LECTURE NOTES ON
AIRPLANE STABILITY AND CONTROL I
Part II

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Delft - The Netherlands

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CHAPTER 7. STEADY, ASYMMETRIC FLIGHT

7.1. Introduction

In the preceding Chapters the equilibrium about the lateral axis and longitudinal control in steady, symmetric flight was studied. In addition, static longitudinal stability was considered since this is usually the most critical condition for dynamic longitudinal stability.

Studies in dynamic lateral stability relate to the asymmetric motions about a steady, symmetric flight condition. There is, however, not one specific 'static lateral stability', characteristic for the asymmetric disturbed motions. These disturbed asymmetric motions appear to be determined by more than just one stability derivative, while the inertial parameters have also an important influence.

In this Chapter the asymmetric forces and moments acting on the airplane are discussed. Next, the lateral equilibrium and lateral control in some steady, asymmetric flight conditions are studied.

7.2. The aerodynamic forces and moments acting on the airplane due to sideslipping, rolling and yawing flight.

The asymmetric degrees of freedom of an airplane are the following, see fig. 7.1:

a. The translation of the airplane center of gravity along the Y-axis of the airplane body axes, positive in the direction of the positive Y-axis. The component of airspeed along the Y-axis is indicated as \( v \). If such a component is present, the airplane is said to be in sideslipping flight. The angle of sideslip \( \beta \) is the angle between the velocity vector \( \mathbf{V} \) of the center of gravity and the plane of symmetry, see fig. 7.2:

\[
\beta = \arcsin \frac{v}{V}
\]

or, at sufficiently small angles of sideslip:

\[
\beta = \frac{v}{V} \text{ (rad)} \tag{7-1}
\]

\( \beta \) is positive if the airplane moves in the direction of the positive Y-axis.
Fig. 7.1: The six degrees of freedom of a rigid airplane.

Fig. 7.2: Definitions of the angle of attack and angle of sideslip.
b. The angular velocity about the X-axis, positive if the right wing moves down. The angular velocity about the X-axis, the rolling velocity, is indicated as \( p \). Often the non-dimensional form \( \frac{pb}{2V} \) is used.

c. The angular velocity about the Z-axis, positive if the airplane's nose moves to the right as seen by the pilot. The angular velocity about the Z-axis, the yawing velocity, is indicated as \( r \). The non-dimensional velocity about the top axis is \( \frac{rb}{2V} \).

The asymmetric motions generate in general an aerodynamic force \( Y \) along the Y-axis, a moment \( L \) about the X-axis and a moment \( N \) about the Z-axis, see fig. 7.3. As noted already in 1.3, asymmetric motions in general do not produce symmetric aerodynamic forces and moments. This will be further explained in 8.1. As a consequence the symmetric aerodynamic forces \( X \) and \( Z \) and the moment \( M \), with a single exception, do not have to be considered.

The rolling moment \( L \) is commonly indicated as \( L' \) in situations where confusion with the lift \( L \) is possible.

The non-dimensional asymmetric forces and moments are defined as:

\[
C_Y = \frac{Y}{\frac{1}{2} \rho V^2 S}
\]

\[
C_L = \frac{L}{\frac{1}{2} \rho V^2 S b}
\]

\[
C_n = \frac{N}{\frac{1}{2} \rho V^2 S b}
\]

(7-2)

where \( C_L \) and \( C_n \) are referenced to the wing span and not, like \( C_m \), to the wing m.a.c.

The non-dimensional aerodynamic force and the moment resulting from sideslipping, rolling and yawing, are determined by the partial derivatives with respect to the non-dimensional asymmetric components of the moment \( \beta \), \( \frac{pb}{2V} \) and \( \frac{rb}{2V} \). These derivatives are:
Fig. 7.3: The asymmetric forces and moments.

a. for sideslipping flight:

\[ C_Y = \frac{\partial C_Y}{\partial \beta}, \quad C_\lambda = \frac{\partial C_\lambda}{\partial \beta}, \quad C_n = \frac{\partial C_n}{\partial \beta} \]

b. for rolling flight:

\[ C_Y = \frac{\partial C_Y}{\partial \frac{pb}{2V}}, \quad C_\lambda = \frac{\partial C_\lambda}{\partial \frac{pb}{2V}}, \quad C_n = \frac{\partial C_n}{\partial \frac{pb}{2V}} \]

c. for yawing flight:

\[ C_Y = \frac{\partial C_Y}{\partial \frac{rb}{2V}}, \quad C_\lambda = \frac{\partial C_\lambda}{\partial \frac{rb}{2V}}, \quad C_n = \frac{\partial C_n}{\partial \frac{rb}{2V}} \]

The derivatives with respect to \( \frac{pb}{2V} \) and \( \frac{rb}{2V} \) are indicated only with the subscript \( p \) and \( r \) respectively, to simplify the notation. The above mentioned partial derivatives are called the **stability derivatives** with respect to angle.
of sideslip, non-dimensional rolling velocity and non-dimensional yawing velocity, as these derivatives are used primarily in relation to the stability of airplane flight.

When using the stability derivatives, it is assumed that the forces and moments vary linearly with $\beta$, $\frac{pb}{2V}$ and $\frac{rb}{2V}$. The additional assumption is made, that in symmetric flight $C_Y$, $C_\alpha$ and $C_n$ are zero. As a result is, for instance in steady, straight sideslipping flight ($p = r = 0$):

$$C_Y = C_{Y*} \beta, \text{ etc.}$$

For small deviations from the symmetric equilibrium condition to which the study of stability refers, this latter assumption is generally acceptable. Commonly this assumption can be used also when considering steady, asymmetric flight at values of $\beta$, $\frac{pb}{2V}$ and $\frac{rb}{2V}$ which are not too large.

During the non-steady motion of the airplane caused by a disturbance from the symmetric equilibrium situation, the accelerations also generate aerodynamic forces and moments. These are usually so small as to be negligible. An exception is only the moment about the top axis caused by an acceleration along the Y-axis. The stability derivative describing this moment, indicated in accordance with the above as $C_{n*}^\beta$, can be compared to the stability derivative $C_{m*}^*\alpha$ for the symmetric motions to be discussed in 8.2.

A deviation from the rule that asymmetric motions cause no symmetric forces and moments, is the change in pitching moment caused by a velocity about the lateral axis. In the study of stability, where small deviations from the symmetric equilibrium situation are considered, this change in $C_m$ is commonly negligible. In steady asymmetric flight, where large angles of sideslip occur, but also during the take-off run and during the landing with cross wind, this pitching moment due to sideslip certainly has to be considered. The effect is expressed quantitatively by the derivative $C_{m*}^\beta$.

Summarizing the above, the non-dimensional asymmetric aerodynamic forces and moments caused by the lateral motions are written as:

$$C_Y = C_{Y*} \beta + C_{Y*} \frac{pb}{2V} + C_{Y*} \frac{rb}{2V}$$

$$C_\alpha = C_{\alpha*} \beta + C_{\alpha*} \frac{pb}{2V} + C_{\alpha*} \frac{rb}{2V}$$

(7-3)
\[ C_n = C_{n_\beta} \cdot \beta + C_{n_p} \cdot \frac{pb}{2V} + C_{n_r} \cdot \frac{rb}{2V} \]

In the following, the underlying mechanisms, the magnitudes and the signs of the stability derivatives are discussed. Fig. 7.52 on page 264 gives a summary of the forces and moments on the airplane due to \( \beta \), \( \frac{pb}{2V} \) and \( \frac{rb}{2V} \). The usual signs of the corresponding stability derivatives are shown as well. Quantitative values of the stability derivatives have been compiled in Table 7.1 for a number of airplanes in various flight conditions, see also ref. 7.68. When it comes to the actual calculation of the stability derivatives, reference will be made to the relevant literature. An extensive list of references on the calculation of the stability derivatives can be found in ref. 7.1. The calculation methods indicated in the literature often rely on empirical data obtained from similar airplane configurations. Calculation methods based entirely on theory result in most instances in insufficiently accurate results. Wind tunnel measurements remain necessary in later design stages to arrive at sufficiently accurate data. This applies in particular to the stability derivatives with respect to angle of sideslip, which have a predominant influence on lateral stability and the lateral control characteristics of the airplane.

7.2.1. The stability derivatives with respect to angle of sideslip

a. The derivative of the lateral force with respect to angle of sideslip, \( C_{\gamma} \)

Sidslipping motion generates an aerodynamic lateral force, usually negative at positive angles of sideslip:

\[ Y = C_{\gamma_\beta} \cdot \beta \cdot \frac{1}{2} \rho V^2 S \]  

(7-4)

where \( C_{\gamma_\beta} \) is negative. The dominant contributions to \( C_{\gamma_\beta} \) are caused by the fuselage and the vertical tailplane, see fig. 7.4. The contribution of the wing is generally negligible, unless the wing has a large angle of sweep. The propulsion system may also give a contribution to \( C_{\gamma_\beta} \) that is not negligible.

The small contribution of the wing may, if necessary, be obtained for low airspeeds using refs. 7.2 to 7.4, 7.8, 7.9 and 7.18.
Fig. 7.4: $C_Y$ as a function of $\beta$, measured on a model of the Fokker F-27.
(gliding flight). (From refs. 7.65, 7.66 and 7.67).

The influence of the compressibility of the air, can be accounted for using
ref. 7.24. Calculation methods applicable to supersonic speeds are given in
refs. 7.2 to 7.4, 7.32, 7.33 and in refs. 2.18 and 2.19.

The contribution of the fuselage arises in the same way as the normal force
on a symmetric fuselage at a non-zero angle of attack. Calculation methods can
be found in refs. 7.2 to 7.4, 7.32, 7.33 and in refs. 2.18 and 2.19.

The contribution of the vertical tailplane can, in analogy with the
gradient of the normal force in the horizontal tailplane in the symmetric case,
be written as:

$$
(C_Y)_v = C_Y \cdot \frac{d\alpha}{d\beta} \cdot \frac{V}{(V)} \cdot \frac{\alpha}{S} \cdot \frac{V}{S} \cdot \frac{S}{S}
$$

(7-5)

Compare this to the corresponding expression for the horizontal tailplane:
\[ \left[ C_{N_{\alpha}} \right] = C_{N_{h}} \cdot \frac{d \alpha_{h}}{d \alpha} \cdot \left( \frac{V}{V_{h}} \right)^{2} \cdot \frac{S_{h}}{S} \]

In (7-5) is \( C_{\alpha} \) the gradient of the normal force on the isolated vertical tailplane referenced to the area \( S_{v} \) of the vertical tailplane and the average local dynamic pressure \( \frac{1}{2} \rho V^{2} \) at the vertical tailplane; \( \alpha_{v} \) is the local angle of attack of the vertical tailplane, measured in the XOY-plane and counted as positive if the airflow comes from the left, see fig. 7.5.

Even when disregarding the sign, \( \alpha_{v} \) is generally not equal to \( \beta \). Due to the presence of the fuselage and the wing-fuselage interference a change in the component of the local velocity of the air parallel to the Y-axis occurs. The resulting change in the direction of the airflow is described using the sideways angle \( \sigma \), comparable to the downwash angle \( \varepsilon \) in symmetric flow. From fig. 7.5 follows:

\[ \alpha_{v} = -(\beta - \sigma) \]

Fig. 7.5: Relation between the angle of attack of sideslip \( \beta \), the sideways angle \( \sigma \), and the angle of attack \( \alpha_{v} \) of the vertical tailplane.
\[ \alpha_v = - (\beta - \sigma) \]  
(7-6)

The corresponding expression for the horizontal tailplane is:

\[ \alpha_h = \alpha - \varepsilon + i_h \]  
(3-24)

From (7-6) follows:

\[ \frac{d\alpha_v}{d\beta} = - \left(1 - \frac{d\sigma}{d\beta}\right) \]  
(7-7)

Substituting (7-7) in (7-5) results in:

\[ (c_{\gamma_v}) = - c_{\gamma_{v\alpha}} \cdot (1 - \frac{d\sigma}{d\beta}) \left(\frac{V}{V} \right)^2 \cdot \frac{S}{S} \]  
(7-8)

As a result of the sign convention for \( \alpha_v \), \( c_{\gamma_{v\alpha}} \) is positive like any normal force gradient. In the quantitative calculation of \( c_{\gamma_{v\alpha}} \), the presence of the horizontal tailplane, sometimes acting as an end plate to the vertical tailplane, has to be taken into account. Due to this effect the effective aspect ratio of the vertical tailplane may be considerably larger than the geometric aspect ratio.

The calculation of \( c_{\gamma_{v\alpha}} \) can be made for subsonic airspeeds using refs. 7.2, 7.3 and 7.46 to 7.51. For supersonic speeds refs. 7.55 to 7.58 provide calculation methods.

If the angle of attack is not too large, \( \left(\frac{V}{V} \right)^2 \) may usually be taken as 1, if the tailplane is not situated in a slipstream. In the contrary case, \( \left(\frac{V}{V} \right)^2 \) is determined in the same way as \( \left(\frac{V}{V} \right)^2 \) for the horizontal tailplane. The magnitude of \( \frac{d\sigma}{d\beta} \) will be discussed in more detail when dealing with \( c_{n_B} \), see page 22.

The propulsion system provides a negative contribution to \( c_{\gamma_{v\alpha}} \), caused by a lateral force in the propeller plane or in the plane of the engine inlet in the case of jet propulsion. This effect is a direct analogy with the normal force generated by a propeller or the engine air inlet if placed under a non-zero angle of attack. Calculation methods to determine this contribution are given in
refs. 7.59 and 7.60.

b. The derivative of the rolling moment with respect to angle of sideslip, $C_{\alpha\beta}$

The stability derivative $C_{\alpha\beta}$ is one of the primary lateral stability derivatives, usually indicated as the 'effective dihedral' of the airplane. The explanation of this name lies in the fact that for conventional airplanes without wing sweep, $C_{\alpha\beta}$ usually can be varied primarily by changing the geometric dihedral of the wing. To obtain desirable lateral control characteristics, it is required that the airplane possesses a negative $C_{\alpha\beta}$. This can be illustrated as follows. Suppose a disturbance causes an angle of roll to the right. Under the influence of the component of gravity along the Y-axis, the airplane starts sideslipping to the right. A negative $C_{\alpha\beta}$ then causes a rolling moment trying to return the airplane to an even keel, without interference by the pilot. In a qualitative sense this explains the desirability of a negative $C_{\alpha\beta}$.

Apart from the wing dihedral, wing sweep and the wing-fuselage interference have a large influence on $C_{\alpha\beta}$. The slipstream influence on $C_{\alpha\beta}$ of propeller driven airplanes can be considerable. The contributions of the fuselage and the tailplane are generally small.

The contribution of the wing-dihedral to $C_{\alpha\beta}$ is caused by the difference in geometric angle of attack of the two halves of the wing arising in sideslipping flight. From fig. 7.6b follows for the right wing without sweep and having not too large a dihedral:

$$V_n = w \cos \Gamma + v \sin \Gamma = \tilde{w} + v \cdot \Gamma$$

where for small values of $\alpha$ and $\beta$, see fig. 7.6a:

$$\tilde{w} \approx V \sin \alpha = V \cdot \alpha$$

and:

$$\tilde{v} \approx V \sin \beta = V \cdot \beta$$

resulting in:

$$V_n = V(\alpha + \beta \cdot \Gamma)$$

(7-9)
(a) Wing with dihedral in sideslipping flight.

(b) The normal velocities $V_{n_l}$ and $V_{n_r}$ at the two sides of the wing.

(c) The angles of attack at the left and right wing.

Fig. 7.6: The origin of the difference in angles of attack at the left and the right wing of a wing with dihedral in sideslipping flight.
The geometric angle of attack of the right wing then is, see fig. 7.6c:

\[ \alpha_{wr} = \arctan \frac{v}{u} = \alpha + \beta \Gamma \]  

(7-10)

In the same manner can be derived for the left wing:

\[ \alpha_{wl} = \alpha - \beta \Gamma \]  

(7-11)

and thus:

\[ \alpha_{wl} = \alpha + \beta \Gamma \]  

(7-12)

The increase in lift on the right wing due to \( \Delta \alpha_{wr} = +\beta \Gamma \) and the decrease in lift on the left wing due to \( \Delta \alpha_{wl} = -\beta \Gamma \) are thus proportional to the geometric dihedral \( \Gamma \).

From the above it can be concluded, that \( C_{\beta w} \) will also be approximately proportional to \( \Gamma \).

Swept wings experience in a sideslip an additional difference in lift between the two halves of the wing, independent of the dihedral. This difference in lift is caused by the difference in the components of airspeed perpendicular to the wing leading edge on the \( \frac{1}{4} \)-chord line.

It can be seen from fig. 7.7, that the difference in lift \( \Delta L \) can be roughly approximated as:

\[ \Delta L = C_L \cdot \frac{1}{4} \rho V^2 \cdot \frac{S}{2} \cdot \{ \cos^2 (\Delta - \beta) - \cos^2 (\Delta + \beta) \} \]  

(7-13)

Further analysis results for small values of \( \beta \) in:

\[ \Delta L = C_L \cdot \frac{1}{4} \rho V^2 S \cdot \sin 2\Delta \cdot \beta \]  

(7-14)

As the rolling moment of the wing (\( C_{\beta w} \)) \( \beta \) will be proportional to \( \Delta L \), to a first approximation, this simple calculation shows that \( C_{\beta w} \) will be approximately proportional to \( C_L \) and to \( \sin 2\Delta \), see fig. 7.8.
Fig. 7.7: The origin of the difference in the velocities over the left and right wing, of a swept wing in sideslipping flight.

Fig. 7.8: The variation of $C_{l_{\beta}}$ with $C_L$ of swept wings.
Flap deflection causes, at constant lift on the entire wing, a concentration of the lift at the center parts of the wing. Swept wings will have a less negative $C_{\beta_w}$ due to flap deflection. The same effect is obtained by applying negative wing twist to swept wings.

The influences of dihedral and sweep on $C_{\beta_w}$ are additive to a first approximation. In summary it is seen from the foregoing that for a swept wing the contribution ($C_{\beta_w}$) is composed of one part proportional to the wing dihedral $\Gamma$ and independent of $C_L$ and wing sweep $\Lambda$, and of one part independent of $\Gamma$ and proportional to $C_L$ and $\sin 2\Lambda$.

A consequence of the variation of $C_{\beta_w}$ with $C_L$ is, that swept wings often have a zero or even negative geometric dihedral, to avoid the unfavourable effects of too large and effective dihedral at large angles of attack. This will be further discussed when studying the dynamic lateral stability. Calculation methods to determine ($C_{\beta_w}$) are presented in refs. 7.2, 7.3, 7.8 and 7.11 to 7.13.

The contribution of the fuselage itself to $C_{\beta_w}$ is very small. The wing fuselage interference in sideslipping flight can cause significant differences in the angles of attack of the two halves of the wing. This again has a strong influence on the rolling moment due to side-slipping. Important to the way in which the rolling moment varies with angle of sideslip is the vertical position of the wing relative to the fuselage, see fig. 7.9. The forward half wing of a low wing airplane experiences a decrease in angle of attack due to the sideslip. This produces a decrease in lift, whereas the receding half wing experiences an increase in angle of attack and thus an increase in lift, due to the presence of the fuselage.

The situation is the reverse for a high wing airplane. A positive angle of sideslip causes for a low wing airplane an extra positive rolling moment, $C_{\beta_w}$ becomes less negative. On the other hand, the $C_{\beta_w}$ of a high wing airplane becomes more negative. The magnitude of the contribution is strongly influenced by the shape of the fuselage and by the ratio of its dimensions to those of the wing.

A detailed description of the influence of the wing-fuselage interference on $C_{\beta_w}$ is given in ref. 7.36. A determination of $C_{\beta_w}$ of the airplane without tailplanes is possible using refs. 7.8 yo 7.13 for subsonic speeds and refs. 7.25 to 7.29 for supersonic speeds.
Fig. 79: The origin of a rolling moment caused by the wing-tail interference in sideslipping flight.
The contribution of the vertical tailplane to \( C_{\beta}^{\lambda} \) is usually not large for conventional airplanes, but may not be neglected for airplanes having a relatively large vertical tailplane. The magnitude of \( C_{\beta}^{\lambda} \) follows from the magnitude and the position of the point of action of the lateral force \( C_{\beta}^{\lambda} \). In the calculation of \( C_{\beta}^{\lambda} \) the choice of the reference frame is important.

In the study of stability, the so-called stability axes are commonly employed, see 8.1. This is a system of airplane body axes, of which the X-axis in the plane of symmetry is chosen in the direction of the undisturbed flow. In this situation the rolling moment due to the side force on the vertical tailplane depends on the angle of attack of the airplane.

From fig. 7.10 follows for \( C_{\beta}^{\lambda} \):

\[
(C_{\beta}^{\lambda} v) = (C_{\beta}^{\lambda} v) \cdot \left( \frac{x_{v} - z_{c} \cdot \alpha}{b} \cos \alpha - \frac{x_{v} - z_{c} \cdot \alpha}{b} \sin \alpha \right) \quad (7-15)
\]

\( x_{v} \) and \( z_{v} \) can be determined using the same references as indicated for the determination of \( C_{v \alpha}^{\lambda} \).

The effective dihedral of a propeller driven airplane having a propeller in front of the wing can experience an important change due to the interference between the slipstream and the wing. Due to the increased dynamic pressure in the slipstream an extra rolling moment is generated in sideslipping flight, see fig. 7.11. At a positive \( \beta \) this rolling moment acts in the right wing down sense, thus giving a positive contribution to \( C_{\beta}^{\lambda} \). The contribution increases with \( T_{c} \) and \( C_{L} \). Deflection of the landing flaps will increase the effective angle of attack over the center part of the wing. It will further increase the effect of the slipstream on \( C_{\beta}^{\lambda} \).

A quantitative calculation of this phenomenon is as yet not possible.

c. The derivative of the yawing moment with respect to the angle of sideslip, \( C_{n \beta}^{\lambda} \)

The stability derivative \( C_{n \beta}^{\lambda} \) is called the static directional stability, as this derivative is geometrically directly comparable to the static longitudinal
Fig. 7.10: The position of the point of action of \((C_Y)_v\) relative to the X-axis and Z-axis in the stability reference point.
stability \( C_{m\alpha} \). For good control characteristics it is desirable that the angle of sideslip causes a moment about the top axis trying to reduce the angle of sideslip. The desired sign of \( C_{m\beta} \) is thus negative.

The magnitude of \( C_{m\beta} \) is determined by a small, positive contribution of the wing, an important destabilizing, negative contribution of the fuselage and a usually large stabilizing, positive contribution of the vertical tailplane, see fig. 7.12.

The contribution of the propulsion system may be either positive or negative.

The contribution to \( C_{m\beta} \) of a wing without sweep is very small.

The contribution of a swept back wing may be important especially at high \( C_L \) values, due to the sideforce acting relatively far behind the center of gravity. This wing contribution may be calculated using the same references as for the determination of the contribution of the wing to \( C_{\beta\text{v}} \).

The negative destabilizing contribution of the fuselage arises in the same way as the corresponding distribution of the fuselage to \( C_{m\alpha} \) in symmetric flight, see refs. 2.16, 2.18, 2.19 and refs. 7.32 to 7.35.

The contribution of the vertical tailplane to \( C_{m\beta} \), expressed in the system of stability axes, follows from 7.10:

\[
(C_{m\beta}) = -(C_{\beta\text{v}}) \cdot \left( \frac{x-x_{\text{c.g.}}}{b} \cos \alpha + \frac{z-z_{\text{c.g.}}}{b} \sin \alpha \right)
\]  

(7-16)

In this expression, \((C_{\beta\text{v}})\) is given by (7-8), see page 9.

At small angles of attack with \( x-x_{\text{c.g.}} = \alpha \) follows from (7-8) and (7-16):

\[
(C_{m\beta}) = C_{\beta\text{v}} \cdot \left( 1 - \frac{d\alpha}{dH} \right) \frac{V^2}{V} \cdot \left( \frac{S_{\text{v}} \alpha}{S \cdot b} \right)
\]  

(7-17)

The corresponding expression for the horizontal tailplane is:

\[
(C_{m\alpha}) = -C_{\alpha\text{h}} \cdot \left( 1 - \frac{d\alpha}{d\alpha} \right) \frac{V^2}{V} \cdot \left( \frac{S_{\text{h}} \alpha}{S \cdot c} \right)
\]  

(4-7)
(a) The part of the wing submerged in the slipstream.

(b) The change in the lift distribution.

Fig. 7.11: The contribution of the slipstream to $C_{l\beta}$. 
Fig. 7.12: $C_n$ as a function of $\beta$, measured on a model of the Fokker F-27. (gliding flight) (From refs. 7.55, 7.66 and 7.67).
The components of the flow velocities in a plane perpendicular to the fuselage axis.

The flow around the fuselage and the vertical tailplane.

Fig. 7.13: The sidewash induced by the fuselage at the vertical tailplane.
For simplicity reasons $l_v$ is often taken as the distance between the ½-chord points on the m.a.c.'s of the wing and the vertical tailplane.
Calculation methods to determine $C_{\alpha}^V$ have been mentioned already on page 9.

The calculation of the sideward gives rise to considerable difficulties in practice as the flow is strongly determined by the interference between flow around the fuselage, the wing and the tailplane. The flow around an isolated fuselage can be presented schematically as in fig. 7.13. A positive angle of sideslip causes a negative induced sideward, or $\frac{d\sigma}{d\beta} < 0$.

Earlier on page 14 the change in lift distribution in a sideslip due to the presence of the fuselage was discussed. The extra lift distribution caused by the interference also induces a cross-flow at the location of the vertical tailplane. The difference in lift on the two halves of the wing causes a difference between the downwash at either side of the fuselage, see fig. 7.14a and b.

The downwash pattern is displaced laterally along with the main flow, over an angle $\beta$ with the X-axis, see fig. 7.14b. The main effect, however, is a circulation about the fuselage. If a low wing airplane sideslips to the right, the circulation is counter-clockwise, see fig. 7.14c. This circulation generates at the vertical tailplane an extra negative sideward. This induced cross-flow is stabilizing, as can also be seen from (7-17): $\Delta \frac{d\sigma}{d\beta} < 0$.

For a high wing airplane in a sideslip to the right, the sideward is positive, the influence of the wing fuselage interference on $(C_{n\beta}^V)$ is thus seen to be destabilizing: $\Delta \frac{d\sigma}{d\beta} > 0$.

Fig. 7.15 shows the results of measurements of the sideward and the static directional stability illustrating the above. From this figure it can also be seen that the sideward is greatly influenced by the presence of the horizontal tailplane. A low horizontal tailplane acts as an end plate to the vertical tailplane. This reduces the sideward considerably. An accurate calculation of the average sideward $\sigma$, or $\frac{d\sigma}{d\beta}$, at the vertical tailplane is not very well possible, although refs. 7.39 to 7.42, 7.45 and 7.54 give calculation methods. In many cases results of systematic measurements on similar configurations are used to determine $\frac{d\sigma}{d\beta}$.

The contribution of the propulsion system to $C_{n\beta}^V$ is caused by the lateral force acting on the propeller or the engine inlet in cross-flow, see fig. 7.16. If the propeller or the engine inlet is situated forward of the center of gravity, this contribution is destabilizing. For propeller-driven airplanes the increased dynamic pressure in the slipstream causes a higher static directional stability, if the vertical tailplane is placed in the slipstream.
Fig. 7.14: The effect of the wing-fuselage interference on the sidewash at the vertical tailplane of a low-wing airplane in sideslipping flight.
Fig. 7.15: The effect of the wing-fuselage interference on the sidewash at the tailplane with and on the derivative $C_{n\beta}$.
(From ref. 7.53).
d. The stability derivative $C_{n_B}^*$

The stability derivative $C_{n_B}^*$ is a measure of the moment about the top axis caused during a change in the angle of sideslip, when the airplane executes an accelerated motion along the Y-axis. The derivative $C_{n_B}^*$ corresponds entirely to the stability derivative $C_{m^*}^*$ for the symmetric motions.

After a sudden change in the angle of sideslip a short interval passes before the changed sideward, caused by the wing-fuselage interference, has reached the vertical tailplane. As a consequence, the change of $C_n$ with $\beta$ does not occur as sudden as the change in $\beta$. This phenomenon is described using the stability derivative $C_{n_B}^*$.

For airplanes having a straight wing of sufficiently large aspect ratio (e.g. $A > 4$ to $5$), $C_{n_B}^*$ is usually neglected. Also in the discussion of the dynamic lateral stability, $C_{n_B}^*$ will not be considered. For airplanes having swept wings of low aspect ratio, $C_{n_B}^*$ has to be determined using the results of systematic measurements on oscillating models, see for instance ref. 7.44.
Fig. 7.17: The pitching moment as a function of the angle of sideslip.
(From ref. 7.52)
e. The pitching moment due to an angle of sideslip, $C_{m_{\beta}^2}$

A sideslip can cause a pitching moment that may be considerable, especially at large angles of sideslip, see fig. 7.17. It can be seen from this figure, as follows also from considerations of symmetry, that the change in pitching moment has the same sign for both positive and negative angles of sideslip. The stability derivative expressing the pitching moment due to a sideslip is, therefore, written as $C_{m_{\beta}^2}$.

This moment is caused by the wing-fuselage interference and the interference between the fuselage and the tailplanes already discussed in relation to $C_{n_{\beta}}$. Ref. 7.36 discusses the influence of the wing-fuselage interference on the pitching moment for combinations of a fuselage with a straight wing and a swept wing in a sideslip. Here also the vertical position of the wing relative to the fuselage is important. Especially at large angles of sideslip the change in pitching moment is determined to a large extent by the contribution of the horizontal tailplane, see ref. 7.52 and fig. 7.17. The tailplane is located in the downwash field behind the wing, modified by the wing-fuselage interference. If the tailplane is mounted low on the fuselage, the field of flow is modified additionally by the fuselage and the vertical tailplane. The influence of the vertical position of the horizontal tailplane of $C_m$ in a sideslip is presented in fig. 7.18. For a quantitative determination of $C_{m_{\beta}^2}$ the designer has to rely on wind tunnel measurements.

7.2.2. The stability derivatives with respect to rolling velocity

If the airplane has an angular velocity about the $X$-axis, the geometric angle of attack of the various wing and tailplane chords varies proportional to the rolling-velocity $p$ and to the distance of the chords to the $X$-axis, see fig. 7.19. For a wing chord at a distance $y$ from the plane of symmetry of the airplane, the change in the geometric angle of attack is:

$$\Delta\alpha = \frac{p \cdot y}{V} = \frac{pb}{2V} \cdot \frac{y}{b/2}$$  \hspace{1cm} (7-18)
Fig. 7.18: The influence of the vertical position of the horizontal tailplane on the variation of $C_m$ in sideslipping flight for a fuselage with tailplanes (From ref. 7.52).

Fig. 7.19: The variation of the local geometric angle of attack at the wing and the tailplane of a rolling airplane.
Here \( y \) is measured in the system of airplane body axes. For simplicity reasons the airplane c.g. is assumed to lie in the plane of symmetry. The non-dimensional rolling velocity \( \frac{pb}{2V} \) is thus equal to the change in geometric angle of attack at the wing tip, where \( y = \frac{b}{2} \). This is also the helix angle of the helix described by the wing tip in rolling flight. Herein lies the reason why the rolling velocity is made non-dimensional through multiplication by \( \frac{b}{2V} \).

*** a. The lateral force derivative with respect to rolling velocity, \( C_{\gamma}^{p} \)

The derivative \( C_{\gamma}^{p} \) is always relatively small and is very often neglected. Only if the wing has a large angle of sweep, or if a relatively large vertical tailplane is used, this derivative has to be taken into account.

The origin of the contribution from a swept wing \( C_{\gamma}^{p} \) will be explained when discussing \( (C_{n}^{p}) \). For swept back wings \( C_{\gamma}^{p} \) is negative.

The contribution of the vertical tailplane to \( C_{\gamma}^{p} \) is caused by the change in angle of attack experienced by the vertical tailplane in rolling flight. A positive rolling velocity gives rise to a negative lateral force:

\[
Y_{v} = (C_{\gamma}^{p}) \cdot \frac{pb}{2V} \cdot \frac{1}{4}pV^{2}S
\]

(7-19)

Herein \( (C_{\gamma}^{p}) \) is thus negative. In the determination of \( (C_{\gamma}^{p}) \) the sideways at the vertical tailplane, induced by the rolling velocity has to be taken into account. This subject will be further discussed when dealing with \( C_{\alpha}^{p} \). Calculation methods can be found in refs. 7.43, 7.51, 7.52, 7.57 and 7.58.

***
(a) The variation in geometric angle of attack.

(b) The variation in effective angle of attack.

Fig. 7.20: The variation of the local geometric and effective angles of attack along the span of a rolling wing.

---

Fig. 7.21: The damping roll as a function of aspect ratio and sweep at various taper ratio's.

(From ref. 7.4)
b. The rolling moment derivative with respect to rolling velocity, $C_{L_p}$

The stability derivative $C_{L_p}$ is a measure of the moment about the X-axis due to the rolling velocity about this axis:

$$L = C_{L_p} \cdot \frac{p_b}{2V} \cdot \frac{1}{4} \rho V^2 S_b$$

(7-20)

In all normal flight conditions this moment opposes the rolling velocity, trying to slow the motion down. $C_{L_p}$ is normally negative and a measure of the roll damping.

The dominant contribution to $C_{L_p}$ is generated by the wing. In some cases, the contribution from the tailplanes has to be taken into account as well. The contributions due to the fuselage, the wing-fuselage interference, and the propeller are negligible.

The contribution of the wing is caused by the change in geometric angle of attack and the resulting change in effective angle of attack along the wing span, see fig. 7.20. An extra anti-symmetric lift distribution is generated, causing a rolling moment opposite to the sense of the rotation. This has a damping effect on the rotation. Calculated values of $C_{L_p}$ for various wing planforms have been collected in fig. 7.21. It can be seen that the roll damping of the wing increases in the absolute sense with the taper ratio, $\lambda$, and the aspect ratio, $A$. The damping decreases with increasing wing sweep.

If the airflow over the wing remains attached, the wing contribution to $C_{L_p}$ is independent of $C_L$ and proportional to the lift gradient $C_{L_\alpha}$. If the lift gradient decreases at large angles of attack due to flow separation, $C_{L_p}$ decreases in the absolute sense as well. At very large angles of attack the down going wing may stall, causing a considerable reduction in wing damping. It is even possible that the influence of the loss in lift caused by the stall of the down going wing, dominates the influence of the decrease in lift due to smaller angle of attack of the up going wing. In that case $C_{L_p}$ is positive.

The lift distribution in rolling flight can be calculated using refs. 7.10 and 7.11. At subsonic speeds the wing contribution to $C_{L_p}$ can be determined with 7.2, 7.3, 7.9, 7.16, 7.17 and 7.19. Ref. 7.21 enables the wing contribution to be determined for angles of attack where the relation between $c_L$ and $\alpha$ of the profiles is non-linear. For supersonic speeds refs. 7.2, 7.3, 7.26, 7.27 and 7.30 are available.
a) The change in the lift distribution on a rolling wing.

b) The change in the downwash behind the rolling wing.

c) The flow at the horizontal and vertical tailplanes due to $p$ and $\Gamma$.

Fig. 7.22: The flow at the tailplanes of a rolling airplane.
The contributions of the horizontal and the vertical tailplane to $C_{p}$, caused by the same mechanism as the wing contribution, see fig. 7.19, is generally much smaller than the wing contribution. Due to the smaller span of the tailplanes, the maximum change in angle of attack is smaller than that of the wing. In addition, the rolling moment on the tailplanes is referenced to $S$ and $b$.

A further decrease in the change of angle of attack at the horizontal and vertical tailplanes is caused by the change in downwash behind the rolling wing. Behind the downgoing wing the downwash increases, it decreases behind the upgoing wing, relative to non-rolling flight, see fig. 7.22. This extra downwash distribution causes a circulation as depicted in fig. 7.22c. This decreases the change in angle of attack along the span of the tailplanes. A further study of this phenomenon can be found in ref. 7.43. For conventional, subsonic airplanes the contributions of the tailplanes to $C_{p}$ can usually be neglected. Modern fighter airplanes and V/STOL-airplanes usually have relatively large tailplanes. Calculation of the tailplane contributions to $C_{p}$ for such airplanes is possible using refs. 7.50, 7.51 and 7.56 to 7.58.

c. The yawing moment derivative with respect to rolling velocity, $C_{n}$

This derivative is determined mainly by the contribution of the wing. Only a relatively large vertical tailplane produces a non-negligible contribution to $C_{n}$. Under normal circumstances $C_{n}$ is negative.

The generation of the wing contribution to $C_{n}$ is illustrated in fig. 7.23. In rolling flight an extra force in X-direction acts on each section of the wing. Since the local angle of attack varies along the wing span, it is essential to study the distribution of this extra force in the stability reference frame. The axes of the reference frame have a fixed direction relative to the airplane. In this reference frame the down going wing experiences an extra forward force due to the increased angle of attack. The up going wing experiences an extra longitudinal force pointing backwards. As a result, a positive rolling velocity causes a negative yawing moment:

$$N = (C_{n})_{p} \cdot \frac{pb}{2V} \cdot \frac{4}{pV^2} \cdot Sb$$  \hspace{1cm} (7-21)

where $(C_{n})_{p} < 0$. 

\[\text{(7-21)}\]
(a) The forces $\Delta C_X$ due to $\Delta \alpha$ of a rolling wing and the resultant yawing moment.

(b) Illustration of the change in the forces in $X_S$- and $Z_S$ directions on the two sides of a rolling wing.

Fig. 7.23: The origin of a negative yawing moment about the $Z_S$-axis of the stability reference frame, for a wing having a positive rate of roll (attached flow).
If flow separation occurs on the down going wing, the drag increases considerably, causing the extra force in $X$-direction to become negative. This renders $C_{n_{p_w}}$ less negative or even positive. This applies also to wings having a sharp leading edge. Due to the absence of the suction peak on the profile nose, the resultant aerodynamic force and the change in the resultant aerodynamic force act here approximately perpendicular to the wing chord. From fig. 7.23b it can be seen, that for normal wings the extra force in $X_s$-direction on the down going wing is positive (pointing forwards) and the extra force on the up going wing acts in the direction of the negative $X_s$-direction. For normal wings $C_{n_{p_w}}$ apparently is always negative.

***

For swept wings the change in resultant aerodynamic force acting on a section of the rolling wing has a component in the $Y$-direction, see fig. 7.24. This lateral force has the same direction on both wing halves. This causes a resultant lateral force $C_{n_{p_w}} \frac{pb}{2V}$. This force generates a second, less important contribution to $C_{n_{p_w}}$, dependent on the c.g. position.

![Fig 7.24: The side force and yawing moment on a rolling, sweptback wing](image-url)
Calculation methods to determine \( C_{y_p} \) and \( C_{n_p} \) are given for subsonic \( p_w \) speeds in refs. 7.2, 7.3, 7.8, 7.9 and 7.20. For supersonic speeds refs. 7.26 and 7.27 apply.

The contribution of the vertical tailplane to \( C_{n_p} \) follows at once from \( C_{y_p} \) and the position of the point of action of \( C_{y_p} \) using an expression in analogy with (7-16). The position of the point of action \( C_{y_p} \) can be determined using the same references as given for the determination of the magnitude of \( C_{y_p} \).

***

### 7.2.3. The stability derivatives with respect to yawing velocity

Before going into a discussion of the derivatives with respect to yawing velocity, \( C_{y_r} \), \( C_{l_r} \) and \( C_{n_r} \) each separately, the motion to be considered needs a further description. The notion of a so-called 'r-motion' is introduced here, in analogy with the q-motion, see 6.3. When performing such an r-motion, the airplane moves along a curved trajectory in the XOY-plane, such that the velocity vector of the center of gravity remains in the plane of symmetry. By definition the angle of sideslip of the airplane remains zero during an r-motion, see fig. 7.25. Apparently, the r-motion causes merely a curvature of the streamlines in the XOY-plane. At the position of the c.g. the radius of curvature is \( R \):

\[
R = \frac{V}{r}
\]

The center of rotation is situated at a distance \( R \) from the c.g. on the positive Y-axis if \( r \) is positive.

The effect of this r-motion is twofold. In the first place, and in analogy with the change in angle of attack caused by the q-motion, a change in the flow direction, measured in planes parallel to the XOY-plane, occurs at all points of the airplane, see fig. 7.26:

\[
\Delta x = \frac{rb}{2V} \cdot \frac{x - x_{c.g.}}{b/2}
\]  

(7-22)

As a consequence the airflow meets the vertical tailplane from the left if \( r \) is positive.
In the second place, the local speed of the airflow changes at all points of the airplane by an amount, see fig. 7.27:

\[ \frac{\Delta V}{V} = -\frac{rb}{2V} \cdot \frac{y}{b/2} \]  \hspace{1cm} (7-23)

As in (7-18), \( y \) is measured here in the stability reference frame. At a positive \( r \), the right wing has a lower airspeed and the left wing a larger airspeed than the airplane center of gravity. Similar differences in airspeed occur in the plane of symmetry during the \( q \)-motion, but they are negligible due to the relatively small dimensions of the airplane in \( Z \)-direction.

According to (7-23), \( \frac{rb}{2V} \) is the non-dimensional change in airspeed occurring at the wing tip. The choice of the factor \( \frac{b}{2V} \) to render the angular velocity non-dimensional thus appears to have a geometric interpretation.

***  a. The lateral force derivative with respect to yawing velocity, \( C_{Y_r} \)

This derivative is usually of small importance. The dominant contribution to \( C_{Y_r} \) is due to the vertical tailplane, \( C_{Y_r} \) is commonly positive.

The contribution of the wing is very small. Calculation methods are given in refs. 7.2, 7.3, 7.8 and 7.9 for subsonic speeds and in refs. 7.26 and 7.27 for supersonic speeds. The contributions of the fuselage and the wing-fuselage interference are negligible, as is the contribution of the horizontal tailplane.

The lateral force on the vertical tailplane due to yawing can be written as:

\[ Y_v = (C_{Y_r}) \cdot \frac{rb}{2V} \cdot \frac{\rho V^2 S}{V} \]  \hspace{1cm} (7-24)

The lateral force is caused by the change in angle of attack \( \Delta \alpha_v \) of the vertical tailplane, see fig. 7.28. Accordingly:

\[ Y_v = C_{\alpha_v} \cdot \frac{rb}{2V} \cdot \frac{\rho V^2 S}{V} \]  \hspace{1cm} (7-25)

\( \Delta \alpha_v \) follows from (7-22) with \( x_v - x_{c.g.} = l_v \):

\[ \Delta \alpha_v = \frac{rb}{2V} \cdot \frac{l_v}{b/2} \]  \hspace{1cm} (7-26)
Fig. 7.25: The pure "r-motion".

\[ R = \frac{V}{r} \]

Fig. 7.26: The change in geometric flow direction at an arbitrary point of the airplane due to an r-motion.

\[ \Delta \alpha = \frac{r b}{2V} \cdot \frac{x - x_{cg}}{b/2} \]
Fig. 7.27: The variation in airspeed in spanwise direction due to an $r$-motion.

$$\frac{\Delta V}{V} = \frac{r b}{2V} \cdot \frac{y}{b/2}$$

Fig. 7.28: The side forces on the vertical tail plane and the propeller due to an $r$-motion.
From (7-24), (7-25) and (7-26) follows after some reduction:

\[ C_{Y_r} = 2 C_{Y_v} \left( \frac{V}{V} \right)^2 \frac{S_{v}}{S} \frac{\ell}{b} \]  \hspace{1cm} (7-27)

From this expression it follows that \( C_{Y_r} \) is positive.

The values of \( C_{Y_r} \) calculated using (7-27) may not be very accurate due to the unknown influence of the wing-fuselage in the curved flow field on the flow direction at the vertical tailplane. For these reasons it is commonly necessary to use the results of wind tunnel measurements on similar configurations to determine \( C_{Y_r} \).

During the \( r \)-motion the propeller experiences on oblique inflow also causing a contribution to \( C_{Y_r} \), see fig. 7.28. In analogy with (7-27), \( C_{Y_r} \) can be written as:

\[ C_{Y_r} = 2 C_{Y_p} \left( \frac{P}{V} \right)^2 \frac{S_{v}}{S} \frac{x_{p} - x_{c.g.}}{b} \]  \hspace{1cm} (7-28)

\( C_{Y_r} \) is usually negative. \( C_{Y_p} \) and \( \left( \frac{P}{V} \right) \) can be determined using refs. 7.59 and 7.60.

b. The rolling moment derivative with respect to yawing velocity, \( C_{\phi_r} \)

The derivative \( C_{\phi_r} \) is determined primarily by the contributions of the wing and the vertical tailplane. The contributions of the fuselage, the wing-fuselage interference, the horizontal tailplane and the propulsion system may usually be neglected. \( C_{\phi_r} \) is always positive.

The contribution of the wing arises due to the differences in airspeed over the two wing halves during the \( r \)-motion, as described previously. At a positive yawing velocity, i.e. a rotation to the right, the left wing generates more lift than the right wing, causing a right wing down, positive, rolling moment:

\[ L = (C_{\phi_r}) \frac{rb}{2V} \frac{1}{2} \rho V^2 S_b \]  \hspace{1cm} (7-29)
This rolling moment, and thus \( (C_{\lambda r}) \), is proportional to the lift coefficient \( \frac{C_{L}}{w} \) and at a given \( C_{\lambda} \), dependent on the lift distribution over the wing span. Tapered wings show a higher lift concentration near the center of the wing, causing an increase in speed over the outer wing to have less effect. As a consequence, \( (C_{\lambda r}) \) decreases with \( \lambda \) at constant \( C_{L} \). The same applies to wings having negative twist and to wings with deflected landing flaps. Swept back wings possess a higher lift concentration on the outer wings. This causes \( (C_{\lambda r}) \) to increase somewhat with \( \lambda \). \( (C_{\lambda r}) \) can be calculated using refs. 7.2, 7.3, 7.8, 7.9 and 7.22 for subsonic speeds and refs. 7.26 and 7.27 for supersonic speeds.

