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Abstract

In this article we discuss the Heston [17] model with stochastic interest rates driven by Hull-White [18] (HW) or Cox-Ingersoll-Ross [8] (CIR) processes. We define a so-called volatility compensator which guarantees that the Heston hybrid model with a non-zero correlation between the equity and interest rate processes is properly defined. Moreover, we propose an approximation for the characteristic function, so that pricing of basic derivative products can be efficiently done using Fourier techniques [12; 7]. We also discuss the effect of the approximations on the instantaneous correlations, and check the influence of the correlation between stock and interest rate on the implied volatilities.

Key words: Heston-Hull-White, Heston-Cox-Ingersoll-Ross, equity-interest rate hybrid products, affine jump diffusion processes.

1 Introduction

Modelling derivative products in Finance usually starts with the specification of a system of Stochastic Differential Equations (SDEs), that correspond to state variables like stock, interest rate and volatility. By correlating the SDEs from the different asset classes one can define so-called hybrid models, and use them for pricing multi-asset derivatives. Even if each of these SDEs yields a closed form solution, a non-zero correlation structure between the processes may cause difficulties for modelling and product pricing. Typically, a closed form solution of the hybrid models is not known, and numerical approximations by means of Monte Carlo (MC) simulation or discretization of the corresponding Partial Differential Equations (PDEs) have to be employed for model evaluation and derivative pricing. The speed of pricing European products is however crucial, especially for the calibration. Several theoretically attractive SDE models, that cannot fulfill the speed requirements, are not used in practice.

The aim of this paper is to define hybrid SDE models that fit in the class of affine jump diffusion processes (AJD), as in Duffie, Pan and Singleton [11]. For processes within this class a closed form solution of the characteristic function exists. Suppose we have given a system of SDEs (without any jumps), i.e.,

$$\mathrm{d}\mathbf{X}_t = \mu(\mathbf{X}_t)\mathrm{d}t + \sigma(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t. \tag{1.1}$$

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This system (1.1) is said to be of the affine form if:

$$\mu(\mathbf{X}_t) = a_0 + a_1\mathbf{X}_t, \text{ for any } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \tag{1.2}$$

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \text{ for arbitrary } (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \tag{1.3}$$

$$r(\mathbf{X}_t) = r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.4}$$

with $r(\mathbf{X_t})$ being an interest rate component. Then, the discounted characteristic function (CF) is of the following form [11]:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds + i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right) = e^{A(\mathbf{u},\tau)+\mathbf{B^T}(\mathbf{u},\tau)\mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure, $\mathbb{Q}$. For a time lag, $\tau := T - t$, the coefficients $A(\mathbf{u}, \tau)$ and $\mathbf{B}^T(\mathbf{u}, \tau)$ have to satisfy the following complex-valued ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u}, \tau) = & -r_1 + a_1^T\mathbf{B} + & \dfrac{1}{2}\mathbf{B}^T c_1\mathbf{B}, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}A(\mathbf{u}, \tau) = & -r_0 + \mathbf{B}^T a_0 + & \dfrac{1}{2}\mathbf{B}^T c_0\mathbf{B}, \end{cases} \tag{1.5}$$

with $a_i, c_i, r_i,\ i = 0, 1$, as in (1.2), (1.3) and (1.4).

In this article we focus our attention specifically on a hybrid model which combines the equity and interest rate asset classes. Brigo and Mercurio [6] have shown that the assumption of constant interest rates in the classical Black-Scholes model [4] can be generalized, and by including the stochastic interest rate process of Hull and White [18], one is still able to obtain a closed form solution for European-style option prices. Originally, the Black-Scholes-Hull-White model in [6] was not dedicated to pricing hybrid products, but to increasing the accuracy for long-maturity options. The model is, however, unable to describe any smile and skew shapes present in the equity markets. It was later improved in [14] and [2], where the two-dimensional stochastic volatility model by Heston [17] was incorporated, however, with the assumption of no correlation between the equity and interest rate processes. Some form of correlation was indirectly modeled by including additional terms in the SDEs.

In [15] the Heston stochastic volatility model was replaced by the Schöbel-Zhu model, while the interest rate was still driven by a Hull-White process (SZHW model). In this model a full matrix of correlations can be directly imposed on the driving Brownian motions. The model is well-defined under the class of AJD processes, but since the SZHW model is based on a Vašiček-type process [24] for stochastic volatility, the volatilities can become negative.

A different approach to modelling equity-interest rate hybrids was presented by Benhamou *et al.* [3], which extended the local volatility framework of Dupire [10] and Derman, Kani [9] and incorporated stochastic interest rates. This idea was further investigated in [16], where the short-rate interest rate was implied from the Stochastic Volatility Libor Market Model, enabling the incorporation of an interest rate smile in the framework.

The contribution of the present paper is two-fold. First of all, we investigate the Heston-Hull-White, and the Heston-Cox-Ingersoll-Ross hybrid models and propose approximations so that we can obtain their characteristic functions. Secondly, we provide numerical results for the resulting approximations and compare them with the full-scale Heston hybrid models.

In Section 2 we discuss the Heston hybrid models with the stochastic interest rate processes. In Section 3 we present the approximations for the Heston Hull-White model and derive the corresponding CF, first assuming a zero correlation between interest rate and volatility. In Section 4 we give an approximation for the Heston-Cox-Ingersoll-Ross model. Section 5 concludes. Numerical results are presented in Sections 3 and 4.

2 Heston Hybrid Models with Stochastic Interest Rate

With state vector $\mathbf{X}_t = [S_t, \sigma_t]$, under the risk-neutral pricing measure, the Heston stochastic volatility model [17], which is our point-of-departure, is specified by the following system of SDEs:

$$\begin{cases} \mathrm{d}S_t = & rS_t\mathrm{d}t + & \sqrt{\sigma_t}S_t\mathrm{d}W_t^x, & S_0 > 0, \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma}_t - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & r_0 > 0, \end{cases} \tag{2.1}$$

with r a constant interest rate, correlation $\mathrm{d}W_t^x\mathrm{d}W_t^\sigma = \rho_{x,\sigma}\mathrm{d}t$, and $|\rho_{x,\sigma}| < 1$. The variance process, σ_t, of the stock S_t is a mean reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $\bar{\sigma} > 0$, and $\gamma > 0$ is the second-order volatility, i.e., the variance of the volatility.

As already indicated in [17], the model given in (2.1) is not in the class of affine processes, whereas under the log transform for the stock, $x_t = \log S_t$, it is. Then, the discounted CF is given by:

$$\phi_{Hes}(u, \mathbf{X}_t, \tau) = \mathrm{e}^{B_x(u,\tau)x_0 + B_\sigma(u,\tau)\sigma_0 + A(u,\tau)}, \tag{2.2}$$

where the functions $B_x(u,\tau)$, $B_\sigma(u,\tau)$ and $A(u,\tau)$ are known in closed form (see [17]).

The CF is explicit, but also its inverse has to be found for pricing purposes. Because of the form of the CF, we cannot get it analytically and a numerical method for integration has to be used, see, for example, [5; 7; 12; 19; 20] for Fourier methods.

2.1 Full-Scale Hybrid Models

A constant interest rate, r, may be insufficient for pricing interest rate sensitive products. Therefore, we extend our state vector with an additional stochastic quantity, i.e.: $\mathbf{X}_t = [S_t, \sigma_t, r_t]$; This model corresponds to a *hybrid stochastic volatility equity model with a stochastic interest rate* process, r_t. In particular, we add to the Heston model the Hull-White (HW) interest rate [18], or the square root Cox-Ingersoll-Ross [8] (CIR) process. The extended model can be presented in the following way:

$$\begin{cases} \mathrm{d}S_t = & r_tS_t\mathrm{d}t + & \sqrt{\sigma_t}S_t\mathrm{d}W_t^x, & S_0 > 0, \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + & \eta r_t^p\mathrm{d}W_t^r, & r_0 > 0, \end{cases} \tag{2.3}$$

where exponent $p = 0$ in (2.3) represents the Heston-Hull-White (HHW) model and for $p = \frac{1}{2}$ it becomes the Heston-Cox-Ingersoll-Ross (HCIR) model. For both models the correlations are given by $\mathrm{d}W_t^x\mathrm{d}W_t^\sigma = \rho_{x,\sigma}$, $\mathrm{d}W_t^x\mathrm{d}W_t^r = \rho_{x,r}$, $\mathrm{d}W_t^\sigma\mathrm{d}W_t^r = \rho_{\sigma,r}$, and κ, γ and $\bar{\sigma}$ are as in (2.1), $\lambda > 0$ determines the speed of mean reversion for the interest rate process; θ_t is the interest rate term-structure and η controls the volatility of the interest rate. We note that the interest rate process in (2.3) for $p = \frac{1}{2}$ is of the same form as the volatility process σ_t.

System (2.3) is not in the affine form, not even with $x_t = \log S_t$. In particular, the symmetric instantaneous covariance matrix is given by:

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} \sigma_t & \rho_{x,\sigma}\gamma\sigma_t & \rho_{x,r}\eta r_t^p\sqrt{\sigma_t} \\ * & \gamma^2\sigma_t & \rho_{r,\sigma}\gamma\eta r_t^p\sqrt{\sigma_t} \\ * & * & \eta^2 r_t^{2p} \end{bmatrix}. \tag{2.4}$$

Setting the correlation $\rho_{r,\sigma}$ to zero would still not make the system affine. Matrix (2.4) is of the linear form w.r.t state vector $[x_t = \log S_t, \sigma_t, r_t]$, if two correlations, $\rho_{r,\sigma}$ and $\rho_{x,r}$, are set to zero[1]. Models with two correlations equal to zero are covered in [21].

Since for pricing equity-interest rate products a non-zero correlation between stock and interest rate is crucial, alternative approximations to the Heston hybrid models need to be formulated, so that correlations can be imposed. Variants are discussed in the sections to follow. These approximate models are evaluated with the help of the Cholesky decomposition of a correlation matrix.

[1] where we assume positive parameters.

We can decompose a given general symmetric correlation matrix, $\mathbf{C}$, denoted by

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ * & 1 & \rho_3 \\ * & * & 1 \end{bmatrix}, \tag{2.5}$$

as $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a lower triangular matrix with strictly positive entries:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \frac{\rho_3 - \rho_2\rho_1}{\sqrt{1-\rho_1^2}} & \sqrt{1-\rho_2^2 - \left(\frac{\rho_3-\rho_2\rho_1}{\sqrt{1-\rho_1^2}}\right)^2} \end{bmatrix}. \tag{2.6}$$

We can rewrite a system of SDEs in terms of the independent Brownian motions with the help of the lower triangular matrix $\mathbf{L}$.

Since our main objective is to derive a closed form of the CF while assuming a nonzero correlation between the equity process, S_t, and the interest rate, r_t, we first assume that the Brownian motions for the interest rate r_t and the volatility σ_t are not correlated.

By exchanging the order of the state variables $\mathbf{X}_t = [x_t, \sigma_t, r_t]$ to $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]$, the HHW and HCIR models in (2.3) have $\rho_1 \equiv \rho_{r,\sigma} = 0$, $\rho_2 \equiv \rho_{x,r} \neq 0$ and $\rho_3 \equiv \rho_{x,\sigma} \neq 0$ in (2.5) and read:

$$\begin{bmatrix} \mathrm{d}r_t \\ \mathrm{d}\sigma_t \\ \mathrm{d}S_t \end{bmatrix} = \begin{bmatrix} \lambda(\theta_t - r_t) \\ \kappa(\bar{\sigma} - \sigma_t) \\ r_t S_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \rho_{x,r}\sqrt{\sigma_t}S_t & \rho_{x,\sigma}\sqrt{\sigma_t}S_t & \sqrt{\sigma_t}S_t\sqrt{1-\rho_{x,\sigma}^2-\rho_{x,r}^2} \end{bmatrix} \begin{bmatrix} \mathrm{d}\widetilde{W}_t^r \\ \mathrm{d}\widetilde{W}_t^\sigma \\ \mathrm{d}\widetilde{W}_t^x \end{bmatrix}. \tag{2.7}$$

In Section 3.3 we also discuss a model with a full matrix of correlations.

2.2 Reformulated Heston Hybrid Models

In the previous section we have seen that for the HHW and HCIR models with a full matrix of correlations given in (2.3), the affinity relations [11] are not satisfied, so that the CF cannot be obtained by standard techniques.

