The contrast-source stress-velocity integral-equation formulation of three-dimensional time-domain elastodynamic scattering problems: A structured approach using tensor partitioning

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The contrast-source stress-velocity integral-equation formulation of three-dimensional time-domain elastodynamic scattering problems is discussed. A novel feature of the formulation is a tensor partitioning of the relevant dynamic stress and the contrast source volume density of deformation rate. The partitioning highlights several features about the structure of the formulation. These can advantageously be incorporated in a computational implementation of the method. An application to the case of a scatterer composed of isotropic material and embedded in an isotropic elastic background medium shows that the corresponding newly introduced constitutive coefficients are more natural as a characterization of the media than the traditional Lamé coefficients.

I. INTRODUCTION

In different regimes of wave scattering, the contrast-source integral-equation formulation with kernels resulting from the Green’s functions of the background medium has proven to be a versatile tool for analyzing and computing the associated wave phenomena. As compared with the scattering of acoustic waves in fluids and the scattering of electromagnetic waves, where in the formulation only scalar and vector quantities are involved, the scattering of elastic waves in solids brings in the extra feature of the presence of tensorial quantities of rank 2, viz., the dynamic stress and the contrast source volume density of deformation rate. In particular, this holds for media that are linearized elastodynamics. Inhomogeneity and arbitrary anisotropy are included in the constitution of the solid. As to the inertia properties of the medium a symmetrical inertia tensor of rank 2 is introduced that generalizes the volume density of mass as it applies to isotropic solids and can accommodate the presence of preferred-direction oriented heterogeneities in a macroscopic mixture theory (spatial averaging over representative elementary domains) that can be used on a scale that the interrogating pulsed elastic waves can sense. The anisotropic elastic properties are represented in the compliance, a symmetrical tensor of rank 4. The theory shows that, in scat-
tering, the compliance is a more natural quantity to character-
ize inhomogeneities than the traditional stiffness tensor. It is
important to notice that in the use of the contrast-source
formulation in inverse scattering problems (detection of all
sorts of inhomogeneities in a structure under elastodynamic
interrogation) it is the values of these constitutive coeffi-
cients that one is after. The analysis presented shows which
fundamental combinations of these coefficients manifest
themselves already in the structure of the governing wave
equations. Also, the relevant Green’s tensors that occur in
any of the wavefield’s integral representations (which are at
the heart of any wave scattering formulation) are constructed
out of these combinations and explicitly show how they occur
in the particle velocity and/or the dynamic stress of the
generated wave motion and thus are accessible to measure-
ment. This feature can help in the design of measurement
sets up in the search of particular constitutive parameters of
the scattering objects. In the processing of the measured
scattering data, the property of “orthogonality in function space”
of the different constituents associated with the tensor par-
tioning is expected to behave rather “independently.” This
could serve as a guideline to the design of processing soft-
ware as far as filtering of noise is concerned.

As the case of a homogeneous, isotropic solid already
shows, the wavespeeds of the different propagating wave
constituents are related to certain combinations of the consti-
tutive coefficients associated with the tension partitioning.
This implies that arrival time measurements and elasto-
dynamic ray-tracing techniques yield additional information to
measured values of pulse shapes (amplitudes, rise times,
time widths, and frequency of oscillating ringing pulses) and
can be used as such in inverse scattering parameter extrac-
tion methods.

Computational algorithms for modeling elastodynamic
wave motion in heterogeneous and anisotropic structures are
notoriously complicated and computationally, time consum-
ing. The tensor partitioning method offers itself as a tool for
breaking up any algorithm for the wave quantities as a whole
into subroutines applying to the separate constituents, after
which the latter’s interaction is programed separately. Such a
procedure would make the computer code more transparent
in its relation to the underlying physics as well as more ef-
cient as far as computation time is concerned.

