BSc report APPLIED MATHEMATICS

"Stationary sets and on the existence of homeomorphisms between them" (Dutch title: "Stationaire verzamelingen en het bestaan van homeomorphismes tussen deze")

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Abstract

Stationary sets are important tools in proofs of properties in sets of uncountable cardinality. In this thesis we look at mapping properties between stationary sets. First, the theory necessary for the construction and evaluation of stationary sets is made. That is the theory of ordinal, cardinal and regular cardinal numbers is build up from the level of knowledge of a mathematics student. Two important theorems for stationary sets, Fodor's theorem and the theorem of Ulam and Solovay, are proven. Next mapping properties of stationary subsets of a regular cardinal κ under measurable functions is looked into. With these properties we construct a necessary condition for the existence of homeomorphisms between stationary sets; they may only differ on a non-stationary set. Lastly the amount of stationary subsets of a regular cardinal κ without a homeomorphism between them is estimated as the cardinality of the power set of κ. We find that there are 2^κ topologically incomparable subsets of κ.

Contents

1. Introduction

> "You need a nice quote? How about: My girlfriend thinks stationery sets are adorable, I think stationary sets are essential" -Thao Nguyen, 2020

In this report we will build the theory for stationary sets and prove some peculiar properties and theorems about these. Stationary sets are much used tools in proving theorems in sets of uncountable cardinality. A property of elements is interesting if it holds on a very large set, that is on some closed unbounded set. Any stationary set is defined as such that they must then also contain many elements with that property while the stationary set itself does not have to be as large, just sufficiently so. As stationary sets inherit many of the properties of elements of closed unbounded sets, they are often used in proofs for forcing arguments. The theory will be build with proofs that double as examples explaining the inner workings.

In Appendix A a brief explanation of the used first-order theory, **ZFC**, is given. Within this theory the first building blocks, the elementary operations and sets, of set theory are constructed for those with an interest in it. The first chapter treats various properties of relations and constructs our theory for ordinal numbers. These numbers are used to make proofs on a well-ordered set easier using isomorphisms between the set and its uniquely corresponding ordinal. Next, chapter 3 treats a special type of ordinals, called cardinal numbers. Cardinals are the special sets that are used to give a measure of the number of elements of a set. The chapter concludes with an explanation on cofinal sets and regular cardinal numbers, finishing the fundamental theory for the research of stationary sets. In chapter 4 we build upon the theory described in the previous chapters to construct closed and unbounded sets from the ordinal numbers. From these thick sets the stationary sets will be developed and explored. To give some sense of application we will prove the Δ-system lemma for regular cardinals. Lastly, in chapter 5 we explore how many topologically incomparable stationary subsets there are in a regular cardinal.

2. Ordinal numbers

Almost everything has a natural ordering. One can think of comparing different times, the alphabetical ordering or even the order in which you read these words. Often we want to say something about orderings in general. With this in mind we build the theory of ordinal numbers.

In this chapter the theory of important relations, the well-orders, and of ordinal numbers will be developed. With this we prove that well-ordered sets can be uniquely represented by some ordinal number. These special "numbers" give a sense to the size of the ordering of a set.

2.1. **Ordering.**

Formally, a relation, R, is a subset of the cartesian product of two sets X,Y. That is, $R \subseteq X \times Y$ is a collection of ordered pairs. We say 'x is related to y' if $\langle x, y\rangle \in R$ and we write $x\,R\,y$. Let A be a set and R a relation with $R \subseteq A \times A$. Then R can have the following properties on A.

Reflexive: $(\forall x \in A)(x\,R\,x)$ "every element is related to itself"
Irreflexive: $(\forall x \in A)(\neg(x\,R\,x))$ "no element is related to itself"
Symmetry: $(\forall x, y \in A)(x\,R\,y \to y\,R\,x)$ "relations are two-sided"
antisymmetry: $(\forall x, y \in A)((x\,R\,y \wedge y\,R\,x) \to x = y)$ "relations are one-sided"
Transitive: $(\forall x, y, z \in A)((x\,R\,y \wedge y\,R\,z) \to x\,R\,z)$ "relations concatenate linearly"
Trichotomy $(\forall x, y \in A)(x = y \vee x\,R\,y \vee y\,R\,x)$ "all sets are related"

We call a relation an **equivalence relation** if it is reflexive, transitive and symmetric. Equivalence relations are used to create groups of elements with some shared property. Some examples include congruent modulo 3 and the existence of a bijection between two sets.

A relation is a **partial ordering on A** if it is reflexive, transitive and antisymmetric. With a partial ordering one can think of the relation $\leq$ on the natural numbers inclusion on sets, '$\subseteq$' and the divisibility of integers, 'a divides b'. Note that for inclusion and divisibility not every two elements are related; two sets can be disjoint and 2 does not divide 3. Partial orders and equivalence relations will become more important in chapter 3.

Lastly, if a relation R is irreflexive, transitive and satisfies trichotomy, we call it a **total or linear ordering on A**. We will denote the pair consisting of a linear order R on the set A with $\langle A, R\rangle$. An example of a linear order is the normal order on $\mathbb{N}$. A linear ordering $\langle A, R\rangle$ is called a **well-order** if it is also well-founded. That is every non-empty subset of A has an R-least element. In symbols this boils down to the following for a linear ordering R:
well-founded: $(\forall B)((B \subseteq A \wedge B \neq \varnothing) \to (\exists x \in B)(\forall y \in B)(x = y \vee x\,R\,y)$

Let us give an example of a well-ordered set.

Example. *Consider the set of integers $\mathbb{Z}$ and the standard ordering '$<$'. This ordering is clearly linear, however not even $\mathbb{Z}$ itself has a minimal element. A well-order R can made by setting $x\,R\,y$ if:*

(1) $x = 0$ *and* $y \neq 0$
(2) $0 < x$ *and* $y < 0$
(3) $0 < x < y$
(4) $y < x < 0$ *(and thus $0 < |x| < |y|$ with $|\cdot|$ the absolute value)*

Before we continue, we make some definitions

Definition 2.1. *Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be well-ordered sets. We call a function $f : A \longrightarrow B$ an* ***isomorphism*** *if it is an order preserving bijection, i.e. a bijection where if $x, y \in A$ with $x\,R\,y$ then also $f(x)\,S\,f(y)$. If such a function exists, we call A and B* ***isomorphic*** *and write $A \cong B$*

Definition 2.2. *Let $\langle A, R\rangle$ be a linear order and $x \in dom(R)$, we define the set of its 'predecessors' ('voorgangers' in Dutch) as*

$$\mathrm{vg}(A, x, R) := \{y \in A \mid y\,R\,x\}$$

When x is an elements of A itself this is called an ***initial segment of A****. If from the context it is clear which relation is used, we will write $\mathrm{vg}(A, x)$ instead of $\mathrm{vg}(A, x, R)$.*

Well-ordered sets are closely related as shown by the following theorem from the book by Azriel Levy [4]:

Theorem 2.3. *If $\langle A, R\rangle$ and $\langle B, S\rangle$ are well-ordered sets then exactly one of the following holds:*

- $A \cong B$
- *A is isomorphic to a unique initial segment of B*
- *B is isomorphic to a unique initial segment of A*

It can be seen that well-orderable sets can be compared via these isomorphisms. However, this does give much useful to work with. For this we will be looking into ordinal numbers.

2.2. **Ordinal numbers.**
We will now look at special kind of sets called **ordinal numbers**. What is so special about ordinal numbers is that they can be used to uniquely describe the order of well-orderable sets. We will take a closer look at the ordering induced by '$\in$' (this can be identified by $\in_A = \{\langle x, y\rangle \in A \times A \,|\, x \in y\}$). It is clear that this relation is irreflexive on any set. For transitivity of the relation we look at a special property of a set.

Definition 2.4. *We call a set A **transitive** if every element of the set is also a subset. In terms of our first-order theory:* $(\forall x, y)((y \in x \wedge x \in A) \to y \in A)$.

A few transitive sets we can easily construct are $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$.

Definition 2.5. *An **ordinal number** is a transitive set which is well-ordered by $\in$.*

Of the four examples of transitive sets above, only the first three are ordinal numbers as the ordering of the fourth set is not transitive, nor has trichotomy.

To talk about sets of ordinal numbers we define the class of ordinal numbers as the collection

$$\mathbf{ON} := \{x \,|\, x \text{ is an ordinal number}\}$$

. We will prove that **ON** is well-ordered by $\in$ and simultaneously prove some important properties of ordinal numbers. We will first show that **ON** itself is transitive.

Proposition 2.6. *Every element of an ordinal number is an ordinal number itself.*

Proof. Let $x \in \mathbf{ON}$ and $y \in x$. It is clear that y is well-ordered by '$\in$', so we only need to prove that y is a transitive set.

Let $z \in y$ and $u \in z$. By transitivity of x we have that $u, z \in x$. Now applying the transitivity of the relation on x, we get $u \in y$. □

Lemma 2.7. *For all ordinals x, y and z, if $x \in y$ and $y \in z$, then $x \in z$.*

Proof. This follows from transitivity of the set z. □

This ensures that **ON** is a transitive class. Next, we will prove that $\langle \mathbf{ON}, \in\rangle$ has trichotomy. For this we will first prove some uniqueness of ordinals.

Lemma 2.8. *Let $x \in \mathbf{ON}$. If $y \in x$, then $y = \mathrm{vg}(x, y)$.*

Proof. From the definition of $\mathrm{vg}(x, y)$ and transitivity of the set x it follows that $\mathrm{vg}(x, y) = y \cap x = y$. □

If we look at the minimal element of an ordinal number, we arrive at a remarkable property:

Corollary 2.9. *If $x \in \mathbf{ON}$ is non-empty then $\varnothing \in x$*

Continuing this we find that some of the smallest ordinals are $\varnothing$, $\{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$. As mentioned earlier, ordinal numbers should uniquely represent well-orderable sets. Different ordinal numbers should hence not be isomorphic.

Proposition 2.10. *If for two ordinal numbers x, y we have $x \cong y$, then $x = y$.*

Proof. Suppose $A, B \in \mathbf{ON}$ are isomorphic and nonempty. Suppose $A \neq B$, without loss of generality (WLOG) we can assume that $A - B \neq \varnothing$. By well-foundedness we can assume that $A - B$ has a minimal element x, then also $x \subseteq B$ by minimality of x. If we would have $x \neq B$, then for $u \in B - x$ minimal we would have $u \subseteq x$ but also for all $z \in x$, $z \in u$ as we can't have $u = z$ or $u \in z$ as this implies that $u \in x$. This would be contradictory as now $x = u \in B$.

Either way, we find $B \subsetneq A$ which gives a contradiction with theorem 2.3. □

Proposition 2.11. *Let $x, y \in \mathbf{ON}$. Then exactly one of the following holds:*

- $x = y$
- $x \in y$
- $y \in x$

Proof. That at most one holds is clear. The remainder is a direct consequence of theorem 2.3 and proposition 2.10. □

The only thing that remains to be proved is the well-foundedness of **ON**.

Proposition 2.12. *Let $C \subseteq ON$ be non-empty, then C has a minimum.*

Proof. Let $z \in C$ be arbitrary. If $C \cap z = \{x \in C \mid x \in z\} = \varnothing$, we must have that z is the minimal element. Now, suppose $C \cap z \neq \varnothing$ and let x be the minimal element of this set. Let $y \in C$ be arbitrary. As trichotomy holds we must have one of the following:

- $y = x$
- $y \in x$
- $x \in y$

If we have $y \in x$, then also, by transitivity of the ordinal z, $y \in z$. This contradicts the minimality of x in z. Either way, we find a minimal element of C. □

As the minimum is unique, we can define the **minimum of A** to be the set $\min(A)$. In this fashion we define two more important elements, the supremum and least strict upperbound.

Definition 2.13. *Let A be a non-empty set of ordinals.*

$$\begin{aligned} \sup(A) &:= \min(\{a \mid (\forall x \in A)(x \in a \vee x = a)\}) \\ \sup^+(A) &:= \min(\{a \mid (\forall x \in A)(x \in a)\}) \end{aligned}$$

Note that the supremum can be an element of the set itself, $\sup^+$, however, can never be an element of the set. Also note that $\sup(A) = \bigcup A$.

We found that **ON** is transitive and is well-ordered by $\in$. So it behaves as a ordinal itself, but **ON** can not be a set as is proved with the next lemma.

Lemma 2.14. **ON** *is not a set.*

Proof. Suppose **ON** is a set. As we showed that it is transitive and well-ordered by $\in$, we get that **ON** is an ordinal number itself. But that would mean that **ON** $\in$ **ON** which gives a contradiction. □

As **ON** is well-ordered we get the following corollary

Corollary 2.15. *Every transitive set of ordinals is an ordinal itself.*

Now that the basic properties of ordinal numbers are clear, we can show its importance.

Theorem 2.16. *Every well-ordered set is isomorphic to a unique ordinal.*

Proof. Uniqueness follows from Proposition 2.10. Let $\langle A, R\rangle$ be a well-ordered set. We first prove that for all $y \in A$ there is some ordinal α isomorphic to $\mathrm{vg}(A, y, R)$.

Suppose not, then there is a minimal y for which $\langle \mathrm{vg}(A, y, R), R\rangle$ is not isomorphic to any ordinal. Set the set of elements of A related to this y as $Y := \mathrm{vg}(A, y, R)$. Note that this set is also well-ordered as every element of Y is an element of A. Every element of Y is isomorphic to an unique ordinal. By the Axiom schema of Replacement (see page 31 in the appendix) with formula ϕ meaning $\mathrm{vg}(Y, x, R) \cong \alpha$, we get some set of ordinals B of which each ordinal is isomorphic to a initial segment given by some $x \in Y$. This set B must also be transitive and hence is an ordinal. But this gives a contradiction, as B must be isomorphic to $\langle Y, R\rangle$.

Repeating the proof above for gives $\langle A, R\rangle$ proves the theorem. □

This theorem makes ordinal numbers live up to their name. An ordinal can be used to uniquely describe the ordering on a well-ordered set $\langle A, R\rangle$. That is if we want to say something about a well-ordered set we can pretend it to be the set of its corresponding ordinal number. We will denote the unique ordinal corresponding to $\langle A, R\rangle$ by $\mathrm{Ord}(\langle A, R\rangle)$. If we, moreover, assume the Axiom of Choice, the well-ordering theorem states that every set has a well-ordering.

Remark. *In the remainder of this report Greek letters are ordinal numbers and we will write $\alpha < \beta$ for $\alpha \in \beta$. Furthermore, we write $\alpha \leq \beta$ for $\alpha < \beta \vee \alpha = \beta$.*

With this theory for ordinal numbers we can prove some fundamental theorems of mathematics.