In order to have a well-defined Heston hybrid model with *indirectly imposed correlation*, $\rho_{x,r}$, we propose the following system of SDEs:

$$\begin{cases} \mathrm{d}S_t = r_t S_t \mathrm{d}t + \sqrt{\sigma_t}S_t\mathrm{d}W_t^x + \Omega_t r_t^p S_t \mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}S_t\mathrm{d}W_t^\sigma, & S_0 > 0, \\ \mathrm{d}\sigma_t = \kappa(\bar{\sigma}-\sigma_t)\mathrm{d}t + \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \\ \mathrm{d}r_t = \lambda(\theta_t - r_t)\mathrm{d}t + \eta r_t^p \mathrm{d}W_t^r, & r_0 > 0, \end{cases} \tag{2.8}$$

with

$$\begin{cases} \mathrm{d}W_t^x \mathrm{d}W_t^\sigma = \hat{\rho}_{x,\sigma}, \\ \mathrm{d}W_t^x \mathrm{d}W_t^r = 0, \\ \mathrm{d}W_t^\sigma \mathrm{d}W_t^r = 0, \end{cases} \tag{2.9}$$

where $p = 0$ for HHW and $p = \frac{1}{2}$ for HCIR. We have included a time-dependent function, Ω_t, and a constant parameter, Δ. Note that we still assume independence between instantaneous short rate, r_t, and the volatility process σ_t, i.e., $\hat{\rho}_{r,\sigma} = 0$.

By exchanging the order of the state variables, to $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]$, system (2.8) is given, in terms of the independent Brownian motions, by:

$$\begin{bmatrix} \mathrm{d}r_t \\ \mathrm{d}\sigma_t \\ \mathrm{d}S_t \end{bmatrix} = \begin{bmatrix} \lambda(\theta_t - r_t) \\ \kappa(\bar{\sigma} - \sigma_t) \\ r_t S_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \Omega_t r_t^p S_t & \sqrt{\sigma_t}S_t(\hat{\rho}_{x,\sigma}+\Delta) & \sqrt{\sigma_t}S_t\sqrt{1-\hat{\rho}_{x,\sigma}^2} \end{bmatrix} \begin{bmatrix} \mathrm{d}\widetilde{W}_t^r \\ \mathrm{d}\widetilde{W}_t^\sigma \\ \mathrm{d}\widetilde{W}_t^x \end{bmatrix}, \tag{2.10}$$

In the lemma below we show that the model (2.8) is equivalent with the full-scale HHW model in (2.3), with a nonzero correlation $\rho_{x,r}$.

Lemma 2.1. *Model (2.8) is a well-defined Heston hybrid model in the sense of Equation (2.3) with non-zero correlation, $\rho_{x,r}$, for:*

$$\begin{cases} \Omega_t = & \rho_{x,r} r_t^{-p}\sqrt{\sigma_t}, \\ \hat{\rho}^2_{x,\sigma} = & \rho^2_{x,\sigma} + \rho^2_{x,r}, \\ \Delta = & \rho_{x,\sigma} - \hat{\rho}_{x,\sigma}, \end{cases} \tag{2.11}$$

where correlation $\hat{\rho}_{x,\sigma}$ is as in model (2.8) and $\rho_{x,\sigma}$ in model (2.3).

Proof. We present the two models (2.3) and (2.8) in terms of the independent Brownian motions, (2.7) and (2.10), respectively. By matching the appropriate coefficients in (2.7) and (2.10), we find that the following relations should hold:

$$\begin{cases} \Omega_t r_t^p S_t = & \rho_{x,r}\sqrt{\sigma_t} S_t, \\ \sqrt{1-\hat{\rho}^2_{x,\sigma}}\sqrt{\sigma_t} S_t = & \sqrt{1-\rho^2_{x,\sigma}-\rho^2_{x,r}}\sqrt{\sigma_t} S_t, \\ (\hat{\rho}_{x,\sigma}+\Delta)\sqrt{\sigma_t} S_t = & \rho_{x,\sigma}\sqrt{\sigma_t} S_t. \end{cases} \tag{2.12}$$

By simplifying (2.12) the proof is finished. □

When including the results (2.11) directly in the main system (2.8) the affinity property of the system would be lost. So, in order to satisfy the affinity constraints, we *approximate* Ω_t by a deterministic time-dependent function, as follows:

$$\Omega_t \approx \rho_{x,r}\mathbb{E}\left(r_t^{-p}\sqrt{\sigma_t}\right) \stackrel{\perp\!\!\!\perp}{=} \rho_{x,r}\mathbb{E}\left(r_t^{-p}\right)\mathbb{E}\left(\sqrt{\sigma_t}\right), \tag{2.13}$$

assuming independence between r_t and σ_t.

Finally, the coefficients for our first approximate model, which we call the Heston-Hybrid-Model-1 (H1), based on the approximations of System (2.8) explained above, are given by:

$$\begin{cases} \Omega_t \approx & \rho_{x,r}\mathbb{E}\left(r_t^{-p}\right)\mathbb{E}\left(\sqrt{\sigma_t}\right), \\ \hat{\rho}^2_{x,\sigma} = & \rho^2_{x,\sigma} + \rho^2_{x,r}, \\ \Delta = & \rho_{x,\sigma} - \hat{\rho}_{x,\sigma}, \end{cases} \tag{2.14}$$

where $\rho_{x,\sigma}$ and $\rho_{x,r}$ are the correlations from System (2.3); $\hat{\rho}_{x,\sigma}$ is the correlation in (2.9).

The approximation for Ω_t in (2.14) consists of two expectations: one with respect to $\sqrt{\sigma_t}$ and another with respect to r_t^p. Since exponent p can only take two values, $p = 0$ or $p = \frac{1}{2}$, $\mathbb{E}\left(r_t^{-p}\right)$ simply equals 1 for $p = 0$, and $\mathbb{E}\left(\sqrt{1/r_t}\right)$ for $p = \frac{1}{2}$. Approximations of these expectations are given in the sections to follow.

In the lemma below we present the affinity of model H1.

Lemma 2.2 (Affinity of the Heston-Hybrid-Model-1)**.** *The model (2.8) in log-equity space, $x_t = \log S_t$, with a constant parameter, Δ, and a deterministic function, Ω_t, as given by (2.14) is of the affine form.*

Proof. We use the log-transform, $x_t = \log S_t$, so that we find, by applying Itô's lemma,

$$\begin{aligned} \mathrm{d}x_t &= \left(r_t\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \Omega_t r_t^p\mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma\right) - \frac{1}{2}\left(\Omega_t^2 r_t^{2p} + \sigma_t\left(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta\right)\right)\mathrm{d}t \\ &= \left(r_t - \frac{1}{2}\Omega_t^2 r_t^{2p} - \frac{1}{2}\left(1+2\hat{\rho}_{x,\sigma}\Delta+\Delta^2\right)\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \Omega_t r_t^p\mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma. \end{aligned}$$

For state vector $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]$, it is clear that according to (1.2) and (1.4) the drift and the interest rate are already in the affine form. The symmetric instantaneous covariance matrix (1.3) is given by:

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} r_t^{2p}\eta^2 & 0 & \eta\Omega_t r_t^{2p} \\ * & \gamma^2\sigma_t & \gamma\hat{\rho}_{x,\sigma}\sigma_t + \gamma\Delta\sigma_t \\ * & * & \sigma_t + \Omega_t^2 r_t^{2p} + \Delta^2\sigma_t + 2\hat{\rho}_{x,\sigma}\Delta\sigma_t \end{bmatrix}. \tag{2.15}$$

Since $p = 0$ or $p = \frac{1}{2}$ the instantaneous covariance matrix (2.15) is of the affine form, i.e., all terms in the covariance matrix are linear combinations of the state vector $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]$. $\square$

2.2.1 The Case $\Delta = 0$

With $\Delta = 0$ in Systems (2.8) and (2.10), the model resembles the one in [14]. There, a constant parameter $\bar{\Omega} = \Omega_t$ was prescribed, and an instantaneous correlation was *indirectly* imposed.

The following lemma, however, shows that this model with $\Delta = 0$ resembles the full-scale HHW and HCIR models only for correlation $\rho_{x,r} = 0$.

Lemma 2.3. *The hybrid model (2.8) with $\Delta = 0$ are full-scale HHW and HCIR models, in the sense of System (2.3), only if the instantaneous correlation between the stock and the interest rate processes in System (2.3) equals zero, i.e., $\rho_{x,r} = 0$.*

Proof. The proof is analogous to the proof of Lemma 2.1. We see from the equalities in (2.11) that the System (2.7) resembles System (2.10) with $\Delta = 0$, only if:

$$\begin{cases} \bar{\Omega} = & \rho_{x,r} r_t^{-p}\sqrt{\sigma_t}, \\ \hat{\rho}_{x,\sigma} = & \rho_{x,\sigma}, \\ \hat{\rho}_{x,\sigma}^2 = & \rho_{x,\sigma}^2 + \rho_{x,r}^2. \end{cases} \tag{2.16}$$

The equations (2.16) only hold for $\rho_{x,r} = 0$. So, we conclude that the HHW and HCIR models with $\Delta = 0$ are not full-scale HHW and HCIR models with a non-zero correlation $\rho_{x,r}$. $\square$

Although the model with $\Delta = 0$ is not a properly defined Heston hybrid model, one can still proceed with the analysis. Parameter $\bar{\Omega}$ was derived based on following equality, see [14], using the definition of the instantaneous correlation,

$$\hat{\rho}_{x,r} = \frac{\mathbb{E}\left(\mathrm{d}S_t \mathrm{d}r_t\right) + \mathbb{E}(\mathrm{d}S_t)\mathbb{E}(\mathrm{d}r_t)}{\sqrt{\sigma_t S_t^2\,\mathrm{d}t + \bar{\Omega}^2 S_t^2\,\mathrm{d}t}\sqrt{\eta^2 r_t^{2p}\,\mathrm{d}t}} = \frac{\eta\bar{\Omega} r_t^{2p} S_t\,\mathrm{d}t}{\sqrt{\sigma_t S_t^2\,\mathrm{d}t + \bar{\Omega}^2 r_t^{2p} S_t^2\,\mathrm{d}t}\sqrt{\eta^2 r_t^{2p}\,\mathrm{d}t}} = \frac{\bar{\Omega} r_t^p}{\sqrt{\sigma_t + \bar{\Omega}^2 r_t^{2p}}}. \tag{2.17}$$

To deal with the affinity issue a constant approximation for $\bar{\Omega}$ was proposed, given by:

$$\bar{\Omega} \approx \frac{\hat{\rho}_{x,r}}{\sqrt{1 - \hat{\rho}_{x,r}^2}}\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}t\right)^{\frac{1}{2}} / \mathbb{E}\left(\frac{1}{T}\int_0^T r_t \mathrm{d}t\right)^p. \tag{2.18}$$

By choosing $\bar{\Omega} = 0$ the model collapses to the well-known Heston-Hull-White model ($p = 0$) or Heston-CIR model ($p = \frac{1}{2}$) with zero correlation $\rho_{x,r}$. This model is included in the numerical experiments of the next section.

In the following sections we derive the characteristic functions for the reformulated Heston hybrid models from Section 2.2, as defined in (2.8). We distinguish the case $p = 0$, which we call the reformulated Heston Hull-White hybrid model, abbreviated by H1-HW, from $p = \frac{1}{2}$ which is the reformulated Heston Cox-Ingersoll-Ross hybrid model (H1-CIR).

3 Reformulated Heston Hull-White Hybrid Model

We start with the derivation of the CF for the reformulated Heston Hull-White hybrid model (H1-HW), with $p = 0$ in (2.8). We first assume that the term-structure for the interest rate θ_t is constant, $\theta_t = \theta$. A generalization is presented in Appendix C.

According to [11], the discounted CF for the H1-HW model is of the following form:

$$\phi(u, \mathbf{X}_\tau, \tau) = \mathrm{e}^{A(u,\tau)+B_x(u,\tau)x_0+B_\sigma(u,\tau)\sigma_0+B_r(u,\tau)r_0}, \tag{3.1}$$

with boundary conditions $B_x(u, 0) = iu$, $B_\sigma(u, 0) = 0$, $B_r(u, 0) = 0$ and $A(u, 0) = 0$, where $\tau := T - t$. Next, we derive the CF for the H1-HW model, with Δ and Ω_t as in (2.14). For this, we need some simplifications for some of the terms appearing.

