{A_{\nu}}{L_z W} \) - the diffusion parameter

\( S = \frac{B_{\nu}}{W} \) - the secondary flow parameter \( \tag{4.6} \)

use has been made of: \( L/(U) = 1 \)

The two terms on the left hand side of eq. (4.5) have to be the dominant terms of the equation in order to justify a depth-averaged formulation (the concentration profile will be close to equilibrium). In that case these terms are of order one; 0(1), while the other terms are \( < 0(1) \).

The scales of the secondary flow component and the eddy diffusion are estimated by using mathematical profile formulations which are based on a logarithmic mainstream velocity profile.

if \( \nu \neq 0 \): \( V = 0.5 \frac{W}{U} \) (length scale of variation of \( \tilde{v} \) is 0.5 \( W \)) \( \tag{4.7a} \)
if \( \nu = 0 \): \( V = 4 \) to \( 6 \frac{A_{\nu}}{R} \), for \( 20 \text{ m/s} < \text{Chezy} < 60 \text{ m/s} \), \( V = u_\zeta (\zeta = 1) \) \( \tag{4.7b} \)

\( \zeta = 0.067 U_{\nu} \), \( \zeta = \text{mean value of a parabolic } \nu \text{ profile} \) \( \tag{4.8} \)

This yields:

\( D = 0.067 \frac{A_{\nu}}{L_z W} \) and ,if \( \nu \neq 0 \); \( S = 0.5 \), or if \( \nu = 0 \); \( S = 4 \) to \( 6 \frac{A_{\nu}}{R} \) \( \tag{4.9} \)

To apply a depth-averaged model \( D \) will have to be of order one; \( D = O(1) \)
The $\epsilon$ parameter will have to be less than order one; $\epsilon \leq 0(0.1)$

The $S$ parameter will be restricted to; $S \leq 0(1)$

The turbulent diffusion terms on the right hand side of eq. (4.5) are of order $< 0(0.01)$, because $D = 0(1)$, $L_z/W \leq 0(0.1)$, $L_z/R \leq 0(0.01)$ and $L_z/L \leq (0.01)$, consequently from an order of magnitude point of view these terms can be omitted.

4.4 Estimate of terms in a laboratory experiment

In Talmon and Marsman (1988) a suspended-load experiment in a curved flume is reported. The bed topography displays a damped harmonic oscillation.

Parameters relevant to the scale analysis are: $L = 4.8$ m, $W = 0.5$ m, $A = 0.048$ m, $R = 4.1$ m, $w_s = 0.0076$ m/s, $U = 0.24$ m/s, $U_s = 0.04$ m/s

The concentration profiles do not have large concentration gradients (the Rouse parameter is estimated to be $Z = 0.28$). Consequently $L_z/A$ is arbitrarily estimated to be equal to: $L_z/A = 0.5$.

This yields:

$$\epsilon = \frac{A U}{L w_s} = 0.32$$

$$D = 0.067 \frac{A u_z}{L w s} = 0.71$$

$S = 0.5$, if $\dot{V} \neq 0$ or $S = 4...6 \frac{A L}{R w} = 0.56$, if $\dot{V} = 0$

Inspection of eq. (4.5) reveals that the coefficient which precedes the $u$ and $v$ components of convective transport is equal to: $\epsilon L_z/A = 0.16 \approx 0(0.1)$.

The coefficient preceding the term representing vertical convective transport (by the $w$ component) is equal to: $\epsilon = 0.32 \approx 0(0.1)$.

The scale analysis applied to this particular experiment reveals that the relative importance of convective transport by the $w$ component is of the same order as by the $u$ and $v$ components. The flow component in $n$-direction contributes to the solution because; $S = 0.5$, which indicates that the effect of deviations of the depth averaged streamlines with the $s$-direction and the
effect of secondary flow is about half of that of changes in main flow direction.

In order to apply a depth-averaged model, from a theoretical point of view, it is desired that the convective transport terms are one order of magnitude smaller than both the fall velocity term and the vertical turbulent diffusion term. In this case the maximum contribution of the convective transport terms is estimated at about 30% of the leading terms. It is expected that this will not be a too serious problem for application of a depth-averaged model, because the scale analysis yields upper bound estimates. Thus only locally these terms will attain their maximal values.

4.5 Estimate of terms in an actual river

In Jackson (1975) the topography of the Helm bend in the Lower Wabash river USA is reported. Part of the sediment is transported as suspended-load. The bed topography displays a point-bar like feature located about 1300 m downstream of the bend entrance. Ikeda and Nishimura (1985) has also used this bend to perform some sample computations, some parameters will be chosen equal to estimates of Ikeda and Nishimura. Parameters relevant to the scale analysis are: \( L = 1300 \text{ m}, W = 250 \text{ m}, A = 3.4 \text{ m}, R = 610 \text{ m}, \omega_s = 0.07 \text{ m/s}, U = 1.25 \text{ m/s}, U_w = 0.125 \text{ m/s} \)

The concentration profiles are expected to have large concentration gradients in vertical direction because the estimated Rouse parameter is equal to: \( z = 1.4 \). Consequently \( L_z/A \) is arbitrarily estimated to be equal to: \( L_z/A = 0.1 \).

This yields:

\[
\epsilon = \frac{A U}{L \omega_s} = 0.03
\]

\[
D = 0.067 \frac{A U_w}{L \omega_s} = 0.70
\]

\[
S = 0.5, \text{ if } \tilde{v} \neq 0 \text{ or } S = 4...6 \frac{A L}{R \omega} = 0.14, \text{ if } \tilde{v} = 0
\]

Inspection of eq. (4.5) reveals that the coefficient which precedes the u and v components of convective transport is equal to: \( \epsilon L_z/A = 0.003 = 0(0.001) \).
The coefficient preceding the term representing vertical convective transport (by the $w$ component) is equal to: $\epsilon = 0.03 = 0(0.01)$

The scale analysis applied to this particular river bend also shows that the relative importance of convective transport by the $w$ component is generally larger than by the $u$ and $v$ components.

The convective transport terms are two orders of magnitude smaller than both the fall velocity term and the vertical turbulent diffusion term. Consequently the use of a depth-averaged model is justified.

4.6 Dominant terms of the transformed convection-diffusion equation

Shape functions for the velocity components, the eddy viscosity and the concentration are introduced:

$$r_u = \frac{\bar{u}}{u}, \quad \text{with: } \frac{1}{a} \int_a^z r_u \, dx = 1, \quad r_u = f(\zeta, \frac{z}{a}, \frac{u}{u_*})$$

$$r_v = \frac{u_{sec}}{u}, \quad \text{with } r_x = u_{sec} + \bar{v} r_u, \quad r_v = f(\zeta, \frac{z}{a}, \frac{u}{u_*})$$

$$r_{\nu_x} = \frac{x}{\nu_h}, \quad \text{with: } r_{\nu_x} = \frac{1}{a} \bar{z}(1 - \bar{z}), \quad r_{\nu_x} = f(\zeta, \frac{z}{a}, \frac{w}{u_*})$$

$$r_c = \frac{c}{\bar{c}}, \quad \text{with: } \int_0^1 r_c \, dz = 1, \quad r_c = f(\zeta, \frac{z}{a}, \frac{w}{u_*})$$  \hspace{1cm} (4.10)

with: $z$ - the vertical coordinate relative to the bed, $z_r$ - the vertical coordinate of the reference level; $z_r = x_r - x_b$

The parameters $r_u$, $r_v$, $r_{\nu_x}$ and $r_c$ are the shape functions for velocity, eddy viscosity and concentration. The parameter $\bar{c}$ is the local depth-averaged concentration. The parameter $\bar{u}$ is the depth-averaged velocity. The parameter $1/r_s$ is the local streamline curvature.

Some examples of these shape functions, in case of a parabolic function of the eddy viscosity for both momentum and mass, are given in fig.4.1.

The integrals of the shape functions for velocity are given by:

$$R_u = \int_0^1 r_u \, dz, \quad \text{with: } R_u = f(\zeta, \frac{z}{a}, \frac{u}{u_*}) \quad \text{and } r_u = -\frac{\partial R_u}{\partial \zeta}$$

$$R_v = \int_0^1 r_v \, dz, \quad \text{with: } R_v = f(\zeta, \frac{z}{a}, \frac{u}{u_*}) \quad \text{and } r_v = -\frac{\partial R_v}{\partial \zeta}$$  \hspace{1cm} (4.11)
in which $R_u$ and $R_v$ are the integrals of the shape functions for the velocity from $\zeta$ up to the water surface.

The estimation of magnitude of terms has shown that turbulent diffusion terms other than due to vertical mixing can be omitted. Maintaining, however, turbulent diffusion terms involving second order derivatives in the mathematical model (which will be based on an asymptotic solution technique) the results become more accurate, Wang (1989). Additional turbulent diffusion terms due to the non-orthogonal cylindrical coordinate system and curvature are neglected. The secondary flow velocity profile may, in most practical cases, be approximated to be linear with $1/r$, instead of $1/r_s$.

This reduces the convection-diffusion equation to:

$$
\frac{\partial c}{\partial t} + r_u \frac{\partial c}{\partial \theta} + (r_v \frac{\partial c}{\partial r} + r_u \frac{\partial c}{\partial \theta}) + \left[ -(\zeta - 1) \frac{\partial h}{\partial \theta} r_u \frac{\partial c}{\partial r} + (\zeta - 1) \frac{\partial h}{\partial r} (r_v \frac{\partial c}{\partial r} + r_u \frac{\partial c}{\partial \theta}) \right] \frac{\partial c}{\partial h_s} +
$$

$$
+ \left\{ \frac{\partial h}{\partial \xi} \frac{\partial c}{\partial \xi} + \frac{\partial h}{\partial \eta} \frac{\partial c}{\partial \eta} \right\} - \frac{\partial c}{\partial \zeta} = \frac{\partial c}{\partial \zeta} + \left[ \frac{\partial h}{\partial \xi} \frac{\partial c}{\partial \xi} + \frac{\partial h}{\partial \eta} \frac{\partial c}{\partial \eta} \right]
$$

$$
+ \left[ \frac{\partial h}{\partial \xi} \frac{\partial c}{\partial \xi} + \frac{\partial h}{\partial \eta} \frac{\partial c}{\partial \eta} \right] - \frac{\partial c}{\partial \zeta}
$$

$$
- \frac{1}{r^2} \frac{\partial (\nu \frac{\partial c}{\partial r})}{\partial r} - \frac{\partial (\nu \frac{\partial c}{\partial \theta})}{\partial \theta} = \frac{1}{r^2} \frac{\partial (\nu \frac{\partial c}{\partial \zeta})}{\partial \zeta} - \frac{w_s \frac{\partial c}{\partial \zeta}}{h \frac{\partial c}{\partial \xi}} - 0
$$

(4.12)
5. MODELLING THE CONCENTRATION FIELD BY ASYMPTOTIC SOLUTION
(THE GALAPPATI METHOD)

5.1 Introduction

The concentration field is approximated by an asymptotic solution. The method of asymptotic solution of the concentration field has been formulated by Galappatti (1983). Galappatti has confined the method to a two-dimensional situation, but indicated that extension to a 3-D model is quite feasible. Wang (1989) has extended the model to 3-D in order to apply it to estuary flow. The present application of the model is directed towards use in river bend flow.

The goal of the method is the formulation of the convection-diffusion equation in depth-averaged variables. To that purpose similarity shape functions are used for eddy viscosity, velocity and the equilibrium concentration.

In this chapter the Galappatti method is applied to the 3-D convection-diffusion equation which has been derived in the preceding chapters. Mathematical expressions for the time and length scales of the depth-averaged equation are derived for both a concentration and a concentration gradient boundary condition at the reference level.

