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A GENERAL LINEAR HYDROELASTICITY THEORY OF FLOATING STRUCTURES MOVING IN A SEAWAY

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The dynamics of an elastic beam floating on the surface of disturbed water has formed the basis of a fairly comprehensive linear theory of hydroelastic behaviour of ships in waves. The existing theory cannot easily be extended to floating vehicles of more complicated shape (such as semi-submersibles), or to fixed offshore structures. A general method is presented, by which finite elements permit any three-dimensional elastic structure to be admitted in a linear hydroelastic theory. Sinusoidal waves provide the excitation of the structure and the fluid flow is three-dimensional. Some examples are given which illustrate the use of the theory and expose behaviour that has not been encountered hitherto.

1. Introduction

When designing a ship or offshore structure, the naval architect has to meet requirements of initial cost, safety, reliability, performance, and so on. The testing of prototypes is usually precluded by cost, yet misjudgements could have horrendous consequences. For this reason, the naval architect tends to rely heavily on semi-empirical rules based on past experience; the result is that designs evolve only slowly from one type of vessel to another. Seldom is a radical departure made in the hull form of a ship, for instance.

The hull of a ship is usually based on considerations of static or quasi-static analyses, whereas in reality the structure will operate in conditions determined by the wind and seaway. This approach may be contrasted with that adopted in aeronautics; from the outset, aeronautics have been closely associated with dynamics.

Traditionally, the behaviour of a moving, floating structure in water has been divided, somewhat artificially, into distinct subjects, each with its own basic assumptions. These specialist branches of naval architecture may be described as follows.

Manoeuvring theory. This relates to the behaviour of a rigid ship in calm water when it is subject to external actions caused by forced motion of the rudder or stabilizer fins, or by selective use of propellers or thrusters. In general, both the inputs and the resultant ship responses are deterministic.

Seakeeping. This describes the responses of a rigid ship, moving or stationary, in regular sinusoidal waves or in a random seaway. The responses are either deterministic or random in form, and for the latter a probabilistic approach is required to determine the behaviour of the vessel.

Structural theory. This is a large subject that is usually based on empirical rules which determine the loading imposed on the structure, and then the use of structural analysis of a static or quasi-static nature.

Hydroelasticity is the study of the behaviour of a flexible body moving through a liquid. When applied to a flexible ship hull or offshore structure, it may be used to determine stresses, motions and distortions under the actions of external fluid loadings arising from the seaway, deflection of the rudder, rotation of a propeller, etc. The theory necessarily embodies a description of the structure concerned and of the fluid actions applied to it. In its most general form, this approach subsumes both manoeuvring and seakeeping theories in the sense that the dynamics of a rigid body is a special example of the more general problem associated with a flexible one.

Even when the structural and hydrodynamic theories are available, the naval architect still has the task of reconciling them. Until comparatively recently, little effort was made at adequate reconciliation and it is probably true to say that there have emerged two apparently distinct disciplines, one to do with structural theory and the other with naval hydrodynamics; this dichotomy is readily perceived in the literature. If, however, the predictions of fluid actions offered by naval hydrodynamics are matched to the representation of the ship hull as some form of elastic structure proposed by the structural analyst, the estimation of ship responses to waves becomes, in effect, a vibration problem.

To assess the dynamical behaviour of a structure of any prescribed form placed in a seaway, it is necessary to determine the forces applied to it by the fluid. This has been the subject of much research, both theoretical and experimental, and there is now a vast literature on naval hydrodynamics (see, for example, Newman 1978).

This paper discusses the dynamics of a flexible structure of arbitrary shape moving in a seaway. It does so, starting from fundamental studies in both hydrodynamics and structural mechanics. The interactions between the fluid and the moving flexible structure are allowed for in the linearized mathematical model. This is based on a three-dimensional description of the structure moving *in vacuo* and a three-dimensional hydrodynamic analysis of the fluid actions, which accounts for forward speed, free surface-wave effects and distortions of the flexible body. No attempt is made to distinguish between 'manoeuvring', 'seakeeping' and 'structural theory' because this general approach unifies all three, while putting them on a sounder footing.

To illustrate the theory, responses are calculated for two different idealized structures.

(*a*) A uniform box beam representing a rudimentary ship. (An actual ship structure could readily be substituted at the expense of greater detail.) This analysis allows a check to be made on the previous hydroelasticity theory of Bishop & Price (1979) in which the assumption of a 'beam-like' structure is fundamental, for hitherto there has been no comparable theory for use with actual ships which can be used for comparison.

(*b*) A semi-submersible in transit, or alternatively, a s.w.a.t.h. (i.e. small water-plane area twin hull).

The last structure travels at an arbitrary heading angle in regular sinusoidal waves and, because it is far from 'beam-like', there is no existing approach by which a hydroelastic analysis may be made.

2. Existing hydroelasticity theory

Hydroelasticity theory has been developed during the past decade by Bishop & Price (1979). It has been based on a linear dynamic analysis of the responses of a flexible ship hull travelling in a seaway. The responses (i.e. motions, distortions, shearing forces, bending moments, twisting moments) have been determined by using techniques of modal analysis. Briefly, the ship's hull is assumed to be 'beam-like' and its dynamic vibration characteristics (*in vacuo* in the absence of damping and external forces) are determined in a 'dry-hull analysis'. By treating the hull as a non-uniform Timoshenko beam and adopting a suitable process for representing the continuous structure as one with finite freedom, a set of principal modes and natural frequencies may be determined.

When the hull is afloat, all structural damping and hydrodynamic forces are treated as external actions applied to the dry hull whose characteristics are now known. The fluid actions may be determined by means of established techniques of naval hydrodynamics, such as one of the well known strip theories. In this way, the responses of the hull in the seaway may be determined.

This approach was first used to determine symmetric responses, namely the 'rigid body' motions of heave and pitch, distortions in the vertical plane and bending moments and shearing forces at any section of the 'beam-like' hull. The theory is described by Bishop & Price (1974) and Bishop *et al.* (1977).

The theory was subsequently extended by Bishop & Price (1976*a*, *b*) and Bishop *et al.* (1980*a*), to account for antisymmetric responses. These include the rigid-body motions of sway, yaw, and roll, together with antisymmetric distortions of coupled twisting and lateral deflection. These distortions are associated with the additional responses of twisting moment, lateral bending moment and lateral shearing force at any section.

This separation of symmetric from antisymmetric responses depends upon the existence of port and starboard symmetry of the hull. The theory has been extended so as to admit unsymmetrical structures such as the hull of an aircraft carrier (see Bishop *et al.* 1986), the relevant structural members of wave energy devices (see Bishop *et al.* 1980), or a ship having an angle of heel, as when cargo shifts (see Bishop *et al.* 1980; Conceicao *et al.* 1984). In these, no appeal can be made to arguments of symmetry for the purposes of simplification because all responses are coupled.

As an alternative to representation of the beam-like hull as a Timoshenko beam (with possible modification to allow for twisting), representation as a Vlasov beam has been suggested by Bishop *et al.* (1983).

The foregoing investigations for flexible ships have dealt with motion through sinusoidal waves and through irregular waves. Slamming may occur in these types of seaway. Then the wave conditions are sufficiently severe for the forward part of the hull to leave the water; impact occurs at the ship's bottom on re-entry and this results in a severe transient vibration of the hull (Bishop *et al.* 1978; Belik *et al.* 1980). The concept of modal analysis has been successfully applied to describe the behaviour of the transient responses after slamming. Computer time simulations of the behaviour of ships travelling in irregular seaways have been made and these show good correlation with measured results from full-scale trials in which ship slamming occurred (Bishop *et al.* 1984; Clarke *et al.* 1984).

In all these investigations the fluid actions have been determined by means of a suitable strip theory or two-dimensional hydrodynamic theory; see, for example, Gerritsma & Beukelman (1964); Vugts (1971); Salvesen *et al.* (1970). This imposes severe limitations on the use of existing hydroelasticity theory. Thus it is not possible to examine the behaviour of non-beam-like flexible structures such as multi-hulls, semi-submersibles, jack-up rigs, fixed structures, etc., either travelling or stationary in a seaway.

This paper discusses a general method which overcomes these objections. It relies on a more complex theoretical model to describe the dynamics of a flexible body of arbitrary shape travelling in a seaway. This new theory covers the rather more rudimentary approaches already mentioned but still relies on a linear structural model, and, for simplicity, a linear hydrodynamic theory. (This latter restriction may be relaxed, and a nonlinear theory adopted instead; but this possibility will not be followed up in this paper.)

Briefly, a linear finite-element approach is used to describe the dynamical behaviour of the three-dimensional dry structure *in vacuo*. The fluid actions associated with the distorting three-dimensional wet structure are determined from a theoretical hydrodynamic model involving translating, pulsating sources and sinks. A modified modal theory is again used and it is shown that a description of the responses of a floating, flexible structure travelling in waves may be determined from this general hydroelasticity theory. In addition, the theory associated with a floating rigid body travelling in a seaway can also be obtained from the present theory.

3. Structural dynamics

When the floating structure is a slender hull, it has hitherto been treated either as a Timoshenko beam or a Vlasov beam. To remove this restriction on the hull shape we shall use a finite-element approach. As before, the vibration characteristics of the structure *in vacuo* will initially be investigated. It will be shown later, when discussing the hydrodynamic fluid actions on the flexible structure, that the theory requires information on the dynamic characteristics of the dry structure.

In outlining the theory for the structure, we shall refer to a flexible body of arbitrary shape which is not fixed at any point. Being free to float without restraint, the structure will possess rigid-body modes as well as modes of distortion. If, instead, the structure is fixed in some way, there will be no rigid-body modes and there will be some modification of the boundary conditions.

3.1. *A simple finite-element approach*

Consider first a single structural element located by reference to 'global axes' $Oxyz$ (which, as we shall see later, may be a frame of equilibrium axes). A local frame of axes may be erected at the element whence positions in it may be identified by means of local coordinates ξ, η, ζ. The nodes of the element suffer generalized displacements $\overline{U}_e = \{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_N\}$, where N denotes the number of nodes of the element concerned and the overbar signifies that the quantities are expressed in the local coordinate system, while the subscript e means that the quantity relates to an element. A *static* displacement $\bar{\boldsymbol{u}} = \{\bar{u}, \bar{v}, \bar{w}\}$ at any point in the continuous structure may be specified approximately in terms of a finite number of displacements at the nodes. That is,

$$\bar{\boldsymbol{u}}(\xi, \eta, \zeta) \equiv \{\bar{u}, \bar{v}, \bar{w}\} = \boldsymbol{N}(\xi, \eta, \zeta)\, \overline{\boldsymbol{U}}_e,$$

where the matrix $\boldsymbol{N}$ contains suitable shape functions of geometric origin prescribed in terms of the local coordinates. (See Zienkiewicz (1977).)

The relation between $\bar{\boldsymbol{u}}$ and $\overline{\boldsymbol{U}}_e$ is more complicated when these quantities are time-dependent, because inertial forces cause distortion of the element concerned. Nevertheless, if a large enough number of elements are used (so the elements are all sufficiently small) the above relation is adequate, even when relating $\bar{\boldsymbol{u}}(\xi, \eta, \zeta, t)$ to $\overline{\boldsymbol{U}}_e(t)$, provided $\overline{\boldsymbol{U}}_e(t)$ is found from the dynamics equations of the structure (Przemieniecki 1968). This standpoint will be adopted henceforth.

(*a*) *Energy considerations*

The strain–displacement relation is

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{b}\overline{\boldsymbol{U}}_e,$$

where $\boldsymbol{b}$ is obtained by differentiation of the matrix $\boldsymbol{N}$. Now the stresses may be found from a generalized form of Hooke's law,

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\chi}\bar{\boldsymbol{\varepsilon}},$$

the matrix $\boldsymbol{\chi}$ being a suitable matrix of elastic constants. (We shall ignore thermal effects.)

The strain energy of the element is given by

$$\varPi_e = \frac{1}{2}\iiint_{\Omega_e} \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}} \bar{\boldsymbol{\sigma}} \,\mathrm{d}\Omega,$$

where $\boldsymbol{\Omega}_e$ is the volume of the element and the superscripted T denotes transposition. That is,

$$\varPi_e = \frac{1}{2}\iiint_{\Omega_e} \bar{\boldsymbol{U}}_e^{\mathrm{T}} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\chi} \boldsymbol{b} \bar{\boldsymbol{U}}_e \,\mathrm{d}\Omega = \tfrac{1}{2}\bar{\boldsymbol{U}}_e^{\mathrm{T}} \bar{\boldsymbol{K}}_e \bar{\boldsymbol{U}}_e,$$

where

$$\bar{\boldsymbol{K}}_e = \iiint_{\Omega_e} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\chi} \boldsymbol{b} \,\mathrm{d}\Omega.$$

The quantity $\bar{\boldsymbol{K}}_e$ is the symmetric stiffness matrix of the element. Similarly, the kinetic energy of the element is

$$T_e = \frac{1}{2}\iiint_{\Omega_e} \rho_{\mathrm{b}} \dot{\boldsymbol{u}}^{\mathrm{T}} \dot{\boldsymbol{u}} \,\mathrm{d}\Omega = \frac{1}{2}\iiint_{\Omega_e} \rho_{\mathrm{b}} (\boldsymbol{N}\dot{\bar{\boldsymbol{U}}}_e)^{\mathrm{T}} (\boldsymbol{N}\dot{\bar{\boldsymbol{U}}}_e) \,\mathrm{d}\Omega,$$

where ρ_{b} is the density of the structural material. It follows that

$$T_e = \tfrac{1}{2}\dot{\bar{\boldsymbol{U}}}_e^{\mathrm{T}} \bar{\boldsymbol{M}}_e \dot{\bar{\boldsymbol{U}}}_e,$$

the quantity

$$\bar{\boldsymbol{M}}_e = \iiint_{\Omega_e} \boldsymbol{N}^{\mathrm{T}} \rho_{\mathrm{b}} \boldsymbol{N} \,\mathrm{d}\Omega$$

being the symmetric mass matrix of the element, or 'consistent mass matrix' (Zienkiewicz & Cheung 1964).

The contribution to the dissipation function from the element is

$$D = \frac{1}{2}\iiint_{\Omega_e} \dot{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{\beta} \dot{\boldsymbol{u}} \,\mathrm{d}\Omega,$$

where $\boldsymbol{\beta}$ is a specific damping matrix. That is,

$$D = \tfrac{1}{2}\dot{\bar{\boldsymbol{U}}}_e^{\mathrm{T}} \bar{\boldsymbol{\beta}}_e \dot{\bar{\boldsymbol{U}}}_e,$$

where $\bar{\boldsymbol{B}}_e$ is the symmetric damping matrix given by

$$\bar{\boldsymbol{B}}_e = \iiint_{\Omega_e} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{\beta} \boldsymbol{N} \,\mathrm{d}\Omega.$$

(*b*) *Equation of motion of an element*

Internal and external actions may be applied simultaneously to the structure. The former are produced by the interaction between adjacent elements, the actions arising at the common nodal points. These are represented by the matrix $\bar{\boldsymbol{E}}_e$. The external applied forces or moments may be associated with gravity, aerodynamic and hydrodynamic actions, moorings, etc. These loadings are either distributed through or over the structure or are concentrated at particular points.

When a structure floats in a fluid, a distributed hydrodynamic pressure field p acts over the wetted surface. From the principle of virtual work, the corresponding generalized force is found to be

$$\bar{\boldsymbol{P}}_e = -\iint_{S_e} p \boldsymbol{N}^{\mathrm{T}} \bar{\boldsymbol{n}} \,\mathrm{d}S = \{\bar{P}_{e1}, \bar{P}_{e2}, \ldots\},$$

where $\bar{\boldsymbol{n}}$ is the outward unit vector on the wetted surface S_e, measured in the local element coordinate system.

A mooring force, a propulsion force or the actions of a mechanical exciter are examples of a concentrated load acting on the structure. If, in the local coordinate system, the column matrix $\boldsymbol{f}_i(t)$ denotes the components of the concentrated load acting at an arbitrary point (ξ_i, η_i, ζ_i), then the generalized force matrix associated with all such concentrated loads, i, may be written as

$$\bar{\boldsymbol{F}}_e = \sum_i \boldsymbol{N}^{\mathrm{T}}(\xi_i, \eta_i, \zeta_i)\boldsymbol{f}_i(t).$$

The equations of motion of an element may be found once the energy functions and generalized forces have been determined. Thus by the use of Lagrange's equation (see Bishop *et al.* 1965) it is found that the matrix equation of motion for an element is

$$\bar{\boldsymbol{M}}_e \ddot{\bar{\boldsymbol{U}}}_e + \bar{\boldsymbol{B}}_e \dot{\bar{\boldsymbol{U}}}_e + \bar{\boldsymbol{K}}_e \bar{\boldsymbol{U}}_e = \bar{\boldsymbol{E}}_e + \bar{\boldsymbol{P}}_e + \bar{\boldsymbol{F}}_e + \bar{\boldsymbol{g}}_e.$$

Here the column matrix $\bar{\boldsymbol{g}}_e$ represents the generalized gravitational forces whose value will be discussed later.

(*c*) *General equation of motion*

It has been found by Bishop & Price (1979) that the hydroelasticity problem of a slender ship hull is much simplified by the use of an 'equilibrium' frame of coordinates Oxyz. Such axes are particularly helpful in the description of the externally applied fluid actions. To use this technique here, it is necessary to construct a matrix $\boldsymbol{L}$ which transforms a displacement $\bar{\boldsymbol{U}}_e$ in the local coordinate system O$\xi\eta\zeta$ to a displacement $\boldsymbol{U}_e$ in the equilibrium coordinate system. That is, a relation

$$\bar{\boldsymbol{U}}_e = \boldsymbol{L}\boldsymbol{U}_e$$

must be found, where $\boldsymbol{L}$ is a band matrix with each diagonal submatrix of the form

$$\boldsymbol{l} = \begin{bmatrix} \cos(x, \xi) & \cos(y, \xi) & \cos(z, \xi) \\ \cos(x, \eta) & \cos(y, \eta) & \cos(z, \eta) \\ \cos(x, \zeta) & \cos(y, \zeta) & \cos(z, \zeta) \end{bmatrix}.$$

That is, the components of $\boldsymbol{l}$ are the direction cosines of the angles formed between the two sets of axes and so $\boldsymbol{l}$ is an orthogonal matrix. It can be shown further that the displacement $\boldsymbol{u} = \{u, v, w\}$ and the normal vector $\boldsymbol{n}$ at any point in the equilibrium axis system, and those of $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{n}}$ in the local axis system, satisfy the relations

$$\bar{\boldsymbol{u}} = \boldsymbol{l}\boldsymbol{u} \quad \text{and} \quad \bar{\boldsymbol{n}} = \boldsymbol{l}\boldsymbol{n}$$

respectively.

If this transformation is applied to all the variables, for example, $\bar{\boldsymbol{P}}_e = \boldsymbol{L}\boldsymbol{P}_e$, the change may be made apparent by removal of the various overbars. The elemental equation of motion is then stated in the equilibrium coordinate system as

$$\boldsymbol{M}_e \ddot{\boldsymbol{U}}_e + \boldsymbol{B}_e \dot{\boldsymbol{U}}_e + \boldsymbol{K}_e \boldsymbol{U}_e = \boldsymbol{E}_e + \boldsymbol{P}_e + \boldsymbol{F}_e + \boldsymbol{g}_e.$$

It is convenient to give the expression for $\boldsymbol{g}_e$ at this point because it is best to take the equilibrium axes with the axis Oz vertical. We assume that it points upwards. The virtual work is given by

$$\delta \boldsymbol{U}_e^{\mathrm{T}} \boldsymbol{g}_e = -\iiint_{\Omega_e} \rho_{\mathrm{b}} g \delta w_e \,\mathrm{d}\Omega,$$

whence

$$\boldsymbol{g}_e = -\iiint_{\Omega_e} \rho_{\mathrm{b}} g \boldsymbol{L}^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} l_3 \,\mathrm{d}\Omega,$$

where

$$l_3 = \{\cos(z, \xi), \cos(z, \eta), \cos(z, \zeta)\}.$$

To obtain a complete solution it is necessary for the conditions of compatibility and equilibrium to be satisfied throughout the structure. The compatibility condition is automatically satisfied if the nodal displacement for an element, $\boldsymbol{U}_e$, is re-labelled suitably so that it is replaced by an identical nodal displacement

$$\boldsymbol{U} = \{\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_n\},$$

defined throughout the whole structure. For equilibrium to prevail throughout the structure at all common nodes between adjacent elements, it is necessary that

$$\sum_e \boldsymbol{E}_{ej} = 0, \quad (j = 1, 2, \ldots, n),$$

where the summation extends over all the elements meeting at the common jth node. Thus the matrix equation of motion for an element may be replaced by a general equation of motion which, in matrix form, is expressible as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{B}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} + \boldsymbol{F} + \boldsymbol{g}.$$

The matrices $\boldsymbol{M}$, $\boldsymbol{B}$ and $\boldsymbol{K}$ contain $n \times n$ submatrices, each of which contains 6×6 elements, n being the number of nodes; they are referred to as the mass, damping, and stiffness matrices respectively. The matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are positive semi-definite or positive definite, depending on the boundary restraints imposed on the structure. The column matrices of generalized forces associated with the loading, i.e.

$$\boldsymbol{P} = \{P_1, P_2, \ldots, P_n\},$$

$$\boldsymbol{g} = \{g_1, g_2, \ldots, g_n\},$$

$$\boldsymbol{F} = \{F_1, F_2, \ldots, F_n\},$$

are of order $(n \times 1)$.

This equation is a generalization of equations of motion discussed by Bishop & Price (1979). When they are derived in this way the matrices $\boldsymbol{M}$, $\boldsymbol{B}$ and $\boldsymbol{K}$ are real and symmetric because they are associated with the dry structure. The fluid loading is simply represented by an external generalized force matrix $\boldsymbol{P}$, but in practice its evaluation may be difficult. Although we shall discuss its form in some detail, it is worth noting some of the difficulties at this stage.

