Hankel-Norm Model Reduction of Time-varying Networks and Systems

A.-J. van der Veen and P. Dewilde

Delft University of Technology
Department of Electrical Engineering
2628 CD Delft, The Netherlands

A procedure to obtain the optimal low degree (generalized) Hankel norm approximant of a bounded causal linear *time-varying* system is described. In the classical Hardy space theory this is known as the "model reduction problem" and it has resulted in solutions which go back to the work of Adamjan, Arov and Krein on the Schur-Takagi interpolation problem. For time-varying systems, the interpolation theory can be extended to cover general, not only Toeplitz, upper operators as well. In this paper, we describe an order-recursive solution to this interpolation problem.

1. INTRODUCTION

In this paper, we consider bounded causal linear time-varying systems, which map input sequences $u \in \ell_2$ to output sequences in ℓ_2 via $y = uT$, and which are described by a transfer operator T

$$T = \begin{bmatrix} \ddots & & \vdots & & \iddots \\ & T_{00} & T_{01} & T_{02} & \\ & & T_{11} & T_{12} & \cdots \\ & \mathbf{0} & & T_{22} & \\ & & & & \ddots \end{bmatrix} . \tag{1}$$

The causality of the system is reflected by the fact that T is upper, because its k-th row corresponds to the response of an impulse at time k. For linear time-invariant (LTI) systems, T is a Toeplitz matrix. In the analysis of the system T, a major role is played by the Hankel operator. For time-varying systems, we define the Hankel operator H_T to correspond to the sequence of operators $\{H_i\}$, where

$$H_i = \begin{bmatrix} T_{i-1,i} & T_{i-1,i+1} & \cdots \\ T_{i-2,i} & T_{i-2,i+1} & \\ \vdots & & \ddots \end{bmatrix} .$$

Each H_i is thus an operator that corresponds to the part of T to the right and strictly above entry T_{ii}. Motivated by abstract system theory, we call the H_i (time-varying) Hankel operators, although they will have a Hankel structure (constant along anti-diagonals) only if T has a Toeplitz structure. Kronecker has shown that for time-invariant systems the *model order* of T–the minimal number of states needed in a state space realization of T, or the number of poles of $T(z)$–is equal to the rank of H_T. It is finite if and only if the system has a rational transfer function. For time-varying systems, the minimal number of states in a non-stationary state realization is in general also time-varying. *The number of states of a minimal realization at time k is equal to the rank of* H_k [1].

This research was supported in part by the commission of the EC under the ESPRIT BRA program 6632 (NANA2).

For model reduction purposes, one is interested in reducing the system order of T, that is, reducing the rank of each H_k. Adamjan, Arov and Krein [2] showed in the LTI case that there is a Hankel matrix of rank n such that the Euclidian norm difference between the original Hankel operator and the approximant is equal to the value of the neglected $(n+1)$-st largest singular value. This approximation leads to what can be called the *optimal reduced system* in Hankel norm. A generalization to time-varying systems is obtained by defining the Hankel norm of the system T as the supremum over the operator (spectral-) norm of each individual Hankel matrix:

$$\| T \|_H \;=\; \sup_i \| H_i \| \,.$$

In terms of this Hankel norm, one can prove the following model reduction theorem (see [3]):

Theorem 1. *Let T be a bounded strictly upper triangular operator with a strictly stable state realization, having a finite number of states at each point, and let $\Gamma = \mathrm{diag}(\gamma_i)$ be a diagonal Hermitian operator which parametrizes the acceptable approximation tolerance ($\gamma_i > 0$). Let H_k be the Hankel matrix of $\Gamma^{-1}T$ at stage k, and suppose that, for all k, the singular values of H_k are uniformly bounded away from 1. Then there exists a strictly upper triangular operator T_a with system order at stage k equal to the number of singular values of H_k that are larger than 1, such that*

$$\| \Gamma^{-1}(T - T_a) \|_H \;\le\; 1\,.$$

Hankel norm approximation theory originates as the solution to the classical Schur-Takagi interpolation problem in the context of complex function theory [2]. The solution can be obtained both via a global *state space* based method, as was extensively studied in the book [4], but also in a recursive fashion, see [5] as a pioneering paper in this respect. With regard to time-varying systems, the classical interpolation problems of Schur or Nevanlinna-Pick can be formulated and solved for upper operators, in a context where diagonals take the place of scalars. A comprehensive treatment can be found in [6], and references therein. The general (state space based) solution to the model reduction problem is submitted for publication in [3]; the present paper is a specialization to solve the underlying interpolation problem for finite upper triangular matrices in an order-recursive fashion.