3.1 Approximation for Stochastic Volatility

We start with the following lemma:

Lemma 3.1. *The expectation $\mathbb{E}(\sqrt{\sigma_t})$, with stochastic process σ_t given in Equation (2.8), can be approximated by:*

$$\mathbb{E}(\sqrt{\sigma_t}) \approx \left(\bar{\sigma}(1 - \mathrm{e}^{-\kappa t}) + \sigma_0 \mathrm{e}^{-\kappa t} - \frac{\gamma^2}{8\kappa} \frac{(1 - \mathrm{e}^{-\kappa t})\,(\bar{\sigma}\mathrm{e}^{\kappa t} - \bar{\sigma} + 2\sigma_0)}{(\bar{\sigma}\mathrm{e}^{\kappa t} - \bar{\sigma} + \sigma_0)}\right)^{\frac{1}{2}} =: \Lambda(t), \tag{3.2}$$

where κ, $\bar{\sigma}$, γ and σ_0 are the parameters given in (2.3).

Proof. Deriving an analytic expression for the expectation of $\sqrt{\sigma_t}$ directly for stochastic volatility process σ_t, as in (2.8), is involved. Therefore, in order to find a first order approximation we apply the so-called *delta method*, see for example [1; 23], which states that a function $\varphi(X)$ can be approximated by a first order Taylor expansion at $\mathbb{E}(X)$, i.e.,

$$\varphi(X) \approx \varphi(\mathbb{E}X) + (X - \mathbb{E}X)\frac{\partial \varphi}{\partial X}(\mathbb{E}X), \tag{3.3}$$

for a given random variable, X, with expectation, $\mathbb{E}X$, and variance, $\mathbb{V}\mathrm{ar}X$, assuming that for $\varphi(X)$ its first derivative with respect to X exists and is sufficiently smooth. Since the variance of $\varphi(X)$ can be approximated by the variance of the right-hand side of (3.3) we have:

$$\begin{aligned} \mathbb{V}\mathrm{ar}(\varphi(X)) &\approx \mathbb{V}\mathrm{ar}\left(\varphi(\mathbb{E}X) + (X - \mathbb{E}X)\frac{\partial \varphi}{\partial X}(\mathbb{E}X)\right) \\ &= \left(\frac{\partial \varphi}{\partial X}(\mathbb{E}X)\right)^2 \mathbb{V}\mathrm{ar}X. \end{aligned} \tag{3.4}$$

Now, by using this result for function $\varphi(\sigma_t) = \sqrt{\sigma_t}$, we find

$$\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}) \approx \left(\frac{1}{2}\frac{1}{\sqrt{\mathbb{E}(\sigma_t)}}\right)^2 \mathbb{V}\mathrm{ar}(\sigma_t) = \frac{1}{4}\frac{\mathbb{V}\mathrm{ar}(\sigma_t)}{\mathbb{E}(\sigma_t)}. \tag{3.5}$$

However, from the definition of variance we also have:

$$\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}) = \mathbb{E}(\sigma_t) - \left(\mathbb{E}\sqrt{\sigma_t}\right)^2. \tag{3.6}$$

and by combining Equations (3.5) and (3.6) we obtain the following approximation:

$$\mathbb{E}(\sqrt{\sigma_t}) \approx \sqrt{\mathbb{E}(\sigma_t) - \frac{1}{4}\frac{\mathbb{V}\mathrm{ar}(\sigma_t)}{\mathbb{E}(\sigma_t)}}. \tag{3.7}$$

Since σ_t is a square root process, as in (2.8), we have

$$\sigma_t = \sigma_0 \mathrm{e}^{-\kappa t} + \bar{\sigma}(1 - \mathrm{e}^{-\kappa t}) + \int_0^t \mathrm{e}^{\kappa(s-t)} \gamma \sqrt{\sigma_s} \mathrm{d}W_s^\sigma. \tag{3.8}$$

The expectation of $\mathbb{E}(\sigma_t)$ now equals

$$\mathbb{E}(\sigma_t) = \sigma_0 \mathrm{e}^{-\kappa t} + \bar{\sigma}(1 - \mathrm{e}^{-\kappa t}), \tag{3.9}$$

whereas for the variance we get:

$$\begin{aligned} \mathbb{E}(\sigma_t^2) &= \mathbb{E}\left(\mathbb{E}(\sigma_t) + \int_0^t \gamma \mathrm{e}^{\kappa(s-t)} \sqrt{\sigma_s} \mathrm{d}W_s^\sigma \right)^2 && (3.10) \\ &= (\mathbb{E}(\sigma_t))^2 + \mathbb{E}\left(\int_0^t \gamma \mathrm{e}^{\kappa(s-t)} \sqrt{\sigma_s} \mathrm{d}W_s^\sigma \right)^2, && (3.11) \end{aligned}$$

which, by Itô's isometry, reads:

$$\begin{aligned} \mathbb{E}(\sigma_t^2) &= (\mathbb{E}(\sigma_t))^2 + \gamma^2 \int_0^t \mathrm{e}^{2\kappa(s-t)} \mathbb{E}(\sigma_s) \mathrm{d}s && (3.12) \\ &= (\mathbb{E}(\sigma_t))^2 + \frac{\gamma^2}{2\kappa} \mathrm{e}^{-2\kappa t}(\mathrm{e}^{\kappa t} - 1) \left(\bar{\sigma} \mathrm{e}^{\kappa t} - \bar{\sigma} + 2\sigma_0 \right). && (3.13) \end{aligned}$$

So the variance, $\mathbb{V}\mathrm{ar}(\sigma_t)$, equals

$$\mathbb{V}\mathrm{ar}(\sigma_t) = \frac{\gamma^2}{2\kappa} \mathrm{e}^{-2\kappa t}(\mathrm{e}^{\kappa t} - 1) \left(\bar{\sigma} \mathrm{e}^{\kappa t} - \bar{\sigma} + 2\sigma_0 \right). \tag{3.14}$$

Now, by substituting Equations (3.9) and (3.14) in (3.7), the lemma is proved. □

Since Lemma 3.1 provides an explicit approximation for Ω_t in (2.14) in terms of a deterministic function for $\mathbb{E}(\sqrt{\sigma_t})$, we are, in principle, able to derive the corresponding CF.

Remark. *We assume that the first-order linear terms around the parameter values in the Taylor expansion give an accurate representation. However, this may not work satisfactory for 'flat' density functions, like those from a uniform distribution. In order to increase the accuracy, higher order terms can be included in the expansion [1]. More discussion on conditions for the delta method to perform well can be found in [23].*

The approximation for $\mathbb{E}(\sqrt{\sigma_t})$ in (3.2) is still non-trivial. In order to find the coefficients of the CF, a routine for numerically solving the corresponding ODEs has to be incorporated. Numerical integration, however, slows down the option pricing engine, and would make the SDE model less attractive. As we aim to find a closed form expression for the CF, we further simplify $\Lambda(t)$ in (3.2). Expectation $\mathbb{E}(\sqrt{\sigma_t})$ can be approximated by a function of the following form:

$$\mathbb{E}(\sqrt{\sigma_t}) \approx a + b\mathrm{e}^{-ct} =: \widetilde{\Lambda}(t), \tag{3.15}$$

with a, b and c constant. Appropriate values for a, b and c in (3.15) can be obtained via an optimization problem of the form, $\min_{a,b,c} ||\Lambda(t) - \widetilde{\Lambda}(t)||_n$, where $||\cdot||_n$ is any n^{th} norm.

We propose here, instead of a numerical approximation for these coefficients, a simple analytic expression in Result 3.2:

Result 3.2. *By matching the functions $\Lambda(t)$ and $\widetilde{\Lambda}(t)$ for $t \to +\infty$, $t \to 0$ and $t = 1$, we find:*

$$\begin{aligned} \lim_{t\to+\infty} \Lambda(t) = \sqrt{\bar{\sigma} - \frac{\gamma^2}{8\kappa}} &= a = \lim_{t\to+\infty} \widetilde{\Lambda}(t), \\ \lim_{t\to 0} \Lambda(t) = \sqrt{\sigma_0} &= a + b = \lim_{t\to 0} \widetilde{\Lambda}(t), \\ \lim_{t\to 1} \Lambda(t) = \Lambda(1) &= a + b\mathrm{e}^{-c} = \lim_{t\to 1} \widetilde{\Lambda}(t). \end{aligned} \tag{3.16}$$

The values a, b and c can now be estimated by:

$$\begin{cases} a = & \sqrt{\bar{\sigma} - \dfrac{\gamma^2}{8\kappa}}, \\ b = & \sqrt{\sigma_0} - a, \\ c = & -\log\left(b^{-1}(\Lambda(1) - a)\right), \end{cases} \tag{3.17}$$

where $\Lambda(t)$ is given by (3.2).

The approximation given in Result 3.2 may give difficulties for $\bar{\sigma} < \frac{\gamma^2}{8\kappa}$ in Equation (3.17) (the expression under the square root then becomes negative). The variance process σ_t is always positive and cannot reach zero if $2\kappa\bar{\sigma} > \gamma^2$ (the Feller condition), which, rewritten, equals $\bar{\sigma} > \frac{\gamma^2}{2\kappa}$. With all the parameters assumed to be positive, this means that, if the Feller condition is satisfied, the approximation in (3.17) is also well-defined.

In order to measure the quality of approximation (3.17) to $\Lambda(t)$ in (3.2), we perform a numerical experiment (see the results in Figure 3.4). For randomly chosen sets of parameters the approximation (3.17) resembles $\Lambda(t)$ in (3.2) very well.

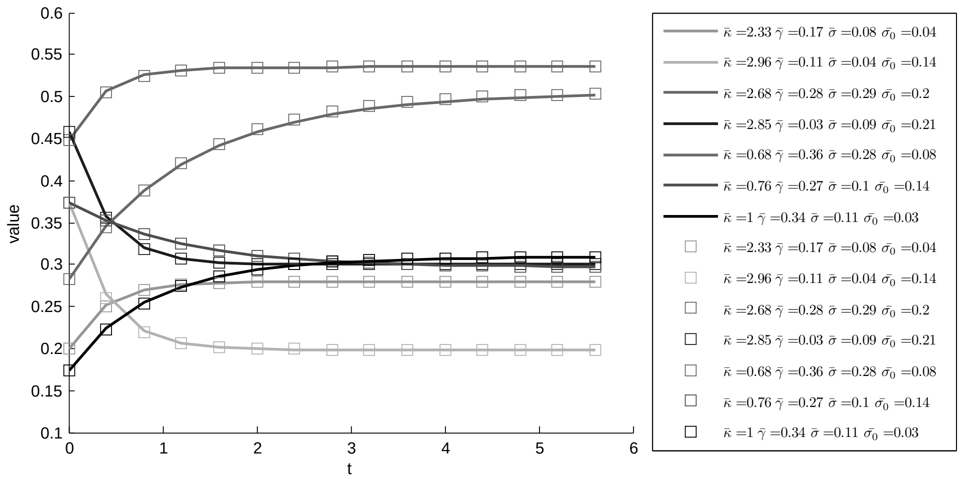

Figure 3.1: The quality of the approximation $\mathbb{E}(\sqrt{\sigma_t}) \approx a + b\mathrm{e}^{-ct}$ (squares) versus Equation (3.2) (continuous line) for 7 randomly chosen parameter sets κ, γ, $\bar{\sigma}$ and σ_0.

Remark. *In order to increase the accuracy of the model it is possible to approximate Ω_t by an appropriate stochastic process. In Section 4.2 we discuss such a model extension for the hybrid Heston-Cox-Ingersoll-Ross model. These results can also be simply adopted to this framework. However, an analytic closed-form CF will be difficult to derive.*

3.2 Characteristic Function for H1-HW Model

In Result 3.2 we have obtained values for the parameters a, b and c. In the derivation of the CF for the H1-HW model we use $\Omega_t = \rho_{x,r}(a + b\mathrm{e}^{-ct})$. The CF for the H1-HW model can now be found in closed form, by the following lemmas:

Lemma 3.3 (The ODEs related to the H1-HW model)**.** *The functions $A(u,\tau)$, $B_x(u,\tau)$, $B_r(u,\tau)$ and $B_\sigma(u,\tau)$ for $u \in \mathbb{R}$ and $\tau \geq 0$ in (3.1) for H1-HW satisfy the following system of*

ODEs:

$$\begin{aligned}
\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0,\\
\frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1-\lambda B_r + B_x,\\
\frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= -\kappa B_\sigma - \frac{1}{2}(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta)B_x(1-B_x)+\frac{1}{2}\gamma^2 B_\sigma^2 + (\gamma\hat{\rho}_{x,\sigma}+\gamma\Delta)B_\sigma B_x,\\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \lambda\theta B_r + \kappa\bar{\sigma}B_\sigma - \frac{1}{2}B_x\Omega_t^2(1-B_x)+\frac{1}{2}\eta^2 B_r^2 + \eta\Omega_t B_x B_r,
\end{aligned}$$

where κ, λ, θ and η correspond to the parameters in Model (2.8), Δ and $\hat{\rho}_{x,\sigma}$ are given in (2.14) and Ω_t as in (2.13), with (3.15).

