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Sampling Graph Signals with Sparse Dictionary Representation

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***Abstract*—Most graph sampling strategies require the signal to be relatively sparse in an alternative domain, e.g. bandlimitedness. When such a condition is violated or approximation demands a large bandwidth, the reconstruction often comes with unsatisfactory results even with large samples. In this paper, we propose an alternative sampling strategy based on a type of overcomplete graph-based dictionary. The dictionary, built from graph filters, has been demonstrated to provide excellent sparse representations for graph signals. We recognize the proposed sampling problem as a coupling between support recovery of sparse signals as well as subset selection for nodes. Thus to approach the problem we further propose a sampling procedure that alternates between these two. The former estimates the sparse support via orthogonal matching pursuit (OMP), which in turn enables the latter to achieve the sampling set selection through the greedy algorithm. Numerical results corroborate the role of key parameters and the effectiveness of the proposed method.**

***Index Terms*—Compressive sensing, graph signal sampling, graph signal processing, signal reconstruction, sparse sensing**

I. Introduction

Sampling strategies are ubiquitous for graph signals over e.g. sensor, social, and biological networks [1]; to name a few. Different from the conventional temporal and spatial sampling, graph sampling requires accounting for the coupling between the signal and the underlying topology. This coupling is often expressed as a prior to obtain a sparse signal representation in an alternative domain [2], [3]. The typical approach in these cases is to consider the eigenvectors of the graph representative matrix, e.g., adjacency or Laplacian, and represent the graph signal as a linear combination of a few eigenvectors that capture most of the energy; i.e. bandlimited representation of graph signals [1], [4], [5].

Several approaches have been proposed for sampling bandlimited graph signals. Followed by [5], the work in [6] developed necessary and sufficient conditions for the exact recovery of bandlimited data. Further studies have proposed sampling methods that are related to experiment design, such as the convex relaxation techniques [4] or the greedy sampling methods [7]. The work [8] introduced the notion of cut-off and maximized it to optimally sample graph signals, whereas the work in [9] proposed aggregation sampling which builds on the fact that each node has access to shifted versions of the signal. To avoid the eigendecomposition cost when working with bandlimited graph signals, the work in [10] considered the signal to be smooth and used Gershgorin discs to optimize a sampling criterion which is based on the smallest eigenvalue bound of the graph Laplacian matrix. The smoothness assumption is related to the bandlimited setting and it is often exhibited in signals that have similar values in adjacent nodes or in piecewise constant signals, i.e., signals with similar values within a cluster of nodes but have arbitrarily value variations between clusters [11]. Other efforts have also been made to facilitate sampling over large graphs. In [8], the author proposed to use the so-called spectral proxies as an alternative to the graph frequencies thus avoiding the computation of the eigenvectors. Probabilistic sampling has also been brought into consideration to reduce the associated computational burden when sampling signals on large graphs [12], [13].

While the approximately-bandlimited assumption is often a safe choice for sampling graph signals, it often leads to non-sparse representations, ultimately, requiring a large number of samples to reconstruct the signal within a prescribed accuracy [8]. When a graph signal is not bandlimited, approximating it as such may lead to reconstruction artifacts that are difficult to mitigate even if almost all nodes are sampled. To still be able to sample graph signals in such situations, we propose a novel dictionary-based graph sampling framework that represents the graph signal as a sparse combination of atoms of a parametric graph dictionary [14]. Since this setting works with an underdetermined system of equations (contrarily to the overdetermined case of bandlimited graph signal sampling), we exploit the sparsity of the signal and introduce a combinatorial ℓ_0-minimization problem for jointly optimizing the sampling matrix and the signal sparse representation. By identifying the two subproblems in this task: (i) sparse recovery and (ii) subset selection, we devise an efficient approach to find a tractable solution for the sampling problem. Numerical experiments demonstrate that the proposed sampling method is able to outperform bandlimited sampling at moderate signal-to-noise ratio (SNR) and number of samples.

II. Preliminaries

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of N nodes and M edges respectively, and $\mathbf{W}$ is the weighted adjacency matrix. The interconnections between nodes are captured by the entries of a symmetric matrix $\mathbf{S}$ known as the graph shift operator (GSO) [3], whose off-diagonal entry $[\mathbf{S}]_{i,j} > 0$ if there exists an edge connecting tuple (i, j), and $[\mathbf{S}]_{i,j} = 0$ otherwise. Throughout the paper, we assume $\mathbf{S}$ is normal and has the eigenvalue decomposition $\mathbf{S} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$. Choices for the GSO are the adjacency matrix $\mathbf{W}$, the graph Laplacian $\mathbf{L}$, and the normalized Laplacian $\mathbf{L}_{\text{n}}$

To each node i a signal $x_i \in \mathbb{R}$ is associated, and the graph signal vector $\mathbf{x} = [x_1, \ldots, x_N]^\top$ collects the values of all nodes. As in the classical setting, filters are the tools to process graph signals [15]. Graph filters are defined as functions of the GSO, i.e. $\mathbf{H}(\mathbf{S}) = h(\mathbf{S})$, and the filtering is performed by multiplying the graph filtering matrix $\mathbf{H}(\mathbf{S})$ with the graph signal $\mathbf{x}$, namely $\mathbf{y} = \mathbf{H}(\mathbf{S})\mathbf{x}$. Specifically, the finite impulse response (FIR) graph filter is defined as matrix polynomials of