2.3. **Induction and recursion.**

Two of the most used principles in mathematics are induction and recursion. Any mathematician has learned of the basic, finite, induction (also known by some as 'baby'-induction). This goes as follows: We first prove a formula $\phi(n)$ for some n. Secondly, we prove that if $\phi(m)$ holds, then also $\phi(m+1)$ for any $m > n$ If we have proven both these claims, we may conclude that $\phi(m)$ holds for any $m \geq n$. This method must be getting a bit boring and limited, thus we will extend it up to transfinite induction.

Theorem 2.17. ***Transfinite induction*** *Let $\phi(\alpha)$ be some formula. If we prove that $\phi(0)$ holds and also that for all α we have*

$$(\forall\beta < \alpha)(\phi(\beta)) \to \phi(\alpha),$$

we may conclude $(\forall\alpha \in \mathbf{ON})(\phi(\alpha))$.

Proof. Let $\mathbf{O} = \{\alpha \,|\, \phi(\alpha)\}$ and suppose $0 \in \mathbf{O}$. We prove that $\mathbf{O} = \mathbf{ON}$. The formula $(\forall\beta < \alpha)(\phi(\beta)) \to \phi(\alpha)$ can be rewritten as $(\forall\beta < \alpha)(\beta \in \mathbf{O}) \to \alpha \in \mathbf{O}$. Suppose there is some ordinal which is not an element of $\mathbf{O}$. Let α be the minimal ordinal not in $\mathbf{O}$. This gives a contradiction as now $\alpha \neq 0$ and $(\forall\beta < \alpha)(\beta \in \mathbf{O})$. We must have that $\mathbf{O} = \mathbf{ON}$ and, therefore, we have $(\forall\alpha)(\phi(\alpha))$. □

Using isomorphisms from any set onto some ordinal, we moreover get the following corollary.

Corollary 2.18. *Transfinite induction works for any well-orderable set.*

Recursion can also use some dusting so lets extend it as well. The proof can be found in any textbook on ordinals. Here $\mathbf{V} = \{x \,|\, x = x\}$ is called the universal set and contains all sets. $\mathbf{V}$ itself is not a set itself as is proven in the appendix.

Theorem 2.19. ***Transfinite recursion on ON*** *If $F : \mathbf{V} \longrightarrow \mathbf{V}$, then there is a unique $G : \mathbf{ON} \longrightarrow \mathbf{V}$ such that $(\forall\alpha)(G(\alpha) = F(G \restriction \alpha))$,*

This represents that we can define an operation recursively. We define the starting operation and then define how the operation depends on the predecessing function values.

2.4. Natural numbers.

In Corollary 2.9 we found that for every non-empty ordinal the empty set is an element. From this we concluded that $\varnothing$ and $\{\varnothing\}$ are the two smallest ordinals. To refrain from working with sets of sets of empty sets we give them the names $0 = \varnothing$ and $1 = \{\varnothing\}$. As the names suggest we will also define the next, larger, ordinals.

Definition 2.20. *For an ordinal α its successor is $\alpha \cup \{\alpha\}$, which we will denote by $\alpha + 1$. We call α a **successor ordinal** if there is a β with $\alpha = \beta + 1$, otherwise α is called a **limit ordinal**.*

To show that this definition indeed gives the next, larger, ordinal, we must first agree that $\alpha \cup \{\alpha\}$ is again an ordinal. For this we simply note that $\alpha \cup \{\alpha\}$ is transitive and apply corollary 2.15.

Proposition 2.21. *$\alpha + 1$ is the successor of α in the sense that there is no γ such that $\alpha < \gamma < \alpha + 1$*

Proof. Suppose there is such a γ. From $\gamma < \alpha \cup \{\alpha\}$ it follows that $\gamma \in \alpha \vee \gamma = \alpha$. Furthermore, as we have $\alpha \in \gamma$, we have a contradiction with the trichotomy of $\mathbf{ON}$ (2.11). □

The definition of successor ordinals works as was intended. The first ordinal numbers we can construct as successors will be called the **natural numbers**.

Definition 2.22. *An ordinal α is a natural number if it contains only successor ordinals and is a successor itself, or*

$$(\forall\beta \leq \alpha)(\beta = 0 \vee (\exists\gamma < \beta)(\beta = \gamma + 1)).$$

We denote the collection of natural numbers as $\omega = \{x \,|\, x \text{ is a natural number }\}$.

Corollary 2.23. *If α is a natural number, then also $\alpha \subset \omega$.*

This corollary represents that the natural numbers are an initial segment of $\mathbf{ON}$. As ω is a transitive collection of ordinals, the question arises if it is an ordinal itself. To show that we must use an axiom from $\mathbf{ZF}$ (see page *Axiom 7*). We rewrite it in our new notation (for non-ordinals the definition of successor is analogous).

Axiom 7 Infinity: There is a set containing the empty set and for every element its successor:

$$(\exists x)(0 < x \wedge (\forall y < x)(y + 1 < x))$$

.

Theorem 2.24. *ω is an ordinal.*

Proof. Let x be a set from the Axiom of Infinity. We first prove that x contains all natural numbers.

Suppose there is a natural number n that is not in x. By definition of x, $n \neq 0$ and thus there is a m with $m + 1 = n$. We must then have that $m \notin x$ and therefore $n - x \neq 0$. Now, we look at the minimal element of $n - x$, which we will call k. Again, as $k \neq 0$ by definition of x, there is an l with $l + 1 = k$. By minimality of k we have $l < x$ which gives a contradiction.

We know for sure that x contains all natural numbers, so we can take $x \cap \mathbf{ON}$.

Using the Axiom schema of Separation (see page 29 in the appendix) we set $\omega = \{y \in x \mid y \text{ is a natural number}\}$. As ω is a transitive set of ordinals the claim follows from corollary 2.15. $\square$

We now have the first couple of ordinal numbers we can make, including the first limit ordinal, ω. We are in a position to define what it means for a set to be **finite** or countable.

Definition 2.25. *A set A is called finite if there exists a $n < \omega$ and a bijection $f : n \longrightarrow A$. Otherwise it is called infinite. A set A is called* ***countable*** *if there is a bijection $f : \alpha \longrightarrow A$ for some $\alpha \leq \omega$. Otherwise it is called uncountable.*

This definition gives some sense of the size of a set. This idea can be expanded upon, which is the theme of the next section.

3. Cardinal numbers

In this chapter we will look at some special ordinal numbers called cardinal numbers. These numbers represent the size or "the number of elements" of a set. For this we first briefly expand our theory for equivalence classes. We will then look at the cardinal numbers and define arithmetics on these. Lastly, a special kind of cardinals called regular cardinals will be looked into.

3.1. **Equivalence classes.**

In subsection 2.1 we defined what an equivalence relation is; a reflexive, transitive and symmetric relation. In this subsection we will look at the properties of such a relation.

Due to the properties of an equivalence relation, we see that elements which are related closed groups under the relation. With this is meant that two different such groups have no elements related with each other. Such an decomposition into disjoint groups is called a partition.

Definition 3.1. *A **partition** of a set A is a collection $\mathcal{F}$ of pairwise disjoint sets such that $\bigcup \mathcal{F} = A$*

Here **pairwise disjoint** means that for all $A, B \in \mathcal{F}$ their intersection is empty. We can think of an equivalence relation as a relation that chops the set into smaller sets of elements with some shared property. These smaller sets are called **equivalence classes**. We will give a few examples to illustrate various equivalence relations and classes.

Example.

- *"Has the same elements" on **V**. Every set forms an equivalence class on its own.*
- *"Is isomorphic to" on well-ordered sets. Every ordinal represents an equivalent class.*
- *"Has the same birthday as" on the group of people. Every date is a equivalence class.*
- *"Is congruent to modulo n", for some natural number n, on the natural numbers. Every distinct number below n represents a different equivalence class.*

For an element a of a well-ordered set $\langle A, R\rangle$ we denote the equivalence class to which a belongs by $[a] = \{x \in A \mid x\, R\, a\}$.

3.2. **Cardinal numbers.**

The **cardinal numbers** are special ordinal numbers that represent the size or amount of elements of a set. For finite sets the size is intuitively clear; one can just "count" the number of elements by hand. For infinite sets, however, it is not always clear how many elements the set really has. For example, if we take the set of natural numbers ω and add one element to it, e.g. take $\omega + 1$, is there really a difference? To answer this question We expand on the idea developed in definition 2.25.

Definition 3.2. *For a well-orderable set A its **cardinality** is the smallest ordinal α with a bijection to A. This ordinal will be denoted by $|A|$. A **cardinal number** is an ordinal α with $\alpha = |\alpha|$*

For the sake of compactness we will introduce some notations.

Definition 3.3. *Let A and B be two sets.*
$A \approx B$ means there is bijection from A to B.
$A \preccurlyeq B$ means there is an injection from A to B.
$A \prec B$ means there is an injection from A to B but no bijection.

It should be clear that $\approx$ defines an equivalence relation on $\mathbf{V}$.

Remark. *Using this notation we can rewrite the definition of the cardinality of a set:*

$$|A| = \min(\{\alpha \mid \alpha \in ON \wedge \alpha \approx A\}) = \min([A] \cap \boldsymbol{ON})$$

The cardinality of a set is therefore the minimum ordinal of the equivalence class generated by $\approx$.

The following result from Cantor, Bernstein, Schröder and Dedekind, which we will give without proof, gives an important relation between $\preccurlyeq$ and $\approx$.

Theorem 3.4 (The Cantor-Bernstein-Schröder Theorem)**.** *If $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a \approx b$.*

The converse of this theorem is clearly also true. With this result we deduce that $\preccurlyeq$ defines a partial order on the equivalence classes of $\approx$ (equality '=' means '$\approx$' in this context). As every equivalence class contains one cardinal, we think of the cardinal as the **representative** of this class.

We can now find our first cardinal numbers.

Proposition 3.5. *ω is a cardinal and every $n \in \omega$ is a cardinal.*

We first prove the following lemma.

Lemma 3.6. *If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.*

Proof. Note that $\beta \leq \alpha$ implies that $\beta \preccurlyeq \alpha$. Furthermore, as we have that $\alpha \approx |\alpha|$, there is a bijection $f : \alpha \longrightarrow |\alpha|$. From $|\alpha| \leq \beta$ we find that there is an injection $g : |\alpha| \longrightarrow \beta$. We get that $g \circ f$ is an injection from α into β, thus $\alpha \preccurlyeq \beta$. From the Cantor, Bernstein and Schröder Theorem (3.4) we conclude that $|\alpha| \approx \alpha \approx \beta \approx |\beta|$. □

Proof. Of proposition 3.5. We first prove by induction that every natural number is a cardinal. That is, we prove $(\forall n \in \omega)(n \not\approx n+1)$. It is easy to see that there is no bijection between 0 and 1. Let $n \in \omega$ and suppose $(\forall k < n)(k \not\approx k+1)$, we show that then also $n \not\approx n+1$.

Suppose $n \approx n+1$, then there is a bijection $f : n \longrightarrow n+1$. Thus, for every $l < n+1$ there is a unique $k \in n$ for which $f(k) = l$. If we look at the set m with $m+1 = n$, we can WLOG assume that $f(m) = n$ as if $f(k) = n$ for some $k \neq m$ we can take the bijection $g : n \longrightarrow n+1$ defined by:

$$g(l) = \begin{cases} f(l) & \text{if } l \neq k, m \\ f(k) & \text{if } l = m \\ f(m) & \text{if } l = k \end{cases}$$

But this gives a contradiction, as $g \restriction m$ is a bijection onto n and hence $m \approx n$. Using induction (2.17), we conclude that there is no bijection from a natural number to its successor.

By lemma 3.6 and the fact that $|\alpha| \leq \alpha$ for any ordinal, we moreover conclude that every natural number is a distinct cardinal. As the only ordinals smaller than ω are the natural numbers, we get as a corollary that ω is also a cardinal. □

We thus find our first cardinals to be the same as our first ordinals. To answer the question asked at the beginning of this subsection; no. One extra element does not make an infinite number of elements larger. A bijection from ω to $\omega+1$ is quickly made: set $f(0) = \omega$ and for every $n \in \omega$ set $f(n)$ to be the unique m with $m+1 = n$.

Creating even larger cardinals will be the theme of the next subsection.

3.3. **Cardinal arithmetic.**

First, we quickly note what is meant by a **sequence** and its limit.

Definition 3.7. *A sequence is a function on the ordinals $s : \alpha \to \mathbf{ON}$ where α is also an ordinal. We denote this by $\langle a_\epsilon \,|\, \epsilon < \alpha\rangle$ where $a_\epsilon = s(\epsilon)$. The collection of all sequences from an ordinal α into another ordinal β is denoted by ${}^\alpha\beta$. That is, ${}^\alpha\beta = \{s \,|\, s : \alpha \longrightarrow \beta\}$.*
*For non-decreasing sequences we define the **limit of a sequence** by $\lim_{\xi\to\alpha} a_\xi = \sup\{a_\xi \,|\, \xi < \alpha\}$. Lastly, we call a sequence **continuous** if for every limit α, $a_\alpha = \lim_{\xi\to\alpha} a_\xi$*

Definition 3.8. *Let κ and μ be two cardinals. We define*

Addition:	$\kappa + \mu$	$= \|\kappa \times \{0\} \cup \mu \times \{1\}\|$
Multiplication:	$\kappa \cdot \mu$	$= \|\kappa \times \mu\|$
Exponentiation:	κ^μ	$= \|{}^\mu\kappa\|$

Most of the properties of 'regular' arithmetics also hold for the cardinal arithmetic. A few of these properties which can be proven constructing simple functions are:

Lemma 3.9.

- *Addition and multiplication are associative, commutative and distributive.*
- $(\kappa \cdot \mu)^\lambda = \kappa^\lambda \cdot \mu^\lambda$.
- $\kappa^{\mu+\lambda} = \kappa^\mu \cdot \kappa^\lambda$.
- $(\kappa^\mu)^\lambda = \kappa^{\mu\cdot\lambda}$

Another important property that is preserved is the following.

Proposition 3.10. *Let κ be any cardinal, then $\kappa < 2^\kappa = |\mathscr{P}(\kappa)|$.*

Proof. We first prove that $2^\kappa = |\mathscr{P}(\kappa)|$. Define the function $f : \mathscr{P}(\kappa) \longrightarrow {}^\kappa 2$ as $f(X) = 1_X$ where 1_X is the indicator function;

$$1_X(x) := \begin{cases} 1 \text{ if } x \in X \\ 0 \text{ if } x \notin X \end{cases}$$

This function is a bijection between $\mathscr{P}(\kappa)$ and ${}^\kappa 2$.