Proof. The proof can be found in Appendix A. □

The following lemma gives the closed form solution for the functions $B_x(u,\tau)$, $B_r(u,\tau)$, $B_\sigma(u,\tau)$ and $A(u,\tau)$ in (3.1).

Lemma 3.4 (Characteristic Function for the H1-HW model)**.** *The solution of the ODE system in Lemma 3.3 can be found analytically and is given by:*

$$\begin{aligned}
B_x(u,\tau) &= iu, && (3.18)\\
B_r(u,\tau) &= (iu-1)\lambda^{-1}(1-\mathrm{e}^{-\lambda\tau}), && (3.19)\\
B_\sigma(u,\tau,\gamma) &= \frac{1}{\gamma^2}\frac{1-\mathrm{e}^{-d\tau}}{1-g\mathrm{e}^{-d\tau}}\left(\kappa-\gamma\hat{\rho}_{x,\sigma}iu-\gamma iu\Delta-d\right), && (3.20)
\end{aligned}$$

with $d=\sqrt{f_1^2-2\gamma^2 f_2}$, $g=\dfrac{f_1-d}{f_1+d}$, κ, θ, η, λ, γ are as in (2.8); Δ and $\hat{\rho}_{x,\sigma}$ are given by (2.14), and a, b and c by (3.17). [2] *For $A(u,\tau,\gamma)$ we have:*

$$A(u,\tau,\gamma)=\lambda\theta\underbrace{\int_0^\tau B_r\mathrm{d}t}_{I_0}-\left(\frac{1}{2}iu+\frac{1}{2}u^2\right)\underbrace{\int_0^\tau \Omega_t^2\mathrm{d}t}_{I_1}+\frac{1}{2}\eta^2\underbrace{\int_0^\tau B_r^2\mathrm{d}t}_{I_2}+iu\eta\underbrace{\int_0^\tau \Omega_t B_r\mathrm{d}t}_{I_3}+\kappa\bar{\sigma}\underbrace{\int_0^\tau B_\sigma\mathrm{d}t}_{I_4},$$

where the integrals I_0, I_1, I_3 and I_4 all admit an analytic solution:

$$\begin{aligned}
I_0 &= \frac{1}{\lambda}(iu-1)\left(\tau+\frac{1}{\lambda}(\mathrm{e}^{-\lambda\tau}-1)\right),\\
I_1 &= \frac{\rho_{x,r}^2}{2c}\left(b^2(1-\mathrm{e}^{-2c\tau})+4ab(1-\mathrm{e}^{-c\tau})+2a^2c\tau\right),\\
I_2 &= \frac{1}{2\lambda^3}(i+u)^2\left(3+\mathrm{e}^{-2\lambda\tau}-4\mathrm{e}^{-\lambda\tau}-2\lambda\tau\right),\\
I_3 &= \frac{\rho_{x,r}(iu-1)}{c\lambda^2(c+\lambda)}\mathrm{e}^{-(c+\lambda)\tau}\left(b\lambda(c-\mathrm{e}^{\lambda\tau}(c+\lambda-\lambda\mathrm{e}^{c\tau}))+ac(c+\lambda)\mathrm{e}^{c\tau}(1+\mathrm{e}^{\lambda\tau}(\lambda\tau-1))\right),\\
I_4 &= \frac{1}{\gamma^2}(f_1-d)\tau-\frac{2}{\gamma^2}\log\left(\frac{1-g\mathrm{e}^{-d\tau}}{1-g}\right),
\end{aligned}$$

and

$$\begin{aligned}
f_1 &= \kappa-\gamma\hat{\rho}_{x,\sigma}iu-\gamma iu\Delta, && (3.21)\\
f_2 &= -\frac{1}{2}(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta)iu-\frac{1}{2}(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta)u^2, && (3.22)
\end{aligned}$$

Proof. The proof is straightforward and is based on solving the Riccati-type ODEs. The general solution given in Appendix B. Derivation has been performed using the package Mathematica. □

[2] Note that both $B_\sigma(u,\tau,\gamma)$, and $A(u,\tau,\gamma)$ have additionally the vol-vol parameter, γ, in their argument lists. This is because in the derivation of the extended model with a full matrix of correlations, to be discussed in Section 3.3, this notation will turn out to be useful.

3.2.1 Numerical Experiment

Here, we perform numerical experiments by which we validate the performance of the new hybrid model (2.8), for $p = 0$. We compare this H1-HW model with the full-scale HHW model (2.3), with respect to implied volatilities and instantaneous correlations. Since the characteristic function of the HHW model in (2.3) is not known, these prices were obtained by Monte Carlo simulation, and compared with the prices obtained by the Fourier inverse of the CF for the H1-HW model, derived in Section 3.2. The Monte Carlo simulation was performed with 50.000 paths and 1.000 time-steps. The parameters were chosen to be: $\theta = 0.03$, $\kappa = 1.2$, $\bar{\sigma} = 0.08$, $\gamma = 0.09$, $\lambda = 1.1$, $\eta = 0.1$, $\rho_{x,v} = -0.7$, $\rho_{x,r} = 0.6$, $S_0 = 1$, $r_0 = 0.08$ and $v_0 = 0.0625$.

Since the approximation for the CF in Lemma 3.4 is heavily dependent on the approximation for Ω_t, we determine the unknown parameters a, b and c first. In Table 3.1 we present the analytically obtained parameters from Result 3.2 and the numerically obtained values by a Levenberg-Marquardt algorithm. The table shows that both approaches generate an approximation with a very small squared-sum-error (SSE). Both approximations are highly satisfactory, and the analytic approximation can safely be used here.

Table 3.1: Values for a, b and c in approximation (3.15). The analytic solution is obtained by Result 3.2.

	Parameters			Error
method	a	b	c	SSE
analytic (3.17)	0.2814	-0.0313	1.2114	1.29E-09
Levenberg-Marquardt	0.2814	-0.0311	1.1347	4.69E-10

In Figure 3.2 we present the implied volatilities for the HHW (obtained by Monte Carlo) and the H1-HW (obtained by the FFT) models. For short and long maturity experiments (left side and right side picture, respectively), we obtain a very good fit of the H1-HW to the full-scale HHW model.

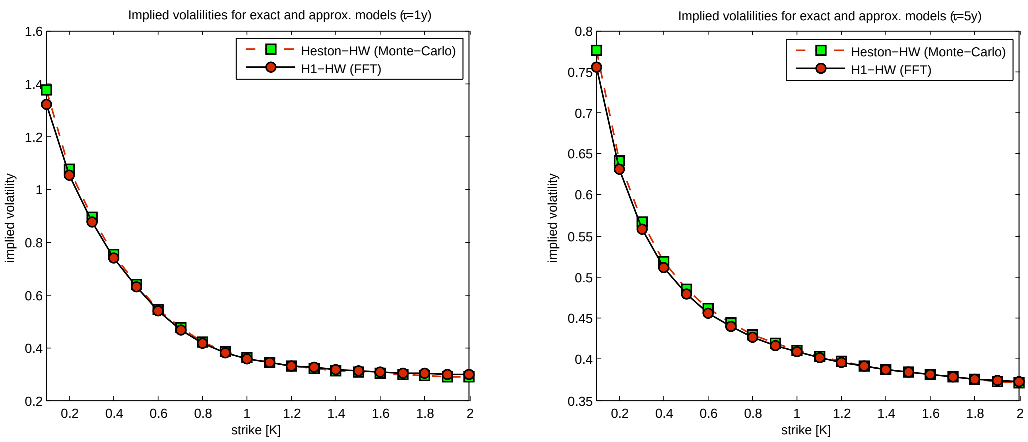

Figure 3.2: The implied Black-Scholes volatilities for the HHW model in (2.3) generated by MC simulation, and the H1-HW model (2.8) obtained by inverting the CF. The parameters are $\theta = 0.03$, $\kappa = 1.2$, $\bar{\sigma} = 0.08$, $\gamma = 0.09$, $\lambda = 1.1$, $\eta = 0.1$, $\rho_{x,v} = -0.7$, $\rho_{x,r} = 0.6$, $S_0 = 1$, $r_0 = 0.08$, $v_0 = 0.0625$, $a = 0.2813$, $b = -0.0311$ and $c = 1.1347$. LEFT: $\tau = 1y$, RIGHT: $\tau = 5y$.

In a next experiment (see Figure 3.3) we examine, by means of a Monte Carlo simulation, the behavior of the instantaneous correlations between stock S_t and the interest rate r_t for three hybrid models, the HHW model (2.3), the model with $\Delta = 0$, as in [14] and constant $\overline{\Omega}$, and the H1-HW model given by (2.8),(2.13),(2.14),(3.15)(3.17). Figure 3.3 shows that for the HHW and the H1-HW model, the instantaneous correlations are stable and oscillate around the exact value, chosen to be 0.6, whereas for the model with $\Delta = 0$ a different correlation pattern is observed. For the latter model, initially the correlation is significantly higher than 0.6, and it decreases in time. These results show that a constant $\bar{\Omega}$ in the model with $\Delta = 0$ may give an average correlation close to the exact value, although the instantaneous correlation is not stable.

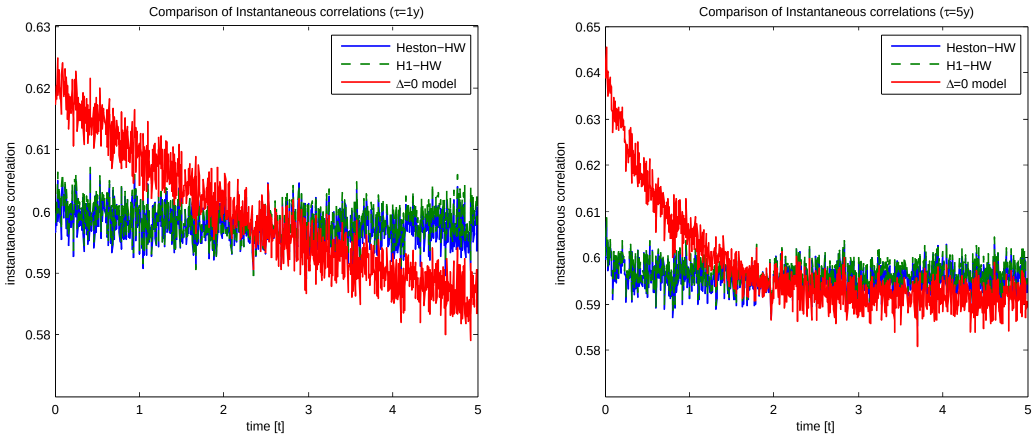

Figure 3.3: The instantaneous correlations for different models. The red line represents the model with $\Delta = 0$ with constant $\bar{\Omega}$, the blue line corresponds to the full-scale HHW model, and the dotted-green line to the H1-HW model. Parameters are the same as in Figure 3.2. LEFT: $\tau = 1y$, RIGHT: $\tau = 5y$.

3.3 Heston Hybrid Model with Full Matrix of Correlations

Following the idea of reformulating the Heston hybrid model presented in Section 2.2 we discuss here the inclusion of the additional correlation, $\rho_{r,\sigma}$, between the interest rate r_t and the stochastic volatility σ_t. We call the resulting model, the reformulated Heston Hull-White Hybrid Model-2, and denote it by H2-HW.

