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Faculteit Elektrotechniek, Wiskunde en Informatica
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AEX-Sparen
(Engelse titel: AEX Savings Account)

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"AEX-Sparen"

(Engelse titel: "AEX Savings Account")

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Abstract

Certain banks offer its customers a new investment product, which is known as AEX-sparen. A minimal amount of 5000 Euro is put into a bank account and this will be returned after four months plus interest. The interest is the same as the AEX-Index has earned in the previous four months, but is maximized to 10 %. If the AEX-Index has gone down after four months, the initial investment of 5000 Euro will be returned, so the customer is protected against the risk of losing money and has a potential of earning a higher percentage after four months than the (annualized) risk-free rate of 2.5%.
It is impossible to predict the stock market with certainty and investments always involve risk. The bank will make advantage of this by guaranteeing the initial investment and still offering a chance to make a nice profit. But who will actually profit more from this, the bank or the customer? In this report, one strategy that the bank might use will be discussed. It is interesting to note that this product can also be bought in real at ABN-Amro. The maximum interest is lower, but the initial investment is still guaranteed.

Sommige banken bieden haar klanten een nieuw investeringsproduct, ook bekend als AEX-sparen. Een minimum bedrag van 5000 Euro wordt op een bankrekening gestort en komt na vier maanden vrij inclusief rente. De rente is gelijk aan het percentage dat de AEX-Index het afgelopen vier maanden is gestegen, maar is gemaximeerd op 10%. Mocht de AEX-Index naar beneden gegaan zijn in de afgelopen vier maanden, dan wordt de volledige inleg teruggegeven. De klant is hiermee beschermd tegen eventuele verliezen maar maakt toch kans om meer te verdienen dan de jaarlijkse nominale spaarrente van 2.5%.
Het is bekend dat het onmogelijk is om de beurs met zekerheid te voorspellen en dat er risico's kleven aan investeringen in de beurs. Daar speelt de bank op in door dit product aan te bieden waarbij de klant beschermd wordt tegen het verlies van de inleg en toch kans maakt op een redelijk rendement. Maar wordt de klant hier nou daadwerkelijk beter van of wordt eigenlijk de bank rijk van dit product? In dit rapport wordt een strategie besproken die de bank kan gebruiken. Wat verder interessant is om op te merken, is dat dit product in het echt aangeschaft kan worden bij ABN-Amro. Weliswaar tegen een lagere maximale rente, maar de inleg is gegarandeerd.

Contents

Preface

Banks and investment funds offer a lot of investment products. Some of them are called *structured products*. The mean goal of a structured product is to make a profit and reducing the risk of potential losses and make it sound like an interesting product to invest in for customers. Structured products can somehow be compared to car insurances, where it is recommended to buy some insurance to protect yourself against certain risks, but buying too much insurance is only expensive and will not pay itself back.

Structured products can be very simple as well as very complex. When the crisis began in November 2008, such complex products existed that nobody understood them anymore.

In this report, a structured product called 'AEX-Sparen' is discussed.
5000 Euro is put into a bank account and during four months the bank will invest this in the AEX-Index. After four months, the 5000 Euro is returned plus interest. The interest is the same as the AEX-Index has earned in the previous four months, but is maximized to 10 %. If the AEX-Index has gone down after four months, the initial investment of 5000 Euro will be returned, so the customer is protected against the risk of losing the initial investment and has a potential of earning a higher percentage after a year than the risk-free rate of 2.5%.

Since it is impossible to predict the stock market and thus investing money can be risky, this might sound like an interesting product to buy instead of creating it yourself, because the bank will guarantee the initial investment.

But is this product really so attractive for a customer? Does the bank not get a free lunch and make significant money which the customer could have earned himself if he had constructed the product himself?

An important question is who will profit more from this product, the bank or the customer.

In this report, an investment strategy that the bank can use is discussed where we try to minimize the risk.

Before discussing this structured product and the investment strategy, we must first explain some theory behind it, varying from Brownian motion to the Black-Scholes option pricing model. All this is discussed in the first part of this report.
In the second part, the structured product is explained and the investment strategy that the bank might use.

For me this is a beautiful combination of Mathematics, Economy and the real financial world, since this product can be bought in real at ABN-Amro. This subject has my special interest, since I also follow courses at Erasmus University Rotterdam in Econometrics and Quantitative Finance, where I learned about analyzing market data such as returns on assets, risk reduction and models for option pricing. It is a good example where abstract Mathematics provides solutions for real world problems. In this report, I will also use some knowledge I obtained during the courses at Erasmus University Rotterdam.

The first part might be hard to understand for people with little or no background in Mathematics. To understand the second part where the investment strategy will be discussed, not much mathematical knowledge is required.

As said before, the central question is who will profit more from this investment product, so the question I will answer in this report is

Why should we buy bread instead of making it?

1 Brownian Motion

In this report, 5000 Euro will be invested in the AEX-Index (real data plotted in figure **??** from 13 Oktober 2009 to 31 march 2011).
One of the goals is to find an optimal investment strategy in the AEX-Index.
Since it is not possible to predict any future value of the AEX-Index given the present or the past, all future increments of the AEX-Index are stochastic. We will assume in this report that the AEX-Index follows the geometric Brownian motion (which is a stochastic process).

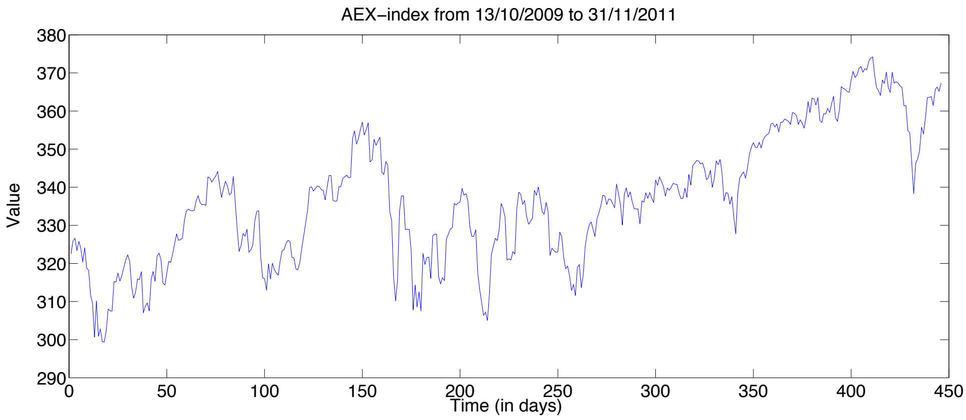

Figure 1: The AEX-Index

In this section, we will first discuss returns on assets and standard Brownian motion, then geometric Brownian motion, some interesting properties because of the stochastic differential equation and the Markov property.

1.1 Returns

We will start with a discussion on returns on assets. Let us first define what we mean by returns. Let $S(t)$ be the asset price at time t and $S(t-1)$ be the asset price at time $t-1$. The arithmetic return of the asset at time t is then given by:

$$r(t) = \frac{S(t) - S(t-1)}{S(t-1)} \tag{1}$$

This is equivalent to

$$1 + r(t) = \frac{S(t)}{S(t-1)} \tag{2}$$

Let us rewrite equation (**??**). By Taylor's expansion we get for small $r(t)$

$$\begin{aligned} \ln\left(\frac{S(t)}{(S(t-1)}\right) &= \ln(1 + r(t)) \\ &\approx r(t) \end{aligned}$$

Rewriting this equation a little yields

$$r(t) \approx \ln(S(t)) - \ln(S(t-1)) \tag{3}$$

This is called the log return instead of the arithmetic return (equation (**??**)).

The arithmetic returns of the AEX-Index data from figure **??** are plotted in figure **??** (from 13 Oktober 2009 to 17 juli 2010).

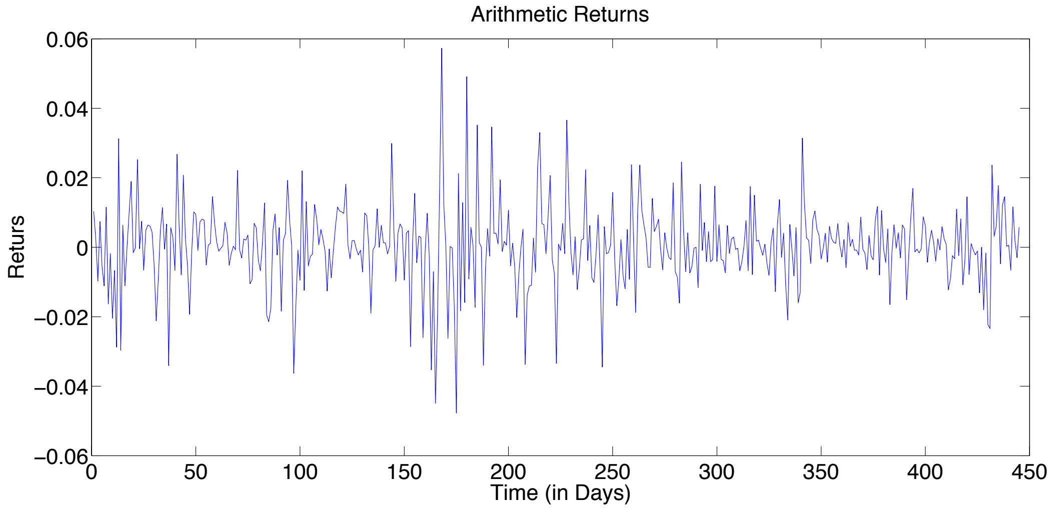

Figure 2: The arithmetic returns

The log returns of the AEX-Index data from figure **??** are plotted in figure **??** (from 13 Oktober 2009 to 17 juli 2010).