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the GSO, i.e. $\mathbf{H}(\mathbf{S}) = \sum_{k=0}^{K} h_k \mathbf{S}^k$, where K is the filter order and $\mathbf{h} = [h_0, ..., h_K]^\top$ are the filter coefficients. The output of an FIR graph filter,

$$\mathbf{y} = \sum_{k=0}^{K} h_k \mathbf{S}^k \mathbf{x} \tag{1}$$

is the weighted sum of the K-hop neighbours' shifted signals $\{\mathbf{S}^k\mathbf{x}\}_k$ around the nodes. Thus the FIR filter captures the local behaviors of graph signals for up to a radius K from a node.

In essence, a PGD forms a dictionary composed of S FIR graph filters, i.e., $\mathbf{H}_{1:S}(\mathbf{S}) = [\mathbf{H}_1(\mathbf{S}), \ldots, \mathbf{H}_S(\mathbf{S})]$, where to each filter $\mathbf{H}_s(\mathbf{S})$ [cf. (1)] it is referred to as a sub-dictionary. Given the link with the FIR graph filters, a PGD is inherently defined by the coefficients of all S sub-dictionaries $\mathbf{h}_{1:S} = [\mathbf{h}_1^\top, \ldots, \mathbf{h}_S^\top]^\top \in \mathbb{R}^{(K+1)S\times 1}$, where $\mathbf{h}_s = [h_{0s}, \ldots, h_{Ks}]^\top$ are the coefficients of the sth sub-dictionary filter (1). Training the PGD dictionary $\mathbf{H}_{1:S}(\mathbf{S})$ is therefore reduced to estimating the filter coefficients $\mathbf{h}_{1:S}$. To identify the latter, we first consider a set of T graph signals collected in the matrix $\mathbf{X} = [\mathbf{x}_1, ..., \mathbf{x}_T] \in \mathbb{R}^{N\times T}$. Then, we aim to find a collection of T sparse vectors $\mathbf{z}_\tau \in \mathbb{R}^{NS}$ with $\tau = 1, ..., T$, each having at most S_0 non-zero entries, i.e., $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_T] \in \mathbb{R}^{NS\times T}$ and the coefficients $\mathbf{h}_{1:S}$. Formally, this task translates to solving the optimization problem

$$\begin{aligned}
\underset{\mathbf{h}_{1:S},\mathbf{Z}}{\text{argmin}} \quad & \|\mathbf{X} - \mathbf{H}_{1:S}(\mathbf{S})\mathbf{Z}\|_F^2 + \gamma\|\mathbf{h}_{1:S}\|_2^2 \\
\text{subject to} \quad & \|\mathbf{z}_\tau\|_0 \le S_0,\ \tau = 1, \ldots, T, \\
& \mathbf{H}_s(\mathbf{S}) = \sum_{k=0}^{K} h_{ks}\mathbf{S}^k, s = 1, \ldots, S, \\
& 0\mathbf{I}_N \preceq \mathbf{H}_s(\mathbf{S}) \preceq \delta\mathbf{I}_N, s = 1, \ldots, S, \\
& (\delta - \delta_1)\mathbf{I}_N \preceq \sum_{s=1}^{S} \mathbf{H}_s(\mathbf{S}) \preceq (\delta + \delta_2)\mathbf{I}_N.
\end{aligned} \tag{2}$$

Problem (2) aims at minimizing the Frobenius norm distance between the signals in $\mathbf{X}$ and their sparse dictionary reconstruction $\mathbf{H}_{1:S}(\mathbf{S})\mathbf{Z}$, while regularizing with a ℓ_2-norm of the coefficients $\gamma\|\boldsymbol{\phi}_{1:S}\|_2^2$. The ℓ_0-norm constraint $\|\mathbf{z}_\tau\|_0$ forces each vector $\mathbf{z}_\tau$ to be at most S_0-sparse. The other constrains impose the sub-dictionaries forming $\mathbf{H}_{1:S}(\mathbf{S})$ to be FIR filters and control the eigenvalues of each filter to be at most δ (third contraint) and that the total sum of eigenvalues to be bounded between $(\delta - \delta_1)$ and $\delta + \delta_2$ for some scalars $\delta, \delta_1, \delta_2$.

Notice that while other non-graph-based dictionaries can be used to sparsely represent graph signals, the approach in (2) is attractive because: (i) it is built by leveraging the coupling between the signal and the underlying graph; (ii) it forces a locality of representation of radius K from a node therefore captures local details; and (iii) it often allows sparser representations for graph signals [14]. Therefore, in the sequel, we will focus on sampling graph signals with sparse representations as per (2). However, the approach here presented can be readily extended to any general sparse signal representation by considering a different dictionary.