5.2 Conversion to an s,n,r coordinate system

From a practical point of view the θ and r coordinates are transformed to coordinates in streamwise and normal directions, s and n respectively. These are given by:

\[ s = r\theta, \quad \partial s = r\partial \theta, \text{ at } r\text{-constant} \]
\[ n = r - R, \quad \partial n = \partial r \]

(5.1)

in which: s = streamwise coordinate
n = lateral coordinate
R = radius of channel curvature at channel axis
The convection-diffusion equation expressed in the s,n,ζ coordinate system becomes:

\[
\frac{\partial c}{\partial \tau} + \frac{\partial}{\partial \tau} \left( \nu \frac{\partial c}{\partial \tau} \right) = \\
- \frac{h}{w} \frac{\partial c}{\partial t} + \frac{\partial}{\partial t} \left[ r_u \frac{\partial c}{\partial s} + (r_v + \frac{\nu}{u} r_u) h \frac{\partial c}{\partial n} \right] + \\
+ \frac{\partial}{\partial s} (\xi - 1) r_u \frac{\partial h}{\partial s} \cdot (\xi - 1) (r_v + \frac{\nu}{u} r_u) \frac{\partial h}{\partial n} \frac{\partial c}{\partial \tau} + \frac{h}{w} [R_u \frac{\partial u}{\partial \tau} + R_v \frac{\partial v}{\partial \tau} + R_n \frac{\partial n}{\partial \tau}] \frac{\partial c}{\partial \tau} + \\
+ \frac{\partial}{\partial s} (\xi - R_u \frac{\partial h}{\partial s} + 2R_v \frac{\partial h}{\partial n} + h \frac{\partial R_u}{\partial \tau} \frac{\partial h}{\partial s} + h \frac{\partial R_v}{\partial \tau} \frac{\partial h}{\partial n} \frac{\partial c}{\partial n} \frac{\partial c}{\partial \tau} + \frac{\partial c}{\partial \tau} = 0
\]

\[
- \frac{h}{w} \frac{\partial c}{\partial s} (\nu \frac{\partial c}{\partial s}) - \frac{h}{w} \frac{\partial c}{\partial n} (\nu \frac{\partial c}{\partial n})
\]

(5.2)

5.3 Asymptotic approximation of the concentration field

The concentration is modelled by: \( c^i = \sum_{j=0}^{j} c^i \) \( \quad (5.3) \)

In which: \( c^i \) = local concentration, approximated to \( j \)th order

\( c^i = i \)th order contribution to \( c^i \)

The leading terms are the upward turbulent diffusion flux and the downward flux due to gravity, the zeroth order contribution to the local concentration \( c_0(s,n,\xi) \) is given by:

\[
\frac{\partial c}{\partial \tau} + \frac{\partial}{\partial \tau} \left( \nu \frac{\partial c}{\partial \tau} \right) = 0
\]

(5.4)

Galappattu assumed that the higher order contributions to the solution do not contribute to the depth-averaged concentration. This assumption minimizes the ratio \( c^i / c_0 \), which facilitates the theoretical solution of the model. Consequently the zeroth order concentration vertical is given by:

\[
c_0 = c_0 \hat{c}
\]

(5.5)
The higher order contributions satisfy: \( \int_0^1 c_i \, d\zeta = 0 \), for \( i > 0 \) \hfill (5.6) \\

To compute the higher order contribution to the solution Galappatti introduced the following operator:

\[
\text{OP}_\zeta[f] = g, \quad \text{with } \text{OP}_\zeta[f] = \frac{\partial f}{\partial \zeta} + \frac{\nu}{\zeta} \left( \frac{\partial f}{\partial s} \right)
\]

\hfill (5.7)

The solution of \( f \) is derived by the following considerations:

At the water surface no vertical fluxes are present, this yields:

\[
f(\zeta - 1) + \left. \frac{\nu}{\zeta} \left( \frac{\partial f}{\partial s} \right) \right|_{\zeta = 1} = 0
\]

\hfill (5.8)

This condition yields, see Galappatti (1983) appendix A,

\[
f(\zeta) - \text{OP}_\zeta^{-1}[g] = \int_\zeta^1 g \, d\zeta + r_c \int_\zeta^1 g \, d\zeta + Br_c
\]

\hfill (5.9)

\( B \) is a constant by which \( f \) is made to satisfy: \( \int_0^1 f \, d\zeta = 0 \)

The operator is linear, this means:

\[
\text{OP}_\zeta^{-1}[ag + \beta h] = \alpha \text{OP}_\zeta^{-1}[g] + \beta \text{OP}_\zeta^{-1}[h]
\]

\hfill (5.10)

Convection and turbulent diffusion due to the spatial non-homogeneous velocity and concentration fields are modelled on basis of the \( c_0(s,n,\zeta) \) concentration field. Turbulent diffusion in \( s \)-direction is neglected. The first order contribution to the local concentration, \( c_1(s,n,\zeta) \), is calculated by:

\[
\frac{\partial c_1}{\partial \zeta} + \left. \frac{\partial}{\partial \zeta} \left( \frac{\nu}{\zeta} \frac{\partial c_1}{\partial s} \right) \right|_{\zeta = 1} = \frac{h}{w_s} \frac{\partial c_0}{\partial t} + \frac{\dot{w}}{w_s} \left[ r_u \frac{\partial c_0}{\partial s} + \left( r_v + \frac{\dot{v}}{u} r_u \right) h \frac{\partial c_0}{\partial n} \right] + \\
+ \frac{\dot{w}}{w_s} \left[ - (\zeta - 1) r_u \frac{\partial h}{\partial s} - (\zeta - 1) \left( r_v + \frac{\dot{v}}{u} r_u \right) \frac{\partial h}{\partial n} \right] \frac{\partial c_0}{\partial \zeta} + \\
+ \frac{\dot{w}}{w_s} \left[ h \frac{\partial R}{\partial s} \frac{\partial c_0}{\partial s} + h \frac{\partial R}{\partial n} \frac{\partial c_0}{\partial n} \right] \frac{\partial c_0}{\partial \zeta} + \frac{\dot{v}}{w_s} h \frac{\partial R}{\partial \zeta} \frac{\partial c_0}{\partial n} 
\]
\[ + \left[ \frac{R_u}{w_s} \frac{\partial \tilde{u}}{\partial s} + R_u \frac{\tilde{u}}{w_s} \frac{\partial \tilde{h}}{\partial s} + \frac{R_v}{w_s} \frac{\partial \tilde{v}}{\partial n} + 2R_v \frac{\tilde{v}}{w_s} \frac{\partial \tilde{h}}{\partial n} \right] \frac{\partial c_0}{\partial \tilde{s}} + \]

\[ + \left[ \frac{R_u}{w_s} \frac{\partial \tilde{v}}{\partial n} + R_u \frac{\tilde{v}}{w_s} \frac{\partial \tilde{h}}{\partial n} \right] \frac{\partial c_0}{\partial \tilde{t}} - \frac{\tilde{h}}{w_s} \frac{\partial c_0}{\partial \tilde{s}} \frac{\partial c_0}{\partial \tilde{s}} - \frac{\tilde{h}}{w_s} \frac{\partial c_0}{\partial \tilde{n}} \frac{\partial c_0}{\partial \tilde{n}} \right] \]

(5.11)

This process, of substitution of the \(c_{i-1}\) contribution in the convective terms and the turbulent diffusion term in \(n\)-direction to compute the \(c_i\) contribution, can be repeated as many times as desired. In view, however, of the approximations underlying the derivation of the convection-diffusion equation in the cylindrical bed following coordinate system, in which products of more than two differentials are discarded, it will be unlikely that approximation of the concentration field by higher orders will improve the results.

The shape of the \(r_c\) profiles is governed by the \(Z\) parameter and the level at which \(\zeta = 0\), (the reference level). The ratio of reference level and local water depth will be assumed constant. In this case only the \(Z\) parameter varies spatially. The \(Z\) parameter is defined by:

\[ Z = \frac{w_s}{\rho Ku_s} \]  

(5.12)

The derivatives of the concentration vertical shape function \(r_c\) are:

\[ \frac{\partial r_c}{\partial t} = \frac{\partial r_c}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = \frac{\partial r_c}{\partial \tilde{s}} \frac{\partial \tilde{s}}{\partial s} = \frac{\partial r_c}{\partial \tilde{n}} \frac{\partial \tilde{n}}{\partial n} \]

(5.13)

These derivatives are neglected by Wang (1989, p. 33), but are included by Galappatti (1983, p. 53). The derivatives of \(Z\) are written as:

\[ \frac{\partial Z}{\partial s} = \frac{\partial Z}{\partial (w_s/u_s)} \frac{\partial (w_s/u_s)}{\partial s} = - \frac{w_s}{u_s} \frac{1}{u_s} \frac{\partial u_s}{\partial s} \frac{\partial Z}{\partial (w_s/u_s)} = - Z \frac{1}{u_s} \frac{\partial u_s}{\partial s} \]

(5.14)

Substitution of these expressions in eq.(5.11) yields eq.(5.15), representing the first order contribution to the concentration. The diffusion terms \(s\)- and \(n\)-direction are truncated. Wang (1989) has demonstrated that from a mathematical point of view it is important to have second order derivatives in the equation. Consequently the dependence of the
turbulent diffusion terms on the derivatives of $r_c$, $r_{\nu x}$ and $h$ are neglected, because these only yield first order derivatives, only second order derivatives are incorporated.

$$c_1 = \text{OP}^{-1} [r_c] \frac{\partial c}{\partial t} + \text{OP}^{-1} [g_2] \frac{h}{u_x} \frac{\partial u_x}{\partial t} \frac{\partial c}{\partial t} +$$

$$+ \text{OP}^{-1} [r_c u_s] \frac{h}{u_s} \frac{\partial u_s}{\partial t} \frac{\partial c}{\partial u_s} + \text{OP}^{-1} [r_c r_{\nu x}] \frac{h}{u_s} \frac{\partial u_s}{\partial t} \frac{\partial c}{\partial u_s} + \text{OP}^{-1} [r_c r_{\nu x c}] \frac{h}{u_s} \frac{\partial u_s}{\partial t} \frac{\partial c}{\partial u_s} +$$

$$- \text{OP}^{-1} [r_c g_2] \frac{h}{u_s} \frac{\partial u_s}{\partial t} \frac{\partial c}{\partial u_s} - \text{OP}^{-1} [r_c g_2] \frac{h}{u_s} \frac{\partial u_s}{\partial t} \frac{\partial c}{\partial u_s} +$$

$$- \text{OP}^{-1} [(r-1)r_u] \frac{\partial c}{\partial t} \frac{\partial c}{\partial u_s} + \text{OP}^{-1} [(r-1)\nu_{\rho \xi}] \frac{\partial c}{\partial t} \frac{\partial c}{\partial u_s} +$$

$$- \text{OP}^{-1} [(r-1)r_u] \frac{\partial c}{\partial t} \frac{\partial c}{\partial u_s} +$$

$$+ \text{OP}^{-1} \left[ \frac{\partial r_{\nu x}}{\partial \xi} \frac{\partial u_s}{\partial t} \right] \frac{\partial f_{\tau}}{\partial \xi} \frac{\partial c}{\partial t} + \text{OP}^{-1} \left[ \frac{\partial r_{\nu x}}{\partial \xi} \frac{\partial c}{\partial t} \right] \frac{\partial f_{\tau}}{\partial \xi} \frac{\partial c}{\partial t} +$$

$$+ \text{OP}^{-1} \left[ \frac{\partial r_{\nu x}}{\partial \xi} \frac{\partial c}{\partial t} \right] \frac{\partial f_{\tau}}{\partial \xi} \frac{\partial c}{\partial t} + \text{OP}^{-1} \left[ \frac{\partial r_{\nu x}}{\partial \xi} \frac{\partial c}{\partial t} \right] \left( \frac{\partial u_s}{\partial t} + \frac{\partial \nu_{\rho \xi}}{\partial \xi} \right) \frac{\partial c}{\partial t} +$$

$$+ \text{OP}^{-1} [r_{\nu x c}] \frac{\partial c}{\partial t} \frac{\partial c}{\partial \xi} + \text{OP}^{-1} [r_{\nu x c}] \frac{\partial c}{\partial t} \frac{\partial c}{\partial \xi} +$$

$$- \text{OP}^{-1} [r_{\nu x c}] \frac{\partial c}{\partial \xi} + \text{OP}^{-1} [r_{\nu x c}] \frac{h^2}{\partial \xi} \frac{\partial c}{\partial \xi} - h^2 \frac{\partial^2 c}{\partial \xi^2}$$

(5.15)

In which $g_2$ is defined by: $g_2 = \frac{\partial r_c}{\partial Z}$

(5.16)
5.4 Simplification in case of constant roughness height

The derivatives of the friction factor $f_*$ can be expressed in derivatives of $h$ in case the roughness height of the bed is constant.