II. DESCRIPTION OF THE CONFIGURATION AND
FORMATION OF THE PROBLEM

Position in the configuration is specified by the coordi-
nates \(x_1, x_2, x_3\) with respect to an orthogonal, Cartesian ref-
ence frame with the origin \(O\) and the three mutually per-
pendicular base vectors \(\hat{e}_1, \hat{e}_2, \hat{e}_3\) of unit length each. In the
indicated order, the base vectors form a right-handed system.
The corresponding position vector is \(x = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3\).
The time coordinate is \(t\). The subscript notation for vectors and
tensors will be used and the summation convention for re-
peated subscripts applies. Lower-case Latin subscripts are
employed for this purpose; they run through the values 1, 2,
and 3. Partial differentiation with respect to \(x_m\) is denoted by
\(\partial_m\); \(\partial_t\) is a reserved symbol for differentiation with respect to
\(t\).

Both the background medium and the scatterer are
assumed to consist of media that are linear, time-invariant,
locally and instantaneously reacting in their elastodynamic
behavior. The background medium has the entire \(\mathbb{R}^3\) as its
support. The support of the scatterer is the bounded domain
\(D^s \subset \mathbb{R}^3\). In the scatterer the elastodynamic constitutive coef-
ficients differ from those applied to the background me-
dium. The physical quantities associated with elastodynamic
wave motion in general are listed in Table I. The generic
form of the coupled elastodynamic stress-velocity wave
equations is taken as

\[
-\Delta_{k,m,p,q} \partial_m \tau_{p,q} + \rho_{k,r} \partial_r \nu_r = f_k, \tag{1}
\]

\[
\Delta_{i,j,p,q} \partial_q v_r - S_{i,j,p,q} \partial_i \tau_{p,q} = h_{i,j}, \tag{2}
\]

where

\[
\Delta_{i,j,p,q} = \frac{1}{2} (\delta_{i,p} \delta_{j,q} + \delta_{j,q} \delta_{i,p}),
\]

with \(\delta_{ij}\) as the symmetrical unit tensor of rank 2 (Kronecker
tensor): \(\delta_{ij} = 1\) for \(i = j\) and \(0\) for \(i \neq j\).

To ensure the uniqueness of the solution to the initial-
value problem associated with Eqs. (1) and (2) (a require-
ment set by the condition of causality of the physical phe-
nomena involved), the constitutive coefficients have to
satisfy the symmetry relations

\[
\rho_{k,r} = \rho_{r,k}, \tag{4}
\]

and

\[
S_{i,j,p,q} = S_{i,j,q,p} = S_{j,i,p,q} = S_{j,i,q,p} = S_{p,i,j}, \tag{5}
\]

while they are to be positive definite, i.e., \(\nu_k \rho_{k,r} \nu_r > 0\) for any
\(\nu_k \neq 0\) and \(\tau_{p,q} S_{i,j,p,q} \tau_{r,s} > 0\) for any \(\tau_{p,q} \neq 0\).

The wavefield quantities are expressed in terms of their
generating source distributions through the relevant Green’s
tensors (point-source solutions) according to the scheme

\[
\left[\begin{array}{c}
-\tau_{p,q} \\
u_r
\end{array}\right](x,t) = \int_{x \in \text{supp}(h_k)} \left[\begin{array}{c}
G^{th}_{p,q,i,j} \\
G^{tf}_{r,i,j}
\end{array}\right](x,x',t) * \left[\begin{array}{c}
h_{i,j} \\
f_k
\end{array}\right](x',t) dV(x') \quad \text{for} \quad x \in \mathbb{R}^3, \tag{6}
\]

where \(\ast\) denotes time convolution. The Green’s tensors sat-
sify the symmetry relations

### Table I. Elastodynamic wavefield, medium, and source quantities.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau_{p,q})</td>
<td>Dynamic stress</td>
</tr>
<tr>
<td>(v_r)</td>
<td>Particle velocity</td>
</tr>
<tr>
<td>(\rho_{k,r})</td>
<td>Coefficient of inertia</td>
</tr>
<tr>
<td>(S_{i,j,p,q})</td>
<td>Compliance</td>
</tr>
<tr>
<td>(f_k)</td>
<td>Volume source density of force</td>
</tr>
<tr>
<td>(h_{i,j})</td>
<td>Volume source density of</td>
</tr>
<tr>
<td></td>
<td>deformation rate</td>
</tr>
</tbody>
</table>
In a homogeneous medium the arguments \( x \) and \( x' \) in the Green’s tensors only occur via their difference \( x - x' \), which makes the integral in Eq. (6) of the spatial convolution type, which property can be computationally useful.