The contribution of the vertical tailplane to \( C_{\lambda r} \) is found from \( (C_{\lambda v}) \) and the position of the point of action of \( (C_{\lambda v}) \):

\[
(C_{\lambda r}) = (C_{\lambda v}) \left( \frac{z - z_{c.g.}}{b} \cos \alpha - \frac{x - x_{c.g.}}{b} \sin \alpha \right) \]  

(7-30)

As \( z_{c.g.} \) is usually positive, \( (C_{\lambda r}) \) is usually positive as well.

c. The yawing moment derivative with respect to yawing velocity, \( C_{n r} \)

\( C_{n r} \) is like \( C_{m q} \) and \( C_{\lambda p} \), a measure of the aerodynamic moment about an axis caused by an angular velocity about the same axis. Under normal circumstances such a moment tries to slow down the motion, which implies that \( C_{n r} \) is normally negative, again like \( C_{m q} \) and \( C_{\lambda p} \). The dominant contribution to \( C_{n r} \) is provided by the vertical tailplane. Additionally, only the wing, the fuselage and the propulsion system deliver non-negligible contributions.

The part provided by the wing is generated by the differences in drag between the two halves of the wing, caused by the differences in local airspeed. Calculation methods for subsonic speeds are found in refs. 7.2, 7.3, 7.8 and 7.9 and in refs. 7.26 and 7.27 for supersonic speeds.

The contribution coming from the fuselage is small, but it may be non-negligible for modern configurations having a relatively large fuselage in relation to the wing. A calculation is possible using ref. 7.33, see also ref. 7.5.

The contribution of the vertical tailplane follows directly from
see (7-27) and the taillength of the airplane, see also fig. 7.28:

\[
(C_{n_r}) = -(C_{Y_r}) \cdot \frac{L_v}{b}
\]  \hspace{1cm} (7-31)

For the determination of \((C_{n_r})\) in the stability reference frame, at large angles of attack, use has to be made of an expression in analogy with (7-16).

Fig. 7.29 gives \(C_{n_r}\) of a model for different values of \(\frac{L_v}{b}\) and \(\frac{S_v}{\bar{S}}\).

The contribution of the propeller to \(C_{n_r}\) follows in a similar way from \((C_{Y_r})\). If the propeller is situated in front of the c.g., \((C_{Y_r})\) is negative. The damping moment is then increased by the propeller, see fig. 7.28. The increased dynamic pressure in the slipstream may additionally cause an increase in the contribution of the vertical tailplane.

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**Fig. 7.29**: The effects of the size of the vertical tailplane and the taillength on \(C_{n_r}\).

(From ref. 7.5)
7.3. The forces and moments due to aileron deflection, rudder deflection and the application of spoilers

For lateral control the capability to generate variable moments about the X-axis and the Z-axis is required. The moment about the X-axis is usually obtained by deflecting ailerons, at high airspeeds the use of spoilers is also quite common. The moment about the Z-axis is derived from rudder deflection.

The deflection of the ailerons or the spoilers generates in addition to the rolling moment L also a lateral force Y and a yawing moment N. Equally, rudder deflection causes not only a yawing moment, but also a lateral force and a rolling moment.

As is the case with the lateral motions of the entire airplane, no symmetric forces and moments are generated in principle.

Measures of the non-dimensional force $C_Y$ and the non-dimensional moments $C_\lambda$ and $C_n$ due to the control surface deflections are again the partial derivatives with respect to aileron deflection $\delta_a$ and the rudder deflection $\delta_r$. These derivatives, called the control derivatives, are:

a. for aileron deflection:

$$ C_{Y_\delta_a} = \frac{\partial C_Y}{\partial \delta_a}, \quad C_{\lambda_\delta_a} = \frac{\partial C_\lambda}{\partial \delta_a}, \quad C_{n_\delta_a} = \frac{\partial C_n}{\partial \delta_a} $$

b. for rudder deflection:

$$ C_{Y_{\delta_r}} = \frac{\partial C_Y}{\partial \delta_r}, \quad C_{\lambda_{\delta_r}} = \frac{\partial C_\lambda}{\partial \delta_r}, \quad C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r} $$

When using the control deflections as measures for the forces and moments, the assumption is made, in analogy with the stability derivatives, that the forces and moments vary linearly with $\delta_a$ and $\delta_r$. For not too large control surface deflections, this assumption is usually quite acceptable. Table 7.1 on page 86 provides, in addition to the stability derivatives, also the control derivatives of some airplanes.

Fig. 7.30 shows once more the positive directions of the control surface and control deflections. The control surface deflections are always measured in a plane perpendicular to the hinge axis of the surface. The deflection $\delta_a$ is defined as:
\[ \delta_a = \delta_{a_{\text{right}}} - \delta_{a_{\text{left}}} \] (7-32)

A positive deflection of the control stick or a positive deflection of the control wheel (to the left) can be seen to produce a positive \( \delta_a \). Equally, a positive deflection of the rudder pedals (left pedal forward) causes a positive \( \delta_r \).

If spoilers are used, derivatives with respect to spoiler deflection \( \delta_s \) are defined, in analogy with the above control deflections:

\[ C_{Y_{\delta_s}} = \frac{\partial C_Y}{\partial \delta_s}, \quad C_{L_{\delta_s}} = \frac{\partial C_L}{\partial \delta_s}, \quad C_{n_{\delta_s}} = \frac{\partial C_n}{\partial \delta_s} \]

---

**Fig. 7.30**: The positive direction of control deflections, control forces, control surface deflections and hinge moments.
A spoiler deflection on the left wing is taken as positive. A positive $\delta_s$ corresponds with a positive deflection of the roll control manipulator. When using the derivatives with respect to $\delta_s$, great care is required, as for many spoiler configurations the forces and moments vary in a strongly non-linear fashion with the spoiler deflection.

The control derivatives are discussed in the following. The use of spoilers for roll control is discussed in 7.3.2. Finally, in 7.3.4 the hinge moments and control forces required for the deflection of the control surfaces are considered.

7.3.1. The derivatives with respect to aileron deflection

a. The lateral force derivatives with respect to aileron angle, $C_{y\delta_a}$

The lateral force caused by an aileron deflection may be taken as zero for straight wings. On swept wings a lateral force does arise as is the case in rolling flight. The derivative $C_{y\delta_a}$ is very small in the absolute sense and is usually neglected.

b. The rolling moment derivative with respect to aileron angle, $C_{\lambda\delta_a}$

This is, of course, the primary control derivative for the ailerons. Due to a positive aileron deflection the lift on the right wing increases, on the left wing the lift decreases. The result is a rolling moment to the left, which is counted as negative, see fig. 7.31:

$$L = C_{\lambda\delta_a} \cdot \delta_a \cdot \frac{1}{2} \rho V^2 S_b$$  \hspace{1cm} (7-33)

$C_{\lambda\delta_a}$ is negative. This control derivative is also called the aileron effectiveness.

If: a. the aeroelastic deformation of the wing is neglected,
   b. the influence of the Mach number is not considered,
   c. no flow separation on the wing in the area of the ailerons occurs,
then $C_{\lambda\delta_a}$ may be considered as independent of $C_L$. 
The magnitude of $C_{L_{\delta_a}}$ depends strongly on the dimensions and the location of the ailerons relative to the wing. Wing sweep and taper also have an important influence on the aileron effectiveness. Under the assumption of constant location and dimension of the ailerons, $C_{L_{\delta_a}}$ decreases in the absolute sense with increasing wing sweep and taper $(\text{lower } \lambda)$, especially if the wing aspect ratio is large.

The calculation of $C_{L_{\delta_a}}$ is possible, using refs. 7.2, 7.3, 7.6, 7.11 and 7.31. For supersonic speeds $C_{L_{\delta_a}}$ can be determined with ref. 7.31. In many instances, measured values of $C_{L_{\delta_a}}$ obtained from flight tests turn out to be smaller in the absolute sense than was expected on the basis of calculations.

c. The yawing moment derivative with respect to aileron angle, $C_{n_{\delta_a}}$

The yawing due to aileron deflection arises, because the drag of the wing having the downward deflected aileron increases, while that of the other side decreases, or perhaps increases to a lesser extent.

As a consequence, a positive aileron deflection causes a positive yawing moment:
\[ N = C_{n_{\delta_a}} \cdot \delta_a \cdot \dot{\rho} V^2 S_b \]  

(7-34)

where \( C_{n_{\delta_a}} \) is positive. To a first approximation \( C_{n_{\delta_a}} \) is proportional to \( C_L \).

Too large a value of \( C_{n_{\delta_a}} \) is undesirable. If, for instance, the pilot wants to initiate a left turn, he deflects the roll control (stick or wheel) to the left to obtain the required negative \( \varphi \) (left wing down). While the airplane must obtain a negative rate of yaw (to the left), the positive aileron deflection generates a positive yawing moment, because \( C_{n_{\delta_a}} \) is positive. This yawing moment causes the airplane to yaw initially to the right. This effect is amplified by the usually positive yawing moment caused by the negative rolling velocity of the airplane, \( C_{n_p} \) was seen to be negative. This initial yaw opposite to the desired direction is usually called the 'adverse yaw' of the airplane.

\[ \delta_\alpha < 0 \]

\[ \delta_\alpha = 0 \]

\[ \delta_\alpha > 0 \]

Fig. 7.32: The Frise-aileron.
C\textsubscript{n5} can be kept small in various ways. One of the means is the application of differential deflections of the ailerons when applying roll control. This implies that the downward deflection of the aileron is made smaller than the upward deflection of the other aileron. The difference in drag is reduced in this way. An other method uses so called Frise ailerons. The shape of the nose of these ailerons is such that the drag of the aileron strongly increases if it is deflected upward, see fig. 7.32 and ref. 7.6.

A calculation of C\textsuperscript{n5} having some accuracy is generally not possible. As a consequence it is always desirable to determine C\textsuperscript{n5} on the basis of experimentally obtained data, see ref. 7.3.

7.3.2. The application of spoilers for roll control

At transonic and supersonic airspeeds roll control of the airplane can often not be effected by ailerons alone. Roll control of such airplanes is obtained by the use of spoilers, or by a combination of spolers and ailerons. The latter solution is the more usual one.

A spoiler is in principle a flap, to be deflected from the upper surface on one side of the wing, to 'spoil' the airflow over that part of the wing, see fig. 7.33a. The local disturbance causes a decrease in lift and thereby generates a rolling moment. A deflection of the spoiler on the left wing is counted as positive. This causes, like an upward deflection of the left aileron, a negative rolling moment, hence C\textsubscript{\lambda5} is negative like C\textsubscript{\lambda5}.

Spoilers are more effective if they are located closer to the wing leading edge. The response of the airplane to spoiler deflection becomes slightly more sluggish in this way. In practice spoilers are located mostly on the rear part of the wing chord.

Various causes make the use of spoilers on high speed airplanes more or less mandatory. In the first place, aileron effectivity decreases due to compressibility effects with increasing Mach number. On the other hand, the effectiveness of spoilers slightly increases with growing Mach number.

A second, even more important phenomenon is the decrease in aileron effectivity due to the wing deformation caused by aileron deflection. Fig. 7.34 shows how a downward deflection of the ailerond causes a torsion of the wing, such that the local angle of attack decreases. This counteracts the increase in lift due to the aileron deflection. Conversely, an upward deflection of the aileron causes an increase of the local angle of attack.
Fig. 7.33: Two types of spoilers.

(a) plate type spoiler.

(b) plugtype spoiler.

Fig. 7.34: The local torsion of a wing cross-section due to the elastic deformation caused by aileron deflection.
The decrease in aileron effectiveness caused in this way may be appreciable especially at high airspeeds, both due to the high dynamic pressure and to the rearward shift of the point of action of the local change in lift caused by the aileron deflection with increasing Mach number. In this way the resultant effect of an aileron deflection on highly swept wings having thin profiles may even be in the sense opposite to the desired one. The effect is called 'reversal of control' and the airspeed from which onward this occurs is the 'reversal speed'. The point of action of the resultant change in lift for spoilers lies more forward on the chord than for ailerons. The wing torsion is thus less and as a result the reversal speed is higher.

Finally and quite apart from the foregoing reasons, it is often not very well possible to use large ailerons on wings having a high sweep angle or a low aspect ratio. The trailing edges of such wings are taken up largely by landing flaps to obtain an acceptably low landing speed.

Spoiler deflection causes not only a reduction in lift. It causes also a strong increase in a drag. A secondary advantage of the use of spoilers is, that due to this drag increase, $C_{n_{\delta_s}}$ is nearly always negative, contrary to $C_{n_{\delta_a}}$. The adverse yaw at the initiation of a turn is thereby decreased or even entirely avoided.

The increase in drag created by the deployment of the spoilers may cause the designer to use the spoilers not only for roll control, but as speed brakes as well. The spoilers are deflected symmetrically for this purpose. Symmetric deflection of spoilers situated in front of the landing flaps, when applied directly after touch down, provides the possibility to drastically decrease the wing lift during the roll out. A larger part of the weight comes to rest on the undercarriage, making the wheel brakes more effective. This may lead to an appreciable reduction in the stopping distance.

A disadvantage of spoilers is their low effectivity at small deflections, caused by re-attachment of the airflow behind the spoiler. So-called 'plug type spoilers', see fig. 7.33b, have better characteristics in this respect, due to the open connection between lower and upper surface of the wing caused by the spoiler deflection. Re-attachment is avoided in this way. At large angles of attack the effectivity of the spoiler may also be low, especially if flow separation occurs at the trailing edge of the wing. A further disadvantage during the approach and landing phases is the decrease in total wing lift caused by the operation of the spoilers. This renders the precise control of the airplane's flight path slightly more difficult.

The hinge moment of a plain flap type spoiler, see fig. 7.33a, may be
appreciable. It is usually a non-linear function of the deflection angle. On the other hand, the hinge moment of a spoiler in the shape of a half circular plate, see fig. 7.33b, may be next to negligible. But the total aerodynamic load on a spoiler of this type may be relatively high.

$C_A^\delta_s$ and $C_n^\delta_s$ can be determined using ref. 7.3. Ref. 7.62 gives theoretical calculation methods to determine $C_A^\delta_s$, while ref. 7.63 contains an extensive list of references on the use of spoilers.

7.3.3. The derivatives with respect to rudder deflection

a. The lateral force derivative with respect to rudder angle, $C_Y^\delta_r$

The rudder deflection causes a lateral force:

$$Y = C_Y^\delta_r \cdot \delta_r \cdot \frac{1}{2} p v^2 S \quad (7-35)$$

A positive $\delta_r$ gives rise to a positive lateral force, see fig. 7.35. $C_Y^\delta_r$ is positive. The lateral force can be written as:

$$Y = C_Y^\delta_r \cdot \delta_r \cdot \frac{1}{2} p v^2 S \quad (7-36)$$

where $C_Y^\delta_v$ is the gradient of the normal — here lateral — force on the vertical tailplane. From (7-35) and (7-36) follows:

$$C_Y^\delta_r = C_Y^\delta_v \left( \frac{v}{v} \right) \cdot \frac{S_v}{S} \quad (7-37)$$

The derivative $C_Y^\delta_v$ is determined in a similar way as the derivative $C_{N_h}^\delta_v$ of the horizontal tailplane, see refs. 7.6, 7.14, 7.15 and 7.52. For supersonic speeds this derivative can be calculated with ref. 7.31.
b. The rolling moment derivative with respect to rudder angle, $C_{\alpha \delta_r}$

A positive lateral force acting on the vertical tailplane, caused by a positive rudder deflection, causes a positive rolling moment:

$$L = C_{\alpha \delta_r} \cdot \delta_r \cdot \frac{1}{2} \rho V^2 S_b$$  \hspace{1cm} (7-38)

where $C_{\alpha \delta_r}$ is positive. In analogy with the contributions of the vertical tailplane ($C_{\alpha \beta_v}$, $C_{\alpha p_v}$ and $C_{\alpha r_v}$), $C_{\alpha \delta_r}$ can be written as:

$$C_{\alpha \delta_r} = C_{\alpha \delta_r} \cdot \left( \frac{z - c_s \cdot \alpha}{b} \cos \alpha_0 - \frac{x - c_s \cdot \alpha}{b} \sin \alpha_0 \right)$$  \hspace{1cm} (7-39)
Large positive values of \( C_{\delta r} \) could be detrimental to good lateral control characteristics. In flight, when accurate lateral control is required, small deviations from the desired heading may be corrected by using rudder deflections, without recourse to the ailerons. This may give a result more quickly than changing the airplane's heading via a change in angle of roll generated by the ailerons. If in such a situation a positive rudder deflection is used to yaw the airplane to the left, a large positive \( C_{\delta r} \) causes an angle of roll to the right. Both such a roll angle is associated with a turn to the right and opposes the desired yawing motion of the airplane.

The effect just mentioned occurs mainly with airplanes having a highly mounted vertical tailplane, where \( z_v - z_{c.g.} \) in (7-39) is large. In these airplanes the rudder and the ailerons may be coupled mechanically such that a positive aileron deflection is generated by a positive rudder deflection. The moment \( C_{\delta a} \) then opposes the moment \( C_{\delta r} \).

c. The yawing moment derivative with respect to the rudder angle, \( C_{n_{\delta r}} \)

The primary effect of a rudder deflection is the yawing moment:

\[
N = C_{n_{\delta r}} \cdot \delta_r \cdot \frac{1}{2} \rho V^2 S_b
\]

(7-40)

\( C_{n_{\delta r}} \) follows from fig. 7.35 at small \( \alpha \) and taking \( x_v - x_{c.g.} = l_v \):

\[
C_{n_{\delta r}} = -C_{V_{\delta r}} \cdot \frac{l_v}{b}
\]

(7-41)

At large angles of attack, \( C_{n_{\delta r}} \) follows from an expression in analogy with (7-16) for \( C_{\nu_{\delta}} \). The control derivative \( C_{n_{\delta r}} \) is negative.

7.3.4. The control forces and hinge moments for lateral control

a. Roll control

Using the same argument as in Chapter 3 for the elevator control, the following expression applies to roll control in analogy with (3-36):
\[ F_a \, ds_a + H_{a_r} \, d\delta_{a_r} + H_{a_\lambda} \, d\delta_{a_\lambda} = 0 \]  

or:

\[ F_a = - \left( \frac{d\delta_{a_r}}{ds_a} \cdot H_{a_r} + \frac{d\delta_{a_\lambda}}{ds_a} \cdot H_{a_\lambda} \right) \]  

The hinge moments \( H_{a_r} \) and \( H_{a_\lambda} \) can be written in the usual way as:

\[ H_{a_r} = C_{h_{a_r}} \cdot \frac{1}{2} \rho V^2 S_{a_r} \, \bar{c}_{a_r} \]

\[ H_{a_\lambda} = C_{h_{a_\lambda}} \cdot \frac{1}{2} \rho V^2 S_{a_\lambda} \, \bar{c}_{a_\lambda} \]

If \( S_{a_r} = S_{a_\lambda} = S_a \) (the area of one single aileron) and:

\[ \bar{c}_{a_r} = \bar{c}_{a_\lambda} = \bar{c}_a \]

the aileron control force in (7-43) becomes:

\[ F_a = - \left( \frac{d\delta_{a_r}}{ds_a} \cdot C_{h_{a_r}} + \frac{d\delta_{a_\lambda}}{ds_a} \cdot C_{h_{a_\lambda}} \right) \cdot \frac{1}{2} \rho V^2 S_a \, \bar{c}_a \]  

(7-44)

If no differential deflection of the ailerons is used, this expression can be further developed in a straightforward manner. Using (7-32):

\[ \delta_{a_r} = - \delta_{a_\lambda} = \frac{\delta_a}{2} \]

or:

\[ \frac{d\delta_{a_r}}{ds_a} = - \frac{d\delta_{a_\lambda}}{ds_a} = \frac{1}{4} \frac{d\delta_a}{ds_a} \]

Substitution in (7-44):
\[ F_a = - \frac{d^2 s}{d a^2} \rho v S \frac{1}{a} \left( C_{h_r} - C_{h_{\lambda}} \right) \]  

where:

\[ C_{h_r} = C_{h_{o_r}} + C_{h_\alpha} \cdot \alpha_r + C_{h_\delta} \cdot \delta_r + C_{h_{\delta_t}} \cdot \delta_{\tau_r} \]

and:

\[ C_{h_{\lambda}} = C_{h_{o_{\lambda}}} + C_{h_\alpha} \cdot \alpha_{\lambda} + C_{h_\delta} \cdot \delta_{\lambda} + C_{h_{\delta_t}} \cdot \delta_{\tau_{\lambda}} \]

In asymmetric flight is \( \alpha_r \neq \alpha_{\lambda} \). Suppose now:

\[ \alpha_r = \alpha + \Delta \alpha \]

Then:

\[ \alpha_{\lambda} = \dot{\alpha} - \Delta \alpha \]

The result is:

\[ \alpha_r - \alpha_{\lambda} = 2 \Delta \alpha \]

In addition, is:

\[ \delta_r - \delta_{\lambda} = \delta \]

\[ \delta_{\tau_r} - \delta_{\tau_{\lambda}} = \delta_{\tau} \]

\[ C_{h_{o_r}} - C_{h_{o_{\lambda}}} = 0 \]

and thus:

\[ C_{h_r} - C_{h_{\lambda}} = C_{h_\alpha} \cdot 2 \Delta \alpha + C_{h_\delta} \cdot \delta + C_{h_{\delta_t}} \cdot \delta_{\tau} \]  

(7-46)
Substitution of (7-46) in (7-45) finally results in:

\[
F_a = - \frac{d\delta}{ds_a} \cdot \frac{1}{2} pV^2 S_a \text{c}_a \left( C_{h_a} \cdot \Delta \alpha_a + C_{h_\delta} \cdot \frac{\delta_a}{2} + C_{h_{\delta_r}} \cdot \frac{\delta_{\delta_r}}{2} \right) \tag{7-47}
\]

The expression between parentheses is \( C_{h_a} \) of a single aileron.

b. Rudder control

The applicable expression for the control force on the rudder pedals is entirely in analogy with (3-36) for the elevator control:

\[
F_r = - \frac{d\delta}{ds_r} \cdot \frac{1}{2} pV^2 S_r \text{c}_r \left( C_{h_\alpha} \cdot \alpha_v + C_{h_\delta} \cdot \delta_r + C_{h_{\delta_r}} \cdot \delta_{\delta_r} \right) \tag{7-48}
\]

Calculation methods for the hinge moment derivatives in (7-47) and (7-48) have been mentioned in 3.4.

7.4. The equilibrium and lateral control in steady, asymmetric flight

Using the equilibrium equations for asymmetric flight, the lateral control characteristics in some important steady, asymmetric flight conditions will be discussed in the following. To simplify the discussion only horizontal flight will be considered. In a condition of steady flight, where by definition the roll angle is constant, the rolling velocity about the X-axis is zero, only if the X-axis - along the c.g. velocity vector - lies in the horizontal plane, see (8-21) of 8.1.

From the above follows, that in steady, asymmetric horizontal flight the rolling velocity is always zero, and also that horizontal flight cannot be exactly steady. The 'steady' rolling flight to be considered in 7.4.5, is an approximated, quasi-steady flight condition giving in a simple manner insight in some important requirements on roll control for all airplanes.

7.4.1. The equations of equilibrium

In the most general case of a horizontal, steady, asymmetric condition of flight, the airplane sideslips and yaws at constant \( \beta \) and \( \frac{rb}{2V} \). The angle of roll
\( \phi \) of the airplane differs from zero but the rolling velocity is zero. To maintain this flight condition, generally the ailerons as well as the rudder are deflected.

On the airplane act not only the aerodynamic force \( Y \) along the \( Y \)-axis and the aerodynamic moments \( L \) and \( N \) about the \( X \)- and \( Z \)-axes respectively, but also the component of the weight along the \( Y \)-axis. Since the pitch angle \( \theta \), measured in the stability reference frame, is zero in horizontal flight, the component of the weight along the \( Y \)-axis is:

\[
W \sin \phi
\]

The combined forces along the \( Y \)-axis cause a centripetal acceleration in the \( Y \)-direction: \( v_r \), see fig. 7.36. The resultant equation for the forces along the \( Y \)-axis, both in the airplane and the stability reference frame, then is:

\[
W \sin \phi + Y = m V r
\]

(7-49)

Fig. 7.36 The forces along the \( Y \)-axis of an airplane in steady, horizontal, asymmetric flight.
whereas the equilibrium of the moments about the X- and Z-axes is expressed by:

\[ L = 0 \]  \hspace{1cm} (7-50)

\[ N = 0 \]  \hspace{1cm} (7-51)

The equations are made non-dimensional by dividing (7-49) by \( \frac{1}{2} \rho V^2 S \), and both (7-50) and (7-51) by \( \frac{1}{2} \rho V^2 S_b \). This results for small values of \( \psi \) in:

\[ C_L \cdot \psi - 4 \mu_b \frac{rb}{2V} + C_Y = 0 \]

\[ C_L = 0 \]  \hspace{1cm} (7-52)

\[ C_n = 0 \]

where:

\[ C_L = \frac{W}{\frac{1}{2} \rho V^2 S} \]

whereas:

\[ \mu_b = \frac{m}{\rho S_b} \]

is the non-dimensional mass parameter for the asymmetric motions, see 8.1. page 118.

If the aerodynamic forces and moments in (7-52) are now expressed in the contributions arising from \( \beta, \frac{rb}{2V}, \delta_a \) and \( \delta_r \), according to 7.2 and 7.3, the resulting asymmetric equilibrium equations for horizontal steady asymmetric flight become:

\[ C_L \cdot \psi + C_Y \cdot \beta + (C_{\beta r} - 4 \mu_b) \cdot \frac{rb}{2V} + C_{Y_a} \cdot \delta_a + C_{Y_r} \cdot \delta_r = 0 \]

\[ C_{L \beta} \cdot \beta + C_{L r} \cdot \frac{rb}{2V} + C_{L a} \cdot \delta_a + C_{L r} \cdot \delta_r = 0 \]  \hspace{1cm} (7-53)

\[ C_n \cdot \beta + C_{n_r} \cdot \frac{rb}{2V} + C_{n_a} \cdot \delta_a + C_{n_r} \cdot \delta_r = 0 \]
Based on the discussions in the previous paragraphs, \( C_{\delta a} \) and \( C_{\delta r} \) will be neglected in the following. In a first approximation used to study the various types of steady flight, \( C_{\delta a} \), \( C_{\delta r} \) and \( C_{\gamma} \) will also be dropped. In the thus simplified form the equilibrium equation reads as:

\[
C_L \Phi + C_{\gamma} \beta - 4\mu_b \cdot \frac{rb}{2V} = 0 \quad (7-54)
\]

\[
C_{\alpha} \beta + C_{\alpha} \frac{rb}{2V} + C_{\delta a} \delta_a = 0 \quad (7-55)
\]

\[
C_{n\beta} \beta + C_{n\beta} \frac{rb}{2V} + C_{n\delta r} \delta_r = 0 \quad (7-56)
\]

Or, using matrices:

\[
\begin{bmatrix}
C_L & C_{\gamma} & -4\mu_b & 0 & 0 \\
0 & C_{\alpha} & C_{\alpha} & C_{\delta a} & 0 \\
0 & C_{n\beta} & C_{n\beta} & 0 & C_{n\delta r} \\
\end{bmatrix} \begin{bmatrix}
\Phi \\
\beta \\
\frac{rb}{2V} \\
\delta_a \\
\delta_r \\
\end{bmatrix} = 0
\]

In 7.4.2 to 7.4.4 the lateral control characteristics of the airplane in various steady, asymmetric flight conditions are studied, using the above equations. Because of the simplifications introduced, the considerations are primarily qualitative in nature, especially for larger deviations from symmetric flight.

7.4.2. Steady, horizontal turns

In the three equations (7-54) to (7-56) five variables occur. This implies, that steady turns at a given airspeed and rate of yaw \( \frac{rb}{2V} \) can be flown in principle at infinitely many combinations of the remaining four variables \( \delta_r \), \( \delta_a \), \( \beta \) and \( \Phi \). Only if one of these four variables has been fixed, the remaining three can be expressed as functions of \( \frac{rb}{2V} \).
Fig. 7.37: Steady turns on ailerons only. North American "Harvard II B," gliding flight. $C_L = 0.31$, $x_{cg} = 0.304\bar{x}$, $V = 78$ m/sec.
(From ref. 7.64)
In the following steady turns are studied in which each of the four variables mentioned are assumed separately to be equal to zero.

a. Turns, using the ailerons only \((\delta_r = 0)\)

Using (7-55) and (7-56) and the condition \(\delta_r = 0\), it follows that the variation of the aileron angles with rate of yaw can be expressed as:

\[
\frac{d\delta}{dr} = \frac{1}{C_{\alpha_{\delta_a}}} \frac{C_{\alpha} \cdot C_{n_r} - C_{\alpha_r} \cdot C_{n_{\beta}}}{C_{n_{\beta}}}
\]

(7-57)

Here \(C_{\alpha_{\delta_a}} < 0\) and \(C_{n_{\beta}} > 0\).

To initiate a turn to the right using the ailerons only, a negative aileron deflection must be given, to obtain a positive rolling velocity and a positive angle of roll. Using the same arguments as in chapter 4, it is desirable in principle, that in the ultimate steady flight condition the aileron control remains deflected in the direction of the initial control deflection. It is thus desirable that 

\[
\frac{d\delta}{dr} < 0.
\]

According to (7-57) the necessary condition to satisfy this requirement is:

\[
C_{\alpha} \cdot C_{n_r} - C_{\alpha_r} \cdot C_{n_{\beta}} > 0
\]

(7-58)

This latter condition corresponds to the condition for spiral stability which is discussed in 8.5 on page 191.

If \(\delta_r = 0\) in the expressions (7-54) and (7-56), the variation of angle of roll with rate of yaw can be derived as:

\[
\frac{d\varphi}{dr} = \frac{C_{n_r}}{C_L} > 0 \quad \text{for } C_L > 0
\]

(7-59)

and also:

\[
\frac{d\beta}{dr} = -\frac{C_{n_r}}{C_{n_{\beta}}} > 0 \quad \text{for } C_{n_r} < 0, \ C_{n_{\beta}} > 0
\]

(7-60)
Fig 7.38: Steady turns on rudder only. North American "Harvard II B", gliding flight. $C_L = 0.31$, $x_{cg} = 0.3045$, $V = 78$ m/sec.
(From ref. 7.64)
In a turn to the right, at $\delta_\alpha = 0$, the airplane has a roll angle to the right ($\varphi > 0$) and the sideslip is towards the inside of the turn ($\beta > 0$). At practical values of $\varphi$ and $\frac{rb}{2V}$ the angle of sideslip usually remains limited to a few degrees. Fig. 7.37 shows the results of some measurements made in steady flight, using the ailerons only.

b. Turns, using the rudder only ($\delta_\alpha = 0$)

Using (7-55) and (7-56) and the condition $\delta_\alpha = 0$, the variation of the rudder angle with rate of yaw is obtained as:

$$\frac{d\delta_r}{dr} = \frac{C_{\lambda\beta}}{C_{n\delta_r}} (C_{\lambda\beta} \cdot C_{n_r} - C_{\lambda r} \cdot C_{n_\beta})$$  \hspace{1cm} (7-61)

Here $C_{n\delta_r} < 0$ and $C_{\lambda\beta} < 0$. It then follows that if the condition (7-58) for spiral stability is satisfied, in a turn to the right the rudder is also deflected to the right ($\delta_r < 0$). In analogy with (7-59) and (7-60) the following expressions for the variations of angle of roll and angle of sideslip hold:

$$\frac{d\varphi}{dr} = \frac{4\mu_b + C_{Y_b} \cdot \frac{C_{\lambda r}}{C_{\lambda\beta}}}{C_L} > 0 \quad \text{(for } C_L > 0)$$ \hspace{1cm} (7-62)

and:

$$\frac{d\beta}{dr} = -\frac{C_{\lambda r}}{C_{\lambda\beta}} > 0 \quad \text{(} C_{\lambda r} > 0, C_{\lambda\beta} < 0)$$ \hspace{1cm} (7-63)

In a turn to the right, using the rudder only, the airplane assumes again a positive angle of roll (to the right) and a positive, usually very small, angle of sideslip. Fig. 7.38 shows the results of some measurements made in steady flight, using the rudder only. As in fig. 7.37, the control surface deflection and the angle of sideslip remain very small, even at angles of roll of ± 40°. As a consequence, the influence of the limited accuracy of the measurements and possible non-linearities is large enough to obscure a clear relation between the slopes of some measured curves and the expressions derived in this paragraph, such as (7-63).
c. Coordinated turns \((\beta = 0)\)

When flying a steady turn, the pilot's aim is in principle to keep the angle of sideslip zero. Then the drag is at a minimum. In addition, the coordinated turn is the most comfortable turn for the passengers.

If \(\beta = 0\), it follows from (7-54) that:

\[
C_{L'}\phi = 4\mu_b \frac{rb}{2v}
\]  

(7-64)

This means that in the coordinated turn the component along the Y-axis of the centripetal acceleration is caused only by the lateral component of the airplane weight. This is true, also for all objects in the airplane. They do not experience a sideward force during the coordinated turn. As a consequence the pilot can also feel if he flies a well coordinated turn.

The variation of the roll angle with rate of yaw follows from (7-64):

\[
\frac{\frac{d\phi}{dt}}{\frac{rb}{2V}} = \frac{4\mu_b}{C_L} > 0 \quad (C_L > 0)
\]  

(7-65)

The required control surface deflections follow from (7-55) and (7-56) with \(\beta = 0\):

\[
\frac{d\delta_a}{d\frac{rb}{2V}} = -\frac{C_{\delta_a}}{C_{\delta_a}} > 0 \quad (C_{\delta_a} > 0 , C_{\delta_a} < 0)
\]  

(7-66)

and:

\[
\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{C_{\delta_r}}{C_{\delta_r}} < 0 \quad (C_{\delta_r} < 0 , C_{\delta_r} < 0)
\]  

(7-67)

In a coordinated turn to the right the airplane has an angle of roll to the right. The rudder is deflected in the desirable negative direction (to the right). But the aileron is deflected to the left, in the direction opposite to that of the initial control deflection. This must be considered in principle a less desirable lateral characteristic which is inevitable in normal airplanes. Fig. 7.39 shows the results of measurements made in steady, coordinated turns. The aileron angle turns out to be very nearly constant.
Quantitative calculations show very simply that the control surface deflections in each of the three types of steady turns just described remain usually small, see also figs. 7.37 to 7.39. It follows then that these flight conditions are not at all critical for the dimensioning of the control surfaces.

In cruising flight and usually also in the approach the differences between the coordinated turn \((\beta = 0)\) and a turn using the ailerons only \((\delta_r = 0)\) is so small as to make it attractive to the pilot to perform the lateral control using the roll control only. If the initiation of a turn is done not too quickly, the adverse yaw of a conventional airplane will be small enough to render the use of the rudder unnecessary. Obviously, if the rudder need not be employed, the control of the airplane is that much easier. This may improve the accuracy of the control of the airplane.

\textbf{d. Flat turns (}\(\psi = 0\))

From (7-54) follows, if \(\psi = 0\):

\[
\frac{d\beta}{drb} = \frac{1}{c_{\gamma} \beta} \cdot 4\mu_b < 0 \quad (C_{\gamma \beta} < 0) \quad (7-68)
\]

Using (7-68) it is simple to derive that:

\[
\frac{d\delta_a}{drb} > 0 \quad \text{and} \quad \frac{d\delta_r}{drb} < 0
\]

In a flat turn to the right, the rudder angle is negative, just as in a turn to the right using the rudder only. In a flat turn the airplane must perform a sideslip to the outside of the turn \((\beta < 0\) if \(r > 0\)) in order to obtain the lateral force required for the centripetal acceleration. Due to the combined sideslipping and yawing motions, a positive rolling moment (to the right) is generated. This requires a positive aileron deflection (to the left) for equilibrium about the X-axis. Fig. 7.40 shows the results of some measurements made in steady, flat turns.
Fig. 7.39: Steady, coordinated turns. North American "Harvard II B", gliding flight. $C_L = 0.31$, $x_{c.g.} = 0.304 \, \xi$, $V = 78 \, \text{m/sec}$.
(From ref. 7.64.)
7.4.3. Steady, straight, sideslipping flight

In general the pilot will try to avoid sideslipping in flight. The intentional use of steady, straight, sideslipping flight is limited to the landing with cross wind and to the approach in flapless airplanes in order to control the flight path angle. A further discussion of steady, straight sideslipping flight is justified, however, because the airplane may enter into a sideslip even without the pilot's intention.

Using the lateral control characteristics in steady, straight, sideslips, it is simple to verify if the effective dihedral, \( C_{\gamma^\beta} \), and the static dimensional stability, \( C_{n^\beta} \), have the correct sign.

These two stability derivatives have a strong influence on the lateral control characteristics in general. They are also two important parameters in obtaining good lateral stability characteristics.

The equilibrium equations for the steady, straight sideslip follow directly from the expressions (7-54) to (7-56), by letting \( \frac{rb}{2V} = 0 \). Neglecting small contributions, the equilibrium equations then become:

\[
\begin{bmatrix}
C_L & C_{\gamma^\beta} & 0 & 0 \\
0 & C_{\gamma^\beta} & C_{\delta a} & 0 \\
0 & C_{n^\beta} & 0 & C_{n\delta r} \\
\end{bmatrix}
\begin{bmatrix}
\varphi \\
\beta \\
\delta_a \\
\delta_r \\
\end{bmatrix} = 0
\]

(7-69)

The variation of the angle of roll with \( \beta \) follows from the first row of (7-69):

\[
\frac{d\varphi}{d\beta} = -\frac{C_{\gamma^\beta}}{C_L} > 0 \quad (C_{\gamma^\beta} < 0, C_L > 0)
\]

(7-70)

Apparently, a positive angle of sideslip is associated with a positive angle of roll. A given angle of sideslip creates a large lateral force \( C_{\gamma^\beta} \beta \), which has to be balanced in steady flight by a component of the weight along the Y-axis.
Fig. 7.40: Steady flat turns, North American "Harvard II B", gliding flight, $C_L=0.31$, $x_{c.g.}=0.304\,\text{ft}$, $V=78\,\text{m/sec}$.
(From ref. 7.64.)
If the absolute value of $C_{Y}^{\beta}$ is large, the equilibrium at a given angle of side-slip necessitates a large roll angle, as can be seen from (7-70).

A suddenly occurring angle of sideslip, caused for instance by a gust, gives also rise to a force along the $Y$-axis and hence to a sideward acceleration. The pilot senses this acceleration as a sideward force exerted upon him by his seat and seat belts. If, again $C_{Y}^{\beta}$ is large in the absolute sense, this force will be relatively large, making it easier for the pilot to detect the presence of the sideslipping motion.

These arguments lead to the conclusion that a large negative value of $C_{Y}^{\beta}$ may be considered as desirable.

To initiate a straight sideslip with a positive $\beta$ (to the right) the airplane has to be given a positive rolling moment (to the right) and a negative yawing moment (to the left). This requires an initial aileron deflection to the right (negative) and an initial rudder deflection to the left (positive).

It is then desirable that in the ultimate steady flight condition the aileron control is also deflected to the right and the rudder control to the left or:

\[
\frac{d\delta_a}{d\beta} < 0 \quad \text{and} \quad \frac{d\delta_r}{d\beta} > 0
\]

The required surface deflections follow from (7-69):

\[
\frac{d\delta_a}{d\beta} = -\frac{C_{\delta a}^{\beta}}{C_{\delta a}^{\beta}}
\]

and:

\[
\frac{d\delta_r}{d\beta} = -\frac{C_{\delta r}^{\beta}}{C_{\delta r}^{\beta}}
\]

As both $C_{\delta a}^{\beta}$ and $C_{\delta r}^{\beta}$ are negative, the required directions of the control $\delta_a$ and $\delta_r$ deflections in steady, straight sideslips are obtained, if:

\[
C_{\delta a}^{\beta} < 0 \quad \text{and} \quad C_{\delta r}^{\beta} > 0
\]
Fig 7.41: Steady, straight, sideslipping flight, North American "Harvard II B", gliding flight. $C_L = 0.31, x_{c.g.} = 0.304\ell, V = 78$ m/sec.

(From ref. 7.64.)
It will be shown in 8.5 that the latter conditions have to be satisfied also to obtain lateral dynamic stability characteristics. Measurement of the control deflections in steady, straight sideslips offers a simple method to determine if these conditions are satisfied. Fig. 7.41 shows some results of measurements, made in steady, straight sideslips.

As a final remark it can be said that the center of gravity position has an influence on the magnitude $C_{n_p}$, in analogy with the influence on $C_{m_{\alpha}}$. As a consequence, the c.g. position has also some influence on the lateral stability control characteristics. The importance of this influence is, however, generally much less than is the case for the longitudinal stability and control characteristics.

***

7.4.4. Steady, straight flight with one or more engines inoperative

For multi-engined airplanes the requirement exists that the airplane can continue flight if one engine becomes inoperative. This imposes requirements on the controllability and usually in particular on the available rudder power.

The most important consequences of the loss of the engine thrust from an engine not mounted in the plane of symmetry are a yawing moment due to the asymmetric thrust distribution and, in the case of a propeller-driven airplane, a rolling moment caused by the resultant asymmetric lift distribution. The yawing moment can be written in non-dimensional form as:

$$C_{n_e} = k \cdot \frac{\Delta T_p}{\frac{1}{2} \rho V^2 S_b} \cdot y_e$$

(7-73)

Here $\Delta T_p$ is the difference in thrust from the operative engine and the drag of the inoperative engine; $y_e$ is the distance from the line of action of the thrust or drag to the plane of symmetry of the airplane. $\Delta T_p$ may be considerably larger than the thrust of the operative engine if the inoperative engine is windmilling in the airstream. In the case of a propeller-driven airplane, $\Delta T_p$ is minimal if the inoperative propeller is feathered, whereby the rotation is stopped.

The factor $k$ in (7-73), which for propeller-driven airplanes may assume values of 1.5 to 2.0 in some instances, expresses the fact that the yawing moment actually occurring may be larger than would result from the changes in thrust only.
Fig. 7.42: The magnitude of $C_{n_e}$ due to an engine failure, as a function of the location and the sense of rotation of the propeller and the vertical position of the wing, measured on a model of a twin-engined propeller-driven airplane. (From ref. 7.61).
Fig. 7.42 shows some results of measurements of $C_{n_e}^n$ confirming this fact. The explanation of this effect lies in the increased lift of the part of the wing submerged in the slipstream. This increase in lift creates an extra downwash behind the wing which is now asymmetric. The downwash behind the operative propeller is larger than the downwash behind the inoperative propeller. In analogy with the airflow behind a high or low mounted wing in sideslipping flight, the differences in downwash create a circulation around the rear fuselage, thereby causing a cross-flow at the vertical tailplane. It is not difficult to verify that this cross-flow or sidewash causes an increase in the disturbing yawing moment.

The experimental evidence in fig. 7.42 shows that the direction of rotation of the propeller and the vertical position of the wing relative to the fuselage have an important influence on the magnitude of $C_{n_e}^n$. Ref. 7.62 discusses this matter in detail. In general it can be concluded, that if the propellers turn clockwise, the loss of the left engine causes the largest disturbing yawing moment $C_{n_e}^n$. This engine is then called the critical engine.

The rolling moment $C_{\lambda_e}^\lambda$ due to an inoperative engine is caused mainly by the difference in lift on both sides of the wing, behind the operative and the inoperative engine. Some results of measurements on $C_{\lambda_e}^\lambda$ are shown in fig. 7.43.

If the right engine cuts out, both a positive $C_{n_e}^n$ and a positive $C_{\lambda_e}^\lambda$ arise. Due to $C_{n_e}^n$ the airplane assumes a positive rate of yaw leading to a negative $\beta$. The rolling moment $C_{\lambda_e}^\lambda$ causes a positive rate of roll and thus a positive roll angle $\phi$. The rolling motion is amplified by the contributions $(C_{\lambda_e}^\lambda \cdot \beta (> 0)$ and $C_{\lambda_e}^\lambda \cdot \frac{rb}{2V} (> 0)$ due to the sideslipping and yawing motions. As a consequence, the roll angle may increase very rapidly.

The motion is counteracted by the yawing moments generated by $p$, $r$ and $\beta$ $(C_{n_p}^n < 0$, $C_{n_r}^r < 0$ and $C_{n_\beta}^\beta > 0)$, and by the application of both a positive rudder deflection and a positive aileron deflection. A new condition of steady, straight flight may now be established, described by the following equilibrium equation in which a few minor contributions have been neglected:
Fig. 7.43: The magnitude of $C_{le}$ due to an engine failure, as a function of the location and the sense of rotation of the propeller and the vertical position of the wing; measured on a model of a twin-engined propeller-driven airplane. (From ref. 7.61).
\[
\begin{bmatrix}
C_L & C_{Y\beta} & 0 & C_{Y\delta_r} \\
0 & C_{\theta\beta} & C_{\lambda\delta_a} & 0 \\
0 & C_{n\beta} & 0 & C_{n\delta_r}
\end{bmatrix}
\begin{bmatrix}
\varphi \\
\beta \\
\delta_a \\
\delta_r
\end{bmatrix}
=
\begin{bmatrix}
0 \\
C_{\lambda_e} \\
C_{\delta_r}
\end{bmatrix}
\tag{7-74}
\]

In flight with asymmetrically distributed engine thrust very large rudder deflections may be required. As a consequence the lateral force caused by \( \delta_r \) cannot be omitted in (7-74).

As the four variables \( \varphi, \beta, \delta_a \) and \( \delta_r \) occur in the equilibrium equations (7-74) there is not just one, single steady flight condition possible, but a whole range of conditions.

In practice two of the many possible conditions have special significance: steady flight at a roll angle equal to zero and the condition at which the angle of sideslip \( \beta \) is equal to zero. If \( \beta = 0 \) the airplane drag is at a minimum, resulting in maximum flight performance.

If the right engine is inoperative, a positive rudder deflection (to the left) is required to balance \( C_{n_e} \). In the flight condition at \( \varphi = 0 \), a positive angle of sideslip is necessary according to the top row of (7-74), to obtain an aerodynamic lateral force to balance the lateral force \( C_{Y_r} \cdot \delta_r \), see fig. 7.44.

The aircraft performs a sideslipping motion in the direction of the inoperative engine.

In the flight condition at \( \beta = 0 \) the pilot must apply a negative roll angle \( \varphi \), assuming again that the right engine is stopped, such that the component of the weight \( C_L \cdot \varphi \) balances the lateral force \( C_{Y_r} \cdot \delta_r \). The wing with the operative engine has to be kept low, see fig. 7.45. In this situation the rudder deflection required for equilibrium is smaller than in the flight condition at \( \varphi = 0 \), where not only \( C_{n_e} \) but also the moment \( C_{n\beta} \cdot \beta \) acting in the same direction have to be balanced, see the bottom row in (7-74). The flight condition at \( \varphi = 0 \) is, therefore, more critical for the determination of the required rudder power than the flight condition at \( \beta = 0 \).