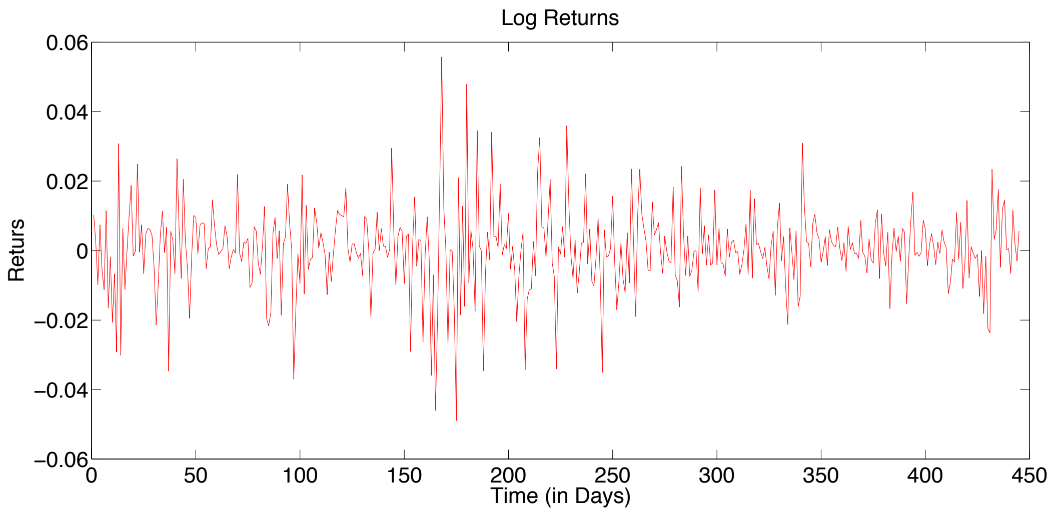

Figure 3: The log returns

The difference between these to figures is best visible in figure ??.

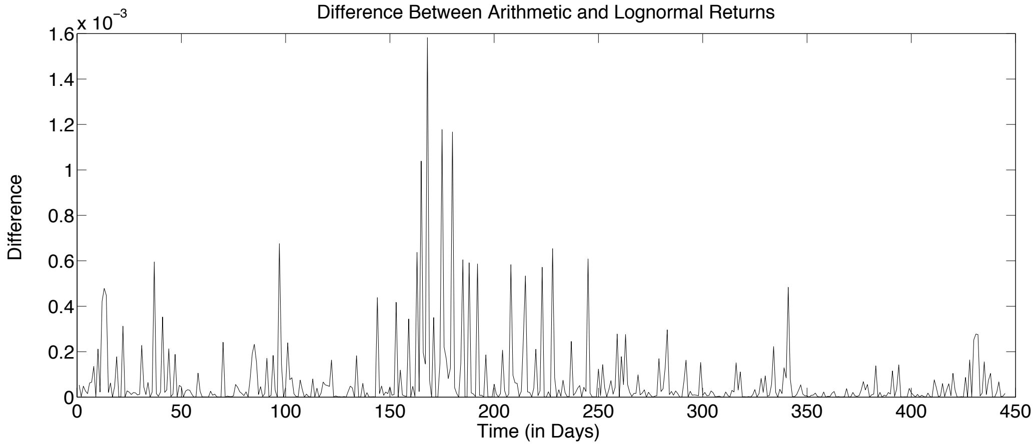

Figure 4: Difference between Artithmetic and Log Normal Returns

As can be seen in figure ??, the difference is very small and this suggests that indeed for small $r(t)$ the relation $\ln(1 + r(t)) \approx r(t)$ holds.

Using log returns instead of arithmetic returns has several advantages. One is that the log return is symmetric, while the arithmetic return is not: positive and negative percent normal returns are not equal.

Before going to Brownian motion, we want to say a little more about log returns.
For these returns, we assume the following relation

$$r(t) \stackrel{d}{=} \mu + \sigma N$$

Where $\mu \in \mathbb{R}$ and $\sigma > 0$ are parameters. N is a standard normal random variable. So the mean and standard deviation of the log returns are μ and σ
A histogram of the returns of the AEX-Index data (see figure ??) together with a plot of the normal density with parameters estimated from the AEX data is found in figure ??.

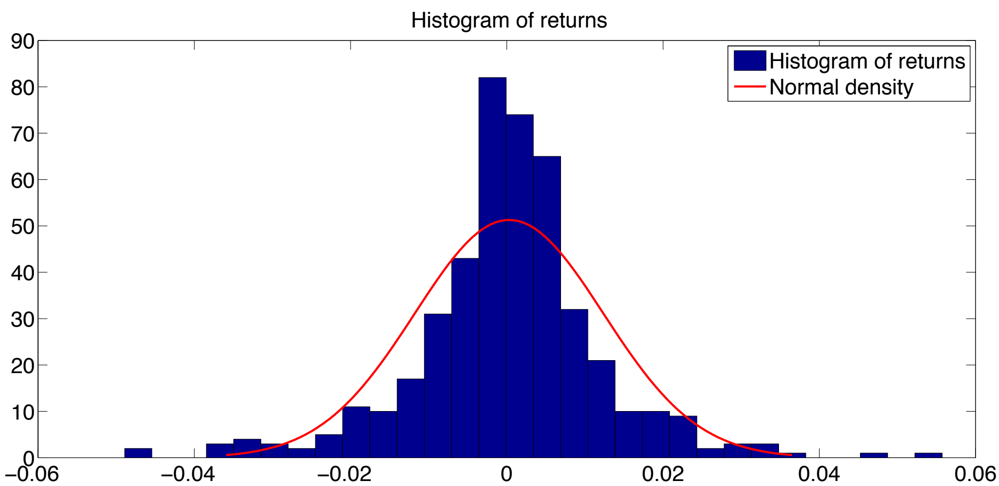

Figure 5: Histogram of returns

The relation $r(t) \overset{d}{=} \mu + \sigma N$ has $\mu = 0$, but the noise term σN is quite big here.
The kurtosis of the log returns is 6.0589 and the skewness is -0.1056. For a normal distribution, the kurtosis is 3.0 and the skewness is 0. A much larger kurtosis implies fat tails and high peaks. Especially high peaks are visible in figure **??**. The distribution of log returns is almost symmetric. This is also implied by the skewness, since it is almost zero.

Now let us look a little closer to $\ln(S(t))$ (equation (**??**)) and see how we can extend it using a so called telescope sum:

$$\begin{aligned}
\ln(S(t)) &= \ln\left(\frac{S(t)}{S(t-1)}\frac{S(t-1)}{S(t-2)}\cdots\frac{S(1)}{S(0)}S(0)\right) \\
&= r(t) + r(t-1) + \ldots + r(1) + \ln(S(0)) \\
&\overset{d}{=} \mu t + \sigma(N_1 + \ldots + N_t) + \ln(S(0)) \\
&\overset{d}{=} \mu t + \sigma B(t) + \ln(S(0))
\end{aligned}$$

Here, $B(t)$ is the Wiener process (which we shall discuss in the next subsection).
Finally, we find for $S(t)$:

$$S(t) \overset{d}{=} S(0)e^{\mu t + \sigma B(t)} \tag{4}$$

1.2 Standard Brownian Motion

Brownian Motion is the name given to the motion exhibited by a small particle that is totally immersed in a liquid or gas. The British botanist Robert Brown discovered this in 1827 when he observed the irregular movement of pollen in water.
This mathematical model has many real-world applications like modeling asset prices (that sounds interesting for this report).

But general or normal Brownian motion has a big disadvantage for modeling asset prices (discussed later). That is why we use a special type of Brownian motion, called *geometric Brownian motion*

Let us first start with the definition of standard Brownian motion (see also Ross, 2010).

Definition: A stochastic process, $\{B(t), t \geq 0\}$ is said to be a Brownian motion process if:

- $B(0) = 0$
- $\{B(t), t \geq 0\}$ has stationary and independent increments
- for every $t > 0, B(t)$ is normally distributed with mean 0 and variance σ^2
- $B(t)$ has continuous sample paths

Remark: A stochastic process, say $\{B(t), t \geq 0\}$, has stationary increments if, for any given $t > 0$, the increments $B(t + s) - B(s)$ are identically distributed for all $s > 0$.

The process given in the definition above is sometimes also called a Wiener process. Two sample paths are plotted in figure **??**.