Suppose that, in terms of independent Brownian motions, we have given the following system of SDEs:

$$\begin{bmatrix} \mathrm{d}r_t \\ \mathrm{d}\sigma_t \\ \mathrm{d}S_t \end{bmatrix} = \begin{bmatrix} \lambda(\theta_t - r_t) \\ \kappa(\bar{\sigma} - \sigma_t) \\ r_t S_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \eta r_t^p & 0 & 0 \\ \Upsilon_t & \Gamma\sqrt{\sigma_t} & 0 \\ \Omega_t r_t^p S_t & (\hat{\rho}_{x,\sigma} + \Delta)\sqrt{\sigma_t} S_t & \sqrt{1 - \hat{\rho}_{x,\sigma}^2 \sqrt{\sigma_t}} S_t \end{bmatrix} \begin{bmatrix} \mathrm{d}\widetilde{W}_t^r \\ \mathrm{d}\widetilde{W}_t^\sigma \\ \mathrm{d}\widetilde{W}_t^x \end{bmatrix}, \tag{3.23}$$

with Γ, Δ, Ω_t and Υ_t unknown functions. Following the proof in Lemma 2.3 we immediately find that System (3.23) and the Heston hybrid model with a full matrix of correlations (2.3) are equivalent if and only if the following equalities hold:

$$\begin{cases} \Upsilon_t = & \gamma\rho_{r,\sigma}\sqrt{\sigma_t}, \\ \Gamma = & \gamma\sqrt{1-\rho_{r,\sigma}^2}, \\ \Omega_t = & \rho_{r,x} r_t^{-p}\sqrt{\sigma_t}, \\ \hat{\rho}_{x,\sigma}^2 = & \rho_{r,x}^2 + \left(\dfrac{\rho_{x,\sigma}-\rho_{r,x}\rho_{r,\sigma}}{\sqrt{1-\rho_{r,\sigma}^2}}\right)^2, \\ \Delta = & \dfrac{\rho_{x,\sigma}-\rho_{r,x}\rho_{r,\sigma}}{\sqrt{1-\rho_{r,\sigma}^2}} - \hat{\rho}_{x,\sigma}. \end{cases} \tag{3.24}$$

Here, $\rho_{r,\sigma}$, $\rho_{x,\sigma}$, $\rho_{r,x}$ are the correlations in the full-scale Heston model defined in (2.3) and $\hat{\rho}_{x,\sigma}$ is the correlation in the Heston hybrid model given in (3.23). Note that in (3.23) we assume zero correlations $\hat{\rho}_{x,r}$ and $\hat{\rho}_{r,\sigma}$, and relate the processes indirectly via representation (3.24).

It is clear that by taking $\rho_{r,\sigma} = 0$ the H2-HW model with a full matrix of correlations collapses to the setup in Lemma 2.1. Moreover, because of the affinity constraints discussed in Lemma 2.2 we are unable to place Υ_t and Ω_t directly in Equation (3.24). Therefore, as before, we use the deterministic approximation $\Upsilon_t \approx \gamma\rho_{r,\sigma}\mathbb{E}(\sqrt{\sigma_t})$. For which Result (3.2) can be used.

The representations of the Heston-Hull-White model in (2.8) and the model in (3.23) with $\hat{\rho}_{r,\sigma} \neq 0$ for $p = 0$ are closely related. The lemma below specifies the relation in terms of the coefficients of the corresponding CF.

Lemma 3.5 (The Heston-Hull-White Hybrid Model-2 with Full Matrix of Correlations)**.** *The solution for the ODEs system for the Heston-Hull-White hybrid model with a full matrix of correlations defined by (3.23), for $p = 0$, is given by:*

$$B_x(u,\tau) = iu, \tag{3.25}$$

$$B_r(u,\tau) = (iu-1)\lambda^{-1}(1-\mathrm{e}^{-\lambda\tau}), \tag{3.26}$$

$$B_\sigma(u,\tau) = B_\sigma(u,\tau,\Gamma), \tag{3.27}$$

with $B_\sigma(u,\tau,\Gamma)$ given in (3.20) and for $A(u,\tau)$ we have:

$$A(u,\tau) = A(u,\tau,\Gamma) + \int_0^\tau B_\sigma(u,s)\Upsilon_s\left(\eta B_r(u,s) + \frac{1}{2}\Upsilon_s B_\sigma + iu\Omega_s\right)\mathrm{d}s, \tag{3.28}$$

where Γ is defined in (3.24), $A(u,\tau,\Gamma)$ is given in (3.21), $\Upsilon_s \approx \gamma\rho_{r,\sigma}\mathbb{E}(\sqrt{\sigma_t})$, $\Omega_t \approx \rho_{r,x}\mathbb{E}(\sqrt{\sigma_t})$ with $\mathbb{E}(\sqrt{\sigma_t}) \approx a + b\mathrm{e}^{-ct}$ and parameters a, b, and c specified in Result 3.2.

This result can be found easily with the help of package Mathematica.

Lemma 3.5 shows that term $A(u,\tau)$ in the CF for the H2-HW model with a full matrix of correlations requires numerical integration of the hyper-geometric function ${}_2F_1$. This integration can be efficiently performed by a basic quadrature routine as already shown in [15].

Since for this model we are able to introduce a full matrix of correlations, we show the effect of varying $\rho_{x,r}$ and $\rho_{r,\sigma}$ in Figure 3.4. For both cases, we investigate the impact of the correlations on the implied at-the-money Black-Scholes volatilities. We notice that both correlations, $\rho_{x,r}$ and $\rho_{x,\sigma}$, have a significant impact on the implied volatilities. Although, their effect is reversed, i.e., for $\rho_{x,r} = 0.3$ the implied volatility is higher than for $\rho_{x,r} = 0$ and lower for $\rho_{x,r} = -0.3$, while higher correlations $\rho_{r,\sigma}$ imply lower implied volatilities, we also see that the effect of correlation $\rho_{x,r}$ is more significant than for $\rho_{r,\sigma}$.

The agreement between the results with the full-scale HHW model and the H2-HW model is very well. The full-correlation model requires a numerical integration routine to be employed, which may be a disadvantage. One can, instead, work with correlation $\rho_{r,\sigma} = 0$ and use the closed form solution for the CF in Section 3.2.1, or use this additional degree of freedom either to increase the accuracy of the model fit to market data, or estimate it externally from historic data.

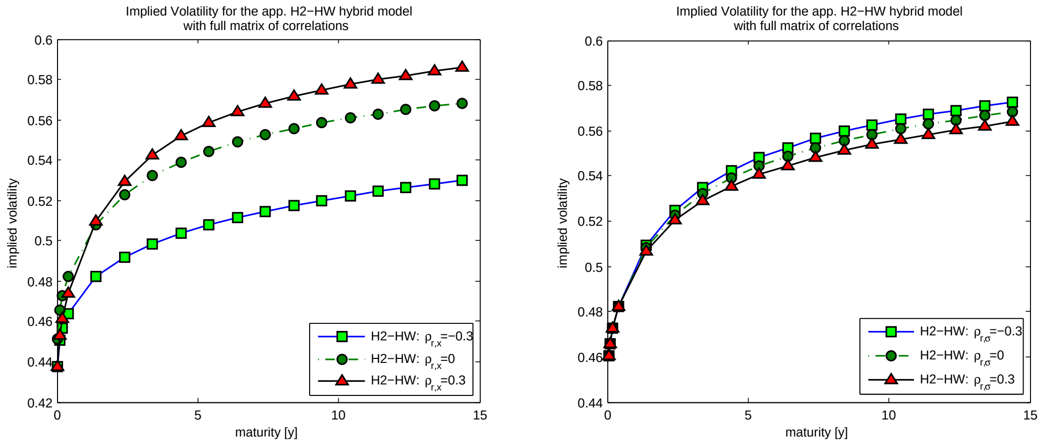

Figure 3.4: The impact of correlations $\rho_{x,r}$ and $\rho_{r,\sigma}$ on the at-the-money implied volatilities. The parameters in both cases were chosen to: $\theta = 0.05$, $\kappa = 0.5$, $\bar{\sigma} = 0.2$, $\gamma = 0.25$, $\lambda = 0.8$, $\eta = 0.15$, $S_0 = 1$, $r_0 = 0.05$ and $v_0 = 0.2$.

4 Reformulated Heston-Cox-Ingersoll-Ross Hybrid Model

We discuss here the CF for the reformulated Heston-Cox-Ingersoll-Ross hybrid model. Since this model is more involved than the Hull-White based models, we dedicate separate sections to the derivation. For the reformulated Heston Hull-White model we have used the approximation $\Omega_t \approx \rho_{x,r}\mathbb{E}(\sqrt{\sigma_t})$, which could be approximated with the help of the function given in (3.15). In the Cox-Ingersoll-Ross case, the approximation for Ω_t involves two processes, i.e.,

$$\Omega_t \approx \rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) \cdot \mathbb{E}(\sqrt{1/r_t}).$$

Although the first expectation can be approximated as in Result 3.2, the second expectation brings some additional complications. Moreover, since integration over Ω_t is needed in the derivation of the CF, we cannot determine an analytic form for the CF.

4.1 Constant Approximation of Function Ω_t

In order to derive the CF analytically, we first assume $\Omega_t \equiv \bar{\Omega}$ to be constant, i.e.:

$$\bar{\Omega} \approx \rho_{x,r}\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}t\right)^{\frac{1}{2}} / \mathbb{E}\left(\frac{1}{T}\int_0^T r_t \mathrm{d}t\right)^{\frac{1}{2}}, \tag{4.1}$$

and we abbreviate this model by H1-CIR.

Since in the H1-CIR model the dynamics for σ_t and r_t are of the same type, the expectations can be calculated in the same manner. Unfortunately, the expectations in the right-hand side of (4.1) are not trivially solved. We use an approximation given by Lemma 4.1:

Lemma 4.1. *For a given square root process σ_t, as in Equation (2.8), the expectation of the integral of σ_t over some interval $[0, T]$, $T > 0$, has a first order approximation given by:*

$$\mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s}\right) \approx \left(\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s\right) - \frac{1}{4}\frac{\mathbb{V}ar\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s\right)}{\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s\right)}\right)^{\frac{1}{2}}. \tag{4.2}$$

Moreover, the expression on the right-hand side of (4.2) admits a closed form solution.

Proof. The first part of the proof results from the delta approximation as in the proof of Lemma 3.1. Now, we show that the right-hand side of Equation (4.2) can be determined analytically. From Equation (3.8) we have:

$$\begin{aligned}\int_0^T \sigma_t \mathrm{d}t &= \int_0^T \left(\sigma_0 \mathrm{e}^{-\kappa t} + \bar{\sigma}(1-\mathrm{e}^{-\kappa t})\right) \mathrm{d}t + \int_0^T \left[\int_0^t \mathrm{e}^{\kappa(s-t)} \gamma \sqrt{\sigma_s} \mathrm{d}W_s^\sigma\right] \mathrm{d}t \\ &= \bar{\sigma}T + \frac{\sigma_0 - \bar{\sigma}}{\kappa}\left(1-\mathrm{e}^{-T\kappa}\right) + \int_0^T \frac{\gamma}{\kappa}\left(1-\mathrm{e}^{\kappa(s-T)}\right)\sqrt{\sigma_s}\mathrm{d}W_s^\sigma,\end{aligned}$$

using Fubini's Theorem. For the expectation we then have:

$$\mu := \mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}t\right) = \bar{\sigma} + \frac{\sigma_0 - \bar{\sigma}}{\kappa T}\left(1-\mathrm{e}^{-\kappa T}\right), \tag{4.3}$$

and for the second moment:

$$\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_s \mathrm{d}s\right)^2 = \mu^2 + \frac{1}{T^2}\int_0^T \mathbb{E}\left[\frac{\gamma}{\kappa}\left(1-\mathrm{e}^{\kappa(s-T)}\right)\sqrt{\sigma_s}\right]^2 \mathrm{d}s, \tag{4.4}$$

with the expectation under the integral equal to:

$$\mathbb{E}\left[\frac{\gamma}{\kappa}\left(1-\mathrm{e}^{\kappa(s-T)}\right)\sqrt{\sigma_s}\right]^2 = \frac{\gamma^2}{\kappa^2}\left(\mathrm{e}^{\kappa(s-T)}-1\right)^2\left(\sigma_0\mathrm{e}^{-\kappa s} + \bar{\sigma}\left(1-\mathrm{e}^{-\kappa s}\right)\right).$$

After integration and simplifying the resulting expression, we obtain:

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{T}\int_0^T \sigma_s \mathrm{d}s\right) = \frac{\gamma^2 \mathrm{e}^{-2T\kappa}}{2T^2\kappa^3}\left[-2\sigma_0 + \bar{\sigma} + 4\mathrm{e}^{T\kappa}\left(\bar{\sigma} + (\bar{\sigma}-\sigma_0)\kappa T\right) + \mathrm{e}^{2T\kappa}\left(2\sigma_0 + \bar{\sigma}(2T\kappa - 5)\right)\right]. \tag{4.5}$$

Now, by the inclusion of the appropriate terms in the delta method approximation, as discussed in the proof of Lemma 3.1, this proof is complete. □

We continue with the derivation of the CF for H1-CIR in the following lemma:

Lemma 4.2 (The ODEs for the H1-CIR model)**.** *The functions $A(u,\tau)$, $B_x(u,\tau)$, $B_r(u,\tau)$ and $B_\sigma(u,\tau)$ for $u \in \mathbb{R}$ and $\tau \geq 0$ in the CF (3.1) satisfy the following system of ODEs for the H1-CIR model with (4.1), (4.2):*

$$\begin{aligned}\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0,\\ \frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 - \lambda B_r + (1 - \frac{1}{2}\bar{\Omega}^2)B_x + \frac{1}{2}\eta^2 B_x^2 + \eta\bar{\Omega}B_r B_x + \frac{1}{2}\bar{\Omega}^2 B_x^2,\\ \frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= -\kappa B_\sigma - \frac{1}{2}(1 + \Delta^2 + 2\hat{\rho}_{x,\sigma}\Delta)B_x(1-B_x) + \frac{1}{2}\gamma^2 B_\sigma^2 + (\gamma\hat{\rho}_{x,\sigma} + \gamma\Delta)B_\sigma B_x,\\ \frac{\mathrm{d}A}{\mathrm{d}\tau} &= \lambda\theta B_r + \kappa\bar{\sigma}B_\sigma.\end{aligned}$$

where κ, $\bar{\sigma}$, η, γ, θ, λ, are as in (2.8), $\bar{\Omega}$ is given in (4.1), (4.2), and Δ and $\hat{\rho}_{x,\sigma}$ are given in (2.14).