(i) $\boldsymbol{P}$ depends on the motions and distortions of the structure.

(ii) Even in a linear formulation of an expression for $\boldsymbol{P}$, matrix symmetry may provide no assistance, particularly if the structure has forward speed.

(iii) All the fluid terms depend on the encounter frequency with which waves meet the moving structure.

(iv) The passage of waves along the surface of the structure complicates the form of the wave excitation.

(v) Nonlinear fluid actions have to be included in the mathematical model of the fluid loading of some fixed structures.

Although these difficulties are not insuperable, the only practical approach to their solution is that of numerical analysis. Once the fluid loading terms have been found, the equations for $\boldsymbol{U}$ can be solved (see, for example, Zienkiewicz & Bettess (1982)). It has been shown by Bishop & Price (1979), however, that the principal coordinates of the dry structure bestow significant advantages in the subsequent analyses.

3.2. *Natural frequencies and principal modes*

If the damping and the forcing terms are ignored, the equation of motion reduces to

$$\boldsymbol{M}\ddot{\boldsymbol{U}}+\boldsymbol{K}\boldsymbol{U}=\boldsymbol{0}.$$

The trial solution

$$\boldsymbol{U}=\boldsymbol{D}\,\mathrm{e}^{\mathrm{i}\omega t}$$

shows that non-trivial amplitude matrices $\boldsymbol{D}$ exist provided the characteristic equation

$$|\boldsymbol{K}-\omega^2\boldsymbol{M}|=0$$

is satisfied. The real and positive eigenvalues ω_r $(r=1, 2, \ldots, m)$ are the natural frequencies and each is associated with a characteristic 'eigenvector'

$$\boldsymbol{D}_r=\{\boldsymbol{D}_{r_1}, \boldsymbol{D}_{r_2}, \ldots, \boldsymbol{D}_{r_m}\},$$

giving the rth principal mode. Here m is the total number of degrees of freedom of the dry structure.

The generalized displacement vector of the rth principal mode at the jth node is

$$\boldsymbol{D}_{r_j}=\{u_r, v_r, w_r, \theta_{x_r}, \theta_{y_r}, \theta_{z_r}\}_j$$

if the elements concerned are to maintain compatibility in displacement and slope. Any one element is associated with more than one node. A submatrix of $\boldsymbol{D}_r$, denoted by $\boldsymbol{d}_r$, may be formed for the one element, so that

$$\boldsymbol{U}_{e_r}=\boldsymbol{d}_r\,\mathrm{e}^{\mathrm{i}\omega_r t}.$$

The rth mode shape at any point in the element is then

$$\boldsymbol{u}_r\equiv\{u_r, v_r, w_r\}=\boldsymbol{l}^{-1}\bar{\boldsymbol{u}}_{e_r}=\boldsymbol{l}^{\mathrm{T}}\boldsymbol{N}\bar{\boldsymbol{U}}_{e_r}=\boldsymbol{l}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{L}\,\boldsymbol{d}_r\,\mathrm{e}^{\mathrm{i}\omega_r t},$$

$\boldsymbol{l}$ being an orthogonal matrix so that $\boldsymbol{l}^{\mathrm{T}}\boldsymbol{l}=\boldsymbol{I}$, the unit matrix.

(*a*) *Rigid-body modes*

The body motions of floating structures are frequently of interest. The body is unanchored and the stiffness matrix $\boldsymbol{K}$ is positive semi-definite so that $|\boldsymbol{K}|=0$. The frequency equation then has six zero roots.

In seakeeping theory the rigid-body modes are conveniently chosen to be the three components of displacement $u_{\mathrm{c}}, v_{\mathrm{c}}, w_{\mathrm{c}}$, of the centre of mass C, and the three components of small rotations θ_{xc}, θ_{yc}, θ_{zc}, about the body axes whose origin is fixed at C. If the body axes are aligned with the equilibrium axes in the equilibrium position it is found (for example, by evaluating a non-zero column of the adjoint of the matrix $\boldsymbol{K}-\omega^2\boldsymbol{M}$) that, in general,

$$\boldsymbol{D}_{r_j} = \begin{bmatrix} 1 & 0 & 0 & 0 & z_j-z_c & -(y_j-y_c) \\ 0 & 1 & 0 & -(z_j-z_c) & 0 & x_j-x_c \\ 0 & 0 & 1 & y_j-y_c & -(x_j-x_c) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}_r u_c \\ {}_r v_c \\ {}_r w_c \\ {}_r\theta_{xc} \\ {}_r\theta_{yc} \\ {}_r\theta_{zc} \end{bmatrix},$$

($r = 1, 2, 3, \ldots, 6$). The quantities ${}_r u_c$, ${}_r v_c$, ${}_r w_c$, ${}_r\theta_{xc}$, ${}_r\theta_{yc}$, ${}_r\theta_{zc}$ are arbitrary constants, and by suitably selecting their values the rigid-body modes may be specified in various ways.

(*b*) *Orthogonality conditions*

Let $\boldsymbol{D}_r$ and $\boldsymbol{D}_s$ be two of the principal modes. Evidently,

$$\boldsymbol{K}\boldsymbol{D}_s = \omega_s^2 \boldsymbol{M}\boldsymbol{D}_s$$

and

$$\boldsymbol{K}\boldsymbol{D}_r = \omega_r^2 \boldsymbol{M}\boldsymbol{D}_r.$$

Pre-multiply the first equation by $\boldsymbol{D}_r^{\mathrm{T}}$ and post-multiply the transpose of the second by $\boldsymbol{D}_s$. By subtraction, the result

$$(\omega_s^2-\omega_r^2)\,\boldsymbol{D}_r^{\mathrm{T}}\boldsymbol{M}\boldsymbol{D}_s = 0$$

is obtained. It follows that

$$\boldsymbol{D}_r^{\mathrm{T}}\boldsymbol{M}\boldsymbol{D}_s = \delta_{rs}\,a_{rs},$$

where δ_{rs} is the Kronecker delta function, defined by

$$\delta_{rs} = \begin{cases} 0 & \text{for } r \neq s, \\ 1 & \text{for } r = s. \end{cases}$$

This relation between $\boldsymbol{D}_r$ and $\boldsymbol{D}_s$ is that of orthogonality.

It will be seen that an alternative statement of the orthogonality principle is

$$\boldsymbol{D}_r^{\mathrm{T}}\boldsymbol{K}\boldsymbol{D}_s = \delta_{rs}\,\omega_r^2\,a_{rs} = \delta_{rs}\,c_{rs}.$$

The quantities a_{ss}, c_{ss} represent generalized mass and generalized stiffness associated with the sth principal mode. Their properties are discussed by various authors (see, for example, Bishop *et al.* 1965). Their values depend upon the scaling of the sth principal mode. The orthogonality relations remain valid when one of the modes is a rigid-body mode, though special care must be taken when both modes are rigid-body modes (see §3.2 (*d*)).

It is convenient to assemble a matrix of principal modes

$$\boldsymbol{D} = [\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots, D_m],$$

each column being a mode. The orthogonality relations may now be expressed as

$$\boldsymbol{D}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{D} = \boldsymbol{a}$$

and

$$\boldsymbol{D}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{D} = \boldsymbol{c},$$

where $\boldsymbol{a}$ and $\boldsymbol{c}$ are generalized mass and stiffness matrices respectively, both symmetric such that $c_{ss} = \omega_s^2\,a_{ss}$.

(*c*) *Principal coordinates*

The total deflection and distortion of the structure may be expressed as the sum of displacements in the principal modes. It follows that the matrix of nodal displacements may be expressed as

$$\boldsymbol{U} = \sum_{r=1}^{m} p_r(t)\, \boldsymbol{D}_r.$$

The displacement at any point is then

$$\boldsymbol{u} = \{u, v, w\} = \sum_{r=1}^{m} p_r(t)\, \boldsymbol{u}_r,$$

where $p_r(t)$ $(r = 1, 2, \ldots, m)$ are a set of principal coordinates.

It follows from this definition of the $p_r(t)$ that

$$\boldsymbol{U} = \boldsymbol{D}\boldsymbol{p},$$

the matrix $\boldsymbol{p}$ being the vector $\{p_1(t), p_2(t), \ldots, p_m(t)\}$. If this expression for $\boldsymbol{U}$ is substituted in the general equation of motion and that equation is pre-multiplied by $\boldsymbol{D}^{\mathrm{T}}$, it is found that

$$\boldsymbol{a}\ddot{\boldsymbol{p}} + \boldsymbol{b}\dot{\boldsymbol{p}} + \boldsymbol{c}\boldsymbol{p} = \boldsymbol{Z} + \boldsymbol{G} + \boldsymbol{\Delta},$$

where the significance of $\boldsymbol{b}$, $\boldsymbol{Z}$, $\boldsymbol{G}$ and $\boldsymbol{\Delta}$ will be examined individually.

The matrix

$$\boldsymbol{b} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{D}$$

represents the damping in terms of the principal coordinates. It is symmetric but not, in general, diagonal. Now damping is not well understood and, *faute de mieux*, it is common to make the assumption that $\boldsymbol{B}$ is expressible in the form

$$\boldsymbol{B} = \alpha\boldsymbol{M} + \beta\boldsymbol{K},$$

where α and β are constants. This makes $\boldsymbol{b}$ diagonal so that the matrix equation of motion reduces to a set of uncoupled scalar equations,

$$a_{rr}\ddot{p}_r + b_{rr}\dot{p}_r + c_{rr}p_r = Z_r + G_r + \Delta_r \quad (r = 1, 2, \ldots, m),$$

provided that $\boldsymbol{a}$ is diagonal (for instance when the structure has no rigid-body motion, a matter which is further discussed in §3.2 (*d*)).

The generalized fluid force corresponding to $\boldsymbol{p}$ is

$$\boldsymbol{Z} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} = \{Z_1, Z_2, \ldots, Z_m).$$

It follows that the rth component, corresponding to p_r, is

$$Z_r = \boldsymbol{D}_r^{\mathrm{T}}\,\boldsymbol{P} = \sum_e \boldsymbol{d}_r^{\mathrm{T}}\,\boldsymbol{P}_e,$$

where the summation is over all elements on the wetted surface.

The other generalized distributed force is that of gravity, namely

$$\boldsymbol{G} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{g} = \{G_1, G_2, \ldots, G_m\}.$$

The term corresponding to p_r is

$$G_r = \boldsymbol{D}_r^{\mathrm{T}} \boldsymbol{g} = \sum_e \boldsymbol{d}_r^{\mathrm{T}} \boldsymbol{g}_e,$$

where the summation is now over all elements.

Finally, the matrix representing concentrated forces at the principal coordinates is

$$\boldsymbol{\Delta} = \boldsymbol{D}^{\mathrm{T}} \boldsymbol{F} = \{\Delta_1, \Delta_2, \ldots, \Delta_m\}.$$

The generalized concentrated force at p_r is

$$\Delta_r = \boldsymbol{D}_r^{\mathrm{T}} \boldsymbol{F} = \sum_e \boldsymbol{d}_r^{\mathrm{T}} \boldsymbol{F}_e,$$

the summation being over all elements that are subject to a concentrated load.

(*d*) *Separation of rigid and distortion modes*

The principal coordinates $\boldsymbol{p}$ fall naturally into two groups, $\boldsymbol{p}_{\mathrm{R}}$ and $\boldsymbol{p}_{\mathrm{D}}$. That is,

$$\boldsymbol{p} = \{\boldsymbol{p}_{\mathrm{R}}, \boldsymbol{p}_{\mathrm{D}}\},$$

where

$$\boldsymbol{p}_{\mathrm{R}} = \{p_1, p_2, \ldots, p_6\}$$

refers to the 'rigid-body' modes and

$$\boldsymbol{p}_{\mathrm{D}} = \{p_7, p_8, \ldots\}$$

refers to the 'flexible-body' or 'distortion' modes. It will be helpful to summarize the results that can be derived simply when this distinction is made.

The matrix equation of motion may be partitioned to give

$$\begin{bmatrix} \boldsymbol{a}_{\mathrm{R}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{a}_{\mathrm{D}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{p}}_{\mathrm{R}} \\ \ddot{\boldsymbol{p}}_{\mathrm{D}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{b}_{\mathrm{D}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_{\mathrm{R}} \\ \dot{\boldsymbol{p}}_{\mathrm{D}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{C}_{\mathrm{D}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{p}}_{\mathrm{R}} \\ \boldsymbol{p}_{\mathrm{D}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_{\mathrm{R}} \\ \boldsymbol{Z}_{\mathrm{D}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_{\mathrm{R}} \\ \boldsymbol{G}_{\mathrm{D}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Delta}_{\mathrm{R}} \\ \boldsymbol{\Delta}_{\mathrm{D}} \end{bmatrix},$$

$\boldsymbol{0}$ being the null matrix. The matrices $\boldsymbol{a}_{\mathrm{D}}$ and $\boldsymbol{c}_{\mathrm{D}}$ are diagonal while $\boldsymbol{b}_{\mathrm{D}}$ is square and symmetric.

If interest centres on rigid-body motions only, the equation

$$\boldsymbol{a}_{\mathrm{R}} \ddot{\boldsymbol{p}}_{\mathrm{R}} = \boldsymbol{Z}_{\mathrm{R}} + \boldsymbol{G}_{\mathrm{R}} + \boldsymbol{\Delta}_{\mathrm{R}}$$

has to be addressed. It will be understood that this equation does not exclude the effect of distortions which arise from the hydrodynamic terms on the right-hand side of the equation.

Case A. Because an eigenvector for a rigid-body mode may be scaled arbitrarily and normalized as desired, we may express the relevant modes in the familiar terms of surge, sway, heave, roll, pitch, and yaw. Then, for the jth node,

$$\begin{aligned}
\boldsymbol{D}_{1_j} &= \{1, 0, 0, 0, 0, 0\}, \\
\boldsymbol{D}_{2_j} &= \{0, 1, 0, 0, 0, 0\}, \\
\boldsymbol{D}_{3_j} &= \{0, 0, 1, 0, 0, 0\}, \\
\boldsymbol{D}_{4_j} &= \{0, -(z_j - z_{\mathrm{c}}), (y_j - y_{\mathrm{c}}), 1, 0, 0\}, \\
\boldsymbol{D}_{5_j} &= \{(z_j - z_{\mathrm{c}}), 0, -(x_j - x_{\mathrm{c}}), 0, 1, 0\}, \\
\boldsymbol{D}_{6_j} &= \{-(y_j - y_{\mathrm{c}}), (x_j - x_{\mathrm{c}}), 0, 0, 0, 1\},
\end{aligned}$$

where (x_c, y_c, z_c) is the centre of mass relative to the chosen axes. The rotation of the structure is then specified by reference to the centre of mass.

The displacements at any point (x, y, z) of the rigid body are

$$\begin{aligned} \boldsymbol{u}_1 &= \{1, 0, 0\}, \\ \boldsymbol{u}_2 &= \{0, 1, 0\}, \\ \boldsymbol{u}_3 &= \{0, 0, 1\}, \\ \boldsymbol{u}_4 &= \{0, -(z-z_c), (y-y_c)\}, \\ \boldsymbol{u}_5 &= \{(z-z_c), 0, -(x-x_c)\}, \\ \boldsymbol{u}_6 &= \{-(y-y_c), (x-x_c), 0\}. \end{aligned}$$

Corresponding to this specification of the rigid-body modes,

$$\boldsymbol{a}_{\mathrm{R}} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{44} & -I_{45} & -I_{46} \\ 0 & 0 & 0 & -I_{54} & I_{55} & -I_{56} \\ 0 & 0 & 0 & -I_{64} & -I_{65} & I_{66} \end{bmatrix}.$$

Here, m is the total mass of the structure and the quantities $I_{44}, I_{45}, \ldots$, are the moments and products of inertia given by

$$a_{46} = -I_{46} = -\iiint_{\Omega} \rho_{\mathrm{b}}(x-x_c)\,(z-z_c)\,\mathrm{d}\Omega = -I_{64} = a_{64},$$

$$a_{45} = -I_{45} = -\iiint_{\Omega} \rho_{\mathrm{b}}(x-x_c)\,(y-y_c)\,\mathrm{d}\Omega = -I_{54} = a_{54},$$

$$a_{56} = -I_{56} = -\iiint_{\Omega} \rho_{\mathrm{b}}(y-y_c)\,(z-z_c)\,\mathrm{d}\Omega = -I_{65} = a_{65},$$

where Ω is the volume of the structure.

For a structure with port and starboard symmetry and an axis system chosen to have its origin and two axes Ox, Oz in this plane of symmetry.

$$a_{45} = 0 = a_{56} \quad \text{but} \quad a_{46} \neq 0.$$

This is in line with the approach that is commonly adopted in seakeeping theory.

Case B. An alternative specification of the rigid-body modes has been used by Bishop & Price (1979) because it simplifies the analysis of distortions. The modes are so scaled that displacement at the stern of a ship is unity; let us take the reference point as $\{x_l, y_l, z_l\}$. The rigid-body modes are then given by

$$\boldsymbol{D}_{1_j} = \{1, 0, 0, 0, 0, 0\},$$

$$\boldsymbol{D}_{2_j} = \{0, 1, 0, 0, 0, 0\},$$

$$\boldsymbol{D}_{3_j} = \{0, 0, 1, 0, 0, 0\},$$

$$D_{4_j} = \left\{0,\ -\frac{z_j-z_c}{z_l-z_c},\ \frac{y_j-y_c}{z_l-z_c},\ \frac{1}{z_l-z_c},\ 0,\ 0\right\},$$

$$D_{5_j} = \left\{\frac{z_j-z_c}{x_l-x_c},\ 0,\ -\frac{x_j-x_c}{x_l-x_c},\ 0,\ \frac{1}{x_l-x_c},\ 0\right\},$$

$$D_{6_j} = \left\{-\frac{y_j-y_c}{x_l-x_c},\ \frac{x_j-x_c}{x_l-x_c},\ 0,\ 0,\ 0,\ \frac{1}{x_l-x_c}\right\}.$$

The displacements at any point (x, y, z) of the rigid body corresponding to these modes are

$$u_1 = \{1, 0, 0\},$$

$$u_2 = \{0, 1, 0\},$$

$$u_3 = \{0, 0, 1\},$$

$$u_4 = \left\{0,\ -\frac{z-z_c}{z_l-z_c},\ \frac{y-y_c}{z_l-z_c}\right\},$$

$$u_5 = \left\{\frac{z-z_c}{x_l-x_c},\ 0,\ -\frac{x-x_c}{x_l-x_c}\right\},$$

$$u_6 = \left\{-\frac{y-y_c}{x_l-x_c},\ \frac{x-x_c}{x_l-x_c},\ 0\right\}.$$

The generalized mass matrix now takes the same form as that in case A but with the quantities $I_{44}, I_{45}, \ldots, I_{66}$ replaced by $I'_{44}, I'_{45}, \ldots, I'_{66}$ where, for example,

$$I'_{46} = \iiint_{\Omega} \rho_b \left(\frac{x-x_c}{x_l-x_c}\right)\left(\frac{z-z_c}{z_l-z_c}\right) d\Omega = \frac{I_{46}}{(x_l-x_c)\,(z_l-z_c)},$$

and so on. Again the rotations are defined with respect to the centre of mass and not the origin of the axes.

Case C. In a more general formulation, the rigid-body modes may be expressed in the form

$$D_{1_j} = \{1, 0, 0, 0, 0, 0\},$$

$$D_{2_j} = \{0, 1, 0, 0, 0, 0\},$$

$$D_{3_j} = \{0, 0, 1, 0, 0, 0\},$$

$$D_{4_j} = \{0,\ -z_j,\ y_j,\ 1,\ 0,\ 0\},$$

$$D_{5_j} = \{z_j,\ 0,\ -x_j,\ 0,\ 1,\ 0\},$$

$$D_{6_j} = \{-y_j,\ x_j,\ 0,\ 0,\ 0,\ 1\},$$

with no reference made to a prescribed point. The displacements at any point (x, y, z) of the rigid body corresponding to these modes are

$$u_1 = \{1, 0, 0\}, \qquad u_4 = \{0,\ -z,\ y\},$$

$$u_2 = \{0, 1, 0\}, \qquad u_5 = \{z,\ 0,\ -x\},$$

$$u_3 = \{0, 0, 1\}, \qquad u_6 = \{-y,\ x,\ 0\}.$$

All rotations are now specified with respect to the origin of axes and not the centre of mass.

The inertia matrix is now

$$\boldsymbol{a}_{\mathrm{R}} = \begin{bmatrix} m & 0 & 0 & 0 & mz_{\mathrm{c}} & -my_{\mathrm{c}} \\ 0 & m & 0 & -mz_{\mathrm{c}} & 0 & mx_{\mathrm{c}} \\ 0 & 0 & m & my_{\mathrm{c}} & -mx_{\mathrm{c}} & 0 \\ 0 & -mz_{\mathrm{c}} & my_{\mathrm{c}} & I''_{44} & -I''_{45} & -I''_{46} \\ mz_{\mathrm{c}} & 0 & -mx_{\mathrm{c}} & -I''_{54} & I''_{55} & -I''_{56} \\ -my_{\mathrm{c}} & mx_{\mathrm{c}} & 0 & -I''_{64} & -I''_{65} & I''_{66} \end{bmatrix}.$$

The moments and products of inertia $I''_{44}, I''_{45}, \ldots, I''_{66}$ are now given by

$$I''_{46} = \iiint_{\Omega} \rho_{\mathrm{b}}\, xz \,\mathrm{d}\Omega,$$

and so forth.

The introduction of these rigid-body modes implies that the matrix equation of motion quoted in §3.2 (c) now has the elemental form

$$\sum_{k=1}^{m} [a_{rk}\ddot{p}_k(t) + b_{rk}\dot{p}_k(t)] + c_{rr}p_r(t) = Z_r(t) + G_r + \Delta_r$$

for $r = 1, 2, \ldots, m$. The damping matrix $\boldsymbol{b}$ is represented by the more general form, which is not necessarily diagonal.