III. Problem formulation

We consider the scenario that a graph signal $\mathbf{x}$ follows a sparse representation w.r.t. the trained PGD $\boldsymbol{\Psi} \in \mathbb{R}^{N\times L}$,

$$\mathbf{x} = \boldsymbol{\Psi}\mathbf{s}_0 \tag{3}$$

where $\mathbf{s}_0$ is an $L \times 1$ vector with sparsity $S_0 \ll L$. Suppose the signal $\mathbf{x}$ is corrupted by an additive Gaussian noise, i.e., $\mathbf{y} = \mathbf{x} + \mathbf{n} = \boldsymbol{\Psi}\mathbf{s}_0 + \mathbf{n}$ where $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_n)$ with the covariance matrix $\boldsymbol{\Sigma}_n = \text{diag}(\sigma_1^2, ..., \sigma_N^2)$. Our goal is to design a sampling strategy that subsamples $\mathbf{y}$ from a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$ and to recover the original $\mathbf{x}$ using (3) such that the distortion is minimized.

To formalize the sampling, consider a binary selection matrix $\mathbf{C}_\mathcal{S}$ drawn from the combinatorial set

$$\mathcal{C}_{\mathcal{S},N} = \{\mathbf{C}_\mathcal{S} \in \{0,1\}^{|\mathcal{S}|\times N} : \mathbf{C}_\mathcal{S}\mathbf{1}_N = \mathbf{1}_{|\mathcal{S}|}, \mathbf{C}_\mathcal{S}^\top\mathbf{1}_{|\mathcal{S}|} \preceq \mathbf{1}_{|\mathcal{S}|}\}. \tag{4}$$

By construction, matrix $\mathbf{C}_\mathcal{S}$ satisfies $\mathbf{C}_\mathcal{S}\mathbf{C}_\mathcal{S}^\top = \mathbf{I}_{|\mathcal{S}|}$, $\mathbf{C}_\mathcal{S}^\top\mathbf{C}_\mathcal{S} = \text{diag}(\mathbf{c})$ and $\mathbf{c} \in \{0,1\}^N$ such that $c_i = 1$ if $v_i \in \mathcal{S}$.[1] Therefore, the sampled noisy signal over the vertex set $\mathcal{S}$ can be expressed as

$$\begin{aligned}
\mathbf{y}_\mathcal{S} &= \mathbf{C}_\mathcal{S}(\mathbf{x} + \mathbf{n}), \\
&= \mathbf{C}_\mathcal{S}\boldsymbol{\Psi}\mathbf{s}_0 + \mathbf{n}_\mathcal{S},
\end{aligned} \tag{5}$$

where $\mathbf{n}_\mathcal{S}$ represents the noise residing on the nodes in $\mathcal{S}$. The sampling and reconstruction now depend on designing $\mathbf{C}_\mathcal{S}$ and estimating $\mathbf{s}_0$ from the observations $\mathbf{y}_\mathcal{S}$. However, different from graph signal sampling using a bandlimited representation, model (5) is underdetermined because the system matrix $\mathbf{C}_\mathcal{S}\boldsymbol{\Psi}$ is wide instead of tall. Hence, conventional sampling techniques cannot be applied anymore.

For such an ill-posed problem, a unique solution of $\mathbf{s}_0$ can be obtained through regularization. To exploit the sparsity of $\mathbf{s}_0$, it is a natural choice to adopt the sparse recovery paradigm [16], which regularizes the ill-posed problem by seeking the sparsest estimate of $\mathbf{s}_0$ that fits the observations. Then, our goal for jointly designing $\mathbf{C}_\mathcal{S}$ and estimating $\mathbf{s}_0$ can be framed as solving the optimization problem

$$\begin{aligned}
\underset{\mathbf{C}_\mathcal{S},\mathbf{s}}{\text{argmin}} \quad & \|\mathbf{s}\|_0 \\
\text{subject to} \quad & \|\mathbf{y}_\mathcal{S} - \mathbf{C}_\mathcal{S}\boldsymbol{\Psi}\mathbf{s}\|_2 \le \epsilon, \\
& \mathbf{C}_\mathcal{S} \in \mathcal{C}_{\mathcal{S},N}.
\end{aligned} \tag{$\text{P}_{J,0}^\epsilon$}$$

which seeks for a vector $\mathbf{s}$ with the minimum number of non-zero entries, while tolerating the reconstructed signal $\mathbf{C}_\mathcal{S}\boldsymbol{\Psi}\mathbf{s}$ on the sampled nodes to deviate from $\mathbf{y}_\mathcal{S}$ for at most ϵ, and constraining $\mathbf{C}_\mathcal{S}$ to be a proper sampling matrix drawn from (4). If it were not for this sampling matrix, problem ($\text{P}_{J,0}^\epsilon$) can be solved with conventional pursuit algorithms for $\mathbf{s}$ [16]. However, matrix $\mathbf{C}_\mathcal{S}$ adds a major difficulty to the problem because of its combinatorial nature, making problem ($\text{P}_{J,0}^\epsilon$) NP-hard. Our goal next is to circumvent this challenge via an alternating minimization to efficiently estimate the sparse vector $\mathbf{s}$ and optimize the sampling matrix $\mathbf{C}_\mathcal{S}$.

