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14th-ORDER HARMONICS IN THE GEOPOTENTIAL FROM ANALYSIS OF SATELLITE ORBITS AT RESONANCE

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SUMMARY

The Earth's gravitational potential is usually expressed as a double series of tesseral harmonics of degree ℓ and order m, and values of coefficients for ℓ and m up to 36 or more are determined in recent models. The individual harmonic coefficients of a particular order m can most accurately be evaluated by analysing the perturbations on satellites which experience mth-order resonance (*ie* repeat their ground track after m revolutions).

Here we use results from analyses of 15 satellite orbits at 14th-order resonance to evaluate 7 pairs of individual coefficients of 14th order and odd degree (ℓ = 15,17,...27), and 6 pairs of coefficients of even degree (ℓ = 14,16,...24). The five most accurate pairs of values, for ℓ = 14,15,16,17 and 19, have a mean standard deviation corresponding to an accuracy of 0.9 cm in geoid height. These values are used as a test of the accuracy of three recent comprehensive models of the geopotential: the agreement is generally satisfactory, the differences being consistent with the expected errors (about 3×10^{-9}) in the models.

We also obtain values for the pair of coefficients of order 28 and degree 28, again with standard deviation equivalent to 0.9 cm in geoid height.

Departmental Reference: Space 643

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1 INTRODUCTION

Several comprehensive models of the Earth's gravitational field have been derived in recent years by expressing the geopotential as a double series of tesseral harmonics and evaluating these harmonic coefficients up to a chosen order and degree. For example, the Goddard Earth Model 10B[1] goes up to order and degree 36 and has about 1300 coefficients; while the 1981 model of Rapp[2] goes to order and degree 180 and has some 32000 coefficients. The best current models define the height of the geoid surface correct to about 150 cm; but most of the individual harmonic coefficients are subject to considerable uncertainty, and more accurate values of the coefficients of a particular order can best be obtained by analysis of resonant satellite orbits. For example, analysis of 15th-order resonant orbits (with the track over the Earth repeating each day after 15 revolutions) has yielded values for coefficients of order 15 and degree 15-23 with an accuracy equivalent to 1 cm in geoid height[3]. If such precision could be achieved for all other orders, there would be a great improvement in the accuracy of the geoid.

Satellites which pass slowly through 14th-order resonance - with tracks repeating each day after 14 revolutions - are unfortunately much less numerous than the 15th-order resonant satellites. However, a solution for individual 14th-order coefficients was obtained[4] in 1978, and the new solution offered in this Report uses the previous satellites together with three new ones which have been specially analysed for the purpose[5,6,7]. The newly derived values for the coefficients extend to a higher degree and have smaller standard deviations than before. The individual values therefore represent a considerable improvement; but there is still room for much more improvement, many further satellites being needed at inclinations between 30° and 45°, and between 55° and 65°.

2 DEFINITIONS

2.1 The geopotential

The longitude-dependent part of the Earth's gravitational potential at an exterior point (r,θ,λ) may be written in normalized form[8] as

$$\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta)\left\{\bar{C}_{\ell m}\cos m\lambda + \bar{S}_{\ell m}\sin m\lambda\right\}N_{\ell m}\ , \qquad (1)$$

where r is the distance from the Earth's centre, θ is co-latitude, λ is longitude (positive to the east), μ is the gravitational constant for the Earth (398600 km^3/s^2), R is the Earth's equatorial radius (6378.1 km), $P_{\ell}^{m}(\cos\theta)$ is the associated Legendre function of order m and degree ℓ, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients. The normalizing factor $N_{\ell m}$ is given by[8]

$$N_{\ell m}^{2} = \frac{2(2\ell+1)(\ell-m)!}{(\ell+m)!}\ , \qquad (2)$$

and coefficients of order $m = 14$ particularly concern us here.

2.2 Lumped harmonics

The theory for the resonance has been given many times[3-5] and will not be repeated here. For each satellite analysed as it passes through the 14th-order resonance, we obtain a numerical value for a 'lumped harmonic', $\bar{C}_m^{q,k}$ or $\bar{S}_m^{q,k}$, given by

$$\bar{C}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{C}_{\ell m} , \qquad \bar{S}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{S}_{\ell m} , \tag{3}$$

where the values of ℓ are such that $(\ell-k)$ is even, and $\ell \geqslant m$, so that ℓ increases in steps of 2 from its minimum permissible value ℓ_0 (which is either m or $m+1$), and the Q_ℓ are constants for a particular satellite, with $Q_{\ell 0} = 1$. Here $m = 14$, so that ℓ_0 is either 14 or 15. Generally the odd-degree lumped harmonics (with $\ell = 15,17,19,\ldots$) are obtained by analysing changes in inclination at resonance, while those of even degree (with $\ell = 14,16,18,\ldots$) come from analysis of eccentricity variations.

The theoretical curves[4,9,10] fitted to the variations of i and e near $\beta{:}\alpha$ resonance are derived from equations with terms of the form $(\bar{C}_m^{q,k}, \bar{S}_m^{q,k}) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (\gamma\Phi - q\omega)$, where ω is the argument of perigee and Φ is the resonance angle, defined by

$$\Phi = \alpha(\omega + M) + \beta(\Omega - \nu) .$$

In equation (4), M is the mean anomaly, Ω the right ascension of the node and ν the sidereal angle. Here $\beta = 14$ and $\alpha = 1$. The parameter γ may take the values $0,1,2,\ldots$ and q the values $0,\pm1,\pm2,\ldots$; but the terms with $\gamma = 1$ and with $q = 0$ and ±1 are usually the largest. Another equation between the parameters is $m = \gamma\beta$, and because $\beta = 14$ here, $m = 14$ if $\gamma = 1$. Finally k is given by $k = \gamma\alpha - q$, so that $k = 1 - q$ here, if $\gamma = 1$.

Each value of a lumped harmonic provides a linear equation of the form (3), and, if the contributing satellites are at distinctly different inclinations, the equations (3) can be solved for individual coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$.

The contributing satellites are listed and discussed in section 3, and the method of solution is described in section 4.

3 THE CONTRIBUTING SATELLITES

3.1 General

The values of the lumped harmonics obtained from analysis of each of the contributing satellites, with the appropriate Q factors, are given in Tables 1, 2 and 3 for $(q,k) = (0,1),(1,0)$ and $(-1,2)$ respectively. Also listed in these Tables are the values of Allan's[10] normalized inclination functions $\bar{F}_{\ell m p_0}$, where $p_0 = \frac{1}{2}(\ell_0 - k)$. Multiplication of the lumped harmonics by the $\bar{F}$ gives a measure of the amplitude of the perturbation that can arise for that inclination. The method of solution (see section 4) requires the standard deviations of some values to be relaxed, usually by a factor of 2, and this is indicated by footnotes to the Tables.

Sections 3.2-3.16 offer brief comments on the data for each of the 15 satellites used, in order of increasing inclination.

3.2 1965-82, inclination 32.0°

This is not a single satellite but a combination of the results obtained by Wagner[11] from analysing 3 fragments from the 1965-82 launch. The results, for $(q,k) = (0,1)$ only, were not used in our previous solution because the $\ell = 25$ and $\ell = 27$ terms for this satellite are likely to be the largest (see Table 1) and the previous solution only extended to $\ell = 21$. Even here, with a 7-coefficient solution, the sd had to be increased to 400, to allow for the neglect of terms with $\ell \geqslant 29$.

3.3 1958 α, inclination 33.2°

This orbit, analysed by Walker[12], was not included previously, for the same reasons. The orbit was of high drag and the values obtained, for $(q, k) = (0, 1)$, are not particularly accurate. However, in view of the dearth of satellites at low inclinations, the values were used. The error due to neglecting harmonics of degree $\ell \geqslant 29$ was considerably less than the standard deviations.