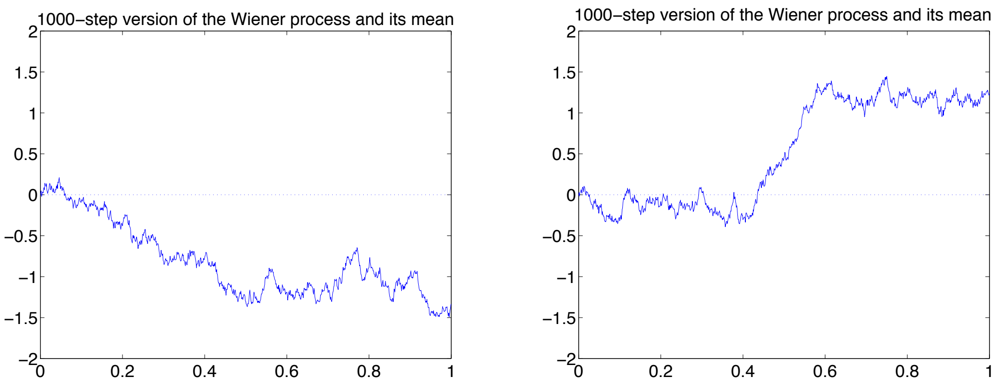

Figure 6: Two sample paths of the Wiener process

As said before, Brownian motion has a big disadvantage for modeling asset prices. As can be seen in figure **??**, Brownian motion can take on negative values. In the real world, assets cannot take on negative values. To solve this problem we introduced *geometric Brownian motion*, the non-negative version of Brownian motion. This is also useful to describe a lognormally distributed stochastic process in continuous time that has independent increments.

Let us now give the definition of geometric Brownian motion.

Definition: If $\{B(t), t \geq 0\}$ is a Brownian motion process with drift μ and volatility (volatility will be discussed in section 3.3) parameter σ, then the process $\{Z(t), t \geq 0\}$ defined by

$$Z(t) = e^{B(t)}$$

is called a geometric Brownian motion.

Taking the exponential of the standard Brownian motion is clear from formula (**??**).

Two sample paths of geometric Brownian motion (red line) with the corresponding standard Brownian motion (blue line) are plotted in figure **??**, to illustrate the difference. Clear from the definition, GBM does not take on negative values.

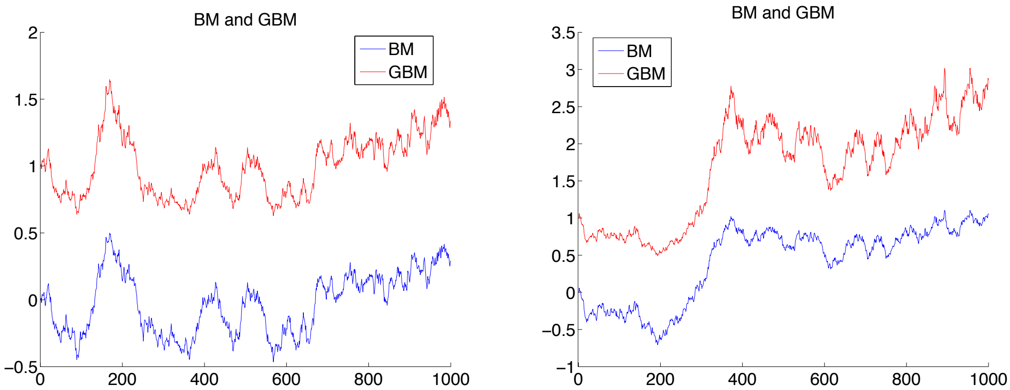

Figure 7: Two sample paths of the standard BM and its exponential geometric BM

1.3 Returns of GBM

Let us consider again returns of geometric Brownian motion. We assumed that the following relation holds

$$r(t) \stackrel{d}{=} \mu + \sigma N$$

Hence, $r(t)$ should be normally distributed with mean $\mu = 0$.
Generating a 10.000 step Wiener process $B(t)$ and taking its exponential $e^{B(t)}$ to obtain the geometric Brownian motion and fitting a normal distribution (with the command histfit in MATLAB) results in figure **??**.

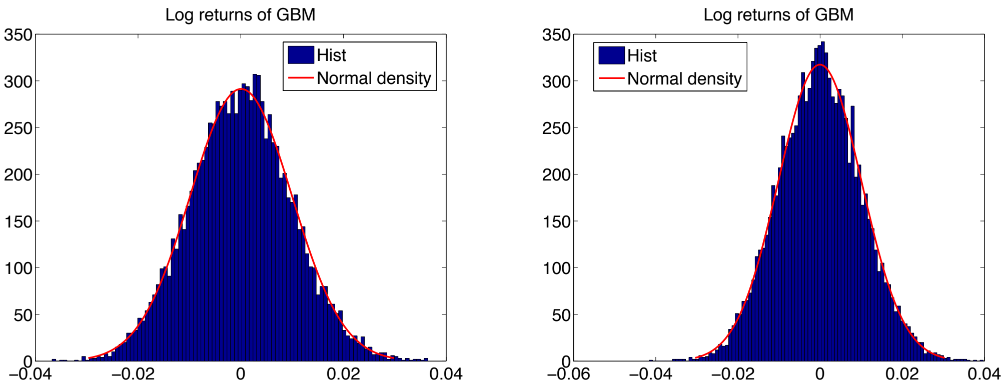

Figure 8: Returns on GBM with a normal distribution fit

As can be seen in figure **??**, it almost seems that indeed the data is standard normally distributed. To test this, we calculate its kurtosis and skewness. Its kurtosis should be around 3 and its skewness around 0.
The skewness of the left distribution in figure **??** is 0.2797 and the kurtosis is 3.1588.
The skewness of the right distribution in figure **??** is 0.8351 and the kurtosis is 3.5008.
The data is not perfectly normally distributed, but the skewness and kurtosis are pretty close to 0 and 3 respectively.

Additionally, the Jarque-Bera test can be performed. For both samples, normality is then accepted.

1.4 The Stochastic Differential Equation

Since it is impossible to predict the AEX-Index with certainty, we assumed that the AEX-Index follows geometric Brownian motion. If there was no uncertainty about the asset price $S(t)$, then the change of the asset price after some time t with a constant growth rate μ would be, according to Alexander (2008)

$$\frac{dS(t)}{dt} = \mu S(t) \tag{5}$$

However, there is uncertainty about the future asset price, so we must add a stochastic differential term $dB(t)$ to (**??**).

Now equation (**??**) becomes

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t) \tag{6}$$

Where μ is the drift rate and σ is the volatiltiy. Equation (**??**) is called a *stochastic differential equation.*

We will go into more detail about the term $dB(t)$, before we will solve equation (**??**). Let us say we want to integrate Brownian motion using ordinary calculus. So we integrate

$$Y(T) = \int_0^T B(t)dB(t) \tag{7}$$

Using ordinary calculus yields

$$Y(T) = \int_0^T B(t)dB(t) = \frac{1}{2}B^2(T) \tag{8}$$

But is this true? Let us calculate the Riemann left sum S_L and the Riemann right sum S_R, defined the following way

$$\begin{aligned} S_L &= \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\ S_R &= \sum_{i=1}^{n} B(t_i)(B(t_i) - B(t_{i-1})) \end{aligned}$$

As we can see, we multiplied S_L with the most left point $B(t_{j-1})$ and S_R with the most right point $B(t_j)$.

Adding and distracting the left sum S_L and the right sum S_R yields

$$\begin{aligned} S_L + S_R &= \sum_{i=1}^{n} \left(B(t_i) - B(t_{i-1})\right)\left(B(t_i + B(t_{i-1})\right) \\ &= \sum_{i=1}^{n} B(t_i)^2 - B(t_{i-1})^2 \\ &= B^2(t) - B^2(0) = B^2(t) \\ S_R - S_L &= \sum_{i=1}^{n} \left(B(t_i) - B(t_{i-1})\right)^2 \end{aligned}$$

Important to notice: $S_R - S_L \neq 0$.

Normally we would expect that $S_R - S_L = 0$ *in the limit*, but a can be seen, this is not true. Before we continue with Brownian motion, let us look closer to the difference between S_L and S_R, defined by

$$S_R - S_L = \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2$$

which is called the *quadratic variation* (again, in the limit of course). Now we will calculate the quadratic variation for a continuous differentiable function f and we will show that this quadratic variation goes to zero as what we actually also wanted for $S_R - S_L$ to use standard calculus for equation (**??**).

Define

- $\Pi = 0 < t_0 < t_1 < \ldots < t_{n-1} < t_n = T$
- $K = \max(t_i - t_{i-1})$

Now, the quadratic variation of f is equal to zero, or

$$S(f) = \lim_{K \to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2 = 0$$

proof:

The Mean Value Theorem says that in each subinterval $[t_{i-1}, t_i]$ there is a point τ_i such that

$$f(t_i) - f(t_{i-1}) = f'(\tau_i)(t_i - t_{i-1})$$

Thus

$$\begin{aligned}
S(f) &= \lim_{K \to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2 \\
&= \lim_{K \to 0} \sum_{i=1}^{n} f'(\tau_i)^2 (t_i - t_{i-1})^2 \\
&\leq \lim_{K \to 0} K \sum_{i=1}^{n} f'(\tau_i)^2 (t_i - t_{i-1}) \\
&= \lim_{K \to 0} K \lim_{K \to 0} \sum_{i=1}^{n} f'(\tau_i)^2 (t_i - t_{i-1}) \\
&= \lim_{K \to 0} K \int_0^T f'(t)^2 dt \\
&= 0
\end{aligned}$$

□

Now that the quadratic variation of a continuous differentiable function is zero, let us look again at Brownian motion.