IV. Alternating Dictionary-Based Sampling

We approach problem ($\text{P}_{J,0}^\epsilon$) through alternating optimizations between $\mathbf{s}$ and $\mathbf{C}_\mathcal{S}$. To be precise, we start with a sampling set $\mathcal{S}_0 \subset \mathcal{S}$, with $|\mathcal{S}_0| \ll |\mathcal{S}|$. The initialization selection set $\mathcal{S}_0$ is defined either randomly or by any other prior. After the noisy data $\mathbf{y}_{\mathcal{S}_0}$ is sampled from the set of nodes $\mathcal{S}_0$, we keep $\mathbf{C}_\mathcal{S} = \mathbf{C}_{\mathcal{S}_0}$, while solving ($\text{P}_{J,0}^\epsilon$) only w.r.t. $\mathbf{s}$ via pursuit algorithms [16]. Then, with the solution $\hat{\mathbf{s}}^{(0)}$ obtained, we update the sampling set to $\mathcal{S}_1 \supset \mathcal{S}_0$ via sparse

[1] Notice that $\mathcal{S}$, $\mathbf{C}_\mathcal{S}$ and $\mathbf{c}$ are equivalent representations of a certain node selection – knowing one of them, the other two are uniquely defined.

sensing techniques [17]. The procedure then repeats with $\mathcal{S}_1$. This alternating update will produce a series of sampling sets $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}$. Without loss of generality, we keep the selection stepsize unitary, i.e., $|\mathcal{S}|_i - |\mathcal{S}|_{i-1} = 1$. At iteration i, we therefore first find the sparsest representation $\hat{\mathbf{s}}^{(i)}$ of the sampled noisy data $\mathbf{y}_{\mathcal{S}_i}$, then sample the nodes to build the selection set $\mathcal{S}_{i+1}$. The algorithm terminates after the selection budget $|\mathcal{S}|$ is reached. We refer to this scheme as alternating dictionary-based sampling (ADBS).

A. Support Estimation

To detail the above procedure, at iteration i, we are given the selection set $\mathcal{S}_i$ and the respective samples $\mathbf{y}_{\mathcal{S}_i}$. First, we construct the corresponding normalized subsampled dictionary $\tilde{\mathbf{\Psi}}_{\mathcal{S}_i}$ by first removing the all-zero columns of $\mathbf{C}_{\mathcal{S}_i}\mathbf{\Psi}$, and then normalizing the remaining ones.

The support of $\mathbf{s}^{(i)}$ is identified by solving the following optimization problem,

$$\begin{aligned} \underset{\mathbf{s}^{(i)}}{\text{argmin}} \quad & \|\mathbf{s}^{(i)}\|_0 \\ \text{subject to} \quad & \|\mathbf{y}_{\mathcal{S}_i} - \tilde{\mathbf{\Psi}}_{\mathcal{S}_i}\mathbf{s}^{(i)}\|_2 \leq \epsilon, \end{aligned} \tag{$\mathrm{P}^{\epsilon}_{\mathcal{S}_i,0}$}$$

which is a typical sparse recovery problem and can be solved via a range of well-developed algorithms [16]. In this work, we use the OMP algorithm to solve problem ($\mathrm{P}^{\epsilon}_{\mathcal{S}_i,0}$). This is due to the consideration that he intention of this paper is to demonstrate the effectiveness of the proposed ADBS paradigm. Using the OMP already suffices and it is one of the simplest and fastest sparse recovery algorithms.

The OMP algorithm is a type of greedy sparse recovery algorithm. For every iteration, the algorithm identifies the (normalized) dictionary atom that is the most correlated to the input observation. Then the contribution from the identified atom is excluded from the input data and the residual is regarded as the next input. Such a process is repeated until the stopping criterion is met. There exist a few different OMP stopping criteria that is commonly used in literature [18]:

1) The algorithm terminates when the number of iteration reaches $k = T_0$, the sparsity.
2) The algorithm terminates when the magnitude of the residual is small, say less then ϵ_2.
3) The algorithm terminates when the correlation between the non-selected atoms are correlated with the residual has a magnitude less than a threshuold, say ϵ_3.

While for criterion 1) the OMP algorithm requires the knowledge of T_0 as prior information, the other two criteria require the thresholds ϵ_2 or ϵ_3 to be tuned in order to get desirable sparse recovery performances. Since in this work we do not focus on the explicit implementation of the sparse recovery algorithm, we choose to implement the stopping criterion 1), as it usually provides better support identification accuracy.