4. Fluid–structure interaction

Figure 1 shows the three right-handed systems of axes which will be used to define the fluid actions. $Ax_0y_0z_0$ is a fixed frame of reference; Oxyz is an equilibrium set of axes (the 'global axes' referred to previously) moving with forward speed $\bar{U}$ and remaining parallel to $Ax_0y_0z_0$; O$'x'y'z'$ is an axis system fixed in the structure at O$'$ such that it coincides with Oxyz in the

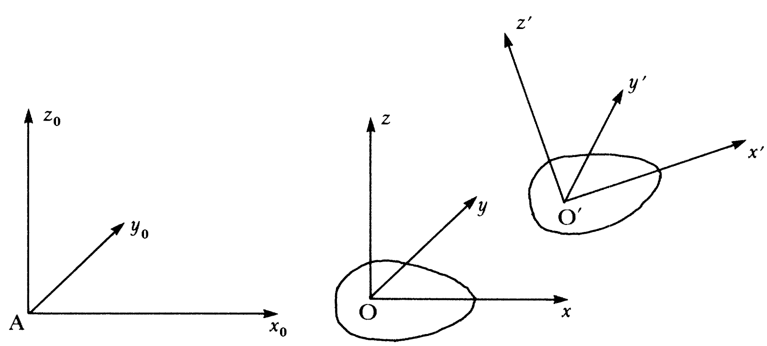


FIGURE 1. Right-handed systems of axes used to define the fluid actions and structural dynamic characteristics.

absence of any disturbance. The origins O, O$'$ are located at a convenient position in the body, usually on the line formed by the intersection of a longitudinal plane of symmetry and the calm water surface.

The structure travels in deep water and moves with a constant speed $\bar{U}$ at a heading angle χ (180°, in head waves) with respect to regular sinusoidal waves of amplitude a, frequency ω

and wave number k. This means that the wavelength is

$$\lambda = 2\pi/k = (2\pi g)/\omega^2$$

and the encounter frequency is (see, for example, Bishop & Price 1979)

$$\omega_e = \omega - (\bar{U}\omega^2/g)\cos\chi.$$

The fluid loading produced by such waves causes deflections in the flexible structure, these deflections being a combination of body motions and distortions. Only the rigid 'body motions' are admitted in theory of seakeeping, accounting for deflections in the first six principal modes of the dry structure. These modes thus form a subset of the infinite number of modes required to describe the dynamic characteristics of the dry structure. By suitably modifying the fluid actions of existing seakeeping theory, it is possible to allow for a flexible structure.

If the fluid is ideal (i.e. inviscid and incompressible) and its flow is irrotational, there exists a potential function $\Phi(x_0, y_0, z_0, t)$ satisfying the Laplace equation $\nabla^2\Phi = 0$ and such that the fluid velocity $V(x_0, y_0, z_0, t)$ is

$$V = \operatorname{grad}\Phi = \nabla\Phi.$$

Newman (1977, 1978) has shown that this potential satisfies the following boundary conditions.

(i) On the free surface, $z_0 = \zeta$,

$$\Phi_{tt} + 2\nabla\Phi\cdot\nabla\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla(\nabla\Phi\cdot\nabla\Phi) + gz_0 = 0,$$

where $\Phi_{tt} = \partial^2\Phi/\partial t^2$, etc., and ζ is the elevation of the wave surface. (Ideally this surface condition is used, but regrettably linearization is required.)

(ii) On the sea bed, $z_0 = -d$,

$$\Phi_{z_0} = 0.$$

(iii) A suitable far-field boundary condition.

(iv) On the moving wetted surface area S of a floating structure at any instant,

$$\partial\Phi/\partial n = \Phi_n = V^s\cdot n,$$

where V^s denotes the local velocity on the wetted surface S and n is the outward unit normal vector into the fluid.

(Although the symbol S is principally associated, as here, with an instantaneous configuration of the wetted surface, it will also be used to denote a time-dependent departure of configuration from the steady or mean configuration $\bar{S}$ associated with the distortions of a flexible body. For example, if the body is totally submerged the wetted surface area of the structure remains practically constant but because of the distortions there still exist states S and $\bar{S}$.)

By means of a simple transformation this total potential may be represented in the equilibrium frame of axes as

$$\Phi(x_0, y_0, z_0, t) = \bar{U}\bar{\phi}(x, y, z) + \phi(x, y, z, t),$$

where $\bar{\phi}$, ϕ denote velocity potentials for the steady motion of the structure in calm water and the unsteady forced motion in waves respectively. In addition, when there is only a steady motion, the velocity of the steady flow relative to the moving equilibrium frame of reference is

$$W = \bar{U}\operatorname{grad}(\bar{\phi} - x)$$

and the body boundary condition (iv) takes the form $W\cdot n = 0$ on the mean wetted surface $\bar{S}$.

4.1. *Principal coordinates and displacements*

The structure in the fluid is excited into a parasitic motion by the waves (i.e. it executes a forced response). According to Rayleigh (1894), any distortion of the structure may be expressed as an aggregate of distortions in its principal modes. That is, the deflection of the floating structure, defined in the equilibrium axis system Oxyz, may be expressed as

$$u(x, y, z, t) = \sum_{r=1}^{\infty} p_r(t)\, u_r(x, y, z),$$

$$v(x, y, z, t) = \sum_{r=1}^{\infty} p_r(t)\, v_r(x, y, z),$$

$$w(x, y, z, t) = \sum_{r=1}^{\infty} p_r(t)\, w_r(x, y, z),$$

where $p_r(t)$ is the rth principal coordinate and u_r, v_r, w_r are the components of deflection in the directions Ox, Oy, Oz of the rth principal mode of the dry hull. These latter functions are defined with respect to the mean equilibrium position of the floating structure in which the axis systems Oxyz, O$'x'y'z'$ initially coincide. Therefore, by adopting a suitable transformation, these mode shapes may be expressed as functions of (x', y', z'), and by representing the principal mode in the vector form

$$\boldsymbol{u}_r(x', y', z') = u_r\,\hat{\imath} + v_r\,\hat{\jmath} + w_r\,\hat{k} = \{u_r, v_r, w_r\},$$

the deflection vector may be written as

$$\boldsymbol{u}(x', y', z', t) = u(x', y', z', t)\,\hat{\imath} + v(x', y', z', t)\,\hat{\jmath} + w(x', y', z', t)\,\hat{k} = \sum_{r=1}^{\infty} p_r(t)\,\boldsymbol{u}_r.$$

The velocity of any point (x', y', z') on the surface of the flexible structure travelling with forward speed $\overline{U}$ can be found from this result. It is given by

$$\boldsymbol{V}^s(x', y', z', t) = \overline{U}\hat{\imath} + \dot{\boldsymbol{u}} = \overline{U}\hat{\imath} + \sum_{r=1}^{\infty} \dot{p}_r(t)\,\boldsymbol{u}_r.$$

In a similar manner, the rotation vector at any point (x', y', z') is given by

$$\boldsymbol{\Theta}(x', y', z', t) = \sum_{r=1}^{\infty} p_r(t)\,\boldsymbol{\theta}_r,$$

where

$$\boldsymbol{\theta}_r(x', y', z', t) = \{\theta_{xr}, \theta_{yr}, \theta_{zr}\} = \tfrac{1}{2}\operatorname{curl}\boldsymbol{u}_r = \frac{1}{2}\left[\left(\frac{\partial w_r}{\partial y'} - \frac{\partial v_r}{\partial z'}\right)\hat{\imath} + \left(\frac{\partial u_r}{\partial z'} - \frac{\partial w_r}{\partial x'}\right)\hat{\jmath} + \left(\frac{\partial v_r}{\partial x'} - \frac{\partial u_r}{\partial y'}\right)\hat{k}\right].$$

The direction of the unit vector $\boldsymbol{n}$ changes because the structure suffers a time-dependent distortion. This direction is unaffected by a pure translation of the whole body and is wholly dependent on rotations of the structure. Thus if $\boldsymbol{n}|_s$, $\boldsymbol{n}|_{\bar{s}}$ denote unit vectors at some point on the wetted surface relating to the disturbed and steady-state conditions respectively,

$$\boldsymbol{n}|_s = \boldsymbol{n}|_{\bar{s}} + \boldsymbol{\Theta} \times \boldsymbol{n}|_{\bar{s}}$$

to a first approximation. If the velocity of the steady flow is to be described by reference to the moving reference frame, O$'x'y'z'$ the description must be modified because it depends on which state, S or $\bar{S}$, is being considered. Thus to a first approximation the variation of $\boldsymbol{W}|_s$ about $\boldsymbol{W}|_{\bar{s}}$ owing to a parasitic deflection $\boldsymbol{u}$ may be expressed as

$$\begin{aligned}\boldsymbol{W}|_s &= [1+(\boldsymbol{u}\cdot\nabla)]\,\boldsymbol{W}|_{\bar{s}} \\ &= [1+(\boldsymbol{u}\cdot\nabla)]\,\{\bar{U}\nabla(\bar{\phi}-x)\}|_{\bar{s}} = \bar{U}\nabla(\bar{\phi}-x)\,|_s.\end{aligned}$$

Now the dry-structure modes $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_6$ defined in §3.2 (d) correspond to the principal coordinates $p_1(t), p_2(t), \ldots, p_6(t)$. Let

$$\boldsymbol{\eta} = \{p_1(t), p_2(t), p_3(t)\}, \quad \boldsymbol{\Omega} = \{p_4(t), p_5(t), p_6(t)\},$$

$$\boldsymbol{\alpha} = \boldsymbol{\eta}+\boldsymbol{\Omega}\times\boldsymbol{r}', \quad \boldsymbol{r}' = \{x', y', z'\}.$$

With this notation, the deflection at x', y', z' is

$$\boldsymbol{u} = \boldsymbol{\alpha}+\sum_{r=7}^{\infty} p_r(t)\,\boldsymbol{u}_r,$$

the rotation is

$$\boldsymbol{\Theta} = \boldsymbol{\Omega}+\sum_{r=7}^{\infty} p_r(t)\,\boldsymbol{\theta}_r,$$

the velocity is

$$\boldsymbol{V}^s = \bar{U}\hat{\imath}+\dot{\boldsymbol{\alpha}}+\sum_{r=7}^{\infty} \dot{p}_r(t)\,\boldsymbol{u}_r,$$

the unit normal is

$$\boldsymbol{n}|_s = \boldsymbol{n}|_{\bar{s}}+\boldsymbol{\Omega}\times\boldsymbol{n}|_{\bar{s}}+\sum_{r=7}^{\infty} p_r(t)\,\boldsymbol{\theta}_r\times\boldsymbol{n}|_{\bar{s}},$$

and the velocity of the steady flow is

$$\boldsymbol{W}|_s = [1+(\alpha\cdot\nabla)]\,\boldsymbol{W}|_{\bar{s}}+\sum_{r=7}^{\infty} p_r(t)\,(\boldsymbol{u}_r\cdot\nabla)\,\boldsymbol{W}|_{\bar{s}}.$$

It is immediately clear that if the structure is rigid, so that there is no distortion possible and $p_r(t) = 0$ for $r \geqslant 7$, then the rigid-body theory of Newman (1977, 1978) results.

Suppose that the structure is such that 'bending' and 'twisting' are identifiable effects and that M_r, V_r, T_r denote the characteristic functions of bending moment, shearing force and twisting moment associated with the dry-hull principal modes. If the structure is placed in fluid the equivalent responses are expressible as

$$M(x, y, z, t) = \sum_{r=7}^{\infty} p_r(t)\,M_r(x, y, z),$$

etc., because no contribution to these loadings arise from the body motions. That is

$$M_1 = 0 = M_2 = \ldots = M_6 = V_1 = V_2 = \ldots = V_6 = T_1 = T_2 = \ldots = T_6.$$

4.2. *Velocity potential*

The unsteady component of the velocity potential function ϕ must include contributions from the distortions of the structure in the fluid as well as the incident and diffracted wave fields. That is, the total potential remains in the form

$$\boldsymbol{\Phi}(x_0, y_0, z_0, t) \equiv \bar{U}\bar{\phi}(x, y, z)+\phi(x, y, z, t),$$

with the unsteady component expressed as

$$\phi(x, y, z, t) = \phi_0(x, y, z, t) + \phi_D(x, y, z, t) + \sum_{r=1}^{\infty} \phi_r(x, y, z, t).$$

The quantities ϕ_0, ϕ_D, ϕ_r denote the incident wave potential, diffracted wave potential and radiation potential arising from the response of the flexible structure.

Because the deflection of the structure may be expressed in the form of a series of distortions in the principal modes, a similar series expression will be adopted for the radiation potentials. That is to say we shall postulate the existence of a series of potentials $\phi_1, \phi_2, \ldots, \phi_6, \phi_7, \ldots$, each corresponding to one of the principal modes of the dry structure and, hence, to one of the principal coordinates. Thus these radiation potentials may be written in the form

$$\phi_r(x, y, z, t) = \phi_r(x, y, z)\, p_r(t)$$

for $r = 1, 2, \ldots, 6, 7, \ldots$.

The unsteady potential for a sinusoidal wave excitation with encounter frequency ω_e thus takes the oscillatory form

$$\begin{aligned} \phi(x, y, z, t) &= \phi(x, y, z)\, e^{i\omega_e t} \\ &= \Big[\phi_0(x, y, z) + \phi_D(x, y, z) + \sum_{r=1}^{\infty} p_r\, \phi_r(x, y, z)\Big]\, e^{i\omega_e t}. \end{aligned}$$

The amplitude of the incident wave potential is

$$\phi_0 = (iga/\omega) \exp\,[kz - ik(x \cos\chi - y \sin\chi)],$$

ϕ_D is the amplitude of the diffracted wave potential, ϕ_r is the amplitude of the radiation potential and the principal coordinates are assumed to be of the form

$$p_r(t) = p_r\, e^{i\omega_e t}$$

in which the amplitude p_r may be complex. In this notation the principal coordinate $p_1(t)$ relates to surge motion, $p_2(t)$ to sway, $p_3(t)$ to heave, $p_4(t)$ to roll, $p_5(t)$ to pitch, $p_6(t)$ to yaw, and $p_7(t)$, $p_8(t), \ldots$, to the distortion responses of the structure.

4.3. *Generalized Timman–Newman relations*

The boundary condition on the instantaneous wetted surface S of the flexible body is

$$\partial \boldsymbol{\Phi}/\partial n = \boldsymbol{V}^s \cdot \boldsymbol{n}.$$

On substituting for $\boldsymbol{\Phi}$ and $\boldsymbol{V}^s$, it is found that

$$\partial \boldsymbol{\Phi}/\partial n = \nabla \boldsymbol{\Phi} \cdot \boldsymbol{n} = (\bar{U} \nabla \bar{\phi} + \nabla \phi) \cdot \boldsymbol{n} = (\bar{U}\hat{\imath} + \dot{\boldsymbol{u}}) \cdot \boldsymbol{n}$$

or

$$\partial \phi/\partial n = (\dot{\boldsymbol{u}} - \boldsymbol{W}) \cdot \boldsymbol{n}$$

on S. However, because quantities may be related to states S and $\bar{S}$ it follows, after neglecting second-order terms in ϕ, $\boldsymbol{u}$ and $\boldsymbol{\Theta}$ that the linearized boundary condition at the wetted surface reduces to

$$\partial \phi/\partial n = [\dot{\boldsymbol{u}} + \boldsymbol{\Theta} \times \boldsymbol{W} - (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{W}] \cdot \boldsymbol{n}$$

on $\bar{S}$ and after further substitutions of these quantities in series forms it follows that

$$\sum_{r=1}^{\infty} [\partial\phi_r/\partial n - \mathrm{i}\omega_e \boldsymbol{u}_r \cdot \boldsymbol{n} - \boldsymbol{\theta}_r \times \boldsymbol{W} \cdot \boldsymbol{n} + (\boldsymbol{u}_r \cdot \nabla)\, \boldsymbol{W} \cdot \boldsymbol{n}]\, p_r\, \mathrm{e}^{\mathrm{i}\omega_e t} = 0$$

on $\bar{S}$. This boundary condition must be satisfied for any arbitrary combination of the quantities p_r. This is always true if the condition is satisfied for each p_r separately, and so

$$\partial\phi_r/\partial n = [\mathrm{i}\omega_e \boldsymbol{u}_r + \boldsymbol{\theta}_r \times \boldsymbol{W} - (\boldsymbol{u}_r \cdot \nabla)\, \boldsymbol{W}] \cdot \boldsymbol{n},$$

on $\bar{S}$ for all $r = 1, 2, \ldots$.

This result is a generalization of the Timman–Newman (Timman & Newman 1962) relation derived previously for the rigid-body modes. This may be verified by discarding all contributions from modes $r = 7, 8, \ldots$. This shows that

$$\partial\phi_r/\partial n = \mathrm{i}\omega_e\, n_r + \bar{U} m_r$$

on $\bar{S}$ for $r = 1, 2, \ldots, 6$, while

$$\boldsymbol{n} = \{n_1, n_2, n_3\}, \quad \boldsymbol{r} \times \boldsymbol{n} = \{n_4, n_5, n_6\}, \quad \boldsymbol{r}' = \{x', y', z'\}$$

$$(\boldsymbol{n} \cdot \nabla)\, \boldsymbol{W} = -\bar{U}(m_1, m_2, m_3), \quad (\boldsymbol{n} \cdot \nabla)\, (\boldsymbol{r}' \times \boldsymbol{W}) = -\bar{U}(m_4, m_5, m_6).$$

Unfortunately, regardless of whether the structure is treated as rigid or flexible, the steady motion problem in calm water (involving $\bar{\phi}$, $\boldsymbol{W}$, etc.,) must be solved before the boundary conditions for the perturbed motion can be defined. It has been shown by Inglis & Price (1980) that this complication greatly increases the difficulty of deriving the linear velocity potentials for a rigid body, If, by way of simplification, it is assumed that the perturbation of the steady flow by the body is negligible, then

$$\boldsymbol{W} = -(\bar{U}, 0, 0) = -\bar{U}\hat{i} = -\boldsymbol{U}.$$

This approximation allows the unsteady motion problem to be solved without prior description of the steady motion in calm water.

4.4. *Summary of linearized boundary conditions*

The linear velocity potentials associated with the flow around the moving flexible body satisfy the following boundary conditions.

(i) On the free surface, the incident, diffracted and radiation potentials ϕ_0, ϕ_D and ϕ_r ($r = 1, 2, \ldots$) respectively satisfy the linearized boundary condition

$$\bar{U}^2 \phi_{xx} - 2\mathrm{i}\omega_e \bar{U} \phi_x - \omega_e^2 \phi + g\phi_z = 0$$

on $z = 0$, where ϕ represents either ϕ_0, ϕ_D or ϕ_r.

(ii) Suitable bottom and radiation conditions at infinite distance from the oscillating, translating structure.

(iii) The incident and diffracted potentials associated with the incoming and outgoing sinusoidal waves satisfy the relation

$$\partial\phi_0/\partial n = -\partial\phi_D/\partial n$$

on $\bar{S}$.

(iv) The radiation potentials are governed by the body boundary condition

$$\partial\phi_r/\partial n = [\mathrm{i}\omega_e \boldsymbol{u}_r + \boldsymbol{\theta}_r \times \boldsymbol{W} - (\boldsymbol{u}_r \cdot \nabla)\, \boldsymbol{W}] \cdot \boldsymbol{n}$$

on $\bar{S}$. For the approximation $\boldsymbol{W} = -\bar{U}\hat{\imath}$ this generalized Timman–Newman relation reduces to

$$\partial\phi_r/\partial n = \mathrm{i}\omega_e(u_r n_1 + v_r n_2 + w_r n_3) + \tfrac{1}{2}\bar{U}[n_3(\partial u_r/\partial z' - \partial w_r/\partial x') - n_2(\partial v_r/\partial x' - \partial u_r/\partial y')]$$

on $\bar{S}$ for each $r = 1, 2, \ldots, 6, 7, \ldots$. Further, if only rigid-body modes ($r = 1, 2, \ldots, 6$) are considered, then

$$\partial\phi_r/\partial n = \mathrm{i}\omega_e n_r + \bar{U}m_r,$$

with

$$m_1 = 0 = m_2 = m_3 = m_4, \quad m_5 = n_3, \quad m_6 = -n_2.$$

4.5. *Pressure distribution*

The fluid pressure acting on the instantaneous wetted surface S during oscillatory motion of the flexible structure may be found from the Bernoulli equation. It is given by

$$p = -\rho[\partial\phi/\partial t + \boldsymbol{W}\cdot\nabla\phi + \tfrac{1}{2}(W^2 - \bar{U}^2) + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + gz].$$

Unfortunately a knowledge of the position of S is necessary if this expression is to be used. Newman (1978) shows that this difficulty may be overcome by relating the pressure on the surface S, i.e. $p\,|_s$, to the pressure on the surface $\bar{S}$, i.e. $p\,|_{\bar{s}}$, by a Taylor series expansion. Thus, for the flexible structure, it follows that

$$p\,|_s = [1 + (\boldsymbol{u}\cdot\nabla) + \tfrac{1}{2}(\boldsymbol{u}\cdot\nabla)^2 + \ldots]\,p\,|_{\bar{s}}.$$

If it is also assumed that the oscillatory motion of the structure and parasitic flow are small, so that the second-order terms of the unsteady component may be neglected, then the linearized form of the pressure on the wetted surface S becomes

$$p\,|_s = -\rho\{\partial\phi/\partial t + \boldsymbol{W}\cdot\nabla\phi + [\tfrac{1}{2}(W^2 - \bar{U}^2) + gz'] + [gw + \tfrac{1}{2}(\boldsymbol{u}\cdot\nabla)\, W^2]\}_{\bar{s}}.$$

This approximation implies that the oscillatory flow and the motion of the structure are linearized but the steady flow due to the steady forward motion remains nonlinear. However, if $\boldsymbol{W} = -\bar{U}\hat{\imath}$, then the pressure expression reduces to

$$p\,|_s = -\rho(\partial\phi/\partial t - \bar{U}\,\partial\phi/\partial x)_{\bar{s}} - g(z' + w)\,|_{\bar{s}}.$$

The orders of magnitude of the terms in this expression for the pressure have been discussed by Price & Wu (1984) for structures with various geometries; those considered are a thin body, a flat body, a general three-dimensional body and a slender body.