Theorem: The quadratic variation of Brownian motion is not equal to zero.
Instead, the quadratic variation of Brownian motion goes to T, or

$$S(K) = \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2 \to T$$

Note that we define $S(K)$ as follows:

$$S(K) = \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2$$

proof:

First, we calculate the expectation and show that this is actually equal to T. Then, we calculate the variance and last we will prove the theorem.

1.

$$\begin{aligned} E(S(K)) &= \sum_{i=1}^{n} E\left((B(t_i) - B(t_{i-1}))^2\right) \\ &= \sum_{i=1}^{n} |t_i - t_{i-1}| \\ &= T \end{aligned}$$

Let us explain why we can make these steps. From the definition, the variance is equal to

$$\begin{aligned} Var((B(t_i) - B(t_{i-1})) &= E\left[(B(t_i) - B(t_{i-1}))^2\right] - \left(E\left[(B(t_i) - B(t_{i-1})\right]\right)^2 \\ &= E\left[(B(t_i) - B(t_{i-1}))^2\right] - \mu^2 \end{aligned}$$

Since the mean $\mu = 0$ (see definition Brownian motion), $E(S(K)) = \sum_{i=1}^{n} Var((B(t_i) - B(t_{i-1}))$ and since this (see definition of Brownian motion) is equal to $\sum_{i=1}^{n} |t_i - t_{i-1}|$, all these partitions add up to T. Hence, the expectation of $S(K)$ does not go to T, but is equal to T.

2.

$$\begin{aligned} Var(S(K)) &\stackrel{d}{=} Cov(S(K), S(K)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov\left((B(t_i) - B(t_{i-1}))^2, (B(t_j) - B(t_{j-1}))^2\right) \\ &= \sum_{i=1}^{n} Cov\left((B(t_i) - B(t_{i-1}))^2, (B(t_i) - B(t_{i-1}))^2\right) \end{aligned}$$

Here, $B(t_i) - B(t_{i-1}) \stackrel{d}{=} \sqrt{t_i - t_{i-1}}\, U$, where $U \sim N(0, 1)$, thus we find

$$\begin{aligned} Var(S(K)) &= \sum_{i=1}^{n} (t_i - t_{i-1})^2 Cov(U^2, U^2) \\ &= 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \end{aligned}$$

3.

$$E\left[\sum_{N=1}^{\infty} (S(K) - T)^2\right] = E\left[\lim_{N \to \infty} \sum_{k=1}^{N} (S(K) - T)^2\right]$$

Using the monotone convergence theorem, we find

$$\begin{aligned} E\left[\sum_{N=1}^{\infty}(S(K)-T)^2\right] &= \lim_{N\to\infty}\sum_{k=1}^{N} E\left[(S(K)-T)^2\right] \\ &= \lim_{N\to\infty}\sum_{k=1}^{N} 2^k 2\left(\frac{T}{2^k}\right)^2 \\ &= T^2\sum_{k=1}^{\infty}\frac{1}{2^{k-1}} \\ &= 2T^2 < \infty \end{aligned}$$

This results in

$$\lim_{k\to\infty}(S(K)-T) = 0$$

Hence,

$$\lim_{k\to\infty} S(K) = T$$

$\square$

Indeed, the quadratic variation of Brownian motion goes to T. From this property, another property can be shown, which is discussed below.

Theorem: Every sample path of Brownian motion has infinite length, in other words, the linear variation goes to infinity, or

$$\sum_{i=1}^{n}|B(t_i)-B(t_{i-1})| \to +\infty$$

proof:

Suppose that B is a function of bounded linear variation and let V denote the total linear variation of B on $[0,T]$, then

$$\begin{aligned} \sum_{i=1}^{n}|B(t_i)-B(t_{i-1})|^2 &\leq \max|B(t_i)-B(t_{i-1})|\sum_{i=1}^{n}|B(t_i)-B(t_{i-1})| \\ &\leq V\cdot\max|B(t_i)-B(t_{i-1})| \end{aligned}$$

Now, B is a continuous function on $[0,T]$ and because we consider a *closed* interval, necessarily uniformly continuous on $[0,T]$, we get

$$|B(t_i)-B(t_{i-1})| \to 0 \text{ as } K \to 0$$

Hence,

$$\sum_{i=1}^{n}|B(t_i)-B(t_{i-1})|^2 \to 0$$

But the quadratic variation does not converge to zero, but to T as we proved earlier. Hence this is a contradiction, so our claim that the linear variation goes to infinity holds.

$\square$

This has two important consequences for Brownian motion:

- Sample paths of $B(t)$ are not differentiable ($B(t)$ is nowhere differentiable)
- Every sample path has infinite length (as stated earlier)

Because $B(t)$ is not differentiable, equation (**??**) does not hold. Equation (**??**) is called the *Ito integral.* But what is a right expression for the Ito integral?

Let us write $B(t_{i-1})$ as

$$B(t_{i-1}) = \frac{1}{2}(B(t_i) + B(t_{i-1})) - \frac{1}{2}(B(t_i) - B(t_{i-1}))$$

The Riemann right sum S_R now becomes

$$\begin{aligned} S_R &= \sum_{i=1}^{n} \frac{1}{2}(B(t_i) + B(t_{i-1}))(B(t_i)) - B(t_{i-1}) - \frac{1}{2}(B(t_i) - B(t_{i-1}))(B(t_i) - B(t_{i-1})) \\ &= \sum_{i=1}^{n} \frac{1}{2}(B(t_i)^2 - B(t_{i-1})^2) - \frac{1}{2}(B(t_i) - B(t_{i-1}))^2 \end{aligned}$$

Now, Using the fact that we found that

$$\sum_{i=1}^{n} \frac{1}{2}(B(t_i)^2 - B(t_{i-1})^2) = B^2(t)$$

And that

$$\frac{1}{2}(B(t_i) - B(t_{i-1}))^2 \to T$$

We get the correct expression for the Ito integral

$$\int_0^T B(t)dW(t) = \frac{1}{2}(B^2(t) - T)$$

1.5 The Markov Property

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant. Brownian motion and geometric Brownian motion are both Markov processes, because their future given the present state is independent of the past.
Let us find an expression for a process $B(t+h)$, the future after h units of time t [1].

$$\begin{aligned} B(t+h) &= B_0 e^{X(t+h)} \\ &= B_0 e^{X(t)+X(t+h)-X(t)} \\ &= B_0 e^{X(t)} e^{X(t+h)-X(t)} \\ &= B(t) e^{X(t+h)-X(t)} \end{aligned}$$

Thus given $B(t)$, the future $B(t+h)$ only depends on the future increment of the Brownian motion, $X(t+h) - X(t)$. But BM has independent increments, so this future is independent of the past; we get the Markov property.

[1] see http://www.columbia.edu/ ks20/FE-Notes/4700-07-Notes-GBM.pdf

2 Option pricing the discrete way: The Binomial Option Pricing Model

In this report, options play a central role in finding an optimal strategy for our investments in the AEX-index. A simple model for pricing an option is the Binomial Model (Alexander 2008). In this model, for a (small) time step Δt, the price of an asset can either move upward or move downward by a constant multiple.
A simple example is the One-Period Binomial model, illustrated in figure **??**

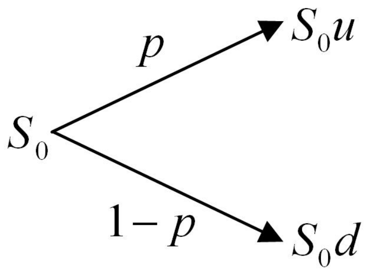

Figure 9: One-Period Binomial Model

At the initial node, we have an asset price denoted by S_0. After a period of Δt, the price can either be uS_0 with $u > 1$ when the price of the asset goes up, or dS_0 with $d < 1$ when the price of the asset goes down.
We make the following assumptions about the transition probabilities (i.e. the probability that the price moves up or down):

- p when it moves up, p is constant and $0 < p < 1$.
- q when the price moves down and $q = 1 - p$

In this section, we will further discuss the binomial model but we will first discuss three important definitions related to the (Binomial) Option Pricing Model and, namely Risk Neutral Valuation (RNV), Arbitrage and Martingales.

2.1 RNV, Arbitrage and Martingales

Risk Neutral Valuation states that the current value of a financial asset is equal to the expected future payoff of the asset discounted at the risk-free rate. This expected value is a martingale:

Definition: In probability theory, a martingale is a stochastic process such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time s, is equal to the observation at that earlier time s.