Contrary to the moments \( C_{n\delta_r} \cdot \delta_r \) and \( C_{n\beta} \cdot \beta \), the disturbing yawing moment \( C_{n_e} \) depends strongly on airspeed. Taking the case of the propeller-driven airplane, the engine power is constant at constant throttle setting.
Fig. 7.44: Steady, straight, single-engined flight at $\phi = 0$.

Fig. 7.45: Steady, straight, single-engined flight at $\beta = 0$. 
To a first approximation this means:

\[ T_p V = \text{constant} \]

From (7-73) then follows, that \( C_{n_e} \) in that case is inversely proportional to \( V^3 \). There will be an airspeed below which it is not possible to balance the yawing moment due to an engine failure, \( C_{n_e} \), see fig. 7.46. This airspeed is called the 'minimum control speed', \( V_{\text{m.c.}} \). When determining \( V_{\text{m.c.}} \) in flight, the regulations stipulate that \( |\phi| \) may not be larger than 5°.

High engine power and low airspeeds, and consequently large values of \( C_{n_e} \) at engine failure, occur during take-off and in the climb immediately after take-off. In the take-off configuration at high \( C_L \)-values, the extra interference yawing moment, expressed by the factor \( k \) in (7-73), is also large. An additional factor is the fact that during take-off the airplane is not allowed to lose height or to obtain large angles of roll.

The importance of the minimum control speed as an element in the choice of the take-off procedure to be followed, depends strongly on the type and location of the engine. Jet-propelled airplanes, having the engines mounted on the rear fuselage, often have a minimum control speed lower than the minimum airspeed for sustained flight.

---

**Fig 7.46**: The minimum control speed.
Fig 7.47: Steady straight sideslipping flight with the right propeller feathered
Fokker F-27, h = 1850 m, C_L = 1.92, x_c.g = 0.28 C, V = 45.3 m/sec
(From ref 431)
For such airplanes the minimum control speed loses its significance. Handling
the airplane on the ground with failed engine may then be an important consider-
ation. For twin-engined, propeller-driven airplanes, however, the minimum
control speed in the take-off configuration is of extreme importance. Fig. 7.47
shows some results of measurements on the Fokker F-27 with the right propeller
feathered.

7.4.5. Steady rolling flight

One of the measures for the manoeuvrability of an airplane is the rate of
roll obtained by giving a certain aileron deflection. The time required to
establish a steady turn, is determined largely by this rate of roll. This is the
reason why requirements exist for the attainable rates of roll. In many cases
this fixes the required aileron power.

If the ailerons are deflected and then maintained in their deflected
position, the airplane will experience at first an angular acceleration about
the X-axis. When the growing damping moment \( C_{\alpha_p} \cdot \frac{pb}{2V} \) has become equal to the
rolling moment \( C_{\alpha} \cdot \delta_a \), the rate of roll will remain constant, if the simplify-
ing assumption is made that during this rolling motion the airplane will neither
sidelslip nor yaw. The resulting flight condition is that of steady rolling
flight. It is possible to demonstrate, that this 'steady' condition is usually
established relatively quickly. This makes it sensible to assess the
manoeuvrability of an airplane by means of the attainable rate of roll.

Under the influence of for instance the component of the weight \( C_L \cdot \phi \) and
the yawing moments due to \( \delta_a \) and \( p \), the airplane will not only roll but also
sidelslip and yaw. This means, that continuous rudder control would be required
to make the airplane perform a rolling flight at constant rate of roll, without
any sideslipping and yawing motions.

A real exactly steady rolling flight is thus not possible. The 'steady'
rolling flight to be discussed below is, therefore, a quasi-steady flight con-
dition in straight flight (\( r = 0 \)) at constant rudder angle (\( \delta_r = 0 \)), at small
angles of roll \( \varphi \) and to the neglect of the angle of sideslip actually
occurring (\( \beta = 0 \)).

For this idealized flight condition the equation for the equilibrium of the
aerodynamic moments about the X-axis is:

\[
C_{\alpha_p} \cdot \frac{pb}{2V} \cdot C_{\alpha} \cdot \delta_a = 0 \tag{7-75}
\]
or:

\[
\frac{p_b}{2V} = -\frac{C_{\lambda p}}{C_{\lambda a}} \cdot \delta_a \tag{7-76}
\]

where both \(C_{\lambda p}\) and \(C_{\lambda a}\) are negative. A positive aileron deflection is thus seen to cause a negative rate of roll.

In the discussion of \(C_{\lambda p}\) and \(C_{\lambda a}\), it was shown that these two derivatives are to a first approximation independent of \(C_L\) or the airspeed. According to (7-76) the rate of roll at a given aileron angle is thus proportional to airspeed, but the non-dimensional rate of roll is independent of \(V\). From early investigations, see ref. 7.7, it appeared that the pilot's assessment of the manoeuvrability of the then considered airplanes, showed a better correlation with \((p_b/2V)_{\text{max}}\) than with \(p_{\text{max}}\). On this basis the requirements on manoeuvrability were expressed using the non-dimensional rate of roll.

As an example, older U.S. military Regulations, see ref. 1.13, required for a fighter airplane:

\[
(p_b/2V)_{\text{max}} > 0.09
\]

and for the other categories of airplanes:

\[
(p_b/2V)_{\text{max}} > 0.07
\]

Since the time when the requirements were formulated the range of possible airspeeds has increased so much, that the non-dimensional rate of roll, \(p_b/2V\), as the only criterion for the manoeuvrability of an airplane about the X-axis, has lost some of its significance.

For V/STOL airplanes capable to operate at very low airspeeds, the rate of roll \(p\) resulting from the above requirements on \(p_b/2V\) would be too low at low values of \(V\), to insure good manoeuvrability. For such airplanes the additional requirement is sometimes used, that in the approach \((p_b/2V)_{\text{max}}\), which can be seen as the maximum vertical speed of the wing tip due to rolling, must be at least 10 ft/sec.
Fig. 7.48: The limitation in the maximal attainable rate of roll due to elastic wing deformation.

Fig. 7.49: Variation of the torsion angle $\varphi$ along the wing span due to aileron deflection, for two locations of the ailerons.
On the other hand, the rolling velocity at supersonic airspeed of a fighter airplane having a small wing span according to the requirement on $\frac{p_b}{2V}$, would be unreasonably large. At those high rates of roll, difficulties with the dynamic stability might occur, as at high rolling velocities the symmetric and the asymmetric airplane motions at high rates of roll may become quite violently unstable. In later U.S. military Regulations, see ref. 1.17, the requirement is given that fighter airplanes in the combat configuration must be capable to attain an angle of roll $\phi = 50^\circ$ within one second. For other airplane configurations and flight conditions, as well as for other categories of airplanes, requirements on non-dimensional rate of roll are maintained.

It was discussed in 7.3.2 that due the elastic deformation of the wing, $C_{\alpha \delta a}$ will decrease in absolute value at high airspeed, or rather at high dynamic pressures. Due to this effect, the rate of roll no longer increases linearly with airspeed at a constant aileron deflection, see fig. 7.48. At high airspeeds the rate of roll will even decrease. The reversal speed is reached when $C_{\alpha \delta a} = 0$ and as a consequence also $p = 0$.

The elastic wing twist may be reduced by locating the ailerons not at the wing tips, but more inboard. The sketch in fig. 7.49 shows that this may considerably reduce the total twist at the wing tip, thereby reducing the loss in rolling moment generated by aileron deflection.

If the ailerons are located inboard, the outer part of the wing, generally possessing less torsional stiffness, will experience no further increase in wing twist. In cases were no satisfactory roll control can be obtained in this way, it may be necessary to use spoilers for roll control, as discussed in 7.3.2.

The maximum achievable rate of roll $p$ may be limited at high airspeeds by the maximum possible roll control force, rather than by the maximum aileron deflection. If no hydraulic servos are applied and if no differential aileron deflections are used, the roll control force can be written according to (7-47), if in addition $\Delta_{t a} = 0$, as:

$$F_a = -\frac{d\delta}{ds_a} \cdot \frac{1}{\rho V^2 c_a} \left[ C_{h_a} \cdot \Delta_{a} + C_{h_\delta} \cdot \frac{1}{2} \delta_a \right]$$

(7-77)

It will be seen, that the required control force strongly increases with airspeed at a given aileron deflection.
In (7-77) $\Delta \alpha_a$ is the average effective change of angle of attack due to rolling over the span of the aileron. This average effective value can be equated to a geometric change in angle of attack at a distance $y_m$ from the plane of symmetry, see fig. 7.50. The expression is:

$$\Delta \alpha_a = \frac{pb}{2V} \cdot \frac{2y_m}{b} \quad (7-78)$$

For a given wing, $y_m$ can be determined using ref. 7.6.

The aileron deflection $\delta_a$ required for a steady, rolling flight at $\beta = 0$ follows from (7-76):

$$\delta_a = -\frac{pb}{2V} \cdot \frac{C_{Lp}}{C_{L\delta_a}} \quad (7-79)$$

Fig 7.50: The average value of $\Delta \alpha$ along the aileron span in steady, rolling flight.
Substitution of (7-78) and (7-79) in (7-77) results in the following expression for the roll control force in a steady, rolling flight at \( \beta = 0 \), as a function of rate of roll \( p \) and airspeed \( V \):

\[
P_a = - \frac{d\delta}{ds_a} \cdot \frac{1}{2} p V^2 S_a \frac{c}{l} \left[ c_{h,\alpha} \frac{2y_m}{b} - c_{h,\delta} \cdot \frac{C_p}{c_{l,\delta}} \right] \frac{pb}{2V}
\]  

or:

\[
P_a = - \frac{d\delta}{ds_a} \cdot \frac{1}{2} p V^2 \cdot \frac{b}{2} S_a \frac{c}{l} \left[ c_{h,\alpha} \frac{2y_m}{b} - c_{h,\delta} \cdot \frac{C_p}{2c_{l,\delta}} \right]
\]  

(7-81)

If for a given airplane the aerodynamic derivatives \( c_{h,\alpha}, c_{h,\delta}, c_{l,p} \) and \( c_{l,\delta} \) may be assumed to remain constant, at a given flight altitude (i.e. value of \( p \)) the roll control force is:

\[
P_a = \text{constant} \cdot p \cdot V
\]  

(7-82)

![Diagram](image)  

Fig. 7.51: Rate of roll \( p \) as a function of \( V \) for various values of \( \delta_a \) and \( F_a \)
In a diagram showing rate of roll as a function of $V$, curves for a constant control force will be hyperbola's, see fig. 7.51.

Due to wing twist, $C_{\delta_a}$ will decrease in absolute value with increasing airspeed, as discussed in 7.3.2. For a real, elastic wing, the rate of roll achieved at a given control force will be smaller than for a rigid wing, just like the rate of roll resulting from a given aileron deflection, see fig. 7.51.
### Table 7.1: Asymmetric characteristic of various airplane types

<table>
<thead>
<tr>
<th>Airplane type</th>
<th>Lockheed 1049C Super Constellation</th>
<th>Lockheed 1049C Super Constellation</th>
<th>Fokker F-27 Friendship</th>
<th>Subsonic passenger plane with four jets</th>
<th>BAC-Sud Concorde</th>
<th>North-American X-15</th>
<th>De Havilland DHC-2 Beaver</th>
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<td>Approach</td>
<td>Cruising flight</td>
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CHAPTER 8. EQUATIONS OF MOTION OF AN AIRPLANE; INTRODUCTION TO THE DYNAMIC STABILITY

8.1. Equations of motion of an airplane

8.1.1. Introduction

The equations of motion of an airplane are derived from Newton's laws from theoretical mechanics, see e.g. refs. 8.1 and 8.2.

1. A body of constant mass is at rest or moves at a constant velocity, unless a force acts on the body.
2. If a force acts on a body, the force is equal to the time derivative of the momentum - i.e. the product of mass and velocity - of the body.
3. If two bodies at rest or moving at constant velocity, exert forces upon one another, the force on the first body is equal in magnitude but opposite in direction of the force of the second body.

If a point mass dm moves with time varying velocity \( \vec{v} \) under the influence of a force \( \vec{dF} \), see fig. 8.1, Newton's second law is expressed by:

\[
\vec{dF} = \frac{d}{dt} (dm \cdot \vec{v})
\]

From the same law follows that the moment of the force about a fixed point is equal to the time derivative of the angular momentum of the body relative to that same fixed point:

\[
d\vec{M} = \frac{d}{dt} (dm \times \vec{v})
\]

In the following, the equations of motion of an airplane are derived on the basis of the above two expressions, by adding up the forces acting on all parts of the airplane as well as the moments of these forces about the center of gravity of the airplane. The internal forces and moments acting between the various parts of the airplane are cancelled in the summation according to Newton's third law. Only the external forces \( \vec{F} \) and moment \( \vec{M} \) on the airplane remain. They can expressed by:

\[
\vec{F} = \int \vec{dF} = \int_{m} \frac{d}{dt} (dm \cdot \vec{v})
\]  

(8-1)
and:

\[ \mathcal{M} = \int \frac{d}{m} \mathcal{M} = \int \frac{d}{m} \left( dm \times \mathbf{V} \right) \]  

(8-2)

Fig 8.1: The body reference frame

To simplify the following discussions, a number of assumptions are introduced: In the first place, the mass of the airplane is assumed constant during the time interval over which the motion is studied. This implies that (8-1) and (8-2) can be rewritten as the following vectorial equations:

\[ \mathbf{F} = \int \frac{dm}{m} \cdot \frac{d\mathbf{V}}{dt} \]

\[ \mathcal{M} = \int \frac{dm}{m} \cdot \frac{d(\mathbf{r} \times \mathbf{V})}{dt} \]

Refs. 8.4, 8.5 and 8.6 discuss how the motions of an airplane or a projectile can be described if the mass cannot be considered constant.
A second assumption is, that the airplane may be considered a rigid body during the motion to be studied.

The position of a rigid body in three-dimensional space is fully determined if six independent coordinates of the body are given. This means, that the airplane has six degrees of freedom during its motion in our three-dimensional space. Correspondingly, the two vectorial equations (8-1) and (8-2) can each be separated into three scalar equations. (8-1) yields three equations, each expressing the forces along one of the three axes of an orthogonal reference frame. (8-2) also produces three scalar equations, each for the moments about one of the axes of the reference frame. Together, these are the six equations determining the motions of the airplane.

When describing the airplane motions, use is made in the following of the system of airplane body axes discussed already in Chapter 1.

The origin of this airplane fixed reference frame is situated in the center of gravity. Apart from this system of body axes a few other reference frames are used in the derivation of the equations of motion. These are the system of stability axes, the earth reference frame and the airplane reference frame. These reference frames have also been discussed in Chapter 1. Table 8.1. on page 235 summarizes the principal characteristics of the above systems of axes.

The use of a reference frame fixed relative to the airplane for the present purpose requires some further comments. Newton's laws upon which the equations (8-1) and (8-2) are based, are valid only if the velocities occurring in these expressions are taken relative to non-moving space, i.e. space which itself is at rest. Strictly speaking, the only space satisfying this condition, is a space at rest relative to the fixed stars in the neighbourhood of our solar system.

It is shown in theoretical mechanics, see e.g. ref. 8.1, that Newton's second law also applies, if the motion of the body of constant mass is described relative to a space or reference frame moving itself at a constant velocity relative to the primary space just defined.

For many purposes, a reference frame fixed relative to the earth, such as the earth reference frame, may be assumed to move at sufficiently constant speed relative to primary space, notwithstanding the rotations of the earth about it's own axis and about the sun. Sometimes the reference frame may even be taken as fixed relative to the air, moving itself at a constant speed relative to the earth.

If motions of long duration or motions of very fast airplanes are studied, the rotation of the earth may no longer be neglected, whereas the earth may also have to be described as a sphere.
In such cases more complete equations of motions apply than are derived below. They are discussed e.g. in ref. 8.3. If, however, the motions of the airplane would be described using only the 'non-moving' earth reference frame, the vectors \( \mathbf{r} \) describing the locations of all mass elements would be time dependent. They would appear as variables in the equations.

In particular the moments and products of inertia would become functions of time. In order to avoid this complication the system of body axes is used, which is fixed relative to the airplane and thus moves along with the airplane. Contrary to the earth reference frame, the system of body axes does not in general move at constant speed relative to the non-moving space. The equations expressing Newton's second law by means of such a moving reference frame require some further elaboration, which is given below. The use of the moving reference frame to describe the motions of an arbitrary rigid body is due to Euler (1707-1783).

8.1.2. Derivation of the equations of motion

The integration over all mass elements of the airplane indicated in (8-1) and (8-2) is performed after writing for the velocity vector \( \mathbf{v} \) of an arbitrary point \( P \) of the airplane:

\[
\mathbf{v}_P = \mathbf{v}_o \cdot \frac{dr}{dt}
\]  

(8-3)

where \( \mathbf{v}_o \) is the velocity of the origin of the system of body axes and the vector \( \mathbf{r} \), which is constant in magnitude, indicates the location of the point \( P \) relative to the origin. Then:

\[
\int \frac{dm}{m} \mathbf{v}_P = \int \frac{dm}{m} \left( \mathbf{v}_o + \frac{dr}{dt} \right)
\]

\[
= m \mathbf{v}_o + \frac{d}{dt} \int \frac{dm}{m} \mathbf{r}
\]

As the origin of the system of body axes is the center of gravity (more precisely: the center of mass), the following expression holds by definition:

\[
\int \frac{dm}{m} \cdot \mathbf{r} = 0
\]

so:
\[ \int \frac{dm}{m} v_p = m \frac{\dot{v}_o}{\dot{t}} \]  

Equation (8-1) is then written as:

\[ F = \int \frac{d}{dt} (dm \cdot v_p) = m \cdot \frac{\dot{v}_o}{\dot{t}} \]

Using Appendix 1, see page 199, the derivative \( \frac{\dot{v}_o}{\dot{t}} \) is expressed in the variations of the components of \( \dot{v}_o \) along the axes of the rotating system of airplane body axes:

\[ \frac{\dot{v}_o}{\dot{t}} = \frac{\delta v}{\delta t} + \Omega \times \dot{v}_o \]

where \( \Omega \) is the total angular velocity of the airplane. Then \( F \) is:

\[ F = m \left( \frac{\delta v}{\delta t} + \Omega \times \dot{v}_o \right) \]

This vectorial equation is separated into its scalar components along each of the airplane body axes, as follows:

\[ F = \hat{i} \cdot F_x + \hat{j} \cdot F_y + \hat{k} \cdot F_z = \]

\[ = m \left\{ \hat{i} \dot{v}_{o_x} + \hat{j} \dot{v}_{o_y} + \hat{k} \dot{v}_{o_z} + \hat{i} (\Omega_y \dot{v}_{o_z} - \Omega_z \dot{v}_{o_y}) + \right. \]

\[ \left. + \hat{j} (\Omega_z \dot{v}_{o_x} - \Omega_x \dot{v}_{o_z}) + \hat{k} (\Omega_x \dot{v}_{o_y} - \Omega_y \dot{v}_{o_z}) \right\} \]

The components of \( \dot{v}_o \) along the airplane body axes are:

\[ v_{o_x} = u, \quad v_{o_y} = v \quad \text{and} \quad v_{o_z} = w \]
The components of $\mathbf{Q}$ along the airplane body axis are:

$$Q_x = p, \quad Q_y = q \quad \text{and} \quad Q_z = r$$

The resulting three scalar force equations then become:

$$F_x = m(\dot{u} + qw - rv)$$
$$F_y = m(\dot{v} + ru - pw)$$
$$F_z = m(\dot{w} + pv - qu) \quad (8-5)$$

Next, the expression (8-2) for the moment is further analyzed. The moment of the force $d\mathbf{F}$ acting on the mass element $dm$ in the point $P$, is relative to the origin $O$:

$$d\mathbf{M} = \mathbf{r} \times d\mathbf{F}$$

$$= \mathbf{r} \times \frac{d\mathbf{v}_P}{dt} \ dm \quad (8-6)$$

The relation between this moment $d\mathbf{M}$ and the contribution $d\mathbf{B}$ of $dm$ to the total angular momentum $\mathbf{B}$ of the complete airplane follows from taking the time derivative of $d\mathbf{B}$. The expression for $d\mathbf{B}$ is:

$$d\mathbf{B} = \mathbf{r} \times \mathbf{v}_P \ dm$$

The time derivative is:

$$\frac{d}{dt} (d\mathbf{B}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v}_P \ dm + \mathbf{r} \times \frac{d\mathbf{v}_P}{dt} \ dm$$

According to (8-3) is at constant magnitude of $\mathbf{r}$ (implying a rigid airplane):

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_P - \mathbf{v}_o$$

and thus:

$$\frac{d}{dt} (d\mathbf{B}) = (\mathbf{v}_P - \mathbf{v}_o) \times \mathbf{v}_P \ dm + \mathbf{r} \times \frac{d\mathbf{v}_P}{dt} \ dm$$
This leads with (8-6) to the following expression for \( \mathcal{M} \):

\[
d\mathcal{M} = \frac{d}{dt} (d\mathcal{B}) - (v_p - v_o) \times v_p \, dm
\]

According to Appendix 1, see page 199:

\[
v_p \times v_p = 0 \quad (8-7)
\]

leading to:

\[
d\mathcal{M} = \frac{d}{dt} (d\mathcal{B}) + v_o \times v_p \, dm
\]

The expression is now integrated over all mass elements of the airplane. The result is:

\[
\mathcal{M} = \frac{d\mathcal{B}}{dt} + v_o \times \int_m v_p \, dm \quad (8-8)
\]

where \( \mathcal{M} \) is the total external moment about the center of gravity and \( \mathcal{B} \) is the total angular momentum about the center of gravity.

But according to (8-4) is:

\[
\int_m v_p \, dm = v_o \, m
\]

This means in (8-8):

\[
\mathcal{M} = \frac{d\mathcal{B}}{dt} + v_o \times v_o \, m
\]

or, using an expression in analogy with (8-7)

\[
\mathcal{M} = \frac{d\mathcal{B}}{dt} \quad (8-9)
\]

This vectorial moment equation is now separated into three scalar equations.
According to Appendix I, see page 199:

\[
\frac{dB}{dt} = \frac{\partial B}{\partial t} + \Omega \times B
\]

and thus:

\[
\dot{\mathcal{M}} = \dot{\mathcal{M}}_x + j \dot{\mathcal{M}}_y + k \dot{\mathcal{M}}_z =
\]

\[
= i \ddot{B}_x + j \ddot{B}_y + k \ddot{B}_z + i (\Omega_y B_z - \Omega_z B_y) + j (\Omega_z B_x - \Omega_x B_z) +
\]

\[
+ k (\Omega_x B_y - \Omega_y B_x)
\]

Finally, the three moment equations, the so-called Euler equations, see also refs. 8.2, 8.7 to 8.9, are:

\[
\dot{\mathcal{M}}_x = \ddot{B}_x + qB_z - rB_y
\]

\[
\dot{\mathcal{M}}_y = \ddot{B}_y + rB_x - pB_z
\]

\[
\dot{\mathcal{M}}_z = \ddot{B}_z + pB_y - qB_x
\]

These moment equations (8-10) are the equivalents to the force equations (8-5) on page 92.

8.1.3. The angular momentum of the airplane

The angular momentum of the airplane relative to the center of gravity is now further analyzed. The angular momentum is expressed by, see also page 92:

\[
\mathcal{B} = \int_r \mathbf{r} \times v_p \, dm
\]

According to (8-3), \( v_p \) is:

\[
v_p = \frac{dr}{dt}
\]
\[ V_p = V_o + \Omega \times r \]

As the airplane is assumed to be rigid, the magnitude of \( r \) is constant, or\[
\frac{\delta r}{\delta t} = 0.\]
This results in:

\[ V_p = V_o + \Omega \times r \]

The angular momentum then is:

\[
\mathbf{B} = \int \mathbf{r} \times (V_o + \Omega \times r) \, dm
\]

\[= \int \mathbf{r} \times V_o \, dm + \int \mathbf{r} \times (\Omega \times r) \, dm\]

Because \( V_o \) is constant for all mass elements, the first integral in the latter expression is:

\[
\int \mathbf{r} \times V_o \, dm = \int \mathbf{r} \, dm \times V_o = 0
\]

and thus:

\[ \mathbf{B} = \int \mathbf{r} \times (\Omega \times r) \, dm \]

and according to Appendix 1, see page 199:

\[ \mathbf{B} = \int \mathbf{r} \times (\Omega \times r) \, dm - \int \mathbf{r} \times (\Omega \times r) \, dm \]

Herein is:

\[ \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2 \]

and:

\[ \Omega \cdot \mathbf{r} = px + qy + rz \]
Consequently:

\[ B = \Omega \int_{m} (x^2 + y^2 + z^2) \, dm - \int_{r} (px + qy + rz) \, dm \]

The scalar components of \( B \) then are:

\[ B_x = p \int (y^2 + z^2) \, dm - q \int xy \, dm - r \int xz \, dm \]
\[ B_y = -p \int xy \, dm + q \int (x^2 + z^2) \, dm - r \int yz \, dm \]
\[ B_z = -p \int xz \, dm - q \int yz \, dm + r \int (x^2 + y^2) \, dm \]

The following moments and products of inertia are now introduced:

\[ I_x = \int (y^2 + z^2) \, dm \]
\[ I_y = \int (x^2 + z^2) \, dm \]
\[ I_z = \int (x^2 + y^2) \, dm \]
\[ J_{yz} = \int yz \, dm \]
\[ J_{xz} = \int xz \, dm \]
\[ J_{xy} = \int xy \, dm \]

Then:

\[ B_x = I_x \cdot p - J_{xy} \cdot q - J_{xz} \cdot r \]
\[ B_y = -J_{xy} \cdot p + I_y \cdot q - J_{yz} \cdot r \]
\[ B_z = -J_{xz} \cdot p - J_{yz} \cdot q + I_z \cdot r \]

or, using matrix notation:

\[
\begin{bmatrix}
B_x \\
B_y \\
B_z
\end{bmatrix} = \begin{bmatrix}
I_x & -J_{xy} & -J_{xz} \\
-J_{xy} & I_y & -J_{yz} \\
-J_{xz} & -J_{yz} & I_z
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]
An additional assumption is now introduced, which is commonly made. It takes the mass distribution of the airplane as symmetric relative to the XOZ-plane; the Y-axis is perpendicular to this plane. Because of the symmetry there is for any mass element dm at the coordinates x, y, z a corresponding mass element at coordinates x, -y, z having an equal mass dm. This implies for the products of inertia \( J_{yz} \) and \( J_{xy} \):

\[
\begin{align*}
J_{yz} &= \int yz \, dm = 0 \\
J_{xy} &= \int xy \, dm = 0
\end{align*}
\]  \hspace{1cm} (8-11)

Using (8-11), the components of the angular momentum finally are:

\[
\begin{align*}
B_x &= I_x \cdot p - J_{xz} \cdot r \\
B_y &= I_y \cdot q \\
B_z &= -J_{xz} \cdot p + I_z \cdot r
\end{align*}
\]  \hspace{1cm} (8-12)

or, using again matrix notation:

\[
\begin{bmatrix}
B_x \\
B_y \\
B_z
\end{bmatrix} =
\begin{bmatrix}
+I_x & 0 & -J_{xz} \\
0 & +I_y & 0 \\
-J_{xz} & 0 & +I_z
\end{bmatrix} \cdot
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

8.1.4. The components of the weight in the system of body axes and the relations between \( p \), \( q \) and \( r \) and \( \dot{\psi} \), \( \dot{\theta} \) and \( \dot{\phi} \)

The attitude of the airplane is determined by giving the attitude of the body axes relative to the earth reference frame. To this end, use is generally made of three angles, in a way used first in principle by Euler. These three angles are called for the airplane:

1. the angle of yaw, \( \psi \)
2. the angle of pitch, \( \theta \)
3. the angle of roll, \( \phi \)

These angles are the results of three subsequent rotations of the body axes, starting from the attitude of the earth reference frame and ending up the desired attitude of the airplane in space. The order in which the rotations are
Fig 8.2: The attitude of the airplane relative to the earth.
applied is important. If the order of the rotations is changed, a different attitude of the airplane results, even if the magnitudes of the rotations remain constant. A second peculiarity of the system of Euler angles is the fact that the rotations are applied about non-orthogonal axes.

Fig. 8.2 indicates how the rotations are made.

In the original situation, see fig. 8.2a, the system of body axes coincides with the earth reference frame. Starting from this situation, the body axes are rotated about the $Z_1$-axes over the angle of yaw, $\psi$. The result is shown in fig. 8.2b. Next, the body axes are rotated about the $Y_2$-axis over the angle of pitch, $\theta$, see fig. 8.2c. Finally, the body axes are rotated a third time, about the $X_3$-axis over the angle of roll, $\phi$. The final result is shown in fig. 8.2d.

The three rotations are thus seen to be applied in sequence about the $Z_1$-, the $Y_2$- and the $X_3$-axis. It is apparent from fig. 8.2d that these three axes are not orthogonal. It is easy to see, that changing the order of the rotations, whilst keeping the magnitude of the rotations constant, results in a different attitude of the airplane. If $\theta = \pm 90^\circ$ and the X-axis points vertically either up or down, the angle of yaw and the angle of roll are indeterminate. The attitude of the airplane can then be described by infinitely many combinations of $\phi$ and $\psi$.

In relation to the equations of motion of an airplane, two questions regarding the airplane's attitude are important, viz:

a. what are the components of the weight along the X-, Y- and Z-axis, pertaining to a given attitude of the airplane,

b. what are the relations between the angular velocities $p$, $q$, and $r$ of the airplane about the body axes and the rates of change of the three attitude angles $\phi$, $\theta$ and $\psi$.

The components of the weight follow from 8.3.

In fig. 8.3b the vertical plane XOZ is shown. In this plane lies the vector $W$ along the $Z_3$-axis. The $Z_3$-axis lies also in this plane, see fig. 8.3c. The components of $W$ along the X- and Z-axes are:

$$W_x = -W \sin \theta$$

$$W_z = +W \cos \theta$$

Next, the YOZ-plane is considered, see fig. 8.3c. The $Z_3$-axis lies also in this plane, see fig. 8.3a. $W_z$ can now be resolved in the components along the Y- and Z-axes:
Symmetric and asymmetric components of the weight.

The X0Z_e-plane.

The Y0Z-plane.

Fig. 8.3: The components of the weight along the body axis.
\[ W_y = W \sin \varphi \]
\[ W_z = W \cos \varphi \]

Summarizing, the components of the weight along the three body axes are:

\[ W_x = -W \sin \theta \]
\[ W_y = W \cos \theta \sin \varphi \]
\[ W_z = W \cos \theta \cos \varphi \]

In fig. 8.3a the three components are shown together. Evidently the components of the weight do not change with the angle of yaw.

To find the relations between \( p, q, r \) and \( \dot{\psi}, \dot{\theta}, \dot{\varphi} \) it has to be remembered, that the rotation \( \psi \) takes place about the \( Z_e \)-axis. The vector of the angular velocity \( \dot{\psi} \) coincides with the \( Z_e \)-axis. The rotation \( \theta \) is about the \( Y_2 \)-axis, the vector \( \dot{\theta} \) is directed along the \( Y_2 \)-axis. Finally the rotation \( \varphi \) is made about the \( Y_3 \)-axis; so \( \dot{\varphi} \) lies along that same \( X_3 \)-axis. Fig. 8.4a shows these three vectors, together with the angular velocity vectors along the airplane body axes.

The relations between the two sets of angular velocities are found, by resolving \( \dot{\psi}, \dot{\theta}, \dot{\varphi} \) in their components along the three body axis. This is done in the figs. 8.4b to d. From fig. 8.4c and d it can be seen, that the components of \( \dot{\varphi} \) are the same as those of the weight:

\[ \dot{\psi}_x = -\dot{\psi} \sin \theta \]
\[ \dot{\psi}_y = \dot{\psi} \cos \theta \sin \varphi \]
\[ \dot{\psi}_z = \dot{\psi} \cos \theta \cos \varphi \]

The components of \( \dot{\theta} \) follow from fig. 8.4b and d:

\[ \dot{\theta}_x = 0 \]
\[ \dot{\theta}_y = \dot{\theta} \cos \varphi \]
\[ \dot{\theta}_z = -\dot{\theta} \sin \varphi \]
Symmetric and asymmetric components of the motion.

The horizontal plane.

The XOZ_e-plane.

The YOZ-plane.

Fig. 8.4: The relations between $\vec{p}, \vec{q}$ and $\vec{r}$ and $\vec{\psi}, \vec{\theta}$ and $\vec{\psi}$. 
From fig. 8.4c follows at once for $\ddot{\phi}$:

\[
\dot{\phi}_x = \dot{\phi} \\
\dot{\phi}_y = 0 \\
\dot{\phi}_z = 0
\]

The angular velocities about the airplane body axes are now obtained by adding up the components of $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ along each of the axes. The result is:

\[
p = -\dot{\phi} \sin \theta + \dot{\psi} + \dot{\phi} \\
q = \dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi \\
r = \dot{\phi} \cos \theta \cos \phi - \dot{\theta} \sin \phi
\]

or, using matrix notation:

\[
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
= 
\begin{bmatrix}
- \sin \theta & 0 & +1 \\
+ \cos \theta \sin \phi & + \cos \phi & 0 \\
+ \cos \theta \cos \phi & - \sin \phi & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix}
\tag{8-14a}
\]

From (8-14) follow the inverse relations

\[
\dot{\psi} = q \sin \phi \cos \theta + r \cos \phi \cos \theta \\
\dot{\theta} = q \cos \phi - r \sin \phi \\
\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta
\]

Using matrix notation:

\[
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix}
= 
\begin{bmatrix}
0 & + \frac{\sin \phi \cos \theta}{\cos \theta} & + \frac{\cos \phi \cos \theta}{\cos \theta} \\
0 & + \cos \phi & - \sin \phi \\
1 & + \sin \phi \tan \theta & + \cos \phi \tan \theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\tag{8-15a}
\]
For small deviations from symmetric flight, when \( \phi \approx 0 \), the following simplified relations hold:

\[
\begin{align*}
    p &= -\dot{\phi} \sin \theta + \dot{\phi} = \ddot{\phi} \\
    q &= \dot{\theta} \\
    r &= \dot{\phi} \cos \theta = \ddot{\theta}
\end{align*}
\]

for small values of \(|\theta|\).

Evidently:

\[ p \neq \ddot{\phi} \]

if:

\[ \theta \neq 0 \]

Finally it should be noted, that thus far the assumptions have tacitly been made that the earth's surface is a flat plane and the angular velocity of the earth in space is negligible in comparison with the angular velocities of the airplane relative to the earth. Under these assumptions, the latter angular rates may also be considered to be relative to non-moving space. The equations which apply to a spherical, rotating earth are discussed, e.g. in ref. 8.3.

8.1.5. The resulting equations of motion of the airplane

The equations of motion derived in the foregoing can now be given their final shape by further elaborating into the forces and moments acting on the airplane. The equations of motion are, according to (8-5) and (8-10):

\[
\begin{align*}
    F_x &= m (\ddot{u} + qw - rv) \\
    F_y &= m (\ddot{v} + ru - pw) \\
    F_z &= m (\ddot{w} + pv - qu) \\
    M_x &= \dot{B}_x + qB_z - rB_y \\
    M_y &= \dot{B}_y + rB_x - pB_z \\
    M_z &= \dot{B}_z + pB_y - qB_x
\end{align*}
\]

The external forces on the airplane are \( \mathbf{W} \) and \( \mathbf{R} \):

1) the components of the weight \( \mathbf{W} \) along the body axis are:
\[ W_x = -W \sin \theta \]
\[ W_y = W \cos \theta \sin \varphi \]
\[ W_z = W \cos \theta \cos \varphi \]

(8-13)

2) the components of the aerodynamic force \( \mathbf{R} \) are:

\[ R_x = X \]
\[ R_y = Y \]
\[ R_z = Z \]

It should be clearly stipulated that \( \mathbf{R} \) includes the forces generated by the propulsion system and the control surfaces.

The total external force on the airplane then is:

\[ \mathbf{F} = \mathbf{W} + \mathbf{R} \]

where the components are:

\[ F_x = -W \sin \theta \quad + X \]
\[ F_y = W \cos \theta \cdot \sin \varphi + Y \]
\[ F_z = W \cos \theta \cdot \cos \varphi + Z \]

(8-16)

The external moment \( \mathbf{M} \) on the airplane, acting about the center of gravity, consists only of the aerodynamic moment. The components are:

\[ M_x = L \]
\[ M_y = M \]
\[ M_z = N \]

(8-17)

In 8.1.3 the components of the angular momentum of the airplane were expressed in (8-12):

\[ B_x = I_x \cdot p - J_{xz} \cdot r \]
\[ B_y = I_y \cdot q \]
\[ B_z = -J_{xz} \cdot p + I_z \cdot r \]

(8-12)
The general equations of motion finally result, if in (8-5) the expression (8-16) and in (8-10) the expressions (8-17) and (8-12) are substituted:

\[
\begin{align*}
F_x &= -W \sin \theta \quad + X = m \left( \ddot{u} + qw - rv \right) \\
F_y &= W \cos \theta \sin \varphi + Y = m \left( \ddot{v} + ru - pw \right) \\
F_z &= W \cos \theta \cos \varphi + Z = m \left( \ddot{w} + pv - qu \right) \\
\mathcal{M}_x &= L = I_x \ddot{r} + (I_z - I_y) qr - J_{xz}(\ddot{r} + pq) \\
\mathcal{M}_y &= M = I_y \ddot{r} + (I_x - I_z) rp + J_{xz}(p^2 - r^2) \\
\mathcal{M}_z &= N = I_z \ddot{r} + (I_y - I_x) pq - J_{xz}(p - r)
\end{align*}
\]  

(8-18)

To these equations should be added the kinematic relations (8-15), expressing the relations between the rates of change of the airplane's attitude angles and the angular velocities about the airplane body axes:

\[
\begin{align*}
\dot{\phi} &= q \frac{\sin \varphi}{\cos \theta} + r \frac{\cos \varphi}{\cos \theta} \\
\dot{\theta} &= q \cos \varphi - r \sin \varphi \\
\dot{\psi} &= p + q \sin \varphi \tan \theta + r \cos \varphi \tan \theta
\end{align*}
\]  

(8-15)

In the equations of motion \( u, v \) and \( w \) are the components of the velocity of the airplane center of gravity - measured along the airplane body axes - relative to a reference frame moving at constant velocity relative to the primary, non-moving space. This latter reference frame may, as was noted in the introduction, be fixed relative to the earth. It is also permissible, however, to fix this reference frame to a volume of air moving at constant (e.g. wind) speed relative to the earth. This latter possibility is used when studying the influence of a constant wing speed on the motions of the airplane relative to the earth, see Appendix 3.

The angular velocities \( p, q \) and \( r \) are the components of the total angular velocity of the airplane relative to primary space, or to the earth, or to a volume of air moving at constant speed relative to earth.

The equations here derived are based on some restrictive assumptions, mentioned already before. They can be summed up as follows:

1. The airplane's mass is constant in the time interval during which the motion is studied.
2. The airplane is a rigid body in the motion under consideration.
3. The mass-distribution of the airplane is symmetric relative to the XOZ-plane.
4. The rotation of the earth in space, as well as the curvature of the earth's surface are negligible.

The equations of motion thus derived describe the most general motions an airplane can perform. The various equilibrium states in which the airplane can fly are equally expressed by these equations. In an arbitrary steady, curved flight condition, the following simplifications apply:

\[
\begin{align*}
\dot{u} &= v = \dot{w} = p = q = r = 0 \\
\dot{\theta} &= \dot{\phi} = 0
\end{align*}
\]  

(8-19)

transforming the equations of motion for an arbitrary steady flight condition into the equilibrium equations:

\[
\begin{align*}
-W \sin \theta + X &= m (qw - rv) \\
+W \cos \theta \sin \phi + Y &= m (ru - pw) \\
+W \cos \theta \cos \phi + Z &= m (pv - qu) \\
L &= (I_y - I_x) qr - J_{xz} pq \\
M &= (I_x - I_z) rp + J_{xz} (p^2 - r^2) \\
N &= (I_z - I_x) pq + J_{xz} qr
\end{align*}
\]  

(8-20)

The kinematic relations (8-15) are for steady flight reduced to:

\[
\begin{align*}
q \sin \phi + r \cos \phi &= \dot{\phi} \cos \theta \\
q \cos \phi - r \sin \phi &= 0 \\
p + (q \sin \phi + r \cos \phi) \tan \theta &= 0
\end{align*}
\]  

(8-21)

From the first and the third of these latter three expressions follows for steady flight:

\[
p = -\dot{\phi} \sin \theta
\]

According to (8-19), \(\dot{\theta}\) and \(\dot{\phi}\) are both zero in steady flight. In these flight
conditions, only the angular velocity \( \dot{\psi} \) may differ from zero. According to 8.1.4, see page 101, the vector \( \dot{\psi} \) is by definition vertical. This leads to the conclusion that in the most general steady flight condition the resultant vector of the total angular velocity \( \Omega \):

\[
\Omega = \dot{\psi}
\]

is vertical.

In the special case of steady, straight, symmetrical flight, commonly used as the initial condition for the calculation of stability and control characteristics, the following simplifications apply, in addition to (8-19):

\[
v = p = q = r = 0
\]

\[
\dot{\psi} = 0, \quad \varphi = 0
\]

The equilibrium equations (8-20) are then reduced to:

\[
-W \sin \theta + X = 0
\]

\[
Y = 0
\]

\[
+W \cos \theta + Z = 0
\]

\[
L = 0
\]

\[
M = 0
\]

\[
N = 0
\]

The kinematic relations (8-21) are entirely eliminated for these elementary flight conditions, as all angular rates are zero by definition.

8.1.6. Linearization of the equations of motion

The equations of motion of an airplane may be used in several different ways. In the most common application a disturbance function is given, such as a time-dependent control surface deflection or a gust velocity. The response of the airplane to such a disturbance has to be calculated. A reverse situation of this problem also occurs. Then a certain non-steady motion of the airplane is given and the control surface deflection as a function of time is asked, causing this motion. In a third application of the equations of motions both the disturbing control surface deflection and the resulting airplane motion have
been measured in flight. The purpose is now to find expressions for the aero-
dynamic forces and moments in the equations of motions, as functions of the
components of the motion and the disturbances.

In the following, attention is given primarily to the first of the above
three types of problems.

A general solution of the equations of motions is prevented by an essentual
difficulty. In the equations (8-18) the aerodynamic forces, X, Y, Z and the
aerodynamic moments L, M and N occur. These variables contribute to determine
the motions of the airplane, but they are at the same time determined by these
motions. In addition, these six aerodynamic variables are determined by the
aerodynamic disturbances acting on the airplane: the control surface deflections
and the atmospheric gusts. For arbitrary airplane motions, the relations between
the components of the motions and the disturbing variables on the one hand and
the aerodynamic forces and moments on the other hand may be highly complicated.
This prohibits in many cases a general solution of the equations of motion
(8-18).

In general, even the simplest non-steady airplane motion can be calculated
only after highly simplifying assumptions on the aerodynamic behaviour of the
airplane have been introduced. Usually a non-steady motion can be determined if
the apriori assumption has been accepted that the motion will deviate 'not too
far' from a condition of steady, straight, symmetric flight.

The permissible magnitudes of the small deviations are those for which linear-
ization of the equations of motion about the steady flight condition still
results in a sufficiently true replication of the actual airplane motions. Many
flight tests have shown, that such a linearization does indeed result in a
practically useful range of motions about the steady flight condition.

The linearization of the equations of motion for small deviations from a
condition of steady, straight, symmetric flight is performed in the following.