The definition of the friction coefficient is: $f_* = \frac{u}{\kappa u_*}$. \hspace{1cm} (5.17)

The terms $\frac{1}{f_*} \frac{\partial f_*}{\partial s}$ and $\frac{1}{f_*} \frac{\partial f_*}{\partial n}$ are related to the Chézy coefficient by:

$$\frac{1}{f_*} \frac{\partial f_*}{\partial s} = \frac{1}{C} \frac{\partial C}{\partial s}, \quad \frac{1}{f_*} \frac{\partial f_*}{\partial n} = \frac{1}{C} \frac{\partial C}{\partial n} \hspace{1cm} (5.18)$$

This yields for constant roughness height $k_s$;

$$\frac{\partial f_*}{\partial s} = \frac{1}{h} \frac{\partial h}{\partial s}, \quad \frac{\partial f_*}{\partial n} = \frac{1}{h} \frac{\partial h}{\partial n} \hspace{1cm} (5.19)$$

The definition of $f_*$ yields:

$$\frac{1}{f_*} \frac{\partial f_*}{\partial s} = \frac{1}{u} \frac{\partial u}{\partial s} - \frac{1}{u_*} \frac{\partial u_*}{\partial s} \hspace{1cm} (5.20)$$

For $k_s$-constant this yields:

$$\frac{1}{u_*} \frac{\partial u_*}{\partial s} = \frac{1}{u} \frac{\partial u}{\partial s} - \frac{1}{f_*} \frac{1}{h} \frac{\partial h}{\partial s} \hspace{1cm} (5.21)$$

In case the relative roughness height is constant, $k_s/a=\text{const.}$, the derivative of $u_*$ is related to $\hat{u}$ only:

$$\frac{1}{u_*} \frac{\partial u_*}{\partial s} = \frac{1}{u} \frac{\partial \hat{u}}{\partial s} \hspace{1cm} (5.21a)$$

Similar expressions can be computed for the derivatives to $t$ and $n$.

5.5 The first order contribution to the concentration

The formulation of the depth-averaged convection-diffusion equation incorporates the following approximations:

The shape functions for $u$ and $v$ are based on similarity profiles.

The shape function of the secondary flow is proportional with the depth and curvature of the streamlines $1/r_s$. The flow does not diverge or converge significantly such that $\partial r_s/\partial r=1$ is valid. The roughness height $k_s$ is assumed constant.

The first order contribution to the concentration is:
\[ c_1 = a_{11} \frac{\partial c}{\partial t} + a_{21} \frac{\partial c}{\partial s} + a_{12} \frac{\partial c}{\partial n} - a_{22} h^2 \frac{\partial c}{\partial s^2} - a_{22} h^2 \frac{\partial c}{\partial n^2} + \]
\[ + b_{11} \frac{1}{f_s} \frac{\partial h}{\partial t} \frac{\partial c}{\partial t} + (b_{21} \frac{1}{f_s} - c_{21} + d_{21} + e_{21}) \frac{\partial \frac{\partial h}{\partial s}}{\partial s} \frac{\partial c}{\partial s} + \]
\[ + (b_{12} \frac{1}{f_s} - c_{12} + 2d_{12} + e_{12}) \frac{\partial \frac{\partial h}{\partial n}}{\partial n} \frac{\partial c}{\partial n} + \]
\[ - b_{11} h \frac{1}{u} \frac{\partial \frac{\partial h}{\partial t}}{\partial t} \frac{\partial c}{\partial t} + (-b_{21} + d_{21}) h \frac{\partial \frac{\partial h}{\partial s}}{\partial s} \frac{\partial c}{\partial s} + (-b_{12} + d_{12}) h \frac{\partial \frac{\partial h}{\partial n}}{\partial n} \frac{\partial c}{\partial n} + \]
\[ + \frac{\ddot{v}}{u} (a_{21} \frac{\partial \frac{\partial h}{\partial s}}{\partial s} + (b_{21} \frac{1}{f_s} - c_{21} + d_{21} + e_{21}) \frac{\partial \frac{\partial h}{\partial s}}{\partial s} \frac{\partial c}{\partial s} + \]
\[ + (-b_{21} + d_{21}) \frac{\partial \frac{\partial h}{\partial s}}{\partial s} \frac{\partial c}{\partial s} + d_{21} \frac{\partial \frac{\partial h}{\partial s}}{\partial s} \frac{\partial c}{\partial s} \]  
(5.22)

In which:
\[ a_{11} = \text{OP}_r \left[ \frac{1}{r_c} \right] \]
\[ a_{21} = \text{OP}_s \left[ \frac{1}{r_c r_u} \right] \]
\[ a_{12} = \text{OP}_s \left[ \frac{1}{r_c r_v} \right] \]
\[ a_{22} = \text{OP}_s \left[ \frac{1}{r_c r_{\xi \chi}} \right] \]
\[ b_{11} = \text{OP}_s \left[ \frac{1}{g_{\xi}} \right] \]
\[ b_{21} = \text{OP}_s \left[ \frac{1}{g_{\xi} r_u} \right] \]
\[ b_{12} = \text{OP}_s \left[ \frac{1}{g_{\xi} r_v} \right] \]

\[ c_{21} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \xi \eta}} \right] \]
\[ d_{21} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \eta \xi}} \right] \]
\[ e_{21} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \xi \xi}} \right] \]
\[ c_{12} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \eta \eta}} \right] \]
\[ d_{12} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \eta \xi}} \right] \]
\[ e_{12} = \text{OP}_s \left[ \frac{\partial r}{\partial \frac{\partial c}{\partial \xi \xi}} \right] \]

These inverse operators are a function of \( \xi \). The parameters \( a_{12}, b_{12}, c_{12}, d_{12} \) and \( e_{12} \) are linear with \( a/r \), because \( r_v \) is a linear function of \( a/r \), eq.(4.10).

The \( b_{11} \frac{\partial h}{\partial t} \) and the \( b_{11} \frac{\partial h}{\partial t} \) terms originate from \( \frac{\partial r}{\partial \xi} \frac{\partial t}{\partial \xi} \).

The \( a_{21}, a_{12}, b_{21} \) and \( b_{12} \) terms originate from \( \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s} \) and \( v \frac{\partial c}{\partial n} \).

The \( a_{22} \) terms originate from \( \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \) and \( v \frac{\partial c}{\partial n} \).

The \( c_{ij} \) terms originate from the non-orthogonality of the coordinate system.

The \( d_{ij} \) and \( e_{ij} \) terms originate from \( w \frac{\partial c}{\partial \xi} \).
In case of constant relative roughness height, \( \frac{k_y}{h} = \text{const.} \), the \( e_{21} \) and \( e_{12} \) operators are zero. Also the terms of eq.(5.27) incorporating \( \frac{1}{f} \) are zero because of eq.(5.21a).

5.6 Boundary conditions

A boundary condition has to be applied at the reference level \( \xi = 0 \).

Either the concentration or the concentration gradient at this level has to be prescribed.

If the concentration is prescribed, it is assumed that the local concentrations near the bed are in equilibrium with local flow conditions near the bed.

If the concentration gradient is prescribed it is assumed that the flux due to turbulent diffusion, near the bed, is equal to the turbulent diffusion near the bed, under equilibrium conditions.

The differential equations for both boundary conditions will be given in the following paragraphs.

5.6.1 The concentration condition

The concentration at equilibrium conditions is denoted by: \( c_e \). The depth averaged concentration at equilibrium conditions is denoted by: \( \bar{c}_e \).

The boundary condition at the reference level, \( \xi = 0 \), to first order approximation, is:

\[
\dot{c}_e (0) = c_0 (0) + c_1 (0)
\]  

(5.24)

This is equal with:

\[
\dot{c}_e - \dot{c}_h = c_1 (0)/\gamma_c (0)
\]

which is equivalent to:

\[
\dot{c}_e = T_{ct} \frac{\partial \bar{c}}{\partial t} + L_{cs} \frac{\partial \bar{c}}{\partial s} + L_{cn} \frac{\partial \bar{c}}{\partial n} - D \frac{\partial^2 \bar{c}}{\partial s^2} - D \frac{\partial^2 \bar{c}}{\partial n^2} + (1 + T_{ht} \frac{1}{h} \frac{\partial h}{\partial t} + L_{hs} \frac{1}{h} \frac{\partial h}{\partial s} + L_{hn} \frac{1}{h} \frac{\partial h}{\partial n} + T_{ut} \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial t} + L_{us} \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial s} + L_{un} \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial n} )\dot{c} +
\]
\[ + \frac{\hat{Y}}{u} \left( \frac{\delta \hat{c}}{\delta n} \right) + \left( L_{hs} \frac{1}{h} \frac{\delta h}{\delta n} + L_{us} \frac{1}{u} \frac{\delta \hat{u}}{\delta n} \right) \hat{c} + L_{vun} \frac{\delta \hat{v}/\hat{u}}{\delta n} \hat{c} \]  \quad (5.25)

The parameters \( T \) and \( L \) are time and length scales of adaptation. The parameter \( D \) is a diffusion coefficient. These parameters are calculated by:

\[ T_{ct} = \frac{a_{11}(0)}{r_c(0)} \frac{h}{w_s} , \quad L_{cs} = \frac{a_{21}(0)}{r_c(0)} \frac{\hat{u} h}{w_s} , \quad L_{cn} = \frac{a_{12}(0)}{r_c(0)} \frac{\hat{u} h}{w_s} , \quad D = \frac{a_{22}(0)}{r_c(0)} h^2 \]

\[ T_{ht} = \frac{1}{f_*} \frac{b_{11}(0)}{r_c(0)} \frac{h}{w_s} \]

\[ L_{hs} = \left( \frac{b_{21}(0)}{r_c(0)} \frac{1}{f_*} - \frac{c_{21}(0)}{r_c(0)} + \frac{d_{21}(0)}{r_c(0)} + \frac{e_{21}(0)}{r_c(0)} \right) \frac{\hat{u} h}{w_s} \]

\[ L_{hn} = \left( \frac{b_{12}(0)}{r_c(0)} \frac{1}{f_*} - \frac{c_{12}(0)}{r_c(0)} + 2 \frac{d_{12}(0)}{r_c(0)} + \frac{e_{12}(0)}{r_c(0)} \right) \frac{\hat{u} h}{w_s} \]

\[ T_{ut} = - \frac{b_{11}(0)}{r_c(0)} \frac{h}{w_s} , \quad L_{us} = \left( - \frac{b_{21}(0)}{r_c(0)} + \frac{d_{21}(0)}{r_c(0)} \right) \frac{\hat{u} h}{w_s} \]

\[ L_{un} = \left( \frac{b_{12}(0)}{r_c(0)} + \frac{d_{12}(0)}{r_c(0)} \right) \frac{\hat{u} h}{w_s} , \quad L_{vun} = \frac{d_{21}(0)}{r_c(0)} \frac{\hat{u} h}{w_s} \]

5.6.2 The concentration gradient condition

The concentration gradient at equilibrium conditions, at the reference level is denoted by: \( \frac{\partial c_e}{\partial \hat{c}} \bigg|_0 \).