In a more formal context [1]: A sequence of random variables $X_0, X_1, ...$ is said to be a martingale sequence if for all $i > 0, E\left[X_i | X_0, ..., X_{i-1}\right] = X_{i-1}$

However, to satisfy the condition of market efficiency, there cannot exist investment strategies that will make a guaranteed risk-free profit. This is called the concept of no arbitrage. Investment strategies involving arbitrage opportunities can best be defined as trading strategies that begin with no money, have zero probability of losing money and have a positive probability of making money (Shreve 2004).

[1] http://en.wikipedia.org/wiki/Martingale_(probability_theory)

Using the definition of martingales, it can be showed that there are no arbitrage opportunities possible, or else the definition of martingales does not hold anymore. This will be proved in the next subsection.

Remark: In the rest of this report, we assume that there exist no arbitrage opportunities.

2.2 One-Period Binomial Model

As said before, the price of the asset can move up with probability p and move down with probability $q = 1 - p$. The expected asset price after one time step where $S(\Delta t)$ is the asset price after a time Δt:

$$E\left[S_0(\Delta t)\right] = puS_0 + qdS_0$$

Now we must choose p such that the expected asset price grows with the risk free rate in order to satisfy RNV.

$$E[S_0(\Delta t)] = S_0(1 + r)$$

Also, in a simple binomial model for pricing an option we want to assume that the underlying price process follows a geometric Brownian motion with constant volatility. In this case the binomial option price will converge to the Black-Scholes-Merton model price as the number of time steps increases and time interval Δt gets smaller and smaller. The question is how we should choose p, u and d so that the discretization in a binomial tree will be consistent with the assumption that the asset price process follows a geometric brownian motion with constant volatility σ and drift equal to the risk free rate of return. We have the following assumptions:

- The start capital at $t = 0$ is equal to X_0
- The asset price at $t = 0$ is equal to S_0
- The risk free rate of return is r
- The number of assets bought at $t = 0$ is K_0
- We look after an interval of Δ_t

Then we have after an interval of Δt if the price has gone up:

$$pK_0uS_0 + p(X_0 - K_0S_0)(1 + r) = pV_u$$

Here, V_u means Value up.
And if the price has gone down:

$$qK_0dS_0 + q(X_0 - K_0S_0)(1 + r) = qV_d$$

Here, V_d means Value down
for $0 < p < 1$ and $q = 1 - p$.

$$\begin{aligned} (1 + r)pX_0 + K_0S_0(pu - (1 + r)p) &= pV_u && (9) \\ (1 + r)qX_0 + K_0S_0(qd - (1 + r)q) &= pV_d && (10) \end{aligned}$$

If we add these two equations together, we get:

$$(1 + r)X_0 + K_0S_0(pu + qd - (1 + r)) = pV_u + qV_d$$

Since the start capital X_0 must grow with the risk-free rate and is equal to $pV_u + qV_d$, the term $K_0S_0(pu + qd - (1 + r)) = pV_u + qV_d$ must be equal to zero. Since we assume that we bought assets which have an asset price greater than zero, K_0 and S_0 are not equal to zero. So we find as a restriction for p, u and d.

$$pu + qd - (1 + r) = 0$$

Or, in terms of p:

$$p = \frac{(1+r)-d}{u-d}$$

and q equals:

$$q = \frac{u-1-r}{u-d}$$

In Shreve (2004), we find that under the RNV, the discounted asset price is a martingale.

proof:

$$\begin{aligned}
E\left[\frac{S_{n+1}}{(1+r)^{n+1}}\ |S_0, S_1, \dots S_n\right] &= E\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\ |S_0, S_1, \dots S_n\right] \\
&= \frac{S_n}{(1+r)^n}\frac{1}{1+r}E\left[\frac{S_{n+1}}{S_n}\ |S_0, S_1, \dots S_n\right] \\
&= \frac{S_n}{(1+r)^n}\frac{pu+qd}{1+r} \\
&= \frac{S_n}{(1+r)^n}
\end{aligned}$$

$\square$

With such a value for p, the asset price will indeed grow at the risk free rate:

$$E\left[S(\Delta t)\right] = S_0\left[\left(\frac{(1+r)-d}{u-d}\right)u + \left(\frac{u-(1+r)}{u-d}\right)d\right] = S_0(1+r)$$

So the return equals:

$$\frac{S_1 - S_0}{S_0} = \begin{cases} \frac{u-1}{1} = u-1 & \text{with probability } p \\ \frac{d-1}{1} = d-1 & \text{with probability q} \end{cases}$$

We can now see again that the asset price will grow exactly with the risk free rate:

$$\begin{aligned}
\frac{u-1}{1}p + \frac{d-1}{1}q &= up + qd - 1 \\
&= r
\end{aligned}$$

Now we know that the expected growth of the asset price is equal to the risk free rate, let is proof that using the definition of martingales, there cannot exist arbitrage.
Suppose that there is an arbitrage opportunity in the binomial model, then we would have:

$$S_0 = 0$$

And thus

$$E\left[S_0\right] = 0$$

But, because there exists an arbitrage opportunity,

$$E\left[\frac{S_n}{(1+r)^n}\ |S_0, S_1, \dots S_n\right] > 0$$

This violates the definition of martingales that for $n = 0, 1, \dots, N$

$$E\left[\frac{S_n}{(1+r)^n}\ |S_0, S_1, \dots S_n\right] = S_0$$

Hence there cannot exist arbitrage opportunities.

3 Option pricing the continuous way: The Black-Scholes-Merton Model

Now we have discussed geometric Brownian motion, the concept of arbitrage and martingales, we are now ready to discuss options.
An option is a contract that gives its holder the right, but not the obligation to buy or sell an asset, subject to certain conditions, within a specified period of time (Black Scholes article). We call it a derivative, because it is a contract on another contract. The underlying contract is for example a stock or a bond.
Buying a call option gives its holder the right (but, of course, not the obligation) to buy the underlying asset, buying a put option gives its holder the right to sell the underlying asset. There are many different types of options, but we only consider European put and call options. The important property of European options is that they can only be exercised at a specific future date, called the date of maturity.

Options are attractive for various reasons. It is for example possible to make a profit if the asset price goes up (buy a call option) or to make a profit when the asset price goes down (buy a put option).
A second reason is that combining different types of options will reap benefits from various types of behavior of the underlying asset. Third, options can be used as a type of insurance to limit potential losses on investments (for example, buy a put option and the underlying asset).

But the important question is: What is the correct price of an option? (see also Fisher & Black, 1973). Fisher and Black derived a model for option pricing to answer this question. In this section, we will briefly discuss this model and at the end of this section, we will be able to price options ourselves.

3.1 Pay-off Diagrams

First, we will look at pay-off diagrams of European call and European put options. Let $S(t)$ be the price of the underlying asset at the date of maturity denoted by T and let K be the strike price or exercise price of a European call option. Then its pay-off function is given by (figure **??**):

$$C = \max(S(T) - K, 0)$$

Since it is not possible to lose more than the strike price K.

There are three scenarios: $S(T) > K$, $S(T) = K$ and $S(T) < K$.

- If $S(T) > K$, the option is In-The-Money (ITM) and the option will always be exercised at the date of maturity (using the assumption that there are no transaction costs).
- If $S(T) = K$, the option is At-The-Money (ATM) and it has no benefits to exercise the option and the option will expire worthless.
- If $S(T) < K$, the option is Out-The-Money (OTM) and the option will expire worthless.

For a European put option with the same K and $S(T)$, the price at the date of maturity is given by (figure **??**):

$$P = \max(K - S(T), 0)$$

If the asset price has gone down, i.e. $S(T) < K$, the option will be exercised. If the asset price has gone up, i.e. $S(T) > K$, the option will expire worthless.

On the vertical axes, the $'P'$ stands for 'payoff' and not for put' or 'call'.

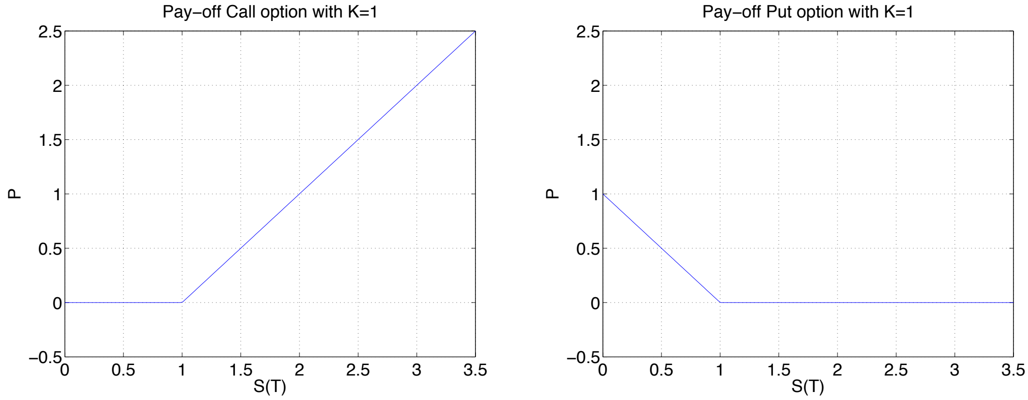

Figure 10: Pay-off diagrams of a Call option (left) and a Put option (right)

3.2 The Black-Scholes formula

The question Fischer Black and Myron Scholes asked themselves was:

Can we systematically determine a fair value of the option at $t = 0$?