Steady, straight flight is characterized by the following conditions, if
the steady condition is characterized by the subscript o:

\[ \begin{align*}
    \dot{u}_o &= 0 & u_o &\neq 0 \\
    \dot{v}_o &= 0 & v_o &= 0 \\
    \dot{w}_o &= 0 & w_o &\neq 0 \\
    \dot{p}_o &= 0 & p_o &= 0 \\
    \dot{q}_o &= 0 & q_o &= 0 \\
    \dot{r}_o &= 0 & r_o &= 0 
\end{align*} \]
\[
\begin{align*}
\dot{\phi}_o &= 0 & \psi_o &= 0 & X_o &= 0 & L_o &= 0 \\
\dot{\theta}_o &= 0 & \theta_o &= 0 & Y_o &= 0 & M_o &= 0 \\
\dot{\varphi}_o &= 0 & \varphi_o &= 0 & Z_o &= 0 & N_o &= 0
\end{align*}
\]

A state of motion differing only little from this steady flight condition is described by:

\[
\begin{align*}
    u &= u_o + du \\
    v &= dv \\
    w &= w_o + dw \\
    \psi &= \psi_o + d\psi \\
    \theta &= \theta_o + d\theta \\
    \varphi &= \varphi_o + d\varphi
\end{align*}
\]

\[
\begin{align*}
    X &= X_o + dX \\
    Y &= dY \\
    Z &= Z_o + dZ \\
    p &= dp \\
    q &= dq \\
    r &= dr
\end{align*}
\]

\[
\begin{align*}
    X_o &= \frac{\dot{u}}{\cos (\theta_o + d\theta)} +\frac{\dot{v}}{\cos (\theta_o + d\theta)} + \frac{\dot{w}}{\cos (\theta_o + d\theta)} - dv
\end{align*}
\]

\[
\begin{align*}
    Y_o &= \frac{\dot{u}}{\cos (\theta_o + d\theta)} +\frac{\dot{v}}{\cos (\theta_o + d\theta)} + \frac{\dot{w}}{\cos (\theta_o + d\theta)} - dv
\end{align*}
\]

Substitution of these expressions in the equations of motion (8-18) results in:

\[
\begin{align*}
    -W \sin (\theta_o + d\theta) + X_o + dX &= m \left\{ \ddot{u} + \dot{d}q \left( w_o + dw \right) - dr \cdot dv \right\} \\
    +W \cos (\theta_o + d\theta) \sin d\psi + dY &= m \left\{ \ddot{v} + \dot{d}r \left( u_o + du \right) - dp \left( w_o + dw \right) \right\} \\
    +W \cos (\theta_o + d\theta) \cos d\varphi + Z_o + dZ &= m \left\{ \ddot{w} + \dot{d}p \cdot dv - dq \left( u_o + du \right) \right\}
\end{align*}
\]

\[
\begin{align*}
    dL &= I_x \ddot{r} + (I_z - I_y) \ddot{d}q + J_{xz} \left( \ddot{r} + dpdq \right) \\
    dM &= I_y \ddot{q} + (I_z - I_x) \ddot{d}p + J_{xz} \left( dp^2 - dr^2 \right) \\
    dN &= I_z \ddot{r} + (I_y - I_x) \ddot{d}p + J_{xz} \left( d^2r + dq^2 \right)
\end{align*}
\]

The kinematic relations now become:

\[
\begin{align*}
    \dot{\psi} &= \frac{\dot{q}}{\cos (\theta_o + d\theta)} \cdot \frac{\sin d\varphi}{\cos (\theta_o + d\theta)} + \frac{\dot{r}}{\cos (\theta_o + d\theta)} \\
    \dot{\theta} &= \frac{\dot{q}}{\cos (\theta_o + d\theta)} \cdot \cos d\varphi - \frac{\dot{r}}{\cos (\theta_o + d\theta)} \cdot \sin d\varphi \\
    \dot{\varphi} &= \frac{\dot{p}}{\cos (\theta_o + d\theta)} + \frac{\dot{d}q}{\cos (\theta_o + d\theta)} \cdot \sin d\varphi + \frac{\dot{d}r}{\cos (\theta_o + d\theta)} \cdot \cos d\varphi \cdot \tan (\theta_o + d\theta)
\end{align*}
\]

Linearization of the equations means that the products of small variables are neglected. In addition, in the first and the third force equation the components of the weight are cancelled in the equilibrium state by the aero-
dynamic forces \( X_o \) and \( Z_o \) existing in that flight condition, see (8-22):

Finally, the following relations are used:

\[
\begin{align*}
\cos d\theta &= 1 \\
\sin d\theta &= d\theta \\
\cos d\phi &= 1 \\
\sin d\phi &= d\phi
\end{align*}
\]

The resulting linearized equations then are:

\[
\begin{align*}
-W \cos \theta_o \cdot d\theta + dX &= m (\dot{u} + dq \cdot w_o) \\
+W \cos \theta_o \cdot d\phi + dY &= m (\dot{v} + dr \cdot u_o - dp \cdot w_o) \\
-W \sin \theta_o \cdot d\theta + dZ &= m (\dot{w} - dq \cdot u_o) \\
\end{align*}
\]

\[
\begin{align*}
dL &= I_x \dot{p} - J_{xz} \dot{r} \\
dM &= I_y \dot{q} \\
dN &= I_z \dot{r} - J_{xz} \dot{p}
\end{align*}
\]

(8-23)

and:

\[
\begin{align*}
\dot{\psi} &= \frac{dr}{\cos \theta_o} \\
\dot{\theta} &= dq \\
\dot{\phi} &= dp + dr \cdot \tan \theta_o
\end{align*}
\]

Next, the variations of the aerodynamic forces \( dX, dY, dZ \) and moments \( dL, dM, dN \) are expressed in the variations of the components of the motion and the control surface deflections. As an example only the variation of the force in the X-direction is considered here:

\[
dX = f (u_o, w_o, du, dv, dw, dp, dq, dr, d\delta_a, d\delta_e, d\delta_r)
\]

This function can be developed in a Taylor series:

\[
dX = \frac{\partial X}{\partial u} \cdot du + \frac{\partial X}{\partial v} \cdot dv + \frac{\partial X}{\partial w} \cdot dw + \frac{\partial X}{\partial p} \cdot dp + \frac{\partial X}{\partial q} \cdot dq + \frac{\partial X}{\partial r} \cdot dr +
\]

\[
+ \frac{\partial^2 X}{\partial u^2} \cdot du^2 + 2 \frac{\partial^2 X}{\partial u \partial v} \cdot du \cdot dv + 2 \frac{\partial^2 X}{\partial u \partial w} \cdot du \cdot dw + \frac{\partial^2 X}{\partial v^2} \cdot dv^2 + \]

\[
+ \frac{\partial^2 X}{\partial v \partial w} \cdot du \cdot dw + \frac{\partial^2 X}{\partial w^2} \cdot dw^2 + \ldots
\]
\[ + 2 \frac{\partial^2 x}{\partial v \partial w} \cdot dv \cdot dw + \ldots \ldots \] 

where the partial derivatives depend in principle on \( u_0 \) and \( w_0 \).

The assumption that the deviations remain small enough to allow the linearization of the equations means, in fact, that terms of second and higher degree may be omitted.

A new, additional assumption is the following one. It is assumed that during the disturbed, non-steady motion the aerodynamic forces and moments - with a few exceptions to be discussed later - are equal to what they would have been, if the motion would have been steady at the same deviations from equilibrium. This means that the influences of linear and angular accelerations and of all higher derivatives with respect to time on the aerodynamic forces and moments are neglected.

The few exceptions just hinted at are the derivatives \( \frac{\partial y}{\partial v} \) and \( \frac{\partial n}{\partial w} \) with respect to the acceleration \( v \) along the Y-axis, see Chapter 7, and the derivatives \( \frac{\partial z}{\partial w} \) and \( \frac{\partial m}{\partial v} \) with respect to the acceleration \( w \) along the Z-axis. Of these four partial derivatives, the two relating the moments \( \frac{\partial n}{\partial v} \) and \( \frac{\partial m}{\partial w} \) and of these two again \( \frac{\partial m}{\partial w} \) - are by far the most important.

Now \( dx \) can be written as:

\[
dx = \frac{\partial x}{\partial u} \cdot du + \frac{\partial x}{\partial v} \cdot dv + \frac{\partial x}{\partial w} \cdot dw + \frac{\partial x}{\partial p} \cdot dp + \frac{\partial x}{\partial q} \cdot dq + \frac{\partial x}{\partial r} \cdot dr + \\
+ \frac{\partial x}{\partial a} \cdot \partial a + \frac{\partial x}{\partial e} \cdot \partial e + \frac{\partial x}{\partial r} \cdot \partial r \quad (8-24)
\]

As a consequence of the assumed symmetry of the airplane and the linearization of the equations of motion a further simplification is possible. Suppose, a change in velocity \(+dv\) along the Y-axis causes an extra force \(+dx\). Then, according to (8-24), a change in velocity \(-dv\) will generate an extra force \(-dx\). Reflecting the situation with \(-dv\) in the plane of symmetry must result in the situation with \(+dv\), see fig. 8.5.

This means, however, that \( dx \) must be equal to zero and, as a consequence, \( X \) cannot vary linearly with \( dv \). Only a variation with \( dv^2, dv^4, dv^6 \) etc. is possible. Such terms of higher degree have already been neglected, see above. This leads to the conclusion that the asymmetric deviation \( dv \) has no influence on the symmetric aerodynamic force \( X \). A similar argument may be given for any of other combinations of symmetric and asymmetric variables.
Evidently, the following rules apply:

a. small asymmetric deviations and disturbances have no influence on the symmetric forces $X$ and $Z$ or on the symmetric moment $M$,

b. small symmetric deviations and disturbances have no influence on the asymmetric force $Y$ or on the asymmetric moments $L$ and $N$.

Expressed in a different way, no \textit{aerodynamic} coupling exists between the symmetric and the asymmetric degrees of freedom, as long as the deviations and disturbances remain small.

A single exception to this rule is the contribution \( \frac{\partial^2 M}{\partial v^2} \cdot dv^2 \) to $dM$, see Chapter 7, which possesses in many practical situations a non-negligible magnitude at values of $dv$ occurring in flight, in flight conditions where no other significant non-linearities are noticeable.

In the following, however, this contribution of $dv^2$ to $dM$ will not be further considered.

The changes in the aerodynamic forces and moments can then be written as follows, using a more compact notation of the partial derivatives:
\[
\begin{align*}
\frac{\text{d}X}{\text{d}t} &= X_u \frac{\text{d}u}{\text{d}t} + X_w \frac{\text{d}w}{\text{d}t} + X_q \frac{\text{d}q}{\text{d}t} + I_\theta \frac{\text{d}\delta}{\text{d}t} \\
\frac{\text{d}Y}{\text{d}t} &= Y_v \frac{\text{d}v}{\text{d}t} + Y_y \frac{\text{d}y}{\text{d}t} + Y_p \frac{\text{d}p}{\text{d}t} + Y_r \frac{\text{d}r}{\text{d}t} + Y_\delta \frac{\text{d}\delta}{\text{d}t} + Y_{\delta_r} \frac{\text{d}\delta}{\text{d}t} \\
\frac{\text{d}Z}{\text{d}t} &= Z_u \frac{\text{d}u}{\text{d}t} + Z_w \frac{\text{d}w}{\text{d}t} + Z_q \frac{\text{d}q}{\text{d}t} + Z_\delta \frac{\text{d}\delta}{\text{d}t} \\
\frac{\text{d}L}{\text{d}t} &= L_v \frac{\text{d}v}{\text{d}t} + L_p \frac{\text{d}p}{\text{d}t} + L_r \frac{\text{d}r}{\text{d}t} + L_\delta \frac{\text{d}\delta}{\text{d}t} + L_{\delta_r} \frac{\text{d}\delta}{\text{d}t} \\
\frac{\text{d}M}{\text{d}t} &= M_u \frac{\text{d}u}{\text{d}t} + M_w \frac{\text{d}w}{\text{d}t} + M_q \frac{\text{d}q}{\text{d}t} + M_\delta \frac{\text{d}\delta}{\text{d}t} \\
\frac{\text{d}N}{\text{d}t} &= N_v \frac{\text{d}v}{\text{d}t} + N_y \frac{\text{d}y}{\text{d}t} + N_p \frac{\text{d}p}{\text{d}t} + N_r \frac{\text{d}r}{\text{d}t} + N_\delta \frac{\text{d}\delta}{\text{d}t} + N_{\delta_r} \frac{\text{d}\delta}{\text{d}t}
\end{align*}
\]

The linearized equations of motion describing the behaviour of the symmetric airplane for small deviations from a condition of steady, straight, symmetric flight can now be summarized by combining (8-23) and (8-25):

\[
\begin{align*}
-W \cos \theta \frac{\text{d}\theta}{\text{d}t} + X_u \frac{\text{d}u}{\text{d}t} + X_w \frac{\text{d}w}{\text{d}t} + X_q \frac{\text{d}q}{\text{d}t} + X_\delta \frac{\text{d}\delta}{\text{d}t} &= m (\dot{u} + dq \dot{w}o) \\
+W \cos \theta \frac{\text{d}\theta}{\text{d}t} + Y_v \frac{\text{d}v}{\text{d}t} + Y_y \frac{\text{d}y}{\text{d}t} + Y_p \frac{\text{d}p}{\text{d}t} + Y_r \frac{\text{d}r}{\text{d}t} + Y_\delta \frac{\text{d}\delta}{\text{d}t} + Y_{\delta_r} \frac{\text{d}\delta}{\text{d}t} &= m (\ddot{v} + dr \dot{u} - dp \dot{w}) \\
-W \sin \theta \frac{\text{d}\theta}{\text{d}t} + Z_u \frac{\text{d}u}{\text{d}t} + Z_w \frac{\text{d}w}{\text{d}t} + Z_q \frac{\text{d}q}{\text{d}t} + Z_\delta \frac{\text{d}\delta}{\text{d}t} &= m (\dot{w} - dq \dot{u}o) \\
L_v \frac{\text{d}v}{\text{d}t} + L_p \frac{\text{d}p}{\text{d}t} + L_r \frac{\text{d}r}{\text{d}t} + L_\delta \frac{\text{d}\delta}{\text{d}t} + L_{\delta_r} \frac{\text{d}\delta}{\text{d}t} &= I \dot{\phi} - J_{xz} \dot{\theta} \\
M_u \frac{\text{d}u}{\text{d}t} + M_w \frac{\text{d}w}{\text{d}t} + M_q \frac{\text{d}q}{\text{d}t} + M_\delta \frac{\text{d}\delta}{\text{d}t} &= I \dot{\psi} \\
N_v \frac{\text{d}v}{\text{d}t} + N_y \frac{\text{d}y}{\text{d}t} + N_p \frac{\text{d}p}{\text{d}t} + N_r \frac{\text{d}r}{\text{d}t} + N_\delta \frac{\text{d}\delta}{\text{d}t} + N_{\delta_r} \frac{\text{d}\delta}{\text{d}t} &= I \dot{\phi} - J_{xz} \dot{\theta} \\
\dot{\phi} &= \frac{\text{dr}}{\cos \theta} \\
\delta &= dq \\
\phi &= dp + dr \tan \theta 
\end{align*}
\]

Evidently, two groups of equations result. In the first group only the symmetric variables occur. These equations describe the symmetric motions of the airplane about the steady flight condition. In the second group only the asymmetric variables occur. These equations describe the asymmetric airplane motions.

If the assumptions previously made are true, no coupling exists between the
symmetric and the asymmetric airplane motions. The three assumptions are:

1. the airplane is asymmetric with respect to the XOZ-plane,
2. only motions about a condition of steady, straight, symmetric flight are considered,
3. the disturbances and the subsequent deviations remain small enough to permit linearization of the equations of motion.

The two separate groups of equations read as follows:

a. the symmetric motions:

\[-W \cos \theta_0 \cdot d\theta + X_d u + X_w q + X_e \delta = m (\dot{u} + dq)\]

\[-W \sin \theta_0 \cdot d\theta + Z_w d\omega + Z_e \delta = m (\dot{w} - dq)\]

\[M_d u + M_w \dot{\omega} + M_e \delta = I_y \dot{q}\]

\[\dot{\delta} = dq\]

(8-26)

b. the asymmetric motions:

\[+W \cos \theta_0 \cdot d\phi + Y_v \dot{v} + Y_p \dot{r} + Y_a \dot{\delta}_a + Y_r \dot{\delta}_r = m(\dot{v} + dr - dp)\]

\[L_v \dot{v} + L_p \dot{p} + L_r \dot{r} + L_a \dot{\delta}_a + L_r \dot{\delta}_r = I_x \dot{p} - J_{xz} \dot{r}\]

\[N_v \dot{v} + N_p \dot{p} + N_r \dot{r} + N_a \dot{\delta}_a + N_r \dot{\delta}_r = I_z \dot{r} - J_{xz} \dot{p}\]

(8-27)

\[\dot{\phi} = \frac{dr}{\cos \theta_0}\]

\[\dot{\phi} = dp + dr \cdot \tan \theta_0\]

8.1.7. The equations of motion in the stability reference frame

The airplane body axes used in the foregoing to describe the motions, have not yet been completely defined. The reference frame is fixed relative to the airplane, but the direction of the X-axis in the plane of symmetry has not yet been defined. Often, the choice depends on the type of motions to be studied.

If the airplane motions about steady, straight, symmetric flight are to be
determined, for instance to investigate the stability of the steady flight condition, it is advantageous to use the so-called 'stability reference frame'. This is a system of body axes, as previously defined, the X-axis of which is parallel to the direction of motion of the center of gravity in the steady initial flight condition. But during the disturbed motion of the airplane the reference frame is fixed relative to the airplane.

This particular choice of the X-axis results in the equations of motion in:

\[
\begin{align*}
\mathbf{u}_0 &= \mathbf{v} \\
\mathbf{w}_0 &= 0
\end{align*}
\]

In addition, the angle of attack in the steady reference condition, measured relative to the X-axis, is zero:

\[
\alpha_0 = 0
\]

and:

\[
\theta_0 = \gamma_0
\]  
(8-28)

A disadvantage of the stability axes becomes apparent, if the disturbed motions about several different equilibrium conditions have to be studied in succession. During each of these disturbed motions the X-axis and the X-axis have a different direction relative to the airplane. This implies, that the aerodynamic variables, the components of the motion and also the moments and products of inertia in each case anew refer to a different attitude of the reference frame relative to the airplane, although that attitude remains invariant during each of the disturbed motions.

In the following, the system of stability axes is used under the assumption that always the motions about a condition of steady, straight, symmetric flight are considered. Then the equations (8-26) and (8-27) apply.

In order to simplify the notation, in the remainder of this Chapter the changes in the variable angles relative to the reference condition are indicated without the letter d, e.g. \( \Theta \) rather than \( d\Theta \), \( \delta_e \) instead of \( d\delta_e \). The magnitude of the relevant variable in steady flight is indicated by the subscript 0. If a certain component of the motion is zero in steady flight, as for instance \( \psi, \phi, p, q, r \), the prefix d in the equations for the disturbed motions can also be omitted, e.g. \( v = dv \) etc. Finally, \( du \) is written as \( u \).

The equations of motions can then be written as follows, if the stability axes are used:
a. symmetric motions

\[-W \cos \theta_0 \cdot \delta + \dot{X}_u \cdot \delta + \dot{X}_w \cdot \delta + X_q \cdot \dot{X}_e \cdot \delta = m \cdot \dot{u} \]

\[-W \sin \theta_0 \cdot \delta + Z_u \cdot \delta + Z_w \cdot \delta + Z_q \cdot \dot{Z}_e \cdot \delta = m (\dot{w} - q \cdot V) \]

\[M_u \cdot \delta + M_w \cdot \delta + M_q \cdot \dot{M}_e \cdot \delta = I_y \cdot \dot{q} \]

\[\dot{\delta} = q \]

b. asymmetric motions

\[+W \cos \theta_0 \cdot \theta + Y_v \cdot \theta + Y_p \cdot \theta + Y_r \cdot \theta + Y_\delta \cdot \theta + Y_\delta \cdot \theta = m (\dot{v} + r \cdot V) \]

\[L_v \cdot \dot{v} + L_p \cdot \dot{p} + L_r \cdot \dot{r} + L_\delta \cdot \dot{\delta} + L_\delta \cdot \dot{\delta} = I_x \cdot \dot{J} - \dot{J} \cdot \dot{x} \cdot \dot{z} \]

\[N_v \cdot \dot{v} + N_p \cdot \dot{p} + N_r \cdot \dot{r} + N_\delta \cdot \dot{\delta} + N_\delta \cdot \dot{\delta} = I_z \cdot \dot{J} - \dot{J} \cdot \dot{x} \cdot \dot{p} \]

\[\dot{\phi} = \frac{r}{\cos \theta_0} \]

\[\dot{\varphi} = p + r \tan \theta_0 \]

8.1.8. The equations of motion in non-dimensional form

The differential equations derived in the foregoing, can be applied directly to the calculation of the symmetric and asymmetric airplane motions about a given condition of steady, straight, symmetric flight as a response to a given disturbance. Very often, however, the equations are used in a non-dimensional form. The reason for this practice is, that the aerodynamic forces and moments in the equations are usually expressed in non-dimensional coefficients. This leads to writing the entire equations in non-dimensional form. To this end, also the inertial characteristics and the time scale are made non-dimensional. One of the several ways in which this can be done, is discussed in the following.

As a starting point, three independent units are chosen, namely a length having the dimension \([\ell]\), a velocity, dimension \([\ell \cdot \ell^{-1}]\), and a mass, dimension \([m]\). The actual values of these units are:
symmetric motions:  

- length  \( \bar{c} \)
- velocity  \( \bar{v} \)
- mass  \( p\bar{S}c \)

asymmetric motions:  

- length  \( b \)
- velocity  \( \bar{v} \)
- mass  \( p\bar{S}b \)

Evidently different units of length are used for the symmetric and the asymmetric motions, namely the mean aerodynamic chord \( \bar{c} \) and the wing span \( b \) respectively. These are also the customary lengths used to make the symmetric moment \( M \) and the asymmetric moments \( L \) and \( N \) respectively non-dimensional. The unit of the mass chosen for both types of motion is the mass of a certain volume of air surrounding the airplane. The volume is the wing area multiplied by the respective unit of length.

With these units, all variables in the equations of motion can be made non-dimensional, according to the schemes in tables 8.2 and 8.3 on pages 236 and 237.

Using these schemes, the way in which the equations of motion are made non-dimensional can now be discussed more in detail.

For the symmetric motions the two force equations, in which each of the terms has the dimension of a force like \( X \) and \( Z \), are made non-dimensional by a division by \( \frac{1}{2} p\bar{v}^2 S \). The moment equation, in which each of the terms has the dimension of a moment like \( M \), is divided by \( \frac{1}{2} p\bar{v}^2 \bar{S} \). Starting from the equations (8-29) the result is:

\[
\frac{W \cos \theta}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{\theta} + \frac{X_u}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{w} + \frac{X_w}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{w} + \frac{X_q}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{e} = \frac{m_u}{\frac{1}{2} p\bar{v}^2 S} \tag{8-31}
\]

\[
\frac{W \sin \theta}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{\theta} + \frac{Z_u}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{w} + \frac{Z_w}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{w} + \frac{Z_q}{\frac{1}{2} p\bar{v}^2 S} \cdot \dot{e} = \frac{m_{\bar{q} \bar{v}}}{\frac{1}{2} p\bar{v}^2 S} \tag{8-32}
\]

\[
\frac{M_u}{\frac{1}{2} p\bar{v}^2 \bar{S}c} \cdot \dot{w} + \frac{M_w}{\frac{1}{2} p\bar{v}^2 S_\bar{c}} \cdot \dot{w} + \frac{M_q}{\frac{1}{2} p\bar{v}^2 S_\bar{c}} \cdot \dot{e} = \frac{I_{\bar{y} \bar{q}}}{\frac{1}{2} p\bar{v}^2 \bar{S}c} \tag{8-33}
\]

\[
\frac{\delta_{\bar{c}}}{\bar{v}} = \frac{q_{\bar{c}}}{\bar{v}} \tag{8-34}
\]

In elaborating the two force equations, use is made of the two equilibrium conditions:
\[ W \sin \theta_0 = X_0 \]
\[ W \cos \theta_0 = -Z_0 \]

Using the previous scheme in table 8.2, these conditions are written as follows: in (8-31):

\[ -\frac{W \cos \theta_0}{\frac{1}{2} \rho v^2 S} = C_{Z_0} \] (8-35)

and in (8-32):

\[ -\frac{W \sin \theta_0}{\frac{1}{2} \rho v^2 S} = -C_{X_0} \] (8-36)

Next, the partial derivatives of the aerodynamic forces and the moment are further elaborated. As an example the contribution \( X_u u \) is considered. It can be written as:

\[ \frac{X_u u}{\frac{1}{2} \rho v^2 S} = \frac{X_u}{\frac{1}{2} \rho v S} \cdot \frac{u}{v} \]

\[ = \frac{X_u}{\frac{1}{2} \rho v S} \cdot \hat{u} \]

As both the left and right hand sides of this latter expression are non-dimensional, like \( \hat{u} \), the quotient \( \frac{X_u}{\frac{1}{2} \rho v S} \) must also be non-dimensional. This quotient is called a stability derivative, written as \( C_{X_u} \):

\[ C_{X_u} = \frac{X_u}{\frac{1}{2} \rho v S} \]

and:

\[ \frac{X_u u}{\frac{1}{2} \rho v^2 S} = C_{X_u} \cdot \hat{u} \]

It should be emphasized, that for the stability derivative \( C_{X_u} \) in particular, but also for \( C \) the order of differentiating with respect to airspeed and non-
dimensionalizing using the factor \( \frac{1}{\rho V^2 S} \), in which the airspeed figures as well, is relevant for the resulting values of \( C^*_X \) and \( C^*_Z \):

\[
C^*_X = \frac{1}{\frac{1}{\rho V^2 S}} \frac{\delta X}{\delta u} = C^*_X \alpha
\]

In entirely the same way:

\[
\frac{X^*_w}{\frac{1}{\rho V^2 S}} = \frac{X^*_w}{\frac{1}{\rho V^2 S}} \cdot \frac{\delta}{\delta V} = C^*_X \cdot \alpha
\]

where the stability derivative \( C^*_X \) is:

\[
C^*_X = \frac{X^*_w}{\frac{1}{\rho V^2 S}} = \frac{\delta C^*_X}{\delta \alpha}
\]

Now the order of differentiating with respect to \( w \) and non-dimensionalizing using the factor \( \frac{1}{\rho V^2 S} \) may be changed, as \( \frac{1}{\rho V^2 S} \) does not contain the factor \( w \).

The expressions for the different stability derivatives are collected in Table 8.4, page 238. The non-dimensional control derivatives are also given in the table.

The non-dimensional equations for the symmetric motions then become:

\[
\begin{align*}
C^*_Z \cdot \theta + C^*_X \cdot \alpha + C^*_X \cdot \alpha & = 2\mu_c \cdot D \cdot \delta_e^u \\
-C^*_X \cdot \theta + C^*_Z \cdot \alpha + C^*_Z \cdot \alpha \cdot D \cdot \alpha + C^*_Z \cdot \alpha + C^*_Z \cdot \alpha & = 2\mu_c \cdot (D \cdot \alpha - D \cdot \theta) \\
C^*_m \cdot \alpha + C^*_m \cdot \alpha + C^*_m \cdot \alpha & = 2\mu_c \cdot (D \cdot \alpha - D \cdot \theta) \\
\frac{\delta \dot{C}}{\dot{V}} & = \frac{\dot{C}}{V}
\end{align*}
\]

The four equations (8-38) are now rewritten in an ordered form. In this way, the following differential equations in \( u \), \( \alpha \), \( \theta \) and \( \frac{\delta \dot{C}}{\dot{V}} \) result:
\[
\begin{align*}
&\left( C_{X_{u}} - 2\mu_c D_c \right) \hat{u} + C_{X_{\alpha}} \cdot \alpha + C_{Z_{o}} \cdot \theta + C_{X_{q}} \cdot \frac{ac}{v} + C_{X_{\delta}} \cdot \delta_e = 0 \\
&\left( C_{Z_{u}} + (C_{Z_{\alpha}} - 2\mu_c D_c) \right) \cdot \alpha - C_{X_{o}} \cdot \theta + (C_{Z_{q}} + 2\mu_c) \cdot \frac{ac}{v} + C_{Z_{\delta}} \cdot \delta_e = 0 \\
&\left( C_{m_{u}} + (C_{m_{\alpha}} + C_{m_{\alpha}^*} D_c) \right) \cdot \alpha + (C_{m_{q}} - 2\mu_c K^2 D_c) \cdot \frac{ac}{v} + C_{m_{\delta}} \cdot \delta_e = 0
\end{align*}
\]

or, using matrices:

\[
\begin{bmatrix}
C_{X_{u}} - 2\mu_c D_c & C_{X_{\alpha}} & C_{Z_{o}} & C_{X_{q}} \\
C_{Z_{u}} & C_{Z_{\alpha}} + (C_{Z_{\alpha}} - 2\mu_c D_c) & -C_{X_{o}} & C_{Z_{q}} + 2\mu_c \\
0 & 0 & -D_c & 1 \\
C_{m_{u}} & C_{m_{\alpha}} + C_{m_{\alpha}^*} D_c & 0 & C_{m_{q}} - 2\mu_c K^2 D_c
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\alpha \\
\theta \\
\frac{ac}{v}
\end{bmatrix}
=
\begin{bmatrix}
-C_{X_{\delta}} \\
-C_{Z_{\delta}} \\
-C_{m_{\delta}}
\end{bmatrix}
\]

with:

\[
\theta = \alpha + \gamma
\]

The left hand side of (8-39a) can also be expressed in the variables \( \hat{u}, \gamma, \theta \) and \( \frac{ac}{v} \):

\[
\begin{bmatrix}
C_{X_{u}} - 2\mu_c D_c & -C_{X_{\alpha}} & C_{X_{\alpha}} + C_{Z_{o}} & C_{X_{q}} \\
C_{Z_{u}} & -C_{Z_{\alpha}} - (C_{Z_{\alpha}} - 2\mu_c D_c) & -C_{X_{o}} + C_{Z_{\alpha}} & C_{Z_{\alpha}} + C_{Z_{q}} \\
0 & 0 & -D_c & 1 \\
C_{m_{u}} & -C_{m_{\alpha}} - C_{m_{\alpha}^*} D_c & C_{m_{\alpha}} & C_{m_{q}} - 2\mu_c K^2 D_c
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\gamma \\
\theta \\
\frac{ac}{v}
\end{bmatrix}
\]

The right hand side of (8-39a) remains unchanged.
The equations for the **asymmetric** motions are made non-dimensional in close analogy with the foregoing. The force equation is again divided by \( \frac{1}{4} \rho V^2 S \) and the two moment equations by \( \frac{1}{4} \rho V^2 S_b \).

The result is:

\[
\frac{W \cos \theta_o \cdot \varphi}{\frac{1}{4} \rho V^2 S} + \frac{Y_v \cdot v + Y_p \cdot p + Y_r \cdot r + \delta_a \cdot \delta_a + \delta_r \cdot \delta_r}{\frac{1}{4} \rho V^2 S} = \frac{m(v+rv)}{\frac{1}{4} \rho V^2 S} \\
\frac{L_v \cdot v + L_p \cdot p + L_r \cdot r + \delta_a \cdot \delta_a + \delta_r \cdot \delta_r}{\frac{1}{4} \rho V^2 S_b} = \frac{I_x \cdot p}{\frac{1}{4} \rho V^2 S_b} \\
\frac{N_v \cdot v + N_p \cdot p + N_r \cdot r + \delta_a \cdot \delta_a + \delta_r \cdot \delta_r}{\frac{1}{4} \rho V^2 S_b} = \frac{I_z \cdot r}{\frac{1}{4} \rho V^2 S_b}
\]

\[ \psi = \frac{r}{\cos \theta_o} \]

\[ \varphi = p + r \tan \theta_o \]

The component of the weight along the Z-axis is now written as:

\[
\frac{W \cos \theta_o}{\frac{1}{4} \rho V^2 S} = \frac{W \cos \gamma_o}{\frac{1}{4} \rho V^2 S} = C_L
\]

Next, the partial derivatives of the aerodynamic force and the two moments are expressed using the non-dimensional stability derivatives according to the scheme of table 8.5, see pages 240 and 241. If in addition, \( \theta_o \) in the kinematic relations is replaced by \( \gamma_o \), using (8-28), the non-dimensional equations for the asymmetric motions are:

\[
C_{L \varphi} = \beta + C_{\varphi} \cdot b \cdot \delta_r + C_{\gamma} \cdot Y_p \cdot 2V \cdot \delta_a \cdot \delta_r = 2 \mu_b (D_b \cdot \beta + \frac{rb}{2V}) \\
C_{L \gamma} \cdot \beta + C_{L \gamma} \cdot p \cdot \frac{pb}{2V} + C_{L \gamma} \cdot r \cdot \frac{rb}{2V} + C_{L \gamma} \cdot \delta_a \cdot \delta_r = 4 \mu_b (K_{XZ} \cdot D_b \cdot \frac{pb}{2V} - K_{XX} \cdot D_b \cdot \frac{rb}{2V})
\]
In level flight \( (\gamma_o = 0) \) the kinematic relations are reduced to:

\[
\dot{\zeta}_{D_b} \psi = \frac{pb}{2V} \quad \dot{\zeta}_{D_b} \varphi = \frac{pb}{2V}
\]

By adding the latter of these two relations:

\[
-\dot{\zeta}_{D_b} \varphi + \frac{pb}{2V} = 0
\]

the equations of motion (8-40) are changed into a set of four first order differential equations in \( \beta, \varphi, \frac{pb}{2V} \) and \( \frac{rb}{2V} \):

\[
\left[ \left( C_{\beta}\beta + C_{\beta\beta} \right) \frac{pb}{2V} + \left( C_{\beta\varphi} \right) \varphi \right] + \left( C_{\varphi} \varphi + C_{\varphi \varphi} \right) \frac{rb}{2V} + \left( C_{\varphi \delta} \delta \right) a + C_{\varphi \delta} \delta_r = 0
\]

\[
-\dot{\zeta}_{D_b} \varphi + \frac{pb}{2V} = 0
\]

\[
\left[ \left( C_{\beta} \beta + C_{\beta \beta} \right) \frac{pb}{2V} + \left( C_{\beta \varphi} \right) \varphi \right] + \left( C_{\varphi} \varphi + C_{\varphi \varphi} \right) \frac{rb}{2V} + \left( C_{\varphi \delta} \delta \right) a + C_{\varphi \delta} \delta_r = 0
\]

\[
\left( C_{\beta} \beta + C_{\beta \beta} \right) \frac{pb}{2V} + \left( C_{\beta \varphi} \right) \varphi + \left( C_{\varphi} \varphi + C_{\varphi \varphi} \right) \frac{rb}{2V} + \left( C_{\varphi \delta} \delta \right) a + C_{\varphi \delta} \delta_r = 0
\]

(8-41)

Using matrix notation, the equations (8-41) are written as:
\[
\begin{bmatrix}
C_Y + (C_Y - 2\mu_b)D_b & C_L & C_Y - \delta_b \\
0 & -\frac{1}{b} & 1 & 0 \\
C_{\alpha} & 0 & C_{\alpha} - 4\mu_b K_z D_b & C_{\alpha} + 4\mu_b K_z D_b \\
C_{n\alpha} + C_{n\alpha} D_b & 0 & C_{n\alpha} + 4\mu_b K_{xz} D_b & C_{n\alpha} - 4\mu_b K_z D_b \\
\end{bmatrix}
\begin{bmatrix}
\beta \\
\phi \\
p_b \\
r_b \\
\end{bmatrix} = \frac{pb}{2V} \frac{rb}{2V}
\]

\[
\begin{bmatrix}
-C_{\delta a} \\
0 \\
-C_{\delta b} \\
-C_{n\delta a} \\
\end{bmatrix}
\begin{bmatrix}
\delta_a \\
0 \\
\delta_b \\
-C_{n\delta a} \\
\end{bmatrix} = \frac{-C_{\delta a}}{\delta_a} + \frac{-C_{\delta b}}{\delta_b} = \frac{-C_{n\delta a}}{C_{n\delta a}} (8-42)
\]

At the end of this paragraph it should be noted, that often – especially in the British literature – the equations of motion are made non-dimensional using a unit of time not equal to \(\frac{C}{V}\), but, \(\tau_c = \mu_c \cdot \frac{C}{V}\) sec for the symmetric motions and \(\tau_b = \mu_b \cdot \frac{b}{V}\) sec rather than \(\frac{V}{C}\) sec for the asymmetric motions. The particular way of making the equations non-dimensional is determined in the first place by the habits of the user. None of the various methods offers clear advantages or disadvantages in comparison with the other possible methods. This explains why only one method has been described in detail here.

8.2. The calculation of the stability and the control derivatives

8.2.1. Introduction

In the following paragraphs attention is given to the methods used to obtain the various stability and control derivatives for the symmetric motions from generalized data available in the literature. The corresponding methods used for the asymmetric motions have been discussed in Chapter 7.

Evidently, the accuracy to which the derivatives can be determined has a direct influence on the accuracy of the calculated non-steady airplane motions.

The stability derivatives can be found not only by calculation from generalized data, but also from measurements. Flight tests with the actual air-
plane can be made, or wind tunnel tests on a model. From comparisons of measured and calculated stability derivatives obtained in different ways, it appears that none of these various ways to obtain the various derivatives gives satisfactory results in all cases of practical interest.

Calculated derivatives may be expected to show an acceptable correspondence with experimentally obtained values, only for conventional airplane configurations. For less conventional configurations, calculated derivatives may be considered only as a very first approximation.

In such situations measured stability derivatives are indispensable for calculations of stability and control characteristics on which some reliance has to be placed.

In the following the various stability derivatives are discussed in turn.

When considering the forces and moments in the initial equilibrium condition, the contributions from the propulsive system are included, when discussing the stability derivatives these contributions are not considered.

Various calculation methods are mentioned, for more detailed discussions and quantitative data reference is made to the literature, see the list of references on page 258. All derivatives, with the exception of \( C_{\alpha} \), \( m_{\alpha} \) and \( n_{\beta} \) are calculated under the assumption that the airflow is always stationary. A summary of the derived expressions is given in table 8.6, on page 242.

8.2.2. The aerodynamic coefficients for the symmetric motion in the initial steady flight condition

For the following discussions the direction of the \( X_s \)-axis of the system of stability axes is defined relative to the airplane, using the fixed and invariable system of the airplane reference axes, see fig. 8.6 and the introductory Chapter 0.

The angle between the negative \( X_r \)-axis and the positive \( X_s \)-axis is \( \alpha_0 \).

Note: This is not the \( \alpha_0 \) meant in (8-28) on page 116. This angle remains constant during the disturbed motion and varies only if another initial, steady flight condition is chosen.

The total aerodynamic force acting on the airplane in symmetric flight is divided in:

a. the thrust of the propulsive system: \( T_p \),

b. the remaining aerodynamic force on the airplane, consisting of the components:
1. the lift, $L$, perpendicular to the direction of the undisturbed flow,
2. the drag, $D$, parallel to the direction of the undisturbed flow.

These three forces are depicted in fig. 8.6. The angle between $T_p$ and the negative $X_r$-axis is $i_p$. In the following it is assumed that $i_p$ is independent of the flight condition and does not change during the disturbed motion, as is the case with $\alpha_o$. Furthermore, $\alpha_o$ and $i_p$ are assumed to be so small, that:

$$\cos (\alpha_o + i_p) \approx 1 \quad \text{and} \quad \sin (\alpha_o + i_p) \approx \alpha_o + i_p$$

The total aerodynamic moment acting on the airplane in steady flight is not divided in contributions explicitly assigned to the thrust, the lift and the drag.

The components $X$ and $Z$ of the total aerodynamic force along the $X_s$-axis and the $Z_s$-axis of the system of stability axes can now be calculated. From fig. 8.6
follows:

\[
X = L \sin \alpha - D \cos \alpha + T_p \\
Z = -L \cos \alpha - D \sin \alpha - T_p \cdot (\alpha_o + \theta_p)
\]

In non-dimensional form, after division by \(\frac{1}{2} \rho V^2 S\):

\[
C_X = C_L \sin \alpha - C_D \cos \alpha + T'_c \\
C_Z = -C_L \cos \alpha - C_D \sin \alpha - T'_c \cdot (\alpha_o + \theta_p)
\]

(8-43)

where \(\alpha\) is now the change in angle of attack relative to the value in the equilibrium situation. In (8-43) is:

\[
T'_c = \frac{T_p}{\frac{1}{2} \rho V^2 S} = \frac{2D^2}{S} \cdot T_c
\]

In the initial, steady flight condition (the equilibrium situation) is \(\alpha = 0\), therefore:

\[
C_{X_o} = -C_D + T'_c \\
C_{Z_o} = -C_L \cdot T'_c \cdot (\alpha_o + \theta_p)
\]

In addition the following relations hold because of (8-36) and (8-35) and \(\alpha = 0\):

\[
C_{X_o} = \frac{W}{\frac{1}{2} \rho V^2 S} \cdot \sin \gamma_o \\
C_{Z_o} = -\frac{W}{\frac{1}{2} \rho V^2 S} \cdot \cos \gamma_o
\]

(8-44)
or:

\[
C_{X_o} = -C_{Z_o} \cdot \tan \gamma_o \\
= +C_L \cdot \tan \gamma_o
\]

If the steady, initial condition is one of steady flight, then according to (8-44):
\[ C_{\chi_0} = 0 \]

and, by consequence:

\[ C_D = T_c \]

For the calculation of the moments reference is made to Chapter 3. For the equations of motion it is important that in the initial steady flight condition the moments are balanced and therefore:

\[ C_{m_0} = 0 \]

8.2.3. The derivatives with respect to airspeed

Experimentally obtained values of the stability derivatives \( C_{X_u}, C_{Z_u} \) and \( C_{m_u} \) are rare. Nearly all measurements of stability derivatives in flight are restricted to the motions at constant airspeed. Measurements in wind tunnels are also invariably made at constant speed. Due to these circumstances it is usually not possible to compare calculated with measured values of \( C_{X_u}, C_{Z_u} \) and \( C_{m_u} \). The expressions for the derivatives with respect to airspeed to be discussed in the following, are given with the caveat, that they have been calibrated against experimental data to an insufficient degree.

a. The derivative \( C_{X_u} \)

According to table 8.4, see page 238, is:

\[ C_{X_u} = \frac{1}{\frac{1}{2}\rho VS} \cdot \frac{\partial X}{\partial V} \]

Also:

\[ X = C_{X_u} \cdot \frac{1}{2}\rho V^2 S \]

From this follows:

\[ \frac{\partial X}{\partial V} = C_{X_u} \rho VS + \frac{\partial C_{X_u}}{\partial V} \cdot \frac{1}{2}\rho V^2 S \]
and, by consequence:

\[ C_{Xu} = 2 \, C_X + \frac{\partial C_X}{\partial V} \cdot V \]  

(8-45)

where:

\[ \frac{\partial C_X}{\partial V} \cdot V = \frac{\partial C_X}{\partial U} \]

see (8-37).

The partial derivative with respect to airspeed is determined for deviations from the steady, initial flight condition in airspeed only, i.e. at \( \alpha = 0 \).

This means in (8-45):

\[ C_X = C_{Xo} \]

and:

\[ C_{Xu} = 2 \, C_{Xo} + \frac{\partial C_X}{\partial V} \cdot V \]  

(8-46)

The starting point for the determination of the partial derivative \( \frac{\partial C_X}{\partial V} \) is the expression for \( C_X \) at \( \alpha = 0 \).

\[ C_X = -C_D + T_c' \]

Differentiating with respect to \( V \) yields:

\[ \frac{\partial C_X}{\partial V} = \frac{\partial T_c'}{\partial V} - \frac{\partial C_D}{\partial V} \]  

(8-47)

b. In analogy with (8-46) it follows for the derivative \( C_{Zu} \):

\[ C_{Zu} = 2 \, C_{Zo} + \frac{\partial C_Z}{\partial V} \cdot V \]  

(8-48)

In the steady, initial condition is:

\[ C_Z = -C_L - T_c' \, (\alpha_o + \alpha_p) \]

After differentiating it follows, in analogy with (8-47):
\[
\frac{\partial C_L}{\partial V} = - \frac{\partial C}{\partial V} (\alpha_o + 1_p) - \frac{\partial C_L}{\partial V}
\]

(8-49)

c. Following (8-46) and (8-48), \( C_{m_u} \) is written as:

\[
C_{m_u} = 2 C_m + \frac{\partial C}{\partial V} \cdot V
\]

Since \( C_{m_o} = 0 \) in the steady, initial condition, \( C_{m_u} \) is reduced to:

\[
C_{m_u} = \frac{\partial C}{\partial V} \cdot V
\]

(8-50)

***

The derivatives \( \frac{\partial C_D}{\partial V} \), \( \frac{\partial C_L}{\partial V} \) and \( \frac{\partial C_m}{\partial V} \) may differ from zero due to various causes.

1. The variation of Mach number with airspeed. Due to the effects of compressibility, the aerodynamic coefficients vary with airspeed at Mach number higher than .6 to .7.

2. The variation of the airplane's aeroelastic deformation with airspeed, or rather with dynamic pressure. This effect is not further considered here.

3. The variation of Reynolds number with airspeed. This influence is entirely neglected for the small variations of airspeed here considered.

4. For a propeller-driven airplane the contribution to the coefficients \( C_L \), \( C_D \) and \( C_m \) made by those parts of the wing and the tailplane submerged in the slipstream, varies with airspeed, via the thrust coefficient \( T_c \). For \( \frac{\partial C_D}{\partial V} \) and \( \frac{\partial C_L}{\partial V} \), the influences of compressibility and slipstream effects are taken separately. The same applies to \( \frac{\partial C_m}{\partial V} \). The partial derivatives with respect to \( V \) then are:
\[
\frac{\partial C_D}{\partial V} = \frac{\partial C_D}{\partial M} \cdot \frac{\partial M}{\partial V} + \frac{\partial C_D}{\partial T_c} \cdot \frac{dT_c'}{dV} \\
\frac{\partial C_L}{\partial V} = \frac{\partial C_L}{\partial M} \cdot \frac{\partial M}{\partial V} + \frac{\partial C_L}{\partial T_c} \cdot \frac{dT_c'}{dV} \\
\frac{\partial C_m}{\partial V} = \frac{\partial C_m}{\partial M} \cdot \frac{\partial M}{\partial V} + \frac{\partial C_m}{\partial T_c} \cdot \frac{dT_c'}{dV}
\]

The expressions (8-46), (8-48) and (8-50) can now be written as:

\[
C_{x_u} = 2 C_{x_o} + \left(1 - \frac{\partial C_D}{\partial T_c}\right) \cdot \frac{dT_c'}{dV} \cdot V - \frac{\partial C_D}{\partial M} \cdot M \tag{8-51}
\]

\[
C_{z_u} = 2 C_{z_o} - \left(\alpha_o + 1 + p\right) \cdot \frac{\partial C_L}{\partial T_c} \cdot \frac{dT_c'}{dV} \cdot V - \frac{\partial C_L}{\partial M} \cdot M \tag{8-52}
\]

\[
C_{m_u} = \frac{\partial C_m}{\partial T_c} \cdot \frac{dT_c'}{dV} \cdot V + \frac{\partial C_m}{\partial M} \cdot M \tag{8-53}
\]

In (8-51), (8-52) and (8-53) the partial derivatives with respect to Mach number are not further considered here. The effects of compressibility, expressed here through the presence of the partial derivatives \(\frac{\partial C_D}{\partial M}\), \(\frac{\partial C_L}{\partial M}\) and \(\frac{\partial C_m}{\partial M}\), are not the subject of these lecture notes. The slipstream effect, expressed through \(\frac{\partial C_D}{\partial T_c}\), \(\frac{\partial C_L}{\partial T_c}\) and the various contributions to \(\frac{\partial C_m}{\partial T_c}\) are not discussed in these lecture notes either.

The derivative \(\frac{dT_c'}{dV}\) is now briefly discussed. The basic assumption is always that during the disturbed airplane motions to be studied, the power setting of the engine(s), i.e. the throttle position and the setting of any other possible control lever, remains unchanged.

A next assumption to be made when determining the variation of \(T_c'\) is, that changes in airspeed always occur in a quasi-steady manner, i.e. so slowly that \(T_c'\) varies with airspeed just as in a series of steady flight conditions. If these assumptions are met, \(\frac{dT_c'}{dV}\) can be calculated in a fairly simple manner. In
the following \( \frac{dT'}{dV} \) is derived for various types of propulsive systems.

1. Jet turbines and rocket motors

In studies on dynamic stability characteristics the usual assumption is, that the thrust of these engines is independent of airspeed at constant throttle setting.

This means:

\[
T_p = T'_c \cdot \frac{1}{2} \rho V^2 S = \text{constant}
\]

This results in:

\[
\frac{dT'_c}{dV} = - \frac{2T'}{V}
\]

At supersonic speeds in particular, the thrust of a jet turbine does vary with airspeed. The correct value \( \frac{dT'_c}{dV} \) then has to be obtained from more detailed calculations or from measurements.