3.4 1967-11G, inclination 40.0°

This is one of the new satellites, recently analysed by King-Hele[6], and the only one having an inclination between 34° and 48°. Values of lumped harmonics are available for $(q, k) = (0, 1)$ and $(1, 0)$. The resonance was rapid and the orbital data were sparse, so the results are not very accurate.

3.5 1973-22A, inclination 48.4°

This was a high-drag orbit analysed by Klokočník[13], and lumped harmonics were available for $(q, k) = (0, 1)$ only.

3.6 1963-26A, inclination 49.7°

This orbit was analysed by Wagner[11], who obtained values for all three pairs of lumped harmonics, that is, for $(q, k) = (0, 1)$, $(1, 0)$ and $(-1, 2)$.

3.7 1964-15A, inclination 51.7°

Gooding[14] determined 210 post-resonant orbits of this satellite, Ariel 4, and obtained values for all three pairs of lumped harmonics.

3.8 1971-106A, inclination 65.7°

This high-drag orbit, analysed by Walker[15], yielded values for lumped harmonics with $(q, k) = (1, 0)$ and $(-1, 2)$.

3.9 1961-15G, inclination 66.8°

This orbit was analysed by Wagner[11], who obtained values for all three pairs of lumped harmonics.

3.10 1971-18B, inclination 69.9°

This was another high-drag orbit, analysed by Hiller[16]. Lumped harmonics were obtained for $(q, k) = (0, 1)$ only.

Table 1

Values of $\bar{C}_{14}^{0,1}$ and $\bar{S}_{14}^{0,1}$ used in the solutions, with values of $Q_{\ell}^{0,1}$

Satellite	i (deg)	Q_{15}	Q_{17}	Q_{19}	Q_{21}	Q_{23}	Q_{25}	Q_{27}	Q_{29}	Q_{31}	$10^9\bar{C}_{14}^{0,1}$	$10^9\bar{S}_{14}^{0,1}$	$\bar{F}_{15,14,7}$	$10^9\bar{F}_{15,14,7}\bar{C}_{14}^{0,1}$	$10^9\bar{F}_{15,14,7}\bar{S}_{14}^{0,1}$
1965-82	32.0	1.0	-5.80	18.64	-41.24	68.27	-87.41	86.29	-62.03	24.99	2762 ± 400†	420 ± 400†	0.001229	3.4 ± 0.5	0.5 ± 0.5
1958	33.2	1.0	-5.61	17.34	-36.55	56.94	-67.25	59.03	-34.19	5.08	6800 ± 5400*	-3300 ± 3000*	0.001805	12.3 ± 9.7	- 6.0 ± 5.4
1967-11G	40.0	1.0	-4.48	10.39	-15.07	13.78	- 6.25	- 1.96	4.91	- 2.23	-65 ± 163	-55 ± 164	0.01272	- 0.8 ± 2.1	- 0.7 ± 2.1
1973-22A	48.4	1.0	-2.996	3.930	- 2.026	- 0.611	1.348	0.046	- 0.740	0.069	-63 ± 52*	-115 ± 28	0.07340	- 4.6 ± 3.8	- 8.4 ± 2.1
1963-26A	49.7	1.0	-2.918	3.559	- 1.313	- 1.334	1.178	0.591	- 0.801	- 0.345	18 ± 18	-82 ± 11	0.09128	1.6 ± 1.6	- 7.5 ± 1.0
1964-15A	51.7	1.0	-2.464	2.280	- 0.174	- 1.070	0.245	0.561	- 0.117	- 0.321	75 ± 46*	-98 ± 23	0.1226	9.2 ± 5.6	-12.0 ± 2.8
1961-15G	66.8	1.0	-0.251	-0.465	- 0.186	0.088	0.170	0.103	0.002	- 0.056	5.5 ± 1.0	-22.5 ± 0.5	0.4900	2.7 ± 0.5	-11.0 ± 0.2
1971-18B	69.9	1.0	0.093	-0.302	- 0.293	- 0.124	0.028	0.094	0.084	0.038	32 ± 44+	-2 ± 20	0.5292	17.0 ± 23.2	- 1.3 ± 10.5
1965-16G	70.1	1.0	0.113	-0.288	- 0.293	- 0.134	0.017	0.088	0.084	0.043	-2.1 ± 0.3	-16.6 ± 0.9	0.5303	- 1.1 ± 0.2	- 8.8 ± 0.5
1970-97B	74.0	1.0	0.472	0.057	- 0.152	- 0.198	- 0.152	- 0.076	- 0.010	0.029	-32 ± 24	-20 ± 33	0.5134	-16.4 ± 12.3	-10.3 ± 16.9
1971-120A	81.2	1.0	0.926	0.743	0.544	0.365	0.221	0.113	0.039	- 0.007	1.4 ± 4.8+	-20.0 ± 1.2	0.2722	0.4 ± 1.3	- 5.4 ± 0.3
1971-120B	81.2	1.0	0.925	0.741	0.542	0.364	0.219	0.112	0.039	- 0.007	-3.7 ± 2.8*	-19.1 ± 1.2	0.2722	- 1.0 ± 0.8	- 5.2 ± 0.3
1970-47B	81.2	1.0	0.923	0.739	0.539	0.360	0.216	0.109	0.037	- 0.008	-1.84 ± 0.62	-19.52 ± 0.48	0.2737	- 0.5 ± 0.2	- 5.3 ± 0.1

* standard deviation ×2 + standard deviation ×4 † increased to cover neglect of Q_{ℓ} for $\ell \geqslant 29$

Table 2

Values of $\bar{C}_{14}^{1,0}$ and $\bar{S}_{14}^{1,0}$ used in the solutions, with values of $Q_{\ell}^{1,0}$

Satellite	i (deg)	Q_{14}	Q_{16}	Q_{18}	Q_{20}	Q_{22}	Q_{24}	Q_{26}	Q_{28}	$10^9\bar{C}_{14}^{1,0}$	$10^9\bar{S}_{14}^{1,0}$	$\bar{F}_{14,14,7}$	$10^9\bar{F}_{14,14,7}\bar{C}_{14}^{1,0}$	$10^9\bar{F}_{14,14,7}\bar{S}_{14}^{1,0}$
1967-11G	40.0	1.0	-10.6	43.9	-106.8	170.3	-178.5	103.7	8.6	1255 ± 1550	2477 ± 1276	0.001268	1.6 ± 2.0	3.1 ± 1.6
1963-26A	49.7	1.0	-7.381	18.736	-22.411	8.705	8.458	-8.538	-3.120	-493 ± 157	538 ± 157	0.01398	-6.9 ± 2.2	7.5 ± 2.2
1964-15A	51.7	1.0	-6.761	15.095	-14.281	1.078	8.115	-2.493	-4.799	-483 ± 128	490 ± 133	0.02053	-9.9 ± 2.6	10.1 ± 2.7
1971-106A	65.7	1.0	-2.601	0.379	1.409	0.568	-0.439	-0.686	-0.303	17 ± 59	-137 ± 350**	0.1690	2.9 ± 9.9	-23 ± 59
1961-15G	66.8	1.0	-2.351	0.026	1.181	0.727	-0.146	-0.564	-0.414	12 ± 7	70 ± 12	0.1886	2.3 ± 1.3	13.2 ± 2.3
1965-16G	70.1	1.0	-1.595	-0.669	0.352	0.673	0.436	0.038	-0.227	17 ± 31*	68 ± 51	0.2598	4.4 ± 8.1	17.8 ± 13.2
1970-97B	74.0	1.0	-0.830	-0.813	-0.343	0.082	0.298	0.308	0.199	-67 ± 53	12 ± 50	0.3547	-24 ± 19	4 ± 18
1971-120B	81.2	1.0	0.172	-0.146	-0.272	-0.292	-0.256	-0.195	-0.128	-34.9 ± 4.7	-12.2 ± 4.0	0.5234	-18.3 ± 2.5	-6.4 ± 2.1
1970-47B	81.2	1.0	0.170	-0.148	-0.273	-0.292	-0.255	-0.193	-0.127	-40.2 ± 2.3	-9.5 ± 2.4	0.5229	-21.0 ± 1.2	-5.0 ± 1.2
1965-81A	87.4	1.0	0.569	0.387	0.270	0.188	0.129	0.086	0.055	-60.0 ± 6.6	-35.4 ± 11.4	0.6075	-36.4 ± 4.0	-21.5 ± 6.9