We need the following background. A time-varying system T as in (1) can act on sequences u whose entries u_i are vectors, instead of scalars, in which case T has entries T_{ij} which are matrices. This corresponds to a multi-input multi-output system. It is essential in our approximation technique that *the number of inputs/outputs of certain transfer operators are allowed to vary in time*, because only this will enable the approximating system to have a varying number of states.

Let I denote an identity operator on such non-uniform sequences of unspecified dimensions. A J-unitary operator Θ is an operator with block decomposition and signature matrices

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \qquad J_1 = \begin{bmatrix} I & \\ & -I \end{bmatrix}, \quad J_2 = \begin{bmatrix} I & \\ & -I \end{bmatrix}. \tag{2}$$

such that $\Theta^* J_1 \Theta = J_2$, $\Theta J_2 \Theta^* = J_1$. These are energy-conservation rules: if $[a_1 \;\; b_1]\Theta = [a_2 \;\; b_2]$, then $a_1a_1^* - b_1b_1^* = a_2a_2^* - b_2b_2^*$. Associated to Θ is an operator Σ such that $[a_1 \;\; b_2]\Sigma = [a_2 \;\; b_1]$ constitutes the same linear relations between the a_i and b_i. Hence Σ satisfies $a_1a_1^* + b_2b_2^* = a_2a_2^* + b_1b_1^*$: it is a unitary operator. However, Σ will only be causal if Θ_{22}^{-1} is upper, which need not be the case.

2. CONVERSION TO AN INTERPOLATION PROBLEM

The key to the proof of theorem 1 is the following conversion to an interpolation problem. Given the original causal system T, we will look for a (non-causal) operator T' such that $E = (T^* - T'^*)\Gamma^{-1}$ is contractive, and the strictly upper part T_a of T' is bounded and has state space dimensions of low order —as low as possible for a given Γ. It is shown in [3] that

$$\| \Gamma^{-1}(T - T_a) \|_H \;=\; \| \Gamma^{-1}(T - T') \|_H \;\le\; \| \Gamma^{-1}(T - T') \| \;\le\; 1\,,$$

so that T_a is a Hankel-norm approximant when T' is an operator-norm approximant.

To find T' we start by determining a factorization of T in the form $T = \Delta^* U$ where Δ and U are upper operators and U is inner: $UU^* = U^*U = I$. This factorization is always possible under the assumptions on T in theorem 1. Next, we look for a causal block-upper J-unitary Θ-operator chosen such that $\left[U^* \;\; -T^*\Gamma^{-1}\right]$, which is block lower, is mapped by Θ to block-upper, *i.e.*, such that

$$\left[U^* \quad -T^*\Gamma^{-1}\right] \Theta \;=\; \left[A' \quad -B'\right] \tag{3}$$

consists of two upper operators. There is an underlying generalized interpolation problem leading to Θ, and the procedure to find Θ and its signature matrices J_1 and J_2 is an extension of the method used in [6] to solve the time-varying Nevanlinna-Pick problem.