2. Piston or turbine engines driving constant speed propellers

In this case, again at constant power setting, the power produced by the engine is approximately independent of airspeed. If the propeller efficiency is also assumed to be constant for relatively small variations in airspeed, then:

\[
T_p \cdot V = \text{constant}
\]

or:

\[
T'_c \cdot V = \text{constant}
\]

This results in:

\[
\frac{dT'_c}{dV} = - 3 \frac{T'_c}{V}
\]

Finally, it should be noted that for any propulsive system in gliding flight \((T'_c = 0)\) the relation holds:
\[
\frac{dT'}{d\nu} = 0
\]

The foregoing expressions for \(\frac{dT'}{d\nu}\) may be summarized as follows:

\[
\frac{dT'}{d\nu} = -k \cdot \frac{T'}{\nu}
\]

where \(k\) is:

a) in the glide \hspace{1cm} k = 0

b) jet turbines at subsonic speeds, rocket motors \hspace{1cm} k = 2

c) piston and turbine engines driving constant speed propellers \hspace{1cm} k = 3

As derived in 8.2.2.:

\[
C_{X_0} = -C_D + T'_c
\]

\[
C_{Z_0} = -C_L + T'_c \left( a_o + i_p \right)
\]

Using these relations, (8-51), (8-52) and (8-53) can finally be written as:

\[
C_{X_u} = -2C_D + T'_c \left[ 2 - k \cdot \left( 1 - \frac{\partial C_D}{\partial T'_c} \right) \right] - \frac{\partial C_D}{\partial M} \cdot M
\] (8-54)

\[
C_{Z_u} = -2C_L + T'_c \left[ (-2 + k)(a_o + i_p) + k \frac{\partial C_L}{\partial T'_c} \right] - \frac{\partial C_L}{\partial M} \cdot M
\] (8-55)

\[
C_{m_u} = -k T'_c \cdot \frac{\partial C_m}{\partial T'_c} + \frac{\partial C_m}{\partial M} \cdot M
\] (8-56)

If \(C_{X_o} = C_L \cdot \tan \gamma_o\) is used, (8-54) is:

\[
C_{X_u} = 2C_L \cdot \tan \gamma_o - k T'_c \cdot \left( 1 - \frac{\partial C_D}{\partial T'_c} \right) - \frac{\partial C_D}{\partial M} \cdot M
\] (8-57)

***
A few specific cases are:

a) Gliding flight at subsonic speed \( k = 0 \), \( \frac{\partial C_D}{\partial M} = \frac{\partial C_L}{\partial M} = \frac{\partial C_m}{\partial M} = 0 \)

\[
\begin{align*}
C_{X_u} &= 2 C_L \tan \gamma_o = -2 C_D \\
C_{Z_u} &= -2 C_L \\
C_{m_u} &= 0
\end{align*}
\]

b) Level flight at subsonic speed for a jet- or rocket-propelled airplane \( (\gamma_o = 0, k = 2, T' = C_D, \frac{\partial C_D}{\partial M} = \frac{\partial C_L}{\partial M} = \frac{\partial C_m}{\partial M} = 0) \)

\[
\begin{align*}
C_{X_u} &= -2 C_D \\
C_{Z_u} &= -2 C_L \\
C_{m_u} &= -2 C_D \cdot \frac{\partial C_m}{\partial T'_c}
\end{align*}
\]

c) Level flight at subsonic speed for a propeller-driven airplane having constant speed propellers \( (\gamma_o = 0, k = 3, T' = C_D, \frac{\partial C_D}{\partial M} = \frac{\partial C_L}{\partial M} = \frac{\partial C_m}{\partial M} = 0) \)

\[
\begin{align*}
C_{X_u} &= -3 C_D \left( 1 - \frac{\partial C_D}{\partial T'_c} \right) \\
C_{Z_u} &= -2 C_L + C_D \left[ - \left( \alpha_o + \frac{\partial C_L}{\partial T'_c} \right) + 3 \frac{\partial C_D}{\partial T'_c} \right] \\
C_{m_u} &= -3 C_D \cdot \frac{\partial C_m}{\partial T'_c}
\end{align*}
\]

8.2.4. The derivatives with respect to angle of attack

A change in angle of attack only is obtained by varying the speed \( w \) along the \( Z_s \)-axis. A pure '\( \alpha \)-motion' is thus seen to be a steady motion where the airplane performs a translation along the \( Z_s \)-axis. The stability derivatives \( C_{X_{\alpha}}, C_{Z_{\alpha}}, \) and \( C_{m_{\alpha}} \) pertaining to this type of motion may be obtained in a relatively
simple way from wind tunnel measurements on a model of the airplane. $C_{\alpha}^x$ and $C_{\alpha}^z$ are derived from a measured or calculated polar curve corresponding to the specific airplane configuration, whereas $C_{m\alpha}^m$ follows from a measured or calculated moment curve $C_{m\alpha}$ - $\alpha$. A discussion of each of the three derivatives is given below.

a. $C_{\alpha}^x$. According to table 8.4, see page 238, is:

$$C_{\alpha}^x = \frac{1}{1\rho SV} \cdot \frac{\delta x}{\delta \omega} \cdot \frac{\delta c_{\alpha}^x}{\delta \alpha}$$

From (8-43) follows that:

$$C_{\alpha}^x = C_L \sin \alpha - C_D \cos \alpha + T_c'$$

Taking the derivative with respect to $\alpha$, leaving $T_c'$ constant, results in:

$$C_{\alpha}^x = C_L \cos \alpha + C_L \sin \alpha + C_D \sin \alpha - C_D \cos \alpha$$

In the steady, initial conditioning $\alpha = 0$, leading to:

$$C_{\alpha}^x = C_L - C_D$$

(8-58)

If $C_{\alpha}$ is expressed using $C_N$ and $C_T$, $C_{\alpha}^x$ can be written as:

$$C_{\alpha}^x = -C_T \cos \alpha - C_N \sin \alpha$$

If a parabolic polar curve is assumed, then:
\[ C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} \]

where \( C_{D_0} \) is the drag coefficient at \( C_L = 0 \). Then \( C_D \) is:

\[ C_D = 2 \frac{C_L}{\pi Ae} \cdot C_L \]

and \( C_{X_\alpha} \) can be written as:

\[ C_{X_\alpha} = C_L \left( 1 - \frac{2C_L}{\pi Ae} \right) \]

Contrary to most stability derivatives, \( C_{X_\alpha} \) is normally positive.

b. \( C_{Z_\alpha} \)

According to table 8.4 on page 238, is:

\[ C_{Z_\alpha} = \frac{1}{\rho V S} \cdot \frac{\partial Z}{\partial \alpha} = -\frac{\partial C_Z}{\partial \alpha} \]

\( C_Z \) has been written in (8-43) as:

\[ C_Z = -C_L \cos \alpha - C_D \sin \alpha - T' \cdot (\alpha_o + i_p) \]

This results in \( C_{Z_\alpha} \):

\[ C_{Z_\alpha} = C_L \sin \alpha - C_L \cos \alpha - C_D \cos \alpha - C_{L_\alpha} \sin \alpha \]

which reduces for \( \alpha = 0 \) to:

\[ C_{Z_\alpha} = -C_{L_\alpha} - C_D \]

As normally \( C_D \ll C_{L_\alpha} \), a simplified expression for \( C_{Z_\alpha} \) is:
\[ C_{Z\alpha} = -C_{L\alpha} \]

The dominant contribution to \( C_{Z\alpha} \) is, of course, provided by the wing. But in some cases the contribution made by the horizontal tailplane is not negligible. Then it may be practical to express \( C_{Z\alpha} \) as in Chapters 3 and 4, see e.g. (4-10):

\[ C_{Z\alpha} = -C_{Nw\alpha} - C_{Nh\alpha} \left( 1 - \frac{d\xi}{d\alpha} \right) \left( \frac{h}{V} \right)^2 \frac{S_h}{S} \]  

(8-60)

Data for a quantitative determination of \( C_{Z\alpha} \) using (8-60) can be found in various places in the literature, see e.g. refs. 2.3 and 8.15.

c. \( C_{m\alpha} \). According to table 8.4, see page 238, is:

\[ C_{m\alpha} = \frac{1}{4\rho V^2} \cdot \frac{\partial M}{\partial w} = \frac{\partial C_m}{\partial \alpha} \]

In Chapter 3 the simplified case is considered where the contributions of the tangential forces to \( C_{m\alpha} \) are neglected and the contributions of the thrust and slipstream effects are also not considered. It was then found for \( C_{m\alpha} \):

\[ C_{m\alpha} = C_{Nw\alpha} \cdot \frac{x_{cg\alpha} - x_w}{c} - C_{Nh\alpha} \left( 1 - \frac{d\xi}{d\alpha} \right) \left( \frac{h}{V} \right)^2 \frac{S_h}{S} \frac{\dot{h}}{c} \]

(8-61)

More general \( C_{m\alpha} \) is the slope of the moment curve \( C_m = C_m(\alpha) \) at the respective angle of attack, at which according to 8.2.2, \( C_{m\alpha} = 0 \). A calculation of \( C_{m\alpha} \) is possible using the literature mentioned in Chapter 3.

8.2.5. The derivatives with respect to pitching velocity

Before discussing the derivatives \( C_{x_q}, C_{z_q} \) and \( C_{m_q} \), the angular motion under consideration will be studied more closely. During the turning, symmetric motion, indicated here as 'q-motion', the center of rotation lies on the top
\[ \Delta \alpha = \frac{x - x_{cg}}{R} = \frac{x - x_{cg}}{c} \cdot \frac{q \bar{c}}{V} \]

Fig. 8.7: The pure \( q \)-motion.

Fig. 8.8: Harmonic \( q \)-motion; \( \alpha = \text{constant}, \theta \equiv \gamma \).
axis through the airplane's center of gravity, see fig. 8.7.
The distance \( R \) to the c.g. is:

\[
R = \frac{V}{q}
\]

At a positive pitching velocity the center of rotation lies on the negative \( Z_s \)-axis, i.e. 'above' the airplane.

In all points of the airplane, with the exception of the c.g. and all other points on the \( Z_s \)-axis, the geometric angle of attack is proportional to the angular velocity and to the distance in \( X_s \)-direction to the center of gravity:

\[
\Delta \alpha = \frac{x - x_{c.g.}}{R} = \frac{x - x_{c.g.}}{c} \cdot \frac{qC}{V} \tag{8-62}
\]

see fig. 8.7.

The magnitude of the local airspeed varies (in principle) also with the angular velocity. This is so, because the local velocity in a given point varies proportional with the distance of that point to the center of rotation. Only in points at the distance \( R \) of the center of rotation - i.e. also in the c.g. - the magnitude of the local velocity does not vary with the rate of pitch. The airplane c.g. is the only point where during this \( q \)-motion neither the magnitude of the velocity, nor the direction, i.e. the geometric angle of attack, vary.

Changes in the magnitude of the local airspeed and the geometric angle of attack at the airplane's c.g. are described as the '\( u \)-motion' and the '\( \alpha \)-motion' respectively.

As far as the airflow at the center of gravity is concerned, the \( q \)-motion causes only a curvature of the streamlines.

If the angular velocity during a \( q \)-motion is not constant, but varies harmonically with time, the airplane describes due to this \( q \)-motion a sinusoidal trajectory, as indicated in fig. 8.8. The most characteristic feature of this trajectory is, that the angle of attack at the airplane's c.g. remains constant during this motion, i.e. \( \dot{\alpha} = 0 \). The center of gravity moves along an undulating trajectory, such that:

\[
\dot{\theta} = q = \dot{\alpha} + \dot{\gamma} = \dot{\gamma}
\]

Both the angle of pitch and the flight path angle will vary harmonically. An entirely different motion, to be distinguished very clearly from the \( q \)-motion, results if the center of rotation lies in the airplane's c.g., see fig. 8.9.
Fig. 8.9: Harmonic combined $q$- and $\alpha$-motion; $q = \dot{\alpha}, \dot{y} =$ constant.

Fig. 8.10: Airplane in a curved flow field.

Fig. 8.11: An equivalent curved airplane in a straight flow field.
Then the angle of attack varies also in the center of gravity and:

\[ \dot{\theta} = q = \dot{\alpha} \quad (8-63) \]

and, by consequence:

\[ \dot{\gamma} = \dot{\theta} - \dot{\alpha} = 0 \quad (8-64) \]

According to (8-64) the center of gravity moves along a straight line, if the airplane rotates while the c.g. is the center of rotation. This motion may be considered a superposition of a q-motion as defined above, and a variable \( \alpha \)-motion, such that the condition (8-63) is satisfied.

This q-motion is here the subject of discussion, the influence on the aero-
dynamic forces and moments due to an acceleration along the top axis during a non-steady \( \alpha \)-motion is discussed in the next paragraph.

The variation of the magnitude of the local airspeed with the distance to
the center of rotation during the q-motion is usually neglected, because of the
small dimensions of the airplane in the \( Z_{s} \)-direction if compared with common
values of \( R \). The only effect of the rotation is then a variation of the
geometric angle of attack in the \( X_{s} \)-direction, expressed by (8-62), see fig.
8.10.

A field of flow equivalent to that of the airplane in curved flow can be
obtained by placing a suitable curved airplane in a field of parallel flow, see
fig. 8.11.

The curvature of the airplane or the wind tunnel model must be such that:

\[ \Delta \alpha = - \frac{dz}{dx} = + \frac{x - x_{c.g.}}{c} \cdot \frac{q \dot{c}}{V} \]

see (8-62), or:

\[ \frac{z - z_{c.g.}}{c} = - \frac{1}{4} \cdot \left( \frac{x - x_{c.g.}}{c} \right)^2 \cdot \frac{q \dot{c}}{V} \]

The curvature is parabolic in the \( X_{s} \)-direction and proportional to \( \frac{q \dot{c}}{V} \). For
theoretical calculations it may be advantageous to use the curved airplane in
parallel flow. If, however, the stability derivatives would have to be measured
using a curved model in parallel flow, it would be necessary to use a separate
model for each value of $\frac{q c}{V}$.

The changes in the forces $X$ and $Z$ and in the moment $M$ caused by the $q$-motion are discussed in the following.

The change in $X$, expressed by the derivative $C_X^q$:

$$C_X^q = \frac{1}{\frac{1}{2} \rho V S c} \cdot \frac{\partial X}{\partial q} = \frac{\partial C_X}{\partial q} \frac{q c}{V}$$

is always neglected. The assumption thus is:

$$C_X^q = 0$$

The two remaining derivatives are $C_Z^q$ and $C_m^q$:

$$C_Z^q = \frac{1}{\frac{1}{2} \rho V S c} \cdot \frac{\partial Z}{\partial q} = \frac{\partial C_Z}{\partial q} \frac{q c}{V}$$

$$C_m^q = \frac{1}{\frac{1}{2} \rho V S c^2} \cdot \frac{\partial M}{\partial q} = \frac{\partial C_m}{\partial q} \frac{q c}{V}$$

Of these two derivatives $C_m^q$ is the most important. For an airplane having a horizontal tailplane, the largest contribution to $C_m^q$ is provided by this tailplane. The change in wing lift caused by the angular velocity, i.e. by the curvature of the flow field, is small. Therefore, no change in downwash, as a reaction to a change in wing lift, is accounted for either. The contribution from the tailplane is calculated as follows.

According to (8-62) the change in angle of attack of the horizontal tailplane due to the rotation is:

$$\Delta \alpha_h = \frac{x_h - x_{c.g.}}{c} \cdot \frac{\dot{q} c}{V}$$

$$= \frac{b}{c} \cdot \frac{\dot{q} c}{V}$$
This change in angle of attack causes a normal force at the tailplane:

\[ \Delta C_{N_h} = C_{N_h \alpha} \Delta \alpha_h \]

and this generates a moment about the airplane's c.g.:

\[ \Delta C_m = -C_{N_h \alpha} \Delta \alpha_h \left( \frac{V}{V'} \right) \frac{S_h f_h'}{S_c} \]

\[ = -C_{N_h \alpha} \left( \frac{V}{V'} \right)^2 \frac{S_h f_h'^2}{S_c^2} \cdot \frac{q_c}{V} \]

As a consequence the contribution of the horizontal tailplane to \( C_{m_q} \) is:

\[ (C_{m_q}) = -C_{N_h \alpha} \left( \frac{V}{V'} \right)^2 \frac{S_h f_h'^2}{S_c^2} \]

and the contribution to \( C_{Z_q} \) is:

\[ (C_{Z_q}) = -C_{N_h \alpha} \left( \frac{V}{V'} \right)^2 \frac{S_h f_h'}{S_c} \]

A rough estimate of \( C_{Z_q} \) of the complete airplane is sometimes taken as:

\[ C_{Z_q} = 2 \left( C_{Z_q} \right) = -2 C_{N_h \alpha} \left( \frac{V}{V'} \right)^2 \frac{S_h f_h'}{S_c} \]

For airplanes having a straight, slender wing \( C_{m_q} \) of the entire airplane is usually approximated by multiplying the contribution of the tailplane with a factor 1.1 to 1.2, in order to account for the influence of the wing. For these conventional airplanes \( C_{m_q} \) can be expressed by:
Finally, a practical remark should be made. Often, quantitative data for the determination of $C_{Z_q}$ and $C_{m_q}$ are taken from publications of the American National Aeronautics and Space Administration (NASA). It is of interest to realize, that NASA - and some other organizations as well - commonly relate the stability derivatives $C_{Z_q}$ and $C_{m_q}$ to $\frac{qC}{2V}$ rather than $\frac{qC}{V}$. In table 8.4, see page 238, correction factors are given to reduce the derivatives according to the NASA definition to the derivatives as used in this text.

In contrast to a 'u'- or 'α-motion', the 'q-motion' changes with a shift in the c.g. position. This can be seen as follows. In the new c.g. the change in angle of attack due to a rotation about the new center of rotation is - of course - again zero. This change in angle of attack at the new c.g. position c.g.2, $\Delta_\alpha_{c.g.2}$ (where $\Delta_\alpha_{c.g.1} = 0$), is the sum of a contribution from a rotation about the old c.g. (c.g.1) and a change in angle of attack due to an α-motion along the $Z_s$-axis. From this consideration $C_{m_q}$ can be calculated in a straightforward manner.

In table 8.6, see page 242, the expressions for the variations of $C_{Z_q}$ and $C_{m_q}$ with the c.g. position are given, based on the above discussion.

8.2.6. The derivatives with respect to the acceleration along the top axis

If the components of airspeed u or w or the rate of pitch q of an airplane experience a sudden change, a certain time interval passes before the pressure distribution over the entire airplane has adjusted to the new flow condition. Usually, changes in the airspeed, i.e. in the component u, occur so slowly that this delayed adjustment is not noticeable. Changes in u are, therefore, always assumed to occur in a quasi-steady fashion. On the other hand, changes in angle of attack, i.e. in the speed component w along the $Z_s$-axis, may occur much more quickly. Such changes are discussed in the following.

In ref. 8.37 Cowley and Clauvert were the first to point to the fact that an acceleration $\dot{w}$ of the airplane in the direction of the top axis has a non-
negligible influence on the longitudinal moment $C_m$. This effect has an important influence on the damping of the symmetric motions. Following Cowley and Glaucert, this effect is usually expressed through a stability derivative $C_m'$. In order to account for the influence of the c.g. on this derivative $C_m'$, it is necessary in principle to know as well the derivative $C_m''$ which is in itself quite important.

As described earlier, see page 111, the relation between the aerodynamic coefficient $C_m$ and the component of the motion $\alpha$ may be expressed in a Taylor series:

$$
\Delta C = \frac{\delta C_m}{\delta \alpha} \cdot \Delta \alpha + \frac{\delta C_m}{\delta \alpha} \cdot \frac{\alpha}{V} \cdot \frac{\alpha}{V} + \frac{\delta^2 C_m}{\delta \alpha^2} \cdot \Delta \alpha \cdot \frac{\alpha}{V} + \frac{\delta^2 C_m}{\delta \alpha \delta \frac{\alpha}{V}} \cdot \left( \frac{\alpha}{V} \right)^2 + \frac{1}{3!} \left\{ \right\} \quad \ldots \quad (8-65)
$$

In principle, the derivatives with respect to $\frac{\alpha}{V}^2$, $\frac{\alpha}{V}^3$ etc. could also be included in this series expansion. Experimental evidence shows, however, that the changes in angle of attack occur slowly enough to neglect any influences of the higher time derivatives than $\alpha$ the first. As usual in the series (8-65) only the linear terms are maintained, under the assumption that the values of $\Delta \alpha$ and $\frac{\alpha}{V}$ remain sufficiently small.

The change in the moment coefficient, $\Delta C_m$, can then be written as:

$$
\Delta C_m = \frac{\delta C_m}{\delta \alpha} \cdot \Delta \alpha + \frac{\delta C_m}{\delta \alpha} \cdot \frac{\alpha}{V} \quad \frac{\alpha}{V}
$$

or, in the common notation used in stability analyses:

$$
C_m = C_{m'} \cdot \alpha + C_{m''} \cdot \frac{\alpha}{V} \quad \frac{\alpha}{V}
$$

(8-66)

The term $C_{m''} \cdot \frac{\alpha}{V}$ is then supposed to express the influence on $C_m$ of the delayed adjustment of the airflow to changes in $\alpha$. 

For airplanes having a horizontal tailplane, this tailplane usually provides the most important contribution to \( C_{m_{\alpha}} \). A simplified explanation of this effect is the following.

In Chapter 3 the angle of attack of the horizontal tailplane in steady, straight flight was expressed by (3-24):

\[
\alpha_h = \alpha - \varepsilon + i_h
\]  

(3-24)

During the accelerated translation along the \( Z_s \)-axis here considered, \( i_h \) remains constant. In accordance with the conventions made in the present chapter, see page 116, the explicit indication that a change in a variable is considered, is omitted. As a consequence \( \alpha_h(t) \) can be written here as:

\[
\alpha_h(t) = \alpha(t) - \varepsilon(t)
\]

The horizontal tailplane is always situated in an air mass having passed the wing a brief time interval \( \Delta t \):

\[
\Delta t = \frac{i_h}{V}
\]

earlier. As a consequence, the downwash angle at the horizontal tailplane is at any time proportional to the wing angle of attack that brief time interval earlier:

\[
\varepsilon(t) = \frac{d\varepsilon}{d\alpha} \cdot \alpha(t - \Delta t)
\]

(8-67)

The angle of attack of the wing at the time \( t - \Delta t \) can be written as:

\[
\alpha(t-\Delta t) = \alpha(t) + \ddot{\alpha} \Delta t + \frac{1}{2!} \alpha \dot{\alpha} \Delta t^2 - \frac{1}{3!} \alpha \Delta t^3 \ldots ...
\]

In this expression the time derivative of \( \alpha \) higher than the first are omitted, as in (8-65). The result is:

\[
\alpha(t-\Delta t) = \alpha(t) - \ddot{\alpha} \Delta t
\]

In (8-67) this leads to:

\[
\varepsilon(t) = \frac{d\varepsilon}{d\alpha} \cdot \alpha(t) - \frac{d\varepsilon}{d\alpha} \cdot \ddot{\alpha} \cdot \frac{i_h}{V}
\]
and:

\[ a_h(t) = a(t) \cdot (1 - \frac{dx}{dx} \cdot \frac{dx}{\alpha} \cdot \frac{\ddot{h}}{V}) \]

The term in the change in \( a_h \) proportional to \( \dot{a} \) is then:

\[ \Delta a_h = + \frac{dx}{dx} \cdot \frac{\ddot{h}}{c} \cdot \frac{\Delta c}{V} \]

This \( \Delta a_h \) produces a force:

\[ \Delta C_z = -C_{h\alpha} \cdot \Delta a_h \cdot \left( \frac{V}{V} \right) \cdot \left( \frac{h}{V} \right) \cdot \frac{S_h}{S_c} \]

and a longitudinal moment:

\[ \Delta C_m = -C_{h\alpha} \cdot \Delta a_h \cdot \left( \frac{V}{V} \right) \cdot \left( \frac{h}{V} \right) \cdot \frac{S_h}{S_c} \cdot \frac{h^2}{S_c} \]

Substituting (8-68) in the latter two expressions results in the desired - approximated - expressions for the stability derivatives:

\[ C_{z\alpha} = -C_{h\alpha} \cdot \left( \frac{V}{V} \right) \cdot \frac{dx}{dx} \cdot \frac{S_h}{S_c} \cdot \frac{h^2}{S_c} \]

\[ C_{m\alpha} = -C_{h\alpha} \cdot \left( \frac{V}{V} \right) \cdot \frac{dx}{dx} \cdot \frac{S_h}{S_c} \cdot \frac{h^2}{S_c} \]

The variation of \( C_{m\alpha} \) with the c.g. position is calculated with:

\[ \Delta C_{m\alpha} = -C_{z\alpha} \cdot \Delta c.g. \cdot \frac{\ddot{c}}{c} \]
In the calculation of $C_{Z\delta}$ and $C_{m\delta}$, attention should be given - as with $C_{Zq}$ and $C_m$ - to the fact that in the literature these derivatives occur referenced to $\frac{\alpha c}{2V}$ rather than to $\frac{\alpha c}{V}$. Correction factors have been given in table 8.4.

8.2.7. The derivatives with respect to the elevator angle

a. $C_{X\delta}$. According to table 8.4, see page 238, is:

$$C_{X\delta} = \frac{1}{\rho V^2 S} \cdot \frac{\delta X}{\delta e} = \frac{\delta C_X}{\delta e}$$

This derivative expresses the variation of airplane drag with the elevator angle. Although the total drag caused by the elevator deflection may be non-negligible, especially at supersonic speed ('trim drag'), the variation in $C_X$ with small variations of $\delta e$ is commonly neglected. Or:

$$C_{X\delta} = 0$$

b. $C_{Z\delta}$. According to table 8.4 is:

$$C_{Z\delta} = \frac{1}{\rho V^2 S} \cdot \frac{\delta Z}{\delta e} = \frac{\delta C_Z}{\delta e}$$

If the airplane has a horizontal tailplane, $C_{Z\delta}$ can also be written more explicitly as:

$$C_{Z\delta} = -C_{N_h\delta} \cdot \left(\frac{h}{V}\right)^2 \cdot \frac{S_h}{S} \quad (8-69)$$

Refs. 8.15 and 8.41 to 8.46 provide data to derive $C_{N_h\delta}$ if the form and the dimensions of the stabilizer and the elevator are known.
For tailless airplanes $C_{Z_δ}$ should preferably be obtained from measurements on a wind tunnel model, see also refs. 8.42, 8.43, 8.45 and 8.46.

c. $C_{m_δ}$. According to table 8.4 is:

$$C_{m_δ} = \frac{1}{\frac{1}{2} \rho V^2 Sc} \cdot \frac{δM}{δa} = \frac{2C_m}{δe}$$

If the airplane has a horizontal tailplane, $C_{m_δ}$ can be simply expressed using $C_{Z_δ}$:

$$C_{m_δ} = C_{Z_δ} \cdot \frac{x_h - x_{c.g.}}{c}$$

or:

$$C_{m_δ} = -C_{N_h_δ} \cdot \left(\frac{V}{V_h}\right)^2 \frac{S_h}{S} \cdot \frac{x_h - x_{c.g.}}{c}$$

$$= -C_{N_h_δ} \cdot \left(\frac{V}{V_h}\right)^2 \frac{S_h^{\dagger} h}{5c}$$

This results in $C_{m_δ}$, if $C_{Z_δ}$ has been previously obtained using (8-69).

Of the three control derivatives $C_{m_δ}$ has the largest effect on the airplane motions. For airplanes having a horizontal tailplane, the influence of $C_{m_δ}$ dominates to such an extent, that very often $C_{m_δ}$ for these airplanes is neglected. This approximation is less applicable to tailless airplanes. The derivative $C_{m_δ}$ for tailless airplanes is derived by the same calculation methods or wind tunnel measurements yielding also $C_{Z_δ}$.

Table 8.6, see page 242, summarizes the expressions for the stability and control derivatives discussed in the previous paragraphs.
8.2.8. The derivatives for the asymmetric airplane motions

The asymmetric aerodynamic derivatives have all been discussed in Chapter 7. Reference is made to that Chapter and the corresponding literature, see page 247.

8.2.9. The symmetric and asymmetric inertial parameters

Reliable quantitative data on moments and products of inertia of airplanes are rare. Quantitative calculations are very time-consuming and may produce rather inaccurate results.

The most reliable inertial parameters of airplanes are those obtained experimentally - usually by oscillating the suspended airplane - on the ground. Details on this experimental technique may be found in refs. 8.48 to 8.51.

Table 8.7, see page 244, gives some symmetrical and asymmetrical inertial data. They have been derived in part from ref. 8.47.

8.3. The stability of an equilibrium condition

8.3.1. Introduction

Once the equations of motion of an airplane have been derived and the required aerodynamic derivatives are known, it is possible to calculate the airplane motions caused by an arbitrary control deflection or external disturbance. In the most general case, if the equations of motion to be used are non-linear, only a numerical integration of the equations is possible.

Many airplane motions, however, may be considered as deviations from a condition of steady, straight, symmetric flight, small enough to permit linearization of the equations about that reference condition. In paragraph 8.6.1 of this Chapter the linearization has been given. Thereafter it is possible to use analytical methods, such as the Laplace-transform to solve the linearized equations. Even then, the electronic computer remains a nearly indispensable tool to carry out the calculations.

If the motions of an airplane about a given equilibrium condition are studied, it is - quite independent of the type of disturbance to be considered - of primary interest to know if the equilibrium is stable. The following discusses how this question is answered.
Stability is a characteristic of any equilibrium condition of any arbitrary system. As indicated already in Chapter 1, the question of the stability of an equilibrium condition relates to the behaviour of the system when it has obtained a small deviation from the equilibrium due to some disturbance, once the disturbance has ceased to act. If, after some time, the system returns to the original equilibrium condition, that condition is stable. If the system does not return to the original condition but deviates increasingly with time, the equilibrium was unstable. Finally, the equilibrium condition is neutrally stable, if the deviation caused by the disturbance neither disappears with time nor keeps increasing.

If the motions of the system under study - here the airplane - about the initial reference condition can be described by linear equations of motion, the stability of the equilibrium condition is independent of the type and magnitude of the disturbance and of the magnitude of the initial deviation from the equilibrium condition. The condition of linearity is not always satisfied. This applies also to the motions of airplanes.

In the following, stability is a characteristic of an arbitrary steady flight condition of an airplane. For the sake of simplicity the 'stability of an airplane' is often referred to, whereas the stability of a certain equilibrium condition of that airplane is actually meant. This is not quite correct, as any particular airplane may be in a stable and sometimes also in an unstable equilibrium condition. The behaviour of the airplane once it has acquired a small deviation from an equilibrium condition is described by the linearized 'deviation equations' (8-39) and (8-41).

In order to simplify the discussions, in the following paragraphs only the longitudinal stability is discussed. The stability criteria to be derived and the methods used to solve the equations of motion are equally applicable to the asymmetric motions to be considered near the end of this Chapter.

The linearized 'deviation equations' for the symmetric motions, where $C_X = 0$, are repeated here to begin the discussion.

\[
\begin{bmatrix}
C_{\alpha} & -2\mu_c D_c & C_{\alpha} & C_{Z_0} & 0 \\
C_{\alpha} & C_{\alpha} + (C_{\alpha} - 2\mu_c) D_c & -C_{X_0} & C_{Z_q} + 2\mu_c \\
0 & 0 & -D_c & 1 \\
C_{m_\alpha} & C_{m_\alpha} + C_{m_\alpha} D_c & 0 & -C_{m_\alpha} - 2\mu_c K_{\alpha} D_c & 0
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{\alpha} \\
\dot{q} \\
\dot{\theta} \\
\dot{\delta}_e
\end{bmatrix}
\begin{bmatrix}
-C_{X_0} \\
-C_{Z_0} \\
-D_c \\
-C_{m_\alpha} \\
-C_{m_\alpha}
\end{bmatrix}
\]

(8-39a)
This is a group of four simultaneous, constant coefficient, linear differential equations of first order. The independent variable in the equations is - in the physical sense - the disturbance variable, i.e. the elevator deflection. The dependent variables are the four components of the motion.

According to the foregoing, the stability of the equilibrium condition is apparent from the airplane's motion, once it has acquired a deviation from the equilibrium situation due to a disturbance and the disturbance has ceased to act.

In the equations of motion this means that the time history of the dependent variable is studied, assuming the independent variables to be all zero and assuming furthermore, that the dependent variables have been given non-zero initial conditions. As the equations are linear, the magnitude of the initial conditions, i.e. the magnitude of the assumed disturbances and the resulting deviations, have no influence on the stability of the equilibrium, as noted before. In the more general case, where the motions are described by non-linear equations, the magnitude of the disturbances may influence the stability.

As a result of the above arguments, the equations to be solved for the disturbed symmetric airplane motions can now be written in the following homogeneous form:

\[
\begin{bmatrix}
    C_{X_\alpha} & -2\mu_c D & C_{Z_\alpha} & 0 \\
    C_{Z_\alpha} & C_{Z_\alpha} + (C_{Z_\alpha} - 2\mu_c)D & -C_{X_\alpha} & C_{Z_\alpha} + 2\mu_c \\
    0 & 0 & -D_c & 1 \\
    C_{m_\alpha} & C_{m_\alpha} + C_{m_\alpha} D & 0 & C_{m_\alpha} - 2\mu_c K_2 D
\end{bmatrix}
\begin{bmatrix}
    \dot{u} \\
    \dot{\alpha} \\
    \dot{\theta} \\
    \dot{\phi} \\
\end{bmatrix} = 0 \quad (8-70)
\]

As the elevator angle has been assumed to remain constant, the stability to be determined from these equations is the so-called 'stability, stick fixed'. The assumption is thus made that the pilot holds the control manipulator and thereby the elevator fixed, such that the elevator angle does indeed remain constant.

A highly realistic alternative is the situation where the pilot has trimmed the control force to zero and takes his hand off the control manipulator. In that case the control mechanism is free during the disturbed motion and the elevator may vary. The actual time history of the elevator angle during the disturbed motion then follows from one or more additional equations of motion describing the behaviour of the control mechanism in the 'stick free' condition.
The stability thus to be determined, is the so-called 'stability, stick free'. The following general discussions of the stability of an equilibrium condition are valid both for the stability, stick fixed and the stability, stick free.

8.3.2. The solution of the equations of motion

The analytical solution of the linear differential equations here considered is based on the transformation of these equations into algebraic equations. This can be done in various ways.

The solution of the above homogeneous equations has the following general form, see e.g. ref. 8.10.

For the symmetric motions:

$$x = A_x e^{\lambda_c s_c}$$  \(\text{(8-71)}\)

where \(x\) represents any of the components of the motion, and the variable \(s_c\) is the time, made non-dimensional, see table 8.2, where:

$$s_c = \frac{V}{c} \cdot t$$

The coefficient \(A_x\) in (8-71) is determined partly by the initial conditions given to the equations of motion. The variable \(\lambda_c\) in (8-71) to be discussed in detail later, can be either real or complex. From the solution (8-71) it can be seen at once if the equilibrium condition under study is stable.

For stability it is both necessary and sufficient that \(x\) goes to zero with increasing time.

The requirement for stability then is:

$$\lim_{s_c \to \infty} x = 0$$

For finite values of the coefficient \(A_x\) it is the variable \(\lambda_c\) that determines entirely and only, if the stability requirement is met. It is, therefore, of primary interest to see how this \(\lambda_c\) is to be found. To this end it is only
necessary to substitute the solution (8-71) of the equations of motion back in these equations. The value or values of $\lambda_c$ turning these equations into equalities are the ones looked for.

Before performing the substitution, it may be realized that in (8-70):

$$D_c x = -\frac{c}{V} \cdot \frac{d}{dt} (A_x, e^{\frac{V t}{c}}) = \lambda_c x$$

$$D_c^2 x = \lambda_c^2 x$$

In the following, the equations for the symmetric motions are further analyzed. The result of the substitution in the equations (8-72) is:

$$
\begin{bmatrix}
C_{x_u} -2\mu_c \lambda_c & C_{x_\alpha} & C_{z_\alpha} & 0 \\
C_{z_u} & C_{z_\alpha} + (C_{z_\alpha} -2\mu_c) \lambda_c & -C_{x_\alpha} & C_{z_q} + 2\mu_c \\
0 & 0 & -\lambda_c & 1 \\
C_{m_u} & C_{m_\alpha} + C_{m_\alpha} \cdot \lambda_c & 0 & C_{m_q} - 2\mu_c \frac{K2^2}{c^2} \lambda_c
\end{bmatrix}
\begin{bmatrix}
A_u \\
A_\alpha \\
A_\theta \\
A_q
\end{bmatrix}
= 0
$$

or, in a more compact notation:

$$\lambda_c s_c \cdot [\Delta] \cdot A = 0$$

where $[\Delta]$ is the matrix and $A$ is the vector with elements $A_u, A_\alpha, \text{ etc.}$
By the substitution the equations of motion change from differential equations into simultaneous, linear, homogeneous algebraic equations. From these equations the non-zero common factor \( e^{\lambda_c t} \), can be omitted without influencing the values of \( \lambda_c \) to be obtained:

\[
[\Delta] \cdot \mathbf{A} = 0
\]

(8-74)

These homogeneous equations are always satisfied by the so-called trivial solution:

\[
\mathbf{A} = 0,
\]

which is a short notation for:

\[
A_u = A_\alpha = A_\theta = A_q = 0
\]

In (8-71) this means:

\[
\hat{u} = \hat{\alpha} = \hat{\theta} = \hat{q} = 0
\]

This is the original equilibrium flight condition, which is evidently a particular case of the disturbed motion.

This trivial solution is not further considered here.

The homogeneous, algebraic equations represented by (8-74) are in general inconsistent.

A simple example may serve to illustrate this fact. Suppose, the following two homogeneous, algebraic equations have been given:

\[
a_{11} x_1 + a_{12} x_2 = 0
\]

\[
a_{21} x_1 + a_{22} x_2 = 0
\]

Or:

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \cdot \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0
\]

From the first equation follows the solution:
\[ x_2 = -\frac{a_{11}}{a_{12}} \cdot x_1 \]

and from the second equation:

\[ x_2 = -\frac{a_{21}}{a_{22}} \cdot x_1 \]

These two solutions are only then consistent, if:

\[ -\frac{a_{11}}{a_{12}} = -\frac{a_{21}}{a_{22}} \]

or:

\[ a_{11}a_{22} - a_{12}a_{21} = 0 \]

Expressed in a different way:

\[
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix} = 0
\]

If the coefficient determinant of the homogeneous, algebraic equations is zero, the equations are no longer inconsistent. But then they have become dependent.

In the case of the airplane motions considered here, the equations (8-74) and (8-73) have a solution only for a few particular values of \( \lambda_c \). These are the values of \( \lambda_c \) resolving the inconsistency of the equations by making the equations dependent. These values of \( \lambda_c \), called the eigenvalues of the differential equations, are found according to the foregoing by equating the determinant of the square matrix \( [\Delta] \) to zero. The determinant of \( [\Delta] \) is called the characteristic determinant. The eigenvalues are then found from:

\[ [\Delta] = 0 \]

This 'characteristic equation' can be written more explicitly as:
For a further study of stability via the eigenvalues it is necessary to expand the above characteristic equation. The result may be expressed as:

$$A \lambda^4_c + B \lambda^3_c + C \lambda^2_c + D \lambda_c + E = 0$$  \hspace{1cm} (8-75)

The four roots of this characteristic equation are the four eigenvalues $\lambda_c$ to be found. According to (8-71) they determine if the original equilibrium condition is stable. The coefficients $A$ to $E$ in (8-75) are given with the more detailed discussion of dynamic longitudinal stability in paragraph 8.4.1. of this Chapter, see page 169.

The resulting eigenvalues may finally be substituted in (8-74). For each $\lambda_c$ follow an arbitrary combination of three out of the four equations the three ratio's - real or complex - of the four components of the motion. They provide further insight into the characteristics of the disturbed motion. But this subject is not further pursued here.

In general the roots of the characteristic equation are all different. After substitution in (8-71) they give the airplane motion after a disturbance from the equilibrium situation. Assuming four different eigenvalues, the solution for the symmetric motions is:

$$\begin{bmatrix}
\dot{\alpha} \\
\dot{\theta} \\
\dot{\omega}
\end{bmatrix} = [A] \begin{bmatrix}
\lambda_{1s_c} \\
\lambda_{2s_c} \\
\lambda_{3s_c} \\
\lambda_{4s_c}
\end{bmatrix}$$

\hspace{1cm} (8-76)
where the matrix \([A]\) can be written more explicitly as:

\[
[A] = \begin{bmatrix}
    A_{u_1} & A_{u_2} & A_{u_3} & A_{u_4} \\
    A_{\alpha_1} & A_{\alpha_2} & A_{\alpha_3} & A_{\alpha_4} \\
    A_{\theta_1} & A_{\theta_2} & A_{\theta_3} & A_{\theta_4} \\
    A_{q_1} & A_{q_2} & A_{q_3} & A_{q_4}
\end{bmatrix}
\]  

Combination of (8-76) and (8-77) results in the solution of the homogeneous differential equations:

\[
\dot{u} = A_{u_1} e^{\lambda_{c_1} s} + A_{u_2} e^{\lambda_{c_2} s} + A_{u_3} e^{\lambda_{c_3} s} + A_{u_4} e^{\lambda_{c_4} s}
\]

\[
\dot{\alpha} = A_{\alpha_1} e^{\lambda_{c_1} s} + A_{\alpha_2} e^{\lambda_{c_2} s} + A_{\alpha_3} e^{\lambda_{c_3} s} + A_{\alpha_4} e^{\lambda_{c_4} s}
\]

\[
\dot{\theta} = A_{\theta_1} e^{\lambda_{c_1} s} + A_{\theta_2} e^{\lambda_{c_2} s} + A_{\theta_3} e^{\lambda_{c_3} s} + A_{\theta_4} e^{\lambda_{c_4} s}
\]

\[
\frac{\dot{q}}{V} = A_{q_1} e^{\lambda_{c_1} s} + A_{q_2} e^{\lambda_{c_2} s} + A_{q_3} e^{\lambda_{c_3} s} + A_{q_4} e^{\lambda_{c_4} s}
\]

Four out of the sixteen constants in \([A]\) are fixed by the four initial conditions, which must be given if the disturbed motion is to be determined completely. The remaining twelve constants then follow from twelve real or complex ratios of the components of the motion, mentioned earlier. From this brief discussion it may be seen, that the constants in \([A]\) have no influence whatsoever on the stability or instability of the equilibrium conditions.

The various possible types of eigenvalues \(\lambda\) and the corresponding motions, the so-called 'eigenmotions' are now studied, under the assumption that all eigenvalues are different.
l. Real eigenvalues

The part of the total airplane response, i.e. the eigenmotion, corresponding to a real eigenvalue $\lambda_c$ is described by an aperiodic exponential function, see fig. 8.12. The equilibrium can only be stable, if all real eigenvalues are negative. Only then is:

$$\lim_{s_c \to \infty} A_x e^{\lambda_c s_c} = 0$$

If $\lambda_c = 0$, then:

$$A_x \frac{\lambda_c}{s_c} e^{\lambda_c s_c} = A_x$$

The eigenmotion corresponding to this particular value is a constant for any of the components of the motion. For $\lambda_c > 0$, all components of the motion tend to go to infinity with increasing time, because for any value of $A_x$:

$$\lim_{s_c \to \infty} A_x e^{\lambda_c s_c} = \pm \infty$$

The speed at which for a given negative, real $\lambda_c$ the corresponding eigenmotion converges to zero, is commonly expressed by two different characteristic times:

a. The time to damp is half amplitude, $T_d$. This is the time interval in which the exponential function decreases to half its original value. This means:

$$x(t + T_d) = \frac{1}{2} x(t)$$

or:

$$\lambda_c \frac{\dot{V}}{c} (t + T_d) = \frac{1}{2} \frac{\dot{V}}{c}$$

$$\lambda_c \frac{V}{c} = \frac{1}{2} \dot{e}$$
Fig. 8.12: Aperiodic motions corresponding to a real eigenvalue $\lambda$. 

(a) Convergent motion, $\lambda < 0$.

(b) Constant deviation, $\lambda = 0$.

(c) Divergent motion, $\lambda > 0$. 
From this follows:

\[
T_\frac{1}{4} = \ln \frac{1}{\lambda_c} \cdot \frac{\bar{c}}{V} = -\frac{0.693}{\lambda_c} \cdot \frac{\bar{c}}{V}
\]

b. The time constant \( \tau \). This is the time interval in which the exponent decreases by 1 and the exponential function itself decreases by the factor \( \frac{1}{e} \). Then:

\[
x(t + \tau) = \frac{1}{e} \cdot x(t)
\]
or:

\[
\lambda_c \frac{V}{c} (t + \tau) \quad \text{or} \quad \frac{\lambda_c}{c} \frac{V}{c} \cdot \tau
\]
e

\[
= \frac{1}{e} \cdot e^{-\frac{1}{\lambda_c} \cdot \frac{\bar{c}}{V} \cdot \tau}
\]
or:

\[
\lambda_c \frac{V}{c} \cdot \tau = -1
\]

\[
\tau = -\frac{1}{\lambda_c} \frac{\bar{c}}{V}
\]

The relations between \( T_\frac{1}{4} \) and \( \tau \) are:

\[
\tau = 1.443 \cdot T_\frac{1}{4}
\]

\[
T_\frac{1}{4} = 0.693 \cdot \tau
\]

Using the time constant, the eigenmotion corresponding to the real \( \lambda_c \) can be written as:

\[
A_c \cdot e^{\frac{\lambda_c}{c} \cdot t}
\]

\[
= A_c \cdot e^{-t/\tau}
\]

If \( \lambda_c \) is positive and an aperiodic divergence results, \( T_\frac{1}{4} \) and \( \tau \) may still be used to indicate the level of divergence. In that case both \( T_\frac{1}{4} \) and \( \tau \) are negative.
In those situations use is often made of the 'time to double amplitude', \(T_2\). It is the time interval in which the exponential function increases to twice its original value:

\[
T_2 = -\frac{T_1}{2}
\]

2. **Complex eigenvalues**

In the case of a complex \(\lambda_c\) is:

\[
\lambda_c = \xi_c + j \eta_c \quad j = \sqrt{-1}
\]

Substituting of such an eigenvalue in (8-71) or (8-72) would lead to complex expressions for the components of the motion. But complex eigenvalues occur always in complex conjugate pairs:

\[
\lambda_{c1,2} = \xi_c \pm j \eta_c
\]

The corresponding constants \(A_{x1}\) and \(A_{x2}\) in (8-71) are also complex conjugate, as the resultant eigenmotion can only be real. This means:

\[
A_{x1,2} = R_x \pm j Q_x
\]

The eigenmotion belonging to the two complex conjugate roots of the characteristic equation, can now be written as:

\[
A_{x1}e^{\lambda_{c1}sc} + A_{x2}e^{\lambda_{c2}sc} = (R_x + j Q_x)e^{(\xi_c + j\eta_c)s_c} + (R_x - j Q_x)e^{(\xi_c - j\eta_c)s_c}
\]

\[
= 2 \sqrt{R_x^2 + Q_x^2} e^{\xi_c s_c} \cos(\eta_c s_c + \arctan \frac{Q_x}{R_x})
\]

(8-78)

This part of the total response of the components of the motion is evidently a damped oscillation.