In the evaluation of the Green’s tensors, also the inverse of the compliance, the stiffness \( C_{p,q,i,j} \) is needed. It is defined through

\[
S_{i,j,m,n}C_{m,n,p,q} = \Delta^{\text{sc}}_{i,j,p,q} \tag{13}
\]

and shares the same symmetry and reciprocity properties as the compliance.

### III. CONTRAST-SOURCE FORMULATION OF THE SCATTERING PROBLEM

The scatterer is irradiated by the action of controlled sources with volume density of force \( f_{k}^{\text{inc}}(x,t) \) and volume source density of deformation rate \( h_{ij}^{\text{inc}}(x,t) \). They are present in the background medium with coefficient of inertia \( \rho_{b}^{k}(x) \) and compliance \( S_{b,i,j,p,q}^{b}(x) \) and generate the incident wavefield \( \{ v_{r}^{\text{inc}}(x,t), u_{r}^{\text{inc}}(x,t) \} \). The domain

\[
D^{\text{inc}} = \text{supp}(f_{k}^{\text{inc}}) \cup \text{supp}(h_{ij}^{\text{inc}}) \tag{14}
\]

is the union of the supports of the generating source distributions.

Starting point for the contrast-source formulation of the scattering problem is writing the constitutive properties of the scatterer as a deviation from the ones of the background medium in which it is embedded. Let

\[
\rho_{k}(x) = \rho_{b}^{k}(x) + \delta\rho_{k}(x) \quad \text{for } x \in D^{\text{inc}}, \tag{15}
\]

\[
S_{i,j,m,n}^{b}(x) = S_{b,i,j,p,q}^{b}(x) + \delta S_{i,j,p,q}(x) \quad \text{for } x \in D^{\text{inc}}, \tag{16}
\]

where

\[
D^{\delta} = \text{supp}(\delta\rho) \cup \text{supp}(\delta S) \tag{17}
\]

is the union of the supports of the contrasts.

Upon introducing the scattered wavefield \( \{ v_{r}^{\text{sc}}, u_{r}^{\text{sc}} \} \) as

\[
v_{r}^{\text{sc}} = v_{r} - v_{r}^{\text{inc}} \tag{19}
\]

accounting for the presence of the scatterer via the introduction of the contrast volume source density of deformation rate

\[
h_{ij}^{\text{sc}} = \delta S_{i,j,p,q}(x) \partial_{p,q}, \tag{20}
\]

and the contrast volume source density of force

\[
f_{k}^{\text{sc}} = -\delta \rho_{k}(x) \partial_{v_{r}}, \tag{21}
\]

the coupled wave equations for the incident and scattered wavefields can be combined to [cf. Eqs. (1) and (2)]

\[
-\Delta_{k,m,p,q}^{\text{inc}} \partial_{m,p,q}^{\text{inc}} + \rho_{b}^{k} \partial_{v_{r}}^{\text{inc}} = f_{k}^{\text{inc}}, \tag{22}
\]

\[
\Delta_{i,j,p,q}^{\text{inc}} \partial_{i,j,p,q}^{\text{inc}} - S_{b,i,j,p,q}^{b} \partial_{i,j,p,q}^{\text{inc}} = h_{ij}^{\text{inc}}, \tag{23}
\]

and the incident and scattered wavefield integral representations to [cf. Eq. (6)]

\[
\begin{bmatrix}
-\Delta^{\text{inc}}_{k,m,p,q}^{\text{inc}}

\end{bmatrix}
\begin{bmatrix}
v_{r}^{\text{inc}}

\end{bmatrix}(x,t) = \int_{x \in D^{\text{inc}}} \begin{bmatrix}
G_{k,p,q}^{\text{inc}}(x,x',t)

G_{r,j,p,q}^{\text{inc}}(x,x',t)

\end{bmatrix}^{(t)} \begin{bmatrix}
v_{r}^{\text{inc}}

\end{bmatrix}(x',t)dV(x') \quad \text{for } x \in \mathbb{R}^{3}, \tag{24}
\]

in which the Green’s tensors are the ones associated with the background medium.