4.6. *Generalized fluid forces*

The rth component of the generalized external force $\boldsymbol{Z}$ acting on the flexible structure which arises from the fluid only may be expressed in the form

$$Z_r(t) = -\iint_S \boldsymbol{n}^{\mathrm{T}}\cdot\boldsymbol{u}_r\, p\,\mathrm{d}S,$$

where $\boldsymbol{n}^{\mathrm{T}}$ denotes the transpose of the matrix representing the unit normal vector pointing out of the structure's surface into the fluid. The integration extends over the instantaneous wetted surface S. This last equation may be shown to be equivalent to the rth generalized fluid force defined in §3.2 (c). That is,

$$Z_r = \boldsymbol{D}_r^{\mathrm{T}} \boldsymbol{P} = \sum_e \boldsymbol{d}_r^{\mathrm{T}} \boldsymbol{P}_e.$$

The results

$$\overline{\boldsymbol{P}}_e = \boldsymbol{L}\boldsymbol{P}_e, \quad \bar{\boldsymbol{n}} = \boldsymbol{l}\boldsymbol{n}, \quad \overline{\boldsymbol{P}}_e = -\iint_{S_e} p\boldsymbol{N}^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \,\mathrm{d}S$$

or

$$\boldsymbol{P}_e = \boldsymbol{L}^{\mathrm{T}}\overline{\boldsymbol{P}}_e = -\boldsymbol{L}^{\mathrm{T}} \iint_{S_e} p\boldsymbol{N}^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \,\mathrm{d}S$$

are found in §§3.1 (b), 3.2 (c) and, from them, it follows that the rth generalized fluid force may be written as

$$Z_r = -\sum_e \boldsymbol{d}_r^{\mathrm{T}} \boldsymbol{L}^{\mathrm{T}} \iint_{S_e} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{l}\boldsymbol{n} p \,\mathrm{d}S.$$

If, now, this expression is transposed and rearranged it is found that

$$\begin{aligned} Z_r &= -\sum_e \iint_{S_e} \boldsymbol{n}^{\mathrm{T}} (\boldsymbol{l}^{\mathrm{T}} \boldsymbol{N} \boldsymbol{L} \boldsymbol{d}_r) \, p \,\mathrm{d}S, \\ &= -\iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r p \,\mathrm{d}S, \end{aligned}$$

as the results in §3.2 show. The summation of the surface integral over the wetted surface of each element is the instantaneous wetted surface area S.

When the steady and the unsteady potential components in the pressure equation are substituted, the contribution from the generalized gravitational force is included, the rth generalized external force may be found in the component form

$$Z_r(t) = \Xi_r(t) + H_r(t) + R_r(t) + \overline{R}_r$$

for $r = 1, 2, \ldots, m$. In this expression, Ξ_r, H_r, R_r, $\overline{R}_r$, denote the rth generalized wave exciting force, radiation force, restoring force, and hydrostatic force respectively.

(a) *Generalized wave forces*

The rth generalized wave exciting force is found, after some algebraic manipulation, to be

$$\begin{aligned} \Xi_r(t) &= \Xi_r \,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t} = (\Xi_{0r} + \Xi_{\mathrm{D}r}) \,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t}, \\ &= \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (\partial/\partial t + \boldsymbol{W} \cdot \nabla) \, (\phi_0 + \phi_{\mathrm{D}}) \,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t} \,\mathrm{d}S. \end{aligned}$$

In this expression, the amplitude

$$\Xi_{0r} = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_{\mathrm{e}} + \boldsymbol{W} \cdot \nabla) \, \phi_0 \,\mathrm{d}S$$

denotes the rth generalized Froude–Krylov contribution, while the amplitude of the rth generalized diffraction force accounting for the scattering of the incident wave owing to the presence of the flexible structure is

$$\Xi_{\mathrm{D}r} = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_{\mathrm{e}} + \boldsymbol{W} \cdot \nabla)\, \phi_{\mathrm{D}}\, \mathrm{d}S.$$

When $\boldsymbol{W} = -\bar{U}\hat{\imath}$, the rth generalized Froude–Krylov contribution reduces to

$$\Xi_{0r} = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r\, \omega \phi_0\, \mathrm{d}S$$

and is independent of forward speed for all $r = 1, 2, \ldots, m$. This contribution reflects the fact that the presence of the structure does not influence the pressure distribution in the incident wave.

(b) Generalized radiation force

The rth generalized radiation force

$$H_r(t) = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r [\partial/\partial t + \boldsymbol{W} \cdot \nabla] \sum_{k=1}^{m} p_k(t)\, \phi_k\, \mathrm{d}S,$$

where m denotes the number of principal coordinates admitted in the analysis. If the rth principal coordinate varies sinusoidally so that

$$p_r(t) = p_r\, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t},$$

the rth generalized radiation force becomes

$$H_r(t) = \sum_{k=1} p_k\, T_{rk}\, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t} = \sum_{k=1}^{m} p_k\, (\omega_{\mathrm{e}}^2 A_{rk} - \mathrm{i}\omega_{\mathrm{e}} B_{rk})\, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t}$$

for $r = 1, 2, \ldots, m$. The coefficients

$$A_{rk} = (\rho/\omega_{\mathrm{e}}^2)\, \mathrm{Re}\left[\iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_{\mathrm{r}} (\mathrm{i}\omega_{\mathrm{e}} + \boldsymbol{W} \cdot \nabla)\, \phi_k\, \mathrm{d}S\right]$$

represent variations that are in phase with the acceleration, while the

$$B_{rk} = (-\rho/\omega_{\mathrm{e}})\, \mathrm{Im}\left[\iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_{\mathrm{e}} + \boldsymbol{W} \cdot \nabla) \phi_k\, \mathrm{d}S\right]$$

terms are in phase with the velocity.

The terms containing the A_{rk} represent the effects of 'added mass' or 'added inertia'. The terms containing the B_{rk}, on the other hand, represent fluid damping. Both of these terms are associated with the rth mode and represent coupled effects owing to oscillatory distortion of unit amplitude in the kth mode. The theory suggests that these coefficients might be determined experimentally by forced oscillation of the flexible structure in a prescribed principal mode of the dry hull, at the arbitrary frequency ω_{e} as the structure travels with constant speed in calm water. That is, the contemporary experimental techniques of oscillatory testing with a planar motion mechanism (which are used to determine the frequency-dependent hydrodynamic coefficients of a rigid-hull model) might be extended to oscillatory testing of a flexible hull.

(*c*) *Generalized restoring force*

In the present mathematical model, the rth generalized restoring force is of the form

$$R_r(t) = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r \, [gw + \tfrac{1}{2}(\boldsymbol{u} \cdot \nabla)\, W^2]\, \mathrm{d}S.$$

Because the displacement at any chosen point in the structure is given by

$$\boldsymbol{u} = \{u, v, w\} = \sum_{k=1}^{m} \boldsymbol{u}_k p_k \,\mathrm{e}^{\mathrm{i}\omega_e t} = \sum_{k=1}^{m} \{u_k, v_k, w_k\} p_k \,\mathrm{e}^{\mathrm{i}\omega_e t},$$

it follows that the rth generalized restoring force may be written as

$$R_r(t) = -\sum_{k=1}^{m} p_k \, C_{rk} \,\mathrm{e}^{\mathrm{i}\omega_e t},$$

where the coefficient

$$C_{rk} = -\rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r \, [gw_k + \tfrac{1}{2}(\boldsymbol{u}_k \cdot \nabla)\, W^2]\, \mathrm{d}S$$

for $r = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, m$.

When $\boldsymbol{W} = -\bar{U}\hat{\imath}$ this coefficient reduces to

$$C_{rk} = -\rho g \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r \, w_k \, \mathrm{d}S,$$

and it can be shown easily that this expression includes a description of the restoring coefficients usually associated with a seakeeping analysis of a slender ship-like structure (i.e. when $r \leqslant 6$, $k \leqslant 6$). The coupling terms between the body and distortion modes are

$$C_{rk} = -\rho g \iint_{\bar{S}} n_r w_k \, \mathrm{d}S \quad (r = 1, 2, 3;\, k = 7, 8, \ldots, m),$$

$$C_{4k} = -\rho g \iint_{\bar{S}} [n_3(y' - y'_G) - n_2(z' - z'_G)]\, w_k \, \mathrm{d}S \quad (k = 7, 8, \ldots, m),$$

$$C_{5k} = -\rho g \iint_{\bar{S}} [n_1(z' - z'_G) - n_3(x' - x'_G)]\, w_k \, \mathrm{d}S \quad (k = 7, 8, \ldots, m),$$

$$C_{6k} = -\rho g \iint_{\bar{S}} [n_2(x' - x'_G) - n_1(y' - y'_G)]\, w_k \, \mathrm{d}S \quad (k = 7, 8, \ldots, m),$$

$$C_{rk} = 0 \quad (k = 1, 2, 6,\, r = 7, 8, \ldots, m),$$

$$C_{r3} = -\rho g \iint_{\bar{S}} \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r \, \mathrm{d}S \quad (r = 7, 8, \ldots, m),$$

$$C_{r4} = -\rho g \iint_{\bar{S}} \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (y' - y'_G)\, \mathrm{d}S \quad (r = 7, 8, \ldots, m),$$

$$C_{r5} = -\rho g \iint \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r (x' - x'_G)\, \mathrm{d}S \quad (r = 7, 8, \ldots, m).$$

The distortion of the body then provides additional restoring forces to the rigid-body motions and vice versa. It is interesting to note that $\boldsymbol{C}$ is not necessarily a symmetric matrix.

(*d*) *Generalized hydrostatic and gravitational forces*

The contributions to the generalized forces of hydrostatic and gravitation effects are independent of all unsteady motions. The rth generalized hydrostatic force takes the form

$$\bar{R}_r = \rho \iint_S \boldsymbol{n}^{\mathrm{T}} \cdot \boldsymbol{u}_r [gz' + \tfrac{1}{2}(W^2 - \bar{U}^2)]\, \mathrm{d}S.$$

This expression contains components arising from the hydrostatic fluid action (gz') and from forces arising from the structure travelling with constant forward speed in calm water ($W^2 - \bar{U}^2$).

The rth gravitational force is

$$G_r = -\rho \iiint_{\forall} \rho_{\mathrm{b}} g w_r' \, \mathrm{d}\forall,$$

where $\forall$ denotes the total volume of material, whose density is ρ_{b}, and $r = 1, 2, \ldots, m$.

4.7. *The generalized equation of motion*

The matrix equation of motion derived in §3.2 (*c*) is

$$\boldsymbol{a}\ddot{\boldsymbol{p}}(t) + \boldsymbol{b}\dot{\boldsymbol{p}}(t) + \boldsymbol{c}\boldsymbol{p}(t) = \boldsymbol{Z}(t) + \boldsymbol{G} + \boldsymbol{\Delta}(t)$$

or, in general,

$$\begin{aligned} \omega_r^2 a_{rr} p_r(t) + \sum_{k=1}^{m} [a_{rk}\ddot{p}_k(t) + b_{rk}\dot{p}_k(t)] \\ &= Z_r(t) + G_r + \Delta_r(t) \\ &= \Xi_r(t) + H_r(t) + R_r(t) + \bar{R}_r + G_r + \Delta_r(t) \\ &= \Xi_r\, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{e}} t} - \sum_{k=1}^{m} [A_{rk}\ddot{p}_k(t) + B_{rk}\dot{p}_k(t)] \\ &\quad - \sum_{k=1}^{m} C_{rk} p_k(t) + \bar{R}_r + G_r + \Delta_r(t) \end{aligned}$$

for $r = 1, 2, \ldots, m$. For a freely floating structure with no concentrated external forces, $\Delta_r(t) = 0$.

(*a*) *Equations of steady motion*

For the flexible structure in calm water, there exists a steady-state solution (i.e. $\omega_{\mathrm{e}} = 0$),

$$p_r(t) = \bar{p}_r,$$

satisfying the equation

$$a_{rr}\,\omega_r^2 \bar{p}_r = -\sum_{k=1}^{m} C_{rk}\bar{p}_k + \bar{R}_r + G_r$$

for $r = 1, 2, \ldots, m$. As shown previously by Bishop & Price (1979) this formulation gives a modal description of structural distortion in still water, trim, sinkage, etc.

(*b*) *General equations of motion of a floating structure*

The generalized linear equations of motion for a freely floating structure moving or stationary in waves, after extraction of the portion accounting for steady-state conditions, may be written in the form

$$\omega_r^2 a_{rr} p_r(t) + \sum_{k=1}^{m} [(a_{rk}+A_{rk})\ddot{p}_k(t) + (b_{rk}+B_{rk})\dot{p}_k(t) + C_{rk} p_k(t)] = \Xi_r \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{e} t},$$

where $r = 1, 2, \ldots, m$. This may be written in the matrix form

$$(\boldsymbol{a}+\boldsymbol{A})\ddot{\boldsymbol{p}}(t) + (\boldsymbol{b}+\boldsymbol{B})\dot{\boldsymbol{p}}(t) + (\boldsymbol{c}+\boldsymbol{C})\boldsymbol{p}(t) = \boldsymbol{\Xi}\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{e} t},$$

which agrees with that found previously by Bishop & Price (1979).

Thus, for a solution

$$\boldsymbol{p}(t) = \boldsymbol{p}\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{e} t},$$

it follows that

$$\boldsymbol{Ip} = [\mathrm{adj}\,\boldsymbol{D}/\det\boldsymbol{D}]\,\boldsymbol{\Xi}$$

where $\boldsymbol{I}$ is the unit matrix,

$$\boldsymbol{D} = -\omega_\mathrm{e}^2(\boldsymbol{a}+\boldsymbol{A}) + \mathrm{i}\omega_\mathrm{e}(\boldsymbol{b}+\boldsymbol{B}) + (\boldsymbol{c}+\boldsymbol{C})$$

and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are dependent on the frequency of encounter ω_e.

Knowing the principal mode shapes of the dry structure and having determined principal coordinates, one may find the displacement at any position in the structure. It is given by

$$\boldsymbol{u}(x, y, z, t) = \sum_{r=1}^{m} \boldsymbol{u}_r(x, y, z)\, p_r\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{e} t}.$$

The bending moments, shearing forces, twisting moments (if such are identified), and any other relevant response may be determined in a similar manner using the appropriate characteristic function of the dry structure.

It is interesting to note that, having no rigid-body modes ($r = 1, 2, \ldots, 6$), a fixed flexible structure produces responses in its distortion modes only. Thus the linear equation of motion remains valid, but only contributions arising from modes $r = 7, 8, \ldots, m$ of the dry structure need be considered.

5. Computations

Singularity distribution methods have provided a successful method of predicting the loadings applied to, and motions of, a rigid ship or offshore structure in waves. In this section, a composite singularity distribution (c.s.d.) method is discussed which allows the unknown singularity (i.e. source) strengths to be determined for a flexible structure having port and starboard symmetry travelling in waves. By using the symmetry of the structure, the diffraction and radiation problems may be solved for sinusoidal waves approaching from any angle.

Brard (1972) has shown that, when a singularity distribution method is used for a surface-piercing structure with forward speed, a line-integral contribution must be included in the expression for the velocity potential at any point $r = (x, y, z)$ in the fluid. That is

$$\phi(x, y, z) = \frac{1}{4\pi}\iint_{\bar{S}} Q(x_1, y_1, z_1)\, G(x, y, z; x_1, y_1, z_1)\,\mathrm{d}S + \frac{\bar{U}^2}{4\pi g}\int_{\bar{C}} Q(x_1, y_1, 0)\, G(x, y, z; x_1, y_1, 0)\, n_1^2(x_1, y_1, 0)\,\mathrm{d}C,$$

where (x_1, y_1, z_1) denotes a point on the wetted surface of the structure, the contour $\bar{C}$ is the intersection of the structure's outer surface and the mean calm water surface, Q, is the source density on the surface of this structure; G is the appropriate Green function.

5.1. *Composite functions*

If the plane of symmetry of the structure is Oxz, the direction cosines of the normal vectors pointing out of the body surface satisfy the relations

$$n_j(x, y, z) = n_j(x, -y, z), \quad (j = 1, 3)$$
$$n_2(x, y, z) = -n_2(x, -y, z).$$

Similarly, the direction cosines for small body rotations are such that

$$n_j(x, y, z) = -n_j(x, -y, z), \quad (j = 4, 6)$$
$$n_5(x, y, z) = n_5(x, -y, z).$$

Composite potential functions may be defined to satisfy the relations

$$\phi_0^{\pm} = \phi_0(x, y, z) \pm \phi_0(x, -y, z),$$
$$\phi_D^{\pm} = \phi_D(x, y, z) \pm \phi_D(x, -y, z),$$
$$\phi_r^{\pm} = \phi_r(x, y, z) \pm \phi_r(x, -y, z),$$

for the incident wave, diffracted wave and radiation potentials $(r = 1, 2, \ldots, m)$ respectively.

If the definition of the velocity potential derived by Brard (1972) is adopted, the composite velocity potential, $\phi_r^{\pm}$, for example, may be expressed as

$$\phi_r^{\pm}(x, y, z) = \frac{1}{4\pi} \iint_{\bar{S}_p} Q_r^{\pm} G^{\pm} \, dS + \frac{\bar{U}^2}{4\pi g} \int_{C_p} Q_r^{\pm} G^{\pm} n_1^2 \, dC,$$

with a derivative

$$\frac{\partial \phi_r^{\pm}}{\partial n}(x, y, z) = \frac{1}{4\pi} \iint_{\bar{S}_p} Q_r^{\pm} \frac{\partial G^{\pm}}{\partial n} \, dS + \frac{\bar{U}^2}{4\pi g} \int_{C_p} Q_r^{\pm} \frac{\partial G^{\pm}}{\partial n} n_1^2 \, dC.$$

In these integrals, $\bar{S}_p$ denotes the mean wetted surface area of the port structure, C_p the line contour along the port structure and the point (x, y, z) lies within the volume of fluid surrounding the port structure or lies on the wetted surface area of the port structure. Similar formulations are valid for $\phi_0^{\pm}$ and $\phi_D^{\pm}$.

In these expressions the composite Green function is defined as

$$G^{\pm} = G(x, y, z; x_1, y_1, z_1) \pm G(x, y, z; x_1, -y_1, z_1),$$

with the properties

$$(\partial/\partial n)\, G(x, y, z; x_1, y_1, z_1) = (\partial/\partial n)\, G(x, -y, z; x_1, -y_1, z_1),$$
$$(\partial/\partial n)\, G(x, -y, z; x_1, y_1, z_1) = (\partial/\partial n)\, G(x, y, z; x_1, -y_1, z_1).$$

The composite source strength is

$$Q^{\pm} = Q(x, y, z) \pm Q(x, -y, z),$$

and a composite boundary condition

$$P^{\pm} = P(x, y, z) \pm P(x, -y, z)$$

is to be defined later.

5.2. *Application of the composite source distribution (c.s.d.) method*

The boundary conditions at the body surface, discussed in §4.4, require that the composite diffraction, incident and radiation potentials satisfy the following relation on the port side (say) of the structure $\bar{S}_{\rm p}$:

$$\partial\phi_{\rm D}^{\pm}/\partial n = -\partial\phi_0^{\pm}/\partial n = -[(\partial/\partial n)\,\phi_0\,(x, y, z) \pm (\partial/\partial n)\,\phi_0\,(x, -y, z)] = P_{\rm D}^{\pm},$$

while the generalized Timman–Newman relation may be cast in the form

$$\partial\phi_r^{\pm}/\partial n = P_r^{\pm} \quad \text{on} \quad \bar{S}_{\rm p}$$

for $r = 1, 2, \ldots, m$.

Now the modal shapes of a dry structure with port–starboard symmetry may be separated into two groups. The symmetric modes are such that

$$(u_r, v_r, w_r)_{\rm st} = (u_r, -v_r, w_r)_{\rm p},$$

$$(\theta_{xr}, \theta_{yr}, \theta_{wr})_{\rm st} = (-\theta_{xr}, \theta_{yr}, -\theta_{zr})_{\rm p};$$

and the antisymmetric modes are such that

$$(u_r, v_r, w_r)_{\rm st} = (-u_r, v_r, -w_r)_{\rm p},$$

$$(\theta_{xr}, \theta_{yr}, \theta_{zr})_{\rm st} = (\theta_{xr}, -\theta_{yr}, \theta_{zr})_{\rm p},$$

where the subscripts st and p relate to the starboard and port sides respectively. In addition, the steady flow $\boldsymbol{W}$ around this symmetric body will also exhibit symmetry such that

$$(W_x, W_y, W_z)_{\rm st} = (W_x, -W_y, W_z)_{\rm p}.$$

From these symmetry properties, the right-hand side of the generalized Timman–Newman relation may be written as

$$P_r^+ = 2[\mathrm{i}\omega_{\rm e}\,\boldsymbol{u}_r + \boldsymbol{\theta}_r \times \boldsymbol{W} - (\boldsymbol{u}_r \cdot \nabla)\,\boldsymbol{W}] \cdot \boldsymbol{n};$$

$$P_r^- = 0$$

for symmetric modes, and

$$P_r^+ = 0;$$

$$P_r^- = 2[\mathrm{i}\omega_{\rm e}\,u_r + \boldsymbol{\theta}_r \times \boldsymbol{W} - (\boldsymbol{u}_r \cdot \nabla)\,\boldsymbol{W}] \cdot \boldsymbol{n}$$

for antisymmetric modes. When the simplification $\boldsymbol{W} = -\bar{U}\hat{\imath}$ is made, these reduce to

$$P_r^+ = 2\mathrm{i}\omega_{\rm e}(u_r n_1 + v_r n_2 + w_r n_3) + 2\bar{U}(\theta_{yr} n_3 - \theta_{zr} n_2);$$

$$P_r^- = 0$$

for symmetric modes and

$$P_r^+ = 0;$$

$$P_r^- = 2\mathrm{i}\omega_{\rm e}(u_r n_1 + v_r n_2 + w_r n_3) + 2\bar{U}(\theta_{yr} n_3 - \theta_{zr} n_2)$$

for antisymmetric modes.