From (3) we have $B' = -U^*\Theta_{12} + T^*\Gamma^{-1}\Theta_{22}$. Define the approximating operator T' as

$$T'^* \;=\; B'\Theta_{22}^{-1}\Gamma \;=\; B'\Sigma_{22}\Gamma\,, \tag{4}$$

then $E = (T^* - T'^*)\Gamma^{-1} = U^*\Theta_{12}\Theta_{22}^{-1}$. Because $\Theta_{12}\Theta_{22}^{-1} = -\Sigma_{12}$ is contractive and U unitary, we infer that $\| E \| \le 1$, so that T'^* is indeed an approximant with an admissible modeling error. It remains to show that the strictly causal part of T' has the stated reduced number of states and to verify the mentioned relation with the Hankel singular values of $\Gamma^{-1}T$. Both follow from the construction in [3]. It is seen from (4) that Θ plays an important role: *the number of anti-causal states of* $\Theta_{22}^{-1} = \Sigma_{22}$ *at time k is precisely equal to the number of states of the approximant* T_a *at this time.*

3. ORDER-RECURSIVE INTERPOLATION

The global state space procedure of [3] obtains, for a given realization of T, an inner factor U and an interpolating Θ. It can be specialized to the case where T is a general upper triangular matrix without an a priori known state structure. The resulting procedure to obtain Θ leads to a generalized Schur recursion which is the subject of the remainder of this paper. Consider a 4×4 matrix T,

$$T = \begin{bmatrix} 0 & t_{12} & t_{13} & t_{14} \\ & 0 & t_{23} & t_{24} \\ & & 0 & t_{34} \\ & & & 0 \end{bmatrix}.$$

For convenience of notation, we may take $\Gamma = I$, and thus seek for T_a (a 4×4 matrix) such that $\| T - T_a \| \le 1$. The interpolation problem is to determine a J-unitary and causal Θ (whose signature will be determined by the construction) such that $[U^* \;\; -T^*]\Theta$ is mapped to upper. For a recursive derivation of an interpolating matrix Θ, we proceed as follows. A trivial choice of U^* such that $\Delta = UT^*$ is upper is one which maps an input vector $u = [u_1, u_2, u_3, u_4]$ into an output vector y with non-uniform dimensions: $y = [[u_1\ u_2\ u_3\ u_4], \cdot, \cdot, \cdot]$. This means that the output sequence of U^* consists of a vector at time 1, and is zero-dimensional at time 2-4. With this choice, the interpolation problem becomes: find a J-unitary and causal Θ such that

$$\left[\begin{array}{cccc|cccc} \boxed{1\quad\quad} & & & & \boxed{0} & & & \\ & 1 & & & -t_{12}^* & 0 & & \\ & & 1 & & -t_{13}^* & -t_{23}^* & 0 & \\ & & & 1 & -t_{14}^* & -t_{24}^* & -t_{34}^* & 0 \end{array}\right] \Theta = \left[\begin{array}{cccc|cccc} * & * & * & * & * & * & * & * \\ & * & * & * & & * & * & * \\ & & * & * & & & * & * \\ & & & * & & & & * \end{array}\right] \qquad (5)$$

where $*$ denotes a possibly non-zero entry. Θ acts on the columns of U^* and $-T^*$. Its operations on U^* are always causal because all columns of U^* correspond to outputs at time $k = 1$. Zeroing the lower triangular part of T^* leads to a recursive algorithm with two types of actions at each stage k:

a. Using each of the columns $k+1 \cdots n$ of U^* in turn, make the last $(n-k)$ entries of the k-th column of U^* equal to 0. In particular, the $k+i$-th column of U^* is used to make the $k+i$-th entry of the k-th column of U^* equal to zero.

b. Using columns $k+1 \cdots n$ of U^*, make the last $(n-k)$ entries of the k-th column of T^* equal to 0.

The operations to do each of these steps are elementary unitary or J-unitary (Givens) rotations that act on two columns at a time and each make one selected entry of the second column equal to zero. The precise nature of the rotations (unitary or J-unitary) is dependent on the data, and discussed below. The signal flow corresponding to this computational scheme of Θ is outlined in figure 1(a), where actually $d_1 \cdots d_4 = 0$. *The scheme is in fact a state realization of* Θ.