For the inequality $\kappa < 2^\kappa$ we consider any function $f : \kappa \longrightarrow \mathscr{P}(\kappa)$ and the set $A = \{\alpha < \kappa \mid \alpha \notin f(\alpha)\}$. This is clearly a subset of κ, thus $A \in \mathscr{P}(\kappa)$. But this set can not be in the range of the function, i.e. there is no $\alpha < \kappa$ with $f(\alpha) = A$ as otherwise, whether or not $\alpha \in A$, we get a contradiction. We get that any $f : \kappa \longrightarrow \mathscr{P}(\kappa)$ can not be surjective. To see that there is an injection, one can take the function $g : \kappa \longrightarrow \mathscr{P}(\kappa)$ mapping each point to its singleton; $g(x) = \{x\}$. Therefore, we have $\kappa < |\mathscr{P}(\kappa)| = 2^\kappa$. □

The power set is thus certainly larger than the original set. Now using this we know that for every cardinal, κ, there is some cardinal larger than κ. Then there must also be a smallest, larger cardinal. Using the Well-ordering Theorem , we can enumerate the infinite cardinals. As the smallest infinite cardinal is ω, we index it as ω_0. From here the next cardinal number will be denoted by ω_1. For limit ordinals α we set $\omega_\alpha = \sup(\{\omega_\beta \mid \beta < \alpha\})$. Generally for an infinite cardinal $\kappa = \omega_\alpha$ we will write $\kappa^+ = \omega_{\alpha+1}$ for the next, larger cardinal.

Remark. *In introductory courses the reader must have heard of the number $\aleph_0$. This number being the cardinality of the natural numbers, we thus have $\aleph_0 = \omega_0$. The alephs are another notation for infinite cardinals and are mostly used when only considering the cardinality of sets.*

As the inequality $\kappa < 2^\kappa$ might not have been too shocking, the following theorem does give a difference between the standard and these infinite arithmetics. The proof of this theorem can be found in many textbooks on basic cardinal numbers.

Theorem 3.11. *Let $\kappa \geq \omega$ be any cardinal, then $\kappa \cdot \kappa = \kappa$.*

Corollary 3.12. *Let $\kappa \geq \omega$ and $\lambda \leq \kappa$ be cardinals, then $\kappa \cdot \lambda = \kappa$*

Proof. Follows from the definition of cardinal multiplication and $\kappa \times \lambda \subseteq \kappa \times \kappa$ □

An infinite cardinal κ is thus so large that κ copies of itself are still as 'large' as one.

But how many smaller sets do you need to "fill" a cardinal? If we for example take the ordinal ω we always have to use ω many smaller sets to fill it. For ω_ω, however, we can also fill it with ω sets of cardinality smaller than ω_ω. For example ω_ω can be made by the union of the set $\{\omega_\alpha \mid n < \omega\}$. This notion will be expanded on in the next subsection.

3.4. **Regular cardinals.**

Lastly, we discuss another measure of size, the **cofinality** of a set. The cofinality of a set can be seen as the smallest size of a set containing arbitrarily large elements or the least amount of elements needed to fill a set. With this in mind we define such a set containing arbitrarily large elements.

Definition 3.13. *Let A be a set of ordinals. $B \subset A$ is said to be **cofinal** in A if B contains maximal elements of A. Or in notation: $(\forall x \in A)(\exists y \in B)(x \leq y)$.*
*We define the **cofinality** of A by*

$$\mathrm{cf}(A) = \min\{\mathrm{Ord}(B) \mid B \textit{ is cofinal in } A\}$$

Corollary 3.14. *Let κ be an ordinal, then the following are equivalent:*

(1) $\mathrm{cf}(\kappa) = \alpha$
(2) α *is the minimal ordinal such that a non-decreasing sequence* $\langle \beta_\xi \mid \xi < \alpha \rangle$ *exists in* κ *with* $\lim_{\xi \to \alpha} \beta_\xi = \kappa$

Both of these characterizations should be remembered as they will be used often in the remainder of this thesis.

It is easy to see that if C is cofinal in B and B is cofinal in A, then C must also be cofinal in A. Hence, the cofinality of a set really gives a minimum, and for this minimum must thus hold that its cofinality is itself. We will give such an ordinal a special name.

Definition 3.15. *An ordinal α is called **regular** if it is equal to its cofinality, that is* $\mathrm{cf}(\alpha) = \alpha$

A noticeable property is that the cofinality is always a regular cardinal.

Lemma 3.16. *Let α be any ordinal, then* $\mathrm{cf}(\alpha)$ *is a regular cardinal.*

Proof. Suppose $\kappa = \mathrm{cf}(\alpha)$ is not a cardinal, we prove it is not regular. Take a bijection $f : \kappa \longrightarrow |\kappa|$ and a cofinal sequence $\langle \beta_\xi \mid \xi < \kappa \rangle$ in α. From this we define a new sequence in α, $\langle \gamma_\xi \mid \xi < |\kappa| \rangle$, by recursion.

- $\gamma_0 = \beta_0$
- $\gamma_{\xi+1} = \sup(\{\gamma_\xi + 1, \beta_{f(\xi)} + 1\})$
- $\gamma_\xi = \sup(\lim_{\mu \to \xi} \gamma_\mu + 1, \beta_{f(\xi)} + 1\})$ for limit ξ

One important thing to note is that $\lim_{\mu\to\xi}\gamma_\mu < \kappa$ for limit ξ as we assumed that $\mathrm{cf}(\alpha) = \kappa > |\kappa| \geq \xi$. Now γ_ξ is non-decreasing and thus its limit must be the same as that of β_ξ which gives a contradiction proving that $\mathrm{cf}(\alpha)$ is a cardinal. For regularity we note that $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$. $\square$

So every regular ordinal must be a cardinal, therefore we will use the term regular cardinal as a synonym for regular ordinal. Next, we will prove maybe the most important property of regular cardinals.

Theorem 3.17. *Let $\kappa \geq \omega$ be a cardinal, then the following are equivalent:*

(1) κ is regular
(2) A set of cardinality κ is not the union of fewer than κ sets of cardinality less than κ

we first prove a small lemma.

Lemma 3.18. *let κ be a cardinal, then the following are equivalent:*

(1) κ is the union of fewer than κ smaller sets
(2) A set of cardinality κ ***is*** *the union of fewer than κ sets of cardinality less than κ*

Proof. Consider a bijection $f : \kappa \longrightarrow A$ and let $\mu < \kappa$. Suppose $A = \bigcup\{A_\xi \,|\, \xi < \mu\}$ where each A_ξ is of cardinality less than κ. Then $\kappa = f[A] = \bigcup\{f[A_\xi] \,|\, \xi < \mu\}$ giving that κ must also be the union of fewer than κ smaller sets as f is bijective. The other case proceeds similarly. $\square$

Proof. of theorem 3.17. By the lemma above we only need to prove the case $A = \kappa$ for (2).

(2)$\to$(1): We prove the contrapositive. Thus we assume that κ is not regular, hence $\mathrm{cf}(\kappa) = \mu < \kappa$. Consider some cofinal sequence $\langle\beta_\xi \,|\, \xi < \mu\rangle$ of κ, then $\kappa = \lim_{\xi\to\mu}\beta_\xi = \bigcup\{\beta_\xi \,|\, \xi < \mu\}$. Thus we find that κ is the union of fewer than κ smaller sets.

(1)$\to$(2): We follow the proof by Jech [3]. We again prove the contrapositive. Let μ be the minimal cardinal such κ is the union of μ sets of cardinality less than κ. Write $\kappa = \bigcup\{\alpha_\xi \,|\, \xi < \mu\}$ and consider the cofinal sequence in κ of length μ, $\langle\kappa_\xi \,|\, \xi < \mu\rangle$ defined by $\kappa_\xi = \sup\bigcup\{\alpha_\nu \,|\, \nu < \kappa\}$. From this we construct a sequence $\langle\alpha_\xi \,|\, \xi < \mu\rangle$ by setting $\alpha_\xi = \mathrm{Ord}(\bigcup\{\kappa_\nu \,|\, \nu < \xi\})$. It is clear that this sequence is non-decreasing. Furthermore, we have $\alpha_\xi \leq \kappa$ as $\bigcup\{\kappa_\nu \,|\, \nu < \xi\} \subseteq \kappa$. We even have strict inequality, consider the case that $\alpha_\xi = \kappa$ for some $\xi < \mu$. Then, for the minimal such ξ, the set $B = \bigcup\{\alpha_\nu \,|\, \nu < \xi\}$ is again a set of cardinality κ with elements of cardinality less than κ. This contradicts the minimality of μ, therefore we must have $\alpha_\xi < \kappa$ for all $\xi < \mu$.

We still have to prove that the limit of the sequence $\alpha := \lim_{\xi\to\mu}\alpha_\xi$ is equal to κ. As $\alpha_\xi < \kappa$ for all $\xi < \mu$, we must have $\alpha \leq \kappa$. Now consider the function $f : \kappa \to \mu \times \alpha$ where $f(\beta) = \langle\xi, \gamma\rangle$ with ξ the minimal ordinal such that $\beta \in \kappa_\xi$ and $\gamma = \mathrm{Ord}(\kappa_\xi \cap \beta)$. This function is injective. If we were to have two ordinals $\beta < \beta'$ with $f(\beta) = \langle\xi, \gamma\rangle$ and $f(\beta') = \langle\xi, \gamma'\rangle$, then $\gamma = \mathrm{Ord}(\kappa_\xi \cap \beta) < \mathrm{Ord}(\kappa_\xi \cap \beta') = \gamma'$. As we have an injection from κ into $\mu \times \kappa$, we have $\kappa \leq |\mu \times \alpha| = \mu \cdot |\alpha|$. Since $\mu < \kappa$ we, furthermore, have $\alpha \geq |\alpha| = \mu \cdot |\alpha| \geq \kappa$ proving the claim. $\square$

Corollary 3.19. *If κ is a regular cardinal and $A \subset \kappa$ then "A is cofinal in κ" and "A has cardinality κ" are equivalent.*

This theorem and corollary state that a regular cardinal can not be broken down into a small number of smaller chunks. That is, when constructing a regular cardinal from non-cofinal sets, you have to take as many as there are points.

An example of what we learn from this theorem is the following:

Corollary 3.20. *Every cardinal of the form $\omega_{\alpha+1}$, is regular.*

Proof. Let $f : \omega_\alpha \to \omega_{\alpha+1}$, then every $f(\alpha)$ is of cardinality at most α. Furthermore, $\mathrm{ran}(f)$ is a collection with at most ω_α elements. We prove that $B := \sup(\mathrm{ran}(f)) = \bigcup\{\mathrm{ran}(f)\}$ has cardinality at most ω_α proving that all cofinal sequences must have cardinality $\omega_{\alpha+1}$.

For each $\xi < \omega_\alpha$ choose a surjective function $f_\xi : \omega_\alpha \longrightarrow \xi$. and define $g : \omega_\alpha \times \omega_\alpha \longrightarrow B$ by $g(\xi, \nu) = f_\xi(\nu)$. This function g is again surjective as f is surjective. Therefore, it follows that $|B| \leq |\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ $\square$

Another property that will be useful for us is that a continuous and increasing sequence preserves cofinality.

Proposition 3.21. *Let $\langle\beta_\xi \,|\, \xi < \alpha\rangle$ be a continuous and strictly increasing sequence. If $\nu < \alpha$ is a limit ordinal, then $\mathrm{cf}(\beta_\nu) = \mathrm{cf}(\nu)$.*

Proof. As the sequence is continuous, the subsequence $\langle \beta_\xi \,|\, \xi < \nu \rangle$ is cofinal in β_ν. We get that $\mathrm{cf}(\beta_\nu) \leq \mathrm{cf}(\nu)$. Define the function $g : \beta_\nu \longrightarrow \nu$ by

$$g(\alpha) := \min(\{\xi \,|\, \alpha \leq \beta_\xi\})$$

Now for $\alpha \in \kappa$, $g(\beta_\alpha) = \alpha$ as for $\xi < \alpha$ we must have that $\beta_\xi < \beta_\alpha$. We find that $\langle g(\alpha) \,|\, \alpha < \beta_\nu \rangle$ is a cofinal sequence in ν. We get that now also $\mathrm{cf}(\nu) \leq \mathrm{cf}(\beta_\nu)$, thus $\mathrm{cf}(\beta_\nu) = \mathrm{cf}(\nu)$. $\square$

4. Stationary sets

In the remainder of this thesis κ will be a regular uncountable cardinal. In this section we define closed unbounded sets and stationary sets. Also the most important properties and theorems of these are derived. Stationary sets are analogous to sets of non-zero measure in measure theory and are a tool that are often used in proofs. To give an example of a use we will derive the Δ-lemma for regular cardinals in subsection 4.3.

4.1. **Closed and unbounded sets.**

One characterization of a closed set is that such a set contains all its limit points. Thus for any sequence of points from the set, the limit must also be in said set. As **ON** is well-founded, and thus has no infinitely decreasing sequences, the only interesting limits are those of limit ordinals. Hence we define **limit points** as such.

Definition 4.1. *Let A be a set of ordinals and ξ an ordinal. ξ is called a **limit point of A** if it is a limit ordinal and arbitrarily large ordinals below ξ belong to A.*
In logical terms: $(\forall\alpha < \xi)(\exists\beta)(\alpha < \beta < \xi \wedge \beta \in A)$ or, equivalently, $sup(A \cap \xi) = \xi$

Definition 4.2. *Let C be a subset of κ.*
*C is called **closed** if it contains all its limit points smaller than κ.*
*C is called **unbounded in κ** if it is not bounded in κ (equivalently cofinal in κ).*

We call closed unbounded sets **club sets** and with this we let **Club(κ)** be the set of closed unbounded subsets of κ. An important property of club sets is that every intersection of fewer than κ club sets is again a club set. For this we first prove a lemma.

Lemma 4.3. *If C and D are club sets, then also $C \cap D$ is closed unbounded.*

Proof. We first prove that it is closed. Let ξ be a limit point of $C \cap D$. Then ξ must also be a limit point of both C and D. As both C and D are closed, $\xi \in C$ and $\xi \in D$ which implies ξ is in the intersection.