Once the incident wavefield has been determined, Eq. (24) as it applies to the scattered wavefield can be used to arrive at integral-equation formulations for the solution of the scattering problem.

### A. Wavefield integral-equation formulation

The wavefield integral-equation formulation follows upon substituting Eqs. (18) and (19) in the left-hand side and Eqs. (20) and (21) in the right-hand side and enforcing the result for \( x \in D^{\text{sc}} \), while considering the wavefield quantities \( \tau_{p,q}(x) \) and \( v_{r}(x) \) for \( x \in D^{\text{sc}} \) as the unknowns. A formulation of this kind has been employed in Ref. 6 to solve elastic wave problems in inhomogeneous media.

### B. Contrast-source integral-equation formulation

The contrast-source integral-equation formulation follows upon substituting Eqs. (18) and (19) in the left-hand side, operating on the resulting upper equation with \( \delta S_{i,j,p,q}(x) \partial_{i} \) and on the lower equation with \( \delta \rho_{k}(x) \partial_{v_{r}} \) and using the resulting relations throughout \( D^{\text{sc}} \), while considering \( h_{ij}^{\text{sc}}(x) \) and \( f_{k}^{\text{sc}}(x) \) as the unknowns.

### IV. THE DECOMPOSITION OF THE ELASTODYNAMIC TENSORS OF RANK 2 INTO THEIR OMNIDIRECTIONAL AND DEVIATORIC PARTS

The symmetrical tensors of rank 2 occurring in elastodynamics can, in a unique manner, be decomposed into their omnidirectional parts and their deviatoric parts in the function space they span. This decomposition highlights particular features of the generated wave motion and can be computationally advantageous. The decomposition is most
effectively carried out through the introduction of a collection of unit tensors of rank 4. In view of two-dimensional, next to 3D, modeling applications, the definitions are given for an arbitrary number \( N (N \geq 2) \) of spatial dimensions. 7

The identity tensor:
\[
\Delta_{i,j,p,q} = \delta_{i,p} \delta_{j,q},
\]
(25)
the symmetrical tensor unit:
\[
\Delta_{i,j,p,q}^+ = \frac{1}{N} (\delta_{i,p} \delta_{j,q} + \delta_{i,q} \delta_{j,p}),
\]
(26)
the omnidirectional part of the symmetrical tensor unit:
\[
\Delta_{i,j,p,q}^\delta = \frac{1}{N} \delta_{i,j} \delta_{p,q},
\]
(27)
the deviatoric part of the symmetrical tensor unit:
\[
\Delta_{i,j,p,q}^{\delta\|} = \Delta_{i,j,p,q}^+ - \Delta_{i,j,p,q}^\delta = \frac{1}{2} (\delta_{i,p} \delta_{j,q} + \delta_{i,q} \delta_{j,p}) - \frac{1}{N} \delta_{i,j} \delta_{p,q}.
\]
(28)

The unit tensors thus introduced all have the unitary properties
\[
\Delta_{i,j,m,n} \Delta_{m,n,p,q} = \delta_{i,j} \delta_{p,q},
\]
(29)
\[
\Delta_{i,j,m,n}^+ \Delta_{m,n,p,q}^+ = \Delta_{i,j,p,q}^+,
\]
(30)
\[
\Delta_{i,j,m,n}^\delta \Delta_{m,n,p,q}^\delta = \delta_{i,j} \delta_{p,q},
\]
(31)
\[
\Delta_{i,j,m,n}^{\delta\|} \Delta_{m,n,p,q}^{\delta\|} = \Delta_{i,j,p,q}^{\delta\|}.
\]
(32)
Furthermore, we have the orthogonality property
\[
\Delta_{i,j,m,n} \Delta_{m,n,p,q}^{\delta\|} = 0.
\]
(33)