B. Node Sampling

Once the estimate $\hat{\mathbf{s}}^{(i)}$ is obtained, the goal next is to leverage its sparsity to update the sampling matrix $\mathbf{C}_{\mathcal{S}_{i+1}}$. At first sight, it may seem we still need to face an underdetermined system. But since we know the support of $\hat{\mathbf{s}}^{(i)}$, we can transform the system into an overdetermined one. Considering $\hat{\mathbf{s}}^{(i)}$ is sufficiently sparse, we can remove the redundant atoms in $\mathbf{\Psi}$ an keep only those $\mathbf{\Psi}^{(i)}_{\text{nnz}}$ that are necessary to represent the estimate, i.e.,

$$\hat{\mathbf{x}}^{(i)} = \mathbf{\Psi}^{(i)}_{\text{nnz}}\hat{\mathbf{s}}^{i}_{\text{nnz}} \tag{6}$$

where $\hat{\mathbf{s}}^{i}_{\text{nnz}}$ is the shorter vector containing the non-zero entries of $\hat{\mathbf{s}}^{(i)}$ and $\mathbf{\Psi}^{(i)}_{\text{nnz}}$ is now of dimensions $N \times \text{nnz}(\hat{\mathbf{s}}^{(i)})$, with $\text{nnz}(\hat{\mathbf{s}}^{(i)})$ denoting the number of non-zero elements in $\hat{\mathbf{s}}^{(i)}$. Therefore, we can write the measurement model for the sampling set $\mathcal{S}_{i+1}$ as

$$\mathbf{y}_{\mathcal{S}_{i+1}} = \mathbf{C}_{\mathcal{S}_{i+1}}\mathbf{\Psi}^{(i)}_{\text{nnz}}\mathbf{s}^{(i)}_{\text{nnz}} + \mathbf{n}_{\mathcal{S}_{i+1}}, \tag{7}$$

where now we work with a tall matrix $\mathbf{\Psi}^{(i)}_{\text{nnz}}$. Notice that in (7) we treat $\mathbf{s}^{(i)}_{\text{nnz}}$ as an unknown variable for which we want to design the sampling matrix $\mathbf{C}_{\mathcal{S}_{i+1}}$ that leads to the best estimate for it. In other words, the sparse solution obtained by solving ($\mathrm{P}^{\epsilon}_{\mathcal{S}_i,0}$) with OMP is only used now to obtain the support of the sparse representation and build the tall matrix $\mathbf{\Psi}^{(i)}_{\text{nnz}}$.

The second step (i.e. node sampling) of the ADBS algorithm consists of updating the sampling set by adding one of the residual nodes that yields the best estimate of $\mathbf{s}^{(i)}_{\text{nnz}}$ through model (7). To describe how good the estimation can be achieved for $\mathbf{s}^{(i)}_{\text{nnz}}$ (and hence for the entire graph signal $\mathbf{x}$), we first use the best linear unbiased estimator (BLUE) [19],

$$\hat{\mathbf{x}}^{(i+1)}_{\text{B}} = \mathbf{\Psi}^{(i)}_{\text{nnz}}\mathbf{\Theta}^{\dagger}_i\mathbf{\Psi}^{(i)\,\text{H}}_{\text{nnz}}\mathbf{C}^{\top}_{\mathcal{S}_{i+1}}\left(\mathbf{C}_{\mathcal{S}_{i+1}}\mathbf{\Sigma}_n\mathbf{C}^{\top}_{\mathcal{S}_{i+1}}\right)^{-1}\mathbf{y}_{\mathcal{S}_{i+1}} \tag{8}$$

with $\mathbf{\Theta}_i = \mathbf{\Psi}^{(i),\text{H}}_{\text{nnz}}\mathbf{C}^{\top}_{\mathcal{S}_{i+1}}\left(\mathbf{C}_{\mathcal{S}_{i+1}}\mathbf{\Sigma}_n\mathbf{C}^{\top}_{\mathcal{S}_{i+1}}\right)^{-1}\mathbf{C}_{\mathcal{S}_{i+1}}\mathbf{\Psi}^{(i)}_{\text{nnz}}$ and $(\cdot)^{\dagger}$ denoting the Moore–Penrose pesudoinverse. We then quantify the estimation performance through its mean square deviation (MSD)

$$\begin{aligned} \text{MSD} &= \mathbb{E}\left[\|\hat{\mathbf{x}}^{(i+1)}_{\text{B}} - \mathbf{x}\|^2_2\right] = \text{tr}\left(\mathbf{\Theta}^{-1}_i\right) \\ &= \text{tr}\left[\left(\mathbf{\Psi}^{(i)\text{H}}_{\text{nnz}}\text{diag}(\mathbf{c}_{i+1})\mathbf{\Sigma}^{-1}_n\mathbf{\Psi}^{(i)}_{\text{nnz}}\right)^{-1}\right], \end{aligned} \tag{9}$$

where the last equality holds since $\mathbf{\Sigma}_n$ is diagonal.

Thus, we can now build $\mathcal{S}_{i+1}$ by minimizing the MSD w.r.t. $\mathbf{c}_{i+1}$, i.e., solving

$$\begin{aligned} \underset{\mathbf{c}_{i+1}}{\text{argmin}} \quad & \text{tr}\left[\left(\mathbf{\Psi}^{(i)\text{H}}_{\text{nnz}}\text{diag}(\mathbf{c}_{i+1})\mathbf{\Sigma}^{-1}_n\mathbf{\Psi}^{(i)}_{\text{nnz}}\right)^{-1}\right] \\ \text{subject to} \quad & \mathbf{c}_{i+1} \in \{0,1\}^{N\times 1},\ \|\mathbf{c}_{i+1}\|_0 = |\mathcal{S}_{i+1}| \\ & \|\mathbf{c}_{i+1} - \mathbf{c}_i\|_0 = 1,\ \mathcal{S}_i \subset \mathcal{S}_{i+1} \end{aligned} \tag{P2}$$

where the last constraints is to indicate that we increase the sampling set by one sample and therefore $\mathcal{S}_{i+1}$ should contain $\mathcal{S}_i$.