The period \(P\) of the oscillation follows from the imaginary part \(\eta_c\) of the two eigenvalues. If the argument of the harmonic function has increased by \(2\pi\), a time \(P\) has elapsed:
Fig 8.13. Periodic motions corresponding to a pair of complex conjugate eigenvalues \( \lambda_{1,2} = \xi \pm j \eta \).
\[ \eta_c \cdot \frac{V}{c} \cdot p = 2\pi \]

or:

\[ p = \frac{2\pi}{\eta_c} \cdot \frac{c}{V} \]

According to (8-78), the amplitude of the oscillation is:

\[ 2\sqrt{R_x^2 + Q_x^2} \cdot \xi_c \cdot e^{sc} \]

Apparently, only the real part \( \xi_c \) of the eigenvalues determines if the amplitude converges to zero with increasing time, see fig. 8.13. For stability it is necessary, that the real parts of all complex eigenvalues are negative. Only then is:

\[ \lim_{S_c \to 0} 2\sqrt{R_x^2 + Q_x^2} \cdot \xi_c \cdot e^{sc} = 0 \]

Various measures of the damping of an oscillation are in use.

a. The time to damp to half amplitude, \( T_\frac{1}{4} \). This characteristic has the same meaning as for the aperiodic motions. \( T_\frac{1}{4} \) is now expressed by:

\[ T_\frac{1}{4} = -\frac{0.693}{\xi_c} \cdot \frac{c}{V} \]

If an oscillation has a negative damping, i.e. it diverges, it means that \( \xi_c > 0 \) and \( T_\frac{1}{4} \) is negative. Usually the time to double amplitude, \( T_2 \), of the oscillation is then given:

\[ T_2 = -T_\frac{1}{4} \]

b. The number of periods, \( C_\frac{1}{4} \), in which the amplitude decreases to half its original value:

\[ C_\frac{1}{4} = \frac{T_\frac{1}{4}}{p} = -\frac{0.693}{2\pi} \cdot \frac{\eta_c}{\xi_c} \]

\[ = -0.110 \cdot \frac{\eta_c}{\xi_c} \]
In analogy with $T_2$ is:

$$C_2 = -C_4$$

c. The logarithmic decrement $\delta$. This is the natural logarithm of the ratio of the oscillation's amplitude in two successive maxima:

$$\delta = \ln \frac{e^{\frac{V}{c}(t + p)}}{e^{\frac{V}{c}t}} = \xi_c \cdot \frac{V}{c} \cdot p$$

Or:

$$\delta = 2\pi \cdot \frac{\xi_c}{\eta_c} = -\frac{0.693}{C_4}$$

d. The damping ratio $\xi$. The complex eigenvalues $\lambda_{c1,2}$ are often written as follows:

$$\lambda_{c1,2} = \{-\xi \omega_o \pm j\omega_o \sqrt{1 - \xi^2}\} \frac{c}{V}$$

(8-79)

Here $\xi$ is the damping ratio and $\omega_o$ is the undamped natural frequency (in rad/sec). If $\xi = 0$, is:

$$\lambda_{c1,2} = \pm j\omega_o \frac{c}{V} = \pm j\eta_c$$

The real part of $\lambda_{c1,2}$ is then zero, the oscillation is accordingly undamped and the angular frequency of the oscillation is $\omega_o$. The relation between the period and the eigenfrequency $\omega_n$ of the damped oscillation is:

$$p = \frac{2\pi}{\omega_n}$$

With $p = \frac{2\pi}{\eta_c} \cdot \frac{c}{V}$ it follows from (8-79) that:

$$\omega_n = \omega_o \sqrt{1 - \xi^2}$$

The relations between $\omega_o$, $\xi$, $\xi$ and $\eta_c$ can be derived to be:
\[ \omega_o = \sqrt{\xi_c^2 + \eta_c^2} \cdot \frac{V}{c} \]

\[ \zeta = \frac{-\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}} \]

If \( 0 < \zeta < 1 \), a damped oscillation occurs, because then \( \lambda_{c1} \) and \( \lambda_{c2} \) are complex and the real part \( -\zeta \omega_o \frac{\xi_c}{V} \) is negative. With increasing \( \zeta \), the damping increases as well. The relations between \( C_4 \), \( \delta \) and \( \zeta \) are:

\[ \delta = -2\pi \cdot \frac{\zeta}{\sqrt{1 - \zeta^2}} \]

and:

\[ C_4 = 0.110 \cdot \frac{\sqrt{1 - \zeta^2}}{\zeta} \]

If \( \zeta = 1 \), a transition occurs from a periodic to an aperiodic motion. The motion is then called critically damped. The characteristic equation in that particular case has two equal roots.

If \( \zeta > 1 \), the motion is aperiodic, as both \( \lambda_{c1} \) and \( \lambda_{c2} \) are then real.

Finally, if \( \zeta < 0 \), the real part of the eigenvalues is positive, see (8-79). The considered equilibrium condition is then unstable. The motions diverge, in a periodic manner if \( -1 < \zeta < 0 \) and aperiodic if \( \zeta < -1 \).

At the end of this review of the stability of an equilibrium condition for the case where all eigenvalues are different, the findings can be summarized as follows.

The equilibrium is stable, if all real eigenvalues and the real parts of the complex eigenvalues are negative. The disturbed motion then converges back to the original equilibrium condition. If at least one real or complex eigenvalue has a positive real part, then the equilibrium condition is unstable. In this case the disturbed motion diverges increasingly from the steady equilibrium flight condition. A transitional situation exists, where no eigenvalue has a positive real part, but a real eigenvalue or the real part of a pair of complex eigenvalues is just equal to zero. In such a situation, a deviation of constant magnitude or amplitude occurs. The equilibrium condition is then called neutrally stable.
Often, the eigenvalues, i.e. the roots of the characteristic equation, are depicted as points in the complex plane. Real eigenvalues are situated on the real axis at a distance of $\lambda_c$ of the origin and complex eigenvalues have as coordinates $(\xi_c \pm j\eta_c)$.

The above stability criterion implies that all eigenvalues must be situated to the left of the imaginary axis. As soon as at least one eigenvalue is placed to the right of the imaginary axes, the equilibrium condition is unstable. If one or more eigenvalues are placed on the imaginary axis — whether or not in the origin of the system axis — and none lies in the right half of the plane, the equilibrium condition is neutrally stable.

The solution of the equations of motion for the case where two or more roots of the characteristic equation are equal, is not further discussed here. This topic is further discussed e.g. in ref. 8.10.

It appears then that what was said previously about distinct eigenvalues is generally true also for multiple eigenvalues. Differences exist only, if multiple eigenvalues lie on the imaginary axis and none to the right of that axis. The equilibrium is then either neutrally stable or unstable, depending on the multiplicity of the relevant eigenvalues and on the rank of the characteristic determinant.

In order to determine the stability of the equilibrium condition according to the foregoing rules, it is always necessary to construct the characteristic determinant and to determine from it the values and types of the eigenvalues.

In contrast with the foregoing, the following paragraph discusses a method to determine the stability without an explicit calculation of the eigenvalues.

8.3.3. Stability criteria

Often a need exists to determine the stability of an equilibrium condition, without resorting at once to the solution of the characteristic equation. Sometimes it is necessary to study the influence on the stability of systematic changes in the system under consideration. In such cases, fruitful use may be made of various so called 'stability criteria'. In this Chapter, only the Routh-Hurwitz criteria will be discussed. A more general stability criterion is that of Lyapunov, of which the Routh-Hurwitz criteria are a particular case, see ref. 8.10.

The Routh-Hurwitz stability criteria

In the foregoing it was shown, that the criterion for stability is that all
real eigenvalues and all real parts of the complex eigenvalues are negative.

Routh, see ref. 8.11 and Hurwitz, see ref. 8.12, derived criteria which the coefficients of an algebraic equation have to satisfy for all real roots and the real parts of the complex roots to be negative. It will be sufficient here, to present only the stability criteria as they apply to a quartic characteristic equation. The quartic is:

\[ a_{\lambda}^4 + b_{\lambda}^3 + c_{\lambda}^2 + d_{\lambda} + e = 0 \]  \hspace{1cm} (8-80)

The coefficients \( a, b, c, d \) and \( e \) are all real. Without loss in generality of the following discussions it is always assumed that:

\[ a > 0 \]

The stability criteria then are:

\[ a > 0, b > 0, c > 0, d > 0, e > 0 \]

and:

\[ BCD - AD^2 - B^2E > 0 \]

The latter expression:

\[ R = BCD - AD^2 - B^2E \]

is often called Routh's discriminant.

If \( a < 0 \), the signs of \( b, c, d, e \) and \( R \) must be also negative to obtain negative real parts of the eigenvalues.

8.4. The symmetric motions in response to a disturbance

As indicated already in 8.3.2, an idea of the airplane motions after a disturbance can be obtained by determining the roots of the characteristic equation (8-75). The general character of the symmetric motions is discussed in the following. To this end, a numerical example is discussed, along the lines presented in 8.3.2.

8.4.1. The complete solution of the equations of motion

According to (8-75) the characteristic equation for the symmetric motions
is:
\[ A\lambda^4_c + B\lambda^3_c + C\lambda^2_c + D\lambda_c + E = 0 \] (8-75)

The coefficients \( A \) to \( E \) are obtained by expanding the characteristic determinant, see 8.3.2. The coefficients \( A \) to \( E \) are:

\[
A = 4\mu_c^2K^2\alpha_{u}^2Z^2\alpha
\]

\[
B = +\mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c) - \mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c) - 2\mu_cK^2\alpha_{u}^2Z^2\alpha
\]

\[
C = \mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c) - \mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c)
\]

\[
D = \mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c) - \mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c)
\]

\[
E = -\mu_c^2(\alpha_{u}^2Z^2\alpha - 2\mu_c)
\]

If the various aerodynamic and inertial parameters are known, the quantitative values of the coefficients \( A \) to \( E \) can be calculated. The eigenvalues \( \lambda_c \) are next obtained as the roots of the quartic characteristic equation (8-75).

Appendix 6 presents a method, from ref. 8.12 see page 258, to find the roots of the characteristic equation for the symmetric motions.

In many practical situations a digital computer may be available to obtain the eigenvalues \( \lambda_c \). Very often the starting point for the calculations is then the characteristic determinant rather than the characteristic equation (8-75). The required program is commonly available as a standard routine. It is then merely necessary to provide the numerical values of the stability derivatives and the inertial parameters to the digital computer.
As an example the symmetric disturbed motions of the Lockheed 1049 C 'Super Constellation' are considered. The required data are:

\[ \begin{align*}
W &= 59020 \text{ kg} \\
S &= 153.5 \text{ m}^2 \\
\bar{c} &= 4.47 \text{ m} \\
\mu_c &= 144 \\
h &= 6900 \text{ m} \\
V &= 145 \text{ m/sec} \\
C_L &= 0.60 \\
K_\gamma^2 &= 1.473
\end{align*} \]

center of gravity position: \( x_{c.g.} = 0.29 \bar{c} \).

Aerodynamic derivatives for the symmetric motions (angles in rad, angular velocities in rad/sec):

\[ \begin{align*}
C_{\alpha} &= 0 \\
C_{\alpha}^u &= -0.105 \\
C_{\alpha}^\alpha &= +0.262 \\
C_{\alpha}^q &= -0.60 \\
C_{\alpha}^u &= -1.277 \\
C_{\alpha}^\alpha &= -6.225 \\
C_m &= 0 \\
C_m^u &= -0.0409 \\
C_m^\alpha &= -0.975 \\
C_m^\alpha &= -1.42 \\
C_m^q &= -3.82 \\
C_m^{\alpha q} &= -5.45 \\
C_m^{\alpha q} &= -18.45
\end{align*} \]

From these data follow the coefficients A to E, according to (8-81):

\[ \begin{align*}
A &= -0.353603261 \times 10^8 \\
B &= -0.275735161 \times 10^7 \\
C &= -0.1140174226 \times 10^6 \\
D &= -0.3909225829 \times 10^2 \\
E &= -0.5942835
\end{align*} \]

From the coefficients A to E the stability can directly be assessed. To this end the stability criteria of Routh-Hurwitz discussed in 8.3.3 are used.
Fig 8.14: The locations of the eigenvalues for the symmetric and asymmetric motions.

(a) Symmetric motions \( \lambda = \lambda_c \frac{v}{c} \)

(b) Asymmetric motions \( \lambda = \lambda_b \frac{v}{b} \)
Routh’s discriminant $R$ follows from $A$ to $E$. It is:

$$R = -0.771769547 \times 10^{13}$$

Taking into account the negative sign of $A$, it follows from the coefficients $A$ to $E$ and $R$, that according to the Routh-Hurwitz stability criteria all eigenvalues $\lambda_c$ have negative real parts. The disturbed motion will be damped, the equilibrium condition is stable.

If the eigenvalues are actually determined, either by the method given in Appendix 6 or using a digital computer, two pairs of complex conjugate eigenvalues result:

$$\lambda_{c1,2} = -0.146 \times 10^{-3} \pm j \times 0.229 \times 10^{-2}$$

$$\lambda_{c3,4} = -0.389 \times 10^{-1} \pm j \times 0.411 \times 10^{-1}$$

Apparently, the disturbed motion is the sum of two periodic motions. The one, corresponding to the eigenvalues $\lambda_{c1,2}$, is a relatively slow, lightly damped oscillation, the other has a much shorter period and is highly damped. Two such oscillations occur in nearly all situations with most categories of conventional airplanes.

They are indicated as the long period oscillation or phugoid and the short period oscillation respectively.

The various characteristics of these motions derived in 8.3.2 are next calculated from the eigenvalues $\lambda_c$. The results are:

**Long period oscillation:** $\lambda_{c1,2} = -0.146 \times 10^{-3} \pm j \times 0.229 \times 10^{-2}$

the period $P = 84.6$ sec ($P = \frac{2\pi}{\eta_c} \cdot \frac{\dot{c}}{\bar{V}}$, see page 165)

the time to damp half amplitude $T_\frac{1}{2} = 146$ sec, $C_\frac{1}{2} = 1.73$ ($T_\frac{1}{2} = -\frac{0.693}{\xi_c} \cdot \frac{\ddot{c}}{\bar{V}}$, see page 164)

the logarithmic decrement $\delta = -0.40$

the undamped natural frequency $\omega_0 = 0.0742$ rad/sec

the damping ratio $\xi = 0.0636$

**Short period oscillation:** $\lambda_{c3,4} = -0.389 \times 10^{-1} \pm j \times 0.411 \times 10^{-1}$

the period $P = 4.71$ sec

the time to damp to half amplitude $T_\frac{1}{2} = 0.55$ sec and $C_\frac{1}{2} = 0.117$
Fig. 8.15: Response curves for a step elevator deflection for the Lockheed 1049 C. "Super Constellation".
the logarithmic decrement $\delta = -5.91$
the undamped natural frequency $\omega_0 = 1.85 \text{ rad/sec}$
the damping ratio $\zeta = 0.68$.

Fig. 8.14a shows the position of the eigenvalues in the complex plane. Here use is made of the non-dimensional variables:

$$\lambda = \lambda_c \cdot \frac{V}{c}$$
$$\xi = \xi_c \cdot \frac{V}{c}$$
$$\eta = \eta_c \cdot \frac{V}{c}$$

Fig. 8.15 shows the time responses of the airplane after some small disturbance. This calculation was made using an analog computer. The actual disturbance used was a step elevator deflection.

In the recordings the long period oscillation is clearly seen, the period is approximately 85 sec. The short period oscillation is best seen in the recording of the rate of pitch $q$ at the larger time scale. The high damping of this motion is also apparent.

8.4.2. The general character of the symmetric motions. Some approximative solutions of the equations of motion

Fig. 8.15 shows some characteristic differences between the long period and the short period oscillations.

The airspeed varies hardly during the short period oscillation.

Due to the long period oscillation, airspeed and pitch angle vary in particular. The angular acceleration about the lateral axis, $\ddot{q}$, is very nearly equal to zero during the long period oscillation. Also during this oscillation the variations of the angle of attack are relatively small if compared with the variations in angle of pitch. As $\theta = \alpha + \gamma$, the changes in flight path angle $\gamma$ will be nearly equal to those of $\theta$.

Based on these characteristic differences, some approximative calculation methods for the short period and the long period oscillations can be given.

The approximate solutions are useful to obtain rapidly and in a simple manner some idea of the various characteristics of the two eigenmotions. The results may also be used for a rough check on the outcome of more detailed automatic calculations.
The accuracy of the approximative calculation methods depends—of course—on the acceptability of the approximations to be used. This varies with the type of airplane considered. Without further arguments the following gives some calculation methods, generally producing satisfactory results for conventional airplanes. It is not uncommon to find that an—apparently—more refined calculation does not produce more accurate results, if compared with the outcome of the complete calculation. A more detailed motivation for the various approximations to be made can be found using the so-called eigenvectors of the eigen-motions. They are not discussed here.

1a) *Short period oscillation, \( V = \text{constant} \)

The approximative calculation of the short period oscillation is based on the assumption that during this motion the airspeed remains constant. This means in the equations of motion, that \( \dot{V} = 0 \) and that the first column in (8-70) disappears. This means also, that the forces in the \( X \)-direction must remain in balance during the entire motion. As a consequence, the \( X \)-equation can be entirely omitted. As an additional simplification, the initial steady flight condition is assumed to be level flight, which means \( \gamma_0 = 0 \) and \( C_{X_0} = 0 \). This latter assumption causes the angle of pitch \( \theta \) to disappear from the \( Z \)- and \( M \)-equations. This allows the kinematic relation

\[
\frac{DC}{V} \theta + \frac{QC}{V} = 0
\]

to be omitted as well. In this way the equations (8-70) are thereby reduced to the following set of simpler equations:

\[
\begin{bmatrix}
C_{Z_\alpha} + (C_{Z_\alpha} - 2\mu_c)D_{\alpha} & C_{Z_\alpha} + 2\mu_c
\\
C_{m_\alpha} + C_{m_\alpha}D_{\alpha} & C_{m_q} - 2\mu_c K_2 D_{\alpha}
\end{bmatrix}
\begin{bmatrix}
\alpha' \\
\frac{qC}{V}
\end{bmatrix}
= 0
\]

The characteristic equation is obtained, in close analogy with the discussion given in paragraph 8.3.2, by equating the characteristic determinant to zero:

\[
\begin{vmatrix}
C_{Z_\alpha} + (C_{Z_\alpha} - 2\mu_c)\lambda & C_{Z_\alpha} + 2\mu_c \\
C_{m_\alpha} + C_{m_\alpha}\lambda & C_{m_q} - 2\mu_c K_2 \lambda
\end{vmatrix}
= 0
\]
Expanding results in:

\[ A\lambda^2_c + B\lambda_c + C = 0 \]

where:

\[ A = 2\mu_c K^2(2\mu_c - C_{Z*}) \]
\[ B = -2\mu_c K_C Z_a - (2\mu_c + C_{Z*}) C_m - (2\mu_c - C_{Z*}) C_m^q \]
\[ C = C_{Z*}^2 C_m - (2\mu_c + C_{Z*}) C_m^q \]

The roots of the characteristic equation are the two eigenvalues \( \lambda_{c1,2} \):

\[ \lambda_{c1,2} = \zeta_c \pm j\eta_c = \frac{-B \pm j\sqrt{4AC - B^2}}{2A} \]

A further simplification is possible by omitting the derivatives \( C_{Z*} \) and \( C_{Z*}^q \). They occur in the equations of motion in combination with the mass parameter \( 2\mu_c \). Usually they are negligible in comparison with \( 2\mu_c \). The coefficients \( A, B \) and \( C \) are then reduced to:

\[ A = 4\mu_c^2 K^2 \]
\[ B = -2\mu_c (K_C^2 C_{Z*} + C_m^q + C_m) \]
\[ C = C_{Z*}^2 C_m - 2\mu_c C_m^q \]

1b) Short period oscillation, \( V = \) constant, rotations in pitch only
(\( \gamma = \) constant)

A further simplifying assumption is, that during the short period oscillation the trajectory of the airplane c.g. is a straight line. If the initial steady flight condition is a level flight, the c.g. moves along a horizontal straight line. This means that the forces in the Z-direction remain in balance during the entire motion. As a consequence the Z-equation can be omitted. The
only remaining motion is the rotation in pitch, only the \( M \)-equation remains. Because \( \gamma = \theta - \alpha = 0 \), \( \alpha \) can be replaced by \( \theta \) and \( D \cdot \alpha \) by \( D \cdot \theta \) in the \( M \)-equation. With \( \frac{D \alpha_c}{c} V = D^2 \theta \) the \( M \)-equation can then be written as:

\[ (C_m + C_a \cdot D + C_a^c \cdot D - 2\mu_c K_c^2 q^2) \cdot \theta = 0 \]

The characteristic equation has the coefficients:

\[ A = -2\mu_c K_c^2 \]
\[ B = C_a + C_a^c \]
\[ C = C_m \]

2a) **Long period oscillation, \( \dot{q} = 0, \alpha = 0 \)**

In this first, coarsest approximation of the long period oscillation the assumption is made that the angle of attack remains constant during this motion. In the equations of motion is as a consequence \( \alpha = 0 \) and also \( \dot{\alpha} = 0 \). This makes the \( \alpha \)-column disappear in (8-70) and one of the equations has become superfluous. Because \( \dot{q} = 0 \), the \( M \)-equation is omitted. The fact that \( \alpha = 0 \), may be seen as a consequence of a relatively high negative value of \( C_m \). In such a case is also very nearly \( C_m = 0 \). If, finally \( C_{\alpha q} \) is neglected relative to \( 2\mu_c \) and again \( C_{X_0} = 0 \), the equations (8-70) now become:

\[
\begin{bmatrix}
C_{X_u} - 2\mu_c D_c & C_{Z_0} & 0 \\
C_{Z_u} & 0 & 2\mu_c \\
0 & -D_c & 1
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\theta \\
\frac{\dot{\alpha}}{V}
\end{bmatrix}
= 0
\]

The characteristic equation is reduced to a quadratic, the coefficients are:
A = -4\mu^2_c

B = 2\mu_c C_{Xu}

C = -C_{zu} \cdot C_{zo}

The result permits a very simple approximation of the period. The undamped natural frequency \( \omega_0 \) and the damping ratio \( \zeta \) follow from the eigenvalues \( \lambda_{c1,2} = \xi_c \pm j \eta_c \) expressed in the coefficients A, B and C.

\[
\omega_0 = \frac{V}{c} \cdot \sqrt{\frac{C}{A}} = \frac{V}{c} \sqrt{\frac{C_{zu} \cdot C_{zo}}{4\mu^2_c}} \text{ (sec}^{-1})
\]

\[
\zeta = \frac{-B}{2\sqrt{AC}} = \frac{C_{Xu}}{2 \sqrt{C_{zu} \cdot C_{zo}}}
\]

For \( C_{Xu}, C_{zo} \) and \( C_{zu} \), the following approximations are introduced, as discussed in the paragraphs 8.2.2 and 8.2.3:

\[
C_{Xu} = -2C_D
\]

\[
C_{zo} = -C_L
\]

\[
C_{zu} = -2C_L
\]

For \( \omega_0 \) and \( \zeta \) then follows:

\[
\omega_0 = \frac{V}{c} \cdot \sqrt{\frac{2C^2_D}{4\mu^2_c}} = \frac{g \sqrt{2}}{V} \text{ (sec}^{-1})
\]

\[
\zeta = \frac{2C_D}{2 \sqrt{2C^2_L}} = \frac{C_D}{\sqrt{2} C_L}
\]
According to this highly simplified calculation the damping of the long period oscillation is determined by the drag to lift ratio $\frac{C_D}{C_L}$. This ratio is usually in the order of 0.1. The damping of the long period oscillation is, therefore, nearly always very low. In that case the period can be written as:

$$p = \frac{2\pi}{\omega_0 \sqrt{1 - \zeta^2}} \approx \frac{2\pi}{\omega_0} \text{ (sec)}$$

With the above result for $\omega_0$, the period of the long period oscillation becomes:

$$p = \frac{2\pi}{g \sqrt{2}} \cdot V = 0.453 \ V \text{ (sec)} \quad (V \text{ in m/sec})$$

According to this approximative calculation, the period in sec. of the long period oscillation is roughly 0.5 times the airspeed in m/sec.

2b) Long period oscillation, $\dot{q} = 0, \ddot{q} = 0$

A second, slightly more refined approximation of the long period oscillation goes less far than the previous one. Variations in $\alpha$ are now permitted. But they are assumed to occur so slowly, that $\dot{\alpha}$ remains negligible. In the equations (8-70) the contributions due to $\frac{D_c}{V}$ and $D_c \alpha$ disappear, but none of the equations can be omitted. If $C_{Zq}$ and $C_{Xq}$ are again neglected, the result is:

$$\begin{bmatrix}
C_{Xq} & -2\mu_c D_c & C_{Xq} & C_{Zq} & 0 \\
C_{Zu} & C_{Zq} & 0 & 2\mu_c & 0 \\
0 & 0 & -D_c & 1 & 0 \\
C_{m} & C_{m} & 0 & C_{mq} & 0 \\
\end{bmatrix} \begin{bmatrix}
\dot{u} \\
\dot{\alpha} \\
\dot{\theta} \\
\dot{q} \\
\end{bmatrix} = 0$$

The coefficients of the characteristic equation, following from an expansion of the characteristic determinant, are:

$$A = 2\mu_c (C_{Zq} \cdot C_{m} - 2\mu_c \cdot C_{m})$$
\[ B = 2\mu_c (C_{x_m} - C_{x_m} \cdot C_{x_m} \cdot C_{x_m} ) + (C_{q_u} \cdot C_{x_m} - C_{x_m} \cdot C_{x_m} \cdot C_{x_m} ) \]
\[ C = C_{q_m} \cdot C_{q_m} - C_{q_m} \cdot C_{q_m} \]

**8.5. The asymmetric disturbed motions**

The equations for the disturbed asymmetric motions are obtained in the homogeneous form from (8-42), see page 124, by omitting the terms due to the control surface deflections:

\[
\begin{bmatrix}
C_{y_\beta} + (C_{y_\beta} - 2\mu_b) D_b & C_{L} & C_{y_p} & C_{y_r} - 4\mu_b \\
0 & -\frac{1}{2} D_b & 1 & 0 \\
C_{L_\beta} & 0 & C_{L_\beta} - 4\mu_b K_{X_\beta} D_b & C_{L_r} + 4\mu_b K_{XZ} D_b \\
C_{n_\beta} + C_{n_\beta} D_b & 0 & C_{n_\beta} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_{Z_\beta} D_b \\
\end{bmatrix}
\begin{bmatrix}
\beta \\
\varphi \\
\psi \\
\theta \\
\end{bmatrix}
= 0
\]

(8-82)

The character of the asymmetric motions is discussed below, using a quantitative example.

In 8.3,2 some characteristic parameters were derived for the symmetric eigenmotions. The same expressions apply also to the asymmetric motions, if the mean aerodynamic chord \( \tilde{c} \) is replaced by the wing span \( b \). The eigenvalues resulting from the characteristic equation are now indicated as \( \lambda_b \).

The expressions for the time to damp to half amplitude and for the period now read as follows:

**Aperiodic motion (\( \lambda_b \) real):**

\[ T_\frac{1}{2} = -\frac{0.693}{\lambda_b} \cdot \frac{b}{V} \]

**Periodic motion corresponding to two complex, conjugate eigenvalues:**
\[ \lambda_{b3,4} = \xi_c \pm j\eta_b \]

\[ P = \frac{2\pi \cdot b}{\eta_b} \cdot \frac{1}{V} \]

\[ T_1 = -0.693 \cdot \frac{b}{\xi_b} \cdot \frac{1}{V} \]

\[ C_1 = \frac{T_1}{P} = -0.110 \frac{\eta_b}{\xi_b} \]

\[ \delta = 2\pi \frac{\xi_b}{\eta_b} = 0.693 \frac{1}{C_1} \]

8.5.1. The solution of the equations of motion.

In Chapter 7 it was stated that \( C_{\gamma,\beta} \) and \( C_{\gamma,\beta} \) are usually neglected. In accordance with this practice, it will be assumed here that:

\[ C_{\gamma,\beta} = C_{\gamma,\beta} = 0 \]

Expanding the characteristic determinant results in a quartic characteristic equation:

\[ A\lambda_b^4 + B\lambda_b^3 + C\lambda_b^2 + D\lambda_b + E = 0 \]

The coefficients \( A \) to \( E \) for the asymmetric motions are derived from the characteristic determinant:

\[ A = 16\mu_b^2 (K_{X,XZ}^2 - K_{Z,XZ}^2) \]

\[ B = -4\mu_b^2 \left\{ 2C_{\gamma,\beta} (K_{X,XZ}^2 - K_{Z,XZ}^2) + C_{\gamma,\beta} K_{X,XZ}^2 + C_{\gamma,\beta} K_{Z,XZ}^2 + (C_{\gamma,\beta} + C_{\gamma,\beta}) K_{X,XZ} \right\} \]
\[ C = 2 \mu_b \left( (c_{\gamma \beta} - c_{\gamma p})^2 + (c_{\alpha \beta} - c_{\alpha p}) \right) X^2 + \\
+ (c_{\gamma \beta} - c_{\gamma p}) X + (c_{\alpha \beta} - c_{\alpha p}) X + \\
+ 4 \mu_c c_{\beta\gamma} K_{\alpha x}^2 + 4 \mu_c c_{\beta\gamma} K_{\alpha x} + \frac{1}{2} (c_{\alpha \beta} - c_{\alpha p}) \} \\
D = -4 \mu_b c_{\alpha L} (c_{\alpha \beta} - c_{\alpha p}) + 2 \mu_b (c_{\alpha \beta} - c_{\alpha p}) + \\
+ 4 \mu_c c_{\beta \gamma} (c_{\alpha \beta} - c_{\alpha p}) + 4 \mu_c c_{\beta \gamma} (c_{\alpha \beta} - c_{\alpha p}) + \\
+ 4 \mu_c c_{\beta \gamma} (c_{\alpha \beta} - c_{\alpha p}) \\
E = c_{\alpha L} (c_{\alpha \beta} - c_{\alpha p}) \\
\]

If the stability derivatives and inertial parameters are known for a certain airplane configuration and flight condition, the coefficients A to E can be calculated. Using one of the common numerical methods, the eigenvalues may be obtained.

Appendix 6 describes a numerical method particularly suitable to calculate the eigenvalues of the asymmetric motions.

As an example the asymmetric disturbed motions of the Lockheed 1049C 'Super-Constellation' are studied. The required data are:

\[ W = 59.020 \text{ kg} \quad h = 6900 \text{ m} \quad \mu_b = 17.219 \]
\[ S = 153.5 \text{ m}^2 \quad V = 145 \text{ m/sec} \quad K_{\alpha}^2 = 0.0283 \]
\[ b = 37.49 \text{ m} \quad C_L = 0.60 \quad K_{\beta}^2 = 0.0471 \]
\[ K_{\alpha x} = 0 \]
\[
\begin{align*}
C_Y &= -0.596 & C_Y &= +0.369 & C_Y &= 0 \\
C_{\beta} &= -0.1374 & C_{\alpha} &= +0.144 & C_{\alpha} &= -0.52 \\
C_{n} &= +0.1173 & C_{n} &= -0.180 & C_{n} &= -0.021 \\
\end{align*}
\]

From these data follow the coefficients A to E of the characteristic equation:

\[
\begin{align*}
A &= 108.8807499 \\
B &= 36.97262208 \\
C &= 10.102745734 \\
D &= 2.48439576 \\
E &= 0.00470448 \\
\end{align*}
\]

As with the symmetric motions, the Routh-Hurwitz stability criteria discussed in 8.3.3 can now be used to see if the disturbed motions are damped and thus if the equilibrium condition is stable.

The coefficients A to E are all positive and Routh's discriminant R is:

\[
R = 249.516873 > 0
\]

It follows that all eigenvalues \( \lambda_b \) must have negative real parts and that the disturbed motion is positively damped.

An explicit determination of the four eigenvalues, see fig. 8.14b on page 171:

\[
\begin{align*}
\lambda_{b1} &= -0.290 \\
\lambda_{b2} &= -0.00191 \\
\lambda_{b3,4} &= -0.0241 \pm j \times 0.279 \\
\end{align*}
\]

The disturbed asymmetric motions consist apparently of two aperiodic eigenmotions and one periodic motion. One of the two aperiodic motions appears to be well damped, the other one is only very lightly damped. The characteristics of the eigenmotions are as follows:
first aperiodic motion \( \lambda_{b1} = -0.290 \)

time to damp to half amplitude \( T_{\frac{1}{2}} = 0.62 \) sec

second aperiodic motion \( \lambda_{b2} = -0.00191 \)

time to damp to half amplitude \( T_{\frac{1}{2}} = 94.0 \) sec

periodic motion \( \lambda_{b3,4} = -0.0241 \pm j \times 0.279 \)

period \( P = 5.83 \) sec

time to damp to half amplitude \( T_{\frac{1}{2}} = 7.45 \) sec

cycles to damp to half amplitude \( C_{\frac{1}{2}} = 1.28 \)

logarithmic decrement \( \delta = -0.541 \)

undamped natural frequency \( \omega_0 = 1.08 \) rad/sec

damping ratio \( \zeta = 0.0865 \)

Figs. 8.16 and 8.17 show the calculated response curves for the airplane, simulated on an analog computer. In this particular case the disturbances are pulse-shaped rudder and aileron deflections.

8.5.2. The general character of the asymmetric motions

From the time responses in figs. 8.16 and 8.17 three eigenmotions may be distinguished, as was evident also from the eigenvalues.

1. A highly damped aperiodic motion

The corresponding eigenvalue is \( \lambda_{b1} \), real and strongly negative. The motion is apparent mostly in the rate of roll. The damping of this motion is due primarily to the damping rolling moment generated by-the wing when rotating about the longitudinal axis, i.e. the moment \( C_p \frac{p_b}{2V} \).

2. An aperiodic motion, in which the airplane sideslips, yaws and rolls

The motion is characterized by the eigenvalue \( \lambda_{b2} \), real lightly positively or negatively damped. Depending on the sign of \( \lambda_{b2} \), this eigenmotion converges or diverges. If the motion diverges (\( \lambda_{b2} > 0 \)), the airplane enters after a disturbance into a sideslipping turn with ever increasing angle of roll and decreasing radius. During this motion the airplane slips towards the inside of the turn, see fig. 8.18. The airplane is then called spirally unstable, as the airplane describes a descending spiral. The eigenmotion is called the spiral motion. If it is damped, the airplane possesses 'spiral stability'.
Fig. 8.16: Response curves for a pulse rudder deflection for the Lockheed 1049C, "Super Constellation".

\[ \delta_r = +0.025 \text{ rad} \]
during 2 sec.

\[ W = 59020 \text{ kg} \]
\[ X_c g = 0.28 \]
\[ V = 145 \text{ m/sec} \]
\[ h = 2900 \text{ m} \]
Fig 8.17: Response curves for a pulse aileron deflection for the Lockheed 1049 C, "Super Constellation". 

\[ \delta_a = +0.025 \text{ rad during 2 sec.} \]

\[ W = 59020 \text{ kg} \]
\[ X_{cg} = 0.28 \text{ m} \]
\[ V = 145 \text{ m/sec} \]
\[ h = 2900 \text{ m} \]
Fig. 8.18: The spiral motion of an airplane.
Fig. 8.19: Characteristics of the Dutch roll motion.

\[ \frac{p_t}{r_t} = \frac{|p|}{|r|} \]
3. A periodic motion, in which the airplane sideslips, yaws and rolls

This eigenmotion is determined by the eigenvalues \( \lambda_{b3,4} = \xi_b \pm j \eta_b \). Fig. 8.19 shows the general character of the motion. Rolling and yawing velocity are presented for the case where the motion is damped \( \xi_b < 0 \). This motion is generally called 'Dutch roll'. The name seems to be inspired by the resemblance to the motion of a skater on the ice.

8.5.3. Application of the Routh-Hurwitz stability criteria to the asymmetric motions

The criteria for stability were discussed already in 8.5.1.

It can be proved, that in the transition from stability to instability the conditions \( E > 0 \) and \( R > 0 \) are the critical stability criteria.

If only \( E \) becomes negative, one real eigenvalue changes sign. This means that one of the aperiodic motions changes from convergence to divergence.

If \( R \) becomes more negative, the real part of the two complex, conjugate eigenvalues changes signs implying periodic divergence. According to the foregoing, the coefficient \( E \) determines the character of the aperiodic spiral motion and from the discriminant \( R \) follows the character of the (periodic) Dutch roll motion.

In the following the stability criteria are used for a further study of the spiral motion and the Dutch roll oscillation of the airplane. To prepare for this discussion, the factors determining the signs of the coefficients A to E are investigated.

As a first approximation, the relatively small derivative \( C_{\gamma p} \) is equated to zero and \( C_{\gamma r} \) is neglected relative to \( 4\mu_r \). For airplanes having a straight wing without a large dihedral these assumptions are acceptable. Also, the coefficient \( K_{\chi Z} \) of the product of inertia is omitted. \( K_{\chi Z} \) is relatively small, relative to \( K^2_\chi \) and \( K^2_Z \).

The remaining stability derivatives determining the magnitude and the signs of the coefficients A to E have the following signs for conventional airplanes in normal flight, according to Chapter 7:

\[
C_{\gamma \beta} < 0; \ C_{\chi p} < 0; \ C_{\eta p} < 0; \ C_{\chi r} > 0; \ C_{\eta r} < 0
\]
The stability derivatives $C_{n\beta}$ and $C_{\beta \gamma}$ are practically the only ones the designer has at his disposal to influence the lateral stability. As discussed in Chapter 7, pleasant control characteristics require that:

$$C_{n\beta} > 0 \quad \text{and} \quad C_{\beta \gamma} < 0.$$

Neglecting $C_{r\gamma}$, $C_{\gamma r}$ and $K_{xz}$ and using the known signs of the other derivatives and the inertial parameters the signs of the coefficients A to E can be established as follows:

$$A = 16\mu_b^3 k^2 x^2 z^2 > 0$$

$$B = -4\mu_b^2 \left[ 2C_{\gamma r} k_r^2 + C_{\beta r} \right] > 0$$

$$C = 2\mu_b \left( C_{\gamma r} C_{\beta r} k_r^2 + 4\mu_b C_{n\beta} k_r^2 + \frac{1}{4}(C_{\gamma r} C_{\beta r} - C_{\gamma r} C_{n\beta}) \right)$$

If $C_{n\beta} > 0$ (the airplane is then statically directionally stable), is $C > 0$.

$$D = -4\mu_b C_{\beta r} k_r^2 + 2\mu_b \left( C_{\gamma r} C_{\beta r} - C_{\gamma r} C_{n\beta} \right) + \frac{1}{4}(C_{\gamma r} C_{\beta r} - C_{\gamma r} C_{n\beta})$$

If $C_{\beta r} > 0$, $C_{\beta r} < 0$ and $C_{n\beta} > 0$, then also is $D > 0$.

$$E = C_{\beta r} (C_{\gamma r} C_{n\beta} - C_{n\beta} C_{\gamma r})$$

At positive values of $C_{\beta r}$, the positive sign of $E$ is determined by the condition:

$$C_{\gamma r} C_{n\beta} - C_{n\beta} C_{\gamma r} > 0$$
It appears that the above requirement for dynamic stability is identical to the one derived in Chapter 7, see paragraph 7.4.2. The latter was based on the criterion for good control characteristics in steady turns using the ailerons or the rudder only.

8.5.4. The spiral motion and the Dutch roll motion, the lateral stability diagram

In the previous paragraph it was stated that positive damping of the spiral motion required that \( E > 0 \) and for damping of the Dutch roll motion, \( R > 0 \).

The magnitudes of \( E \) and \( R \) are influences to a large extent by the effective dihedral \( \alpha_\beta \) and the static directional stability \( C_{n_\beta} \).

\( \alpha_\beta \) can be varied by changing the geometric dihedral of the wing; \( C_{n_\beta} \) is modified by varying the surface \( S_v \) of the vertical tailplane and the taillength \( l_v \). But it should be remembered that changing \( S_v \) and/or \( l_v \) influences also some other stability derivatives, such as \( C_{n_\gamma} \) and \( C_{y_\beta} \).

The effect of changes in \( C_{n_\beta} \) and \( \alpha_\beta \) on \( E \) and \( R \) is depicted graphically in the so-called lateral stability diagram, along the axes \(-C_\beta \) and \( C_{n_\beta} \) are plotted, see fig. 8.20.

![Fig 8.20: The lateral stability diagram.](image-url)
For a particular airplane, the boundaries \( E = 0 \) and \( R = 0 \) are plotted for a given airplane configuration and flight condition in the lateral stability diagram. The remaining stability derivatives are assumed to remain constant, or to be adjusted to the external shape of the airplane corresponding to each combination of \( C_{\alpha}^\beta \) and \( C_{n}^\beta \).

The two curves \( E = 0 \) and \( R = 0 \) indicate the boundaries of areas in the diagram corresponding to spiral stability and a damped Dutch roll oscillation.

The diagram indicates for any combination of \( C_{\alpha}^\beta \) and \( C_{n}^\beta \) the state of the dynamic lateral stability for the airplane configuration and flight condition under study.

Only the most common case, where \( C_{\alpha}^\beta < 0 \) and \( C_{n}^\beta > 0 \) is further discussed.

Using the lateral stability diagram, the spiral motion is considered first. The criterion for spiral stability is:

\[
\frac{C_{\alpha}^\beta}{C_{n}^\beta} \cdot C_{n}^\beta \cdot C_{\alpha}^\beta > 0
\]

It follows, that spiral stability can be increased in two ways, viz. by decreasing \( C_{n}^\beta \) or by increasing \( -C_{\alpha}^\beta \). Because other requirements on the control characteristic have to be met as well, a decrease in \( C_{n}^\beta \) may be hardly acceptable. A value of \( C_{\alpha}^\beta \) too large negative may not be acceptable either, as will be discussed below.

The effect of these conflicting requirements is, that it may be difficult to choose the external form of the airplane such that it is spirally stable in all required airplane configurations and flight conditions. The divergence from a spirally unstable equilibrium flight condition often occurs relatively slowly. In order to avoid even less desirable characteristics, a certain slight degree of spiral instability may be more or less unavoidable.

The curve for \( R = 0 \), determining stability of the Dutch roll motion, appears to depend largely on the relative density \( \mu_b = \frac{m}{\rho S_B} \) of the airplane. The curve for \( R = 0 \) moves upward in the lateral stability diagram with increasing \( \mu_b \). This means, that for given aerodynamic characteristics (stability derivatives) the damping of the Dutch roll motion decreases with increasing \( \mu_b \).

In general, the Dutch roll motion of airplanes having a low wing loading, flying at relatively low altitudes are relatively well damped. A characteristic value of the damping for such airplanes is \( \zeta \approx 0.15 \) or \( C_{\zeta} \approx 0.7 \).

For the Dutch roll motion it does not suffice to determine only the sign of
R, because good flying qualities imply a certain minimal damping of the Dutch roll motion.

This minimal required damping has been made a function of the period of the motion, see ref. 1.17. This may be explained as follows.

If the period is short, the reactions of the pilot will be too slow to stabilize the motion by suitable control deflections. In such a case a relatively high damping or a low value of \( T \), or \( C \), is required. If the period is longer, the damping of the motion may be lower, because in such a situation the pilot has less difficulty to improve the damping of the airplane's motions via his control deflections.

However, the requirement relating the period and the minimal acceptable time to damp to half amplitude has turned out to provide insufficient guarantee for good flying qualities.

More recent requirements, see e.g. ref. 1.17, stipulate that the damping of the Dutch roll has a certain minimal value depending on the roll to yaw ratio \( \frac{|p|}{|r|} \) occurring during the Dutch roll motion. The interpretation of this ratio \( \frac{|p|}{|r|} \) is shown in fig. 8.19, see page 188.

8.5.5. Some approximative solutions of the equations of motion

1. The heavily damped aperiodic rolling motion

The aperiodic rolling moment may be approximated by assuming that the airplane can only roll about the longitudinal axis. This is permissible because this motion has often disappeared before the other eigenmotions of the airplane have really started. If only the rolling motion is considered, in the equations (8-82) of paragraph 8.5 the angle of sideslip \( \beta \), and the non-dimensional rate of yaw \( \frac{r_b}{2V} \) disappear as variables. The equations for the lateral forces and the yawing moments can be omitted.