Proof. The proof is very similar to the proof in Appendix A. □

The next lemma shows that the CF for the H1-CIR model can be found analytically.

Lemma 4.3 (CF for the H1-CIR model)**.** *The solution for the ODEs system in Lemma 4.2 can be found explicitly and is given by:*

$$\begin{aligned} B_x(u,\tau) &= iu, && (4.6)\\ B_r(u,\tau) &= \frac{2i+2u-\bar{\Omega}^2u+i\bar{\Omega}^2u^2+iu^2\eta^2}{2(\bar{\Omega}u\eta+i\lambda)}\left(\mathrm{e}^{\frac{1}{2}(2i\bar{\Omega}u\eta-2\lambda)\tau}-1\right), && (4.7)\\ B_\sigma(u,\tau) &= \frac{1}{\gamma^2}\cdot\frac{1-\mathrm{e}^{-d\tau}}{1-g\mathrm{e}^{-d\tau}}\left(\kappa-\gamma\hat{\rho}_{x,\sigma}iu-\gamma iu\Delta-d\right), && (4.8) \end{aligned}$$

with $d=\sqrt{f_1^2-2\gamma^2f_2}$, $g=\dfrac{f_1-d}{f_1+d}$ *and the remaining parameters are as in Lemma 4.2. For* $A(u,\tau)$ *we have:*

$$A(u,\tau)=\lambda\theta\underbrace{\int_0^\tau B_r\mathrm{dt}}_{I_1}+\kappa\bar{\sigma}\underbrace{\int_0^\tau B_\sigma\mathrm{dt}}_{I_2}, \qquad (4.9)$$

where integrals I_1 *and* I_2 *are given by:*

$$\begin{aligned} I_1 &= \frac{2+u\left(-2i+\bar{\Omega}^2(i+u)+u\eta^2\right)}{2\left(\bar{\Omega}u\eta+i\lambda\right)^2}\mathrm{e}^{-\lambda\tau}\left(\mathrm{e}^{i\bar{\Omega}u\eta\tau}+\mathrm{e}^{\lambda\tau}(\lambda\tau-i\bar{\Omega}u\eta\tau-1)\right), && (4.10)\\ I_2 &= \frac{1}{\gamma^2}(f_1-d)\tau-\frac{2}{\gamma^2}\log\left(\frac{1-g\mathrm{e}^{-d\tau}}{1-g}\right), && (4.11) \end{aligned}$$

with

$$\begin{aligned} f_1 &= \kappa-\gamma\hat{\rho}_{x,\sigma}iu-\gamma iu\Delta, && (4.12)\\ f_2 &= -\frac{1}{2}(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta)iu-\frac{1}{2}(1+\Delta^2+2\hat{\rho}_{x,\sigma}\Delta)u^2, && (4.13) \end{aligned}$$

Proof. The proof is straightforward, based on solving Riccati type ODEs with the general solution given in Appendix B, and obtained via Mathematica. □

Now, as in the case of the H1-HW model, we compare the performance of the H1-CIR model with the full-scale Heston Hybrid Model in (2.3) with $p=\frac{1}{2}$.

4.1.1 Numerical Experiment

We first investigate by a numerical experiment the quality of the approximation of the CF for the H1-CIR model, given in Lemma 4.3, compared to the exact solution from the HCIR model from (2.3). For the HCIR model (2.3) Monte Carlo simulation is used. The Monte Carlo simulation was again performed with 50.000 paths and 1.000 time-steps. Figure 4.1 shows that for $\tau=1$ and $\tau=5$ the H1-CIR model gives highly satisfactory implied volatilities. We can observe, however, that for out-of-the money options the approximation gives slightly lower values than the HCIR model.

Figure 4.2 presents the instantaneous correlations, and shows that for constant Ω_t the average correlation is close to that of the full-scale HCIR model. However, it is not stable in time. This leads us to consider an improved approximation for the Ω_t-term in the next section.

4.2 Stochastic Approximation of Function Ω_t

For the reformulated Heston-CIR model, function Ω_t is of the following form:

$$\Omega_t=\rho_{x,r}\sqrt{\frac{\sigma_t}{r_t}}, \qquad (4.14)$$

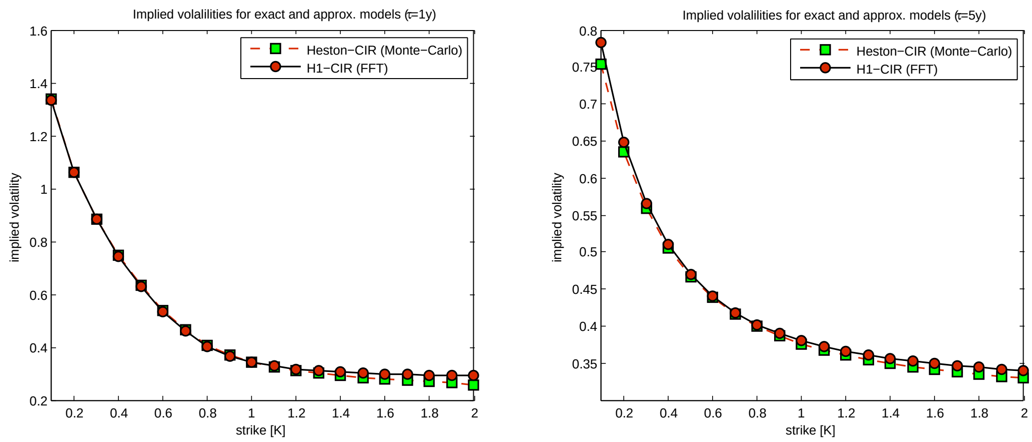

Figure 4.1: The implied Black-Scholes volatilities for the HCIR model in (2.3) generated by MC simulation, and Model (2.8) for $p = \frac{1}{2}$ by inverting the CF given in Lemma 4.3. Parameters are as presented in Figure 3.2. LEFT: $\tau = 1y$, RIGHT: $\tau = 5y$.

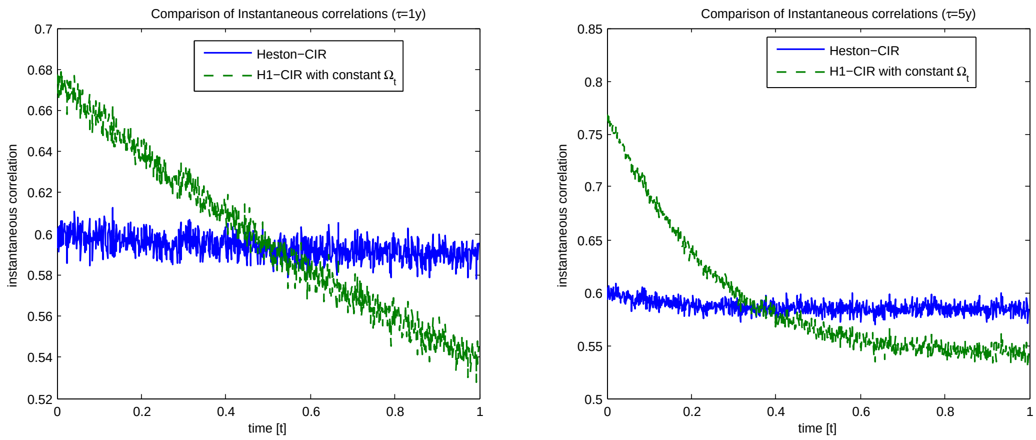

Figure 4.2: The instantaneous correlations for the HCIR and the H1-CIR models. The green line represents Model (2.8) with constant $\bar{\Omega}$ in Equation (4.1) and the blue line corresponds to the full-scale HCIR model. Parameters are as presented in Figure (4.1). LEFT: $\tau = 1y$, RIGHT: $\tau = 5y$.

for which the symmetric instantaneous covariance matrix reads,

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} (1+\rho_{x,r}^2+\Delta^2+2\Delta\hat{\rho}_{x,\sigma})\sigma_t & \hat{\rho}_{x,\sigma}\gamma\sigma_t+\Delta\gamma\sigma_t & \eta\rho_{x,r}\sqrt{\sigma_t}\sqrt{r_t} \\ * & \gamma^2\sigma_t & 0 \\ * & * & \eta^2 r_t \end{bmatrix}. \quad (4.15)$$

For state vector $\mathbf{X}_t = [x_t, \sigma_t, r_t]$ covariance matrix (4.15) includes the terms $\sqrt{\sigma_t}\sqrt{r_t}$ and is thus not affine. Our main goal here is to find an improved approximation of the instantaneous covariance matrix but with the affinity of the model preserved. We call this model the H2-CIR model.

By applying Itô's lemma to $v_t^* := \sqrt{\sigma_t}$ and to $R_t^* := \sqrt{r_t}$, with σ_t and r_t of the same form, we find, for v_t^*:

$$\mathrm{d}v_t^* = \left[-\frac{\gamma^2}{8v_t^*} + \frac{\kappa}{2}\left(\frac{\bar{\sigma}}{v_t^*} - v_t^*\right)\right]\mathrm{d}t + \frac{\gamma}{2}\mathrm{d}W_t^\sigma, \quad v_0^* = \sqrt{\sigma_0}. \quad (4.16)$$

Although the drift in (4.16) is relatively involved, the diffusion term is free of state variables. This suggests that the SDE above can be approximated by some normal process.

Since in the proof of Result 3.2 we have found a highly satisfactory closed-form approximation for $\mathbb{E}(\sqrt{\sigma_t})$ and $\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$, we are able to derive a stochastic process for which the first two moments match. For this purpose we propose the following dynamics for the square of the square-root process $\mathrm{d}v_t \approx \mathrm{d}v_t^*$, and for $dR_t \approx dR_t^*$:

$$\begin{aligned} \mathrm{d}v_t &= \mu_t^v \mathrm{d}t + \psi_t^v \mathrm{d}W_t^\sigma, \quad v_0 = \sqrt{\sigma_0}, \\ \mathrm{d}R_t &= \mu_t^R \mathrm{d}t + \psi_t^R \mathrm{d}W_t^r, \quad R_0 = \sqrt{r_0}, \end{aligned}$$

with some deterministic, time-dependent functions μ_t^v, μ_t^R, ψ_t^v and ψ_t^R. We choose these time-dependent functions so that the first two moments match. Therefore, for the expectation and the variance we find:

$$\begin{cases} \mathbb{E}(\sqrt{\sigma_t}) = & v_0^* + \int_0^t \mu_s^v \mathrm{d}s, \\ \mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}) = & \int_0^t (\psi_s^v)^2 \mathrm{d}s. \end{cases} \tag{4.17}$$

By simplifying the unknown functions μ_t^v and ψ_t^v in (4.17) we have:

$$\begin{cases} \mu_t^v = & \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\sqrt{\sigma_t}), \\ \psi_t^v = & \sqrt{\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})}. \end{cases} \tag{4.18}$$

Using the results from Section 3.1 the expectation $\mathbb{E}(\sqrt{\sigma_t})$ and the variance $\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$ can be approximated by an appropriate Taylor expansion. For both quantities μ_t^v and ψ_t^v we find an approximation $\mu_t^v \approx -cb\mathrm{e}^{-ct}$ with b and c given in Result 3.2 and

$$\psi_t^v \approx \left[\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbb{V}\mathrm{ar}(\sigma_t)}{\mathbb{E}(\sigma_t)}\right)\right]^{\frac{1}{2}} = \left(\frac{\gamma^2\mathrm{e}^{-\kappa t}\left(\bar{\sigma}^2\left(\mathrm{e}^{\kappa t}-1\right)^2 - \left(3-4\mathrm{e}^{\kappa t}+\mathrm{e}^{2\kappa t}\right)\bar{\sigma}\sigma_0 + 2\sigma_0^2\right)}{8\left(\left(\mathrm{e}^{\kappa t}-1\right)\bar{\sigma}+\sigma_0\right)^2}\right)^{\frac{1}{2}}. \tag{4.19}$$

Notice that higher order approximations for μ_t^v and ψ_t^v can be easily included.