Problems of the form in (P2) are standard within the sparse sensing framework. However, since we have the set inclusion constraint $\mathcal{S}_i \subset \mathcal{S}_{i+1}$ and need to increase the size of the sampling set, approaching the latter via greedy methods is a natural choice [4]. As a result, we can directly apply the greedy heuristic to problem (P2) by adding to the sampling set the residual node $n \in \bar{\mathcal{S}}_i = \mathcal{V} \setminus \mathcal{S}_i$ that minimizes the MSD. Alternatively, other criteria used in experimental design that exhibit amenable properties for greedy selection (e.g., submodularity [20]), such as the (pseudo) log-determinant criterion

$$f_i(n) = -\log\det\left[\mathbf{\Theta}_i(\mathcal{S}_i \cup \{n\}) + \xi\mathbf{I}\right] \tag{10}$$

Algorithm 1 ADBS algorithm for problem $(\mathrm{P}^{\epsilon}_{J,0})$

Objective: Minimize $(\mathrm{P}^{\epsilon}_{J,0})$ via alternate minimization.
Initialize $\mathcal{S}_0$ randomly, set iteration count $i = 0$
while $|\mathcal{S}_i| \leq |\mathcal{S}|$ **do**
    Build $\tilde{\mathbf{\Psi}}_{\mathcal{S}_i}$ by removing the all-zero columns of $\mathbf{C}_{\mathcal{S}_i}\mathbf{\Psi}$ and normalizing the columns.
    Update $\mathbf{s}$ by solving $(\mathrm{P}^{\epsilon}_{\mathcal{S}_i,0})$ through the OMP algorithm.
    Build $\mathbf{\Psi}^{(i)}_{\text{nnz}}$ by columns of $\mathbf{\Psi}$ corresponding to the non-zero elements in $\hat{\mathbf{s}}^{(i)}$.
    Update $\mathbf{C}_{\mathcal{S}_{i+1}}$ via the greedy algorithm.
    $i \leftarrow i + 1$
end while
Use $\mathbf{y}_{\mathcal{S}}$ to estimate the original signal $\mathbf{x}$ through (8).

can be used, where the term $\xi\mathbf{I}$ with $\xi \ll 1$ is to avoid rank deficiency of $\mathbf{\Theta}_i$. We considered the log-determinant critetion in the numeral experiments.

The node sampling step concludes iteration i of the ABDS algorithm. All the steps are summarized in Algorithm 1. We would like to remark that the interplay between the support estimation and node selection does not necessarily have any submodularity guarantee even if a submodular function is used in (P2). This is because the support estimated from $(\mathrm{P}^{\epsilon}_{\mathcal{S}_i,0})$ may change between iterations; especially in the earlier ones. However, when the number of samples becomes large enough and the support does not change, using submodular functions in (P2) may come with near-optimal guarantees [20]. A deeper analysis of the latter will be done in future work.

V. Numerical Experiments

This section presents numerical experiments to corroborate the proposed approach and compare with the baseline graph-bandlimited sampling. We considered the scenario in [14] comprising a random sensor graph of $N = 100$ nodes and a PGD with $S = 4$ subdictionaries and FIR filters of order $K = 5$. We generate the signal $\mathbf{x}$ through the linear combination of four random atoms with uniformly distributed coefficients in $[0, 1]$; hence $S_0 \leq 4$. We corrupt the data with different SNRs in $[-5\text{dB}, 30\text{dB}]$. The parameter ξ in (10) is set to 10^{-3}. The initial selection set is built at random and contains two nodes out of the 100 available. We run R Monte-Carlo simulations, and we measure the reconstruction performance through the relative error, defined as $\|\mathbf{x}_r - \hat{\mathbf{x}}_r\|_2^2 \,/\, \|\mathbf{x}_r\|_2^2$, where $\mathbf{x}_r$ and $\hat{\mathbf{x}}_r$ denotes the r-th true signal and reconstruction respectively. We compare the proposed ADBS approach (OMP-greedy for short) with:

1) OA-greedy: Oracle algorithm that knows the $S_0 = 4$ true atoms (instead of estimating through OMP) and adopts a greedy solution to sample the nodes. Allows comparing with the best performance we can achieve with greedy sampling.
2) OMP-Rand: Estimates the support with OMP but samples nodes uniformly at random. Shows the effectiveness of the greedy sampling de-factorizing the impact of the support.
3) Standard bandlimited-based greedy selection with a bandwidth containing 90% of the energy. Contrasts the ADBS with a baseline solution.

Cardinality sampling set. We first analyze the impact of the number of sampled nodes, thus we consider the true dictionary as known and fix the SNR to 30dB. Fig. 1(a) depicts the the reconstruction performance of the compared methods for different $|\mathcal{S}|$. In contrast to the bandlimited sampling, the OMP-greedy and OMP-rand converge to the OA-greedy lower bound in terms of the median, which confirms that the proposed method can indeed address the concerned sampling that bandlimited approach struggles to handle.