The boundary condition, to first order approximation, is formulated by:

\[ \frac{\partial c_e}{\partial \hat{r}} \bigg|_0 = \frac{\partial c_0}{\partial \hat{r}} \bigg|_0 + \frac{\partial c_1}{\partial \hat{r}} \bigg|_0 \]  \quad (5.27)

At equilibrium, the flux by turbulent diffusion is equal to the flux by gravity. Assuming equilibrium at reference level yields:

\[ \frac{\partial c_e}{\partial \hat{r}} \bigg|_0 = - \frac{h w_s}{\nu c_e} - \frac{h w_s}{\nu c_e} \frac{c_e}{r_c(0)} \hat{c}_e \]  \quad (5.28)
The zeroth order part of the solution is also based on the balance of turbulent diffusion and the flux due to gravity. The zeroth order concentration gradient is:

\[
\frac{\partial c_0}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} c_0 = -\frac{h}{\nu} \frac{\omega}{\chi} c_c(0) \tilde{c} 
\]  

(5.29)

The concentration gradient of \( c_1 \) at reference level will have to be formulated in derivatives of the \( a_{ij} \), \( b_{ij} \), \( c_{ij} \), \( d_{ij} \), and \( e_{ij} \) profile functions at reference level. The profiles originate from the inverse operator, eq.(5.23)

According to Ribberink (1986) and Wang (1984) the OP[ ] operator has the following mathematical property:

\[
\frac{\partial f}{\partial \zeta} = -\frac{h}{\nu} \frac{\omega}{\chi} \left( f + \int_0^1 g \, d\zeta \right), \text{ with } \text{OP}[f] = g 
\]

(5.30)

For example for \( f = a_{11} \) this yields:

\[
\frac{\partial a_{11}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{11}(0) + \int_0^1 \text{OP}[a_{11}] d\zeta) = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{11}(0) + \int_0^1 \chi d\zeta) = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{11}(0) + 1) 
\]

(5.31)

The remaining derivatives are equal to:

\[
\frac{\partial a_{21}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{21}(0) + \alpha_s), \quad \frac{\partial a_{12}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{12}(0) + \alpha_n), \\
\frac{\partial a_{22}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (a_{22}(0) + \alpha_d), \\
\frac{\partial b_{11}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (b_{11}(0) + \beta_s), \quad \frac{\partial b_{21}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (b_{21}(0) + \beta_n), \\
\frac{\partial b_{12}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (b_{12}(0) + \beta_n), \\
\frac{\partial c_{21}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (c_{21}(0) + \gamma_s), \quad \frac{\partial c_{12}}{\partial \zeta} \bigg|_0 = -\frac{h}{\nu} \frac{\omega}{\chi} (c_{12}(0) + \gamma_n) 
\]

(5.32)
\[ \frac{\partial d_{21}}{\partial \xi} \bigg|_0 = -\frac{\hat{\nu}_x}{\nu} (d_{21}(0) + \delta_s) , \quad \frac{\partial d_{12}}{\partial \xi} \bigg|_0 = -\frac{\hat{\nu}_x}{\nu} (d_{12}(0) + \delta_n) \]

\[ \frac{\partial e_{21}}{\partial \xi} \bigg|_0 = -\frac{\hat{\nu}_x}{\nu} (e_{21}(0) + \epsilon_s) , \quad \frac{\partial e_{12}}{\partial \xi} \bigg|_0 = -\frac{\hat{\nu}_x}{\nu} (e_{12}(0) + \epsilon_n) \]

in which:

\[ \alpha = \int_0^1 \frac{d r}{d \xi} d \xi = 1 , \quad \alpha_s = \int_0^1 \frac{d r}{d \xi} d \xi , \quad \alpha_n = \int_0^1 \frac{d r}{d \xi} d \xi , \quad \alpha_d = \int_0^1 \frac{\nu_x}{d \xi} d \xi \]

\[ \beta = \int_0^1 \frac{\partial r}{d \xi} d \xi , \quad \beta_s = \int_0^1 \frac{\partial r}{d \xi} d \xi , \quad \beta_n = \int_0^1 \frac{\nu_x}{d \xi} d \xi \]

\[ \gamma_s = \int_0^1 (r-1) \frac{\partial r_s}{d \xi} d \xi , \quad \gamma_n = \int_0^1 (r-1) \frac{\partial r_n}{d \xi} d \xi \]

\[ \delta_s = \int_0^1 \frac{\partial r}{d \xi} d \xi , \quad \delta_n = \int_0^1 \frac{\nu_x}{d \xi} d \xi \]

\[ \epsilon_s = \int_0^1 \frac{\partial r_s}{d \xi} d \xi , \quad \epsilon_n = \int_0^1 \frac{\partial r_n}{d \xi} d \xi \]

(5.33)

In this case the convection-diffusion equation including the concentration gradient condition is also given by eq. (5.25), however, the time and length scales are different. Inserting the above gradients yields the following scales:

\[ T_{ct} = \frac{a_{11}(0)}{r_c(0) + \frac{1}{r_c(0)}} \frac{\tilde{h}}{\tilde{w}_s} \]

\[ L_{cs} = \frac{a_{21}(0) + \alpha_s \tilde{w}}{r_c(0)} \frac{\tilde{w}_s}{\tilde{w}_s} \]

\[ L_{cn} = \frac{a_{12}(0) + \alpha_n \tilde{w}}{r_c(0)} \frac{\tilde{w}_s}{\tilde{w}_s} \]

\[ D = \frac{a_{22}(0) + \alpha_d}{r_c(0)} \tilde{h}^2 \]

\[ T_{ht} = \frac{1}{f_{*}} \frac{r_c(0)}{r_c(0)} \frac{\tilde{h}}{\tilde{w}_s} \]

\[ L_{hs} = \frac{b_{21}(0) + \beta_s}{r_c(0)} \frac{1}{f_{*}} \frac{c_{21}(0) + \gamma_s}{r_c(0)} + \frac{d_{21}(0) + \delta_s}{r_c(0)} + \frac{e_{21}(0) + \epsilon_s}{r_c(0)} \frac{\tilde{w}}{\tilde{w}_s} \]

\[ L_{hn} = \frac{b_{12}(0) + \beta_n}{r_c(0)} \frac{1}{f_{*}} \frac{c_{12}(0) + \gamma_n}{r_c(0)} + \frac{d_{12}(0) + \delta_n}{r_c(0)} + 2 \frac{e_{12}(0) + \epsilon_n}{r_c(0)} \frac{\tilde{w}}{\tilde{w}_s} \]

(5.34)
\[ T_{ut} = -\frac{b_{11}(0) + \beta h_x}{r_c(0)} \frac{w_s}{w_s}, \quad L_{us} = \left( -\frac{b_{21}(0) + \beta s}{r_c(0)} + \frac{d_{21}(0) + \delta s}{r_c(0)} \right) \frac{\bar{u}_h}{w_s}, \]
\[ L_{un} = \left( -\frac{b_{12}(0) + \beta n}{r_c(0)} + \frac{d_{12}(0) + \delta n}{r_c(0)} \right) \frac{\bar{u}_h}{w_s}, \quad L_{un} = \frac{d_{21}(0) + \delta s}{r_c(0)} \frac{\bar{u}_h}{w_s}. \]

5.6.3 The equilibrium concentration at reference height

Both boundary conditions are formulated in terms of the depth averaged equilibrium concentration \( \bar{c}_e \). The depth-averaged equilibrium concentration is related to the equilibrium transport rate by:

\[ s_e = \alpha \frac{h \bar{c}_e}{a} \tag{5.35} \]

This transport rate can be related to the depth-averaged velocity by an Engelund Hansen type of transport formula:

\[ s_e = a_s u_s \tag{5.36} \]

with \( a_s \) and \( b_s \) empirical constants.

In that case the equilibrium depth-averaged concentration becomes:

\[ \bar{c}_e = a_s u \frac{(b_s - 1)}{h \alpha} \tag{5.37} \]

5.7 Computation of characteristic time and length-scales

The time and length-scales of the model have been calculated on basis of the following similarity profiles of velocity and concentration:

\[ r_u = 1 + \frac{1}{f_x^*} (\ln \frac{z}{a} + 1) \tag{5.38} \]

\[ r_v = -\frac{a}{\kappa^2} \left( 2 \int_0^a \frac{\ln(z/a)}{z/a-1} \, dz + \frac{1}{f_x^*} \int_0^a \frac{\ln^2(z/a)}{z/a-1} \, dz \right) + 2 \left( \frac{2}{f_x^*} \right) \left( \frac{2}{f_x^*} \right) \left( \frac{2}{f_x^*} \right) \ln \frac{z}{a} \tag{5.39} \]
\[ r_c = B \left( \frac{z_r}{a-z_r} \right) \frac{a-z}{z} \]  

(5.40)

In which use have been made of the eddy-viscosity for momentum:

\[ \nu_t = \kappa \frac{a}{a} (1 - \frac{z_r}{a}) a \, \nu_x \]  

(5.41)

The eddy viscosity for mass is given by: \[ \nu_x = \beta \, \nu_t \]  

(5.42)

with: \( z_r \) - reference height  
\( Z = \frac{\omega_s}{(\beta \kappa u_x)} \)  
\( u_x = \sqrt{\rho_b} \)  
\( \tau_b \) - the bed shear-stress  
\( B \) - a constant satisfying: \( \int_0^1 r_c \, d\xi = 1 \)

The derivatives of \( r_c \) to \( \xi \) and \( Z \) are:

\[ \frac{\partial r_c}{\partial \xi} - \frac{\partial r_c}{\partial Z} \frac{\partial Z}{\partial \xi} = Z \frac{-a}{z(1-z/a)} r_c \]  

(5.43)

\[ \frac{\partial r_c}{\partial Z} = r_c \left( \ln \frac{1-\xi}{\xi} + \int_0^{1} \frac{1-\xi}{\xi} \ln \frac{1-\xi}{\xi} \, d\xi + \int_0^{1} \frac{1-\xi}{\xi} Z \, d\xi \right) \]  

(5.44)

The first order contribution to the concentration is computed by applying the inverse Galappatti operator \( 0 P^{-1} \), eq.(5.23) to the functions (5.38) 

... (5.44).

The result are concentration profiles which represent the perturbation of the concentration profile compared to the equilibrium shape. In figure 4.1 the equilibrium shapes profiles are shown, for \( C = 35 \, m^3/s, z_r/a = 0.05 \) and \( Z = 0.4 \). In figure 5.1 the perturbation profiles \( a_{ij}, \, b_{ij}, \, c_{ij} \) and \( d_{ij} \) are shown, for the same parameters. To apply the bed boundary condition the value of the perturbation profiles at reference level is necessary. The values of \( a_{ij}, \, b_{ij}, \, c_{ij} \) and \( d_{ij} \) at reference level as a function of the \( Z \) parameter are given in figure 5.2 for \( z_r/a = 0.05 \) and \( C = 35 \, m^3/s \). In case the gradient boundary condition is applied the integrals of the profiles functions are to be calculated too. In figure 5.3 these integrals, \( a \) to \( \delta \), are given as a function of \( Z \), for \( C = 35 \, m^3/s \) and \( z_r/a = 0.05 \).
The characteristic time and length-scales $T_{ct}$, $L_{cs}$ and $L_{cn}$ of the model are given in figure 5.4 in case of the concentration boundary condition, and in figure 5.5 in case of the gradient boundary condition, also for $C = 35 \, m^3/s$ and $z_L/a = 0.05$. In appendix A the remaining scales are also given.