* standard deviation ×2   ** standard deviation ×10

Table 3

Values of $\bar{C}_{14}^{-1,2}$ and $\bar{S}_{14}^{-1,2}$ used in the solutions, with values of $Q_{\ell}^{-1,2}$

Satellite	i (deg)	Q_{14}	Q_{16}	Q_{18}	Q_{20}	Q_{22}	Q_{24}	Q_{26}	Q_{28}	$10^9\bar{C}_{14}^{-1,2}$	$10^9\bar{S}_{14}^{-1,2}$	$\bar{F}_{14,14,6}$	$10^9\bar{F}_{14,14,6}\bar{C}_{14}^{-1,2}$	$10^9\bar{F}_{14,14,6}\bar{S}_{14}^{-1,2}$
1963-26A	49.7	1.0	-4.869	7.254	-2.939	-3.131	2.589	1.691	-1.740	-29 ± 87*	-145 ± 320*	0.05695	-1.7 ± 5.0	-8.3 ± 18.1
1964-15A	51.7	1.0	-4.340	5.254	-0.604	-3.034	0.625	1.963	-0.216	-48 ± 71	100 ± 70	0.07669	-3.7 ± 5.4	7.7 ± 5.4
1971-106A	65.7	1.0	-1.008	-0.812	-0.041	0.458	0.451	0.143	-0.149	-186 ± 160**	66 ± 42	0.3542	-66 ± 57	23 ± 15
1962-15G	66.8	1.0	-0.826	-0.818	-0.208	0.304	0.432	0.246	-0.021	-41 ± 12	30 ± 6	0.3802	-15.6 ± 4.6	11.4 ± 2.3
1965-16G	70.1	1.0	-0.299	-0.637	-0.495	-0.168	0.116	0.245	0.219	-38 ± 25	-11 ± 43	0.4625	-17 ± 12	-5 ± 20
1979-97B	74.0	1.0	0.182	-0.209	-0.369	-0.358	-0.246	-0.103	0.019	-77 ± 54	-67 ± 80*	0.5466	-42 ± 29	-37 ± 44
1971-120B	81.2	1.0	0.623	0.444	0.307	0.193	0.102	0.032	-0.017	-81.4 ± 20.8†	-34.9 ± 4.8	0.6224	-50.7 ± 12.9	-21.7 ± 3.0
1970-47B	81.2	1.0	0.622	0.442	0.304	0.191	0.100	0.030	-0.018	-51.8 ± 3.4	-27.2 ± 3.0	0.6224	-32.2 ± 2.1	-16.9 ± 1.9
1965-81A	87.4	1.0	0.512	0.333	0.237	0.179	0.140	0.114	0.094	-52.3 ± 8.0	12.1 ± 34.4*	0.5831	-30.5 ± 4.6	7.1 ± 20.0

* standard deviation ×2 † standard deviation ×4 ** standard deviation ×10

3.11 1965-16G, inclination 70.1°

This satellite was analysed in Ref 4: good values were obtained for all three pairs of lumped harmonics.

3.12 1970-97B, inclination 74.0°

This was a high-drag satellite, but a recent analysis of the 14th-order resonance[7] has yielded values - though inevitably not very accurate ones - for all three pairs of lumped harmonics. This satellite replaced 1973-82A, also at inclination 74.0°, which was used previously[4].

3.13 1971-120A, inclination 81.2°

This was an analysis by Wagner[11] of the orbit before reaching resonance, and values of lumped harmonics were obtained for (q, k) = (0, 1) only. The standard deviation for $\bar{C}_{14}^{0,1}$ was multiplied by 4 to avoid a clash with the results from 1971-120B (see below) and keep their weighted residuals similar.

3.14 1971-120B, inclination 81.2°

This satellite, analysed in Ref 4, yielded values for all three pairs of lumped harmonics. The results were of very good accuracy, but their reliability is slightly open to question, because the analysis did not begin until after the satellite had passed resonance. The orbit was almost identical to that of 1971-120A, and the standard deviation for $\bar{C}_{14}^{0,1}$ was doubled to avoid a clash between them.

3.15 1970-47B, inclination 81.2°

This new analysis, recently completed by Walker[5], is the most accurate of all the 14th-order resonance analyses, being determined from 208 orbits over 4 years centred on resonance. The results are not only valuable in themselves but also as a test of the two 'half resonances' for 1971-120A and 1971-120B at the same inclination. The result of the test is most satisfactory. The values for 5 of the 6 lumped harmonics from 1970-47B and 1971-120B agreed to within the sum of their standard deviations.

3.16 1965-81A, inclination 87.4°

This satellite, analysed in Ref 4, was at an inclination where the term which usually dominates the change in i - the (q, k) = (0, 1) term - has very little effect. Consequently the satellite only gave values for the (q, k) = (1, 0) and (-1, 2) terms, as recorded in Tables 2 and 3.

4 THE ODD-DEGREE SOLUTIONS

4.1 Procedure

The analyses in section 3 provided 13 equations for 14th-order C-coefficients of odd degree,

$$\bar{C}_{15,14} + Q_{17}^{0,1}\,\bar{C}_{17,14} + Q_{19}^{0,1}\,\bar{C}_{19,14} + \ldots = \bar{C}_{14}^{0,1} \,, \qquad (5)$$

and 13 similar equations for the S-coefficients, as listed in Table 1.

In the past we have added constraint equations of the form

$$\bar{C}_{\ell,14} = 0 \pm 10^{-5}/\ell^2 \tag{6}$$

where the quantity $10^{-5}/\ell^2$ derives from 'Kaula's rule of thumb' for the magnitude of a coefficient of degree ℓ, and we have solved equations (5) and (6) by least squares.

However, now that the coefficients of low degree are becoming quite well determined, it is unhelpful to use equation (6) for the lowest-degree harmonics, because in effect equation (6) is an instruction to keep the value as near zero as possible - an instruction which is counter-productive if the value is known to be large. An alternative plan would be to fix the well-known first few values and solve for the remainder. However, this is too inflexible a procedure, and we have opted for another possibility, namely replacing the zero in the first few equations (6) with the value derived from the first solution, that is the solution utilizing equation (6); then we make a further iteration with $\bar{C}_{\ell,14}$ set equal to the value from the first solution, and with its standard deviation twice that of the first solution. And so on. The residuals for the specified coefficients tend to zero after two iterations, and these equations are then eliminated from the degrees of freedom. This procedure is bringing in outside information - the knowledge that the first few values are well-determined - to replace the 'rule of thumb' expressed in equation (6). The number of coefficients chosen for this iterative procedure was three - those of degree 15, 17 and 19 - because they had much lower standard deviations than the later coefficients. However the choice is not critical, and the values obtained for the individual coefficients do not change significantly when four coefficients are treated in this way.

As in our recent determinations of 15th-order harmonics[3], we have relaxed the assumed standard deviations of any values of $(C,S)^{0,1}_{14}$ for which the weighted residuals exceed a chosen value, taken here as 1.3. (The weighted residual is the residual divided by the assumed sd.) These relaxations - usually doublings, sometimes quadruplings - are indicated in Table 1; of the 26 values, five are relaxed by a factor of 2 and two by a factor of 4. The choice of 1.3 as the limit is not critical, but is merely a convenient level with the particular set of weighted residuals which arise.

4.2 Results

When the C and S equations were solved for 3, 4, ... 8 coefficients, the following values were obtained for the measure of fit ε :

Number of coefficients	3	4	5	6	7	8
C equations	3.05	2.71	1.75	1.19	0.93	0.92
S equations	1.60	0.71	0.70	0.66	0.66	0.65

As usual, ε^2 is defined as the sum of the squares of the weighted residuals, divided by the number of degrees of freedom. The weighted residual is the residual divided by the sd given in Table 1.