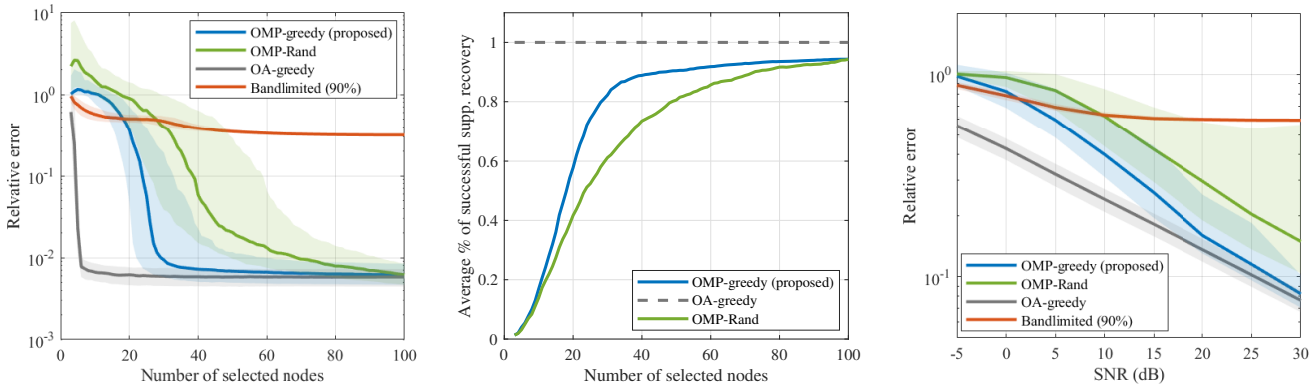
The result of OMP-greedy has a typical trend - the relative error initially reaches a slow-decaying "plateau", followed by a rapid decrease after around 30 observations. The plateau can be a result of initial lack of observations, which makes it considerably hard for the OMP to estimate the sparse support correctly. Once sufficient observations are supplied, the more correct supports can be identified. The evidence can also be found in Fig. 1(b), which illustrates the rate of successful support recovery for different methods averaged over simulations. For OMP-greedy, the recovery rate is initially very low but soon reach its turning points roughly also at around $N = 30$. The random selection follows a similar pattern as the OMP-greedy does, but reaches the rapid-decrease stage slower (at around $N = 45$), suggesting that greedily sampled data not only gives better reconstruction but also benefits the sparse recovery.

SNR. Fig 1(c) compares reconstruction error for the concerned methods at $|\mathcal{S}| = 50$ under different SNRs, for which $R = 10^3$ tests were conducted. We can observe that the SNR does has a significant impact on the proposed method. Under low SNRs (e.g. less than around 2dB), the proposed scheme perform worse than the bandlimited sampling, which may due to the fact the OMP's performance degrades significantly in highly noisy conditions, whereas for the bandlimited case the sampling matrix are fixed. Under higher SNRs, the proposed method improves rapidly and start to converge to the OA-greedy results after around 20dB. Furthermore, the proposed method performs uniformly better than random sampling in terms of the medium and the spread of the errors, again proving the greedy method to be a more effective selection scheme in the proposed ADBS framework.

Trained dictionary. Lastly, we evaluate the proposed method when the true dictionary is unknown. We generate 2600 data samples corrupted by noise with SNR $= 30$ dB, from which 600 are used for training the dictionary while the rest for testing. The training procedure follows the approach from [14] but now with noisy data, and results in a 2.57% dictionary representation error of training error. Fig 2 depicts the reconstruction errors as a function of $|\mathcal{S}|$ for both true (oracle) and trained dictionaries. As shown, when the number of selected nodes are low, the OMP-induced noise dominates. After around 30 observations, both mediums start to drop rapidly and converges to their steady-state errors. The difference between the trained and oracle dictionary cases in steady-state is around 3%, which indicates the main source of error to be dictionary training. Such shows the reliability of the proposed method with trained dictionary, as no significant excessive error is resulted.

VI. Conclusions

This work proposed a sampling strategy for graph signals that enjoy sparse dictionary representations. This sparse representation is of interest when the graph signal does not



(a) Comparison of selection methods (b) Average support recovery rate support (c) Impact of SNR

Figure 1. Performance comparison of different sampling methods: (a) relative error of different selection methods vs. the number of observations; (b) average percentage of the correctly recovered supp(s) by OMP for the same results in (a); (c) relative error for different SNRs. For (a) and (c), the solid lines indicates the medium, whereas the shaded area marks the 25%-75% percentiles (i.e. inter-quartile range).