For any tensor \( T_{p,q} \) of rank 2 we then have
\[
\Delta_{i,j,p,q} \Delta_{p,q} = T_{i,j},
\]
(34)
while we define its symmetrical part by
\[
T_{i,j}^+ = \Delta_{i,j,p,q}^+ T_{p,q},
\]
(35)
its omnidirectional part by
\[
T_{i,j}^\delta = \Delta_{i,j,p,q}^\delta T_{p,q},
\]
(36)
and its deviatoric part by
\[
T_{i,j}^{\delta\|} = \Delta_{i,j,p,q}^{\delta\|} T_{p,q}.
\]
(37)
Evidently,
\[
T_{i,j} = T_{i,j}^+ + T_{i,j}^{\delta\|},
\]
(38)
and
\[
T_{i,j}^\delta T_{i,j}^{\delta\|} T_{p,q} = T_{i,j} T_{i,j}^\delta = 0.
\]
(39)

Equations (38) and (39) imply that \( T_{i,j}^\delta \) and \( T_{i,j}^{\delta\|} \) are mutually orthogonal constituents of \( T_{i,j} \) in the function space spanned by \( T_{i,j}^\delta \).

With the decomposition, Eqs. (1) and (2) can be rewritten as (note that now \( N = 3 \))
\[
- \partial_m (\tau_{\alpha\beta,m}^\delta + \tau_{\alpha\beta,m}^{\delta\|}) + \rho_k r \partial_l v_{r} = f_k,
\]
(40)

In general, Eq. (41) does not decompose into separate equations for the separate constituents. However, such a decomposition does take place for the class of media for which (no coupling between the mutually orthogonal constituents)
\[
\Delta_{i,j,m,n} S_{m,n,r,s} \Delta_{r,s,p,q}^\delta = \Delta_{i,j,m,n} S_{m,n,r,s}^\delta = 0.
\]
(42)

With
\[
S_{i,j,p,q}^\delta = \Delta_{i,j,m,n} S_{m,n,r,s} \Delta_{r,s,p,q}^\delta
\]
(43)
and
\[
S_{i,j,p,q}^{\delta\|} = \Delta_{i,j,m,n} S_{m,n,r,s}^{\delta\|}
\]
(44)
we have in such a case
\[
S_{i,j,p,q} = S_{i,j,p,q}^\delta + S_{i,j,p,q}^{\delta\|},
\]
(45)
in which the two constituents on the right-hand side are mutually orthogonal. As a consequence, Eq. (41) decomposes into
\[
\Delta_{i,j,m,n}^\delta \partial_m v_r - S_{i,j,p,q}^\delta \partial_{p,q}^\delta = h_{i,j}^\delta
\]
(46)
and
\[
\Delta_{i,j,m,n}^{\delta\|} \partial_m v_r - S_{i,j,p,q}^{\delta\|} \partial_{p,q}^{\delta\|} = h_{i,j}^{\delta\|}.
\]
(47)

The decomposition has also consequences for the field integral representation (6). Through the decomposition, Eq. (6) can be reformulated as
\[
\begin{align*}
\mathbf{x}(t) &= \int_{\mathbf{x} \in \text{supp}(h,f)} \left[ G_{p,q,ij}^\delta, G_{p,q,ij}^{\delta\|}, G_{p,q,k}^\delta, G_{p,q,k}^{\delta\|} \right] \\
& \quad \times v_r(x) \cdot (x',t)dV(x') \\
& \quad \forall x \in \mathbb{R}^3,
\end{align*}
\]
(48)
in which
\[
G_{p,q,ij}^\delta = \Delta_{p,q,r,s}^\delta G_{r,s,m,n}^\delta
\]
(49)
\[
G_{p,q,ij}^{\delta\|} = \Delta_{p,q,r,s}^{\delta\|} G_{r,s,m,n}^{\delta\|}
\]
(50)
\[
G_{p,q,k}^\delta = \Delta_{p,q,r,s}^\delta G_{r,s,k}^\delta
\]
(51)
\[
G_{p,q,k}^{\delta\|} = \Delta_{p,q,r,s}^{\delta\|} G_{r,s,k}^{\delta\|}
\]
(52)
\[
G_{p,q,ij}^{\delta\|} = \Delta_{p,q,r,s}^{\delta\|} G_{r,s,m,n}^{\delta\|}
\]
(53)
\[
G_{p,q,k} = \Delta_{p,q,r,s}^\delta G_{r,s,k}
\]
(54)
\[
G_{r,s,ij}^\delta = G_{r,s,m,n}^\delta
\]
(55)
\[
G_{r,s,ij}^{\delta\|} = G_{r,s,m,n}^{\delta\|}
\]
(56)
The decomposition of \( S_{i,j,p,q} \) holds in particular for the case of isotropic media. Here, it leads to the introduction of con-

\[\text{de Hoop et al.: Contrast-source stress-velocity integral-equation}\]

stitutive coefficients that are related to the traditional Lamé coefficients but are particular combinations of them that show up in the contrast-source integral-equation formulation of the scattering problem.