For the C equations, the value of ε decreases substantially for each increase in the number of coefficients up to 7, but there is no significant change when more than 7 are evaluated. At least 7 coefficients are needed because the Q values are large for the three low-inclination satellites in Table 1. For the S equations it happens that the 5th, 6th and 7th coefficients are all small and consequently there is a choice of 4, 5, 6 or 7 coefficients for the solution: the 7-coefficient solution was adopted, so as to conform with that of the C equations.

The values of the odd-degree C and S coefficients from the 7-harmonic solutions are listed in Table 4. Our previous solutions for odd-degree 14th-order harmonics were for four C and four S coefficients only. These previous solutions agree with the new values to well within the sum of their standard deviations. This agreement is not unexpected, as many of the same satellites are included: however, the agreement is valuable in showing that the previous neglect of higher harmonics did not upset the reliability of the results. We hope a similar conclusion applies here.

In Table 4 the standard deviations become larger as the degree of the coefficients increases. For the first three C and the first three S coefficients the average sd is 1.4×10^{-9}, which is equivalent to less than 1 cm in geoid height. For the 4th and 5th coefficients, the equivalent average accuracy is near 2 cm in geoid height; for the 6th and 7th, it is 3 cm. (Curiously, the accuracy of the S coefficients is much better than that of the C coefficients throughout.)

Table 4

Seven-coefficient solution for harmonics of odd degree

Degree ℓ	$10^9\,\bar{C}_{\ell,14}$	$10^9\,\bar{S}_{\ell,14}$
15	4.6 ± 1.5	-24.7 ± 0.8
17	-17.7 ± 2.3	17.9 ±1.1
19	-0.2 ± 1.8	-8.5 ± 0.7
21	17.0 ± 4.8	-10.8 ± 2.6
23	6.8 ± 4.2	-0.6 ± 2.7
25	-15.9 ± 5.7	1.7 ± 2.9
27	16.4 ± 6.5	4.6 ± 3.9

The weighted residuals, for the 13 satellite equations and the relevant constraint equations (6), are given in Table 5. Equations for which the standard deviations were relaxed (as shown in Table 1) are marked with 'R'.

Fig 1 shows the variation of the lumped harmonics with inclination, after each has been multiplied by $\bar{F}_{15,14,7}$. This product of $\bar{F}$ and $\bar{C}$ (or $\bar{F}$ and $\bar{S}$) gives a better indication of the amplitude of the variation in inclination caused by 14th-order harmonics at any particular inclination. In Fig 1, the fitting of the curve from the solution of Table 4 is quite satisfactory, though there are gaps in the inclination coverage which demand to be filled. Good satellites at inclinations between 30° and 45°, between 55° and 65°, and near 75° and 90°, are much needed.

Table 5

Weighted residuals for the 13 satellite equations and the four constraint equations in the solution of Table 4

Satellite	Residuals for: $\bar{C}^{0,1}_{14}$	$\bar{S}^{0,1}_{14}$	Constraint equations ℓ	$\bar{C}_{\ell,14}$	$\bar{S}_{\ell,14}$
1965-82	0.25R	0.15R	21	-0.75	0.48
1958 α	0.91R	-1.18R	23	-0.36	0.03
1967-11G	-0.31	0.02	25	0.99	-0.11
1973-22A	-1.17R	-1.00	27	-1.20	-0.33
1963-26A	0.16	0.51			
1964-15A	0.70R	-0.66			
1961-15G	-0.01	0.06			
1971-18B	0.78R	0.72			
1965-16G	0.01	-0.06			
1970-97B	-1.06	-0.14			
1971-120A	0.67R	-0.38			
1971-120B	-0.67R	0.36			
1970-47B	0.07	-0.01			

5 THE EVEN-DEGREE SOLUTIONS

5.1 Procedure

The procedure was similar to that adopted for the odd-degree coefficients. For even degree, we have the 10 equations for (q,k) = (1, 0) given in Table 2, of the form

$$\bar{C}_{14,14} + Q_{16}^{1,0}\,\bar{C}_{16,14} + Q_{18}^{1,0}\,\bar{C}_{18,14} + \ldots = \bar{C}_{14}^{1,0}\,, \tag{7}$$

and 10 equations with S instead of C. We also have the 9 equations for (q, k) = (-1, 2) given in Table 3, of the form

$$\bar{C}_{14,14} + Q_{16}^{-1,2}\,\bar{C}_{16,14} + Q_{18}^{-1,2}\,\bar{C}_{18,14} + \ldots = \bar{C}_{14}^{-1,2}\,, \tag{8}$$

and 9 equations with S instead of C.

To these 38 equations, 19 for C and 19 for S, we add constraint equations of the form (6) and then replace the first few of these by the iterative procedure for well-determined low-degree coefficients. For the even-degree harmonics, 'few' is taken as 'two', because the first two coefficients are much more accurately determined than the later ones. The standard deviations of the lumped harmonics were relaxed so as to ensure that none of the weighted residuals exceeded 1.35: of the 38 standard deviations, two were relaxed by a factor of 10, one by a factor of 4 and five by a factor of 2, as Tables 2 and 3 show.

5.2 Results

When the equations were solved for 3, 4...7 coefficients the following values of ε were obtained:

Number of coefficients	3	4	5	6	7
C equations	1.30	1.12	1.12	0.79	0.79
S equations	1.04	0.78	0.68	0.67	0.67

Clearly, the best choice here is 6 coefficients, and Tables 2 and 3 show that the neglect of the 7th coefficient (of degree 26) is justified, because if the value of $\bar{C}_{26,14}$ is taken as $0.5 \times 10^{-5}/26^2$ $(= 7 \times 10^{-9})$, the value of $Q_{26}\bar{C}_{26,14}$ is considerably less than the sd of the appropriate lumped harmonic, for all the satellites.

The values of the even-degree $\bar{C}$ and $\bar{S}$ coefficients given by the 6-harmonic solution are listed in Table 6.

In assessing our previous solution for even-degree coefficients, we commented that "there are discrepancies for $\bar{C}_{\ell,14}$" and that "further work... is needed". This conclusion is confirmed by the new results, which, although within one standard deviation of the previous results for $\bar{S}$, are significantly different from the previous results for $\bar{C}$ - in particular for $\bar{C}_{18,14}$ and $\bar{C}_{22,14}$. The reason for the change is the new value for $\bar{C}_{14}^{-1,2}$ from 1970-47B, which is considerably smaller than the value used previously, from 1971-120B: the standard deviation of the latter had to be increased by a factor of 4 in the new solution.

The standard deviations in the new solution are on average about half those in the previous solution, and there is every reason to suppose that the new solution is much better than the old.

The weighted residuals, for the 38 satellite equations and the relevant constraint equations (6), are given in Table 7. Again the values for which the standard deviations were relaxed are marked with 'R'.