To show that $C \cap D$ is unbounded, we construct an increasing sequence. Let $\alpha < \kappa$, since C is unbounded there is some $\alpha_0 > \alpha$ in C. Similarly, there is an $\alpha_1 > \alpha_0$ in D. Continuing this, one finds an increasing sequence $\langle \alpha_n \,|\, n < \omega \rangle$ with $\alpha_0, \alpha_2, \ldots \in C$ and $\alpha_1, \alpha_3, \ldots \in D$. If we let β be the limit of this sequence, then $\alpha < \beta < \kappa$ and β is in both C and D by construction. $\square$

Theorem 4.4. *The intersection of a family $\mathscr{F}$ of club sets with $|\mathscr{F}| < \kappa$ is again a club set.*

Proof. We prove by induction on $\gamma < \kappa$ of the length of a sequence $\langle C_\alpha \,|\, \alpha < \gamma \rangle$. Because of lemma 4.3, the induction step holds for successor ordinals. We thus only need to prove the case were γ is a limit ordinal.

Assume it holds for all $\alpha < \gamma$. We can replace each C_α in the sequence by $\bigcap_{\xi \leq \alpha} C_\xi$ and get a decreasing sequence with the same intersection. We thus have the sequence $C_0 \supseteq C_1 \supseteq \ldots$ of club sets. Let $C = \bigcap_{\alpha<\gamma} C_\alpha$ be its intersection. A similar argument as in the proof of lemma 4.3 gives that C is closed. We will prove the unboundedness of C also in a similar way as in the lemma. Let $\alpha < \kappa$, by unboundedness of the C_ξ for $\xi < \gamma$, we can construct a sequence $\langle \alpha_\xi \,|\, \xi < \gamma \rangle$ with $\alpha < \alpha_0 \in C_0$ and $\alpha_\xi \in C_\xi$ with $\alpha_\xi > \sup\{\alpha_\nu \,|\, \nu < \xi\}$ for each other ordinal. If we let β be the limit of this sequence, then we have $\alpha < \beta < \kappa$. Also, as every C_ξ is closed, $\beta \in C_\xi$ for each $\xi < \gamma$. Hence $\beta \in C$ $\square$

From the theorem it can be concluded that closed unbounded sets are so large that even an intersection of large families of club sets is still club. The definition of club sets can be generalized to cofinal subsets of κ.

Definition 4.5. *Let S be cofinal in κ. We define:*

$$\begin{aligned}
\mathrm{Club}(S) &:= \{A \subset S \,|\, A \textit{ is relatively closed and is cofinal in } \kappa\} \\
\mathscr{M}(S) &:= \{A \subset S \,|\, A \textit{ or } S - A \textit{ contains a member of } \mathrm{Club}(S)\} \\
\mathscr{M}_+(S) &:= \{A \subset S \,|\, A \textit{ contains a member of } \mathrm{Club}(S)\}
\end{aligned}$$

Here relatively closed in S means it is closed in the space S. For example if A is club in κ and α is one of its limit points then $A - \{\alpha\} \in \mathrm{Club}(\kappa - \{\alpha\})$. We will find in subsection 5.1 that $\mathscr{M}(S)$ forms a σ-algebra on S. For this reason, members of $\mathscr{M}(S)$ are called the measurable subsets of S.

If we look at the subsets of κ, two types that jump out are

(1) Sets containing some club set
(2) Sets which are contained in the complement of some club set

These two make up the set $\mathscr{M}(\kappa)$. However, there is a third type:

(3) Sets intersecting every club set but which do not contain a club set.

Mary Ellen Rudin found the existence of such sets in ω_1 [6]. Below is the proof given by David Lutzer [5] which reduces the amount of choosing.

Theorem 4.6. ***Rudin*** *There exists a subset of ω_1 of type (3).*

Proof. of the existence of sets of type (3). Suppose no subset of type (3) exists. Every subset must then be of type (1) or (2). As $|\omega_1| \leq |\mathbb{R}|$ we can fix an injection $f : \omega_1 \longrightarrow [0, \rightarrow) \subset \mathbb{R}$. For every natural number $n \geq 1$ there is a countable collection $\mathscr{I}(n)$ of intervals of $[0, \rightarrow)$ such that:

- The usual ordering on $\mathbb{R}$ gives a well-ordering of each $\mathscr{I}(n)$,
- $\bigcup \mathscr{I}(n) = [0, \rightarrow)$,
- Every interval has length $1/n$.

For example, we can make such a collection by setting $\mathscr{I}(n) := \{[\frac{k}{n}, \frac{k+1}{n}) \,|\, k < \omega\}$ as a collection of half-closed intervals. Each of these collections can be ordered by '$<$' by comparing the minimal point of the set.

We first prove that, for every n, there is some $J \in \mathscr{I}(n)$ for which $C_J := f^{-1}[J]$ is of type (1). If not, then there is some n for which for every interval $J \in \mathscr{I}(n)$ we have that $f^{-1}[J]$ is of type (2). Then the set $C_J := \omega_1 - f^{-1}[J]$ must be of type (1). Now, let $D := \bigcap\{C_J \,|\, J \in \mathscr{I}(n)\}$. By theorem 4.4 this is again a set of type (1) and is, therefore, not empty. This is contradictory as for any $\delta \in D$ and $J \in \mathscr{I}(n)$, $\delta \in C_J = \omega_1 - f^{-1}[J]$. But then $f(\delta)$ does not belong to any J even though $\mathscr{I}(n)$ covers $[0, \rightarrow)$. Therefore, the claim must hold for any $n < \omega$. Moreover, if all the $J \in \mathscr{I}(n)$ are disjoint, we also have that there is only one C_J of type (1).

By our claim and as $\mathscr{I}(n)$ is well-ordered, we can now set J_n as the smallest interval in $\mathscr{I}(n)$ such that $C_n := f^{-1}[J_n]$ is of type (1). Again using theorem 4.4 we know that $E := \bigcap\{C_n \,|\, n \geq 1\}$ is also of type (1). Now let $\alpha, \beta \in E$ be distinct, then, as f is injective, we have $0 < |f(\alpha) - f(\beta)|$. Moreover, by construction of E, we must have $\alpha, \beta \in C_n = f^{-1}[J_n]$ for each n. This gives us that $|f(\alpha) - f(\beta)| \leq \mathrm{diam}(J_n) \leq \frac{1}{n}$ for every n. This gives a contradiction as E can contain at most 1 point and is therefore not of type (1). We conclude that there must be a set of type (3). $\square$

The special property of sets of type (3) of intersecting every club set is worthy of its own name. We will call them **stationary sets**.

Definition 4.7. *$S \subset \kappa$ is called stationary if it intersects all club sets, i.e., if $S \cap C \neq \varnothing$ for every $C \in \mathrm{Club}(\kappa)$.*

The sets of type (3) are even more special as their complement must also intersect every club set.

Definition 4.8. *$S \subset \kappa$ is called* ***bistationary*** *if both S and $\kappa - S$ are stationary.*

The set of stationary subsets of κ will be denoted by $\mathscr{S}(\kappa)$. One thing we can immediately see is that $\mathscr{M}_+(\kappa) \subset \mathscr{S}(\kappa)$.

Corollary 4.9. *If S is (bi)stationary and C is club, then $S \cap C$ is again (bi)stationary.*

Proof. For club sets C and D, $C \cap D$ is again club and so $S \cap (C \cap D)$ must be (bi)stationary. $\square$

For the intersection of two stationary sets we can not make such a claim. The intersection of two bistationary sets can even be empty. It should not be a surprise that if S is stationary that also

$$\mathrm{Club}(S) = \{S \cap C \,|\, C \in \mathrm{Club}(\kappa)\}.$$

For this equality it is really needed that S is stationary as otherwise the intersection might not be cofinal. For $\mathscr{M}(S)$ and $\mathscr{M}_+(S)$ such a characterization also exists (see 5.3). But for this we need to collect some more theorems. From theorem 4.4 we get the following corollary:

Corollary 4.10. *The union of a family $\mathscr{F}$ of non-stationary sets with $|\mathscr{F}| < \kappa$ is non-stationary.*

The definition of stationary sets can also be generalized to cofinal subsets of κ.

Definition 4.11. *Let S be cofinal in κ.*

$$\mathscr{S}(S) = \{A \subset S \,|\, A \text{ is stationary in } \kappa\}$$

We have now defined stationary sets and the ideas behind them. Next we will look into some important properties of stationary sets.

4.2. **Stationary sets and important theorems.**

One very important property about stationary sets is a result known as "Fodor's theorem" or the "Pressing down lemma".

Theorem 4.12. **[Fodor]** *let S be a subset of κ, then the following conditions are equivalent for S:*

(1) S is stationary

(2) For every function $f : S \longrightarrow \kappa$ with $f(x) < x$ for all $x \in S - \{0\}$, there is an $\alpha \in \kappa$ with $|f^{-1}[\{\alpha\}]| = \kappa$

If moreover the axiom of choice holds (1) and (2) are also equivalent to:

(3) For every function $f : S \longrightarrow \kappa$ with $f(x) < x$ for all $x \in S - \{0\}$, there is an $\alpha \in \kappa$ with $f^{-1}[\{\alpha\}] \in \mathscr{S}(S)$

Remark. *Such a function as in (2) and (3) which only maps downwards is called a "**regressive**" or "pressing-down" function.*

To prove this, we first prove the "Diagonal intersection lemma".

Lemma 4.13. *Let $\langle C_\alpha \,|\, \alpha < \kappa \rangle$ be a sequence of club sets. The diagonal intersection of C_α,*

$$\triangle C_\alpha := \{\xi < \kappa \,|\, \xi \in \bigcap_{\alpha < \xi} C_\alpha\},$$

is again a club set.

Proof. Replacing each C_α by $\bigcap_{\xi \leq \alpha} C_\xi$, which is club by theorem 4.4, we may assume $C_\alpha \subseteq C_\beta$ whenever $\alpha > \beta$. Moreover, we get the same diagonal intersection $\triangle C_\alpha = C$. We first show that C is closed. Let α be a limit point of C and let $\xi < \alpha$. We set $X := \{\mu \in C \,|\, \xi < \mu < \alpha\}$, then surely $X \subset C_\xi$. As X must be cofinal in α and C_ξ is club, we have $\alpha = \sup(X) \in C_\xi$. As we now have that $\alpha \in C_\xi$ for all $\xi < \alpha$, we have by definition of C that $\alpha \in C$ and therefore C is closed.

To show unboundedness of C, we again construct a sequence $\langle \alpha_n \,|\, n < \omega \rangle$ with limit in C. Let $\beta_0 \in C_0$ be the least member such that $\beta_0 > \alpha$ and for each n let $\beta_{n+1} \in C_{\beta_n}$ be the least member such that $\beta_{n+1} > \beta_n$. Let β be the limit of this sequence and $\xi < \beta$. Surely there is some n for which $\xi < \beta_n$. Because $\beta_k \in C_{\beta_n}$ whenever $k > n$ and C_{β_n} is club, we have $\beta \in C_{\beta_n}$. Therefore, as $C_{\beta_n} \subseteq C_\xi$, we have $\beta \in C_\xi$. As $\xi < \beta$ was arbitrary, we conclude $\alpha < \beta \in C$. This gives us that C must also be unbounded. $\square$

Proof. of theorem 4.12 We follow the proof in Levy [4].
(3) $\to$ (2). This is trivial as every stationary set is unbounded.
(2) $\to$ (1). We will prove $\neg(1) \to \neg(2)$ by constructing a regressive function for which $f^{-1}[\{\alpha\}]$ is a bounded subset for all α. Assume S to be non-stationary, then there is some club set C disjoint from S. Define $f : S \to \kappa$ by

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq \min(C), \\ \sup(C \cap \alpha) & \text{if } \alpha > \min(C). \end{cases}$$

For $\alpha > \min(C)$, clearly $\gamma := \sup(C \cap \alpha) \leq \alpha$. As $C \cap \alpha$ is a non-empty bounded subset of the closed set C, we must have $\gamma \in C$. Since $\alpha \in S$ and $C \cap S = \varnothing$, we also have $\gamma \neq \alpha$. In either case we conclude $f(\alpha) < \alpha$ and f is a regressive function. To see that (2) fails, we note that for every α, $f^{-1}[\alpha] \subseteq \min(C - \alpha)$ which is clearly bounded.

(1)$\to$(2). Let S be stationary and f regressive. Assume that (2) does not hold. For every $\xi < \kappa$ we can define a club C_ξ by

$$C_\xi := \kappa - \sup{}^+(f^{-1}[\{\xi\}]).$$

This can be done as every set $f^{-1}[\{\xi\}]$ is bounded. Now, the diagonal intersection $C := \triangle C_\xi$ is again a club set. By Corollary 4.9 the set $S \cap C$ is stationary (in particular non-empty). This will give us a contradiction.

Let $\alpha \in S \cap C - \{0\}$. For this α we must have that $f(\alpha)$ is defined. We also have $\alpha \in C_\xi$ for all $\xi < \alpha$. But then, $\alpha \notin f^{-1}[\{\xi\}]$ and, more specifically, $f(\alpha) \neq \xi$ for all $\xi < \alpha$. This gives a contradiction as we now have that $f(\alpha) \geq \alpha$ but f is regressive.

(1)$\to$(3) If (3) fails to hold, then the set $f^{-1}[\{\alpha\}]$ is non-stationary for all $\alpha < \kappa$. Using the axiom of choice, we take the sequence $\langle C_\xi \,|\, \xi < \kappa \rangle$ by choosing C_ξ to be some club set disjoint from $f^{-1}[\{\xi\}]$. Following the proof of (1)$\to$(2) we again find a contradiction. $\square$

Remark. *This is where "stationary" comes from. In 1953 Bloch [1] called a set A stationary if it is unbounded and for every regressive function f on A, $\lim_{x\to\kappa, x\in A} f(x) < \kappa$. Therefore, there must also be some function value that occurs unboundedly often.*

We have seen that stationary sets are not necessarily as big as club sets. However, they still offer a lot of space to work in. Another important theorem for stationary sets is by Ulam and Solovay.

Theorem 4.14. **[Ulam and Solovay]** *Every stationary subset of κ is the union of κ disjoint stationary subsets.*

We follow the proof given in Jech [3]. To prove this theorem we first prove two lemmas. The first lemma is a weaker result.

Lemma 4.15. *Let $\lambda < \kappa$ be a regular cardinal. The set $E_\lambda := \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) = \lambda\}$ is stationary and every stationary subset of E_λ is the union of κ disjoint stationary sets.*

Proof. To see that E_λ is stationary one can consider some club set C and the unique isomorphism $f : \kappa \longrightarrow C$. Now by proposition 3.21 we have $\mathrm{cf}(f(\lambda)) = \mathrm{cf}(\lambda) = \lambda$. Then, $f(\lambda) \in E_\lambda$ and we get $C \cap E_\lambda \neq \varnothing$.