V. BACKGROUND MEDIUM AND SCATTERER WITH ISOTROPIC CONSTITUTIVE PROPERTIES

For isotropic, lossless media, the constitutive coefficients take the form

\[ \rho_L(x) = \rho(x) \delta_{k,r}, \]

where \( \rho(x) \) is the volume density of mass [{\( \rho(x) > 0 \) in view of the uniqueness conditions}],

\[ S_{i,j,p,q}(x) = S^\delta_{i,j,p,q}(x) + S^\gamma_{i,j,p,q}(x), \]

with

\[ S^\delta_{i,j,p,q}(x) = \Lambda(x) \Delta^\delta_{i,j,p,q} \]

and

\[ S^\gamma_{i,j,p,q}(x) = M(x) \Delta^\gamma_{i,j,p,q}, \]

where \( \Lambda(x) \) and \( M(x) \) [with \( \Lambda(x) > 0 \) and \( M(x) > 0 \) in view of the uniqueness conditions] are the compliance coefficients and related to the Lamé coefficients \( \lambda \) and \( \mu \) via

\[ \frac{1}{\Lambda} = 3\lambda + 2\mu, \]

\[ \frac{1}{M} = 2\mu. \]

The corresponding stiffness is given by

\[ C^\delta_{p,q,i,j}(x) = C^\delta_{p,q,i,j}(x) + C^\gamma_{p,q,i,j}(x), \]

with

\[ C^\delta_{p,q,i,j}(x) = \frac{1}{\Lambda(x)} \Delta^\delta_{p,q,i,j} \]

and

\[ C^\gamma_{p,q,i,j}(x) = \frac{1}{M(x)} \Delta^\gamma_{p,q,i,j}. \]

For the scattering problem, the constitutive properties of the background medium are specified by

\[ \rho_b(x) = \rho_b(x) \delta_{k,r}, \]

\[ S^b_{i,j,p,q}(x) = \Lambda^b(x) \Delta^\delta_{i,j,p,q} + M^b(x) \Delta^\gamma_{i,j,p,q}, \]

and the contrast constitutive properties by

\[ \delta \rho_L(x) = \delta \rho(x) \delta_{k,r}, \]

\[ \delta S_{i,j,p,q}(x) = \delta \Lambda(x) \Delta^\delta_{i,j,p,q} + \delta M(x) \Delta^\gamma_{i,j,p,q}. \]

With this, the contrast source densities become

\[ \delta \rho^{c,v}_{i,j} = \delta \rho^{c,v}_{i,j} + \delta \rho^{c,v}_{i,j}, \]

with

\[ \delta \rho^{c,v}_{i,j} = \delta \Lambda(x) \delta \tau^c_{i,j}, \]

\[ \delta \rho^{c,v}_{i,j} = \delta \Lambda(x) \delta \tau^v_{i,j}. \]

VI. THE GREEN’S TENSORS FOR A HOMOGENEOUS, ISOTROPIC, LOSSLESS BACKGROUND MEDIUM

Although the decomposition discussed in Sec. IV leads to the introduction of elastic constitutive coefficients that are directly related to the contrast source densities of deformation rate that occur in the elastodynamic scattering problem and the Green’s tensors do decompose accordingly, the wave speeds occurring in them prove still to be related to combinations of the newly introduced coefficients and not to them separately. To show this, we give in this section the expressions for the Green’s tensors pertaining to a homogeneous, isotropic, lossless background medium with the constitutive coefficients \( \Lambda = \Lambda_0, \ M = M_0, \) and \( \rho = \rho_0. \) In the expressions, the wavespeed \( c_p \) of compressional waves and the wavespeed \( c_s \) of shear waves occur as well as the \( P \)-wave and \( S \)-wave constituents of the Green’s tensor \( G_{r,k}(x-x',t) \) of the elastodynamic wave equation. The latter tensor follows from