In general, the option value is approximately equal to the option price if the date of maturity is very far in the future.
On the other hand, The option value is close to zero or even zero if it reaches almost its date of maturity (considering At-The-Money options).
Normally, the value of an option declines as its maturity date approaches, if the value of the stock does not change.
Before deriving a fair model for the option value in terms of the underlying asset, Black and Scholes first made some important assumptions (see also Fisher & Black, 1973)

- The risk-free interest rate is constant through time
- The asset price follows a random walk in continuous time with a variance rate proportional to the square of the asset price.
- No dividends will be paid
- Only European options exist (only those which can be exercised at the date of maturity)
- There are no transaction costs in buying or selling the stock or the option
- It is possible to borrow any fraction of the price of a security to buy it or hold it at the risk-free interest rate
- There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date

Remark: Short selling means that assets are sold which are not currently owned by the seller, but are promised to be delivered [2].

Under these assumptions, the value of the option will depend only on the price of the underlying asset and time and on variables taken to be known constants.
The derivation of the Black Scholes formula can be found in The Journal of Political Economy, but the final option pricing formula is:

[2] see for example http://en.wikipedia.org/wiki/Short_(finance)

$$C(S,t) = SN(d_1) - Ke^{r(T-t)}N(d_2) \tag{11}$$

With d_1 equal to:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And d_2 is equal to:

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Or, equivalently

$$d_2 = d1 - \sigma\sqrt{T-t}$$

For the convenience of this formula, we will define all the symbols used once more:

- $N(d_1)$ and $N(d_2)$ are $N(0,1)$ distribution functions
- S is the price of the underlying asset
- K is the strike price
- T the date of maturity (in years)
- t the date at which the option price needs to be calculated
- r is the risk-free rate
- σ is the volatility

There are some remarks on the option pricing model. First, in equation (**??**), the option price as a function of the underlying asset price does not depend on the expected return of the asset. However, the expected return of the option does depend on the expected return of the asset. Second, the faster the asset price rises, the faster the option price will rise through the functional relationship. Third, the maturity T appears in the formula only multiplied by the interest rate r or the variance σ^2. Thus, an increase in maturity has the same effect on the value of the option as an equal percentage increase in both r and σ^2.

3.3 Volatility

Volatility can best be described as (Hull 2009):

A measure of the uncertainty of the return realized on an asset

As we saw in the previous section, in the Black-Scholes option pricing formula, S, K, r and T are known. The only parameter that cannot be observed directly is the volatility σ. But it is a crucial parameter to determine the value of an option. There are two different techniques to determine the volatility:

- Implied Volatility
- Historical Volatility

With the presence of an option market, it is possible to extract the volatility given the option value of a certain call or put option and knowing S, K, r and T. Once we found σ, it is possible using the Black-Scholes formula to determine other option values. This technique of computing σ is called *implied volatility.*

It is also possible to determine the volatility in a different way, for example when there is no option market available. Given the behavior of an asset, it is possible to estimate σ. This technique is called *historical volatility.*

These two techniques will be very briefly discussed

3.3.1 Implied Volatility

In this report, we will use the Black-Scholes-Merton model to determine the option value. Consider a case where there is an option market available [3]. Then is it possible to consider an option value with a given asset price S, strike price K, risk-free rate r and date of maturity T. Now only one parameter is unknown: the volatility σ. Using the Black-Scholes-Merton model, the volatility can be obtained. For further reading about this technique with some examples, see Higham (2004) page 131-136. A numerical iteration technique (Newton-Raphson) will be used to iterate to the right σ .

3.3.2 Historical Volatility

Implied volatility is a technique that is widely used to determine σ. There is a second approach that estimates the volatility from the previous behavior of the asset.
There are only two things that need to be present"

- A model for the behavior of the asset price that involves σ (for example Brownian motion or a Time Series)
- Access to asset prices for all times up to the present

Now we can fit σ in the model to the observed data. A value σ arising from this general procedure is called a historical volatility estimate. This technique is further discussed in detail in Higham (2004) page 203-209.
Using this technique for the data of the AEX-Index (visible in figure **??**) yields figure **??**

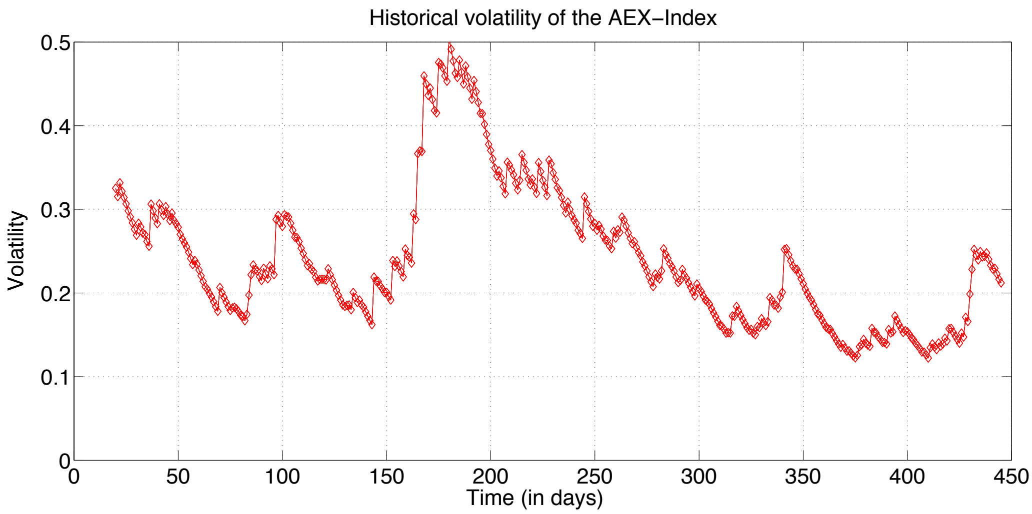

Figure 11: Historical Volatility of the AEX-Index from 13 October 2009 to 31 march 2011

As can be observed in figure **??**, the volatility σ is not constant in time, but changes over time! This is a crucial feature of the stock market. In the next section, where we shall describe the investment product AEX-Sparen, we will assume that we have a constant volatility, but in reality that is certainly not the case.

[3] see for example http://www.iex.nl/Index-Koers/12272/AEX-index/opties-expiratiedatum.aspx

4 Description Of Product

During four months, the bank will invest 5000 Euro in the AEX-Index and after four months the 5000 Euro will be returned to the customer plus interest. The interest is the same as the AEX-Index has earned in the previous four months, but is maximized to 10%. If the AEX-Index has gone down after four months, the initial investment of 5000 Euro will be returned without interest.

The strategy is created with the idea that the initial investment must be guaranteed at virtually no risk, so the initial investment can *always* be paid back, no matter what happens to the AEX-Index.
With this strategy, the bank has the advantage that whatever happens, the initial investment can *always* be paid back.

An interesting question is of course how realistic this strategy is. Can we construct this our selves better than the bank and get rich tomorrow? This is discussed in the sections 'Discussion' and 'Conclusion'.

4.1 Investment Strategy

Let us go back to the original problem. Two things are important:

- The initial of 5000 Euro must *always* be returned to the client
- An interest must be paid to the client equal to what the AEX-Index has earned in the previous four months, *maximized* to 10 %

Our first concern is the initial investment. How can the bank guarantee 5000 Euro? It would be possible to put it in an old sock, wait four months and pay it back. This way, if the AEX-Index has gone up, the bank will lose money and does not have a chance to make any profit itself.

Here, bonds can solve the first issue. Let us consider zero-coupon bonds. Normally, zero-coupon bonds pay interest at the date of maturity, precisely one year after the bonds are bought. But for the moment, consider that the bond pays interest after four months, equal to our investment period.
It is reasonable to assume that the annual interest received from a bond is 5% [4]. This is higher than the annual risk-free interest rate of 2.5%, so some risk is involved. It would be possible that the company from which the bank bought bonds went bankrupt, in which case the full investment is lost. Let us assume that the company does not go bankrupt.