The scales reflecting the non-uniform depth-averaged concentration field are $T_{ct}$, $L_{cs}$ and $L_{cn}$. The scales reflecting the spatial gradients of the local depth and the local depth-averaged velocity are $L_{hs}$, $L_{hn}$, $L_{us}$, $L_{un}$ and $L_{vun}$. The local depth-averaged equilibrium concentration depends on the local depth and the local depth-averaged velocity.

In appendix A the time and length scales of the model are presented for $0.05 < z_L/a < 0.35$ and $20 < C < 65 \, m^3/s$ as function of $Z$, with $0.05 < Z < 1$. Both concentration and the gradient condition are presented.

The magnitude of the scales corresponding to the gradient boundary condition are generally up to about 3 times larger than those of the concentration condition.

The $L_{hn}$ and $L_{un}$ length scales are the most sensitive to the Chézy coefficient (fig. A6), because these involve the secondary flow profile which, for at least $z/a < 0.3$, is sensitive to the Chézy coefficient.

The choice of reference height affects nearly all scales.

Non-orthogonality is reflected in the $L_{hs}$ and $L_{hn}$ scales, its effect is most pronounced for the smaller reference heights (fig. A4, A5, A9 and A10).

The governing equation in case of stationary flow can be written as:

$$
\dot{c} + \lambda_s \frac{\partial c}{\partial s} + \lambda_n \frac{\partial c}{\partial n} - \frac{D}{A} \frac{\partial^2 c}{\partial s^2} - \frac{D}{A} \frac{\partial^2 c}{\partial n^2} = \beta_e \dot{c}_e
$$

(5.45)

In which: $\lambda_s = \frac{L_{cs}}{A}$, $\lambda_n = \frac{L_{cn}}{A}$, $\dot{c}_e = \frac{\dot{v} L_{cs}}{A}$ and $\beta_e = \frac{1}{A}$

(5.46)

with: $A = 1 + L_{hs} \frac{1}{h} \frac{\partial h}{\partial s} + L_{hn} \frac{1}{h} \frac{\partial h}{\partial n} + L_{us} \frac{1}{u} \frac{\partial u}{\partial s} + L_{un} \frac{1}{u} \frac{\partial u}{\partial n} +$ 

$$
+ \frac{\dot{v}}{u} (L_{hs} \frac{1}{h} \frac{\partial h}{\partial n} + L_{us} \frac{1}{u} \frac{\partial u}{\partial n}) + L_{vun} \frac{\partial v}{\partial n}
$$

(5.47)
The adaptation lengths $\lambda_s$ and $\lambda_n$ of the model strongly depend on $L_{cs}$ and $L_{cn}$. From computation of these scales it is, however, clear that these scales are mainly affected by the choice of boundary condition and also by the choice of reference level.

The effect of turbulent diffusion is small. Turbulent diffusion is included because it has some mathematical advantages (Wang, 1989).
6. TRANSPORT AND ENTRAINMENT

6.1. Introduction

On basis of the computed depth-averaged concentration field, eq. (5.45), transport rates in s- and n-direction are to be computed. Mathematical expressions for these transport rates are derived in sections 6.2, 6.3 and 6.4. Entrainment at reference level has to be computed numerically by the expressions given in section 6.5. The change of bed level is governed by the sediment continuity equation, section 6.6.

6.2. Transport in s-direction

The depth-integrated suspended sediment transport in s-direction is defined by:

\[ S_{sus} = h \int_{0}^{1} (uc - \nu \tau_{s}) \frac{\partial \tilde{c}}{\partial s} \, ds \]  \hspace{1cm} (6.1)

In which: \( S_{sus} \) is the depth integrated transport of suspended sediment in s-direction.

The convective transport is equal to: \( h \int_{0}^{1} uc \, ds \)

The first order solution for the concentration is used: \( c = c_{o} + c_{1} \)

This yields, together with the substitution for \( c_{1} \), eq. (5.22) (the diffusion part of eq. (5.22) is neglected):

\[ h \int_{0}^{1} u(c_{o} + c_{1}) \, ds = \]

\[ = \alpha_{s} \tilde{h} \tilde{u} \tilde{c} + h \left( \alpha_{s11} \frac{\partial \tilde{c}}{\partial t} + \alpha_{s21} \frac{\partial \tilde{u}}{\partial s} \frac{\partial \tilde{c}}{\partial s} + \alpha_{s12} \frac{\partial \tilde{u}}{\partial n} \frac{\partial \tilde{c}}{\partial n} + \right. \]

\[ + \beta_{s11} \frac{1}{f_{s}} \frac{\partial \tilde{h}}{\partial t} \tilde{c} + \left( \beta_{s21} \frac{1}{f_{s}} \gamma_{s21} + \epsilon_{s21} \epsilon_{s21} \right) \frac{\partial \tilde{h}}{\partial s} \frac{\partial \tilde{c}}{\partial s} + \right) \]
+ \left( \beta_{s12} \frac{1}{f^*} - \gamma_{s12} + 2\delta_{s12} + \epsilon_{s12} \right) \frac{\partial h}{\partial s} \frac{\partial c}{\partial t} + \\
- \beta_{s11} \frac{h}{w} \frac{\partial u}{\partial s} \frac{\partial c}{\partial t} + \left( -\beta_{s21} + \delta_{s21} \right) \frac{h}{w} \frac{\partial u}{\partial s} \frac{\partial c}{\partial n} + \left( -\beta_{s12} + \delta_{s12} \right) \frac{h}{w} \frac{\partial u}{\partial s} \frac{\partial c}{\partial n} + \\
+ \frac{\gamma}{u} \left( \alpha_{s21} \frac{b}{w} \frac{\partial c}{\partial n} + \beta_{s21} \frac{1}{f^*} + \gamma_{s21} + \delta_{s21} + \epsilon_{s21} \right) \frac{\partial h}{\partial s} \frac{\partial c}{\partial n} + \\
+ \left( -\beta_{s21} + \delta_{s21} \right) \frac{h}{w} \frac{\partial u}{\partial s} \frac{\partial c}{\partial n} \right) (6.2)

In which:

\begin{align*}
\alpha_{s11} &= \int_{0}^{1} r_{u} a_{11} d\xi \\
\beta_{s11} &= \int_{0}^{1} r_{u} b_{11} d\xi \\
\alpha_{s21} &= \int_{0}^{1} r_{u} a_{21} d\xi \\
\beta_{s21} &= \int_{0}^{1} r_{u} b_{21} d\xi \\
\alpha_{s12} &= \int_{0}^{1} r_{u} a_{12} d\xi \\
\beta_{s12} &= \int_{0}^{1} r_{u} b_{12} d\xi \\
\gamma_{s21} &= \int_{0}^{1} r_{u} c_{21} d\xi \\
\delta_{s21} &= \int_{0}^{1} r_{u} d_{21} d\xi \\
\epsilon_{s21} &= \int_{0}^{1} r_{u} e_{21} d\xi \\
\gamma_{s12} &= \int_{0}^{1} r_{u} c_{12} d\xi \\
\delta_{s12} &= \int_{0}^{1} r_{u} d_{12} d\xi \\
\epsilon_{s12} &= \int_{0}^{1} r_{u} e_{12} d\xi
\end{align*} (6.3)

The transport in s-direction by turbulent diffusion is equal to:

\[-h \int_{0}^{1} \nu_{cX} \frac{\partial c}{\partial s} \, d\xi\]

This term is very small in comparison with the convective transport term in s-direction, see section 6.4. Consequently this term will be omitted.
6.3. Transport in n-direction

The depth-integrated suspended sediment transport in n-direction is defined by:

\[ S_{\text{sus n}} = h \int_0^1 (v_c - \nu_t x_\chi \partial c/\partial n) \, \text{d}x \]  \hspace{1cm} (6.4)

In which: \( S_{\text{sus n}} \) - the depth integrated transport of suspended sediment in n-direction

The convective transport is equal to: \( h \int_0^1 v_c \, \text{d}x \)

The first order solution for the concentration is used: \( c = c_0 + c_1 \)
This yields, together with the substitution for \( c_1 \), eq.(5.22), and \( v = r_u \dot{v} + r_v \dot{u} \).

\[ h \int_0^1 v(c_0 + c_1) \, \text{d}x = \]

\[ = \alpha_s h \dot{v} \dot{c} + h \nu \left( \alpha_s \frac{h \partial c}{w_s \partial t} + \alpha_s \frac{uh \partial c}{w_s \partial s} + \alpha_s \frac{uh}{w_s} \frac{\partial c}{\partial n} + \right) \]

\[ + \beta_s \frac{1}{f_s} \frac{\partial h}{\partial t} \dot{c} + \left( \beta_s \frac{1}{f_s} - \gamma_s \delta_{s21} + \delta_{s21} \right) \frac{\partial h}{w_s} \frac{\partial c}{\partial s} + \]

\[ + \left( \beta_s \frac{1}{f_s} - \gamma_s \delta_{s12} + 2 \delta_{s12} \right) \frac{\partial h}{w_s} \frac{\partial c}{\partial n} + \]

\[ - \beta_s \frac{1}{w_s} \frac{\partial u}{\partial t} \dot{c} + (-\beta_s \delta_{s21}) \frac{h \partial u}{w_s} \dot{c} + (-\beta_s \delta_{s12}) \frac{h \partial u}{w_s} \frac{\partial c}{\partial n} + \]

\[ + \dot{v} \left( \alpha_s \frac{uh \partial c}{w_s \partial n} + \beta_s \frac{1}{f_s} - \gamma_s \delta_{s21} + \delta_{s21} \right) \frac{\partial h}{w_s} \frac{\partial c}{\partial n} + \]

\[ + (-\beta_s \delta_{s21}) \frac{h \partial u}{w_s} \frac{\partial c}{\partial n} + \delta_s \frac{h \partial u}{w_s} \frac{\partial c}{\partial n} + \right) + \]

\[ + \alpha_n h \dot{u} \dot{c} + h \nu \left( \alpha_n \frac{h \partial c}{w_s \partial t} + \alpha_n \frac{uh \partial c}{w_s \partial s} + \alpha_n \frac{uh}{w_s} \frac{\partial c}{\partial n} + \right) \]
\[
\begin{align*}
+ \beta_{n11} \frac{1}{f_w^*} \frac{\partial h}{\partial t} \hat{c} + (\beta_{n21} \frac{1}{f_w^*} - \gamma_{n21}^+ \delta_{n21}^+ \epsilon_{n21}) \frac{\partial}{\partial s} \hat{c} + \\
+ (\beta_{n12} \frac{1}{f_w^*} - \gamma_{n12}^+ 2 \delta_{n12}^+ \epsilon_{n12}) \frac{\partial}{\partial n} \hat{c} + \\
- \beta_{n11} \frac{h}{w} \frac{\partial u}{\partial t} \hat{c} + (\beta_{n21}^+ \delta_{n21}) \frac{h}{w} \frac{\partial u}{\partial s} \hat{c} + (\beta_{n12}^+ \delta_{n12}) \frac{h}{w} \frac{\partial}{\partial n} \hat{c} + \\
+ \frac{\partial}{\partial t} \left( \alpha_{n21} \frac{\partial}{\partial n} \frac{\partial \hat{c}}{\partial n} + (\beta_{n21} \frac{1}{f_w^*} - \gamma_{n21}^+ \delta_{n21}^+ \epsilon_{n21}) \frac{\partial}{\partial n} \hat{c} \right) + \delta_{n21} \frac{\partial}{\partial n} \frac{\partial v}{\partial u} \hat{c} \\
\right) 
\end{align*}
\] 