In the remaining equation for the rolling moments the angle of roll \( \varphi \) does not occur, so the kinematic relation \( -4D_b \varphi + \frac{p_b}{2V} = 0 \) is no longer needed. The rolling moment equation then reads:

\[
\left(C_f - 4\mu_b K_{D_b}\right) \frac{p_b}{2V} = 0
\]

It is easy to see, that this expression gives the real eigenvalue:
\[
\lambda_{b1} = \frac{C_L p}{4 \mu_b K_X^2}
\]

2a. The Dutch roll motion, \( \varphi = \frac{p_b}{2V} = 0 \)

For a large category of conventional airplanes a certain approximation of the Dutch roll motion results, if the rolling component is discarded in the Dutch roll motion. If \( \varphi \) and \( \frac{p_b}{2V} \) are set at zero, the \( \varphi \)- and \( \frac{p_b}{2V} \)-columns in the equations of motion disappear and the rolling moment equation can be omitted, as the rolling moments have to remain in balance. Only the \( Y \)- and the \( N \)-equations remain. As usual, \( C_{Yr} \) and \( C_{n\beta} \) are neglected, whereas \( C_{Yr} \) is insignificant relative to \( 4 \mu_b \):

\[
\begin{bmatrix}
C_{Yr} & -2 \mu_b D_b \\
\beta & 0 \\
C_{n\beta} & C_{n\beta} - 4 \mu_b K_b^2 Z_b \\
0 & C_{n\beta} - 4 \mu_b K_b^2 Z_b
\end{bmatrix}
\begin{bmatrix}
\beta \\
\frac{p_b}{2V}
\end{bmatrix} = 0
\]

The coefficients of the quadratic characteristic equation are now:

\[
A = 8 \mu_b^2 K_b^2 Z_b
\]

\[
B = -2 \mu_b (C_{n\beta} + 2 K_b^2 C_{Yr})
\]

\[
C = 4 \mu_b C_{n\beta} + C_{Yr} \cdot C_{n\beta} Z_b
\]

From these coefficients, \( \omega_o, \zeta, \alpha \) and \( T_r \) can be easily obtained.

2b. The Dutch roll motion, \( \varphi = \frac{p_b}{2V} = 0 \), yawing rotations only

A more restrictive assumption is, that the trajectory of the airplane c.g. during the oscillation is a straight line. This means that the course angle \( \times \) is constant, where: \( \times = \beta + \psi = 0 \) or \( \beta = -\psi \).

It implies also, that the \( Y \)-equation is superfluous. The remaining yawing moment equation is further simplified by the fact that \( \frac{p_b}{2V} = \frac{1}{4} D_b \psi \) and \( \beta = -\psi \).

The result is:
\[-C_{n_{\beta}} + \frac{1}{4} C_{n_{r} D_{b}} - 2\mu_{b} k_{2} D_{b}^{2} \] 
\[\phi = 0\]

The coefficients A, B and C then are:

\[A = -2\mu_{b} k_{2}^{2}\]

\[B = \frac{1}{4} C_{n_{r}}\]

\[C = -C_{n_{\beta}}\]

The expressions for \(\omega_{o}\) and \(\zeta\) are obtained very simply as:

\[\omega_{o} = \frac{V}{b} \cdot \sqrt{\frac{C}{A}} = \frac{V}{b} \cdot \sqrt{\frac{C_{n_{\beta}}}{2\mu_{b} k_{2}^{2}}}\]

\[\zeta = -\frac{R}{2\sqrt{AC}} = -\frac{C_{n_{r}}}{4\sqrt{2\mu_{b} k_{2}^{2} C_{n_{\beta}}}}\]

3. The slow, aperiodic spiral motion

This is usually a very slow eigenmotion, in which the airplane sideslips, yaw and rolls. It is, therefore, permissible to approximate the motion by assuming all linear and angular accelerations to be negligible. In the equations of motion this means:

\[D_{b} \dot{\beta} = D_{b} \frac{p_{b}}{2V} = D_{b} \frac{r_{b}}{2V} = 0\]

In addition to \(C_{r}\), also \(C_{p}\) is neglected.
\[
\begin{bmatrix}
C_\beta & C_L & 0 & -\frac{4}{3} \mu_b \\
0 & -\frac{4}{3} \mu_b & 1 & 0 \\
C_\beta & 0 & C_{\beta p} & C_{\beta r} \\
C_{n\beta} & 0 & C_{n p} & C_{n r}
\end{bmatrix}
\begin{bmatrix}
\beta \\
\phi \\
p_b \\
\frac{2V}{r_b}
\end{bmatrix}
= 0
\]

Expanding the characteristic denominator results in:

\[
\lambda_{b,4} = \frac{2C_L(C_{\beta p} C_{n r} - C_{\beta r} C_{n p})}{C_{\beta p} (C_{\beta r} C_{n p} + 4\mu_b C_{n r}) - C_{\beta r} (C_{\beta p} C_{n r} + 4\mu_b C_{n p})}
\]

It is easy to verify, that the dominator is negative, if all stability derivatives have their normal signs. For convergence of this motion it is necessary that \(\lambda_{b,4} < 0\), which means:

\[
C_L (C_{\beta p} C_{n r} - C_{\beta r} C_{n p}) > 0
\]

This corresponds to the requirement \(E > 0\), derived in 8.5.3 on page 190.

4. The Dutch roll motion and the aperiodic rolling motion, \(\phi \neq 0\), c.g. moves along a straight line; \(\beta = -\phi\)

If, as under 2b, the assumption is made that the c.g. moves along a straight line, but in addition the rolling motion is permitted, only the \(Y\)-equation disappears. The following equations in \(\phi\) and \(p_b\) result:

\[
\begin{bmatrix}
-C_{\beta r} + \frac{4}{3} C_{\beta p} + 2\mu_b C_{n p} \frac{\kappa^2}{\delta_b} \\
0 & -\frac{4}{3} \mu_b \frac{\kappa^2}{\delta_b} \\
-C_{n\beta} + \frac{4}{3} C_{n p} - 2\mu_b \frac{\kappa^2}{\delta_b} \\
C_{n p} + 4\mu_b \frac{\kappa^2}{\delta_b}
\end{bmatrix}
\begin{bmatrix}
\phi \\
p_b \\
\frac{2V}{r_b}
\end{bmatrix}
= 0
\]
The characteristic equation is a cubic:

\[ A\lambda_b^3 + B\lambda_b^2 + C\lambda_b + D = 0 \]

The coefficients are then:

\[ A = 4\mu_b^2(\kappa^2 - \kappa_{xz}^2) \]

\[ B = -\mu_b \left\{ (C_{\lambda r n_p} + C_{\kappa_{xz} n_p})\kappa_{xz} + C_{\kappa_{xz} n_p} \kappa_{xz} + C_{\kappa_{xz} n_p} \kappa_{xz} \right\} \]

\[ C = 2\mu_b (C_{\lambda_{\beta xz} n_p} + C_{\kappa_{xz} n_p}) + (C_{\lambda_{\beta xz} n_p} - C_{\kappa_{xz} n_p}) \]

\[ D = \frac{1}{4} (C_{\lambda_{\beta xz} n_p} - C_{\kappa_{xz} n_p}) n_p \]

Evidently no explicit analytical solution to this cubic can be given. Using the approximation for the eigenvalue of the motion in roll as discussed under 1, the characteristic equation may be reduced to a quadratic.
Fig. 8.21: A vector $\mathbf{a}$ and two equal vectors $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} = \mathbf{b}$$

Fig. 8.22: Addition of two vectors.

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$
APPENDIX 1

Some rules for the use of vectors

The following gives a brief review of the various rules applicable to the use of vectors.

1. Definition

A vector is a variable characterized by a magnitude and a direction.

Examples of vectors are: a displacement, velocity, acceleration, force and moment.

A vector is graphically represented by an arrow, the length of which is a measure of the magnitude of the vector and the direction indicates the direction of the vector. The notation of a vector commonly is a letter overlined or underlined, sometimes by an arrow: \( \vec{a}, \overline{a}, \) of \( a, \) fig. 8.21.

The counterpart of a vector is a scalar, a variable having a magnitude only and no direction.

Examples are: work, power, temperature.

Various mathematical operations can be performed with vectors, such as addition, multiplication and also differentiation with respect to time. These operations are discussed and subtraction of vectors.

2. Equality, addition and subtraction of vectors

Two vectors are equal if they have the same magnitude and the same direction, see fig. 8.21.

The addition of two vectors is performed using the parallelogram rule, see fig. 8.22.

Evidently is:

\[ a + b = b + a \]

and also:

\[ (a + b) + c = a + (b + c) = a + b + c \]
Fig. 8.23: Two opposite vectors.

Fig. 8.24: Subtraction of vectors.

Fig. 8.25: The component of a vector along a line.

Fig. 8.26: A set of unit vectors.
The addition of vectors thus is a cumulative as well as an associative operation.

The opposite \(-a\) of a vector \(a\) is a vector having the same magnitude as \(a\), but the opposite direction, see fig. 8.23.

A vector \(b\) is subtracted from a vector \(a\) by adding \(-b\) to \(a\), see fig. 8.24.

3. Components of a vector, unit vectors

The component of a vector \(a\) along a line \(l\) is \(b\), if the end points of \(b\) are obtained by the perpendicular projection of the end points of \(a\) on \(l\), see fig. 8.25.

A unit vector along a line is a vector directed along that line having a length defined as the unit. If \(e\) is the unit vector along \(l\), \(b\) can be written as:

\[
\mathbf{b} = e \cdot \mathbf{b}
\]

A set of perpendicular unit vectors consists of three unit vectors, each directed along one of the axes of a perpendicular reference frame. Right handed reference frames are commonly used, see fig. 8.26. This means that a rotation of the positive X-axis towards the positive Y-axis over the smallest angle would drive a right handed screw in the direction of the positive Z-axis.

An arbitrary vector \(a\) can then be resolved in three components \(a_x\), \(a_y\) and \(a_z\) along the three axes. Where:

\[
a = i \cdot a_x + j \cdot a_y + k \cdot a_z
\]

if \(i\), \(j\) and \(k\) are the three unit vectors.

4. The scalar product of two vectors

The scalar product of two vectors \(a\) and \(b\) is written as:

\[
c = a \cdot b
\]

c is a scalar of magnitude:

\[
c = a \cdot b \cdot \cos \theta
\]
where $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Special cases are:

$$
\theta = 0 \quad \mathbf{a} \cdot \mathbf{b} = ab \quad \text{and} \quad \mathbf{a} \cdot \mathbf{a} = a^2 \\
\theta = \frac{\pi}{2} \quad \mathbf{a} \cdot \mathbf{b} = 0 \\
\theta = \pi \quad \mathbf{a} \cdot \mathbf{b} = -ab
$$

From this follow the scalar products of the unit vectors:

$$
\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\
\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0
$$

The scalar product has the distributive property, i.e.:

$$
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
$$

From the foregoing follows:

$$
\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} \cdot a_x + \mathbf{j} \cdot a_y + \mathbf{k} \cdot a_z)(\mathbf{i} \cdot b_x + \mathbf{j} \cdot b_y + \mathbf{k} \cdot b_z) \\
= \mathbf{i} \cdot a_x \cdot \mathbf{i} \cdot b_x + \mathbf{i} \cdot a_x \cdot \mathbf{j} \cdot b_y + \ldots . \\
= a_x b_x + a_y b_y + a_z b_z
$$

5. The vectorial product

The vectorial product of two vectors is written as:

$$
\mathbf{c} = \mathbf{a} \times \mathbf{b}
$$

$\mathbf{c}$ is a vector having the following properties:

1. the magnitude of $\mathbf{c}$ is:

$$
\mathbf{c} = ab \sin \theta
$$

2. the direction of $\mathbf{c}$ is perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$ and is given by the right hand rule, see fig. 8.27.
Fig. 8.27: The vectorial product of two vectors.

Special cases are:

\[
\begin{align*}
\theta &= 0 & \mathbf{a} \times \mathbf{b} &= 0 \quad \text{and} \quad \mathbf{a} \times \mathbf{a} &= 0 \\
\theta &= \frac{\pi}{2} & \mathbf{c} &= \mathbf{ab} \\
\theta &= \pi & \mathbf{a} \times \mathbf{b} &= 0
\end{align*}
\]

Also:

\[
\begin{align*}
\mathbf{a} \times \mathbf{b} &= -\mathbf{b} \times \mathbf{a} \\
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\
\mathbf{i} \times \mathbf{j} &= \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}
\end{align*}
\]

The vectorial product is distributive, like the scalar product:

\[
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}
\]

From the foregoing follows:
\[ \mathbf{a} \times \mathbf{b} = (ix + iy + kz) \times (ib + jy + kb) = \]
\[ = i(ax + ib) + j(a + yb) + k(ax + yb) = \]
\[ = i(ax - az) + j(ay - az) + k(ay - ay) = \]
\[ \begin{vmatrix}
  i & j & k \\
  a & a & a \\
  x & y & z \\
  b & b & b \\
\end{vmatrix} \]

The rule for the vectorial product of three vectors is:

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \]

as may be verified by expanding the right hand side of this expression.

Some applications of vectors in mechanics

6. The moment of a force

The moment \( \mathbf{M} \) of a force \( \mathbf{F} \) about a point \( O \) is, if \( \mathbf{r} \) is the vector connecting \( O \) with an arbitrary point on the line of action of \( \mathbf{F} \):

\[ \mathbf{M} = \mathbf{r} \times \mathbf{F} \]

see fig. 8.28.

7. The velocity of a point of a rotating body

The velocity \( \mathbf{V} \) of a point \( P \) of a rotating body is:

\[ \mathbf{V} = \omega \times \mathbf{r} \]

if \( \omega \) is the angular velocity of the body and \( \mathbf{r} \) is the vector connecting an arbitrary point on the axis of rotation (the line of action of \( ***** \) with \( P \), see fig. 8.29.
Fig. 8.28: The moment of a force.

Fig. 8.29: The velocity of a point of a rotating wing.

Fig. 8.30: The time-derivative of a vector.
If, in addition, the point $O$ has a velocity $V_o$, the velocity $P$ is

$$V = V_o + \omega \times r$$

8. The time derivative of a vector

Suppose a vector $a$ is given, where $a$ varies in time.

If one end point of the vector is kept in a fixed position, the other, free, end describes a curve, which is called here the hodograph, see fig. 8.30.

The time derivative of $a$ is then defined as:

$$\frac{da}{dt} = \lim_{\Delta t \to 0} \frac{a(t + \Delta t) - a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta a}{\Delta t}$$

After taking the limit, $\Delta a$ is tangent to the hodograph in the point $P$. The magnitude of the limit is:

$$\lim_{\Delta t \to 0} \frac{\Delta a}{\Delta t}$$

This is the velocity of $P$ along the hodograph. The conclusion then is, that the time derivative of a vector is equal to the velocity of its free end point along the hodograph.

Application of this rule results in the following special cases.

1. $\frac{d}{dt} (a + b) = \frac{da}{dt} + \frac{db}{dt}$

2. $\frac{d}{dt} (m \cdot a) = m \frac{da}{dt} + a \cdot \frac{dm}{dt}$ if $m$ is a scalar function of $t$

3. $\frac{d}{dt} (a \cdot b) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$
4. \( \frac{d}{dt} (a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt} \)

5. \( \frac{da}{dt} = i \cdot \frac{d}{dt} x + j \cdot \frac{d}{dt} y + k \cdot \frac{d}{dt} z \) if \( i, j, k \) each have a fixed direction

9. The time derivative of a vector in a rotating reference frame

Sometimes the time derivative of a vector is demanded, expressed in the variations of its components along the axes of a rotating reference frame. The unit vectors \( i, j, \) and \( k \) in the rotating frame then have variable directions. According to the foregoing, the time derivative of a vector \( a \) becomes:

\[
\frac{da}{dt} = \frac{d}{dt} (i \cdot a_x + j \cdot a_y + k \cdot a_z) =
\]

\[
= \frac{di}{dt} \cdot a_x + \frac{dj}{dt} \cdot a_y + \frac{dk}{dt} \cdot a_z + i \cdot \frac{da_x}{dt} + j \cdot \frac{da_y}{dt} + k \cdot \frac{da_z}{dt}
\]

Suppose, the angular velocity of the rotating reference frame is \( \omega \), see fig. 8.31. Then the hodograph of the end point \( P \) of \( i \) is a circle in a plane perpendicular to \( \omega \). The velocity of \( P \) is also the time derivative of \( i \).
According to the foregoing this can be written as:

\[ \frac{di}{dt} = \omega \times i \]

and also:

\[ \frac{dj}{dt} = \omega \times j \quad \text{and} \quad \frac{dk}{dt} = \omega \times k \]

Then the derivative of \( \mathbf{a} \) finally is:

\[ \frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + \omega \times \mathbf{a} \]

where:

\[ \frac{\partial \mathbf{a}}{\partial t} = \mathbf{i} \cdot \frac{da_x}{dt} + \mathbf{j} \cdot \frac{da_y}{dt} + \mathbf{k} \cdot \frac{da_z}{dt} \]

10. The acceleration of a point, expressed in the components of the motion along the axes of a rotating reference frame

Let a fixed reference frame \( O'O'X'Y'Z' \) be given, as well as a moving frame \( OXYZ \). Let the velocity of the origin \( O \) of the moving frame be \( \mathbf{v}_O \), and the angular velocity of this frame \( \omega \), see fig. 8.32.

The acceleration of the point \( P \) is demanded, expressed on its motion relative to the moving frame and the motion of the moving frame relative to the fixed reference frame.

In fig. 8.32 is:

\[ \mathbf{r}' = \mathbf{r}'_0 + \mathbf{r} \]

The velocity of \( P \) relative to the fixed reference frame \( O'O'X'Y'Z' \) is:

\[ \mathbf{v}' = \frac{d\mathbf{r}'}{dt} = \mathbf{v}'_0 + \frac{d\mathbf{r}}{dt} \]

\[ = \mathbf{v}_0 + \frac{d\mathbf{r}}{dt} \]
Fig. 8.32: Motions relative to a fixed and a rotating reference frame.

According to the foregoing, this is equal to:

\[ \mathbf{v} = \mathbf{v}_o + \frac{\partial \mathbf{r}}{\partial t} + \omega \times \mathbf{r} \]

The acceleration of \( P \) relative to the fixed reference frame is:

\[ a = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \omega \times \mathbf{v} \]

\[ = \frac{\partial \mathbf{v}_o}{\partial t} + \omega \times \mathbf{v}_o + \frac{\partial^2 \mathbf{r}}{\partial t^2} + \omega \times \frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \omega}{\partial t} (\omega \times \mathbf{r}) + \omega \times (\omega \times \mathbf{r}) \]

\[ = \frac{\partial \mathbf{v}_o}{\partial t} + \omega \times \mathbf{v}_o + \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \omega \times \frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \omega}{\partial t} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \]

Expanding \( a \) in its components along the axes of the moving OXYZ-reference frame results in:
\[ a_x = \ddot{v}_o + \omega_y v_o z - \omega_z v_o y + x + 2 \omega_y z - 2 \omega_y \dot{y} + x(\omega_y^2 + \omega_z^2) + y(\omega_x \omega_y - \omega_z) + z(\omega_x \omega_z + \omega_y) \]

\[ a_y = \ddot{v}_o + \omega_z v_o x - \omega_x v_o z + y + 2 \omega_x z - 2 \omega_x \dot{z} + + x(\omega_x \omega_y + \omega_z) + y(\omega_z^2 + \omega_x^2) + z(\omega_y \omega_z - \omega_x) \]

\[ a_z = \ddot{v}_o + \omega_x v_o y - \omega_y v_o x + z + 2 \omega_x \dot{y} - 2 \omega_x \dot{y} + x(\omega_x \omega_z + \omega_y) + y(\omega_y \omega_x + \omega_z) - z(\omega_x^2 + \omega_y^2) \]

If, in particular, \( \mathbf{P} \) does not move relative to the moving OXYZ-reference frame, \( |\mathbf{r}| \) is constant and so are \( x, y \) and \( z \).

Then:

\[ a_x = \ddot{v}_o + \omega_v v_o z - \omega_z v_o y + x(\omega_y^2 + \omega_z^2) + y(\omega_x \omega_y - \omega_z) + z(\omega_x \omega_z + \omega_y) \]

\[ a_y = \ddot{v}_o + \omega_z v_o x - \omega_x v_o z + x(\omega_x \omega_y + \omega_z) + y(\omega_z^2 + \omega_x^2) + z(\omega_y \omega_z + \omega_x) \]

\[ a_z = \ddot{v}_o + \omega_x v_o y - \omega_y v_o x + x(\omega_x \omega_z - \omega_y) + y(\omega_y \omega_z + \omega_x) - z(\omega_x^2 + \omega_y^2) \]
APPENDIX 2

Some rules for the use of matrices

A matrix is a collection of numbers arranged in an rectangular array of \( m \) rows and \( n \) columns. An element in a matrix is indicated as \( a_{ij} \), where \( i \) is the number of the row and \( j \) designates the column in which the element figures.

In the description of physical systems three types of matrices commonly occur. These are:

1. The square matrix \((m = n)\), e.g. \( m = 5 \):

\[
[A] = [a_{ij}] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix} \quad (A2-1)
\]

2. The column matrix or vector \((n = 1)\):

\[
[x_{i1}] = \begin{bmatrix}
x_{11} \\
x_{21} \\
x_{31} \\
\vdots \\
x_{n1}
\end{bmatrix} = x \quad (A2-2)
\]

3. The row matrix \((m = 1)\):

\[
[y_{1j}] = [y_{11} \ y_{12} \ y_{13} \ \cdots \cdots \ \ y_{1n}]
\quad (A2-3)
\]

In a square matrix the elements \( a_{ij} \) for which \( i = j \) together are the main diagonal of the matrix.

A determinant \(|a_{ij}|\) (also indicated as \(|A|\)) of a square matrix \([a_{ij}]\) is a sum of terms, obtained by multiplying each element of a row or a column with its minor. The minor of an element \( a_{ij} \) is a determinant of lower order, derived from
by omitting both the i-th row and the j-th column of that element. The determinant \( |A_{ij}| \), the minor of the element \( a_{ij} \), multiplied by \((-1)^{i+j}\) is called the **cofactor** of \( a_{ij} \).

The determinant of a square matrix is equal to the sum of the products of all the elements of a row or a column and their respective cofactors. As an example, the determinant of a 3 x 3 is expanded here.

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11} \begin{vmatrix}
  a_{12} & a_{13} \\
  a_{22} & a_{23}
\end{vmatrix}
- a_{12} \begin{vmatrix}
  a_{11} & a_{13} \\
  a_{21} & a_{23}
\end{vmatrix}
+ a_{13} \begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix}
\]

The three minors are expanded in an identical manner.

The result is:

\[
|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})
\]

(A2–4)

**Various types of matrices**

A particular case of a square matrix is the **diagonal matrix**, the elements of which are zero, except those on the main diagonal.

A diagonal matrix, the non-zero elements of which are all equal, is called a **scalar matrix**; a scalar matrix, the non-zero elements of which are 1 is called the **identity matrix**, written as \([I]\), \([e]\) or \([E]\).

The product of an identity matrix and a vector is equal to that factor:

\[
[I] \cdot x = x
\]

(A2–5)

**A null matrix** or trivial is a matrix, all elements of which are zero.

**A transposed matrix** \([a_{ij}]^T\) corresponding to the matrix \([a_{ij}]\) is obtained by interchanging the rows and the columns:
\[ [a_{ij}]^T = [a_{ji}] \]

In particular, a transposed column matrix or vector is a row matrix and vice versa.

A square matrix is transposed by reflection about the main diagonal.

A symmetric matrix, characterized by the relation \( a_{ij} = a_{ji} \), results after transposing in the original matrix.

A cofactor matrix results by replacing the elements of a matrix by their cofactors. The cofactor matrix of \([a_{ij}]\) then is \([A_{ij}]\).

An adjoint matrix is the transposed cofactor matrix:

\[ \text{adj} \cdot [a_{ij}] = [A_{ij}]^T = [A_{ji}] \]

An inverse matrix is the adjoint matrix divided by the determinant of the original matrix:

\[ [a_{ij}]^{-1} = \frac{\text{adj} [a_{ij}]}{|A|} = \frac{[A_{ij}]^T}{|A|} = \frac{[A_{ji}]}{|A|} \]

The product of the inverted matrix \([a_{ij}]^{-1}\) and the original matrix \([a_{ij}]\) is the identity matrix:

\[ [a_{ij}]^{-1} \cdot [a_{ij}] = [I] \]

The elements \(c_{ij}\) of the inverted matrix:

\[ [c_{ij}] = [a_{ij}]^{-1} \]

then are:

\[ c_{ij} = \frac{|A_{ij}|}{|A|} \quad (A2-8) \]

Operations using matrices

Equality. A m x n matrix and a p x q matrix are equal only, if each element of one matrix is equal to the corresponding element of the other matrix. This
implies, that a necessary but not a sufficient condition is: \( m = p \) and \( n = q \).

The sum of a matrix \([A]\) and a second matrix \([B]\) having equal numbers of rows and columns respectively, is a third matrix \([C]\) having the same numbers of rows and columns. The elements of \([C]\) are the sums of the corresponding elements of \([A]\) and \([B]\).

The product of two matrices is defined only, if the number of columns in the left hand matrix is equal to the number of rows in the right hand matrix. Multiplication is not a commutative operation, i.e. in general is:

\[
[A] \cdot [B] \neq [B] \cdot [A]
\]

The product of a \(m \times n\) matrix \([A]\) and a \(n \times p\) matrix \([B]\) is a \(m \times p\) matrix \([C]\). Each element \(c_{ij}\) is the sum of the products of the elements of the \(i\)-th row of \([A]\) and those of the \(j\)-th column of \([B]\). Such a multiplication is simplified by writing the operation \([A] \cdot [B] = [C]\) as follows:

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
\end{bmatrix}
= 
\begin{bmatrix}
    a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\
    a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}
\end{bmatrix}
\]

(A2-9)

As an example, this multiplication is shown numerically for two \(3 \times 3\) matrices:

\[
\begin{bmatrix}
    1 & 0 & 2 \\
    2 & 2 & 0 \\
    0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 2 & 0 \\
    0 & 3 & 1 \\
    1 & 0 & 2
\end{bmatrix}
= 
\begin{bmatrix}
    5 & 4 & 2 \\
    6 & 7 & 1 \\
    1 & 2 & 4
\end{bmatrix}
\]

Using the scheme of (A2-9), a few special cases may be considered in more detail as follows:
The product of a vector and a row matrix:

\[
\begin{bmatrix}
  b_{11} & b_{12} & b_{13} & \cdots & b_{1n}
\end{bmatrix}
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n}
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n}
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n}
  \vdots & \vdots & \vdots & \ddots & \vdots
  a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\]  

(A2-10)

The product of a row matrix and a vector:

\[
\begin{bmatrix}
  b_j \\
  a_i 
\end{bmatrix} = \sum a_i b_j
\]  

(A2-11)

The product \(\sum a_i b_j\) is called the scalar product.

The solution of simultaneous, linear equations

A set of simultaneous linear equations:

\[
\begin{align*}
  r_{11} x_1 + r_{12} x_2 + r_{13} x_3 &= b_1 y \\
  r_{21} x_1 + r_{22} x_2 + r_{23} x_3 &= b_2 y \\
  r_{31} x_1 + r_{32} x_2 + r_{33} x_3 &= b_3 y
\end{align*}
\]

may be written as the product of a matrix \([R]\) and a vector \(x\).

This product is equal to \(b\cdot y\):

\[
[R] x = b \cdot y
\]  

(A2-12)

or, more explicitly:

\[
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix} \cdot y
\]  

(A2-13)
Multiplication of both the left and the right hand sides of (A2-12) with the inverted matrix \([R]^{-1}\) results in:

\[
[R]^{-1} \cdot [R] \cdot x = [R]^{-1} \cdot b \cdot y
\]  \hspace{1cm} (A2-14)

As, see (A2-7) and (A2-5):

\[
[R]^{-1} \cdot [R] = [I]
\]

and:

\[
[I] \cdot x = x
\]

it follows from (A2-14):

\[
x = [R]^{-1} \cdot b \cdot y
\]  \hspace{1cm} (A2-15)

If \([R]^{-1} = [C]\), according to (A2-6):

\[
[C] = \frac{[R]_{ij}}{|R|}
\]

From (A2-14) and (A2-15) then follows:

\[
x = [R]^{-1} \cdot b \cdot y = \frac{[R]_{ij}}{R} \cdot b \cdot y
\]

The explicit values of \(x_1, x_2\) and \(x_3\) may be obtained using (A2-8). This calculation, the application of Cramer's rule, results in the explicit values of \(x_i\) according to:

\[
x_i = \frac{|B_i|}{|r_{ij}|} \cdot y
\]

where \(|B_i|\) is the determinant following from \(|r_{ij}|\), obtained by replacing in \(|r_{ij}|\) the i-th column by the column matrix \(b\). Applying this rule to (A2-13) results in:
The equations of motion of a system such as the airplane are often written in the form:

\[ \dot{\mathbf{x}} = [A] \mathbf{x} + [B] \mathbf{u} \]  \hspace{2cm} (A2-16)

In (A2-16), \( \dot{\mathbf{x}} \) is the column matrix containing the time derivatives of the components of the motion \( x_1, x_2 \) etc. Using the identity matrix \([I]\) and the operator \( D = \frac{d}{dt} \), (A2-6) can be written with (A2-5) as:

\[ [[A] - d [I]] \mathbf{x} = - [B] \mathbf{u} \]  \hspace{2cm} (A2-17)

Comparing (A2-17) with (A2-12) results in:

\[ [A] - D [I] = [R] \]  \hspace{2cm} (A2-18)

\[ - [B] = b \]  \hspace{2cm} (A2-19)

Expanding (A2-18) gives:

\[
\begin{bmatrix}
a_{11} - D & a_{12} & a_{13} \\
a_{21} & a_{22} - D & a_{23} \\
a_{31} & a_{32} & a_{33} - D \\
\end{bmatrix}
\begin{bmatrix}
r_{11} \\
r_{21} \\
r_{31} \\
\end{bmatrix}
= 
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33} \\
\end{bmatrix}
\]  \hspace{2cm} (A2-20)

It can be seen from (A2-20), that the equations of motion of the airplane can be written in the form of (A2-16), if these equations are of the first order and if each contains only one time derivative occurring on the main diagonal of the matrix \([R]\).
APPENDIX 3

The motions of the airplane relative to the earth

In addition to the study of the equations of motion derived in 8.1, this Appendix briefly considers how the airplane's motion relative to the earth are described. The assumptions are made that u, v and w are the components of the airspeed \( \mathbf{V} \) of the airplane c.g. relative to a volume of ambient air, moving itself at a constant speed \( \mathbf{V}_g \), relative to the earth.

In the more general case the velocity \( \mathbf{V}_g \) of the air relative to the earth or ground may be considered the sum of:

a. the average wind speed \( \mathbf{V}_g \) during the motion under study.

b. the time dependent deviation \( \Delta \mathbf{V} \) of \( \mathbf{V}_g \) relative to the average wind speed \( \mathbf{V}_g \).

At an arbitrary instant is:

\[
\mathbf{V}_g = \mathbf{V}_g + \Delta \mathbf{V}_g
\]

Here the time dependent deviation \( \Delta \mathbf{V}_g \) is assumed to be zero. As a consequence, the variations in the airplane's motion due to the fluctuations in the wind speed need not be further discussed here.

This means, that in the following is:

\[
\mathbf{V}_g = \mathbf{V}_g
\]

The speed of the airplane relative to the ground along the axes of the earth reference frame is required. The \( X_e \)-axis of the reference frame is assumed to point North.

If the speed of the airplane relative to the ground is indicated \( \mathbf{V}_e \), then \( \mathbf{V}_e \) is the sum of the airspeed \( \mathbf{V} \) relative to the surrounding air, and the (wind) speed \( \mathbf{V}_g \) of the air relative to the ground:

\[
\mathbf{V}_e = \mathbf{V} + \mathbf{V}_g
\]

The components of \( \mathbf{V}_e \) are: \( V_{e\text{north}} \), \( V_{e\text{east}} \), and \( V_{e\text{vert}} \). The constant components of the average wind speed \( \mathbf{V}_g \) are: \( V_{g\text{north}} \), \( V_{g\text{east}} \), and \( V_{g\text{vert}} \). The vertical velocities are taken positive if directed upward.
Fig. 8.33: The contribution of $u$, $v$, and $w$ to $V_{\text{north}}$, $V_{\text{east}}$ and $V_{\text{vert}}$. 

(a) The vertical plane through the $X$-axis

(b) The $YOZ$-plane

(c) The horizontal plane
From the figs. 8.33 a to c it can be deduced that the components of \( \mathbf{V} \) along the axes of the earth reference frame are:

\[
V_{\text{north}} = \{u \cos \theta + (v \sin \phi + w \cos \phi) \sin \theta\} \cos \phi +
- (v \cos \phi - w \sin \phi) \sin \psi
\]

\[
V_{\text{east}} = \{u \cos \theta + (v \sin \phi + w \cos \phi) \sin \theta\} \sin \psi +
+ (v \cos \phi - w \sin \phi) \cos \psi
\]

\[
V_{\text{vert}} = u \sin \theta - (v \sin \phi + w \cos \phi) \cos \theta
\]

The components of the velocity \( \mathbf{V}_e \) relative to the earth - the ground speed - then are:

\[
V_{e,\text{north}} = \{u \cos \theta + (v \sin \phi + w \cos \phi) \sin \theta\} \cos \phi +
- (v \cos \phi - w \sin \phi) \sin \psi + V_{g,\text{north}}
\]

\[
V_{e,\text{east}} = \{u \cos \theta + (v \sin \phi + w \cos \phi) \sin \theta\} \sin \psi +
+ (v \cos \phi - w \sin \phi) \sin \psi + V_{g,\text{east}}
\]

\[
V_{e,\text{vert}} = u \sin \theta - (v \sin \phi + w \cos \phi) \cos \theta + V_{g,\text{vert}}
\]

The displacement of the airplane center of gravity relative to the earth is obtained by integrating the three components of the ground speed with respect to time.
APPENDIX 4

The influence of rotating masses on the equations of motion

When deriving the angular momentum of the airplane in Chapter 8, the assumption was made that the coordinates $x$, $y$ and $z$ of all mass elements are constant, as the airplane was considered to be a rigid body. In many cases the airplane has one or more rotating masses, such as propellers or rotors, either helicopters rotors or the rotating masses of jet engines. Due to their own rotation, these rotors may provide an appreciable contribution to the angular momentum. In the following, this contribution is obtained, starting from the assumption that the axis of rotation of the rotating mass does not change its attitude relative to the body axes. The angular velocity $\Omega_r$ about this axis of rotation is taken to be constant.

The situation to be considered is shown in fig. 8.34.

The position of an arbitrary mass element $dm$ in the point $P$ is:

$$r = r_0 + R$$

Fig. 8.34. The contribution of a moving rotor to the angular momentum of the airplane.
where \( \mathbf{r}_0 \) locates the center of mass \( Q \) of the rotor. The velocity of \( dm \) now is:

\[
\mathbf{v}'_p = \mathbf{v}_p + \mathbf{r}_r \times \mathbf{R}
\]

where \( \mathbf{v}_p \) as before is the velocity of \( dm \), if the rotor does not rotate at the angular velocity \( \mathbf{\Omega}_r \). The contribution of \( dm \) to the total angular momentum of the airplane is:

\[
\begin{align*}
\mathbf{d}\mathbf{B} &= \mathbf{d}m \times \mathbf{v}'_p \\
&= \mathbf{d}m \times \mathbf{v}_p + \mathbf{d}m \times (\mathbf{\Omega}_r \times \mathbf{R}) \\
&= \mathbf{d}\mathbf{B}_1 + \mathbf{d}\mathbf{B}_2
\end{align*}
\]

Here \( \mathbf{d}\mathbf{B}_1 \) is the contribution to the angular momentum, as derived in Chapter 8. The additional part, due to the angular velocity of the rotor, is:

\[
\begin{align*}
\mathbf{d}\mathbf{B}_2 &= \mathbf{d}m \times (\mathbf{\Omega}_r \times \mathbf{R}) \\
&= \mathbf{d}m \times \mathbf{r}_0 \times (\mathbf{\Omega}_r \times \mathbf{R}) + \mathbf{d}m \times (\mathbf{\Omega}_r \times \mathbf{R})
\end{align*}
\]

In the integration across the entire rotor, \( \mathbf{r}_0 \) and \( \mathbf{\Omega}_r \) in the first term at the right hand of the latter expression are constants, so:

\[
\int \mathbf{d}m \times \mathbf{r}_0 \times (\mathbf{\Omega}_r \times \mathbf{R}) = \mathbf{r}_0 \times \mathbf{\Omega}_r \times \int \mathbf{d}m \times \mathbf{R} = 0
\]

This integral is equal to zero, as \( \mathbf{R} \) denotes the position of the mass element \( dm \) relative to the center of mass \( Q \) of the rotor. As a result, \( \mathbf{d}\mathbf{B}_2 \) may now be written as:

\[
\begin{align*}
\mathbf{d}\mathbf{B}_2 &= \mathbf{d}m \times (\mathbf{\Omega}_r \times \mathbf{R}) \\
&= \mathbf{d}m \left\{ \mathbf{\Omega}_r (\mathbf{R} \times \mathbf{R}) - \mathbf{R} (\mathbf{\Omega}_r \times \mathbf{R}) \right\} \\
&= \mathbf{d}m \left\{ \mathbf{\Omega}_r (R_x^2 + R_y^2 + R_z^2) - R (\mathbf{\Omega}_r \times \mathbf{R}_x) + \mathbf{\Omega}_r \times \mathbf{R}_y + \mathbf{\Omega}_r \times \mathbf{R}_z \right\}
\end{align*}
\]

The required addition to the angular momentum of the airplane is:

\[
\Delta \mathbf{B} = \int \mathbf{d}\mathbf{B}_2
\]

Expansion results in the components of \( \Delta \mathbf{B} \):
\[
\Delta R_x = \Omega_{r_x} \int (R^2_y + R^2_z) \, dm - \Omega_{r_y} \int R_y \, dx \int R_z \, dx dm - \Omega_{r_z} \int R_z \, dx \int R_y \, dx dm
\]

\[
\Delta R_y = -\Omega_{r_x} \int R_y \, dx \int R_z \, dx dm + \Omega_{r_y} \int (R^2_x + R^2_z) \, dm - \Omega_{r_z} \int R_y \, dy \int R_x \, dy dm
\]

\[
\Delta R_z = -\Omega_{r_x} \int R_z \, dx \int R_y \, dx dm - \Omega_{r_y} \int R_z \, dy \int R_x \, dy dm + \Omega_{r_z} \int (R^2_x + R^2_y) \, dm
\]

In these expressions, the inertial parameters of the rotor are taken about axes parallel to the airplane body axes, as it applies to the inertial parameters of the airplane itself. The moments of inertia are:

\[
I_{r_x} = \int (R^2_y + R^2_z) \, dm
\]

\[
I_{r_y} = \int (R^2_x + R^2_z) \, dm
\]

\[
I_{r_z} = \int (R^2_x + R^2_y) \, dm
\]

and the products of inertia:

\[
J_{r_yz} = \int R_y R_z \, dm
\]

\[
J_{r_xz} = \int R_x R_z \, dm
\]

\[
J_{r_xy} = \int R_x R_y \, dm
\]

and thus:

\[
\begin{bmatrix}
\Delta R_x \\
\Delta R_y \\
\Delta R_z
\end{bmatrix} =
\begin{bmatrix}
+I_{r_x} & -J_{r_xy} & -J_{r_xz} \\
-J_{r_yx} & +I_{r_y} & -J_{r_yz} \\
-J_{r_zx} & -J_{r_zy} & +I_{r_z}
\end{bmatrix}
\begin{bmatrix}
\Omega_{r_x} \\
\Omega_{r_y} \\
\Omega_{r_z}
\end{bmatrix}
\]

According to (8-9), \(\Delta R\) provides an extra contribution in the moment equation: \(\Delta M = \frac{d\Delta R}{dt}\). As the angular velocity \(\Omega_r\) of the rotor is constant, \(\Omega_{r_x} \), \(\Omega_{r_y}\) and \(\Omega_{r_z}\) are constant as well. The additions to the three scalar moment equations (8-10) then are:
\[ \Delta \mathbf{M}_x = q \Delta B_z - r \Delta B_y \]
\[ \Delta \mathbf{M}_y = r \Delta B_x - p \Delta B_z \]
\[ \Delta \mathbf{M}_z = p \Delta B_y - q \Delta B_x \]

The moment equations thus completed are:

\[ \mathbf{M}_x = \ddot{B}_x + q (B_z + \Delta B_z) - r (B_y + \Delta B_y) \]
\[ \mathbf{M}_y = \ddot{B}_y + r (B_x + \Delta B_x) - p (B_z + \Delta B_z) \]
\[ \mathbf{M}_z = \ddot{B}_z + p (B_y + \Delta B_y) - q (B_x + \Delta B_x) \]

Taking into account the contributions of the rotor, the moment equations (8-18) finally are:

\[ L = I_x \ddot{r} + (I_z - I_y) qr - J_{xz} (\dddot{r} + pq) + \Delta B_z q - \Delta B_y r \]
\[ M = I_y \ddot{q} + (I_x - I_z) rp + J_{xz} (p^2 - r^2) + \Delta B_x r - \Delta B_z p \]
\[ N = I_z \ddot{r} + (I_y - I_z) pq - J_{xz} (p r) + \Delta B_y p - \Delta B_x q \]
APPENDIX 5

The influence of the attitude of the reference frame relative to the airplane on the magnitude of the moments and products of inertia

The moments and products of inertia of the airplane depend not only on the mass distribution of the airplane, but also on the chosen directions of the axes of the reference frame relative to the airplane.

Due the symmetry of the airplane two of the three products of inertia are zero, see 8.1.3:

\[ J_{yz} = \int yz \, dm = 0 \]  
\[ J_{xy} = \int xy \, dm = 0 \]

This result does not depend on the directions of the two body axes in the plane of symmetry.

Suppose now, that for an arbitrary direction of the X-axis the magnitudes of the three moments of inertia and the remaining product of inertia are given. The direction of the X-axis is given by the angle \( \epsilon + \eta_1 \) between the X-axis and the negative \( \mathcal{X}_r \)-axis, see fig. 8.35. This fixes also the direction of the Z-axis. If the X-axis has a different angle \( \epsilon + \eta_2 \) with the negative \( \mathcal{X}_r \)-axis, see fig. 8.35, and the first and second reference frames are indicated by the subscripts 1 and 2 respectively, the following relations hold:

\[ x_2 = x_1 \cos (\eta_2 - \eta_1) + z_1 \sin (\eta_2 - \eta_1) \]
\[ y_2 = y_1 \]
\[ z_2 = -x_1 \sin (\eta_2 - \eta_1) + z_1 \cos (\eta_2 - \eta_1) \]

This leads to the following moments and products of inertia, relative to the new reference frame:

\[ I_{x_2} = I_{x_1} \cos^2(\eta_2 - \eta_1) + I_{z_1} \sin^2(\eta_2 - \eta_1) - J_{xz_1} \sin^2(\eta_2 - \eta_1) \]
\[ I_{y_2} = I_{y_1} \]
\[ I_{z_2} = I_{x_1} \sin^2(\eta_2 - \eta_1) + I_{z_1} \cos^2(\eta_2 - \eta_1) + J_{xz_1} \sin^2(\eta_2 - \eta_1) \]
\[ J_{xz2} = \frac{1}{4} (I_{x1} - I_{z1}) \sin 2(\eta_2 - \eta_1) + J_{xz1} \cos 2(\eta_2 - \eta_1) \]  

(A5-1)

Fig. 8.35. The angles between the principal inertial axes and the airplane reference axes.

As could be expected, the moment of inertia about the X-axis does not depend on the directions of the body axes in the plane of symmetry.

For one particular direction of the X-axis, and correspondingly of the Z-axis, the product of inertia \( J_{xz} \) is zero. The axes having these directions are called the principal inertial axes in the plane of symmetry. They are indicated as \( X_0 \) and \( Z_0 \) axes and have an angle \( \varepsilon \) with the negative \( X_r \) and \( Z_r \) axes, see fig. 8.35.

As the products of inertia \( J_{xy} \) and \( J_{yz} \) are always zero, the Y-axis is always a principal inertial axis. This follows also directly from the fact that the Y-axis is perpendicular to the plane of symmetry.

If for an arbitrary direction \( \varepsilon + \eta \) of the X-axis the inertial parameters are known, the angle \( \eta \) of these axes with the principal inertial axes follow from (A5-1) by equating \( J_{xz2} \) and \( \eta_2 \) to zero:

\[ \tan 2\eta = \frac{2 J_{xz}}{I_x - I_z} \]
Using the principal moments of inertia $I_{x_0}$ and $I_{z_0}$, which can be calculated from (A5-1) for a given value $\eta$ by letting $\eta_2 = 0$, the inertial parameters for an arbitrary direction $\eta$ of the body axes are:

$$I_x = I_{x_0} \cos^2 \eta + I_{z_0} \sin^2 \eta$$

$$I_y = I_y_0$$

$$I_z = I_{x_0} \sin^2 \eta + I_{z_0} \cos^2 \eta$$

$$I_{xz} = \frac{1}{2} (I_{x_0} - I_{z_0}) \sin 2\eta$$
APPENDIX 6

Computational methods to solve the characteristic equations for the symmetric and asymmetric motions

Solving the quartic for the symmetric motions

In many cases this equation can be solved using the following method, see ref. 8.13.
The characteristic equation is:

\[ A\lambda_c^4 + B\lambda_c^3 + C\lambda_c^2 + D\lambda_c + E = 0 \]  \hspace{1cm} (A6-1)

Dividing by A:

\[ F_4\lambda_c^4 + F_3\lambda_c^3 + F_2\lambda_c^2 + F_1\lambda_c + F_0 = 0 \]  \hspace{1cm} (A6-2)

where:

\[ F_4 = \frac{A}{A} = 1 \]

\[ F_3 = \frac{B}{A} \]

\[ F_2 = \frac{C}{A} \]

\[ F_1 = \frac{D}{A} \]

\[ F_0 = \frac{E}{A} \]

Using an iterative procedure, (A6-2) is divided by a quadratic factor in \( \lambda_c \):

\[ \lambda_c^2 + a\lambda_c + \beta = 0 \]  \hspace{1cm} (A6-3)

Then:

\[ F_4\lambda_c^4 + F_3\lambda_c^3 + F_2\lambda_c^2 + F_1\lambda_c + F_0 = (\lambda_c^2 + a\lambda_c + \beta)(\lambda_c^2 + C\lambda_c + C_2) = 0 \]  \hspace{1cm} (A6-4)

A first approximation of \( a \) and \( \beta \) is obtained by letting:

\[ \beta = \frac{F_0}{F_2} \]  \hspace{1cm} (A6-5)
and:

\[ \alpha = \frac{F_1 - \beta F_3}{F_2} \quad \text{(A6-6)} \]

Using (A6-5) and (A6-6), \( C_2 \) and \( C_3 \) are calculated:

\[ C_3 = F_3 - \alpha \quad \text{(A6-7)} \]

and:

\[ C_2 = F_2 - \alpha C_3 - \beta \quad \text{(A6-8)} \]

The second approximation of \( \alpha \) and \( \beta \) is:

\[ \beta = \frac{F_2}{C_2} \quad \text{(A6-9)} \]

and:

\[ \alpha = \frac{F_1 - \beta C_3}{C_2} \quad \text{(A6-10)} \]

Next, \( C_2 \) and \( C_3 \) are determined again, from (A6-7) and (A6-8):

\[ C_3 = F_3 - \alpha \quad \text{(A6-11)} \]

and:

\[ C_2 = F_2 - \alpha C_3 - \beta \quad \text{(A6-12)} \]

Then \( \alpha \) and \( \beta \) are calculated again, etc. As a result, the quartic is separated into two quadratic factors which can be solved each separately. If the roots of these two factors are sufficiently different, as is commonly the case for the symmetric motions, this iterative procedure will converge quite rapidly.