\[ (c_p^2 - c_s^2) \delta \partial_t \partial_s G_{r,k}(x-x',t) + c_s^2 \partial_s \partial_s G_{r,k}(x-x',t) \]

\[ - \delta \partial_t G_{r,k}(x-x',t) = - \partial_r \delta \delta(x-x',t), \]

and is obtained as \( c_s^2 \) (Sec. 13.5)

\[ G_{r,k}(x-x',t) = \frac{\delta}{C^b_s} \frac{1}{4\pi|x-x'|} \]

\[ + \partial_s \partial_t \left[ \frac{(t-|x-x'|/c_p)H(t-|x-x'|/c_p)}{4\pi|x-x'|} \right] \]

\[ - \frac{(t-|x-x'|/c_p)H(t-|x-x'|/c_p)}{4\pi|x-x'|} \]

for \( x \neq x', \)

where \( H(t) \) denotes the Heaviside unit step function. Note that, in view of the homogeneity of the medium, the Green’s tensors depend on \( x \) and \( x' \) only via their difference \( x - x'. \) Expressed in terms of \( \Lambda_0, \ M_0, \) and \( \rho_0 \) we further have

\[ c_p = \left[ \frac{2}{3M_0} + \frac{1}{3\Lambda_0} \right]^{1/2} \rho_0 \]

and

\[ c_s = \left[ \frac{1}{2M_0\rho_0} \right]^{1/2}. \]

The Green’s tensors occurring in Eq. (48) can now be expressed in terms of \( G_{r,k}(x-x',t) \) and its derivatives. From the particle-velocity elastodynamic wave equation with point force excitation (18) we directly obtain

\[ G_{r,k}(x-x',t) = \frac{1}{\rho_0} \partial_t G_{r,k}(x-x',t). \]
Using Eq. (20), we obtain, noting that \( \partial_i' = -\partial_i \) for the case of homogeneous media,

\[
G_{x,i,j}^{\delta}(x - x', t) = -\frac{1}{\rho_0 \lambda_0} \Delta_{p,q,n,r}^{\delta} \partial_i \partial_i' G_{r,k}(x - x', t) \tag{81}
\]

and

\[
G_{x,i,j}^{\delta}(x - x', t) = -\frac{1}{\rho_0 \lambda_0} \Delta_{p,q,n,r}^{\delta} \partial_i \partial_i' G_{r,k}(x - x', t). \tag{82}
\]

Finally, Eq. (21) yields

\[
G_{x,i,j}^{\delta}(x - x', t) = -\frac{1}{\rho_0 \lambda_0} \Delta_{p,q,n,r}^{\delta} \partial_i \partial_i' \Delta_{r,i,j'}^{\delta} \partial_i' \partial_i G_{r,k}(x - x', t), \tag{83}
\]

In the right-hand sides of the expressions for the Green’s tensors in terms of the tensor of rank 2 of the elastodynamic wave equation spatial derivatives up to order 4 acting on the elementary scalar Green’s functions of the wave equation occur. A list of these derivatives can be found in Ref. 2 (Sec. 13.3).

VII. CONCLUSION

It is shown that for a large class of elastodynamic scattering problems the decomposition of the pertaining tensors of rank 2 (dynamic stress and volume source density of deformation rate) into their omnidirectional and deviatoric constituents has a number of advantages. First of all, the contrast in elastic properties of the scatterers, in particular, those in the case of scatterers composed of isotropic medium present in a background with isotropic properties, admits a more natural representation than the traditional one in terms of the relevant Lamé coefficients. Since the newly introduced contrast coefficients directly occur in the contrast-source densities that are representative for the scattering phenomena, any sensitivity analysis in the associated inverse scattering problems becomes more transparent. The introduction of the relevant projection tensors of rank 4 also highlights the structure of the relevant (also decomposed) Green’s tensors, which structure can serve as a guideline to the computational implementation of the contrast-source integral-equation method.

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