If body motions only are considered, then

$$P_r^+ = \begin{cases} 2\mathrm{i}\omega_\mathrm{e}\, n_r, & (r = 1,\ 3) \\ 0, & (r = 2,\ 4,\ 6) \\ 2\mathrm{i}\omega_\mathrm{e}(zn_1 - xn_3) + 2\bar{U}n_3, & (r = 5); \end{cases}$$

and

$$P_r^- = \begin{cases} 0, & (r = 1,\ 3,\ 5) \\ 2\mathrm{i}\omega_\mathrm{e}\, n_2, & (r = 2) \\ 2\mathrm{i}\omega_\mathrm{e}(yn_3 - zn_2), & (r = 4) \\ 2\mathrm{i}\omega_\mathrm{e}(xn_2 - yn_1) - 2\bar{U}n_2, & (r = 6). \end{cases}$$

5.3. *Discretization in the c.s.d. method*

Because the Green function formulation for the appropriate type of singularity source (i.e. pulsating, translating, or pulsating and translating) and the sinusoidal incident wave velocity potential are known, the only unknowns occurring in the set of equations discussed in the previous section are the source strengths $Q_\mathrm{D}^\pm$ and $Q_r^\pm$. These functions can only be found by numerical means and, to this end, the equations must be 'discretized' in some way.

Several possible procedures are available, and one of the first practical approaches is that of Hess & Smith (1962), in which the wetted surface of the structure is represented by a large number of quadrilateral elements, N. The source strength over each of the elements is assumed to be constant and so an integral equation is replaced by a set of linear algebraic equations, to solve for the values of the source strength on the elements. For example, the body boundary condition given by the generalized Timman–Newman relation may be discretized to form the set of algebraic equations

$$-2\pi Q_r^\pm(\boldsymbol{\lambda}_k) + \sum_{i=1}^{N} Q_r^\pm(\boldsymbol{\lambda}_i)\left[\iint_{\Delta S_i} \frac{\partial G^\pm(\boldsymbol{\lambda}_k, \boldsymbol{\lambda}_i)}{\partial n(\boldsymbol{\lambda}_k)}\,\mathrm{d}S + \frac{\bar{U}^2}{g}\int_{\Delta C_i} \frac{\partial G^\pm(\boldsymbol{\lambda}_k, \boldsymbol{\lambda}_i)}{\partial n(\boldsymbol{\lambda}_k)}\, n_1^2(\lambda_i)\,\mathrm{d}C\right] = 4\pi P_r^\pm(\lambda_k)$$

for $r = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, N$ where $\boldsymbol{\lambda}_k (= x_k, y_k, z_k)$ denotes the position of the field point which is now on the wetted surface, and $\boldsymbol{\lambda}_i (= x_i, y_i, z_i)$ denotes a source point which is also on the wetted surface. In this expression, N denotes the number of elements on the port side of the structure and ΔS_i is the area of the ith element. The quantity ΔC_i is the length of the edge of the ith element piercing the water surface, which is zero if there is no such piercing.

This set of algebraic equations may be written in the matrix form

$$\boldsymbol{a}\boldsymbol{Q}_r^\pm = \boldsymbol{P}_r^\pm$$

for $r = 1, 2, \ldots, m$, where $\boldsymbol{a}$ is the matrix of influence coefficients of order $N \times N$ and $\boldsymbol{Q}_r^\pm$ is a column matrix of order N for the composite radiation source strengths.

A solution for $\boldsymbol{Q}_r^\pm$ may be found for any distortion mode provided that the modal shapes $\boldsymbol{u}_r$ and $\boldsymbol{\theta}_r$ in $\boldsymbol{P}_r$ are known at the centre of each surface panel used to define the wetted surface area of the port section of the structure. Because of the way the solution of the general problem has been posed, the discretization of the dry structure for the finite-element calculation is completely independent of the discretization adopted to discuss the fluid actions. When the finite-element method is used to obtain the principal modes, however, there is the additional requirement to calculate the modal displacement $\boldsymbol{u}_r$ and the modal rotation $\boldsymbol{\theta}_r$ at each panel of the wetted surface $\bar{S}_{\mathrm{p}}$, by using the data on the modal shapes at the nodes.

In a similar manner, the unknown diffraction source strengths $Q^{\pm}_{\mathrm{D}}$ may be determined from the set of algebraic equations

$$\boldsymbol{a}\boldsymbol{Q}^{\pm}_{\mathrm{D}} = \boldsymbol{P}^{\pm}_{\mathrm{D}},$$

where $\boldsymbol{Q}^{\pm}_{\mathrm{D}}$ is a column matrix of order N.

In general, the composite Green's function may be expressed as

$$G^{\pm}(x_k, y_k, z_k; x, y, z) = \frac{1}{R} \pm \frac{1}{R_1} + f(x_k, y_k, z_k; x, y, z) \pm f(x_k, y_k, z_k; x, -y, z),$$

where

$$R = [(x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2]^{\frac{1}{2}},$$

$$R_1 = [(x-x_k)^2 + (y+y_k)^2 + (z-z_k)^2]^{\frac{1}{2}},$$

and the function $f(\;)$ depends on the type of source distribution considered (see, for example, Wehausen & Laitone (1960)). It will be seen that the term $1/R$ is singular when $\boldsymbol{r}_k$ coincides with $\boldsymbol{r}$ but, as proved by Hess & Smith (1967), in the limit as $\boldsymbol{r}_k$ approaches the element ΔS_i

$$\iint_{\Delta S_i} \frac{\partial}{\partial n}\left(\frac{1}{R}\right) \mathrm{d}S \to -2\pi,$$

and this accounts for the existence of the first term in the set of algebraic equations just discussed. It is interesting to note that the value of this limit is independent of the shape of the panel element.

In the calculations the centroid of each panel element $\boldsymbol{r}_k$ is used as the point at which the normal velocity and derivative of the Green function are evaluated and the composite source strength determined. Because $\boldsymbol{r}_k$ will never be in the water line, the line integral over ΔC_k is not singular and so the coefficient -2π arising from the singular nature of $1/R$ remains unchanged.

The numerical approach adopted to obtain solutions for the velocity potentials is a generalization of the methods used in previous investigations by Inglis & Price (1980, 1981, 1982*a*, *b*) and by Inglis (1980). While refinements and modifications from other investigations – for example, those of Hogben & Standing (1974), Faltinsen (1976) and Hess & Smith (1962) – are included in the numerical procedures, in all these studies the body is assumed to be rigid. Price & Wu (1982, 1983) have developed the c.s.d. method to evaluate the unknown velocity potentials associated with rigid-body motions of a mono-hull or multi-hull vessel, but their procedures have been extensively extended to permit the velocity potentials ϕ_{D}, ϕ_r ($r = 1, 2, \ldots, 6, 7, \ldots, m$) to be determined for a flexible structure moving in waves.

5.4. *Generalized fluid forces*

The symmetry properties of the structure allow the expressions for the generalized fluid forces derived in §4.6 to be cast into more appropriate forms for numerical evaluation. For example, the added-mass and damping coefficients in §4.6 (*b*) may be expressed as

$$A_{rk} = (\rho/\omega_{\mathrm{e}}^2)\,\mathrm{Re}\,(T_{rk})$$

and

$$B_{rk} = -(\rho/\omega_{\mathrm{e}})\,\mathrm{Im}\,(T_{rk})$$

respectively for $r = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, m$, the function T_{rk} being

$$T_{rk} = \rho \iint_{\bar{S}_\mathrm{p}} \boldsymbol{n}^\mathrm{T} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_\mathrm{e} + \boldsymbol{W} \cdot \nabla)\, \phi_k^{\pm}\, \mathrm{d}S,$$

where ϕ^+ is taken for symmetric modal shapes and ϕ^- taken for antisymmetric modal shapes. But the results of §§3.2, 4.6 show that this integral may be further modified to

$$T_{rk} = \rho \sum_{e \subset S_\mathrm{p}} \iint_{\bar{S}_\mathrm{e}} (\boldsymbol{n}^\mathrm{T} \boldsymbol{l}^\mathrm{T}\, \boldsymbol{NLd}_r)\, (\mathrm{i}\omega_\mathrm{e} + \boldsymbol{W} \cdot \nabla)\, \phi_k^{\pm})\, \mathrm{d}S$$

for $r = 1, 2, \ldots, m$, the integration being performed over the wetted surface $\bar{S}_\mathrm{e}$ of a structural element and the summation performed over all the elements adjacent to the water within the port section of the structure. This formulation implies that $A_{rk} = 0 = B_{rk}$ if one of the rth and kth modal shapes is symmetric while the other is antisymmetric, or vice versa.

If a similar procedure is followed, the Froude–Krylov and diffraction forces of §4.6 (a) may be expressed for $r = 1, 2, \ldots, m$, as

$$\mathit{\Xi}_{0r} = \rho \iint_{\bar{S}_\mathrm{p}} \boldsymbol{n}^\mathrm{T} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_\mathrm{e} + \boldsymbol{W} \cdot \nabla)\, \phi_0^{\pm}\, \mathrm{d}S$$

and

$$\mathit{\Xi}_{\mathrm{D}r} = \rho \iint_{\bar{S}_\mathrm{p}} \boldsymbol{n}^\mathrm{T} \cdot \boldsymbol{u}_r (\mathrm{i}\omega_\mathrm{e} + \boldsymbol{W} \cdot \nabla)\, \phi_\mathrm{D}^{\pm}\, \mathrm{d}S$$
$$= \rho \sum_{e \subset S_\mathrm{p}} \iint_{\bar{S}_\mathrm{e}} (\boldsymbol{n}^\mathrm{T} \boldsymbol{lNLd}_r)\, (\mathrm{i}\omega_\mathrm{e} + \boldsymbol{W} \cdot \nabla)\, \phi_\mathrm{D}^{\pm}\, \mathrm{d}S$$

respectively. Here the positive superscript is used when the rth modal shape is symmetric, and the negative superscript is used when the rth modal shape is antisymmetric.

Finally, when $\boldsymbol{W} = -\bar{U}\hat{\imath}$ the restoring coefficient §4.6 (c) reduces to the form

$$C_{rk} = -2\rho g \iint_{\bar{S}_\mathrm{p}} \boldsymbol{n}^\mathrm{T} \cdot \boldsymbol{u}_r\, w_k\, \mathrm{d}S$$
$$= -2\rho g \sum_{e \subset S_\mathrm{p}} \iint_{\bar{S}_\mathrm{e}} (\boldsymbol{n}^\mathrm{T} \boldsymbol{l}^\mathrm{T} \boldsymbol{NLd}_r)\, (\boldsymbol{l}_3^\mathrm{T} \boldsymbol{NLd}_k)\, \mathrm{d}S,$$

when both the rth and kth modal shapes are symmetric or antisymmetric, and $C_{rk} = 0$ when one of the rth and kth modal shapes is symmetric while the other is antisymmetric and vice versa. The matrix $\boldsymbol{l}_3^\mathrm{T}$ denotes the third row of the transposed matrix $\boldsymbol{l}$, defined in §3.1 (c).

Thus a method has been proposed and developed by which the responses of a flexible structure travelling in waves may be found.

6. Numerical examples

While no general analytical criteria were established to confirm the convergence of results, the convergence was checked by considering the influence of discretization, mesh size, etc. In all cases, it was found that results were satisfactorily convergent. A selection of results will be presented which relate to an idealized uniform mono-hull structure of ship-like proportions and a multi-hull structure (i.e. a semi-submersible or small water-plane area twin hull s.w.a.t.h.).

6.1. *The Green function*

To demonstrate the numerical procedures adopted, figure 2 shows the components of the composite Green function at a field point (x, y, z) arising from a source located at (x_1, y_1, z_1) and travelling with forward speed $\bar{U} = 7\ \mathrm{m\ s^{-1}}$. It is seen that this result is well behaved over the entire frequency range. The calculation confirms theoretical predictions in the sense that as the frequency of encounter increases the real part of the Green function tends to a constant value while the imaginary part tends to zero.

6.2. *Uniform mono-hull*

Figure 3 shows the form of the tubular ship-like hull structure considered. The wetted surface area is described by 208 panel elements and the ship travels with a forward speed of $\bar{U} = 7\ \mathrm{m\ s^{-1}}$ (Froude number $Fn = 0.203$), in sinusoidal head waves of unit amplitude.

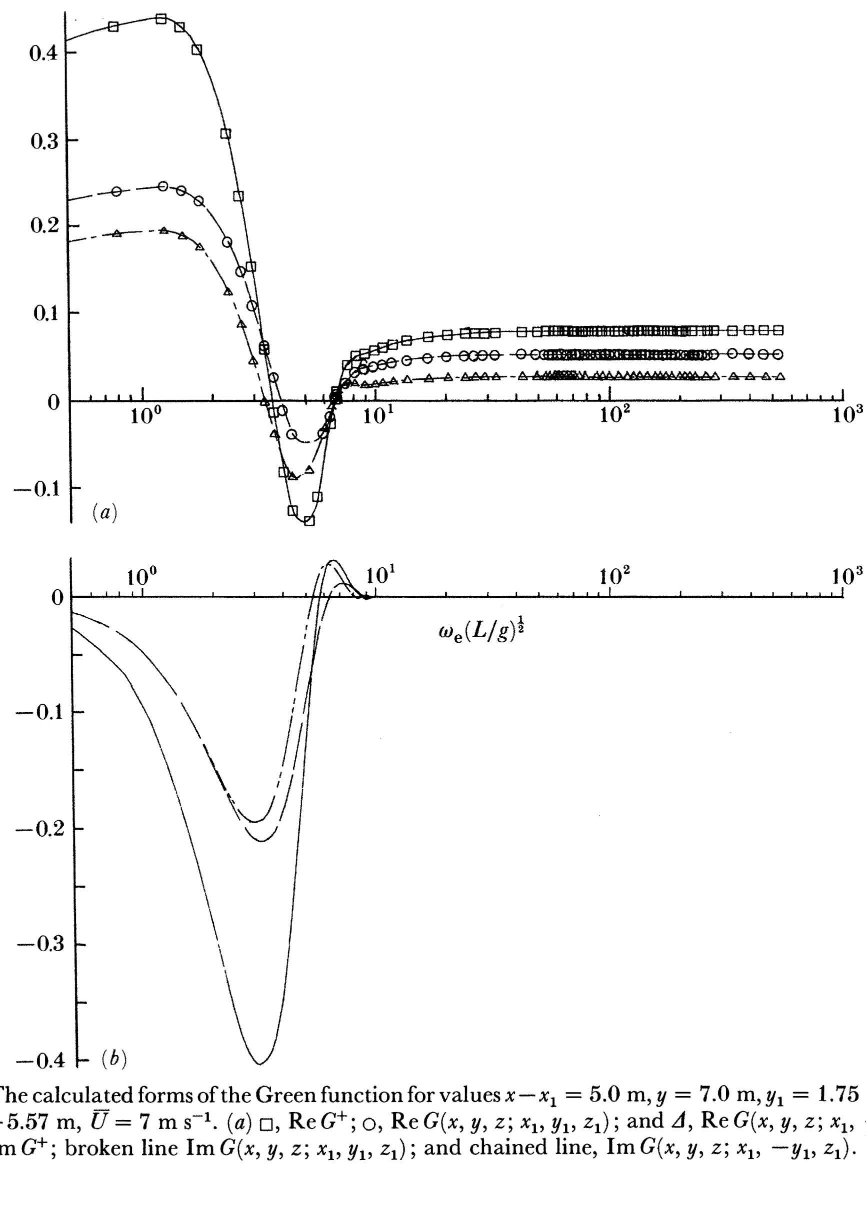


FIGURE 2. The calculated forms of the Green function for values $x - x_1 = 5.0$ m, $y = 7.0$ m, $y_1 = 1.75$ m, $z = -4.18$ m, $z_1 = -5.57$ m, $\bar{U} = 7\ \mathrm{m\ s^{-1}}$. (*a*) □, $\mathrm{Re}\,G^+$; ○, $\mathrm{Re}\,G(x, y, z;\, x_1, y_1, z_1)$; and Δ, $\mathrm{Re}\,G(x, y, z;\, x_1, -y_1, z_1)$. (*b*) Full line, $\mathrm{Im}\,G^+$; broken line $\mathrm{Im}\,G(x, y, z;\, x_1, y_1, z_1)$; and chained line, $\mathrm{Im}\,G(x, y, z;\, x_1, -y_1, z_1)$.

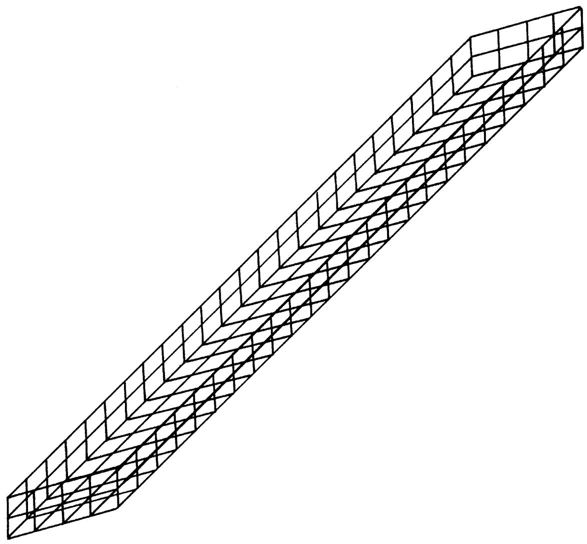


FIGURE 3. The mean wetted surface of the uniform tubular ship represented by 208 panel elements. The ship travels with a Froude number $F_n = 0.203$ ($\bar{U} = 7$ m s^{-1}).

Figure 4 shows a selection of the principal modes that were used to define $P_{\mathrm{D}}^{\pm}$, $P_r^{\pm}$ in calculations of the composite source strengths for each mode $r = (1, 2, ..., 6, 7, ...)$. Generalized hydrodynamic coefficients, generalized wave-exciting forces, principal coordinates and responses (i.e. of deflections, bending moments, and shearing forces) were obtained over a wide range of frequencies. A typical selection of results are given in figure 5.

Whenever possible, these results were compared with results obtained from the program UCLMARS developed by Bishop *et al.* (1977) for the two-dimensional hydroelastic problem. In this earlier approach, the strip theory developed by Gerritsma & Beukelman (1964) was used in conjunction with a simple Euler beam representation of the structure. A comparison of the findings obtained from these two totally different methods reveals an impressive degree of agreement.

6.3. *Multi-hull*

We consider next a multi-hull structure travelling with forward speed $\bar{U} = 6$ m s^{-1} ($F_n = 0.223$) in regular sinusoidal waves. The form of this vessel is illustrated in figure 6 and it raises many interesting problems of both a structural and hydrodynamic nature. Thus the structure may be likened to a tuning fork, with some properties quite unlike those of a conventional ship, and a number of unusual hydrodynamic interactions are brought to light. (In this latter connection, it is worth noting that a slight modification of the theory permits hydrodynamic interactions between passing ships of the same or differing sizes to be studied. And if one hull is made infinitely long and deep it is possible to investigate hydrodynamic interactions between a ship and a bank or dock. While these problems, and others, lie outside the scope of this paper, they are amenable to solution by slight alterations of the theory proposed.)

Figure 7 shows a selection of dry hull principal modes associated with the flexible multi-hull. Modes up to and including $r = 16$ were calculated, though it is impractical to record all details of these modes in this paper so only a limited selection is given.

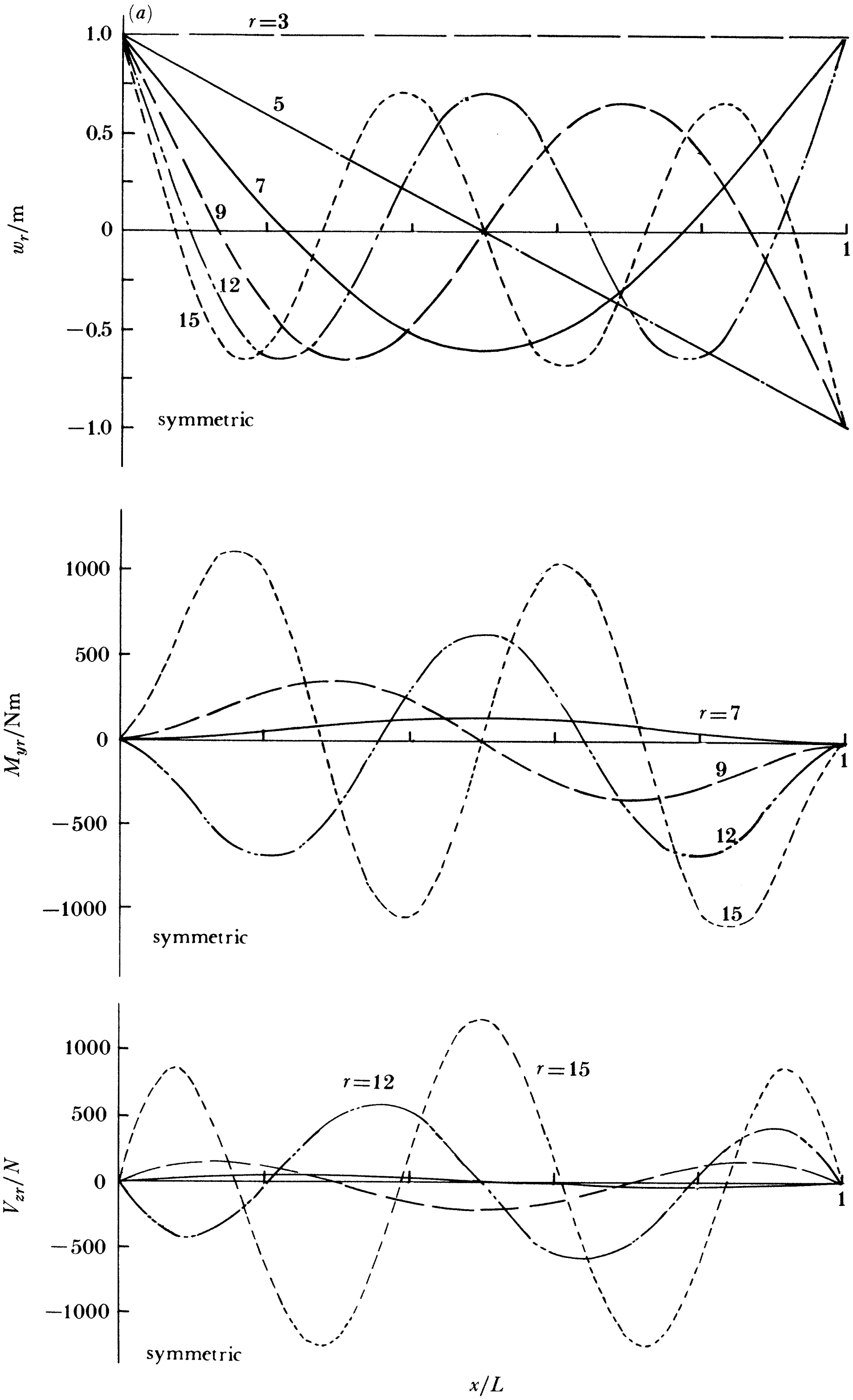


FIGURE 4. Dry modal characteristics of the uniform ship for (*a*) symmetric modes of vertical deflection w_r(m), modal bending moment M_{yr}(Nm), modal shearing force V_{zr}(N) and (*b*) horizontal deflection v_r(m), modal horizontal bending moment M_{zr}(Nm) and modal torsional rotation θ_{xr}.