Table 6

Six-coefficient solution for harmonics of even degree

Degree ℓ	$10^9\ \bar{C}_{\ell,14}$	$10^9\ \bar{S}_{\ell,14}$
14	-40.8 ± 1.2	-5.1 ± 0.9
16	-16.6 ± 1.6	-35.1 ± 1.9
18	-8.0 ± 3.8	-1.4 ± 2.7
20	12.2 ± 2.9	-11.7 ± 2.5
22	-4.6 ± 4.2	7.6 ± 3.2
24	-19.3 ± 4.7	2.0 ± 3.2

Table 7

Weighted residuals for each satellite equation and the four constraint equations in the solution of Table 6

Satellite	Residuals for:				Constraint equations		
	$\bar{C}_{14}^{1,0}$	$\bar{C}_{14}^{-1,2}$	$\bar{S}_{14}^{1,0}$	$\bar{S}_{14}^{-1,2}$	ℓ	$\bar{C}_{\ell,14}$	$\bar{S}_{\ell,14}$
1967-11G	0.07	-	-0.01	-	18	0.26	0.05
1963-26A	0.32	0.69R	-0.22	-0.99R	20	-0.49	0.47
1964-15A	-0.77	-0.45	0.66	-0.36	22	0.22	-0.37
1971-106A	-0.09	-0.98R	-0.60R	0.71	24	1.11	-0.11
1961-15G	0.01	-0.69	0.09	-0.12			
1965-16G	1.06R	0.03	0.29	-0.52			
1970-97B	-0.68	-0.69	-0.37	-0.71R			
1971-120B	0.98	-1.33R	-0.44	-1.13			
1970-47B	-0.31	0.57	0.35	0.75			
1965-81A	-1.00	0.04	-0.73	1.07R			

The variations of the lumped coefficients with inclination are shown in Figs 2 and 3, after multiplication by the appropriate $\bar{F}$ factor: again the product $\overline{FC}$ (or $\overline{FS}$) gives a measure of the strength of the perturbations in the orbital elements for each inclination. Since the fittings of the points in Figs 2 and 3 are simultaneous, a failure to fit a point on the upper curve may be due to the need to fit a point on the lower curve and vice-versa. The $\bar{C}_{14}^{-1,2}$ curves are quite similar in form to the $\bar{C}_{14}^{1,0}$, but displaced by about 7° in inclination. The same applies for S.

To illustrate this effect, the curves are replotted superposed in Fig 4, with the inclination shifted by 7° for (q,k) = (-1,2). The agreement is very close, and it is quite illuminating to assume that one curve is being fitted to all the points, both the triangles and the circles. If we take this 'conspectus' view, the inclination coverage could be regarded as almost complete from 40° to 95°, apart from a gap near 45°. Thus it is probable that the even-degree solutions are more reliable than those of odd degree, although of slightly poorer nominal accuracy because the eccentricity is more difficult to analyse than the inclination. Also, when looked at in this combined form, the curves fit the points most convincingly.

6 COMPARISONS WITH COMPREHENSIVE GEOPOTENTIAL MODELS

Table 8 gives our values of 14th-order geopotential coefficients and those from three recent comprehensive models, the Goddard Earth Model 10B[1], the 1981 model of Rapp[2], and the European Model GRIM3-L1[17]. Standard deviations are not available for GEM 10B or GRIM3-L1. Rapp does give sds, but they are ignored for the moment, to preserve uniformity in Table 8.

The errors in the comprehensive models are believed to be of order 3×10^{-9} for the coefficients in Table 8. If so, and if we accept the validity of our standard deviations, our values should be generally more accurate than those of the comprehensive models for ℓ = 14, 15, 16, 17 and 19. So it is of interest to compare the coefficients for these five values of ℓ. The mean numerical difference between our values of these coefficients and the corresponding values in each of the comprehensive models is denoted by d and given at the end of the Table. Also, in the last row of the Table, we give the mean difference D between our values and the corresponding values in the comprehensive models, averaged over all the 13 values of ℓ for which we have obtained results.

The mean values for d in the three comprehensive models are 3.5, 3.3 and 4.5×10^{-9} for GEM 10B, Rapp 1981 and GRIM 3-L1 respectively, while our mean sd is 1.4×10^{-9}. So the results of the comparisons are consistent with the assumption that the comprehensive models have errors of order 3×10^{-9} in these coefficients. However, it should be remembered that the three models share some common data with each other and with our data, so the comparisons are in no way conclusive.

When all 13 coefficients are included in the comparison, the mean differences D are 4.0, 4.4 and 5.4×10^{-9} for GEM 10B, Rapp 1981 and GRIM3-L1 respectively, while our average sd is 2.9×10^{-9}. Again the results are consistent with errors of order 3×10^{-9} in the comprehensive models; again GRIM3-L1 differs from our values more than the other two models; and again the comparison cannot be regarded as conclusive because of the shared data.

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Table 8

Values of the 14th-order coefficients in four solutions: (1) Ours (Tables 4 and 6) (2) GEM 10B, (3) Rapp 1981 and (4) GRIM3-L1

	$10^9 \bar{C}_{\ell,14}$				$10^9 \bar{S}_{\ell,14}$			
ℓ	Ours	GEM	Rapp	GRIM	Ours	GEM	Rapp	GRIM
14	-40.8 ± 1.2	-51.9	-49.8	-56.3	-5.1 ± 0.9	-4.6	-5.7	-7.5
15	4.6 ± 1.5	4.0	4.0	3.8	-24.7 ± 0.8	-24.0	-24.0	-24.9
16	-16.6 ± 1.6	-17.9	-20.0	-24.0	-35.1 ± 1.9	-37.5	-37.5	-40.5
17	-17.7 ± 2.3	-15.7	-15.5	-17.6	17.9 ± 1.1	11.5	12.2	12.5
18	-8.0 ± 3.8	-7.4	-10.4	-8.2	-1.4 ± 2.7	-10.8	-12.6	-17.1
19	-0.2 ± 1.8	-5.7	-5.7	-5.6	-8.5 ± 0.7	-12.4	-11.5	-10.9
20	12.2 ± 2.9	13.5	12.1	5.7	-11.7 ± 2.5	-11.1	-11.6	-12.8
21	17.0 ± 4.8	19.9	20.3	12.0	-10.8 ± 2.6	9.2	3.2	7.2
22	-4.6 ± 4.2	7.7	8.1	8.7	7.6 ± 3.2	7.5	8.3	10.5
23	6.8 ± 4.2	6.5	7.4	8.8	-0.6 ± 2.7	-5.2	-1.2	7.4
24	-19.3 ± 4.7	-19.2	-18.9	-21.6	2.0 ± 3.2	3.4	4.0	0.2
25	-15.9 ± 5.7	-19.2	-24.4	-21.6	1.7 ± 2.9	10.4	11.5	5.7
26	-	4.0	2.1	5.4	-	0.9	6.5	6.8
27	16.4 ± 6.5	16.6	7.8	12.8	4.6 ± 3.9	8.8	11.0	8.7
d		4.1	4.1	5.8		2.8	2.5	3.2
D		3.2	4.4	5.2		4.8	4.4	5.5

Rapp gives very small standard deviations ($< 1 \times 10^{-9}$) with his values for ℓ = 15, 17 and 19. For ℓ = 17 and 19 the agreement between his values and ours is little better than average - the mean difference is 4.1×10^{-9}. But for ℓ = 15 , the agreement is very close, his C value being 4.0 ± 0.5 as compared with our 4.6 ± 1.5 , and his S value being -24.0 ± 0.3 as compared with our -24.7 ± 0.8. The ℓ = 15 values for GEM 10B and GRIM3-L1 also agree well with our results. Thus it seems highly probable that values of the $(C,S)_{15,14}$ coefficients are now securely established: the values $10^9 \bar{C}_{15,14} = 4.2 \pm 0.4$ and $10^9 \bar{S}_{15,14} = -24.4 \pm 0.5$ are within one sd of the values from all four sources.