\text{(6.5)}

In which:

\[
\begin{align*}
\alpha_{n11} = \int_0^1 r_v a_{11}^d \delta \\
\beta_{n11} = \int_0^1 r_v b_{11}^d \\
\alpha_{n21} = \int_0^1 r_v a_{21}^d \\
\beta_{n21} = \int_0^1 r_v b_{21}^d \\
\alpha_{n12} = \int_0^1 r_v a_{12}^d \\
\beta_{n12} = \int_0^1 r_v b_{12}^d \\
\gamma_{n21} = \int_0^1 r_v c_{21}^d \\
\delta_{n21} = \int_0^1 r_v d_{21}^d \\
\epsilon_{n21} = \int_0^1 r_v e_{21}^d \\
\gamma_{n12} = \int_0^1 r_v c_{12}^d \\
\delta_{n12} = \int_0^1 r_v d_{12}^d \\
\epsilon_{n12} = \int_0^1 r_v e_{12}^d 
\end{align*}
\] 

\text{(6.6)}

The transport in n-direction by turbulent diffusion is equal to:

\[
-h \int_0^1 \nu_t x \frac{\partial \hat{c}}{\partial n} d \delta = -\alpha_{ah} h \frac{\partial \hat{c}_0}{\partial n}
\] 

\text{(6.7)}

The magnitude of this zeroth order contribution is comparable with the contribution by the first order convective terms, see section 6.4.
6.4. Order of magnitude estimates

A same type of order of magnitude analysis as described in section 4.2 is employed. In the transport equations eq. (6.2) and eq. (6.5) the coefficients \( \alpha_{ij}, \alpha_{nij} \) appear. These coefficients have been calculated numerically. Maximum values of these coefficient are given below:

\[
\begin{align*}
\text{range of computation:} & \quad 0.05 < z < 1.0 \\
& \quad 20 < c < 65 \text{ m/s} \\
& \quad 0.05 < z_{ref} < 0.35
\end{align*}
\]

\[
\begin{align*}
\max \alpha_s & \approx 1.2, \quad \max \alpha_n \approx 0.6 \text{ a/r} \\
\max \alpha_{s1} & \approx -0.02, \quad \max \alpha_{n1} \approx -0.4 \text{ a/r} \\
\max \alpha_{s21} & \approx -0.02, \quad \max \alpha_{n21} \approx -0.4 \text{ a/r} \\
\max \alpha_{s12} & \approx -0.02 \text{ a/r}, \quad \max \alpha_{n12} \approx -0.6 \text{ a/r}^2 \\
\max \beta_{s1} & \approx 0.02, \quad \max \beta_{n1} \approx 0.3 \text{ a/r} \\
\max \beta_{s21} & \approx 0.01, \quad \max \beta_{n21} \approx 0.3 \text{ a/r} \\
\max \gamma_{s21} & \approx -0.04, \quad \max \gamma_{n21} \approx -0.6 \text{ a/r} \\
\max \gamma_{s12} & \approx -0.01 \text{ a/r}, \quad \max \gamma_{n12} \approx -1.4 \text{ a/r}^2 \\
\max \delta_{s21} & \approx 0.05, \quad \max \delta_{n21} \approx 0.7 \text{ a/r} \\
\max \delta_{s12} & \approx 0.01 \text{ a/r}, \quad \max \delta_{n12} \approx 2.0 \text{ a/r}^2
\end{align*}
\]

An estimate of the magnitude of convective transport in s-direction yields \( f_z = \text{constant} = 5, h = a \):

\[
h \int_0^1 u(c_0+c_1) \, dx =
\]

\[= 1.2 A\alpha'u'c + \quad | \text{zeroth order} \]

\[+ \epsilon A\alpha'u' \left\{ -0.02 \frac{\partial c}{\partial t'} - 0.02 u'a' \frac{\partial c}{\partial x'} - 0.02 \frac{\partial c}{\partial y'} + \frac{\partial c}{\partial x'} \right\} + \]

\[+ 0.02 \frac{\partial c}{\partial t'} + (0.01 - 0.04 + 0.05) u' \frac{\partial a'}{\partial x}, c + \]

\[+ (0.03 - 0.01 + 0.02) \frac{\partial c}{\partial x'} + (-0.03 + 0.01) \frac{\partial c}{\partial y'}, c + \]

\[+ (-0.01 + 0.01) a' \frac{\partial c}{\partial x'}, c + \]

\[+ (-0.03 + 0.01) \frac{\partial a'}{\partial y}, c + \]
+ 0.5 \epsilon \frac{A}{L} \frac{U}{W} \cdot \frac{x'}{u'} \left[ -0.02 u'a' \frac{\partial c}{\partial y'} + (0.01 \frac{1}{f_*} + 0.04 + 0.05) u' \frac{\partial a'}{\partial y'}, \frac{c}{c} + \right.
\left. + (-0.01+0.05) a' \frac{\partial u'}{\partial y'}, \frac{c}{c} + 0.05 u'a' \frac{\partial v'}{\partial y'}, \frac{c}{c} \right] \quad (6.8)

In which: \( \epsilon = \frac{A}{L} \frac{U}{W} \) = a small parameter, see eq.(4.6)
\( a', x', y', u', v' \) = normalized variables, see eq.(4.4)
\( A, L, W, U \) = characteristic length and velocity scales, see eq.(4.4)

Inspection of eq.(6.8) shows that the ratio of first order and zeroth order contribution is of order: \( O(0.1\epsilon) \). Consequently the first order contribution can be omitted.

An estimate of the magnitude of convective transport in n-direction yields (\( f_* \) = constant = 5, \( h = a, \nu = r_u \nu + r_v \nu \)):

\[
h \int_0^1 v (c_0 + c_1) d\tau = \frac{W}{L} \cdot 0.6 \frac{AUa'v'c}{c} + \frac{A}{R} \left\{ 0.6 \frac{AUa'u'c}{c} + \right.
\left. + \epsilon \frac{AUa'u'}{u'} \left[ -0.4 a' - 0.4 \frac{\partial c}{\partial x'} + 0.6 \frac{\partial c}{\partial y'} + \right. \right.
\left. + 0.3 \frac{1}{\frac{1}{f_*} + 0.6 + 0.7} u' \frac{\partial a'}{\partial x'}, \frac{c}{c} + \right.
\left. + (1.5 \frac{1}{f_*} + 1.4+ 4.0) \frac{A}{R} \frac{u}{U} \frac{\partial a'}{\partial y'}, \frac{c}{c} - 0.3 \frac{1}{u' \\partial \tau'}, \frac{c}{c} + \right.
\left. + (-0.3+0.7) a' \frac{\partial u'}{\partial x'}, \frac{c}{c} + (-1.5+2.0) \frac{A}{R} \frac{u}{U} a' \frac{\partial u'}{\partial y'}, \frac{c}{c} \right] + \right.
\left. + 0.5 \epsilon \frac{AUa'u'}{u'} \left[ -0.4 u'a' \frac{\partial c}{\partial y'} + (0.3 \frac{1}{f_*} + 0.6 + 0.7) u' \frac{\partial a'}{\partial y'}, \frac{c}{c} + \right. \right.
\left. + (-0.3+0.7) a' \frac{\partial u'}{\partial y'}, \frac{c}{c} + 0.7 u'a' \frac{\partial v'}{\partial y'}, \frac{c}{c} \right] \right\} \quad (6.9)
Inspection of eq. (6.9) shows that the ratio of first order and zeroth order contribution due to the secondary flow is of order \( O(\epsilon) \);

An estimate of the magnitude of the transport in s-direction by eddy diffusion yields \((h = a, \nu_{tX} = 0.05 \, au_x)\):

\[-h \int_0^1 \nu_{tX} \frac{\partial c}{\partial s} \, ds = -h \int_0^1 \nu_{tX} \frac{\partial c_0}{\partial s} \, ds = 0.05 \, \epsilon \frac{w}{U} \frac{u}{U} \frac{A}{W} \int_0^1 \frac{\partial c}{\partial s} \, ds (6.10)\]

This is several orders of magnitude smaller than the convective transport in s-direction

An estimate of the magnitude of transport in n-direction by eddy diffusion yields \((h = a)\):

\[-h \int_0^1 \nu_{tX} \frac{\partial c}{\partial n} \, ds = -h \int_0^1 \nu_{tX} \frac{\partial c_0}{\partial n} \, ds =
-\frac{A}{R} \frac{A}{U} \epsilon 0.05 \frac{w}{U} \frac{u}{U} \int_0^1 \frac{\partial c}{\partial y'} \, ds (6.11)\]

To investigate the order of magnitude of this term the examples of sections 4.2.1 and 4.2.2 are used.

Helm bend: \(-h \int_0^1 \nu_{tX} \frac{\partial c}{\partial n} \, ds = \frac{A}{R} \frac{A}{U} \epsilon 0.45 \int_0^1 \frac{\partial c}{\partial y} \, ds\)

Suspended-load experiment run 1: \(-h \int_0^1 \nu_{tX} \frac{\partial c}{\partial n} \, ds = \frac{A}{R} \frac{A}{U} \epsilon 0.22 \int_0^1 \frac{\partial c}{\partial y} \, ds\)

These examples show that magnitude of the zeroth order eddy diffusion term is comparable with the first order convective term. Consequently this term has to be maintained.

Concluding:
Suspension sediment transport in s-direction is to be modelled by:

\[h \int_0^1 u \, c_0 \, ds = \alpha_s h u c (6.12)\]
in which:
\[ \alpha_s \text{ - the integral of the profile functions of } u \text{ and } c \text{ (eq.5.33)} \]

Suspended sediment transport in n-direction is to be modelled by:
(in which use has to be made of eq.(6.5) and (6.7))

\[ S_{\text{sus n}} = h \int_0^1 (\nu(c_0+c_1) - \nu \frac{\partial c_0}{\partial n}) d\xi \] \hspace{2cm} (6.13)

6.5. Entrainment at reference level

The entrainment at reference level is given by the continuity equation of suspended sediment:

\[ E = \frac{\partial S_{\text{sus s}}}{\partial s} + \frac{\partial S_{\text{sus n}}}{\partial n} + \frac{1}{R} S_{\text{sus n}} \] \hspace{2cm} (6.14)

in which: \( E \) - entrainment (upward fluxes are positive)

To achieve a first order approximation of the entrainment the convective transport in s-direction is to be approximated to zeroth order, turbulent transport in s-direction can be omitted, convective transport in n-direction is to be approximated to first order, turbulent transport in n-direction is to be approximated to zeroth order. This yields:

\[ E = \frac{\partial}{\partial s} (h \int_0^1 u \ c_0 \ d\xi) + \frac{\partial}{\partial n} (h \int_0^1 (\nu(c_0+c_1) - \nu \frac{\partial c_0}{\partial n}) d\xi) + \]

\[ + \frac{1}{R} \ h \int_0^1 (\nu(c_0+c_1) - \nu \frac{\partial c_0}{\partial n}) d\xi \] \hspace{2cm} (6.15)

The integrals in eq.(6.15) are to be computed by eq.(6.12), (6.5) and (6.7).
6.6. Bed level changes

The exchange of sediment between the bed-load layer and the region of suspended sediment contributes to the bed-load sediment balance. This balance finally determines whether sedimentation or erosion of the bed takes place.

The change of bed level is governed by the sediment continuity equation of the bed-load layer:

$$(1 - \Gamma) \frac{\partial s}{\partial t} = -E - \frac{\partial S_{\text{bed } s}}{\partial s} - \frac{\partial S_{\text{bed } n}}{\partial n} - \frac{1}{R} S_{\text{bed } n}$$

(6.16)

in which: $\Gamma$ = porosity of the bed
$S_{\text{bed } s}$ = the transport rate in s-direction
$S_{\text{bed } n}$ = the transport rate in n-direction

The bed-load transport rate can be formulated by:

$$S_{\text{bed } s} = a b \tilde{u}_s$$

(6.17)

The bed-load transport rate in n-direction can be modelled similar to the bed-load model of Olesen (1987).

$$S_{\text{bed } n} = \tan \psi S_{\text{bed } s}$$

(6.18)

with: $\tan \psi$ = the direction of sediment transport

The direction of sediment transport depends on the direction of the bed shear-stress and the transversal slope of the bed:

$$\tan \psi = \tan \delta + G(\theta) \frac{\partial s}{\partial n}$$

(6.19)

with: $\tan \delta$ = the direction of the bed shear-stress
$G(\theta)$ = coefficient which weights the effect of the transversal slope of the bed on the direction of sediment transport
7. DISCUSSION

The theoretical basis for a depth-averaged numerical model which will compute the concentration field and suspended sediment transport in rivers of constant width is founded by the present investigation.