Let $S \subset E_\lambda$ be stationary. For every $\alpha \in S$ we choose an increasing sequence $\langle \beta^\alpha_\xi \,|\, \xi < \lambda\rangle$ with $\lim_{\xi\to\lambda} \beta^\alpha_\xi = \alpha$. We claim that there is some ξ such that for all $\nu < \kappa$ the following is stationary.

$$W_\nu := \{\alpha \in S \,|\, \beta^\alpha_\xi \geq \nu\}$$

Thus, for some ξ and any threshold ν, a stationary number of elements of S has its sequence above this threshold. These sets can be refined to get the wanted sets. First we prove the claim.

Suppose the claim does not hold. We must have that for every ξ there is a ν_ξ and a club set C_ξ such that $W_{\nu_\xi} \cap C_\xi = \varnothing$. Then, we must have that $\beta^\alpha_\xi < \nu_\xi$ for all $\alpha \in C_\xi \cap S$. Now, let $\nu := \sup(\nu_\xi)$ and $C := \bigcap_\xi C_\xi$. We have that $\beta^\alpha_\xi < \nu$ for all $\alpha \in C \cap S$ and all ξ. This gives a contradiction as $C \cap S$ is unbounded, but $\alpha = \lim_{\xi\to\lambda} \beta^\alpha_\xi \leq \nu$, but .

We will now refine the W_ν. Let ξ be such that W_ν is stationary for all $\nu < \kappa$. Surely the function $f(\alpha) := \beta^\alpha_\xi$ is regressive on any W_ν. If we now use Fodor's theorem (4.12) on each W_ν, we find, for each ν, a $\gamma_\nu \geq \nu$ with $S_\nu := f^{-1}(\gamma_\nu) \in \mathscr{S}(W_\nu)$. It should be clear that $\gamma_\nu \neq \gamma_{\nu'}$ iff $S_\nu \cap S_{\nu'} = \varnothing$. We can construct a cofinal sequence $\langle \Gamma_\nu \,|\, \nu < \kappa\rangle$ of different γ_ν by recursion. Set $\Gamma_0 = \gamma_0$ and for each α let $\Gamma_\alpha = \gamma_{g(\alpha)}$ where $g(\alpha) := \sup^+(\{\Gamma_\xi \,|\, \xi < \alpha\})$. Now, if $\alpha < \beta$ we must have $\Gamma_\alpha < \Gamma_\beta$; for $\nu \in S_{g(\alpha)}$ and $\tau \in S_{g(\beta)}$ we have $f(\nu) = \Gamma_\alpha < g(\beta) \leq f(\tau) = \Gamma_\beta$. WLOG we can assume that $\bigcup S_\nu = S$, as otherwise the missing points can be added to one of the S_ν. Lastly, by regularity of κ we find that $|\{S_\nu \,|\, \nu < \kappa\}| = |\{\gamma_\nu \,|\, \nu < \kappa\}| = \kappa$ ☐

This lemma is used to prove the following result.

Corollary 4.16. *If $T \subseteq \{\alpha \,|\, \mathrm{cf}(\alpha) < \alpha\}$ is stationary, then it is the union of κ disjoint stationary subsets.*

Proof. We have $T \subseteq \bigcup\{(E_\lambda - \{\lambda\}) \,|\, \mathrm{cf}(\lambda) = \lambda < \kappa\}$. Furthermore, as cf is a regressive function on T, using Fodor (4.12) there is some λ such that the set $T \cap E_\lambda$ is also stationary. ☐

Lemma 4.17. *Suppose there is some stationary set S consisting of only regular uncountable cardinals. Then the set $T = \{\alpha \in S \,|\, S \cap \alpha \notin \mathscr{S}(\alpha)\}$ is stationary.*

Proof. We show that T intersects all club sets. Let C be club and consider the set of its limit points, C'. This set must also be closed as every limit point of C' is also a limit point of C. next, we prove that C' is unbounded. Take some point $\xi \in C'$. By unboundedness of C there is some increasing sequence $\langle \alpha_n \,|\, n \in \omega\rangle$ in C whit $\alpha_0 > \xi$. The limit of this sequence must be a limit point of C and hence also be in C'.

As C' is club, we have that $S \cap C'$ is non-empty. Now, $S \cap C'$ also has a minimal element α. Since α is a limit point of C, we have that $C \cap \alpha \in \mathrm{Club}(\alpha)$. Furthermore, as $\alpha \in S$ it is regular. Therefore, using the same reasoning as above, also $(C \cap \alpha)' \cap \alpha = C' \cap \alpha \in \mathrm{Club}(\alpha)$. By minimality of α in $S \cap C'$, we find that $(S \cap \alpha) \cap (C' \cap \alpha) = \varnothing$. We conclude $S \cap \alpha \notin \mathscr{S}(\alpha)$ and hence $\alpha \in T \cap C$. ☐

Proof. of theorem 4.14 Let $S \in \mathscr{S}(\kappa)$ and define $\Omega := \{\alpha \in S \,|\, \mathrm{cf}(\alpha) = \alpha\}$. We split two cases:

(1) Ω is **non**-stationary
(2) Ω is stationary

In the first case we must have that $S-\Omega$ is stationary. Corollary 4.16 gives us that $S-\Omega$ is the union of κ disjoint stationary subsets.

In the second case we consider $T := \{\alpha \in \Omega \mid \Omega \cap \alpha \notin \mathscr{S}(\alpha)\}$, which is stationary by lemma 4.17. We follow the idea of the proof of lemma 4.15. For each $\alpha \in T$ we choose a strictly increasing continuous sequence $\langle \beta^\alpha_\xi \mid \xi < \alpha\rangle$ with $\alpha = \lim_{\xi\to\alpha}\beta^\alpha_\xi$ and such that $\beta^\alpha_\xi \notin T$ for all α and ξ. We again claim that there is an ξ such that for all ν the following is a stationary set:

$$W_\nu = \{\alpha \in T \mid \beta^\alpha_\xi \geq \nu\}.$$

From this claim it should be clear how to finish the proof.

Suppose the claim does not hold, we work towards a contradiction. We must have that for every ξ there is a ν_ξ and a club set C_ξ such that $W_{\nu_\xi} \cap C_\xi = \varnothing$. Then, for all $\alpha \in C_\xi \cap T$, we have that $\beta^\alpha_\xi < \nu_\xi$ iff β^α_ξ exists. Set $C := \triangle C_\xi$, this set is club by the diagonal intersection lemma (4.13). We must have $\beta^\alpha_\xi < \nu_\xi$ for all $\alpha \in C \cap T$ and all ξ smaller than α . Likewise we construct the set $D := \triangle D_\xi$ where $D_\xi := \{\alpha \in C_\xi \mid \nu_\xi < \alpha\}$. As each C_ξ is club, we almost get for free that D should also be club, furthermore, D is a subset of C. We, however, end up with something weird now. The set $D\cap T$ should be stationary (and thus contains at least two elements of T). Let $\gamma < \alpha$ be two elements of $D \cap T$, We must have for all $\xi < \gamma$, that $\beta^\alpha_\xi < \nu_\xi < \gamma$ and therefore $\beta^\alpha_\gamma \leq \gamma$. As the sequence β^α_ξ is strictly increasing and continuous, and as γ must be regular, we get $\beta^\alpha_\gamma = \gamma$ (see proposition 3.21). This is contradictory as we had $\beta^\alpha_\xi \notin T$ for all α and ξ. □

Note that, as all these stationary sets are disjoint, we must have that all these sets are bistationary. To give an example of what we can prove with the theory build so far, we will prove an important and much used theorem; the Δ-system lemma.

4.3. An example: the Δ-system lemma.

In this subsection an example is given of an use of stationary sets. We will be proving a generalization of the Δ-system lemma. The original Δ-system lemma states that every uncountable collection of finite sets contains an uncountable collection of sets with constant intersection. A so-called δ-system

Definition 4.18. *A collection of sets $\mathscr{A}$ is called a **Δ-system**, if there is some set R with $A \cap B = R$ for all distinct $A, B \in \mathscr{A}$. The set R is called the kernel of the Δ-system (which can be empty).*

The sets in $\mathscr{A}$ can be seen as the union of a 'petal' and the kernel. Because of this a Δ-system is sometimes called a sunflower. We will first define the limit of exponentiation of a limit cardinal by itself; $\kappa^{<\kappa} := \sup(\{\kappa^\lambda \mid \lambda \text{ is a cardinal and } \lambda < \kappa\})$. Note that $\kappa \leq \kappa^{<\kappa} \leq \kappa^\kappa$ and, hence, $\kappa \leq \kappa^\kappa$ with possibly a strict inequality.

Theorem 4.19. **[Δ-system lemma for regular cardinals]**. *Let κ be a regular cardinal and assume $\kappa^{<\kappa} = \kappa$. Let $\mathscr{A}$ be a collection of sets of cardinality less than κ with $|\mathscr{A}| = \kappa^+$. Then there exists a subset $\mathscr{B} \subseteq \mathscr{A}$ with $|\mathscr{B}| = \kappa^+$ which is a Δ-system.*

Proof. We construct the set $\mathscr{B}$. We can well-order $\bigcup \mathscr{A}$ in type κ^+, that is there is an isomorphism between $\bigcup\mathscr{A}$ and κ^+. We can thus pretend that $\mathscr{A}$ is a set of subsets of κ^+. We can, therefore, enumerate $\mathscr{A}$ as $\{A_\xi \mid \xi < \kappa^+\}$. Consider the function $f : \{\xi \in \kappa^+ \mid \mathrm{cf}(\xi) = \kappa\} \longrightarrow \kappa^+$ defined as $f(\xi) := \sup(A_\xi \cap \xi)$. As each $A_\xi \cap \xi$ must be bounded in ξ, this function is regressive. By Fodor's theorem (4.12) we get a stationary subset S of κ^+ and a $\beta < \kappa^+$ with $f(\alpha) = \beta$ on S.

Now we look at the set $C := \{\nu \mid (\forall \xi < \nu)(\sup(A_\xi) < \nu)\}$ and prove that this set is club in κ^+. Set $C_\xi := \{\nu \mid \sup(A_\xi) < \nu\}(= [\sup(A_\xi)+1, \kappa^+))$, this set is clearly club as each A_ξ is bounded in κ^+. Using the diagonal intersection lemma (4.13), we know that $\triangle C_\xi = \{\nu \mid (\forall \xi < \nu)(\sup(A_\xi) < \nu)\} = C$ is again club in κ^+. Therefore, the set $T := S \cap C$ must also be stationary in κ^+.

For each $\xi, \nu \in T$ we have that $A_\xi \cap A_\nu \subseteq [0, \beta)$. For our kernel we will thus search in this segment. For $R \subseteq \beta$, set $T_R := \{\alpha \in T \mid A_\alpha \cap \alpha = R\}$. We have $T = \bigcup\{T_R \mid R \subseteq \beta \wedge |R| < \kappa\}$ as $|A_\xi| < \kappa$ for every $\xi < \kappa^+$. Set $[\beta]^{<\kappa} := \{R \mid R \subset \beta \wedge |R| < \kappa\}$, we claim that $|[\beta]^{<\kappa}| \leq \kappa < \kappa^+$. This claim together with 4.10 gives that for some R, T_R must be stationary. Therefore, defining $\mathscr{B} := \{A_\xi \mid \xi \in T_R\}$ gives the wanted collection. Lastly, we will prove the claim that $[\beta]^{<\kappa}$ has cardinality at most κ.

Set $[\beta]^\lambda := \{R \mid R \subset \beta \wedge |R| = \lambda\}$, then we can write $[\beta]^{<\kappa} = \bigcup\{[\beta]^\lambda \mid \lambda < \kappa \text{ is a cardinal}\}$. We first show that each $[\beta]^\lambda$ has cardinality $|\beta|^\lambda$. Let $F : [\beta]^\lambda \longrightarrow {}^\lambda\beta$ be defined by setting $F(X)$ to be a bijection between λ and X. Clearly F is an injection and hence $|[\beta]^\lambda| \leq |{}^\lambda\beta| = |\beta|^\lambda \leq \kappa^\lambda$. Now, as $\kappa^{<\kappa} \geq \kappa^\lambda$ for each cardinal $\lambda < \kappa$, we must have that $|[\beta]^{<\kappa}| = |\bigcup\{[\beta]^\lambda \mid \lambda < \kappa \text{ is a cardinal}\}| \leq \kappa \cdot \kappa^{<\kappa} = \kappa\cdot\kappa = \kappa$ which proves the claim. □

Remark. *If κ is a regular cardinal and we assume $2^\kappa = \kappa^+$ (so surely given the "global continuum hypothesis") we always have $\kappa^{<\kappa} = \kappa$.*

5. On the existence of homeomorphisms

In this chapter we will be following chapters 4 and 5 of the article of van Douwen and Lutzer [7]. As a recurring theme in this report, we will again be comparing sets with one another. Concretely, we determine some conditions for stationary and bistationary sets to be of different topological types. That is, we find sufficient conditions such that there is no homeomorphism between (bi)stationary sets.

5.1. **Measurable functions.**

Instead of finding homeomorphisms themselves, we will look at more general functions, the measurable functions. We will see that every continuous function on a cofinal subset is also measurable.

First, we fulfill our promise to prove that $\mathscr{M}(S)$ is a σ-algebra on cofinal sets S. It should be clear that $\mathscr{M}(S)$ contains the whole set S and is closed under taking complements. So we only need to prove it is closed under countable intersections. If S is stationary, this follows from proposition 4.4 as $\kappa > \omega$. If S is non-stationary, it follows from the following equivalences:

Proposition 5.1. *Let S be a cofinal subset of κ, then the following are equivalent.*

(1) $S \notin \mathscr{S}(\kappa)$
(2) $\mathrm{Club}(S)$ *has two disjoint members*
(3) $\mathscr{M}(S) = \mathscr{P}(S)$

We first prove a lemma.

Lemma 5.2. *Let $S \subset \kappa$ be cofinal and non-stationary. Then for all cofinal subsets T of S there is some $D \in \mathrm{Club}(S)$ which is a discrete subset of T.*

Proof. As S is non-stationary, it is disjoint from some club set C. For each $\gamma \in C$ set $I_\gamma := (\gamma, \overline{\gamma})$ where $\overline{\gamma} := \min(C - (\gamma+1))$. Now, we must have that $\bigcup I_\gamma = \kappa - C$. As $T \subset \kappa - C$ and $|C| = |T| = \kappa$, we must have that the set $\{I_\gamma \,|\, \gamma \in C\}$ has cardinality κ. Many of these sets, I_γ, may be empty and thus not useful to us. This can be circumvented by setting $\Gamma := \{\gamma \in C \,|\, \gamma + 1 \notin C\}$, which is cofinal, and taking the set $\{I_\gamma \,|\, \gamma \in \Gamma\}$.