7 HARMONICS OF ORDER 28

Only one of the analyses of the 14th-order resonance - that of 1970-47B by Walker[5] - has yielded well-determined values for lumped harmonics of order 28. However, even with this satellite alone, it is possible to gain some information about one pair of individual coefficients, those of degree 28 and order 28, because it happens by good luck that the Q factors beyond the first two are so small as to be negligible, and the equations for the lumped harmonics from 1970-47B are of the form[5]

$$\bar{C}_{28,28} + 0.305\,\bar{C}_{30,28} = (7.1 \pm 1.1) \times 10^{-9} \qquad (9)$$

$$\bar{S}_{28,28} + 0.305\,\bar{S}_{30,28} = (1.5 \pm 1.1) \times 10^{-9} . \tag{10}$$

So if values can be assigned for the (30,28) coefficients, we can obtain a value of the (28,28) coefficients. The values of the (30,28) coefficients in the three comprehensive models are given below:

	GEM 10B	Rapp 1981	GRIM3-L1
$\bar{C}_{30,28}$	-17.4	-9.1 ± 1.6	-6.3
$\bar{S}_{30,28}$	-34.1	-18.5 ± 1.7	-6.7

The value of $\bar{S}_{30,28}$ from GEM 10B is extraordinarily large: indeed it is the largest of the 896 coefficients in the GEM 10B field for $20 < \ell \leqslant 36$, and is equivalent to $3 \times 10^{-5}/\ell^2$. Consequently the GEM values must be treated with suspicion, and we take the mean of the Rapp and GRIM, namely $10^9\,\bar{C}_{30,28} = -7.7$ and $10^9\,\bar{S}_{30,28} = -12.6$, with the sd which has already survived the test of comparison in Table 8, namely 3×10^{-9}. When these values are substituted into equations (9) and (10), we obtain

$$\left.\begin{aligned} 10^9\,\bar{C}_{28,28} &= 9.4 \pm 1.4 \\ 10^9\,\bar{S}_{28,28} &= 5.3 \pm 1.4 \end{aligned}\right\} \tag{11}$$

These values are compared with those from the comprehensive models in Table 9, where GEM 10B and GRIM3-L1 have arbitrarily been allocated standard deviations of 3×10^{-9}.

Table 9

Values of $\bar{C}_{28,28}$ and $\bar{S}_{28,28}$

	Ours	GEM 10B	Rapp 1981	GRIM3-L1
$10^9\,\bar{C}_{28,28}$	9.4 ± 1.4	9.4 ± 3	6.5 ± 1.9	10.5 ± 3
$10^9\,\bar{S}_{28,28}$	5.3 ± 1.4	8.6 ± 3	5.7 ± 2.6	4.7 ± 3

It will be seen that all the values are in agreement to within the sum of their standard deviations.

Thus the new and largely independent values of $(C,S)_{28,28}$ deducible from the analysis of 1970-47B lend support to the values of the three models. (The fact that the models are consistent with each other may merely reflect their use of common terrestrial gravity data.)

8 CONCLUSIONS

We have used the values of lumped 14th-order harmonics obtained from analysis of 15 satellites, as described in section 3.2 - 3.16, to determine individual harmonic coefficients of order 14. For the odd-degree coefficients (degree 15, 17, 19, ...) there are 13 equations for individual C coefficients and 13 for S coefficients. These equations are solved to give values for 7 pairs of individual coefficients, of degree 15, 17...27, as recorded in Table 4. For the even-degree coefficients (degree 14, 16, 18,...) there are 19 equations available for the C coefficients and 19 for the S coefficients. These are solved for 6 pairs of individual coefficients, of degree 14, 16,...24, as recorded in Table 6.

In these solutions the most accurate values are those of degree 14, 15, 16, 17 and 19, which have average standard deviations of 1.4×10^{-9}, equivalent to 0.9 cm in geoid height. The average standard deviation of our coefficients, for all 13 pairs of coefficients, is 2.9×10^{-9}, equivalent to 1.8 cm in geoid height.

When the five most accurate of our values are compared with the corresponding values in three existing comprehensive geopotential models, the mean differences are between 3.3 and 4.5×10^{-9}: this is consistent with the assumption that the models are accurate to about 3×10^{-9} for 14th order, although the comparison is not conclusive because there are some data in common. Table 8 provides a detailed comparison.

Values for the C and S coefficients of degree and order 28 have also been obtained, based on the recent analysis[5] of satellite 1970-47B, and are given in equations (11): again the nominal accuracy is equivalent to 0.9 cm. The corresponding values from the three comprehensive models, which for order 28 rely on data independent of ours, are in satisfactory agreement.

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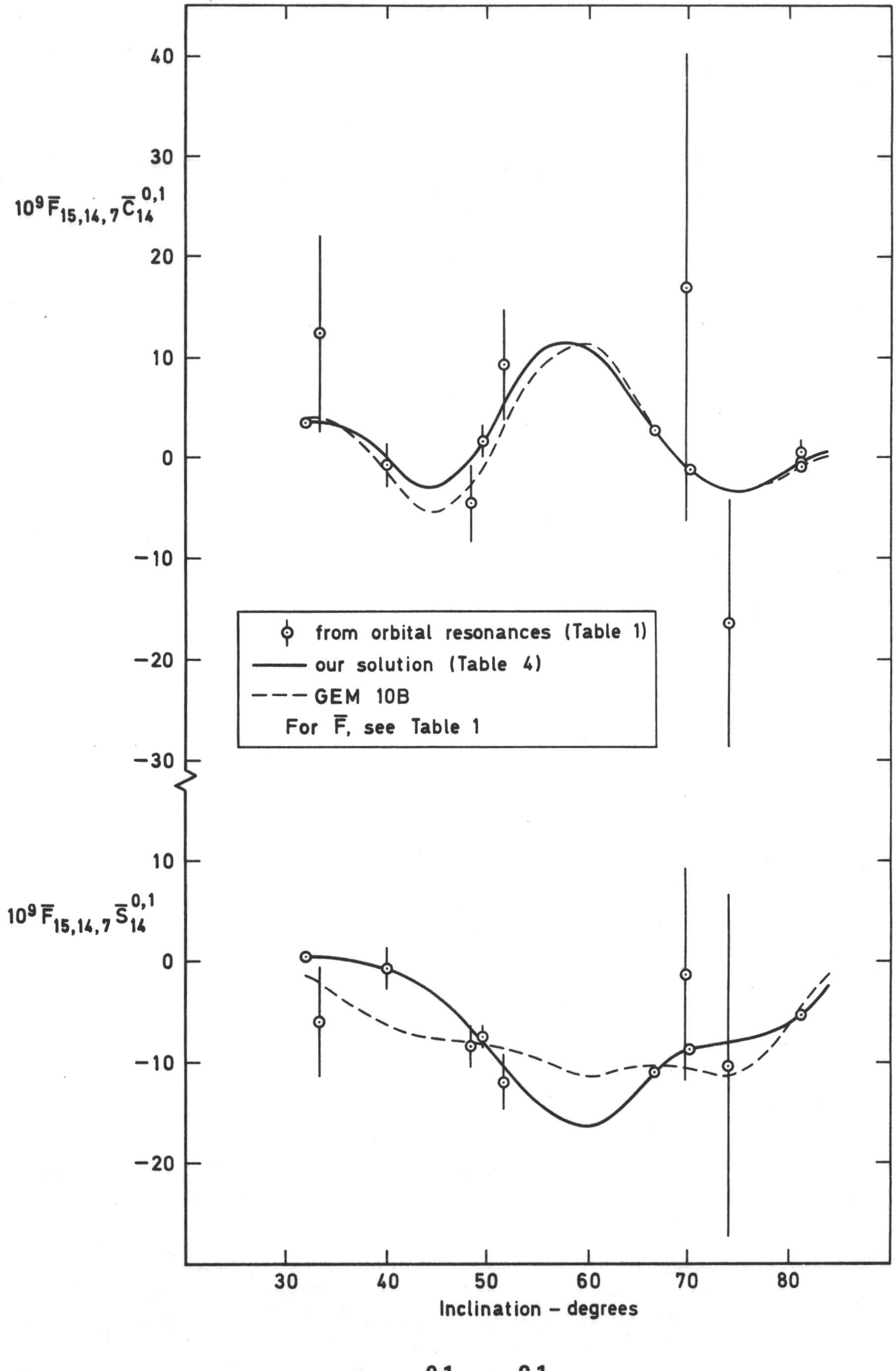


Fig 1 Variation of $\overline{C}_{14}^{0,1}$ and $\overline{S}_{14}^{0,1}$ with inclination