Based on the same theory the present model and the model of Wang (1989) are comparable. Many of the theoretical aspects of Wang's model are incorporated in the present model. An overview of the similarities and differences of both models is given in table 7.1.

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<thead>
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<th>Wang</th>
<th>present model</th>
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<td>numerical grid (2-DH)</td>
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<td>cylindrical</td>
</tr>
<tr>
<td>coordinate system</td>
<td>rectangular bed-f.</td>
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<td>non-orthogonality terms</td>
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Asymptotic approximation

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<th>Galappatti's solution</th>
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<td>C-D eq. : yes</td>
<td>yes</td>
</tr>
<tr>
<td>approximation</td>
<td>S_s eq. : no, 0&lt;sup&gt;th&lt;/sup&gt; order</td>
<td>no, 0&lt;sup&gt;th&lt;/sup&gt; order</td>
</tr>
<tr>
<td></td>
<td>S_n eq. : yes</td>
<td>yes</td>
</tr>
<tr>
<td>horizontal</td>
<td>C-D eq. : yes</td>
<td>yes</td>
</tr>
<tr>
<td>diffusion</td>
<td>S_s eq. : no</td>
<td>no</td>
</tr>
<tr>
<td></td>
<td>S_n eq. : yes, 0&lt;sup&gt;th&lt;/sup&gt; order</td>
<td>yes, 0&lt;sup&gt;th&lt;/sup&gt; order</td>
</tr>
<tr>
<td>∂c/∂t, ∂c/∂s, ∂c/∂n</td>
<td>excluded</td>
<td>included</td>
</tr>
</tbody>
</table>

The application of a model determines to a large extend its mathematical constituents. Wang's model is directed to estuary flow, this allows some simplifications in the model which are questionable in river bend flow. The present model is formulated in a cylindrical bed-following coordinate system, whereas Wang's model is formulated in a rectangular bed-following coordinate system.
The convective terms are extended with two additional terms which are due to non-orthogonality of the vertical and surfaces of $\gamma$-constant, these terms are not considered by Wang (1989, pg. 31). The effect of curvature is reflected in the $v$ and $w$ velocity components. The $w$ component is not considered by Wang (1989). The turbulent diffusion terms are extended with terms due to non-orthogonality and curvature.

With the aid of an order of magnitude analysis, the convection-diffusion equation is simplified. From an order of magnitude point of view turbulent diffusion terms, except turbulent diffusion in vertical direction, can be omitted. The second order derivatives originating from the turbulent diffusion terms are, however, retained because these yields more realistic results when a first order asymptotic solution technique is employed. All the convective terms are retained.

In the present application of Galappatti’s asymptotic approximation technique the concentration is approximated to first order. The zeroth order solution of the concentration consist of equilibrium shape profiles which are based on a local balance of turbulent upward flux and the downward flux due to the weight of sediment. The first order contribution is primarily due to the convective terms. Second order derivatives related to the turbulent diffusion terms are included. These serve to avoid unrealistic behaviour of the first order solution. The latter being of mathematical nature (Wang, 1989).

Wang has paid special attention to the asymptotic approximation technique. He has developed a generalized approximation theory in which some test functions are employed. The Galappatti (1983) solution is obtained by choosing a specific set of test functions. Wang has chosen another set, one which yield a simplified sediment transport equation in main stream direction (but not in transverse direction). In case of the problem of a rigid-bed, movable-bed interface Wang’s model is more realistic.

In the derivation of the asymptotic solution Wang has neglected the spatial derivatives of the equilibrium concentration profile $(r_c)$, these are included in the present model (eq. 5.13).
Wang and Ribberink (1989) have investigated the general limits of application of a first order approximated convection-diffusion equation. To that purpose they have formulated a theoretical solution for the two dimensional convection-diffusion equation in the x-z plane. They concluded that the application of a first order approximated model is limited by the following constraints:

- The first order approximation of the concentration field should accurately approximate the theoretical solution. Starting from an arbitrary concentration profile the adaptation length of the depth averaged concentration should exceed $L = aU/u_X$, because this is the length of a region in which higher order terms constitute significantly to the solution. This condition will not be too serious in river bends because concentration profiles already have near equilibrium shapes.

- The applicability of the first order approximation is limited by $Z < 0.9$ à 1.0, because for larger $Z$ the adaptation of the mean concentration deviates too much from the theoretical solution.

Computations of the time and length scales of the convection-diffusion equation show that these depend on the choice of the reference level, the $Z$ parameter and the Chézy coefficient. The relative reference level $Z^*/a$ is assumed constant in the model. The local $Z$ parameter will vary significantly (proportional with $U^0/U_0$, at constant $C$). The local $C$ coefficient on the contrary will generally vary only slightly.

Computations (Appendix A) show that the choice of boundary condition at reference level has a dominant effect on the scales of the model. Applying the concentration gradient condition yields scales which magnitude is larger than in case of the concentration condition. The scales are also different functions of $Z^*/a$ and $Z$. At present (1989) it is unknown which boundary condition is to be applied in river bend flow.

The non-constant water depth and the non-uniform flow distribution yield additional terms in the depth-averaged convection-diffusion equation (eq. 5.25 and 5.45). These terms are: $L_{hs} \frac{1}{h} \partial h / \partial s$, $L_{hn} \frac{1}{h} \partial h / \partial n$, $L_u \frac{1}{u} \partial u / \partial s$ and $L_u \frac{1}{u} \partial u / \partial n$

In case of the laboratory experiment (Talmon and Marsman, 1988) it will be shown that these additional terms attain significant values. The maximum bed slopes are estimated to be: \( \partial a / \partial s = 0.02 \) (upstream of pool and point-bar),
\[ \frac{\partial \delta h}{\partial n} = 0.15 \text{ (at pool and point-bar). The ratio of the main flow velocity and the fall velocity is: } \frac{\bar{V}}{\bar{w}} = 32. \text{ The ratio of the mean water depth and the bend radius is: } a/R = 0.011. \text{ Using the length scale computations as depicted in fig. A4 and A5 the estimates of these terms become:} \]

\[ L_{hs} \frac{\partial \delta h}{\partial s} = -1 \times 32 \times 0.02 = -0.6 \quad (L_{hs} \frac{\bar{w}}{\bar{v}} = -1, \text{ fig. A4}) \]

\[ L_{hn} \frac{\partial \delta h}{\partial n} = -2 \times 32 \times 0.011 \times 0.15 = -0.1 \quad (L_{hn} \frac{\bar{w}}{\bar{v}} a/R = -2, \text{ fig. A5}) \]

These terms are important because their values have to be compared with a value of 1 (eq. 5.25 and 5.47). No velocity measurements are available, consequently the other two terms are more difficult to estimate.

The effect of non-orthogonality of the coordinate system is reflected in the \( L_{hs} \) and \( L_{hn} \) length scales through the \( c_{ij} \) contributions (eq. 5.26, 5.34 and 5.23). In Appendix A the effect of non-orthogonality on \( L_{hs} \) and \( L_{hn} \) is calculated. Neglect of non-orthogonality yields \( L_{hs} \) values (fig. A9) which are about half the value than when orthogonality is included (fig. A4). Neglect of non-orthogonality yields, for the smaller reference heights, \( L_{hn} \) values (fig. A10) which are quite different than when orthogonality is included (fig. A5). Consequently the effect of non-orthogonality has to be accounted for in river bend flow.

The effect of the vertical velocity component is reflected in the \( L_{hs}', L_{hn}', L_{us}, \) and \( L_{un} \) length scales through the \( d_{ij} \) and \( e_{ij} \) contributions (eq. 5.26, 5.34 and 5.23). This component depends on the actual bed topography and flow distribution. Inspection of eq. 5.26, 5.34, fig. 5.2 and 5.3 reveals that \( d_{ij} \) (and \( \delta_{ij} \)) contributions are of equal magnitude as the other contributions (\( e_{ij} = 0 \) in case of a constant roughness coefficient). This demonstrates that the effect of the vertical velocity component has to be included in river bend flow. Especially at laboratory conditions these effects could be important to the solution, because of enhanced secondary flow and consequently enhanced bed level slope.

To compute transport rates of suspended sediment and subsequently entrainment at reference level, the suspended sediment transport rate in \( s \)-direction is approximated to the zeroth order in the convective terms. Turbulent diffusion in \( s \)-direction is omitted. The suspended sediment
transport rate in n-direction is approximated to the first order in the convective terms and turbulent diffusion is approximated to the zeroth order.

The coefficients in the mathematical expressions for the depth-averaged transport rates are a function of the reference level and the local Chézy and Z parameter values (eq. 6.2 ... 6.7). Together with the calculated depth-averaged concentration field, which also depends on the boundary condition at reference level, it is concluded that the depth-averaged transport rates are significantly affected by:

- The choice of reference level.
- The boundary condition at reference level.
- The Rouse suspension parameter Z.
- The Chézy roughness coefficient.

The depth-averaged convection-diffusion equation and suspended sediment transport equation are formulated in terms of local $\dot{c}$, $a$, $\dot{u}$ and $\dot{v}$, and derivatives of these variables to $t$, $s$ and $n$. Depth-averaged river bend morphological models, which are formulated in terms of $a$, $\dot{u}$ and $\dot{v}$, can be extended with this equation to incorporate suspended sediment.

To implement suspended sediment transport in a depth-averaged morphological model, the following main computation steps have to be added:

- Computation of the concentration field to the first order: eq.(5.45). (The zeroth order concentration field is substituted in the convective terms and in the turbulent diffusion terms)
- Computation of suspended sediment transport in s- and n-directions: eq. 6.12 and 6.13. (zeroth order approximation of the convective term in s-direction, first order approximation of the convective terms in n-direction, zeroth order approximation of turbulent diffusion in n-direction)
- Computation of the entrainment at the reference level: eq.(6.14)

The choice of boundary condition, reference level and the transport formula are likely to dominate the prediction of the concentration field. Investigation of the sensitivity of the solution to different choices of parameters, together with a calibration and a validation of parameters can be achieved by the following methods:

- Laboratory measurements on a river bend model
- Numerical computation of the axi-symmetrical case
- Computation of the analytical solution by linearization of the equations
8. CONCLUSIONS

A two dimensional depth-averaged formulation of suspended sediment transport in a river bend is possible. This is achieved by considering the three dimensional convection-diffusion equation in a cylindrical bed-following coordinate system and subsequent application of Galaappatti's (1983) asymptotic approximation technique. By numerical solution of the resulting equation it will be possible to compute the concentration field.

The concentration field is modelled by a first order asymptotic solution of the convection-diffusion equation. The zeroth order contribution to the solution is given by equilibrium shape concentration verticals, the first order contribution is primarily due to the convection by the flow. The magnitude of terms is analysed by using prototype and laboratory data. This yields the following conclusions:

- The vertical velocity component has to be included, especially for the simulation of laboratory experiments.
- The vertical velocity component, induced by the secondary flow and the non-constant depth, will generally be smaller than the fall velocity of sediment, this justifies the application of the asymptotic solution technique.
- Additional terms due to non-orthogonality of the coordinate system have to be accounted for.
- Computations show that the time and length scales of the model, which are in fact the coefficients of the depth-averaged convection-diffusion equation, are significantly affected by:
  - The choice of reference level.
  - The boundary condition at reference level.
  - The Rouse suspension parameter Z.
  - The Chézy roughness coefficient.