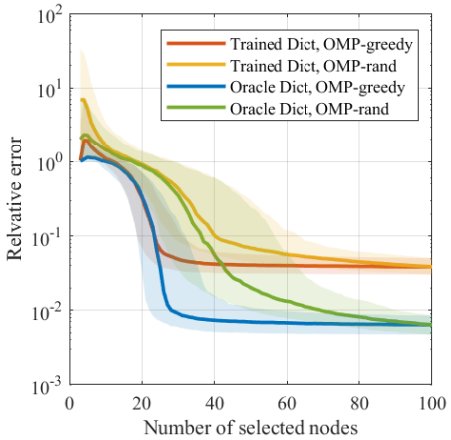


Figure 2. Performance comparison for using trained and exact dictionary. Solid lines and shaded areas indicates the medium and inter-quartile ranges.

satisfy the smoothness or the bandlimitedness prior, thus cannot facilitate effective sampling. We instead resort to the PGD, whose atoms are columns of FIR graph filters. We then proposed a sampling approach for this underdetermined system that relies on the interplay between support estimation and subset selection. Starting with a small given sampling set (e.g., random), we solve a trimmed-dictionary representation problem via OMP to estimate the support. This support is then used to increase the sampling set, which is in turn used to update the support estimate. This alternating procedure is repeated until the desired number of samples is reached. One of the main limitations we have observed is the impact of the initial sampling set. Future research will consider improving upon this aspect to allow for better performance.

REFERENCES

[1] S. Chen, R. Varma, A. Sandryhaila, and J. Kovačević, "Discrete signal processing on graphs: Sampling theory," *IEEE Transactions on Signal Processing*, vol. 63, pp. 6510–6523, Dec. 2015.

[2] Y. Tanaka, Y. C. Eldar, A. Ortega, and G. Cheung, "Sampling signals on graphs: From theory to applications," 2020.

[3] A. Ortega, P. Frossard, J. Kovačević, J. M. F. Moura, and P. Vandergheynst, "Graph signal processing: Overview, challenges, and applications," *Proceedings of the IEEE*, vol. 106, no. 5, pp. 808–828, 2018.

[4] P. Di Lorenzo, S. Barbarossa, and P. Banelli, "Sampling and recovery of graph signals," *arXiv preprint arXiv:1712.09310*, 2017.

[5] M. Tsitsvero, S. Barbarossa, and P. Di Lorenzo, "Signals on graphs: Uncertainty principle and sampling," *IEEE Transactions on Signal Processing*, vol. 64, no. 18, pp. 4845–4860, 2016.

[6] I. Pesenson, "Sampling in paley-wiener spaces on combinatorial graphs," *Transactions of the American Mathematical Society*, vol. 360, no. 10, pp. 5603–5627, 2008.

[7] L. F. O. Chamon and A. Ribeiro, "Greedy sampling of graph signals," *IEEE Transactions on Signal Processing*, vol. 66, no. 1, pp. 34–47, 2018.

[8] A. Anis, A. Gadde, and A. Ortega, "Efficient sampling set selection for bandlimited graph signals using graph spectral proxies," *IEEE Transactions on Signal Processing*, vol. 64, no. 14, pp. 3775–3789, 2016.

[9] A. G. Marques, S. Segarra, G. Leus, and A. Ribeiro, "Sampling of graph signals with successive local aggregations," *IEEE Transactions on Signal Processing*, vol. 64, no. 7, pp. 1832–1843, 2016.

[10] Y. Bai, F. Wang, G. Cheung, Y. Nakatsukasa, and W. Gao, "Fast graph sampling set selection using gershgorin disc alignment," *IEEE Transactions on Signal Processing*, vol. 68, pp. 2419–2434, 2020.

[11] S. Chen, R. Varma, A. Singh, and J. Kovačević, "Representations of piecewise smooth signals on graphs," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2016, pp. 6370–6374.

[12] G. Puy, N. Tremblay, R. Gribonval, and P. Vandergheynst, "Random sampling of bandlimited signals on graphs," *Applied and Computational Harmonic Analysis*, vol. 44, no. 2, pp. 446 – 475, 2018.

[13] N. Perraudin, B. Ricaud, D. I. Shuman, and P. Vandergheynst, "Global and local uncertainty principles for signals on graphs," *APSIPA Transactions on Signal and Information Processing*, vol. 7, pp. e3, 2018.

[14] D. Thanou, D. I. Shuman, and P. Frossard, "Learning parametric dictionaries for signals on graphs," *IEEE Transactions on Signal Processing*, vol. 62, no. 15, pp. 3849–3862, 2014.

[15] D. I. Shuman, S. K Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83–98, 2013.

[16] M. Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*, Springer Publishing Company, Incorporated, 1st edition, 2010.

[17] S. P. Chepuri and G. Leus, "Sparse sensing for statistical inference," *Foundations and Trends® in Signal Processing*, vol. 9, no. 3–4, pp. 233–368, 2016.

[18] J. A. Tropp and S. J. Wright, "Computational methods for sparse solution of linear inverse problems," *Proceedings of the IEEE*, vol. 98, no. 6, pp. 948–958, 2010.

[19] S. M. Kay, *Fundamentals of statistical signal processing*, Prentice Hall PTR, 1993.

[20] E. Tohidi, R. Amiri, M. Coutino, D. Gesbert, G. Leus, and A. Karbasi, "Submodularity in action: From machine learning to signal processing applications," *IEEE Signal Processing Magazine*, vol. 37, no. 5, pp. 120–133, 2020.