Fig 2

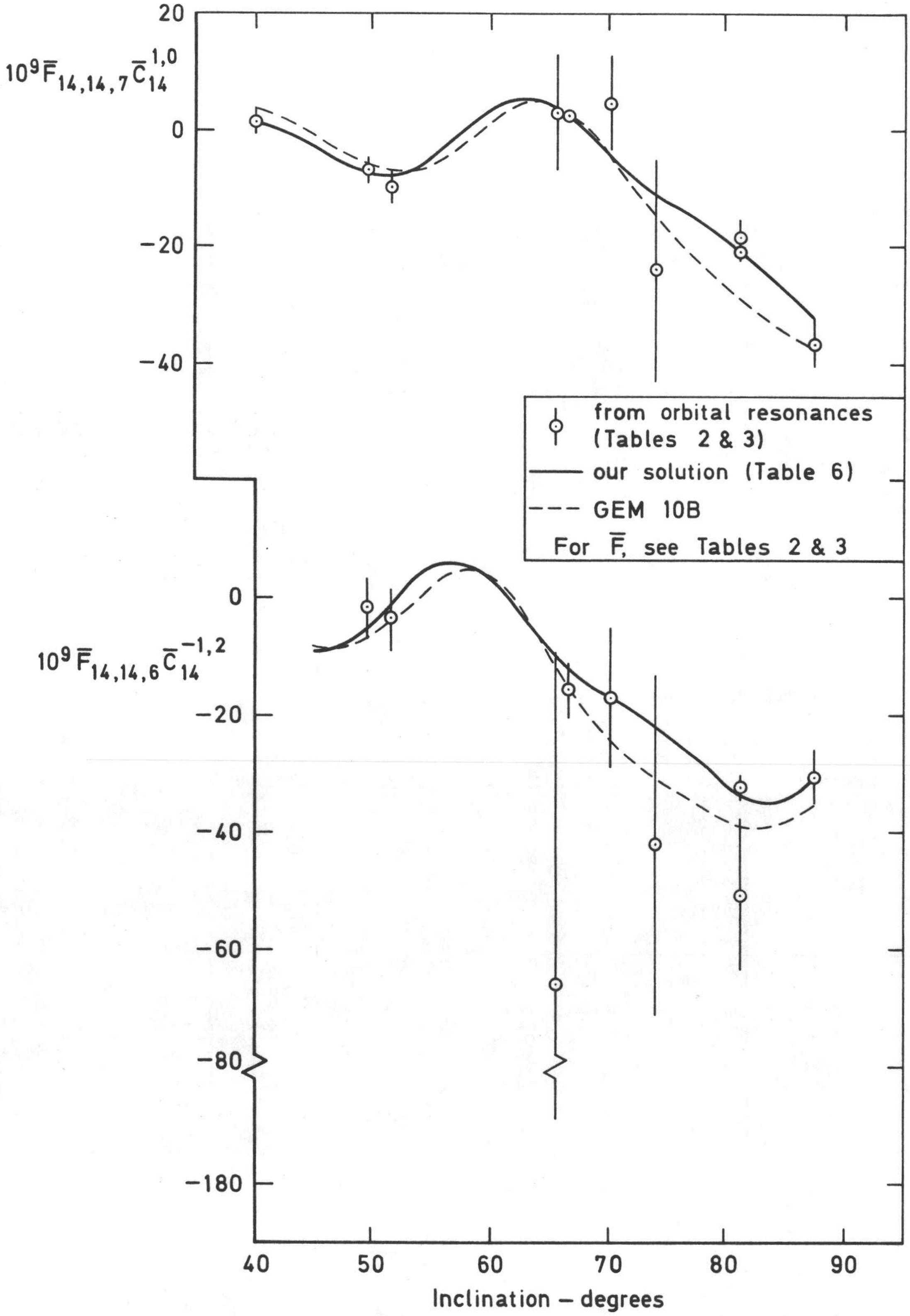


Fig 2 Variation of $\bar{C}_{14}^{1,0}$ and $\bar{C}_{14}^{-1,2}$ with inclination

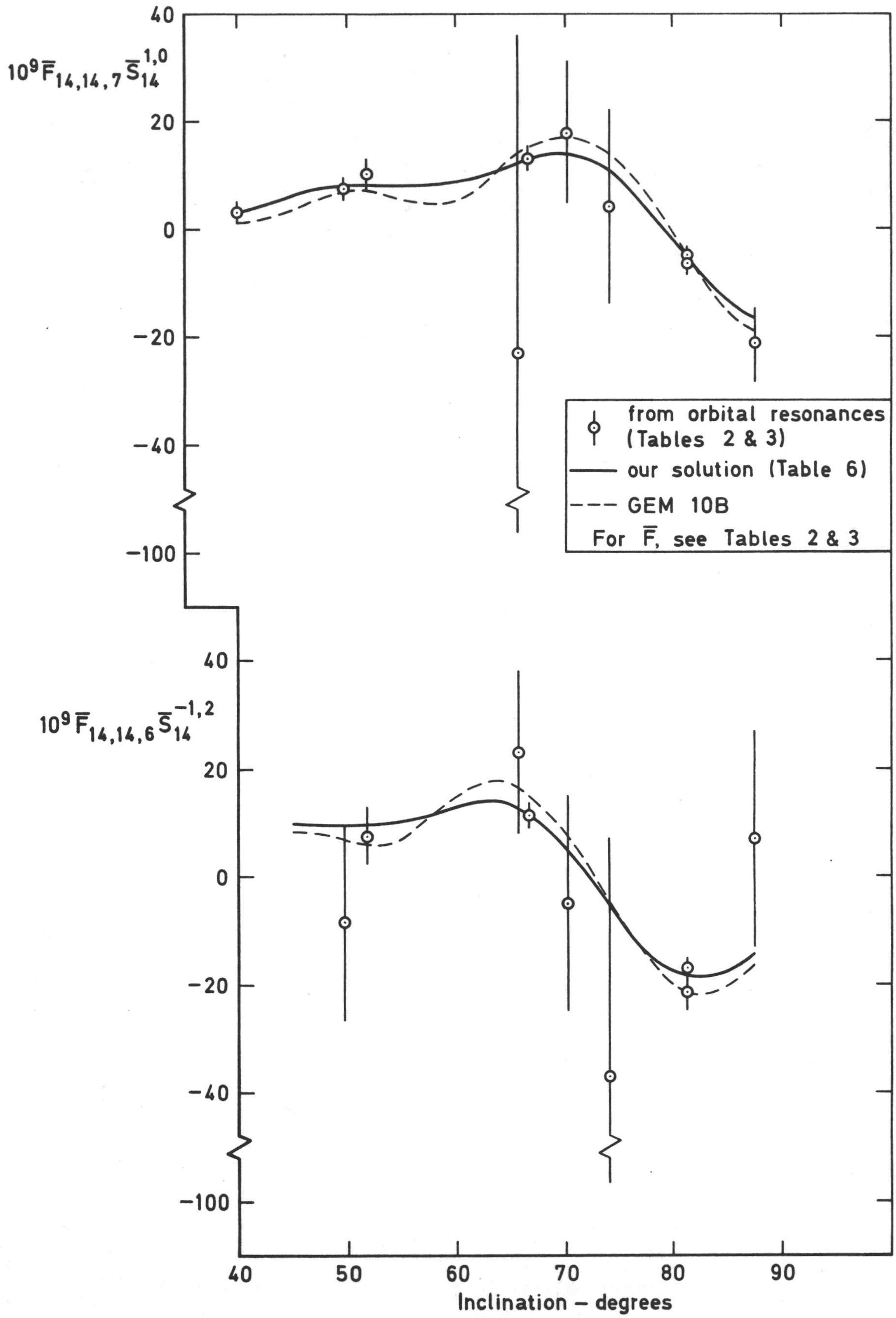
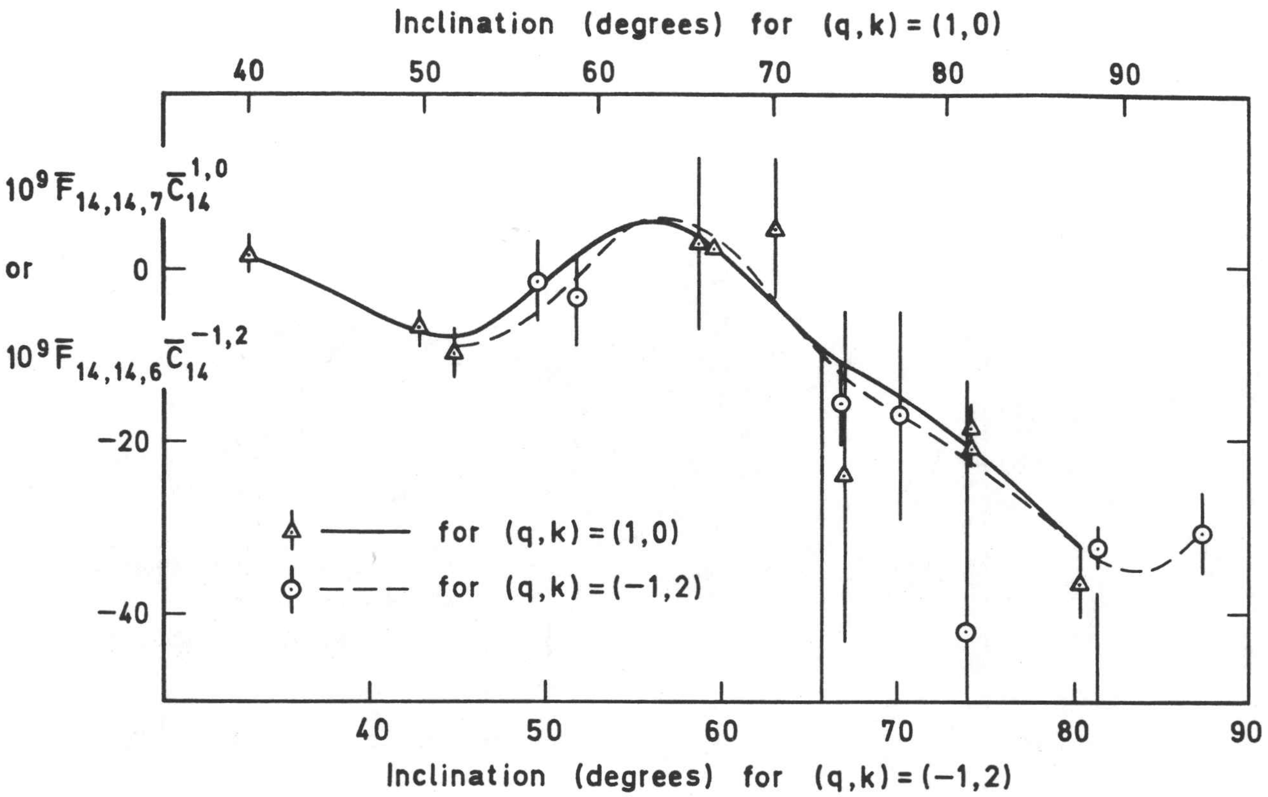