The interest paid after four months is equal to 1.67%. In order to ensure 5000 Euro after four months, the bank must buy bonds for the exact amount of the Net Present Value (NPV), equal to

$$\text{NPV} = \frac{5000}{1+r}$$

Here, $r = 0.0167$, so

$$\text{NPV} = \frac{5000}{1+0.0167} = 4918 \text{ Euro}$$

Hence, the bank must buy for 4918 Euro bonds to ensure the 5000 Euro after four months.
Note that it does not matter in which company the bank invests, it does not have to hold any relation with the AEX-Index!
Now, there is $5000 - 4918 = 82$ Euro left to invest in the AEX-Index. We will read the second important point:

The interest is equal to what the AEX-Index has earned, maximized to 10 %

Since the AEX-Index follows the geometric Brownian motion, its initial starting point at $t = 0$ is equal to 1. Recall section 1 for further details about geometric Brownian motion. The pay-off must increase linear from 1 to 1.1 and then be horizontal. The pay-off of the investment is visible in figure **??**

[4] See for example http://www.iex.nl/Koersen/Obligaties.aspx

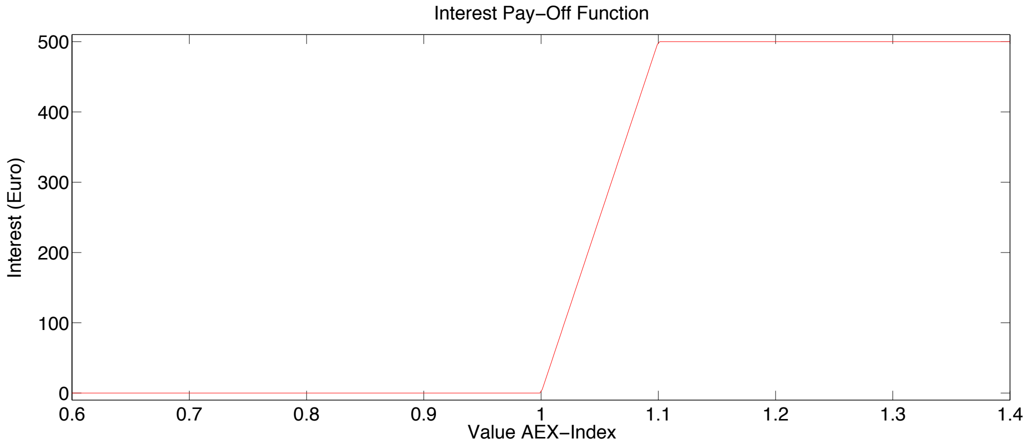

Figure 12: Pay-off Function from interest

On the interval [0.6 1.1], this looks like the pay-off function from a call option. This pay-off is also visible in figure **??**. When the AEX-Index hits the value of 1.1, the interest remains constant at 500.

How can we construct this? Buy call options with strike price $K = 1$, using the 82 Euro interest from the bond and write the exact same amount of call options with strike price $K = 1.1$. This way, the exact same pay-off *construction* is made. Note that the call options bought are At-The-Money and that the call options sold are Out-The-Money.
Let us calculate the option price, using the Black-Scholes-Merton model, discussed in section 3.
Recall that the option price is given by

$$C(S,t) = SN(d_1) - Ke^{r(T-t)}N(d_2)$$

With d_1 equal to:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And d_2 is equal to:

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Or, equivalently

$$d_2 = d1 - \sigma\sqrt{T-t}$$

Using matlab, we find that

$$\begin{aligned} \text{BlackScholes}(\text{Call}, S, K, r, T, \sigma) &= \text{BlackScholes}(\text{Call}, 1, 1, 0.025, 1/3, 0.15) \\ &= 0.0387 \text{ Euro} \end{aligned}$$

As said before, $S = K = 1$. We also assumed that $r = 0.025$ and that there is a four month investment period, hence $T = 1/3$. The volatility σ is estimated with historical volatility (see section 3 for further details). The graph we constructed is once more visible in figure **??**
We chose one of the lowest volatility points, which is $\sigma = 0.15$.
In order to let the interest grow from 0 to 500 if the AEX-Index increases from 1 tot 1.1, it is clear that precisely 5000 call options must be bought (this does not depend on T or σ) .

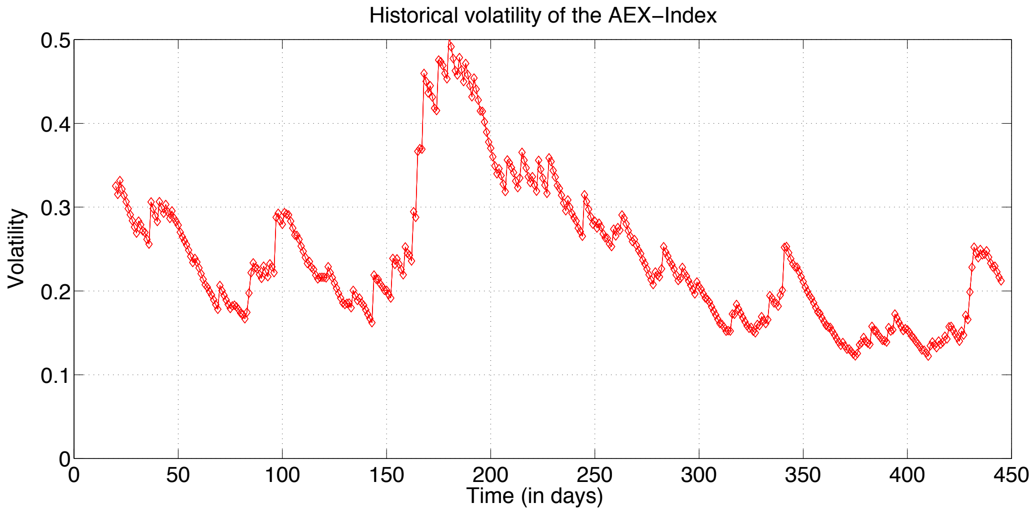

Figure 13: Historical Volatility from 13 october 2009 to 31 march 2011

To buy 5000 call options with a price of 0.0387 Euro, we need 193.52 Euro. With 82 Euro, only 2119 call options can be bought. Hence there are also 2119 call options written with a strike price of $K = 1.1$ and an option price of

$$\text{BlackScholes}(1, 1.1, 0.025, 1/3, 0.15 = 0.0075$$

Of course, S, r, T and σ are the same as for the call options that are bought. This yields an option premium received by the bank of $2119 \cdot 0.0075 = 15.89$ Euro
This can also be invested in call options, the total amount of call options bought is 2.529.
The payoff function (blue) is given in figure **??**, as well the pay-off we needed and constructed in figure **??** (red)

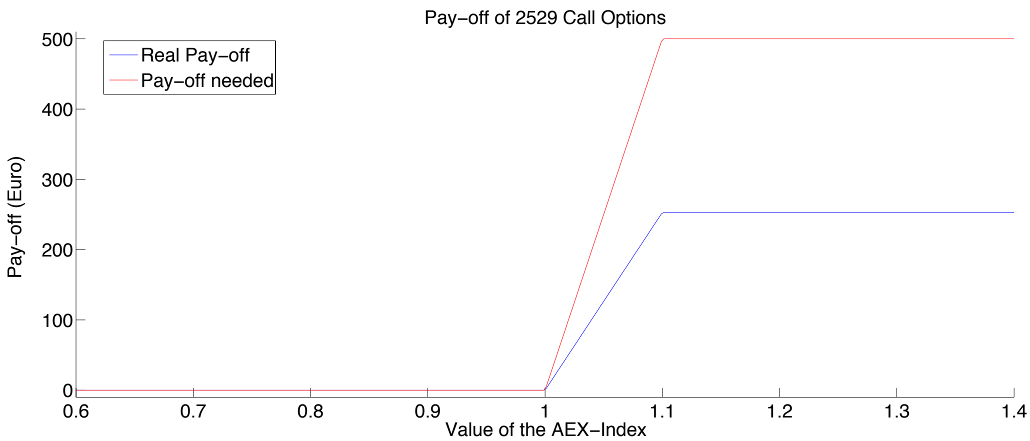

Figure 14: Pay-off Function from interest the real interest and the required interest

As can be seen in figure **??**, we do net get as much pay-off as we wanted. This has the simple reason that the interest that must be paid is far to high.
Also, if the volatility is lowered, the option price drops, as shown in figure **??**. Here, $S = K = 1$, $r = 0.025$ and $T = 1/3$

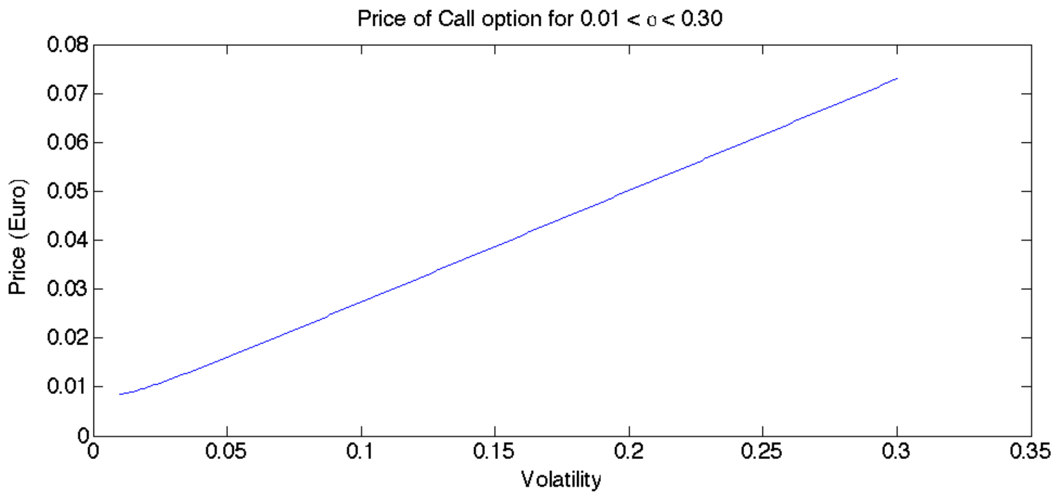

Figure 15: Price of call option with given volatility

But it is not possible to simply lower the volatility. As can be read in section three, the volatility is obtained from the market data using historical volatility or implied volatility.