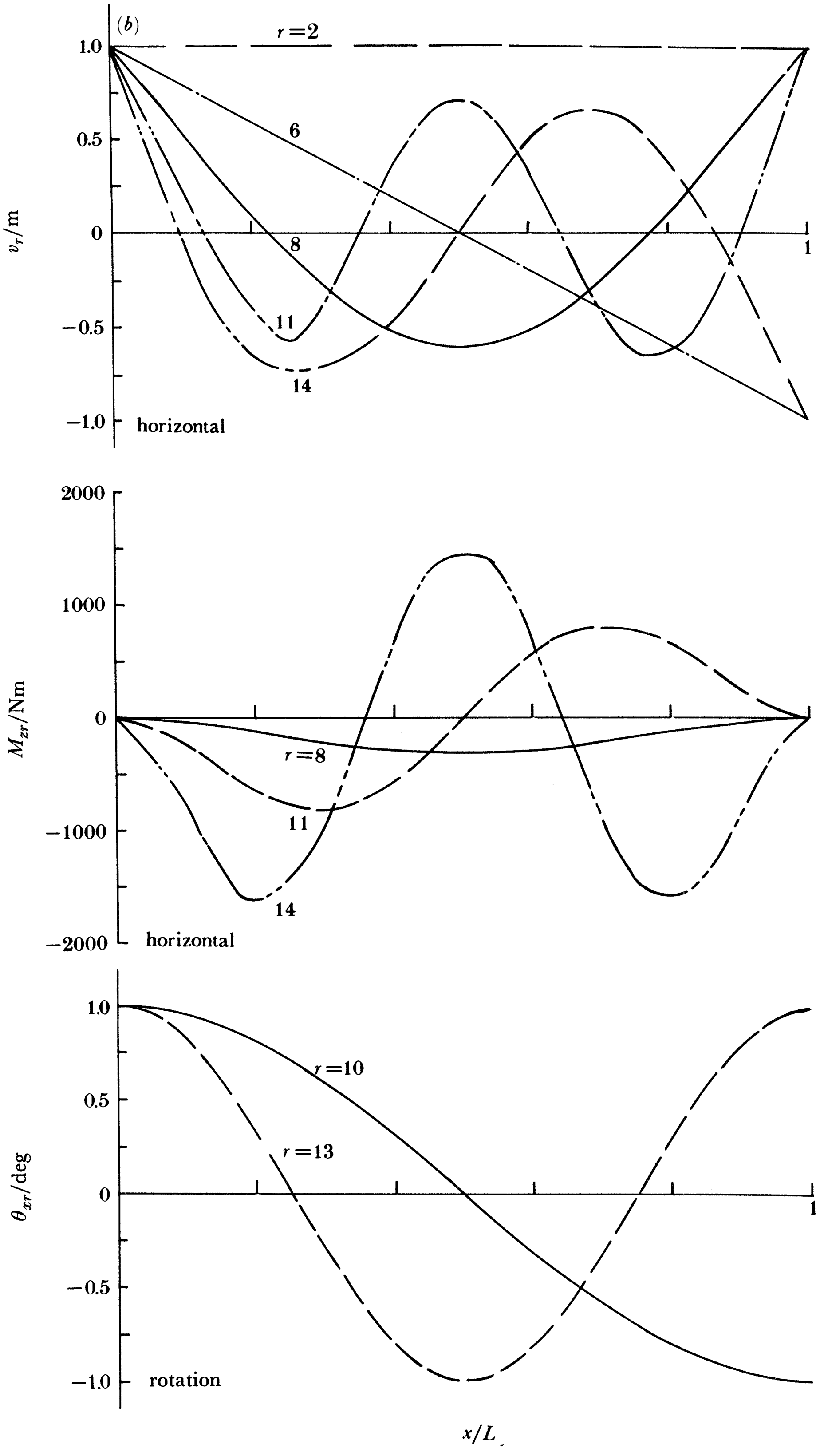


FIGURE 4 (b). For description see facing page.

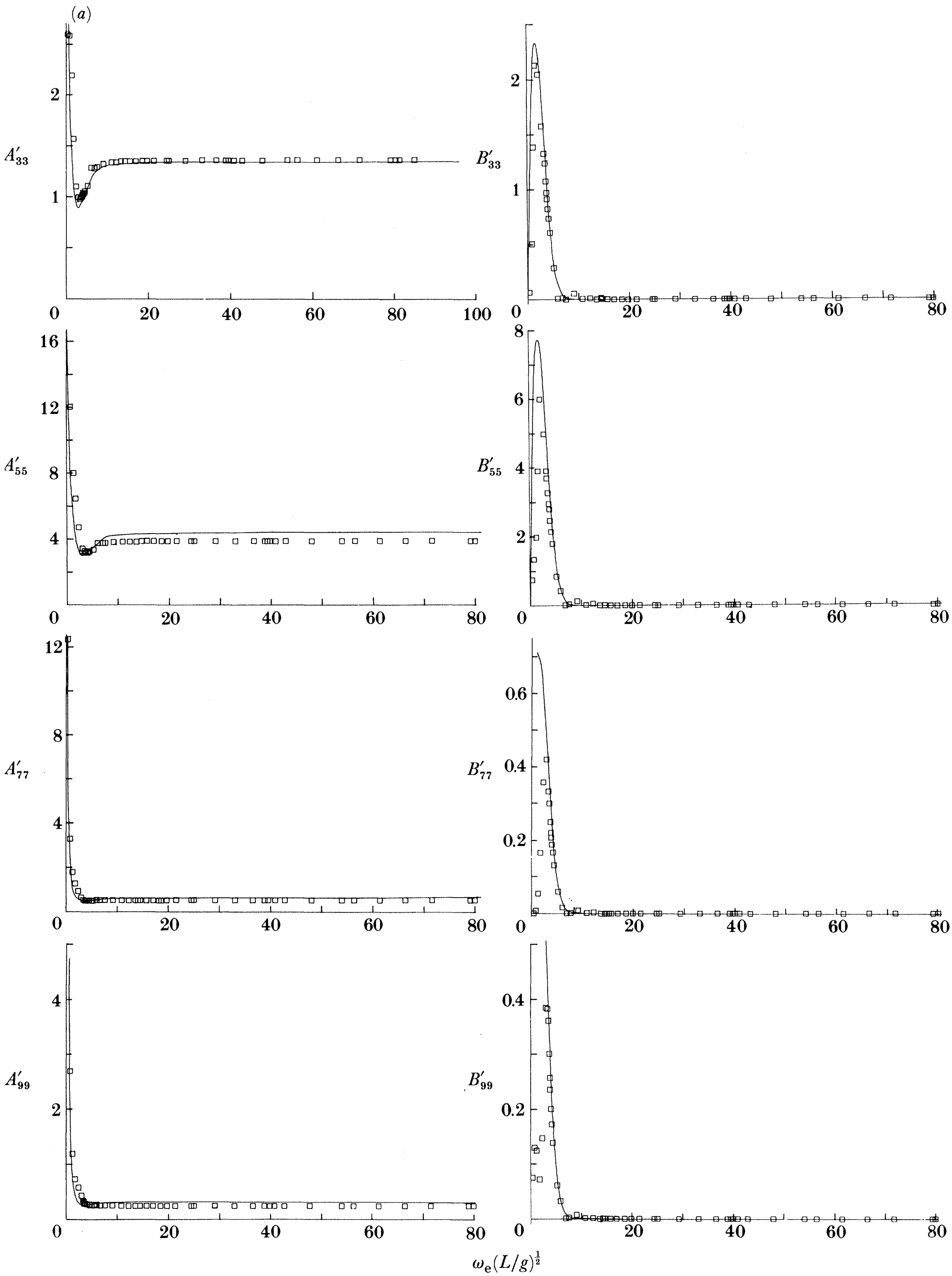


FIGURE 5 (*a*). For description see page 414.

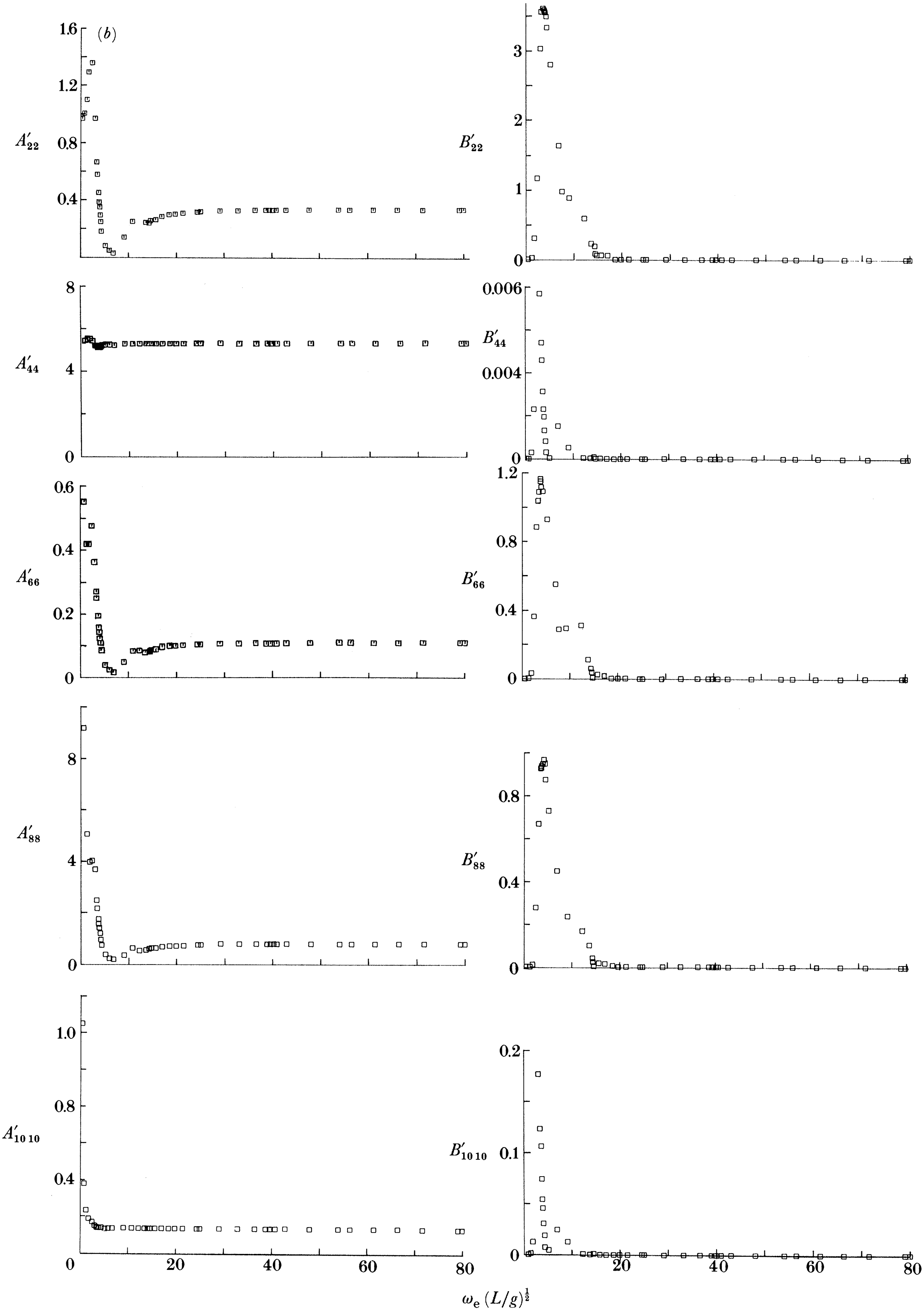


FIGURE 5 (b). For description see page 414.

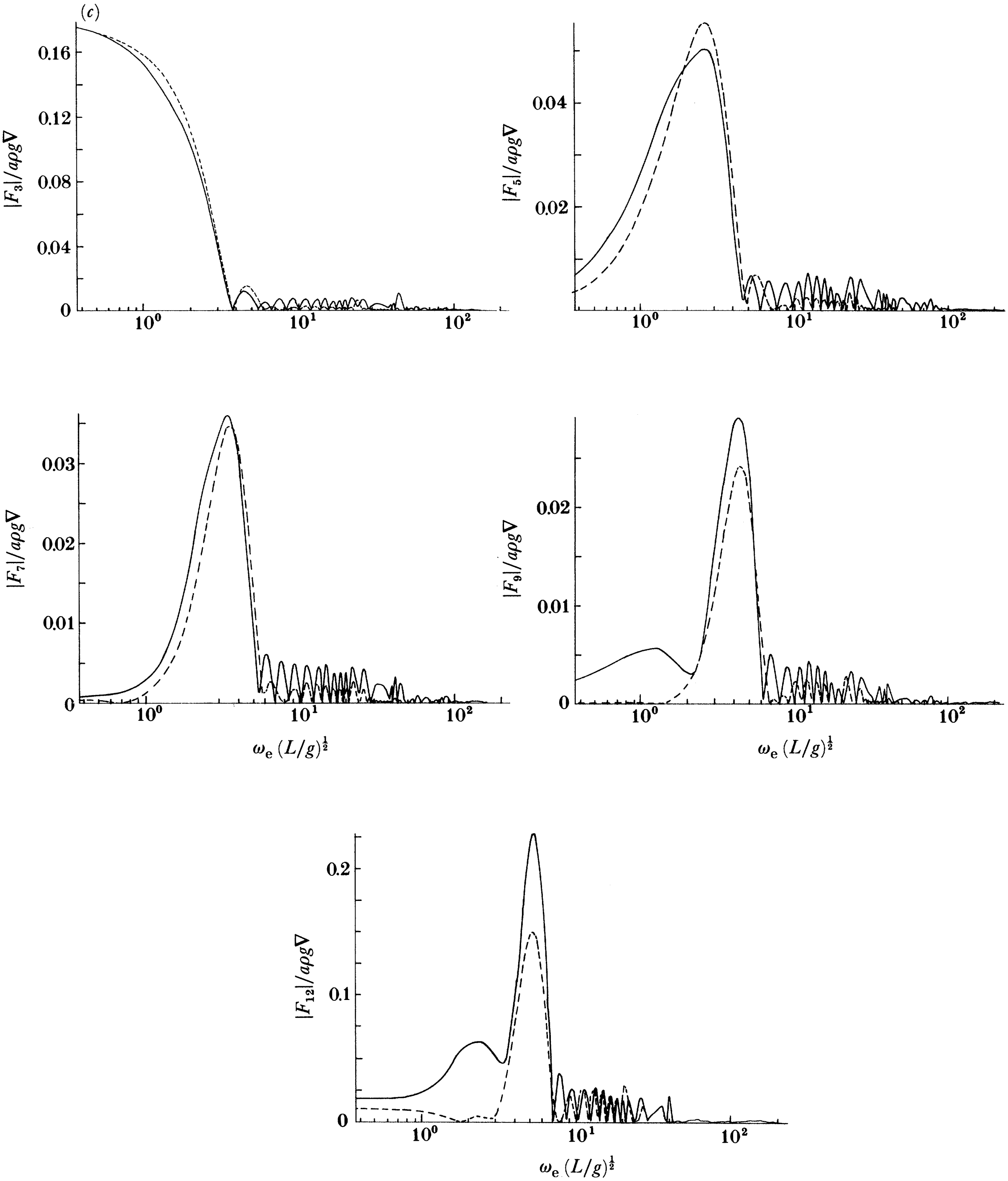


FIGURE 5 (*c*). For description see page 414.

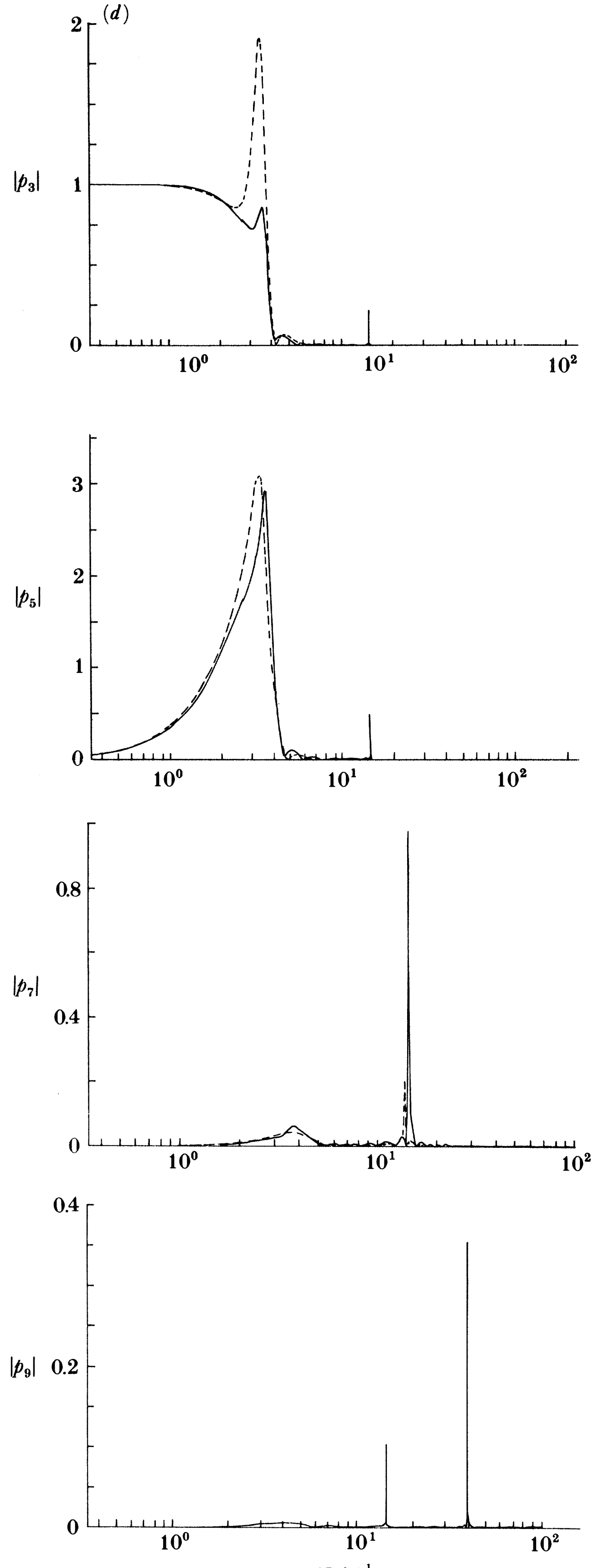


FIGURE 5 (d). For description see page 414.