Fig 3 Variation of $\bar{S}_{14}^{1,0}$ and $\bar{S}_{14}^{1,2}$ with inclination

Fig 4



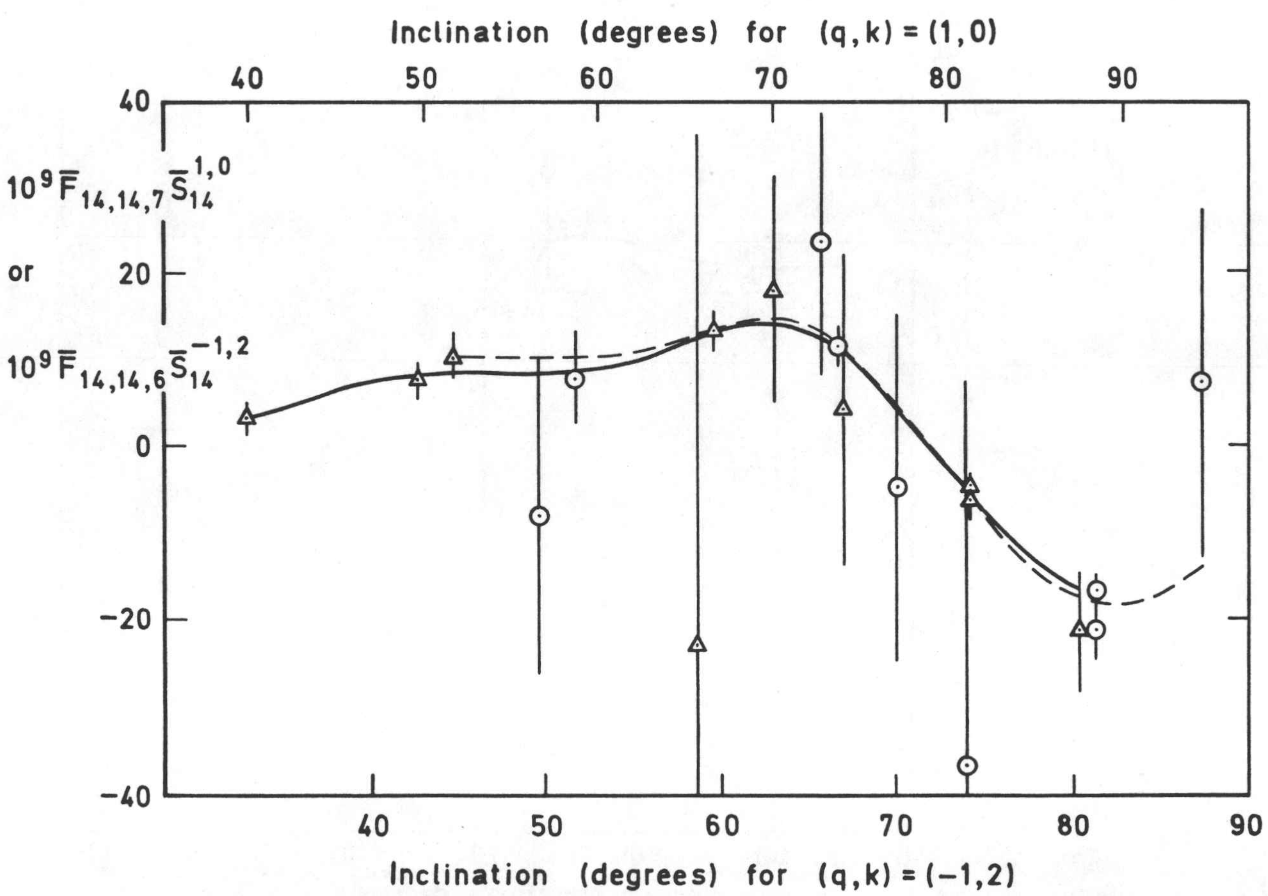


Fig 4 The curves of Figs 2 and 3 'compressed' into one by a 7° shift in inclination

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17. Abstract

The Earth's gravitational potential is usually expressed as a double series of tesseral harmonics of degree ℓ and order m, and values of coefficients for ℓ and m up to 36 or more are determined in recent models. The individual harmonic coefficients of a particular order m can most accurately be evaluated by analysing the perturbations on satellites which experience mth-order resonance (*ie* repeat their ground track after m revolutions).

Here we use results from analyses of 15 satellite orbits at 14th-order resonance to evaluate 7 pairs of individual coefficients of 14th order and odd degree (ℓ = 15,17,...27), and 6 pairs of coefficients of even degree (ℓ = 14,16,...24). The five most accurate pairs of values, for ℓ = 14,15,16,17 and 19, have a mean standard deviation corresponding to an accuracy of 0.9 cm in geoid height. These values are used as a test of the accuracy of three recent comprehensive models of the geopotential: the agreement is generally satisfactory, the differences being consistent with the expected errors (about 3×10^{-9}) in the models.

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