Define $\Gamma_0 := \{\gamma \in \Gamma \,|\, I_\gamma \cap T \neq \varnothing\}$. As T is a cofinal subset of $\kappa - C$ and must intersect I_γ for arbitrarily large γ, we have $|\Gamma_0| = \kappa$. By choosing for each $\gamma \in \Gamma_0$ some $d_\gamma \in I_\gamma \cap T$ we construct $D := \{d_\gamma \,|\, \gamma \in \Gamma_0\}$. $D \subseteq T$ and $|D| = \kappa$ are clear. We finish this proof by proving that D is discrete by showing it has no limit points in S.

Let d be some limit point of D. Then, there is some increasing sequence $\langle d_{\gamma_\xi} \,|\, \xi < \mathrm{cf}(d)\rangle$ of points in D such that $\lim_{\xi \to \mathrm{cf}(d)} d_{\gamma_\xi} = d$. But this would mean that $\langle \overline{\gamma_\xi} \,|\, \xi < \mathrm{cf}(d)\rangle$ is a sequence in C with the same limit. As C is closed, we must have $d \in C$. Now certainly we have $d \notin S$. □

Proof. of proposition 5.1. We prove (1)↔(2) and (1)↔(3).
(1)→(2): By lemma 5.2 there is some discrete $D \in \mathrm{Club}(S)$. From this set we can construct two disjoint cofinal sets D_1, D_2. As $\kappa + \kappa = \kappa$ we look at the set $\kappa \times \{0,1\}$, we will 'define' odd and even numbers this way. We define the bijection $f : \kappa \times \{0,1\} \longrightarrow \kappa$ by recursion. Set $f(0,0) = 0$ and for $(\alpha, i) \neq (0,0)$ define

$$f(\alpha, i) := \begin{cases} \sup^+(\{f(\beta,j) \,|\, \beta < \alpha \wedge j = 0,1\}) & \text{if } i = 0 \\ f(\alpha,0)+1 & \text{if } i = 1 \end{cases}$$

If we now order D in type κ as $\{d_\xi \,|\, \xi < \kappa\}$, we can construct the sequences of odd and even indexes. With this in mind, we define $D_1 := \{d_{f(\xi)} \,|\, \xi \in \kappa \times \{0\}]$ and $D_2 := \{d_{f(\xi)} \,|\, \xi \in \kappa \times \{1\}]$, these are two disjoint discrete sets.

(2)→(1): Let $C, B \in \mathrm{Club}(S)$ be disjoint, then $\overline{C} \cap \overline{B} \in \mathrm{Club}(\kappa)$ but $S \cap (\overline{C} \cap \overline{B}) = \varnothing$.

(1)→(3): Let $T \in \mathscr{P}(S)$ be arbitrary. If $|T| < \kappa$ take $S - T$. Now lemma 5.2 yields that $T \in \mathcal{M}(S)$.

(3)→(1): Suppose S is stationary. By the theorem of Ulam and Solovay (4.14), we have that there is a bistationary subset T of S. But then $T \notin \mathscr{M}(S)$ and we get that $\mathscr{P}(S) \neq \mathscr{M}(S)$ □

Now that we know that $\mathscr{M}(S)$ is a σ-algebra, we come back to the characterization of the measurable sets. We will prove that the algebra on S coincides with the algebra inherited from the whole set, κ.

Proposition 5.3. *Let S be cofinal, then $\mathscr{M}(S) = \{S \cap M \,|\, M \in \mathscr{M}(\kappa)\}$.*

Proof. Set $\mathscr{M}_S := \{S \cap M \,|\, M \in \mathscr{M}(\kappa)\}$. If $S \notin \mathscr{S}(\kappa)$ then by proposition 5.1(3) the inclusion $\mathscr{M}_S \subseteq \mathscr{M}(S)$ is clear. If S is stationary it follows directly from the fact that for every club C, $S \cap C \in \mathrm{Club}(S)$ (see also page 16).

For the other inclusion, let $M \in \mathscr{M}(S)$. Then, M either includes or completely misses some $C \in \mathrm{Club}(S)$. In the first case M is also disjoint from $\overline{C} \in \mathrm{Club}(\kappa)$. Therefore, $M \in \mathscr{M}(\kappa)$ and $M = S \cap M \in \mathscr{M}_S$. In the second case we have $M \cup \overline{C} \in \mathscr{M}(\kappa)$ and so $M = S \cap (M \cup \overline{C}) \in \mathscr{M}_S$. □

From this proof we can also extract a characterization for $\mathscr{M}_+(S)$.

Corollary 5.4. *If $S \in \mathscr{S}(\kappa)$ then $\mathscr{M}_+(S) = \{S \cap M \mid M \in \mathscr{M}_+(\kappa)\}$.*

Using proposition 5.1 we can prove our first result. That is, being stationary is a topological property.

Proposition 5.5. *Let $S \in \mathscr{S}(\kappa)$ and $h : S \longrightarrow \kappa$ be a homeomorphism onto some subset of κ. Then also $h[S]$ is stationary.*

Proof. As $h[S]$ must be cofinal, we can use proposition 5.1(2). Let $C, B \in \mathrm{Club}(h[S])$ and suppose $C \cap B = \varnothing$, then we also have $h^{-1}[C] \cap h^{-1}[B] = \varnothing$. We prove that this gives a contradiction as the preimage of a club set is again club.

Let α be a limit point of $h^{-1}[C]$ (in S), then there is some sequence $\langle a_\xi \mid \xi < \mathrm{cf}(\alpha)\rangle$ of elements of $h^{-1}[C]$ with limit α. By continuity we have $h(\alpha) = \lim_{\xi \to \mathrm{cf}(\alpha)} h(\alpha_\xi)$. This is a point of C as C is closed in $h[S]$. We get a contradiction as $h^{-1}[C], h^{-1}[B]$ are two disjoint club sets in S. Therefore, there can no disjoint sets in $\mathrm{Club}(h[S])$, implying that $h[S]$ must be stationary. □

This is just the tip of the iceberg, as we will see that even more general functions preserve stationary sets. For this reason we define measurable functions.

Definition 5.6. *Let S and T be cofinal sets and $f : S \longrightarrow T$. We call f*
measurable *if $(\forall C \in \mathscr{M}(T))(f^{-1}[C] \in \mathscr{M}(S))$*
strongly measurable *if it is measurable and moreover for all $y \in T$ the fiber $(f^{-1}[\{y\}])$ is non-stationary.*
a ***measurable isomorphism*** *if f is bijective and both f and its inverse f^{-1} are measurable.*

Note that this definition for a measurable function coincides with the usual definition, given that $\mathscr{M}(S)$ is a σ-algebra. To see that this indeed gives the wanted generalization, we consider the following proposition.

Proposition 5.7. *Let S be a cofinal subset of κ and $f : S \longrightarrow \kappa$ be continuous, then:*

(1) f is measurable
(2) if $|f[S]| = \kappa$, then f is strongly measurable.

Proof. (1): As f is continuous we must have, for any club C, that $f^{-1}[C]$ is closed in S. Hence, $f^{-1}[C] \in \mathscr{M}_+(S)$ and measurability follows from this.

(2): Set $T := f[S]$ and let $y \in T$. As the set $[y+1, \kappa) \cap T$ is club in T and f is continuous, we must also have that $D := f^{-1}[[y+1, \kappa) \cap T]$ is club in S. We must have that $\overline{D} \cap f^{-1}[\{y\}] = \varnothing$, hence the fiber $f^{-1}[\{y\}]$ cannot be stationary. As y was arbitrary, no fiber can be stationary and f must be strongly measurable. □

Remark. *Note that this does not violate theorem 4.12. A continuous function on κ with cofinal image can not be regressive.*

A homeomorphism on a cofinal set must certainly be a strong measurable function. In order to fully make use of strongly measurable functions, we will prove some more characterizations of strongly measurable functions.

Theorem 5.8. *Let S be stationary and suppose the function $f : S \longrightarrow \kappa$ has a cofinal image T. The following are equivalent:*

(1) $f^{-1}[A] \in \mathscr{M}_+(S)$ whenever $A \in \mathscr{M}_+(T)$
(2) f is strongly measurable
(3) If A is a stationary subset of S then its image is again stationary
(4) There is a $C \in \mathrm{Club}(S)$ consisting of fixed points of f.

For the proof of this theorem we first need a lemma.

Lemma 5.9. *Let $\mathscr{B}$ be a family of pairwise disjoint non-stationary subsets of κ and suppose there is some stationary set S in its union. Then there is some $C \in \mathrm{Club}(S)$ with $|B \cap C| \leq 1$ for each $B \in \mathscr{B}$.*

Proof. Let $x \in S$, then there is an unique $B(x) \in \mathscr{B}$ that contains x. For each x we also choose a club set $C(x)$ which is disjoint from $B(x)$. Now using the diagonal intersection lemma (4.13) and the fact that S is stationary, whereby $\mathrm{Club}(S) = \{S \cap C \,|\, C \in \mathrm{Club}(\kappa)\}$, we get that the set $C := \{x \in S \,|\, x \in C(y)$ for each $y \in S$ with $y < x\}$ is club in S.

Suppose there is some $B \in \mathscr{B}$ for which $|B \cap C| > 1$. Then there are at least two points, $x_1 < x_2$, in $B \cap C$. As $x_2 \in C$, we have $x_2 \in \bigcap\{C(x) \,|\, x \in S \text{ and } x < x_2\} \subset C(x_1)$ so that $x_2 \in B \cap C(x_1)$. But this gives a contradiction as $x_1 \in B$ tells us that $B = B(x_1)$ which forces $B \cap C(x_1)$ to be empty. $\square$

Proof. of theorem 5.8 We prove (1)→(2)→(3)→(4)→(1).

(1)→(2): As $T - \{y\} \in \mathscr{M}_+(T)$ for all $y \in T$, also all sets $f^{-1}[T - \{y\}]$ belong to $\mathscr{M}_+(S)$ for all such y. Certainly these sets are disjoint from the fibers $f^{-1}[\{y\}]$. Now, as S is stationary, the fibers can not be stationary.

(2)→(3): As f is strongly measurable, we have that $f^{-1}[\{y\}]$ is non-stationary for each $y \in T$. Furthermore, $S = f^{-1}[T]$ which enables us to use lemma 5.9. We get some $C \in \mathrm{Club}(S)$ with $|f^{-1}[\{y\}] \cap C| \leq 1$ for each $y \in T$, then f must be injective on C. Let $A \in \mathscr{S}(S)$ and suppose $f[A] \notin \mathscr{S}(T)$. As f is injective on C and $A \cap C$ is stationary (thus cofinal) its image, I, must also be cofinal. From proposition 5.1(3) we get that $\mathscr{M}(I) = \mathscr{P}(I)$. We prove that this results in a contradiction as then $\mathscr{M}(A \cap C) = \mathscr{P}(A \cap C)$. Let $B \in \mathscr{P}(A \cap C)$, then its image is in $\mathscr{M}(I)$. By proposition 5.3 there is some $M \in \mathscr{M}(\kappa)$ with $f[B] = I \cap M$. As f is measurable we get that $f^{-1}[M \cap T] \in \mathscr{M}(\kappa)$, and so $B = (A \cap C) \cap f^{-1}[M \cap T]$ implying that $B \in \mathscr{M}(A \cap C)$. from proposition 5.1 we get a contradiction. Hence $f[A]$ must also be stationary.

(3)→(4): We will show that the sets $M_1 = \{x \in S \,|\, f(x) < x\}$ and $M_2 = \{x \in S \,|\, f(x) > x\}$ are non-stationary, giving us that there is some set $C \in \mathrm{Club}(S)$ with $C \cap (M_1 \cup M_2) = \varnothing$. We first prove that M_1 is non-stationary. By Fodor's theorem (4.12), we only need to show that $f^{-1}[\{y\}]$ is non-stationary for each $y \in f[M_1]$. If for some y the set $f^{-1}[\{y\}]$ where to be stationary, we get by assumption of (3) that $f[f^{-1}[\{y\}]] = \{y\}$ is stationary, which is certainly not the case.

To prove that M_2 is non-stationary we consider the function $m : f[M_2] \to M_2$ defined by $m(y) := \min(M_2 \cap f^{-1}[\{y\}])$ (the minimal element of M_2 mapping to y). Then certainly m is regressive and injective, by Fodor's theorem we know that $f[M_2]$ is non-stationary and by (3) neither is M_2.

(4)→(1): Let $M \in \mathscr{M}_+(T)$. Then there is some $B \in \mathrm{Club}(T)$ with $B \subset M$. Now, as S is stationary, $C \cap \overline{B} \in \mathrm{Club}(S)$ and thus C is stationary. As for all $x \in C$, $f(x) = x$, we have that $C \cap \overline{B} = C \cap B \subset C \cap A \subset f^{-1}[A]$. Now we get that $f^{-1}[A] \in \mathscr{M}_+(S)$. $\square$

Corollary 5.10. *Let S be stationary and $f : S \longrightarrow T$ be strongly measurable, then also $f[S] \in \mathscr{S}(\kappa)$.*

Proof. Let $T = f[S]$, by theorem 5.8(3) we only need to prove that $|T| = \kappa$. If we suppose that $|T| < \kappa$, we also have that $S = \bigcup\{f^{-1}[\{y\}] \,|\, y \in T\}$. Thus, S is the union of fewer than κ non-stationary sets. This gives a contradiction by 4.4. $\square$

This shows that strong measurable functions also preserve stationary sets. Next, we will show that there are many topologically incomparable bistationary sets.

5.2. **Constructing incomparable bistationary sets.**

We begin this subsection by proving a theorem. Note that, if S is stationary, $S - T \notin \mathscr{S}(\kappa)$ implies that $S \cap T \in \mathscr{S}(\kappa)$.

Theorem 5.11. *Let S, T be cofinal subsets of κ. The following are equivalent:*

1. *$S - T \notin \mathscr{S}(\kappa)$.*
2. *There exists a continuous function from S onto a cofinal subset of T.*
3. *There is a bijective measurable map from S onto T.*
4. *There is a strongly measurable map from S into T.*

Proof. As the case $S \notin \mathscr{S}(\kappa)$ is not interesting to us, we will not prove it here. So suppose $S \in \mathscr{S}(\kappa)$. As (2)→(4) and (3)→(4) should be clear, it suffices to prove (1)→(2), (1)→(3) and (4)→(1).