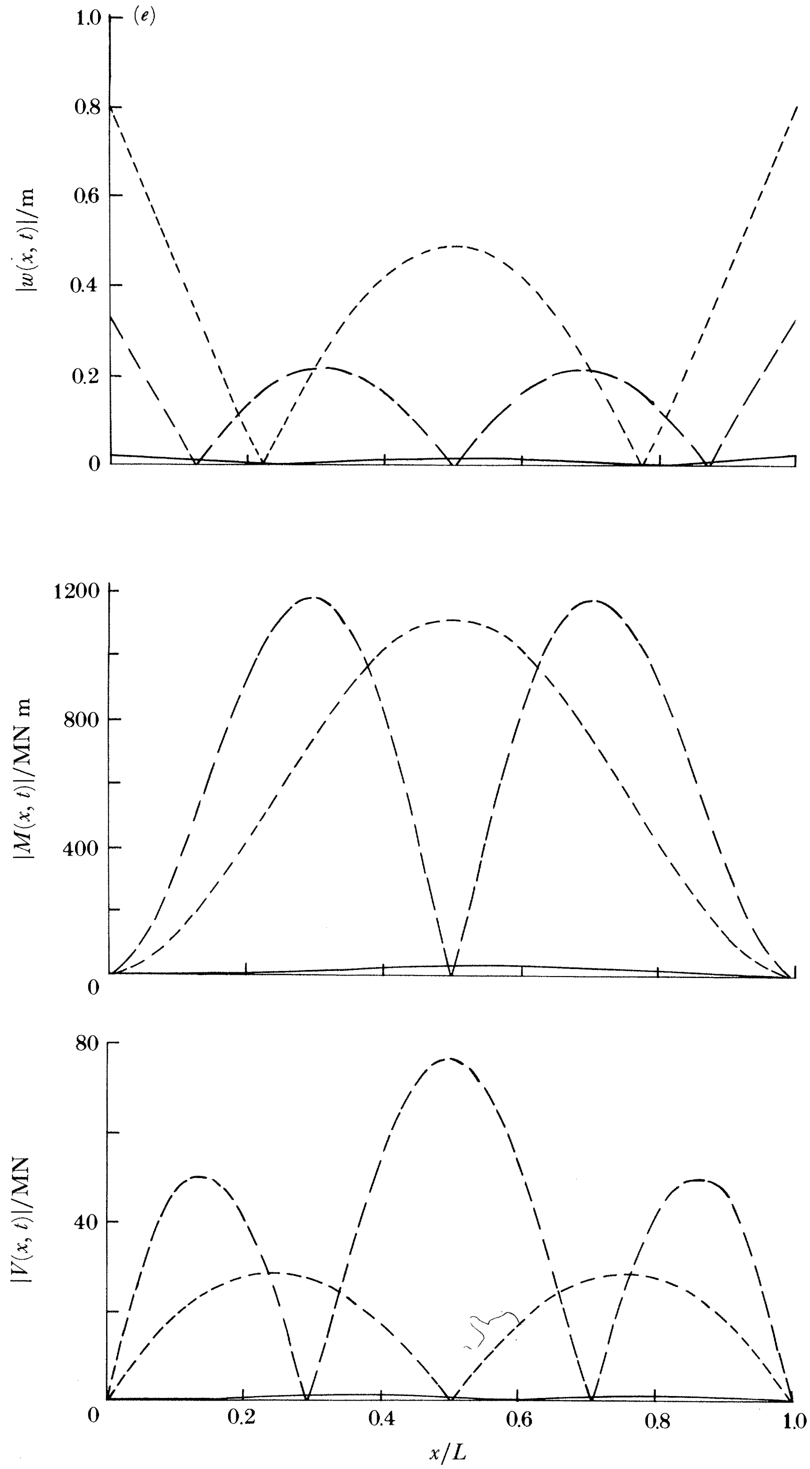
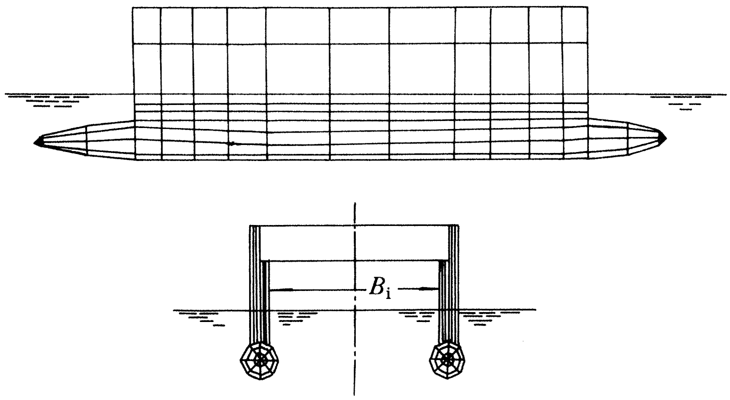


FIGURE 5. Selection of typical data for the uniform ship. (*a*) Non-dimensional generalized added-mass coefficient $A'_{rr} = A_{rr}/\rho\nabla$ and damping coefficient $B'_{rr} = B_{rr}/[\rho\nabla(g/L)^{\frac{1}{2}}]$ for the symmetric modes $r = 3, 5, 7, 9$ (□, present three-dimensional theory; —, strip-beam theory). (*b*) A'_{rr} and B'_{rr} for the antisymmetric modes $r = 2, 4, 6, 8, 10$. (*c*) Amplitudes of the non-dimensional generalized wave-exciting force $F_r/a\rho g\nabla$ for modes $r = 3, 5, 7, 9, 12$ when the uniform ship travels with Froude number 0.203 in sinusoidal head waves of unit amplitude (——, present three-dimensional theory; ---, two-dimensional strip-beam theory). (*d*) The variations in the amplitudes of the principal coordinates $|p_r|$ ($r = 3, 5, 7, 9$) for the uniform ship proceeding at $F_n = 0.203$ in head waves (——, present three-dimensional theory; ---, two-dimensional strip-beam theory). (*e*) The amplitudes of vertical deflection $|w(x, t)|$, bending moment $|M(x, t)|$ and shearing force $|V(x, t)|$ distributed along the length of the uniform ship travelling at $F_n = 0.203$ in head waves, calculated for three values of the frequency of encounter ω_e (——, $\omega_e = 0.804$ rad s^{-1}; ---, $\omega_e = 4.14$ rad s^{-1}; — — —, $\omega_e = 11.39$ rad s^{-1}).

(*a*)



(*b*)

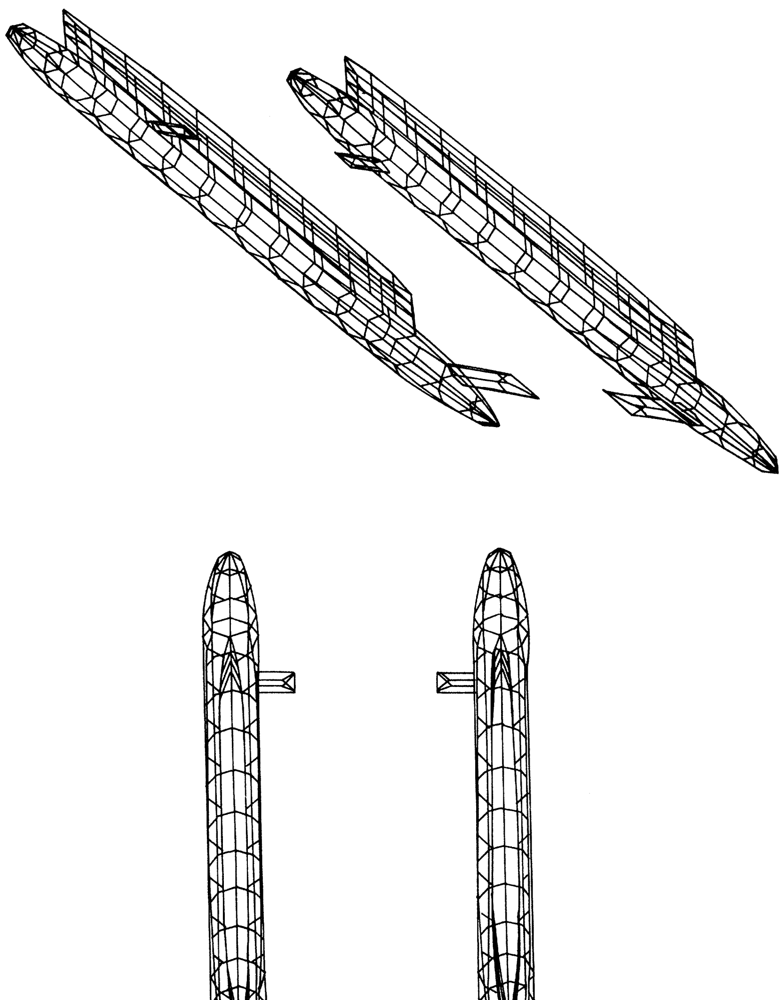


FIGURE 6. (*a*) The geometric form of the multi-hull s.w.a.t.h. (i.e. small water-plane area twin hull) model and (*b*) the mesh used to describe the wetted area of the s.w.a.t.h. model as viewed at different angles. A total of 432 panels were distributed over the hulls.

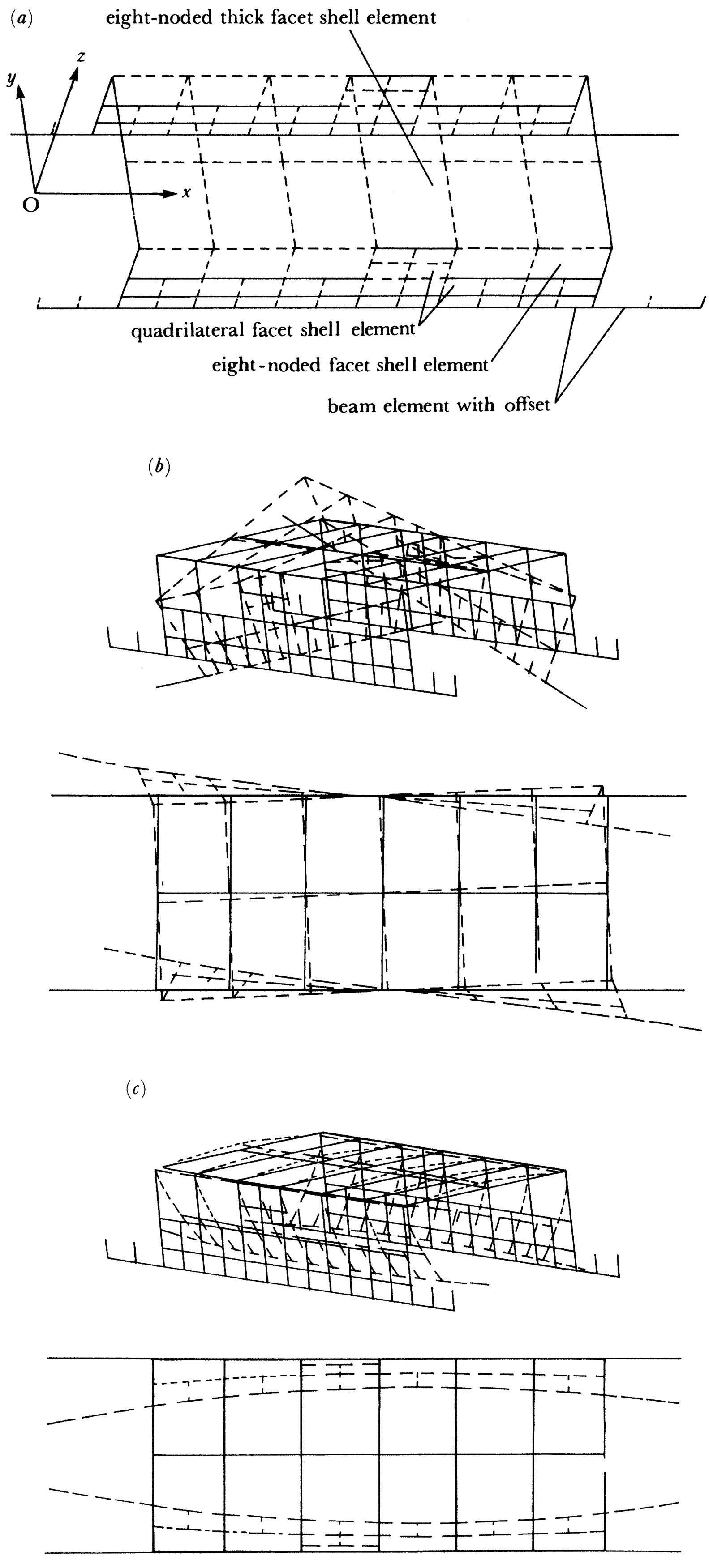
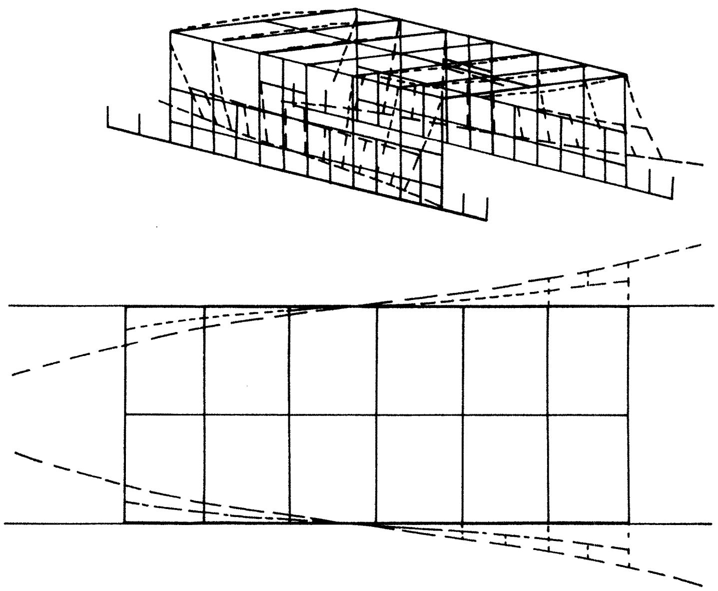


FIGURE 17. (a–c). For description see page 419.

(d)



(e)

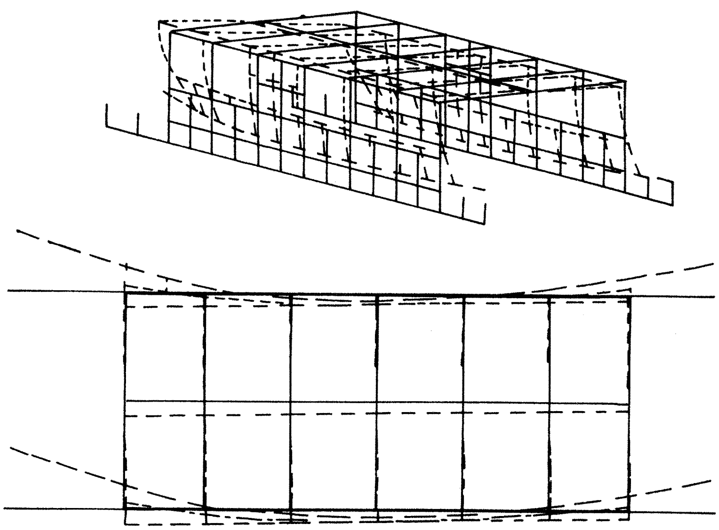


FIGURE 7 (*d*, *e*). For description see page 419.

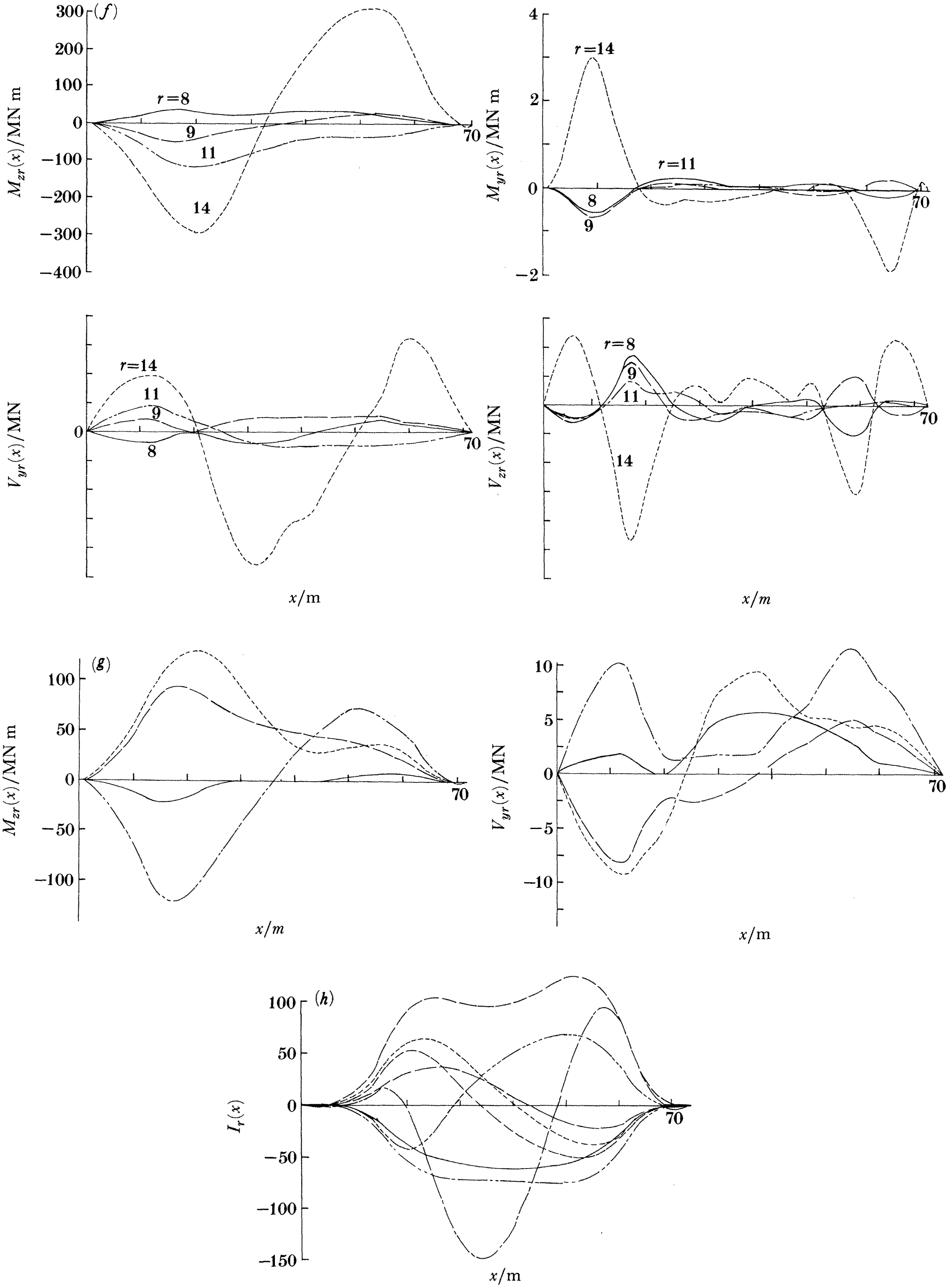


FIGURE 7 (f–h). For description see facing page.

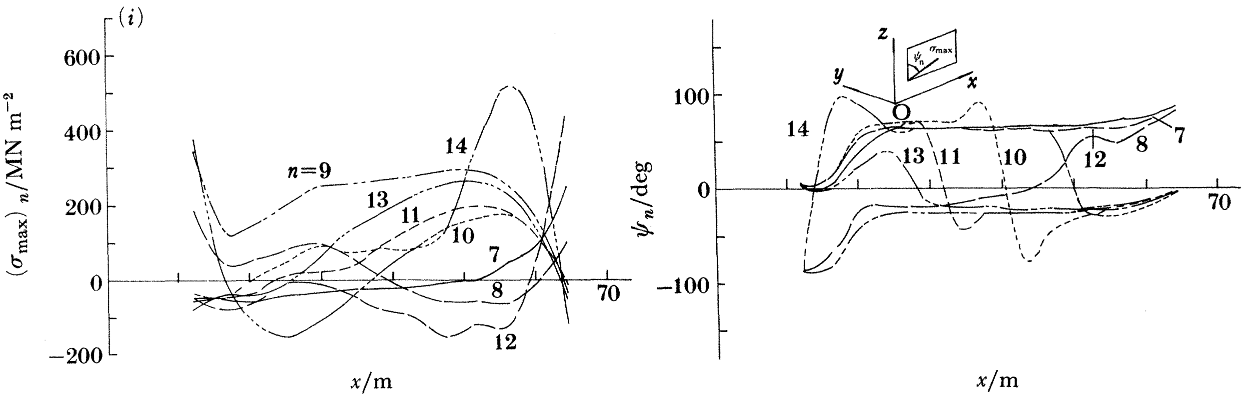


FIGURE 7. (a) Finite-element idealization of the dry s.w.a.t.h. model structure; (b) $r = 7$, antisymmetric mode shape, $\omega_7 = 9.52$ rad s^{-1}; (c) $r = 8$, symmetric mode shape, $\omega_8 = 9.67$ rad s^{-1}; (d) $r = 9$, symmetric mode shape, $\omega_9 = 12.28$ rad s^{-1}; (e) $r = 10$, antisymmetric mode shape, $\omega_{10} = 16.72$ rad s^{-1}; (f) the dry modal bending moments and shearing forces associated with the symmetric modes acting along the shear centre line of the port hull of the s.w.a.t.h. model; (g) the dry modal bending moment and shearing force associated with the antisymmetric modes acting along the shear centre line of the port hull of the s.w.a.t.h. model; (h) the dry modal torsional moments acting at the cross section of the port hull of the s.w.a.t.h. model; (i) the variation in the amplitudes $(\sigma_{max})_n$ and directions ψ_n of the maximum modal stresses acting on the outer surface of the port strut along the fount with the main dry hull (see also figure 8e,f).

Generalized hydrodynamic coefficients, wave forces, responses, etc., were determined, and some of these are shown in figure 8. In general, the generalized hydrodynamic coefficients A_{rk}, B_{rk} display complicated variations, with distinct peaks or jumps in the frequency range $1.5 < \omega_e(L/g)^{\frac{1}{2}} < 10.0$. In a detailed analysis for this frequency range, it was observed that considerable fluid–structure interaction arises between the two pontoons. For example, when $\omega_e(L/g)^{\frac{1}{2}} = 4.7$, the added-mass coefficients for the symmetric modes change rapidly from a local maximum value to a local minimum (or negative) value, while the corresponding damping coefficients attain peak values. At this frequency

$$\lambda_e/B_i = 1.01,$$

where $\lambda_e = 2\pi g/\omega_e^2$ and $B_i(= 0.28L)$ is the distance between the two struts at the waterline. Moreover a wave pattern is generated between the two hulls with a maximum wave height at the centre line and a wavelength conforming with the above conditions.

At the dimensionless encounter frequency $\omega_e(L/g)^{\frac{1}{2}} = 3.3$, the added-mass and damping coefficients associated with antisymmetric modes show similar proportions, but now

$$\lambda_e/B_i = 2.05.$$

This frequency relates to an antisymmetric wave pattern having a maximum elevation at one pontoon and a minimum elevation at the other, with a zero crossing in the vicinity of the centre line.