Now, based on the assumption that the dynamics for $\mathrm{d}(\sqrt{\sigma_t})$ and $\mathrm{d}(\sqrt{r_t})$ can be approximated by basic stochastic processes, we find, by Itô's lemma that $z_t = \sqrt{\sigma_t}\sqrt{r_t}$ has a simple form as well, given by:

$$\mathrm{d}z_t = \left(\mu_t^R v_t + \mu_t^v R_t\right)\mathrm{d}t + \psi_t^v R_t \mathrm{d}W_t^v + \psi_t^R v_t \mathrm{d}W_t^r. \tag{4.20}$$

With these additional variables the state vector $\mathbf{X}_t$ under the log-stock process $x_t = \log S_t$ needs to be expanded to $\mathbf{X}_t = [x_t, \sigma_t, r_t, v_t, R_t, z_t]$, which gives rise to the following system of SDEs:

$$\begin{cases} \mathrm{d}x_t = & \left(r_t - \frac{1}{2}\left(\rho_{x,r}^2 + 1 + 2\hat{\rho}_{x,\sigma}\Delta + \Delta^2\right)\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \rho_{x,r}\sqrt{\sigma_t}dW_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma}-\sigma_t)\mathrm{d}t + \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + \eta\sqrt{r_t}\mathrm{d}W_t^r, \\ \mathrm{d}v_t = & \mu_t^v\mathrm{d}t + \psi_t^v\mathrm{d}W_t^\sigma, \\ \mathrm{d}R_t = & \mu_t^R\mathrm{d}t + \psi_t^R\mathrm{d}W_t^r, \\ \mathrm{d}z_t = & \left(\mu_t^R v_t + \mu_t^v R_t\right)\mathrm{d}t + \psi_t^v\sqrt{r_t}\mathrm{d}W_t^\sigma + \psi_t^R\sqrt{\sigma_t}\mathrm{d}W_t^r, \end{cases} \tag{4.21}$$

with Δ, $\hat{\rho}_{x,\sigma}$ and the other coefficients as in (2.11). Now, the symmetric instantaneous covariance matrix is given by:

$$\begin{bmatrix} \xi_1\sigma_t & \xi_2\sigma_t & \eta\rho_{x,r}\sqrt{\sigma_t}\sqrt{r_t} & \psi_t^v\xi_3\sqrt{\sigma_t} & \rho_{x,r}\psi_t^R\sqrt{\sigma_t} & \xi_4\psi_t^v\sqrt{\sigma_t}\sqrt{r_t}+\rho_{x,r}\psi_t^R\sigma_t \\ * & \gamma^2\sigma_t & 0 & \psi_t^v\gamma\sqrt{\sigma_t} & 0 & \gamma\psi_t^v\sqrt{\sigma_t}\sqrt{r_t} \\ * & * & \eta^2 r_t & 0 & \psi_t^R\eta\sqrt{r_t} & \eta\psi_t^R\sqrt{\sigma_t}\sqrt{r_t} \\ * & * & * & (\psi_t^v)^2 & 0 & (\psi_t^v)^2\sqrt{r_t} \\ * & * & * & * & (\psi_t^R)^2 & (\psi_t^R)^2\sqrt{\sigma_t} \\ * & * & * & * & * & (\psi_t^v)^2 r_t+(\psi_t^R)^2\sigma_t \end{bmatrix}, \tag{4.22}$$

with constants: $\xi_1 = (1+\rho_{x,r}^2+\Delta^2+2\Delta\hat{\rho}_{x,\sigma})$, $\xi_2 = (\gamma\hat{\rho}_{x,\sigma}+\Delta\gamma)$, $\xi_3 = (\Delta+\hat{\rho}_{x,\sigma})$, $\xi_4 = (\hat{\rho}_{x,\sigma}+\Delta)$.

In the current form the covariance matrix does not satisfy the affinity constraints. However, with $v_t = \sqrt{\sigma_t}$, $R_t = \sqrt{r_t}$ and $z_t = \sqrt{\sigma_t}\sqrt{r_t}$ the symmetric matrix reads:

$$\begin{bmatrix} \xi_1\sigma_t & \xi_2\sigma_t & \eta\rho_{x,r}z_t & \psi_t^v\xi_3 v_t & \rho_{x,r}\psi_t^R v_t & \xi_4\psi_t^v z_t+\rho_{x,r}\psi_t^R\sigma_t \\ * & \gamma^2\sigma_t & 0 & \psi_t^v\gamma v_t & 0 & \gamma\psi_t^v z_t \\ * & * & \eta^2 r_t & 0 & \psi_t^R\eta R_t & \eta\psi_t^R z_t \\ * & * & * & (\psi_t^v)^2 & 0 & (\psi_t^v)^2 R_t \\ * & * & * & * & (\psi_t^R)^2 & (\psi_t^R)^2 v_t \\ * & * & * & * & * & (\psi_t^v)^2 r_t+(\psi_t^R)^2\sigma_t \end{bmatrix}, \tag{4.23}$$

which, since ψ_t^v is a deterministic time-dependent function, is now affine.

Since the system of SDEs (4.21) is of the affine form, we can derive the corresponding CF:

$$\phi_{H2-CIR}(u,\mathbf{X}_\tau,\tau) = \mathrm{e}^{B_x(u,\tau)x_0+B_\sigma(u,\tau)\sigma_0+B_r(u,\tau)r_0+B_v(u,\tau)v_0+B_R(u,\tau)R_0+B_z(u,\tau)z_0}, \tag{4.24}$$

with $v_0 = \sqrt{\sigma_0}$, $R_0 = \sqrt{r_0}$, $z_0 = \sqrt{v_0}\sqrt{r_0}$, and the functions $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$, $B_v(u,\tau)$, $B_R(u,\tau)$ and $B_z(u,\tau)$ satisfy the ODEs given by the lemma below.

Lemma 4.4 (The ODEs related to the H2-CIR model)**.** *The functions $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$, $B_v(u,\tau)$, $B_R(u,\tau)$ and $B_z(u,\tau)$ for $u \in \mathbb{R}$ and $\tau > 0$ in (4.24), satisfy:*

$$\begin{aligned}
\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0,\\
\frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}\xi_1 B_x(B_x-1)-\kappa B_\sigma+\xi_2 B_x B_\sigma+\frac{1}{2}\gamma^2 B_\sigma^2+\rho_{x,r}\psi_t^R B_x B_z+\frac{1}{2}(\psi_t^R)^2 B_z^2,\\
\frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1+B_x-\lambda B_r+\frac{1}{2}\eta^2 B_r^2+\frac{1}{2}(\psi_t^v)^2 B_z^2,\\
\frac{\mathrm{d}B_v}{\mathrm{d}\tau} &= \mu_t^R B_z+\psi_t^v\xi_3 B_x B_v+\gamma\psi_t^v B_\sigma B_v+\rho_{x,r}\psi_t^R B_x B_R+(\psi_t^R)^2 B_R B_z\\
\frac{\mathrm{d}B_R}{\mathrm{d}\tau} &= \mu_t^v B_z+\psi_t^R\eta B_r B_R+(\psi_t^v)^2 B_v B_z\\
\frac{\mathrm{d}B_z}{\mathrm{d}\tau} &= \eta\rho_{x,r}B_x B_\sigma+\xi_4\psi_t^v B_x B_z+\gamma\psi_t^v B_\sigma B_z+\eta\psi_t^R B_r B_z\\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \kappa\bar{\sigma}B_\sigma+\lambda\theta B_r+\mu_t^v B_v+\mu_t^R B_R+\frac{1}{2}(\psi_t^v)^2 B_v^2+\frac{1}{2}(\psi_t^R)^2 B_R^2.
\end{aligned}$$

where μ_t^v, μ_t^R, ψ_t^v, ψ_t^R are specified in (4.18), constants ξ_1, ξ_2, ξ_3 and ξ_4 are given as in (4.22), and the remaining parameters correspond to ones given in system (4.21).

Proof. The proof is very similar to the proof in Appendix A. □

We see that the system of the ODEs given in Lemma 4.4 is very difficult to solve analytically. To find the solution we have used an explicit Runge-Kutta method [13; 22], *ode45* from the Matlab package. Numerical results are presented in the next subsection.

Remark. *Note that the extension of the H2-CIR model to the case of a full matrix of correlations is a trivial exercise.*

Our approach of transforming a non-affine system of SDEs to an affine system is based on the assumption that the expectation and variance of the square-root process, i.e., $\mathbb{E}(\sqrt{\sigma_t})$ and $\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$, are accurate approximations. Our results are obtained by expanding the processes around their means, but the framework presented is still applicable with higher order terms of Taylor expansion included, or when exact representations are used.

4.2.1 Numerical Experiment

Finally, we perform numerical experiments to verify the accuracy of the approximations in H2-CIR. Figure 4.3 shows an excellent fit, in terms of the implied volatilities, of the new H2-CIR model to the full-scale Heston-CIR model. Also, the instantaneous correlations presented in Figure 4.4 give very satisfactory results, although for longer maturities some inaccuracies can be observed. In Table 4.1 we present the time needed for obtaining the plain vanilla option prices. The table shows that although the system of ODEs given in (4.4) needs to be solved numerically, the time needed for obtaining European put or call prices, by the COS pricing method [12], is significantly less than a second. The tolerance which was varied in the table was the tolerance for the Riccati ODE solves. The pricing of the options by means of the COS method, a method based on Fourier cosine series expansions, was performed with a fixed number of 250 terms, which guaranteed sufficiently accurate option prices (up to machine precision).

The key for fast evaluation of the ODEs lies in vectorizing the system.

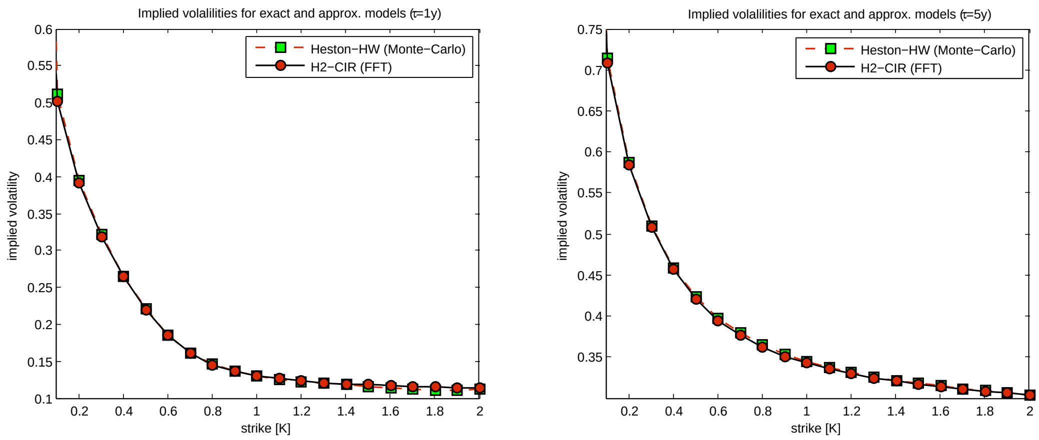

Figure 4.3: The implied Black-Scholes volatilities for the HCIR model in (2.3) and the H2-CIR, FFT-based model, presented in (4.21). Parameters are as presented in Figure 3.2. LEFT: $\tau = 1$, RIGHT: $\tau = 5$.

Table 4.1: Time in seconds for pricing a call option based on an explicit Runge-Kutta method combined with the COS method [12].