(1)→(2): As $S - T$ is non-stationary, there is some club set C which is disjoint from it. Then $S \cap C \subset T$ and $S \cap C \in \mathrm{Club}(S)$ by proposition 5.3. We construct our function by setting $f(x) = \min(\{y \in S \cap C \,|\, x \leq y\})$. This function is cofinal with its image in T. To see it is also continuous take some limit point β of S. Then, if $(\alpha, \beta) \cap (S \cap C) = \varnothing$ for some α, f is constant on $(\alpha, \beta]$. Otherwise, there is some cofinal sequence in β of points in $S \cap C$ and, as $S \cap C$ is relatively closed, $S \cap C$ must also contain β.

(1)→(3): There is some club set C disjoint from $S-T$. As $S\cap C\in \text{Club}(S)$, also the set of its limit points, B, is in Club(S). Now we have $|(S\cap C)-B|=\kappa$, as every limit point must have a limiting sequence. Furthermore, we also have $|T-B|=\kappa$ as $S\cap C\subset S\cap T$. Therefore there exists some bijection between S and T, having B as set of fixed points. By theorem 5.8(d) we have that this function must also be strongly measurable.

(4)→(1): This follows almost directly from proposition 5.8 as there is some $C\in\text{Club}(S)$ consisting of fixed points, but then $(S-T)\cap\overline{C}=\varnothing$. As S is stationary $\overline{C}$ must be club and thus $S-T$ can not be stationary. □

As a consequence of this, we get that being bistationary is preserved under measurable isomorphisms.

Corollary 5.12. *Let S be bistationary and $f:S\longrightarrow T$ be a measurable isomorphism onto some $T\subseteq\kappa$. Then also T is bistationary.*

Proof. Write $T=f[S]$ and note that f^{-1} is also a measurable isomorphism. By 5.11(3) we find that both $S-T$ and $T-S$ are non-stationary. From the second we find that $(\kappa-S)-(\kappa-T)$ is non-stationary. Putting this together, we get that both $S\cap T\subset T$ and $(\kappa-S)\cap(\kappa-T)\subset(\kappa-T)$ are stationary. □

This does not hold for strongly measurable mappings. For example, one can take S to be some bistationary subset and $T=\kappa$. Then, by 5.11(2), there exists a bijective strongly measurable map from S onto κ. But surely κ is not bistationary.

The following theorem does give sufficient conditions for preservence of bistationary sets.

Theorem 5.13. *Let S and T be cofinal subsets of κ, then the following are equivalent:*

(1) There is a measurable isomorphism between S and T
(2) There are strongly measurable maps $f:S\longrightarrow T$ and $g:T\longrightarrow S$
(3) $S\triangle T\notin\mathscr{S}(\kappa)$

Proof. (1)→(2) is clear and the equivalence of (2) and (3) follows directly from 5.11(1). The only implication left to prove is (3)→(1). If S and T are non-stationary, (1) holds for any bijection due to proposition 5.1(3). So WLOG we can assume that S is stationary, then T must be stationary as well. Let C be a club set disjoint from $S\triangle T$. We must have that the points of intersection of C with S and T are shared, hence $S\cap C=T\cap C=(S\cap T)\cap C$. If we set D as the set of limit points of $(S\cap T)\cap C$, then we must have $D\in\text{Club}(S)\cap\text{Club}(T)$. Furthermore, as reasoned in the proof of 5.11, we have $|S-D|=|T-D|=\kappa$. Therefore, we can define a bijection between S and T having D as club set consisting of fixed points. By theorem 5.8(1) we have that this bijection is ,moreover, a measurable isomorphism. □

So for two stationary sets to be homeomorphic, we must surely have that they differ only on a non-stationary set. As another consequence of this, if two (bi)stationary sets are disjoint then there is no homeomorphism between them.

Corollary 5.14. *If S and T are disjoint stationary sets then there is no strongly measurable mapping from S into T.*

Proof. Follows directly from 5.11 and the fact that $S-T=S$ and $T-S=T$. □

This gives rise to the question how many topologically incomparable stationary sets there are. The following theorem tells us that there are as many as there are subsets of κ.

Theorem 5.15. *For any stationary subset $B\subseteq\kappa$ there is a family $\mathscr{B}$ of bistationary subsets of B such that:*

(1) $\mathscr{B}$ has cardinality 2^κ.
(2) If $S\neq T$, then there is no strongly measurable map between S and T.

Proof. By the Theorem of Ulam and Solovay(4.14) there is some family of disjoint stationary subsets of B. Let $\mathscr{D}$ be such a family. we can write it as the union of two disjoint families of the same type (recall that $|\kappa\times\kappa|=\kappa$). In other words, $\mathscr{D}=\{A_\alpha\,|\,\alpha<\kappa\}\cup\{B_\alpha\,|\,\alpha<\kappa\}$ where $A_\alpha\neq A_\beta, B_\alpha\neq B_\beta$ for $\alpha\neq\beta$ and $A_\alpha\neq B_\beta$ for any α and β.

For each $P\in\mathscr{P}(B)$ we define $S(P):=(\bigcup\{A_\alpha\,|\,\alpha\in P\})\cup(\bigcup\{B_\alpha\,|\,\alpha\in\kappa-P\})$. Furthermore, we set $\mathscr{B}:=\{S(P)\,|\,P\in\mathscr{P}(B)\}$ and prove it is the wanted family. If $P\notin\{\varnothing,B\}$, then each $S(P)$ contains a stationary set and is disjoint from another, hence $S(P)$ is bistationary. Now if $P,Q\in\mathscr{P}(B)$ are distinct, then there is an α such that $\alpha\in P-Q$ or $\alpha\in Q-P$. If $\alpha\in P-Q$, then $A_\alpha\subset S(P)-S(Q)$. In the other case, $\alpha \in Q - P$, there is a $B_\alpha \subset S(P) - S(Q)$. Either way we know, $S(P)$ and $S(Q)$ are distinct, hence $|\mathscr{B}| = |\mathscr{P}(B)| = \kappa$. Also $S(P) - S(Q) \in \mathscr{S}(B)$ from which (2) follows by theorem 5.11(4). $\square$

6. Conclusion

In this thesis we constructed the theory for stationary sets and showed sufficient conditions for stationary sets to be of different topological types.

First, the basic theory for ordinal and cardinal numbers was constructed and some importance of the characterization of ordered sets with these was shown. Furthermore, regular cardinals were looked into and their property of being true to their size. That is every regular cardinal can not be decomposed into fewer, smaller sets. Using this theory the closed unbounded subsets of a regular cardinal κ were defined and two important theorems were proved; the diagonal intersection lemma and that the intersection of fewer than κ club sets is again club.

These tools gave us everything we needed for looking at the properties of stationary sets. First the existsence of bistationary sets was proved in ω_1 by following the proof of Rudin. A widely used characterization of stationary sets is Fodor's theorem, which states that a set is stationary iff any regressive function on the set has a function value that occurs a stationary amount of times. This theorem gave us the tools to prove many theorems, for example we proved the Δ-system lemma for regular cardinals. Furthermore, the existence of bistationary sets was proven for arbitrarily large uncountable regular cardinals by proving the theorem by Ulam and Solovay. This theorem, moreover, gives us tools to construct κ disjoint bistationary sets from one stationary set.

In the remainder of this thesis mapping properties of stationary sets were explored leading to some necessary conditions on the existence of measurable functions and homeomorphisms between stationary sets. We found that the property of being a (bi)stationary set is preserved under strongly measurable maps. Furthermore, we found that for the existence of strongly measurable maps between stationary sets their difference is vital. That is if for such a mapping to exist between two stationary sets S and T, T must contain almost all of S such that $S - T$ is non-stationary. Moreover, for a homeomorphism to exist between them it is also necessary for S to contain almost all of T. We conclude that S and T can only be homeomorphic if they differ only on a non-stationary set.

Appendices

Appendix A. Set theory

"Mathematics is the language with which god has written the universe"-Galileo Galilei

But what is the language of mathematics?
Almost all of mathematics can be explained through 'collections of objects', often satisfying some property. These collections are called sets. The branch of mathematics that studies sets is called set theory.

In this chapter we will follow the lecture notes by K.P. Hart [2] and set up the theory of axiomatic set theory. First the formal language is described, after which we will state the axioms by Zermelo with an addition by Fraenkel. In the remaining of this chapter the axioms will be elaborated on building our theory in the progress.

A.1. The Language of set theory.

The theory in which we will build our set theory is a first order theory with equality. In short we have limited set of symbols for relations from which we can make formulas on a domain (in this thesis the domain will be sets). We use the following symbols:

- brackets: '(' and ')'; specifying the order of operations
- negation '$\neg$'; 'It is not true that'
- conjunction '$\wedge$'; 'and' or 'also'
- the quantifier '$\exists$'; 'there exists'
- a symbol for equality '$=$'
- a symbol for 'element of' '$\in$'

Furthermore, variables are used and are denoted by letters, possibly with indexes, such as w_i and A.

Using these symbols one can create **formulas**. A formula is a sequence of symbols for which it makes sense to ask if the formula is true. We can construct formulas in the following way:

- for every two variables w_i and w_j, $w_i \in w_j$ and $w_i = w_j$ are formulas
- if ϕ is a formula then also $\neg(\phi)$ is a formula
- if ϕ and ψ are formulas then also $(\phi) \wedge (\psi)$ is a formula
- if ϕ is a formula and w_i is a variable then also $(\exists w_i)(\phi)$ is a formula.

Now that we know how to construct formulas we immediately make some abbreviations for often used (and well known) formulas:
Inequality: $w_i \neq w_j$ abbreviates $\neg(w_i = w_j)$
Not an element of: $w_i \notin w_j$ abbreviates $\neg(w_i \in w_j)$
Disjunction: $(\phi) \vee (\psi)$ abbreviates $\neg((\neg(\phi)) \wedge (\neg(\psi)))$ 'or'
Implication: $(\phi) \to (\psi)$ abbreviates $(\neg(\phi)) \vee (\psi)$ 'if ... then ...'
Biconditional: $(\phi) \leftrightarrow (\psi)$ abbreviates $((\phi) \to (\psi)) \wedge ((\psi) \to (\phi))$ 'if and only if' or in short 'iff'
Universal quantifier: $(\forall w_i)(\phi)$ abbreviates $\neg((\exists w_i)(\neg(\phi)))$ 'for all'
Often we want to 'bound' our quantifiers as such our variables are from a specific set. This is done by the following abbreviation: $(\exists w_i \in A)(\phi)$ abbreviates $(\exists w_i)(w_i \in A \wedge (\phi))$
(then $(\forall w_i \in A)(\phi)$ abbreviates $(\forall w_i)(w_i \in A \to \phi)$). Lastly, $(\exists w_1, w_2, \dots, w_n)$ abbreviates $(\exists w_1)(\exists w_2)\dots(\exists w_n)$ with similar meaning for the universal quantifier.

To make equality true to its meaning two axioms are required:

- $(\forall x)(x = x)$
- $(\forall x, y)((x = y) \to (\psi \to \psi'))$ where ψ' is a formula created from a formula ψ by changing 0 or more occurrences of x by y

The **scope** of a quantifier is the range where it has an effect in. For example, the scope of the quantifier $(\exists x)$ in $(\exists x)(\phi)$ is ϕ, thus the enclosed brackets after the quantifier. We also define what it means for a variable to be free. A variable x is called **free** when it is not in the scope of any quantifiers $\forall x$ or $\exists x$ and it is called bound otherwise. In other words, a variable is bound when you can substitute something for it <u>without</u> changing the meaning.

To illustrate this we look at the following formula: $(\exists x)(y \in x) \vee (\forall y)(y = x)$. In the first subformula, $(\exists x)(y \in x)$, x is in the scope of $\exists x$ thus x is bound, furthermore y is free in this subformula. In the second subformula, $(\forall y)(y = x)$, y is the bound variable and x is free. We can thus rewrite our formula as $(\exists z)(y \in z) \vee (\forall v)(v = x)$ without changing its meaning.

Sometimes we want to omit some brackets, therefore we make some conventions about the order in which the operations will be evaluated.

The order we will use is as follows:

(1) negation $\neg$
(2) conjunction $\wedge$ and disjunction $\vee$
(3) the quantifiers $\exists$ and $\forall$
(4) implication $\rightarrow$ and biconditional $\leftrightarrow$

where the operations are listed in decreasing order. For example, $((\phi) \vee (\psi)) \leftrightarrow (\neg((\neg(\phi)) \wedge (\neg(\psi))))$ becomes $(\phi) \vee (\psi) \leftrightarrow \neg(\neg(\phi) \wedge \neg(\psi))$ which is more readable. Now that we have set our language and conventions we can move on and look at the set of axioms.

A.2. **The axioms by Zermelo and Fraenkel.**
In the theory constructed by Zermelo and Fraenkel we have 7 axioms and 2 infinite lists of axioms, so called axiom schemas. The axioms are listed below:

Axiom 1 *Extensionality*: If two sets have the same elements they are equal:

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Axiom 2 *Regularity*: Every non-empty set contains an element containing no elements of the set:

$$(\forall x)((\exists y)(y \in x) \rightarrow (\exists y)(y \in x \wedge \neg(\exists z)(z \in x \wedge z \in y)))$$

Axiom 3 *Schema of Separation*: Let ϕ be any formula with all free variables among $x, z, w_1, w_2, \ldots, w_n$ and y not free in ϕ. There exists, for every set x, a set containing exactly those elements of x satisfying ϕ:

$$(\forall x)(\forall w_1)\ldots(\forall w_n)(\exists y)(\forall z)(z \in y \leftrightarrow (z \in x \wedge \phi))$$

Axiom 4 *Pairing*: For every two sets there exists a set with those sets as elements:

$$(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$$

Axiom 5 *Union*: For every set x there is a set for which every element of an element of x is an element:

$$(\forall x)(\exists y)(\forall z)(\forall u)((u \in z \wedge z \in x) \rightarrow u \in y)$$

Axiom 6 *Schema of Replacement*: For any formula ϕ with all free variables among $x, y, A, w_1, \ldots, w_n$ (B not free in ϕ !) the following is an axiom:

$$(\forall A)(\forall w_1)\ldots(\forall w_n)((\forall x \in A)(\exists! y)(\phi) \rightarrow (\exists B)(\forall x \in A)(\exists y \in B)(\phi))$$

Axiom 7 *Infinity*: There is a set containing the empty set and for every element its successor:

$$(\exists x)(\varnothing \in x \wedge (\forall y \in x)(y \cup \{y\} \in x))$$

Axiom 8 *Power set*: For any set x there is a set that contains every subset of x:

$$(\forall x)(\exists y)(\forall z)(z \subseteq x \rightarrow z \in y)$$

The theory constructed with the above axioms is often called **ZF**. Moreover, adding the Axiom of Choice below gives the theory **ZFC**.