Mathematical expressions for the depth-averaged transport in main flow and transverse direction are formulated. These are affected by the same parameters as the depth-averaged convection-diffusion equation. By an order of magnitude analysis the significant terms are identified. Sediment exchange with the bed-load layer is to be calculated by use of the continuity equation for suspended sediment.
The model will have to be tested by use of measurements and numerical computations.
ACKNOWLEDGEMENT

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- Prof. Dr. M. de Vries for adopting the project, for sharing his experience on river morphology and for his contribution to the realization of the final draft.
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Doctoral Thesis, Delft University of Technology
(also: ISSN 0169-6548 Communications on Hydraulic and Geotechnical Engineering, Delft University of Technology, Faculty of Civil Engineering)
FIG. 2.1 DEFINITION SKETCH

FIG. 3.1 COORDINATE SYSTEM
Illustration of nonorthogonal coordinates showing that $\tau_3$ and $\eta^3$ need not be identical, but that $\tau_3$ will be orthogonal to $\eta^1$ and $\eta^3$. The shadow behind $\eta^3$ indicates that it is not in the surface $\hat{x}^2 = \text{const}$.
Z parameter = 0.4, reference height \( z_r / a = 0.05 \), Chézy = 35.0 m\(^2\)/s

VERTICAL PROFILES OF VELOCITY, EDDY VISCOSITY
AND EQUILIBRIUM CONCENTRATION

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FIG. 4.1
$z/a$

$z/a$

$z/a$

$z/a$

$\alpha_{11}$

$\alpha_{21}$

$\alpha_{12}$

$r/a$

$Z$ parameter $= 0.4$, reference height $z/r = 0.05$, Chézy $= 35.0$ m$^{0.5}$/s

PERTURBATION PROFILES

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FIG. 5.1 a
FIG. 5.1 b

Perturbation Profiles

$Z$ parameter = 0.4, reference height $z_0 = 0.05$, Chezy = 33.0 m/s
FIG. S.1 C

Perurbation Profiles

Z parameter = 0.4, Reference height Z/a = 0.05, Climate = 35°C, Wind = 4 m/s
\[ z_r/a = 0.05, \quad C = 35 \text{ m}^3/\text{s} \]

\( a_{ij}, b_{ij}, c_{ij}, d_{ij} \) AT REFERENCE LEVEL

FIG. 5.2

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$z_r/a = 0.05, \quad C = 35 \text{ m}^{0.5}/\text{s}$

COEFFICIENTS FOR USE WITH THE GRADIENT CONDITION

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FIG. 5.3
FIG. 5.4 SOME SCALES AT CONCENTRATION CONDITION

FIG. 5.5 SOME SCALES AT GRADIENT CONDITION

\[ z_r/\alpha = 0.05, \quad C = 35 \text{ m}^{0.5}/\text{s} \]
APPENDIX A:

Scales of the depth-averaged convection-diffusion equation

The time and length scales of the depth-averaged convection diffusion equation have been calculated for both a concentration and a concentration gradient condition. The Chézy coefficient is assumed constant in the model region, and $\dot{V}/\dot{u}$ is neglected. The scales are computed for $C = 20; 25; 30; 35; 40; 45; 50; 55; 60$ and $65 \text{ m}^4/\text{s}$. Relative reference heights of $z_f/a = 0.05; 0.10; 0.15; 0.20; 0.25; 0.30$ and $0.35$ are applied.

In the computations parabolic distributions of the exchange coefficients for momentum and mass are used. These coefficients differ by a constant, see eq.(5.42). An example of the velocity and equilibrium concentration profiles corresponding to these exchange coefficients is given in fig. 4.1.

For the concentration condition the values of the perturbation profiles $a_{ij}, b_{ij}, c_{ij}$ and $d_{ij}$, eq.(5.23), at the reference level have been calculated. Some examples of these profiles are given in fig. 5.1. An example of the value at $z=0$, as a function of $Z$, is given in fig. 5.2. For the concentration gradient condition the values of the $\alpha_s, \alpha_n$ etc. integrals, eq.(5.33) are also computed.

To compute the perturbation profiles $a_{ij}, b_{ij}, c_{ij}$ and $d_{ij}$, the inverse operator $\mathcal{O}^{-1}$, eq.(5.9), is applied to the functions given by eq.(5.23). These functions consist of several terms, see eq.(5.38) to (5.44). The operator is applied to each individual term. This is justified because the operator is linear. Some of the terms can be integrated mathematically, the others are solved numerically. The constant $B$ of each term is computed by taking the integral of the inverse operator which has to satisfy: $\int_0^1 \mathcal{O}^{-1} d\tau = 0$. The $\alpha_s, \alpha_n$ etc integrals are computed by numerical integration of the functions given by eq.(5.33).
The time and length scales are computed by eq. (5.26). for the concentration condition and by eq. (5.34) for the concentration gradient condition.

The results are given in fig. A1 to A10. Given the Chezy value the results generally depend on the choice of boundary condition and the choice of reference level.

In fig. A1 the length scale \( L_{cs} \), involving the derivative of the concentration with streamwise coordinate, is given for \( C = 35 \text{ m}^2/\text{s} \). For other \( C \) values results are practical the same. The gradient condition yields the largest scales, increasing reference level yields smaller scales. The time scale \( T_{ct} \) is not shown, but is nearly identical with \( L_{cs}/U \).

In fig. A2 the length scale \( L_{cn} \), involving the derivative of the concentration in radial direction, is given for \( C = 35 \text{ m}^2/\text{s} \). The scale is rather large in case of the gradient condition. (notice the difference of scale of the vertical axes). The scale is also quite sensitive to the choice of reference level. In fig. A3 the \( L_{cn} \) scale is given for \( z_r/a = 0.10 \). From this figure it is concluded that the \( L_{cn} \) scale is also sensitive to the Chézy coefficient.

In fig. A4 the length scale \( L_{ns} \), involving the derivative of the local depth with streamwise coordinate, is given for \( C = 35 \text{ m}^2/\text{s} \). The magnitude of this scale in case of the gradient condition is circa 2 à 4 times larger than in case of the concentration condition.

In fig. A5 the length scale \( L_{nn} \), involving the derivative of the local depth with streamwise coordinate, is given for \( C = 35 \text{ m}^2/\text{s} \). In case of the gradient condition this scale is quite sensitive to the choice of \( z_r/a \). In case of the concentration condition the scale is sensitive to \( z_r/a \) for the following parameter combinations: \( z_r/a < 0.15 \) and \( z_r/a > 0.5 \). In fig. A6 the dependance on the Chézy coefficient is shown for \( z_r/a = 0.10 \). The magnitude of the scale increases with the Chézy coefficient.
In fig. A7 and A8 the length scales \( L_{us} \) and \( L_{un} \) are shown. The \( L_{us} \) scales are approximately 2 times smaller than the \( L_{hs} \) scales. The \( L_{un} \) scale depends weakly on \( z_r/a \) and \( C \). The dependence on \( C \) is not shown.

The effect of the non-orthogonality terms is incorporated in the \( L_{hs} \) and \( L_{hn} \) scales. In fig. A9 and A10 these scales are shown in case the non-orthogonality terms are neglected. Comparison with fig. A4 and A5 shows that the effect of these terms is significant for \( z_r/a < 0.25 \).

In fig. All the diffusion parameter is shown.

For nearly all parameter combinations the \( L_{hs}', L_{hn}, L_{us} \) and \( L_{un} \) length scales are negative. This means that the governing equation at stationary conditions becomes:

\[
\frac{\partial c_e}{\partial t} = (1 - \alpha_1) \frac{\partial c}{\partial t} + \alpha_2 \frac{\partial^2 c}{\partial s^2} + \alpha_3 \frac{\partial c}{\partial n} \left( \frac{\partial^2 c}{\partial s^2} - \alpha_4 \frac{\partial^2 c}{\partial n^2} \right)
\]

in which:
\[
\begin{align*}
0.05 \frac{\partial h}{\partial s} < \alpha_2 &< 0.95 \frac{\partial h}{\partial s} \\
-0.8 \frac{\partial h}{\partial s} a/r < \alpha_3 &< 1.2 \frac{\partial h}{\partial s} a/r \\
0 < \alpha_4 &< 0.5 h^2
\end{align*}
\]

With \( \alpha_2 \) and \( \alpha_3 \) depending on \( Z, C \), the choice of \( z_r/a \) and the choice of boundary condition.

and:
\[
\begin{align*}
\alpha_1 > 0, & \quad \text{for } \frac{\partial h}{\partial s}', \frac{\partial h}{\partial n'}, \frac{\partial \bar{u}}{\partial s} \text{ and } \frac{\partial \bar{u}}{\partial n} > 0 \\
\alpha_1 < 0, & \quad \text{for } \frac{\partial h}{\partial s}', \frac{\partial h}{\partial n'}, \frac{\partial \bar{u}}{\partial s} \text{ and } \frac{\partial \bar{u}}{\partial n} < 0
\end{align*}
\]

With \( \alpha_1 \) depending on \( \frac{\partial h}{\partial s}', \frac{\partial h}{\partial n'}, \frac{\partial \bar{u}}{\partial s}, \frac{\partial \bar{u}}{\partial n} \), \( Z, C \), the choice of \( z_r/a \) and the choice of boundary condition.
\[ \frac{w_s}{L_{cs}} = \frac{\dot{u}}{h} \]

- Concentration condition
- Gradient condition

\[ C = 35 \, \text{m}^2/\text{s} \]

FIG. A1

THE LENGTH SCALE \( L_{cs} \)

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\[ \frac{w_s}{\bar{u} h} = r \sqrt{\frac{L_{cn}}{a}} \]

**CONCENTRATION CONDITION**

\[ z_r/a = 0.05 \]
\[ z_r/a = 0.35 \]

**GRADIENT CONDITION**

\[ z_r/a = 0.05 \]
\[ z_r/a = 0.35 \]

\[ C = 35 \text{ m/s}^{0.3} \]

Notice different vertical scale

THE LENGTH SCALE \( L_{cn} \)

FIG. A2

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\[ \frac{w_s}{\bar{u}_h} \]

**CONCENTRATION CONDITION**

\[ z_r/a = 0.35 \]
\[ z_r/a = 0.05 \]

\[ \frac{w_s}{\bar{u}_h} \]

**GRADIENT CONDITION**

\[ z_r/a = 0.35 \]
\[ z_r/a = 0.05 \]

\[ C = 35 \text{ m/s} \]

**THE LENGTH SCALE** \[ L_{hs} \]

**FIG. A4**

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CONCENTRATION CONDITION

GRADIENT CONDITION

$C = 35 \text{ m/s}$
CONCENTRATION CONDITION

GRADIENT CONDITION

$z_r/a = 0.10$

THE LENGTH SCALE $L_{hn}$

FIG. A6

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CONCENTRATION CONDITION

GRADIENT CONDITION

C = 35 m/s

THE LENGTH SCALE $L_{us}$
CONCENTRATION CONDITION

GRADIENT CONDITION

\[ C = 35 \text{ m/s} \]

THE LENGTH SCALE \( L_{un} \)
CONCENTRATION CONDITION

GRADIENT CONDITION

\[ \frac{w_s}{L_{hs}} \]

\[ \frac{z_r}{a} = 0.35 \]

\[ \frac{z_r}{a} = 0.05 \]

\[ C = 35 \text{ m/s} \]
THE LENGTH SCALE $L_{hn}$ WITHOUT NON-ORTHOGONALITY TERMS (compare fig. A5)

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FIG. A10
$D \frac{1}{h^2}$

CONCENTRATION CONDITION

$z_r/a = 0.05$
$z_r/a = 0.35$

GRADIENT CONDITION

$z_r/a = 0.05$
$z_r/a = 0.35$

$C = 35 \text{ m/s}$

THE DIFFUSION COEFFICIENT $D$

FIG. A11

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