Similar variations to these are apparent in all the hydrodynamic coefficients at particular values of encounter frequencies. A possible reason for this is to be found in a localized wave pattern that is carried along between the two hulls, perhaps akin to a standing wave pattern. Although such a localized wave pattern can be numerically generated between the two hulls and may be given a physical identity, there is always the possibility that because of the

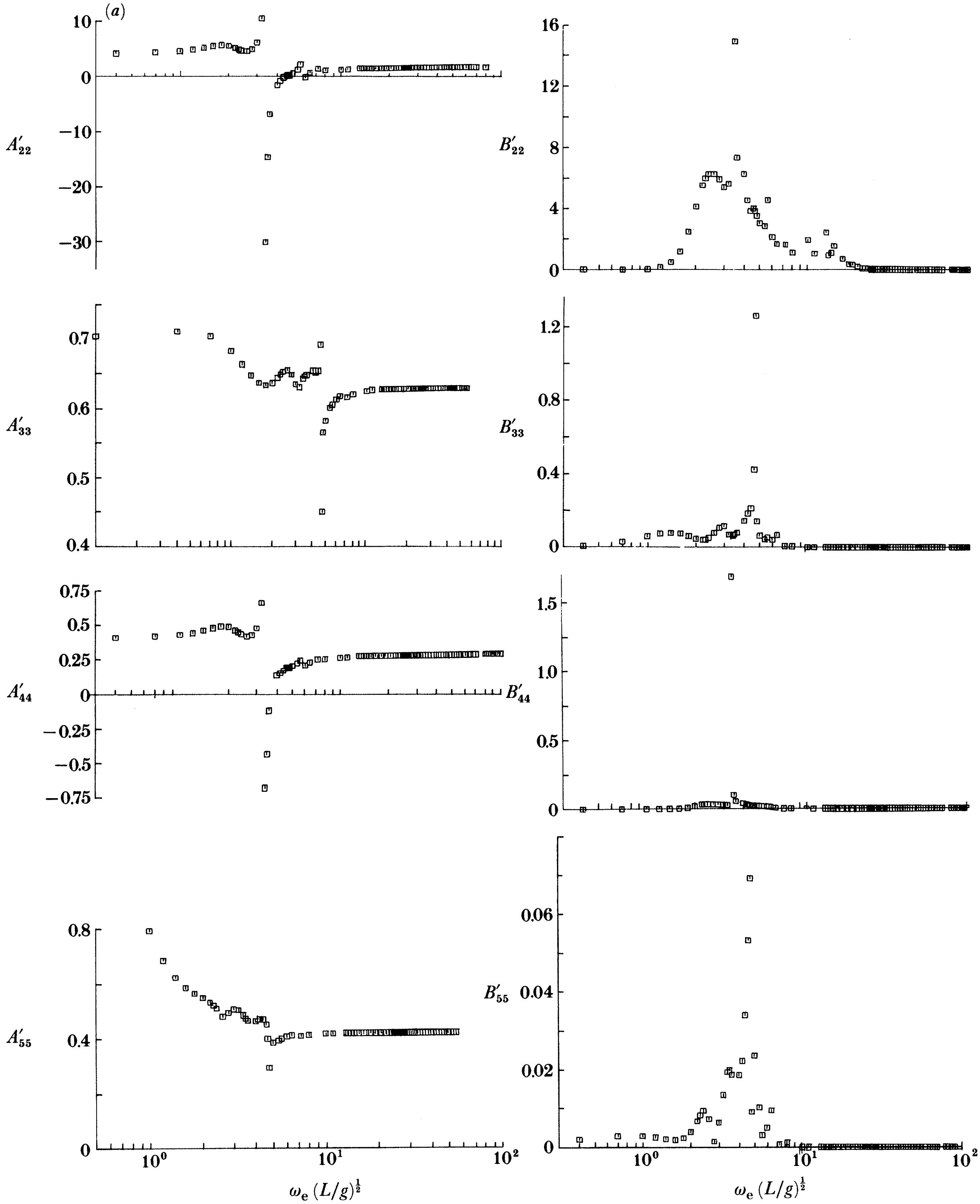


FIGURE 8 (*a*). For description see page 424.

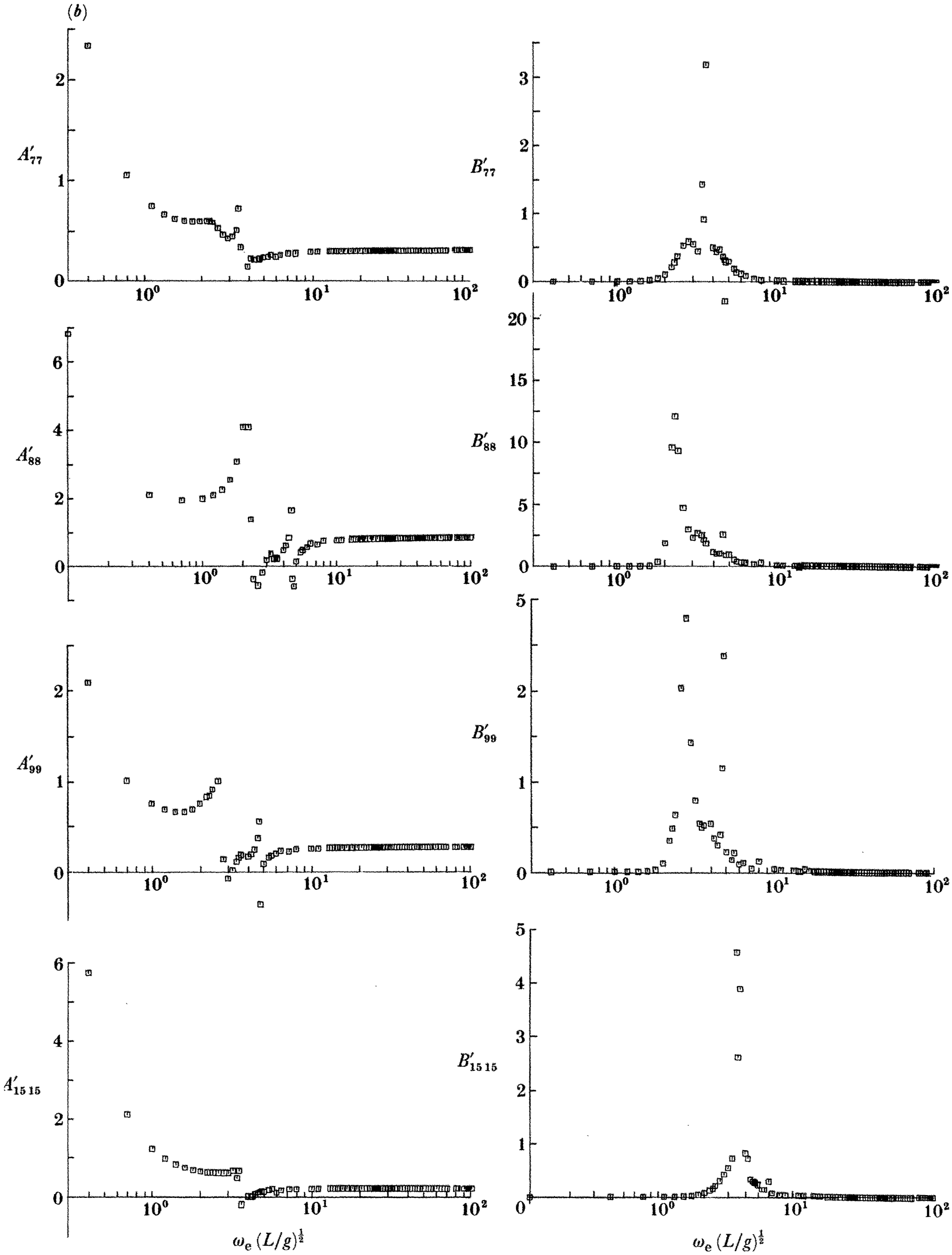


FIGURE 8 (*b*). For description see page 424.

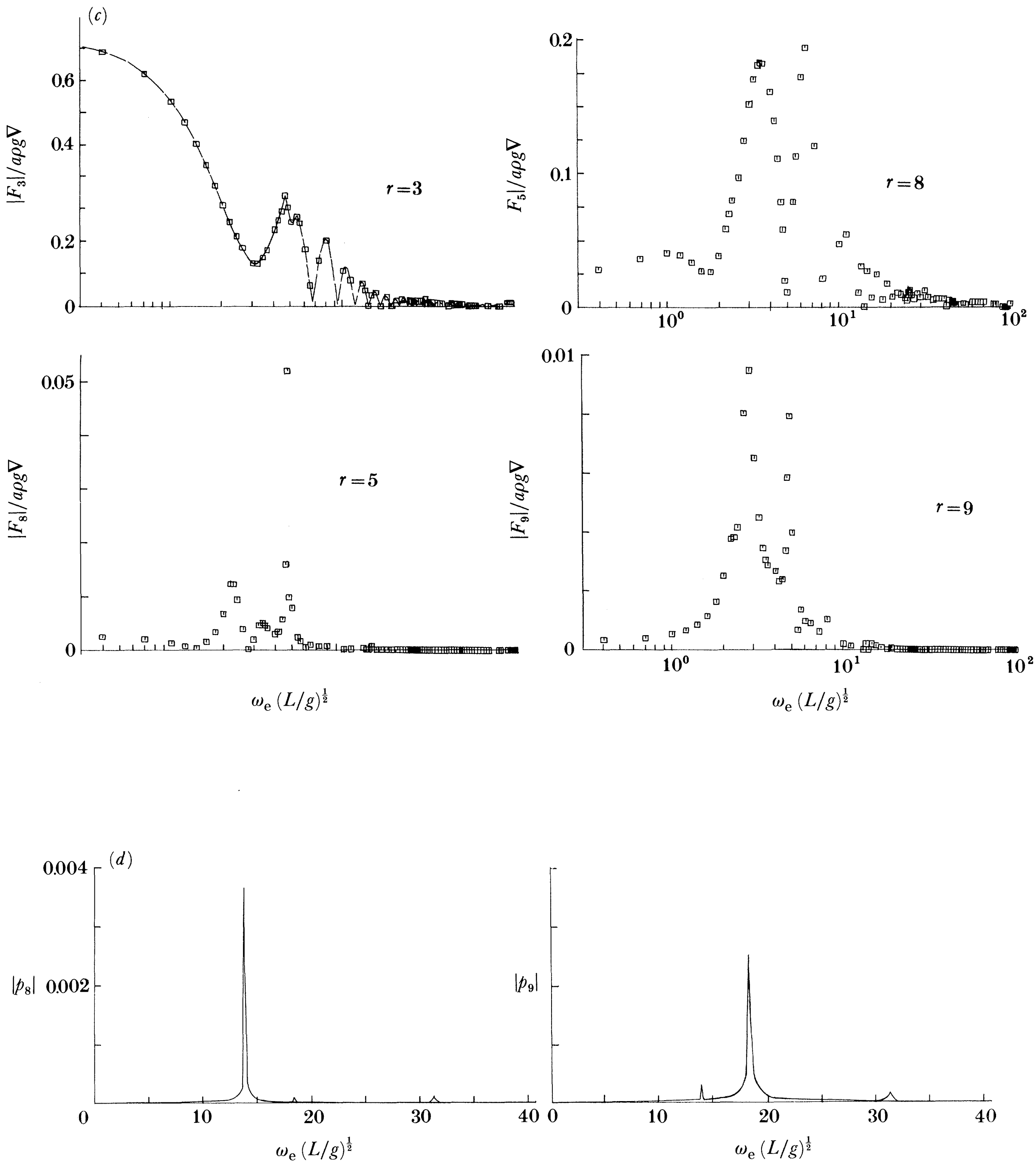


FIGURE 8 (c, d). For description see page 424.

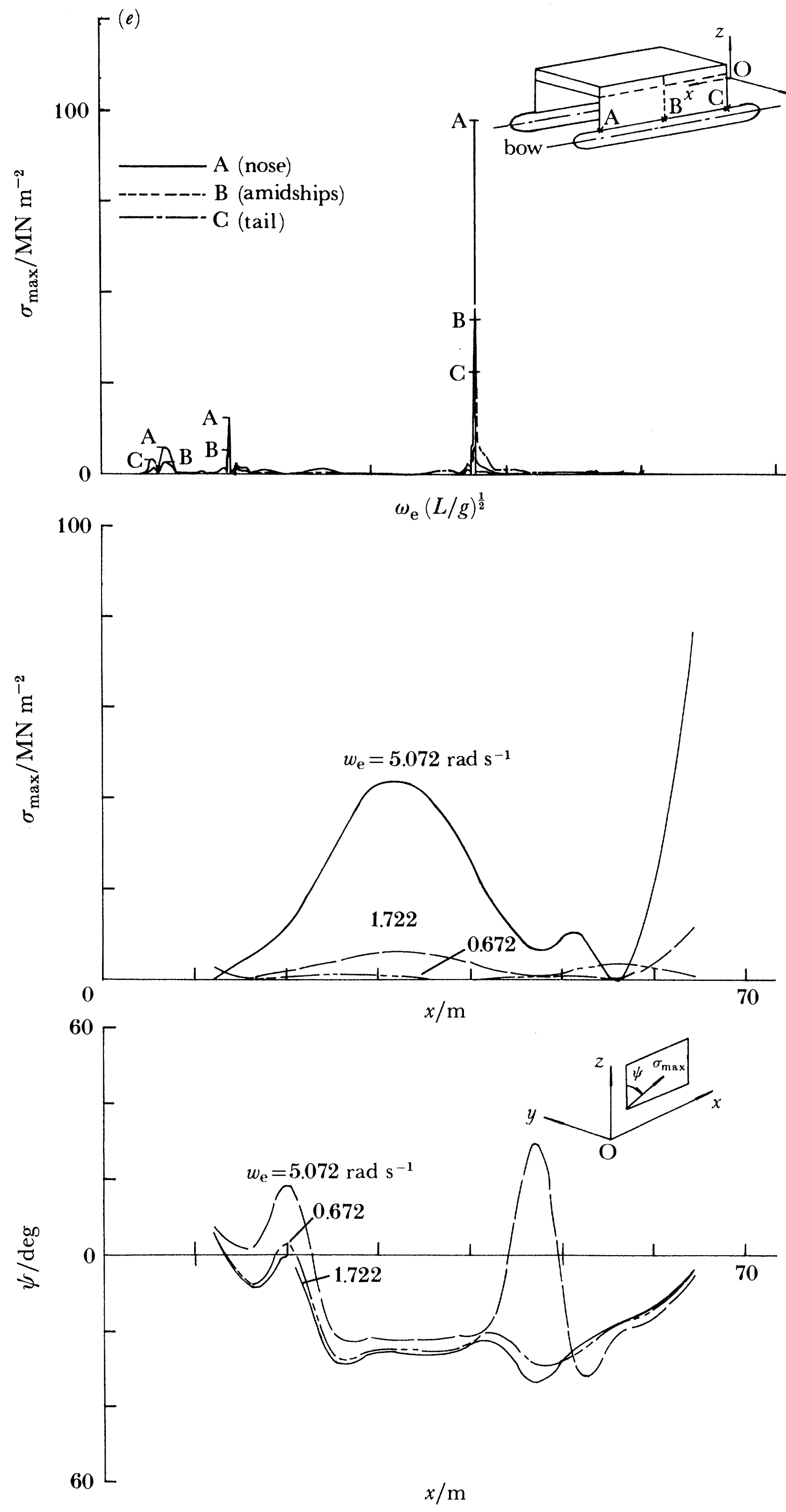


FIGURE 8 (*e*). For description see page 424.

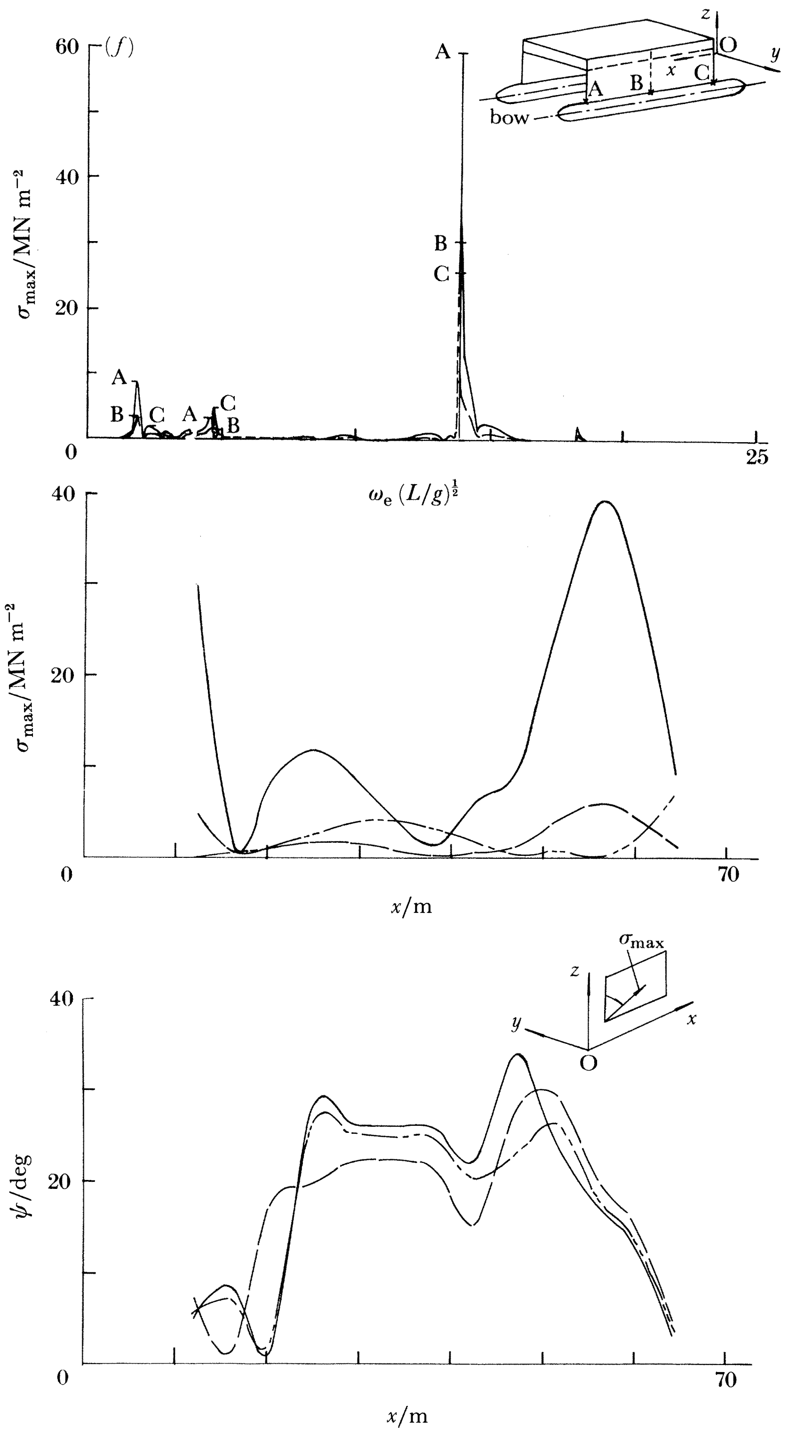


FIGURE 8. Selection of typical data for the s.w.a.t.h. model. (*a*) Non-dimensional generalized added-mass coefficient $A'_{rr} = A_{rr}/\rho\nabla$ and damping coefficients $B'_{rr} = B_{rr}/[\rho\nabla(g/L)^{\frac{1}{2}}]$ for the rigid-body modes $r = 2, 3, 4, 5$. (*b*) A'_{rr} and B'_{rr} for the distortion modes $r = 7$ (antisymmetric), $r = 8$, $r = 9$ (both symmetric), $r = 15$ (antisymmetric). (*c*) Amplitudes of the non-dimensional generalized wave-exciting force $F_r/a\rho g\nabla$ for modes $r = 3, 5, 8, 9$, when the s.w.a.t.h. travels with Froude number $F_n = 0.223$ in sinusoidal head waves of unit amplitude. (*d*) The variations in the amplitudes of the principal coordinates $|p_8|$ and $|p_9|$ for the s.w.a.t.h. proceeding at $F_n = 0.223$ in head waves. The resonance frequencies being $(\omega_e)_8 = 5.06$ rad s^{-1} and $(\omega_e)_9 = 6.72$ rad s^{-1}. (*e*) Variations of the maximum stress (σ_{max}) and direction ψ with frequency and position on the outer surface of the port strut along the four with the main hull ($F_n = 0.223$, head waves). (*f*) Variations of the maximum stress (σ_{max}) and direction ψ with frequency and position on the inner surface of the port strut along the join with the main hull ($F_n = 0.223$, head waves).

numerical procedures adopted the variations in the results may be caused by the occurrence of so-called 'irregular frequencies' in the calculation (see, for example, John (1949, 1950)). That is, the irregular frequencies and standing-wave frequency occur at nearly coincident values.

7. Conclusions

Based on the techniques of structural dynamics and hydrodynamics, a general three-dimensional linear hydroelasticity theory has been developed. It has been used to investigate the behaviour of a flexible mono-hull and a flexible multi-hull floating structure travelling in waves.

No distinction need be made between body motion and distortions for the purposes of calculation, because the theory allows all deflections to be placed on the same footing in the linear structural theory. Although the assumption of linearity has also been made in the hydrodynamic analysis this restriction may be lifted and a nonlinear hydrodynamic theory used if desired. The effect of using a nonlinear theory will, of course, introduce greater difficulty in calculating the variations of the principal coordinates of the dry structures.

The theory has already been used with other types of structures (for example, a jack-up rig in transit, partly and fully immersed vibrating plates, etc.), in which fluid–structure interaction is important. In addition, the theory can be modified easily to apply to fixed structures in which body motions do not occur, so that only the distortion modes ($r = 7, 8, \ldots$) are relevant.

The theory and the numerical techniques need not necessarily be restricted to the solution of marine problems. With slight modification they may be adapted to other engineering fields in which there is a fluid–structure interaction (for example, in the determination of stresses in flexible arteries owing to blood flow).

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