Accuracy	**Maturity**				
	$\tau = 0.5$	$\tau = 1$	$\tau = 2$	$\tau = 5$	$\tau = 10$
10^{-1}	7.83E-2	7.86E-2	7.94E-2	8.50E-2	8.46E-2
10^{-2}	7.78E-2	7.80E-2	8.38E-2	8.48E-2	8.90E-2
10^{-5}	8.33E-2	8.97E-2	1.05E-1	1.34E-1	1.62E-1

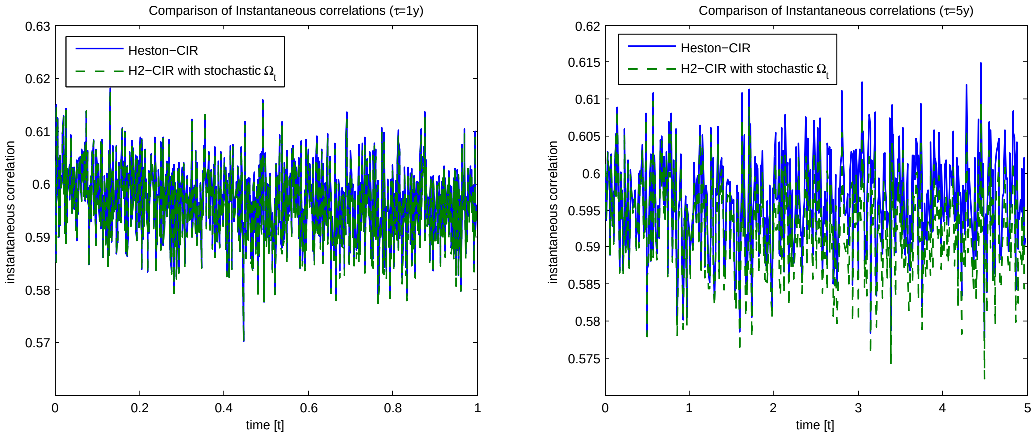

Figure 4.4: The instantaneous correlations for the HCIR (2.3), and the H2-CIR model in (4.21).

5 Conclusions

In this article we have presented an extension of the Heston model by stochastic interest rates. We have focused our attention on two hybrid models, the Heston-Hull-White and the Heston-Cox-Ingersoll-Ross models.

By a series of approximations, we placed variants of the Heston-Hull-White model in the framework of affine diffusion processes. With the assumption of a zero correlation between short rate and volatility we found a closed-form approximation for the characteristic function. For the case with a full matrix of correlations, a semi-closed form for the characteristic function resulted, in which one integral needed to be solved numerically. The approximations in the models have been validated by comparing the implied volatilities and instantaneous correlations to the full-scale Heston-Hull-White model. Restrictions for the accuracy of the various approximations have also been presented.

Suitable approximations of the Heston-Cox-Ingersoll-Ross model lead to semi-closed form solutions for the characteristic function. The most sophisticated version is based on a transformation of the 3D Heston-CIR model to a 6D representation. This reformulated CIR-based model gave a good fit to the full-scale hybrid equivalent model.

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A Proof of Lemma 3.3

Proof.

$$\sigma(\mathbf{X_t})\sigma(\mathbf{X_t})^T = \begin{bmatrix} \eta^2 & 0 & \eta\Omega_t \\ 0 & \gamma^2\sigma_t & \gamma\rho_{x,\sigma}\sigma_t + \gamma\Delta\sigma_t \\ \eta\Omega_t & \gamma\rho_{x,\sigma}\sigma_t + \gamma\Delta\sigma_t & \sigma_t + \Omega_t^2 + \Delta^2\sigma_t + 2\rho_{x,\sigma}\Delta\sigma_t \end{bmatrix}, \tag{A.1}$$

which gives:

$$c_0 = \begin{bmatrix} \eta^2 & 0 & \eta\Omega_t \\ 0 & 0 & 0 \\ \eta\Omega_t & 0 & \Omega_t^2 \end{bmatrix}, \tag{A.2}$$

and

$$c_1 = \begin{bmatrix} (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,\gamma^2,0) & (0,\gamma\rho_{x,\sigma}+\gamma\Delta,0) \\ (0,0,0) & (0,\gamma\rho_{x,\sigma}+\gamma\Delta,0) & (0,1+\Delta^2+2\rho_{x,\sigma}\Delta,0) \end{bmatrix}, \tag{A.3}$$

$$a_0 = \begin{bmatrix} \lambda\theta & \kappa\bar{\sigma} & -\frac{1}{2}\Omega_t^2 \end{bmatrix}, \quad a_1 = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\kappa & 0 \\ 1 & -\frac{1}{2}(1+\Delta^2+2\rho_{x,\sigma}\Delta) & 0 \end{bmatrix}. \tag{A.4}$$

Now,

$$\frac{1}{2}\mathbf{B}^T c_1 \mathbf{B} = \begin{bmatrix} 0 \\ F_1 \\ 0 \end{bmatrix}, \tag{A.5}$$

with,

$$F_1 = \frac{1}{2}\gamma^2 B_\sigma^2 + (\gamma\rho_{x,\sigma} + \gamma\Delta)B_\sigma B_x + \frac{1}{2}(1+\Delta^2+2\rho_{x,\sigma}\Delta)B_x^2. \tag{A.6}$$

So, the system of ODEs to be solved is of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} B_r \\ B_\sigma \\ B_x \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{r_1} + \underbrace{\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\kappa & -\frac{1}{2}(1+\Delta^2+2\rho_{x,\sigma}\Delta) \\ 0 & 0 & 0 \end{bmatrix}}_{a_1^T}\begin{bmatrix} B_r \\ B_\sigma \\ B_x \end{bmatrix} + \frac{1}{2}\mathbf{B}^T c_1 \mathbf{B}, \quad \text{(A.7)}$$

and for $A(u,\tau)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = \begin{bmatrix} B_r & B_\sigma & B_x \end{bmatrix}\begin{bmatrix} \lambda\theta \\ \kappa\bar{\sigma} \\ -\frac{1}{2}\Omega_t^2 \end{bmatrix} + \frac{1}{2}\mathbf{B}^T\begin{bmatrix} \eta^2 & 0 & \eta\Omega_t \\ 0 & 0 & 0 \\ \eta\Omega_t & 0 & \Omega_t^2 \end{bmatrix}\mathbf{B}, \quad \text{(A.8)}$$

By simplifications the proof is finished. □

B Riccati Equations

Let us consider the following system of ODEs:

$$\begin{cases} \dfrac{df(t)}{dt} &= a_0 g(t) \\ \dfrac{dg(t)}{dt} &= b_0 + b_1 g(t) + b_2 g^2(t). \end{cases} \quad \text{(B.1)}$$

with initial conditions: $f(0) = f_0$ and $g(0) = g_0$. The solution for the system above is given by:

$$\begin{cases} g(t) &= g_0 + \dfrac{(-b_1 + d - 2b_2 g_0)\left(1-\mathrm{e}^{-dt}\right)}{2b_2(1-g\mathrm{e}^{-dt})} \\ f(t) &= f_0 + \dfrac{a_0}{2b_2}\left((-b_1-d)t - 2\log\left(\dfrac{1-g\mathrm{e}^{-dt}}{1-g}\right)\right). \end{cases} \quad \text{(B.2)}$$

with: $d = \sqrt{b_1^2 - 4b_0b_2}$, $g = \dfrac{-b_1 - d - 2g_0b_2}{-b_1 + d - 2B_0b_2}$.

C Including the Interest Rate Term Structure

As it was already mentioned, in deriving the characteristic function, $\phi(u, \mathbf{X}_\tau, \tau)$, we have assumed that θ_t in the interest rate process was constant and equal to θ. The inclusion of the term-structure in the Hull-White model is straightforward, but for the CIR model a time-dependent θ_t cannot be obtained analytically. Moreover, in [6] it was explained that numerical estimates do not guarantee a positive term-structure. Therefore we restrict ourselves to the HW model here.

In the system of SDEs (2.8) the interest rate process is then given by the following SDE:

$$\mathrm{d}r_t = \lambda\left(\theta_t - r_t\right)\mathrm{d}t + \eta \mathrm{d}W_t^r, \quad \text{(C.1)}$$

with parameters $\lambda > 0$, $\eta > 0$, and time-dependent function θ_t. In [6] the function θ_t was chosen to exactly fit the term-structure of interest rates observed in the market. One can easily show that in the case of the Hull-White model for a given instantaneous forward rate, one has

$$f(t,T) \stackrel{def}{=} -\frac{\partial \log P(t,T)}{\partial T},$$

for $t > 0$, $T > 0$ and $P(t,T)$ is a zero coupon bond, which pays 1 unit at maturity T. The function θ_t can now be expressed as

$$\theta_t = f(0,t) + \frac{1}{\lambda}\frac{\partial f(0,t)}{\partial t} + \frac{\eta^2}{2\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right). \quad \text{(C.2)}$$

In order to decompose the SDEs in (2.8) so that the term-structure can be inserted, we define: $r_t := \widetilde{r}_t + \psi_t,$ with $r_0 = 0$ and $\psi_t = f(0,t) + \frac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-\lambda t}\right)^2$.

Since the log-stock process, x_t, is defined in terms of the short rate, r_t, we have:

$$\mathrm{d}x_t = \left(\widetilde{r}_t + \psi_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x. \tag{C.3}$$

Therefore, we also define $x_t := \widetilde{x}_t + \zeta_t,$ with $\zeta_t = \int_0^t \psi_s \mathrm{d}s$. For a state vector, $\mathbf{X_t} = [x_t, \sigma_t, r_t]$, and a corresponding vector $\mathbf{u} = [u_1, u_2, u_3]$, we check the impact of the decompositions on the CF:

$$\begin{aligned}
\phi(\mathbf{u}, \mathbf{X}_t, t, T) &\overset{def}{=} \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathrm{e}^{i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right)\\
&= \mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s + i\mathbf{u}^T[\zeta_T, 0, \psi_T]}\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T \widetilde{r}_s\mathrm{d}s}\mathrm{e}^{i\mathbf{u}^T\mathbf{X}_T^*}|\mathcal{F}_t\right)\\
&= \mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s + i\mathbf{u}^T[\zeta_T, 0, \psi_T]}\mathrm{e}^{A(\mathbf{u},\tau)+[x_0,\sigma_0,r_0]\mathbf{B}(\mathbf{u},\tau)}.
\end{aligned}$$

The coefficients $\mathbf{B}(\mathbf{u},\tau) = [B_x(\mathbf{u},\tau), B_\sigma(\mathbf{u},\tau), B_r(\mathbf{u},\tau)]^{\mathrm{T}}$ are known, so by taking $\mathbf{u} = [0,0,0]$, we simply get:

$$\phi(\mathbf{0}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\cdot 1|\mathcal{F}_t\right) = \mathrm{e}^{-\int_t^T \psi_s ds}\mathrm{e}^{A(\mathbf{0},\tau)+[x_0,\sigma_0,r_0]\mathbf{B}(\mathbf{0},\tau)}. \tag{C.4}$$

The left-hand side of Equation (C.4) is the definition of a zero-coupon bond. Therefore, since for $\widetilde{r}_0 = 0$, $B_x(0,\tau) = 0$, and $B_\sigma(0,\tau) = 0$, we set $t = 0$ and get:

$$P(0,T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}\cdot 1|\mathcal{F}_0\right) = \mathrm{e}^{-\int_0^T \psi_s\mathrm{d}s + A(0,\tau)}, \tag{C.5}$$

where

$$A(0,T) = \frac{1}{2}\eta^2\int_0^T B_r(0,s)^2\mathrm{d}s = \frac{\eta^2}{2\lambda^3}\left(-\frac{3}{2} - \frac{1}{2}\mathrm{e}^{-2\lambda T} + \lambda T\right).$$

Finally, we obtain the reformulated Heston-Hull-White CF, with term-structure, as:

$$\phi(0, \mathbf{X}_\tau, \tau) \quad = \quad \mathrm{e}^{A(u,\tau)+B_x(u,\tau)x_0+B_\sigma(u,\tau)\sigma_0+\Theta(u,\tau)},$$

with $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$, $A(u,\tau)$ given in Lemma 3.4, for $\theta = 0$, and

$$\Theta(u,\tau) = (1-iu)\left(\log P(0,\tau) + \frac{\eta^2}{2\lambda^3}\left(\frac{\tau}{\lambda} + 2\left(\mathrm{e}^{-\lambda\tau}-1\right) - \frac{1}{2}\left(\mathrm{e}^{-2\lambda\tau}-1\right)\right)\right).$$