If it were possible that a 10% interest could be earned every three months, than on a year basis 30% could be made, with a guarantee of the initial investment. In the real world, this might sound too good to be true.

So, in order to guarantee an interest of 50 Euro for every percent the AEX-Index increases, a higher option premium must be received. Now, we will buy 5000 call options, but we will write the options with a strike price *lower* than 1.1. This is visible in figure **??**

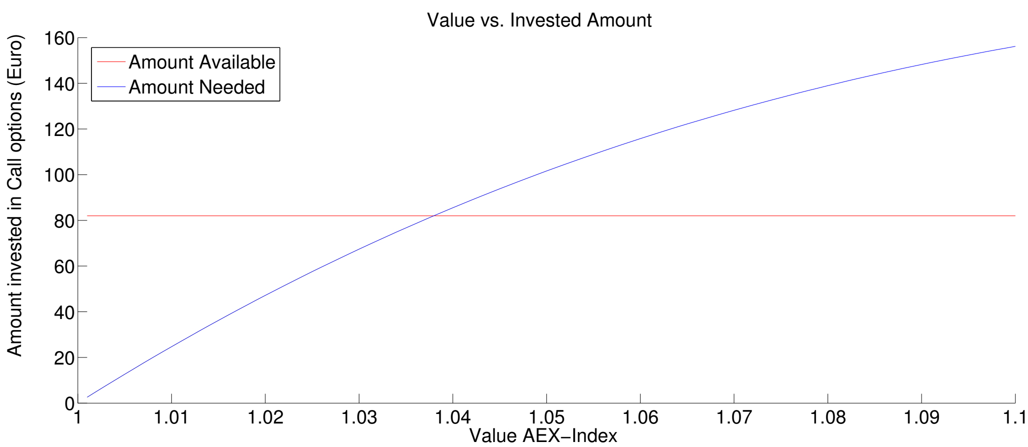

Figure 16: Value vs. Invested Amount

Figure **??** must be interpreted as follows:

- We *buy* 5000 Call options for 193,50
- We *write* 5000 Call options at a strike price between $K = 1$ and $K = 1.1$
- The option premium *received* is subtracted from 193,50
- The net investment is now equal to 193,50 - option premium, represented by the *blue* line
- The *red* line is the net investment we could do, namely 82 Euro
- Once the blue line is *below* the red line, the investment is possible

It is clear from figure **??** that a maximum interest that can be guaranteed by the bank is 3.7% (also calculated with matlab). This is considerably lower than 10%.
Note that the bank has not made any profit itself yet.

How realistic is this? As said before, ABN-Amro also offers the *exact* same product, but the investment period is 6 months instead of 4 months and the maximum interest received is 3%.
As is visible in figure **??**, with this construction it is possible to guarantee even more than 3% after a shorter period. So what ABN-Amro offers is pretty realistic and they can make a profit. This is calculated later.

What the bank also can do is borrow the 5000 Euro to an another customer, which is paying an interest up to 8% or even more (consider for example mortgages).

Let us now calculate the case of ABN-Amro. ABN-Amro still needs to buy 5000 call options for a strike price $K = 1$ and will sell 5000 call options for a strike price $K = 1.03$ (that is why they offer 3% interest). The price of one call option with $K = 1$ and $T = 1/2$ is

$$\text{BlackScholes(Call,}1, 1, 0.025, 1/2, 0, 15 = 0.0485$$

The total investment costs $5000 \cdot 0.0485 = 242.50$ Euro.
The price of one call option with $K = 1.03$ and $T = 1/2$ is

$$\text{BlackScholes(Call,}1, 1.03, 0.025, 1/2, 0, 15) = 0.0346$$

The total option premium received is 173.03 Euro.
The total amount available for investment is

$$5000 - \frac{5000}{1.025} = 121.95 \text{ Euro}$$

Their net investment is equal to $242.50 - 173.03 = 69.47$ Euro, which is lower than the available amount.

Hence, the bank will make a net profit of 52.48 at no risk!

5 Hedging

Normally, structured products are about hedging and reducing risk at the same time. With this product, not much risk is involved for the bank. If the AEX-Index has gone up, the call options will always pay-out and the bank makes 32.70 Euro. If the AEX-Index has gone down, the options will expire worthless and the bank still earns 100.96 Euro.
The only risk involved is that the company from which the bonds are will go bankrupt. There is no specific strategy for reducing that risk which is interesting for this report, since we assumed that the company does not go bankrupt.

6 Conclusion and Advice

The original exercise was to find a strategy that would guarantee the initial investment and would be able to pay interest if the AEX-Index had gone up. The strategy in short is:

- Buy zero-coupons bond with an annual interest of 5% and which will pay back exactly 5000 Euro, interest included.
- Determine the Net Present Value and subtract this from the 5000 Euro
- Buy 5000 call options
- Write 5000 call options with such a strike price that the investment - option premium received is lower than the amount available

This way, the initial investment is guaranteed by the zero-coupon bond and the interest is guaranteed by the call options which the bank bought. The bank can offer this product, since it will receive an option premium from the written options and in the case of ABN-Amro, the total interest from the bond was higher than the amount needed for call options, so the remaining amount also goes to ABN-Amro.

It was not possible to demand 10% after four months, since the cost of that strategy is 193.50 Euro and only a total of 82 Euro interest from bonds is received. The maximum percentage possible with 82 Euro 3.7%, assuming that the volatility is pretty low and around 0.15.

As could be seen in figure **??**, often the volatility is higher, making the options a lot more expensive. If the same strategy is applied, even less interest can be guaranteed by the bank.
But still, a potential of 3.7% interest every four months is a lot more than the very low risk-free interest rate of 2.5%. I contacted some people who deal with products like 'AEX-Sparen' all day and they said that the strategy presented in this report is the one they use.

My advice to the customer is to construct this product himself. It sounds attractive, a guarantee of the initial investment, but much of potential gains is lost. If the customer had bought these bonds himself and would have bought the maximum amount of call options that he could and not writing any, *if* the AEX-Index rises hard, huge profits are made. The customer is now limited to a very low interest rate of 3.7% every four months or in the case of ABN-Amro, only 3% every six months.

Consider again the AEX-Index at 300 points from figure **??**. If the customer had bought 6 call options (here, $K = 300!$) for 82 Euro himself, after 6 months he would have made a profit of

$$6 \cdot (360 - 300) = 360 \text{ Euro}$$

Which is 439% on the investment and 7.2% on the 5000 Euro!

The original question *Why would we buy bread instead of making it?* is best answered by:

Making this product (the bread) yourself does not add any risk since the bond will guarantee your initial investment. Your potential profits can be much higher than what the bank offer, especially if you enlarge your investment amount and construct intelligent option products yourself. Or, if you are willing to take at least *some* risk to lose a maximum of 1.000 Euro. even greater profits can be made. Constructing those products is for the reader.

7 Discussion

The question is of course how realistic this product is. I contacted somebody who deals with these structured products like 'AEX-Sparen' each day. It turns out that their strategy is the same as described in this report and they offer [5] a lower interest than possible, because they want to make a profit themselves of course. Their profit (or the costs they charge) is immediately received, since they write and sell options at the moment they receive the 5000 Euro and they can use this profit to do other things. They do not have to pay much attention to this product anymore, since the bond will pay itself back at the date of maturity and the option can also only be exercised at the date of maturity. It is for the bank minimal effort and still a guaranteed profit!

The question is of course why banks offer investment products like these. Potentially, they need clients who bring in money so the bank can borrow this money to other clients again and the bank needs to find ways to attract customers who bring in money.
Since the stock market cannot be predicted, since people hate losing money and since I believe many people think that there is only a possibility to make a profit if the stock market goes up, this is an ideal investment product. It might feel like the customer is smarter than the bank, because when the AEX-Index decreases, they might think the bank has huge losses they have to pay back to them. With this very simple product, all of this is of course not true, but nowhere in the prospectus can be found what the strategy of the bank is, that they will make a guaranteed profit and have virtually no risk offering this product.

Also, as could be seen in this report, relatively huge profits can be made with relatively small investment amounts. It would be very interesting to see if the potential profit would be much higher if the bank was willing to take some risk on losing (a part of) the 5000 Euro, or the customer if he was willing to lose a maximum of say 30% of 40% of the initial investment.

Lastly, the choice of the volatility is not motivated in this report. Assume that the volatility is much higher than 0.15, then higher fluctuations in the AEX-Index occur. It might be smart to consider other strategies like the straddle, where At-The-Money call and put options are bought. These generate a profit if the AEX-Index increases or decreases a lot.

[5] see http://www.abnamro.nl/nl/prive/sparen/aex_spaarrekening/introductie.html

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