Axiom 9 *Choice*: Every family of non-empty sets has a 'choice function':

$$(\forall x)(\varnothing \notin x \rightarrow (\exists f)((f : x \rightarrow \bigcup x) \wedge (\forall y \in x)(f(y) \in y)))$$

The Axiom of Choice is often considered separately because it is nonconstructive. This means that it tells us that there exists some 'choice function' but does not tell us how to construct it. Furthermore, some consequences of the Axiom of Choice are very counterintuitive as some would think of those as false.

In the upcoming paragraphs we will slowly build our theory by considering some of the above axioms. Moreover, some theorems will be proved with the axioms to build the general theory.

A.3. **Building our theory.**
First, we must prove that there are sets. Otherwise we can not use our theorems for anything substantial. Often another axiom, $(\exists x)(x = x)$, is added to the set of axioms of ZF for this purpose. As all sets are equal with itself, this axiom guarantees the existence of a set without essentially adding more. Instead of adding another axiom we can also look at the Axiom of Infinity as this also suggests the existence of at least one set. This is, however, quite rigorous as even more is implied by this axiom.

Now that we argued that at least one set exists we look at our first real axiom.

Axiom 1 *Extensionality*: If two sets have the same elements they are equal:

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

This axiom gives a connection between equality,'=', and 'element of,'∈'. Now, the frequent applied method for proving equality can be used: if for two sets x and y we prove that every element of x belongs to y and vice versa, we can say they must be equivalent.

Some new notation is introduces for this. We say that **x is a subset of y** if

$$(\forall z)(z \in x \rightarrow z \in y)$$

In other words: all elements of x are elements of y. We denote this by $x \subseteq y$. We can also prove that the reverse of axiom 1 also holds:

Lemma A.1. *Let x and y be two sets then:*

$$(x = y) \rightarrow (\forall z)(z \in x \wedge z \in y)$$

Proof. Suppose $x = y$ and let z be arbitrary. From the axioms for equality we find, using $z \in x$ for ϕ, $z \in x \rightarrow z \in y$. As z was arbitrary we can make a generalization of this statement giving us the formula $(\forall z)(z \in x \rightarrow z \in y)$. □

It is now established that equality of sets is the same as having the same elements.

The main use for the second axiom, the Axiom of Regularity, for us is that there can not be a set containing itself as one of its elements. That is, if some set $x = \{x\}$ were to exist then it violates the second axiom.

We move on to the next axiom which gives a tool for constructing subsets whose elements satisfy a specific formula.

Axiom 3 *Schema of Separation*: Let ϕ be any formula with all free variables among $x, z, w_1, w_2, \dots, w_n$ and y not free in ϕ. There exists, for every set x, a set containing exactly those elements of x satisfying ϕ:

$$(\forall x)(\forall w_1)\dots(\forall w_n)(\exists y)(\forall z)(z \in y \leftrightarrow (z \in x \wedge \phi)) \tag{$*$}$$

Note that with this tool you can only get subsets of a given set and not construct a whole new set. This is to ensure no paradoxes appear, such as *Russell's paradox*: "Does the set containing all sets not containing itself, contain itself?".

One important property we can prove is that, for given $x, w_1, \dots, w_n$ the set y in $(*)$ is unique.

Proposition A.2. *Given $x, w_1, \dots, w_n$ in axiom 3, if u and y both satisfy $(*)$, then $u = y$*

Proof. Suppose both y and u satisfy $(*)$. Let $z \in u$, then we must also have $z \in x$ and $\phi(z)$ by construction of u. But this is exactly the condition for which $z \in y$, thus $u \subseteq y$. Interchanging u and y in the proof gives the other inclusion and hence the sought equality. □

Corollary A.3. *Let x be a set, then for every formula containing one free variable z, $\{z \in x \,|\, \phi(z)\}$ is a set.*

Using the formula $y \neq y$, which is false for all sets y, corollary A.3 tells us that there exists a set without any elements. This very special set will be called the **empty set** and denoted by $\varnothing$. As the set with which $\varnothing$ can be created is arbitrary and $\varnothing$ is unique in each such set (A.2), we moreover know that this set is a subset of any set. We call a set non-empty if it is different from the empty set (and thus has elements). Using *Russell's paradox* mentioned above we can prove that there is no set that contains all sets.

Theorem A.4. *There is no universal set* $\mathbf{V} := \{x \,|\, x = x\}$

Proof. Suppose $\mathbf{V}$ is a set. We prove that this gives a contradiction. As we assumed that $\mathbf{V}$ is a set, we can use the Axiom of Separation to construct a new set: $R = \{x \,|\, x \notin x\}$ or the set of sets not containing itself. Suppose $R \notin R$, then R is a set not containing itself hence $R \in R$. However, $R \in R$ also implies R

should not contain itself. This gives $R \in R \leftrightarrow R \notin R$ which would make our theory non-consistent (this would result in that any formula is true). Thus this set R can not exist and neither can $\mathbf{V}$. $\square$

To construct sets with specific properties we really have to take subsets to be sure we end up with a set again. That is, collections of the form $\{x \mid \phi(x)\}$ can not be taken as a set without proof.

Using the Axiom schema of Separation, we can construct some more well known operations on sets. The **intersection** $(x \cap y)$ and **difference** $(x - y)$ of two sets are easily found using formulas $u \in y$ and $u \notin y$ respectively. We can also construct the intersection over a (non-empty) set x: $\bigcap x = \{u \mid (\forall A \in x)(u \in A)\}$, but this requires some more work.

Theorem A.5. *The intersection over a non-empty set is again a set*

Proof. Let x be a non-empty set, then there is a set A' such that $A' \in x$. By the Axiom schema of Separation we can construct the set $S := \{u \in A' \mid (\forall A \in x)(u \in A)\}$. We prove that $S = \bigcap x$. $S \subseteq x$ is clear. Let $u \in \bigcap x$ then by definition we have $(\forall A \in x)(u \in A)$ and hence surely $u \in A'$ which implies that $u \in S$. We get that x is also a subset of S, resulting in $\bigcap x$ being a set. $\square$

Remark. *We can not define the intersection over the empty set as above, as this would yield the universal set* $\mathbf{V}$*.*

For the union of sets we do not have the right tools. Therefore, we will add two new axioms. The second axiom is written in our new notation:

Axiom 4 Pairing: For every two sets there exists a set with those sets as elements:

$$(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$$

Axiom 5 Union: For every set x there is a set for which every element x is a subset:

$$(\forall x)(\exists y)(\forall z)((z \in x) \to z \subseteq y)$$

Using the Axiom of Separation we can define the **pair** $\{x, y\}$ as the unique set containing exactly the sets x and y. Moreover $\{x, x\} = \{x\}$ is called a **singleton** (just pick x for both x and y in the *axiom of pairing*). Using the *Axiom of Union* we can define the unique set $\bigcup x = \{y \mid (\exists z \in x)(y \in z)\}$ of elements of elements of x. This set is called **the union of x**. We can now define the "union of two sets" using what we constructed; $x \cup y = \bigcup\{x, y\}$. Lastly, the **symmetric difference** of two sets is defined as $x \triangle y = (x - y) \cup (y - x)$.

A.4. Relations and functions.

Now that some ways are constructed to create new sets from old ones, we will focus on forming some sets with important properties and "operations".

If we look at the pair $\{x, y\}$, we notice that there is no difference with the pair $\{y, x\}$; the order of the elements is not important. An ordered pair is defined (by Kuratowski) to be the set

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

In an ordered pair the order of the elements is important, as stated by the following proposition.

Proposition A.6. *Two ordered pairs, $\langle x, y\rangle$ and $\langle u, v\rangle$ are the same if and only if $x = u$ and $y = v$.*

Before we continue, we first prove a lemma.

Lemma A.7. $\{x, y\} = \{x, v\}$ *if and only if* $y = v$

Proof. If y is equal to x we must also have that v is equal to x, since $v \in \{x, v\} = \{x, y\} = \{x\}$. Now, if y is unequal to x we have must have that y equals v. In both cases we must have that $y = v$ $\square$

Proof. of proposition A.6 The only if part is clear. Let $\langle x, y\rangle$ and $\langle u, v\rangle$ be two ordered pairs and suppose $\langle x, y\rangle = \langle u, v\rangle$. As we have $\{x\} \in \langle x, y\rangle = \langle u, v\rangle = \{\{u\}, \{u, v\}\}$, we must have $\{x\} = \{u\}$ or $\{x\} = \{u, v\}$. From the first follows that $x = u$. The second also gives $x = u$, as $u \in \{u, v\} = \{x\}$. In both cases we thus must have $x = u$. Now, we only need to prove that if $\{\{x\}, \{x, y\}\} = \{\{x\}, \{x, v\}\}$, then $y = v$. By applying lemma A.7 we get $\{x, y\} = \{x, v\}$. Applying the lemma once more we get $y = v$ as desired. $\square$

For a function we need to have a domain, on which points the function acts, and a co-domain, to which the points correspond. To make sure that for any function we have a co-domain we will be adding another axiom from our list. In order to make the formulation more compact we first introduce the notation for uniqueness. We say that for a formula ϕ **there exists a unique** y **satisfying** $\boldsymbol{\phi}$, or $(\exists! y)(\phi(y))$, if for every x which satisfies ϕ we must have $x = y$, or

$$((\exists x)(\phi(x))) \wedge ((\forall x)(\forall y)((\phi(x) \wedge \phi(y)) \rightarrow (x = y)))$$

Axiom 6 *Schema of Replacement*: For any formula ϕ with all free variables among $x, y, A, w_1, \ldots, w_n$ and B not free in ϕ the following is an axiom:

$$(\forall A)(\forall w_1, \ldots, w_n)((\forall x \in A)(\exists! y)(\phi) \rightarrow (\exists B)(\forall x \in A)(\exists y \in B)(\phi))$$

This axiom schema states, in some manner, that any 'function' $\phi(x, y)$, in x on the set A, must have a co-domain B. Using this axiom and ordered pairs we construct the **Cartesian product** of two sets:

$$A \times B = \{\langle x, y\rangle \,|\, x \in A \wedge y \in B\}$$

We now show that this is a set:

Proposition A.8. *If A and B are two sets, then $A \times B$ is again a set*

Before we prove this proposition we first prove the following lemma:

Lemma A.9. *Let ϕ and A be as in the axiom of replacement and let $w_1, \ldots w_n$ be given. Then there exists a unique set B such that*

$$(\forall y)(y \in B \leftrightarrow (\exists x)(x \in A \wedge \phi))$$

Proof. By the Axiom schema of Replacement we have $(\exists C)(\forall x \in A)(\exists y \in C)(\phi)$. Using the Axiom schema of Separation with formula ψ meaning $(\exists x)(x \in A \wedge \phi)$, we get that there exists some unique set B such that

$$(\forall y)(y \in B \leftrightarrow (y \in C \wedge (\exists x)(x \in A \wedge \phi)))$$

We show that the requirement $y \in C$ can be dropped Now, let y be such that $(\exists x)(x \in A \wedge \phi)$. By assumption on A and ϕ this y is unique in A. Furthermore, as there is some $z \in C$ satisfying ϕ, we must have $y = z$. From this we conclude that $y \in C$. As y was arbitrary, we get the implication $(\forall y)((\exists x)(x \in A \wedge \phi) \rightarrow (y \in C))$. The requirement $y \in C$ may therefore be dropped in the definition of B. □

From this lemma we can deduce that an entity of the form $\{f(x) \,|\, x \in A\} = \{y \,|\, (\exists x \in A)(y = f(x))\}$ is a set for any set A and given sets $f(x)$.

Proof. of proposition A.8. We prove the claim by construction. Let $x \in A$ be fixed. For every $y \in B$ the ordered pair $\langle x, y\rangle$ is unique by proposition A.6. We have:

$$(\forall y \in B)(\exists! z)(z = \langle x, y\rangle)$$

Using lemma A.9 where ϕ is $z = \langle x, y\rangle$, we know there is a unique set of ordered pairs corresponding to $x \in A$; $p(x, B) := \{z \,|\, (\exists y \in B)(z = \langle x, y\rangle)\}$. Now, again using lemma A.9 with ϕ being $z = p(x, B)$, we can define the set $p(A, B) := \{y \,|\, (\exists x \in A)(y = p(x, B))\} = \{p(x, A) \,|\, x \in A\}$. The set of sets of ordered pairs with first element in A and the second in B. We can now define $A \times B$ as the union over $p(A, B)$. □

Definition A.10. *A **relation** is a set of ordered pairs. For any relation R we set*

$$\mathrm{dom}(R) = \{x \,|\, (\exists y)(\langle x, y\rangle \in R\})$$

and

$$\mathrm{ran}(R) = \{y \,|\, (\exists x)(\langle x, y\rangle \in R)$$

as respectively the domain and range. Moreover for $\langle x, y\rangle \in R$ we shall write $x \, R \, y$

Two examples of relations are '$<$' and '$\equiv \mod 2$' on the natural numbers.

Remark. *To prove that $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets, construct two formulas $1^{st}(z)$ and $2^{nd}(z)$ who 'map' an ordered pair z to its first or second component. We must moreover have $R \subseteq \mathrm{dom}(R) \times \mathrm{ran}(R)$.*

The following sets we define without proof.

Definition A.11. *Let R be a relation.*
*The **restriction** of R to a set C is*

$$R \restriction C = \{\langle x, y\rangle \in R \mid x \in C\}$$

*The **image** of R under C*

$$R[C] = \{y \mid (\exists x \in C)(x\, R\, y)\}$$

*The **composition** of two relations R and S*

$$R \circ S = \{\langle x, z\rangle \mid (\exists y)(x\, R\, y \wedge y\, S\, z)\}$$

*The **inverse** of R*

$$R^{-1} = \{\langle y, x\rangle \mid x\, R\, y\}$$

We also define a very special relation, a function.

Definition A.12. *A **function** f is a relation with the following property:*

$$(\forall x \in dom\, f)(\exists! y \in ran\, f)(\langle x, y\rangle \in f)$$

In other words: every point has only one function value.

For functions we write $f(x)$ for the value y for which $\langle x, y\rangle \in f$. Often we will write that "f is a function from A to B" or "$f : A \longrightarrow B$". With this we mean that $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

If f^{-1} is also a function, and thus every point is mapped uniquely, we call f **injective**. In the case where $ran\, f = B$ we call f **surjective**. Lastly, f is called **bijective** if it is both surjective and injective.

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