An elementary (Givens) rotation θ such that $\theta^* j_1 \theta = j_2$ (j_1 and j_2 are 2×2 signature matrices) is defined by $[u\ \ t]\theta = [*\ \ 0]$, where u, t are scalars. Initially, one would consider θ of a traditional J-unitary form:

$$\theta_1 = \begin{bmatrix} 1 & -s \\ -s^* & 1 \end{bmatrix} \frac{1}{c^*}, \qquad cc^* + ss^* = 1,\ c \neq 0, \qquad \text{with } j_1 = j_2 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$

which satisfies $\theta^* j_1 \theta = j_2$. However, since $|s| < 1$, a rotation of this form is appropriate only if $|u| > |t|$. In the recursive algorithm, this will be the case only if $TT^* < I$ which corresponds to a 'definite' interpolation problem. Our situation is more general. If $|u| < |t|$, we require a rotational section of the form

$$\theta_2 = \begin{bmatrix} -s & 1 \\ 1 & -s^* \end{bmatrix} \frac{1}{c^*}, \qquad \theta_2^* \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \theta_2 = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}.$$

The different signature pairs of θ_2 reflect that the signature of the 'energy' of the output vector of such a section is reversed: if $[a_1\ \ b_1]\theta_2 = [a_2\ \ b_2]$, then $a_1 a_1^* - b_1 b_1^* = -a_2 a_2^* + b_2 b_2^*$. Because this signature can be reversed at each elementary step, we will have to keep track of it to ensure that the resulting global Θ-matrix is J-unitary with respect to a certain signature. Thus assign to each column in $[U^*\ \ -T^*]$ a signature ($+1$ or -1), which is updated after each elementary operation, in accordance to the type of rotation. Initially, the signature of the columns of U^* is chosen $+1$, and those of $-T^*$ are chosen -1, together defining $J_1 = [I \dotplus -I]$. In the process, several combinations (in fact: six) of input- and output signatures can occur, leading to six types of (J-) unitary elementary processors. These form the processors in figure 1(a), where signature is indicated with '$+$' and '$-$'.

Figure 1. Computational structure of a recursive solution to the interpolating problem. (a) Θ operator, with elementary rotations of mixed type (both circular and hyperbolic); (b) one possible corresponding Σ-operator, with circular elementary rotations.

Keeping track of the signature at each intermediate step ensures that $\Theta^* J_1 \Theta = J_2$, where J_2 is some (unsorted) signature matrix, that is given by the signatures of the columns of the final resulting upper triangular matrices. A solution to the interpolation problem $[U^* \;\; -T^*]\Theta = [A' \;\; -B']$ is then obtained by *sorting* the columns of the resulting upper triangular matrices according to their signature, such that all positive signs correspond to A' and all negative signs to B'. The columns of Θ are sorted likewise. This gives a signature J_2 as in (2).

We can associate, as usual, with each J-unitary rotation a corresponding unitary rotation. Upon making these replacements, arrows with negative signature are reversed, and a (non-computable) realization of the unitary matrix Σ corresponding to Θ is obtained (figure 1(b)). $\Theta_{22}^{-1} = \Sigma_{22}$ is the transfer from right to left. The rotations which caused an upward arrow across a dotted line are shaded. These are the *anticausal states* of Σ_{22}, and result in states for the approximant T_a as mentioned in section 2.

References

[1] A.J. van der Veen and P.M. Dewilde, "Time-Varying System Theory for Computational Networks," in *Algorithms and Parallel VLSI Architectures, II* (P. Quinton, Y. Robert, eds.), pp. 103–127, Elsevier, 1991.

[2] V.M. Adamjan, D.Z. Arov, and M.G. Krein, "Analytic Properties of Schmidt Pairs for a Hankel Operator and the Generalized Schur-Takagi Problem," *Math. USSR Sbornik*, vol. 15, no. 1, pp. 31–73, 1971.

[3] P.M. Dewilde and A.J. van der Veen, "On the Hankel-Norm Approximation of Upper-Triangular Operators and Matrices," *submitted to Integral Equations and Operator Theory*, 1992.

[4] J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, vol. 45 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, 1990.

[5] D.J.N. Limebeer and M. Green, "Parametric Interpolation, H_∞-Control and Model Reduction," *Int. J. Control*, vol. 52, no. 2, pp. 293–318, 1990.

[6] P. Dewilde and H. Dym, "Interpolation for Upper Triangular Operators," in *Operator Theory: Advances and Applications*, vol. OT 56, pp. 153–260, Birkhäuser Verlag, Basel, 1992.