Solving the quartic for the asymmetric motions

This iterative method is also taken from ref. 8.13.

The characteristic equation is:
\[ A\lambda_b^4 + B\lambda_b^3 + C\lambda_b^2 + D\lambda_b + E = 0 \]

The method is based on the following formula. If an approximated value \( \alpha_1 \) of one of the real roots \( \lambda_b \) is known, an improved approximation \( \alpha_2 \) is obtained from:

\[ \alpha_2 = \alpha_1 - \frac{\Delta}{\left(\frac{d\Delta}{d\lambda_b}\right)} \quad (A6-13) \]

In (A6-13), \( \Delta \alpha_1 \) is the value of the test function:

\[ A\lambda_b^4 + B\lambda_b^3 + C\lambda_b^2 + D\lambda_b + E = \Delta \quad (A6-14) \]

if \( \lambda_b = \alpha_1 \).

From (A6-14) follows:

\[ \frac{d\Delta}{d\lambda_b} = 4A\lambda_b^3 + 3B\lambda_b^2 + 2C\lambda_b + D \quad (A6-15) \]

In (A6-13), \( \left(\frac{d\Delta}{d\lambda_b}\right) \) is the value of (A6-15) if \( \lambda_b = \alpha_1 \).

There is an advantage in taking as a first approximation for \( \alpha_1 \):

\[ \alpha_1 = 0 \]

Then \( \Delta \alpha_1 = E \) and \( \left(\frac{d\Delta}{d\lambda_b}\right) = D \), so:

\[ \alpha_2 = -\frac{E}{D} \quad (A6-16) \]

Using the \( \alpha_2 \) thus obtained, \( \Delta \alpha_2 \) and \( \left(\frac{d\Delta}{d\lambda_b}\right) \) are determined from (A6-14) and (A6-15), respectively, where upon (A6-13) results in the next approximation \( \alpha_3 \).
\[ \alpha_3 = \alpha_2 - \frac{\Delta \alpha_2}{\left( \frac{d \Delta}{d \lambda_b} \right) \alpha_2} \]

This iterative procedure proves to converge quite well. The root found in this way is the eigenvalue corresponding to the spiral motion. This follows at once from the first approximation (A6-16). Usually, \( E \) is small relative to \( D \), where \( E \) may be either positive or negative.

The quartic can now be divided by \((\lambda_b - \lambda_{b1})\) and the second real root \( \lambda_{b2} \) can be solved from the resulting cubic.

\( \lambda_{b2} \) may be determined using the same method. This root corresponds to the aperiodic rolling motion and is strongly negative.

Further in this Appendix it will be shown that a good first approximation \( a'_{1i} \) of \( \lambda_{b2} \) is:

\[ a'_{1i} = -\frac{B}{A} \quad (A6-17) \]

The quartic can now be written as follows:

\[ \lambda_b^4 + \frac{B}{A} \lambda_b^3 + \frac{C}{A} \lambda_b^2 + \frac{D}{A} \lambda_b + \frac{E}{A} = 0 \quad (A6-18) \]

which can be separated in a quadratic and two linear factors:

\[ (\lambda_b^2 + a\lambda_b + b) (\lambda_b + c) (\lambda_b + d) = 0 \quad (A6-19) \]

In (A6-19) are \( c = -\lambda_{b1} \) and \( d = -\lambda_{b2} \).

The roots \( \lambda_{b3,4} \) can now be obtained simply from (A6-18) and (A6-19).

An approximative analytical solution of the characteristic equation for the asymmetric motions

To simplify the discussion, the characteristic equation is written as:
\[ \lambda_b^4 + B' \cdot \lambda_b^3 + C' \cdot \lambda_b^2 + D' \cdot \lambda_b + E' = 0 \]  
\[ \text{(A6-20)} \]

This equation is solved by writing:

\[ (\lambda_b^2 + a \lambda_b + b) (\lambda_b^2 + c \lambda_b + d) = 0 \]  
\[ \text{(A6-21)} \]

where:

\[ B' = a + c \quad C' = ac + b + d \]  
\[ D' = ad + bc \quad E' = bd \]  
\[ \text{(A6-22)} \]

Elimination of \( c \) and \( d \) from (A6-22) results in:

\[ a = \frac{B'}{2} - \sqrt{\left(\frac{B'}{2}\right)^2 - C' + b + \frac{E'}{b}} \]  
\[ \text{(A6-23)} \]

\[ a = \frac{b^2 B' - b D'}{b^2 - E'} \]  
\[ \text{(A6-24)} \]

It is now possible to derive an approximative analytical solution of (A6-20) using (A6-22) to (A6-24).

To this end, it is realized that for the asymmetric motions, \( a \) and \( E' \) are commonly small relative to \( b \).

From (A6-24) then follows:

\[ b = \frac{D'}{B'} \]  
\[ \text{(A6-25)} \]

With this value of \( b \), (A6-23) yields:

\[ a = \frac{B'}{2} - \sqrt{\left(\frac{B'}{2}\right)^2 - C' + \frac{D'}{B'} + \frac{E' B'}{D'}} \]  
\[ \text{(A6-26)} \]

or, with good approximation: \( (\sqrt{1 + p} \approx 1 + \frac{1}{2} p) \):
\[ a = \frac{2}{b'} \left( \frac{C'}{b'^2} - \frac{D'}{b'^3} - \frac{E'}{b'^4} \right) \]  
(A6-27)

In (A6-21) is:

\[ \lambda_{b3,4} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \]  
(A6-28)

Note: As can be seen from (A6-28), the roots resulting from (A6-28) correspond to a Dutch roll motion. In accordance with this fact, they are indicated as \( \lambda_{b3,4} \) as in 8.5.1.

As \( \left(\frac{a}{2}\right)^2 \ll b \), (A6-28) results in:

\[ \lambda_{b3,4} = -\left( \frac{C'}{b'^2} - \frac{D'}{b'^3} - \frac{E'}{b'D'} \right) \pm j \sqrt{\frac{D'}{b'}} \]  
(A6-29)

With \( b' = \frac{B}{\sqrt{A}} \) etc. this leads to:

\[ \lambda_{b3,4} = -\frac{A}{B} \left( \frac{C}{B^2} - \frac{AD}{B^3} - \frac{E}{B^4} \right) \pm j \frac{\sqrt{D}}{B} \]  
(A6-30)

In relation with (A6-30) it should be noted that the real part of the complex, conjugate roots \( \lambda_{b3,4} \), corresponding to the Dutch roll motion, can be written as:

\[ \frac{A}{B^3D} \cdot \left( BCD - AD^2 - B^2E \right) = \frac{A}{B^3D} \cdot R \]

where \( R \) is 'Routh's discriminant'.

The two remaining roots are obtained, using (A6-22):

\[ c = b' - a; \quad d = \frac{E'}{b} \]  
(A6-31)

The two roots \( \lambda_{b1,2} \) are, according to (A6-21):
\[ \lambda_{b1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - d} \]  

(A6-32)

As \( a \ll b' \), \( \lambda_{b1,2} \) can then be approximated by:

\[ \lambda_{b1,2} = -\frac{b'}{2} \pm \sqrt{\left(\frac{b'}{2}\right)^2 - \frac{E'B'}{D'}} \]  

(A6-33)

or:

\[ \lambda_{b1,2} = -\frac{b}{2A} \pm \sqrt{\left(\frac{b}{2A}\right)^2 - \frac{EB}{AD'}} \]  

(A6-34)
<table>
<thead>
<tr>
<th>Name</th>
<th>Origin</th>
<th>X-Axis</th>
<th>Y-Axis</th>
<th>Z-Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Body axes:</strong></td>
<td><strong>airplane center of gravity.</strong></td>
<td><strong>X-axis:</strong> in the plane of symmetry, in a direction fixed relative to the airplane, to be further defined on a case by case basis, positive X-axis points forward.</td>
<td><strong>Y-axis:</strong> perpendicular to the plane of symmetry, positive Y-axis points to the right (starboard).</td>
<td><strong>Z-axis:</strong> perpendicular to the XOY-plane, positive Z-axis points downward in normal flight.</td>
</tr>
<tr>
<td>Stabilty axes:</td>
<td><strong>as in 1.</strong></td>
<td><strong>X-axis:</strong> as in 1., but parallel to the velocity vector of the center of gravity in the steady flight preceding the disturbed motion; fixed to the airplane during the disturbed motion.</td>
<td><strong>Y-axis:</strong> as in 1.</td>
<td><strong>Z-axis:</strong> as in 1.</td>
</tr>
<tr>
<td><strong>3. Earth axes:</strong></td>
<td>fixed relative to earth, coinciding with the airplane’s center of gravity at the beginning of the motion to be studied.</td>
<td><strong>X-axis:</strong> in the horizontal plane through the origin, in a fixed direction, e.g. pointing North.</td>
<td><strong>Y-axis:</strong> in the horizontal plane through the origin, perpendicular to the X-axis, positive Y-axis is rotated 90 degrees to the right relative to the X-axis, if viewed in the direction of the positive Z-axis.</td>
<td><strong>Z-axis:</strong> vertical, positive Z-axis points to the center of the earth.</td>
</tr>
<tr>
<td>Airplane reference axes:</td>
<td><strong>in an arbitrary but fixed and invariable point of the airplane.</strong></td>
<td><strong>X-axis:</strong> parallel to the plane of symmetry, in a direction fixed and invariable relative to the airplane, positive X-axis points rearward.</td>
<td><strong>Y-axis:</strong> perpendicular to the plane of symmetry positive Y-axis points to the left (port).</td>
<td><strong>Z-axis:</strong> perpendicular to the XOY-plane, positive Z-axis points upwards in normal flight.</td>
</tr>
</tbody>
</table>

**Table 8.1. Reference frames.**
<table>
<thead>
<tr>
<th>Dimensional parameter</th>
<th>Dimension</th>
<th>Divisor</th>
<th>Non-dimensional parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$[t]$</td>
<td>$\ddot{c}/\nu$</td>
<td>$s_c = \frac{\nu}{c} \cdot t$</td>
</tr>
<tr>
<td>$\frac{dt}{dt}$</td>
<td>$[t]^{-1}$</td>
<td>$\nu/\bar{c}$</td>
<td>$D_c = \frac{\nu}{c} \cdot \frac{dt}{ds_c}$</td>
</tr>
<tr>
<td>$\frac{d^2}{dt^2}$</td>
<td>$[t]^{-2}$</td>
<td>$\nu^2/\bar{c}^2$</td>
<td>$D_c^2 = \frac{\nu^2}{c^2} \cdot \frac{d^2}{dt^2} \cdot \frac{d^2}{ds_c^2}$</td>
</tr>
<tr>
<td>$u$</td>
<td>$[\nu] \cdot [t]^{-1}$</td>
<td>$\nu$</td>
<td>$G = \frac{u}{\nu}$</td>
</tr>
<tr>
<td>$w$</td>
<td>$[\nu] \cdot [t]^{-1}$</td>
<td>$\nu$</td>
<td>$a = \frac{w}{\nu}$</td>
</tr>
<tr>
<td>$q$</td>
<td>$[t]^{-1}$</td>
<td>$\nu/\bar{c}$</td>
<td>$\frac{\ddot{c}}{\nu}$</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>$[\nu] \cdot [t]^{-2}$</td>
<td>$\nu^2/\nu^2$</td>
<td>$D_c \bar{u} = \frac{\ddot{u}}{\nu} \cdot \frac{\bar{c}}{\nu} = \frac{\ddot{u}}{\nu}$</td>
</tr>
<tr>
<td>$\bar{w}$</td>
<td>$[t] \cdot [t]^{-2}$</td>
<td>$\nu^2/\nu^2$</td>
<td>$D_c \bar{w} = \frac{\ddot{w}}{\nu} \cdot \frac{\bar{c}}{\nu} = \frac{\ddot{w}}{\nu}$</td>
</tr>
<tr>
<td>$\ddot{q}$</td>
<td>$[t]^{-2}$</td>
<td>$\nu^2/\nu^2$</td>
<td>$D_c \ddot{q} = \ddot{q} \frac{\nu^2}{\nu^2}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$[\nu]$</td>
<td>$\nu \bar{c}$</td>
<td>$u_c = \frac{m}{\nu \bar{c}}$</td>
</tr>
<tr>
<td>$I_Y$</td>
<td>$[m] \cdot [t]^{-2}$</td>
<td>$\nu \bar{c} \bar{c}^2$</td>
<td>$u_c K_Y^2 = \frac{I_Y}{\nu \bar{c}^3}$</td>
</tr>
<tr>
<td>$k_Y$</td>
<td>$[\nu]$</td>
<td>$\bar{c}$</td>
<td>$K_Y = \frac{k_Y}{\bar{c}}$</td>
</tr>
<tr>
<td>$X$</td>
<td>$[m] \cdot [t] \cdot [t]^{-2}$</td>
<td>$\nu \bar{c} \bar{c} \nu^2/\nu^2$</td>
<td>$C_X = \frac{X}{\nu \bar{c} \nu^2}$</td>
</tr>
<tr>
<td>$Z$</td>
<td>$[m] \cdot [t] \cdot [t]^{-2}$</td>
<td>$\nu \bar{c} \bar{c} \nu^2/\nu^2$</td>
<td>$C_Z = \frac{Z}{\nu \bar{c} \nu^2}$</td>
</tr>
<tr>
<td>$M$</td>
<td>$[m] \cdot [t] \cdot [t]^{-2}$</td>
<td>$\nu \bar{c} \bar{c} \nu^2/\nu^2$</td>
<td>$C_M = \frac{M}{\nu \bar{c} \nu^2}$</td>
</tr>
</tbody>
</table>
Table 8.3. Non-dimensional parameters in the equations of motion.
Asymmetric motions.

<table>
<thead>
<tr>
<th>Dimensional parameter</th>
<th>Dimension</th>
<th>Divisor</th>
<th>Non-dimensional parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>([t])</td>
<td>( b/v )</td>
<td>( s_b = \frac{v}{b} \cdot t )</td>
</tr>
<tr>
<td>( \frac{d}{dt} )</td>
<td>([t]^{-1})</td>
<td>( v/b )</td>
<td>( b_b = \frac{v}{b} \cdot \frac{d}{dt} )</td>
</tr>
<tr>
<td>( \frac{d^2}{dt^2} )</td>
<td>([t]^{-2})</td>
<td>( v^2/b^2 )</td>
<td>( b_b = \frac{b}{b} \cdot \frac{d^2}{dt^2} )</td>
</tr>
<tr>
<td>( v )</td>
<td>([v]^{-1})</td>
<td>( v )</td>
<td>( s_b = \frac{v}{v} )</td>
</tr>
<tr>
<td>( p )</td>
<td>([p]^{-1})</td>
<td>( 2v/b )</td>
<td>( b_b = \frac{2p}{2v} )</td>
</tr>
<tr>
<td>( r )</td>
<td>([r]^{-1})</td>
<td>( 2v/b )</td>
<td>( b_b = \frac{2r}{2v} )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>([\phi]^{-2})</td>
<td>( v^2/b^2 )</td>
<td>( b_b = \frac{\theta^2}{2v} \cdot \frac{d^2}{dt^2} )</td>
</tr>
<tr>
<td>( \theta )</td>
<td>([\theta]^{-2})</td>
<td>( v^2/b^2 )</td>
<td>( b_b = \frac{\theta^2}{2v} \cdot \frac{d^2}{dt^2} )</td>
</tr>
<tr>
<td>( x )</td>
<td>([x] )</td>
<td>( \phi )</td>
<td>( b_b = \frac{x}{\phi} )</td>
</tr>
<tr>
<td>( l_x )</td>
<td>([l] )</td>
<td>( \phi )</td>
<td>( b_b = \frac{l_x}{\phi} )</td>
</tr>
<tr>
<td>( l_y )</td>
<td>([l] )</td>
<td>( \phi )</td>
<td>( b_b = \frac{l_y}{\phi} )</td>
</tr>
<tr>
<td>( J_{xx} )</td>
<td>([l] )</td>
<td>( \phi )</td>
<td>( b_b = \frac{J_{xx}}{\phi} )</td>
</tr>
<tr>
<td>( k_x )</td>
<td>([k] )</td>
<td>( b )</td>
<td>( k_x = \frac{k_x}{b} )</td>
</tr>
<tr>
<td>( k_t )</td>
<td>([k] )</td>
<td>( b )</td>
<td>( k_t = \frac{k_t}{b} )</td>
</tr>
<tr>
<td>( k_{xx} )</td>
<td>([k] )</td>
<td>( b^2 )</td>
<td>( k_{xx} = \frac{k_{xx}}{b^2} )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>([\gamma] )</td>
<td>( \frac{v}{b} \cdot \frac{v^2}{b^2} )</td>
<td>( c_T = \frac{\gamma}{\frac{v}{b} \cdot \frac{v^2}{b^2}} )</td>
</tr>
<tr>
<td>( L )</td>
<td>([l] )</td>
<td>( \phi )</td>
<td>( c_L = \frac{L}{\phi} )</td>
</tr>
<tr>
<td>( c_n )</td>
<td>([c] )</td>
<td>( \phi )</td>
<td>( c_n = \frac{c_n}{\phi} )</td>
</tr>
</tbody>
</table>
Table 8.4. Stability derivatives and moments and products of inertia for the symmetric motions in various notation systems. Conversions between the notation systems.

<table>
<thead>
<tr>
<th>Definition</th>
<th>American notation</th>
<th>British notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{X_u} = \frac{1}{2} g v s \cdot \frac{3X}{u} \cdot \frac{\partial C_X}{\partial q}$</td>
<td>$C_{X_u} = 2(x_u - C_{q2}')$</td>
<td>$C_{X_u} = 2(z_u - C_{q2})$</td>
</tr>
<tr>
<td>$C_{Z_u} = \frac{1}{2} g v s \cdot \frac{3Z}{u} \cdot \frac{\partial C_Z}{\partial q}$</td>
<td>$C_{Z_u} = 2(z_u - C_{q2})$</td>
<td>$C_{Z_u} = 2(z_u - C_{q2})$</td>
</tr>
<tr>
<td>$C_{M_u} = \frac{1}{2} g v s \cdot \frac{3M}{u} \cdot \frac{\partial C_M}{\partial q}$</td>
<td>$C_{M_u} = 2(\frac{t}{m_c}) m_u$</td>
<td>$C_{M_u} = 2(\frac{t}{m_c}) m_u$</td>
</tr>
<tr>
<td>$C_{X_a} = \frac{1}{2} g v s \cdot \frac{3X}{a} \cdot \frac{\partial C_X}{\partial q}$</td>
<td>$C_{X_a} = c x_w$</td>
<td>$C_{X_a}$</td>
</tr>
<tr>
<td>$C_{Z_a} = \frac{1}{2} g v s \cdot \frac{3Z}{a} \cdot \frac{\partial C_Z}{\partial q}$</td>
<td>$C_{Z_a} = \frac{t}{m_c} z_w$</td>
<td>$C_{Z_a} = \frac{t}{m_c} z_w$</td>
</tr>
<tr>
<td>$C_{M_a} = \frac{1}{2} g v s \cdot \frac{3M}{a} \cdot \frac{\partial C_M}{\partial q}$</td>
<td>$C_{M_a} = \frac{t}{m_c} x_q$</td>
<td>$C_{M_a} = \frac{t}{m_c} x_q$</td>
</tr>
<tr>
<td>$C_{X_q} = \frac{1}{2} g v s \cdot \frac{3X}{q} \cdot \frac{\partial C_X}{\partial q}$</td>
<td>$C_{X_q} = x_{q2}'$</td>
<td>$C_{X_q} = \frac{t}{m_c} x_q$</td>
</tr>
<tr>
<td>$C_{Z_q} = \frac{1}{2} g v s \cdot \frac{3Z}{q} \cdot \frac{\partial C_Z}{\partial q}$</td>
<td>$C_{Z_q} = \frac{t}{m_c} z_q$</td>
<td>$C_{Z_q} = \frac{t}{m_c} z_q$</td>
</tr>
<tr>
<td>$C_{M_q} = \frac{1}{2} g v s \cdot \frac{3M}{q} \cdot \frac{\partial C_M}{\partial q}$</td>
<td>$C_{M_q} = \frac{t}{m_c} m_q$</td>
<td>$C_{M_q} = \frac{t}{m_c} m_q$</td>
</tr>
<tr>
<td>$C_{X_e} = \frac{1}{2} g v s \cdot \frac{3X}{e} \cdot \frac{\partial C_X}{\partial q}$</td>
<td>$C_{X_e} = x_{e2}'$</td>
<td>$C_{X_e} = 2x_n$</td>
</tr>
<tr>
<td>$C_{Z_e} = \frac{1}{2} g v s \cdot \frac{3Z}{e} \cdot \frac{\partial C_Z}{\partial q}$</td>
<td>$C_{Z_e} = \frac{t}{m_c} z_n$</td>
<td>$C_{Z_e} = \frac{t}{m_c} z_n$</td>
</tr>
<tr>
<td>$C_{M_e} = \frac{1}{2} g v s \cdot \frac{3M}{e} \cdot \frac{\partial C_M}{\partial q}$</td>
<td>$C_{M_e} = \frac{t}{m_c} m_n$</td>
<td>$C_{M_e} = \frac{t}{m_c} m_n$</td>
</tr>
</tbody>
</table>
Table 8.4. (continued)

<table>
<thead>
<tr>
<th>Definition</th>
<th>American notation</th>
<th>Britisch notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_c = \frac{m}{\nu_S \lambda_0} )</td>
<td>( \nu_c = \frac{\lambda}{c \nu} \nu_1 )</td>
<td>( \nu_c = \frac{\lambda}{c \nu} \nu_1 )</td>
</tr>
<tr>
<td>( K_T^2 = \frac{L}{\nu c^2} )</td>
<td>( K_T^2 = \frac{\lambda T^2}{c \nu} )</td>
<td></td>
</tr>
</tbody>
</table>

1) Based on standard specification in V995.
2) Provided that \( C_m = 0 \).
3) The derivatives according to the American definition are indicated with an accent. Only the deviating definitions are stated.
4) \( \bar{c} \) is the British notation for the m.a.c., \( \bar{\lambda}_1 = \bar{c} \) is the mean aerodynamic chord, \( \bar{\lambda}_2 = \bar{\lambda}_t \) is the taillength \( \lambda_h \).
5) Though according to the British notation \( C_m' = \frac{M}{qS} K_1' \), even so \( m_w \neq \frac{dC_m}{d\alpha} \).
Table 8.5. Stability derivatives and moments and products of inertia for the asymmetric motions in various notation systems.

Conversions between the notation systems.

<table>
<thead>
<tr>
<th>Definitions</th>
<th>British notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{Y_B} = \frac{1}{\rho VSB} \cdot \frac{\partial Y}{\partial y} ) = ( \frac{3C_Y}{\rho} )</td>
<td>( C_Y = 2y_v )</td>
</tr>
<tr>
<td>( C_{L_B} = \frac{1}{\rho VSB} \cdot \frac{\partial L}{\partial y} ) = ( \frac{3C_L}{\rho} )</td>
<td>( C_L = \ell_v )</td>
</tr>
<tr>
<td>( C_{N_B} = \frac{1}{\rho VSB} \cdot \frac{\partial N}{\partial y} ) = ( \frac{3C_N}{\rho} )</td>
<td>( C_N = n_v )</td>
</tr>
<tr>
<td>( C_{Y_B} = \frac{1}{\rho VSB} \cdot \frac{\partial Y}{\partial \beta} ) = ( \frac{3C_Y}{\rho} \cdot \frac{b}{V} )</td>
<td>( C_Y = 2y_p )</td>
</tr>
<tr>
<td>( C_{N_B} = \frac{1}{\rho VSB^2} \cdot \frac{\partial N}{\partial \beta} ) = ( \frac{3C_N}{\rho} \cdot \frac{b}{V} )</td>
<td>( C_N = n_p )</td>
</tr>
<tr>
<td>( C_{Y_F} = \frac{2}{\rho VSB} \cdot \frac{\partial Y}{\partial \phi} ) = ( \frac{3C_Y}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_Y = 2y_r )</td>
</tr>
<tr>
<td>( C_{L_F} = \frac{2}{\rho VSB^2} \cdot \frac{\partial L}{\partial \phi} ) = ( \frac{3C_L}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_L = \ell_r )</td>
</tr>
<tr>
<td>( C_{N_F} = \frac{2}{\rho VSB} \cdot \frac{\partial N}{\partial \phi} ) = ( \frac{3C_N}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_N = n_r )</td>
</tr>
<tr>
<td>( C_{Y_R} = \frac{2}{\rho VSB^2} \cdot \frac{\partial Y}{\partial \theta} ) = ( \frac{3C_Y}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_Y = 2y_r )</td>
</tr>
<tr>
<td>( C_{L_R} = \frac{2}{\rho VSB^2} \cdot \frac{\partial L}{\partial \theta} ) = ( \frac{3C_L}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_L = \ell_r )</td>
</tr>
<tr>
<td>( C_{N_R} = \frac{2}{\rho VSB^2} \cdot \frac{\partial N}{\partial \theta} ) = ( \frac{3C_N}{\rho} \cdot \frac{b}{2V} )</td>
<td>( C_N = n_r )</td>
</tr>
</tbody>
</table>
Table 8.5. (continued)

<table>
<thead>
<tr>
<th>Definitions</th>
<th>British notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_n^r = \frac{2}{lv_s v_s^2} \cdot \frac{\partial N}{\partial r} = \frac{3c_N}{2r_b}$</td>
<td>$c_n^r = n_r$</td>
</tr>
<tr>
<td>$c_{r_0}^a = \frac{1}{lv_s v_s} \cdot \frac{2V}{\partial a} = \frac{3c_y}{2a}$</td>
<td>$c_{r_0}^a = 2 Y_z$</td>
</tr>
<tr>
<td>$c_{r_0}^a = \frac{1}{lv_s v_s} \cdot \frac{3L}{\partial a} = \frac{3c_L}{2a}$</td>
<td>$c_{r_0}^a = l_z$</td>
</tr>
<tr>
<td>$c_{r_0}^a = \frac{1}{lv_s v_s} \cdot \frac{2N}{\partial a} = \frac{3c_N}{2a}$</td>
<td>$c_{r_0}^a = n_z$</td>
</tr>
<tr>
<td>$c_{r_0}^a = \frac{1}{lv_s v_s} \cdot \frac{3L}{\partial a} = \frac{3c_L}{2a}$</td>
<td>$c_{r_0}^a = l_z$</td>
</tr>
<tr>
<td>$c_{r_0}^a = \frac{1}{lv_s v_s} \cdot \frac{2N}{\partial a} = \frac{3c_N}{2a}$</td>
<td>$c_{r_0}^a = n_z$</td>
</tr>
<tr>
<td>$\nu_b = \frac{m}{\rho S_b}$</td>
<td>$\nu_b = \frac{1}{2} \nu_2$</td>
</tr>
<tr>
<td>$K_x^2 = \frac{I_x}{mb^2}$</td>
<td>$K_x^2 = i_x$</td>
</tr>
<tr>
<td>$K_z^2 = \frac{I_z}{mb^2}$</td>
<td>$K_z^2 = i_z$</td>
</tr>
<tr>
<td>$K_{xz} = \frac{I_{xz}}{mb^2}$</td>
<td>$K_{xz} = i_{xz}$</td>
</tr>
</tbody>
</table>

$\lambda_1$ is the British notation for $b$.

$\lambda_2$ is the British notation for $b$. 

Table 8.6. Simplified formulas for the calculation of stability and control derivatives and coefficients in the initial, steady flight condition for the symmetric motions. (Without the effects of propellers and jets, aero-elasticity and compressibility of the air).

\[
C_{\theta} = - C_D + T_c \cos(\alpha + \gamma) = \frac{W}{\frac{1}{2} \rho V^2 S} \sin \gamma \approx - C_D + T_c
\]

\[
C_{\phi} = - C_L - T_c \sin(\alpha + \gamma) = - \frac{W}{\frac{1}{2} \rho V^2 S} \sin \gamma \approx - C_L
\]

\[
C_{m_\theta} = 0
\]

\[
C_{m_\phi} = 2 C_L. \tan \gamma
\]

\[
C_{m_\phi} = - 2 C_L
\]

\[
C_{m_\phi} = 0
\]

\[
C_{X_{\alpha}} = C_L - C_D = C_T \cos \alpha - C_{N_{\alpha}} \sin \alpha; \quad \text{if} \quad C_D = \frac{c_L}{m A e}; \quad C_{X_{\alpha}} = C_T(1 - \frac{2 C_L}{m A e})
\]

\[
C_{Z_{\alpha}} = C_L - C_D = C_T \alpha - C_{N_{\alpha}} \alpha - C_{V_{\alpha}} \approx - C_{N_{\alpha}}
\]

\[
C_{m_{\alpha}} = C_{N_{\alpha}} \frac{2 V h}{S c} \frac{V h}{S c}
\]

\[
C_{X_q} = 0
\]

\[
C_{Z_q} = 2 C_L \left( \frac{-h}{V} \right) \frac{V}{h} \frac{h}{S c}
\]

\[
C_{m_q} = 1, \quad 1,2 \left( C_{m_q} \right) = 1, \quad 1,2 \quad C_{N_{h_{\alpha}}} \left( \frac{v}{h} \right) \frac{h}{S c^2}
\]

\[
C_{X_{\dot{\alpha}}} = 0
\]

\[
C_{Z_{\dot{\alpha}}} = - C_{N_{h_{\alpha}}} \frac{V}{h} \frac{h}{S c}
\]

\[
C_{m_{\dot{\alpha}}} = - C_{N_{h_{\alpha}}} \frac{V}{h} \frac{h}{S c^2}
\]
Table 8.6. (continued)

\[ C_x^\delta = 0 \]
\[ C_{z^\delta} = -C_{N_{h^\delta}} \cdot \frac{v_h}{V} \frac{S_h}{S} \]
\[ C_{m^\delta} = -C_{N_{h^\delta}} \cdot \frac{v_h}{V} \frac{S_h f_h}{Sc} \]

Influence of the c.g. position on some stability derivatives.

\[ C_{m_{a2}} = C_{m_{a1}} - C_{m_{a1}} \frac{x_{cg_2} - x_{cg_1}}{c} \]
\[ C_{z_{q2}} = C_{z_{q1}} - C_{z_{q1}} \frac{x_{cg_2} - x_{cg_1}}{c} \]
\[ C_{m_{q2}} = C_{m_{q1}} - C_{m_{q1}} \frac{x_{cg_2} - x_{cg_1}}{c} - C_{m_{a1}} \frac{x_{cg_2} - x_{cg_1}}{c} \]
\[ = C_{m_{q1}} - (C_{z_{q1}} + C_{m_{a1}}) \frac{x_{cg_2} - x_{cg_1}}{c} + C_{z_{a}} \frac{x_{cg_2} - x_{cg_1}}{c} \]
\[ = C_{m_{q1}} - C_{z_{a}} \frac{x_{cg_2} - x_{cg_1}}{c} \]
Table 8.7. Moments and products of inertia of various airplane types, partly derived from ref. 8.47.

<table>
<thead>
<tr>
<th>Manufacturer Type Name</th>
<th>Category</th>
<th>W in kg</th>
<th>S in m²</th>
<th>c in m</th>
<th>b in m</th>
<th>Iₓ in kgm²</th>
<th>Iᵧ in kgm²</th>
<th>Iž in kgm²</th>
<th>Kₓ</th>
<th>Kᵧ</th>
<th>Kž</th>
</tr>
</thead>
<tbody>
<tr>
<td>North American B25 &quot;Mitchell&quot;</td>
<td>Bomber</td>
<td>12,300</td>
<td>56.7</td>
<td>2.95</td>
<td>20.6</td>
<td>6.920</td>
<td></td>
<td></td>
<td>0.798</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Douglas D.C.3 &quot;Dakota&quot;</td>
<td>Passenger plane</td>
<td>10,450</td>
<td>91.9</td>
<td>3.50</td>
<td>29</td>
<td>12.450</td>
<td></td>
<td></td>
<td>0.978</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lockheed 1049 &quot;Super Constellation&quot;</td>
<td>Passenger plane</td>
<td>54,430</td>
<td>153.3</td>
<td>4.47</td>
<td>37.50</td>
<td>170.000</td>
<td></td>
<td></td>
<td>1.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boeing B 47 &quot;Stratojet&quot;</td>
<td>Bomber</td>
<td>52,150</td>
<td>132.7</td>
<td>3.96</td>
<td>35.4</td>
<td>148.500</td>
<td>318,000</td>
<td>0.15</td>
<td>0.22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fokker F27 Friendship</td>
<td>Passenger plane</td>
<td>16,200</td>
<td>70</td>
<td>2.58</td>
<td>29</td>
<td>17.600</td>
<td>29.790</td>
<td>47.390</td>
<td>0.1124</td>
<td>1.650</td>
<td>0.1848</td>
</tr>
<tr>
<td>Douglas D.C. 9</td>
<td>Passenger plane</td>
<td>33,300</td>
<td>86</td>
<td>3.62</td>
<td>26.6</td>
<td>38.600</td>
<td>112.000</td>
<td>143.000</td>
<td>0.128</td>
<td>1.61</td>
<td>0.246</td>
</tr>
<tr>
<td>Douglas D.C. 8</td>
<td>Passenger plane</td>
<td>94,000</td>
<td>268.1</td>
<td>7.0</td>
<td>43.4</td>
<td>418.000</td>
<td>615,000</td>
<td>921,000</td>
<td>0.1535</td>
<td>1.152</td>
<td>0.228</td>
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<tr>
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<td>11.2</td>
<td>1518</td>
<td>8190</td>
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<td>11.5</td>
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<td>Re-Entry vehicle Rocket Glider</td>
<td>3650</td>
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<td>4.9</td>
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<td>2.550</td>
<td>0.239</td>
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<td>Siebel 204 D 1</td>
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<td>5600</td>
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</tr>
</tbody>
</table>
REFERENCES

Chapter 7. Steady, asymmetric flight

General

7.1. H.H.B.M. Thomas
State of the art on estimation of derivatives.

7.2. Anon.
Data Sheets Aerodynamics, Vols, I, II and III,
Royal Aeron. Society, London.

7.3. D.E. Hoak
J.W. Carlson
USAF Stability and Control Handbook,

7.4. J.P. Campbell
M.O. McKinney

7.5. W.E. Cotter Jr.
Summary and analysis of data on damping in yaw and pitch for a number of airplane models.

7.6. T.A. Toll
Summary of lateral control research.

7.7. R.R. Gilruth
W.N. Turner
Lateral control required for satisfactory flying qualities based on flight tests of numerous airplanes.

Wings

7.8. H.A. Pearson
R.T. Jones
Theoretical stability and control characteristics of wings with various amounts of taper and twist.
7.9. T.A. Toll
M.J. Queijo
Approximate relations and charts for low-speed
stability derivatives of swept wings.

7.10. J. Bird
Some theoretical low-speed span loading
characteristics of swept wings in roll and
sideslip.
NACA Rep. 969, 1950

7.11. J. de Young
Theoretical antisymmetric span loading for
wings of arbitrary plan form at subsonic
speeds.

7.12. M.J. Queijo
Theoretical span load distributions and
rolling moments for sideslipping wings of
arbitrary plan form in incompressible flow.

7.13. E.C. Polhamus
W.C. Sleeman Jr.
The rolling moment due to sideslip of swept
wings at subsonic and transonic speeds.

7.14. J. de Young
Spanwise loading for wings and control sur-
faces of low aspect ratio.

7.15. J. de Young
Theoretical symmetric span loading due to flap
deflection for wings of arbitrary plan form at
subsonic speeds.

7.16. W.J.G. Pinsker
A semi-empirical method for estimating the
rotary rolling moment derivatives of swept and
slender wings.
7.17. E.C. Polhamus

A simple method of estimating the subsonic lift and damping in roll of swept-back wings.

7.18. A. Goodman
J.D. Brewer

Investigation at low-speeds of the effect of aspect ratio and sweep on static and yawing stability derivatives of untapered wings.

7.19. A. Goodman
G.H. Adair

Estimation of the damping in roll of wings through the normal flight range of lift coefficient.

7.20. A. Goodman
L.E. Fisher

Investigations at low speeds of the effect of aspect ratio and sweep on rolling stability derivatives of untapered wings.

7.21. A.P. Martina

Method for calculating the rolling and yawing moments due to rolling for unswept wings with or without flaps or ailerons by use of non-linear section lift data.
NACA T.N. 2937, 1953.

7.22. J.P. Campbell
A. Goodman

A semi-empirical method for estimating the rolling moment due to yawing of airplanes.

7.23. F.M. Rogallo
J.C. Lowry
J. Fishel

Lateral-control devices suitable for use with full-span flaps.

7.24. L.R. Fisher

Approximate corrections for the effects of compressibility on the subsonic stability derivatives of swept wings.
7.25. F. E. Purser
An approximation to the effect of geometric dihedral on the rolling moment due to sideslip for wings at transonic and supersonic speeds.
NACA R.M. L52 B01, 1952.

K. Margolis
Theoretical stability derivatives of thin sweptback wings tapered to a point with swept-back or sweptforward trailing edges for a limited range of supersonic speeds.

7.27. A. L. Jones
A. Alksne
A summary of lateral stability derivatives calculated for wing plan forms in supersonic flow.

7.28. K. Margolis
W. L. Sherman
M. E. Hannah
Theoretical calculation of the pressure distribution, span loading, and rolling moment due to sideslip at supersonic speeds for thin sweptback tapered wings with supersonic trailing edges and wing tips parallel to the axis of wing symmetry.
NACA T.N. 2898, 1953.

7.29. W. L. Sherman
K. Margolis
Theoretical calculations of the effects of finite sideslip at supersonic speeds on the span loading and rolling moment for families of thin sweptback tapered wings at an angle of attack.
NACA T.N. 3046, 1952.

7.30. S. H. Harman
L. Jeffreys
Theoretical lift and damping-in-roll of thin wings with arbitrary sweep and taper at supersonic speeds. Supersonic leading and trailing edges.
7.31  R.A. Lagerstrom  
      M.E. Graham  
Linearized theory of supersonic control surfaces.  

Influence of the fuselage

7.32. H.J. Allen  
      E.W. Perkins  
A study of effects of viscosity on flow over slender inclined bodies of revolution.  

7.33. A.H. Sacks  
Aerodynamic forces, moments, and stability derivatives for slender bodies of general cross section.  

7.34. W.R. Bates  
Static stability of fuselages having a relatively flat cross section.  
NACA T.N. 3429, 1955

7.35. E.C. Polhamus  
      K.P. Spreemann  
Subsonic wind-tunnel investigation of the effects of fuselage afterbody on directional stability of wing-fuselage combinations at high angles of attack.  
NACA T.N. 3896, 1956.

7.36. W. Jacobs  
Lift and moment changes due to the fuselage for a yawed aeroplane with unswept and swept wings.  

7.37. H. Mirels  
Aerodynamic of slender wings and wingbody combinations having swept trailing edges.  

7.38. W.A. Tucker  
      R.O. Piland  
Estimation of the damping in roll of supersonic-leading-edge wing-body combinations.  
7.39. W. Jacobs
The influence of the induced sidewind on the efficiency of the vertical tail. A simplified method for calculation.
7.40 W. Jacobs
The induced sidewind behind swept wings at subsonic velocities.
7.41. W. Jacobs
Theoretical and experimental investigations of interference effects of delta wing-vertical tail combinations with yaw.
7.42. W. Jacobs
E. Truckenbrodt
Der induzierte Seitenwind von Flugzeugen.
Ing. Archiv, Bd. 21, 1953, S. 1/22.
(in German).
7.43. W.H. Michael Jr.
Analysis of the effects of wing interference on the tail contributions to the rolling derivatives.
7.44. L.R. Fischer
H.S. Fletcher
Effect of lag of sideward on the vertical-tail contribution to oscillatory damping in yaw of airplane models.
7.45. W.H. Michael Jr.
Investigation of mutual interference effects of several vertical tail-fuselage configurations in sideslip.
7.46. D.L. Lyons
P.L. Bisgood
An analysis of the lift slope of aerofoils of small aspect ratio including fins, with design charts for aerofoils and control surfaces.
<table>
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<th>No.</th>
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<tr>
<td>7.47</td>
<td>S. Katzoff, W. Mutterperl</td>
<td>The end-plate effect of a horizontal-tail surface on a vertical-tail surface.</td>
<td>NACA T.N. 797, 1941</td>
</tr>
<tr>
<td>7.48</td>
<td>H.E. Murray</td>
<td>Wind-tunnel investigation of end-plate effects of horizontal tails on a vertical tail compared with available theory.</td>
<td>NACA T.N. 1050, 1946</td>
</tr>
<tr>
<td>7.50</td>
<td>M.J. Queijo, D.R. Riley</td>
<td>Calculated subsonic span loads and resulting derivatives of unswept and 45° sweptback tail surfaces in sideslip and in steady roll.</td>
<td>NACA T.N. 3245, 1945</td>
</tr>
<tr>
<td>7.51</td>
<td>K.W. Booth</td>
<td>Effect of horizontal-tail chord on the calculated subsonic span loads and stability derivatives of isolated unswept tail assemblies in sideslip and steady roll.</td>
<td>NASA Memo 4-1-59 L. 1959</td>
</tr>
<tr>
<td>7.52</td>
<td>E.C. Polhamus</td>
<td>Some factors affecting the variation of pitching moment with sideslip of aircraft configurations.</td>
<td>NACA T.N. 4016, 1958</td>
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<tr>
<td></td>
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<td>(in German)</td>
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</tbody>
</table>
7.54. P.J. Bobbitt
Tables for the rapid estimation of downwash and sidewash behind wings performing various motions at supersonic speeds.

7.55. J.C. Martin
F.S. Mavestuto Jr.
Theoretical forces and moments due to sideslip of a number of vertical tail configurations and supersonic speeds.
NACA T.N. 2412, 1951.

7.56. P.J. Bobbitt
D.A. Malvestuto Jr.
Estimation of forces and moments due to rolling for several slender-tail configurations at supersonic speeds.
NACA T.N. 2955, 1953.

7.57. K. Margolis
P.J. Bobbitt
Theoretical calculations of the pressures, forces and moments at supersonic speeds due to various lateral motions acting on thin isolated vertical-tails.

7.58. K. Margolis
M.H. Elliot
Theoretical calculations of the pressures, forces and moments due to various lateral motions acting on tapered sweptback vertical tails with supersonic leading and trailing edges.

Influence of the propeller

7.59. H.S. Ribner
Formulas for propellers in yaw and charts of the side-force derivative.

7.60. H.S. Ribner
Propellers in yaw.
7.61. J. Mannée
Wind tunnel investigation of the influence of the aircraft configuration on the yawing and rolling moment of a twin-engined, propeller-driven aircraft with one engine inoperative.
N.L.L. Rapport A 1508 B, 1963
(See also N.L.L. Rapport A 1508 A)

Spoilers

7.62. R.W. Franks
The application of a simplified lifting-surface theory to the prediction of the rolling effectiveness of plain spoiler ailerons at subsonic speeds.

7.63. J.G. Lowry
Data on spoiler-type ailerons.

Test results of NLR and the Department of Aerospace Engineering, Delft University of Technology

7.64. Anon.
Verslag der vliegproeven met het vliegtuig FT-404, type North-American 'Harvard II B', Deel d: Statische stuurstanddwarssstabiliteit.
N.L.L. Rapport V. 1422 d, 1950
(in Dutch)

7.65. C.L. Spigt
A. de Gelder
Windtunnelmetingen aan een model van de romp van de Fokker F-27 'Friendship'.
VTH M41, 1959.
(in Dutch)

7.66. C. Spoon
Windtunnelmetingen aan een model van de vleugel van de Fokker F-27 'Friendship'.
VTH M46, 1960.
(in Dutch)
<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
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</table>
Chapter 8. Equations of motion of an airplane; introduction to the dynamic stability

Literature on the formulation of the equations of motion

8.1. S.W. McCuskey
     An introduction to advanced dynamics.

8.2. J.L. Synge
     B.A. Griffith
     Principles of mechanics.

8.3. O.H. Gerlach
     De vergelijkingen voor de bewegingen van een
     vliegtuig om een bolvormige, roterende aarde.
     Technische Hogeschool Delft, Vliegtuigbouw-
     (in Dutch)

8.4. L.J. Meriam
     Momentum equations for variable mass.
     Aerospace Engineering, July 1962, p. 52/53,
     p. 84/87.

8.5. M.V. Barton
     The effect of variation of mass on the dynamic
     stability of jet-propelled missiles.

8.6. N. Rott
     L. Pottsepp
     Simplified calculation of the jet-damping
     effects.

8.7. H. Goldstein
     Classical Mechanics.
     U.S.A., 1951.

8.8. W.F. Osgood
     Mechanics.

8.9. F.B. Hildebrand
     Advanced calculus for engineers.
Literature on the stability of an equilibrium

8.10. J.G. Malkin
Theorie der Stabilität einer Bewegung.
(in German)

8.11. E.J. Routh
Dynamics of a system of Rigid Bodies.

8.12. A. Hurwitz
Über die Bedingungen, unter welchen eine
Gleichung nur Wurzeln mit negativen reellen
Teilen besitzt.
Mathematische Werke, Band II, Zahlentheorie
und Geometrie, S. 533/545.
Verlag Emil Birkhäuser und Cie., Basel, 1933.
(Orig. publication: Mathematische Annalen, Bd.
46, 1895, S. 273-284).
(in German)

8.13. H.R. Hopkin
Routine computing methods for stability and
response investigations on linear systems.

Review of literature on the computation of stability derivatives for symmetric airplane motions

a. General publications

State of the art on estimation of derivatives.

8.15.
Data Sheets, Aerodynamics.

b. Wings in steady flow

8.16. I.H. Abbot
Summary of airfoil data.
A.E. von Doenhoff
L.S. Stivers Jr.


The lift and pitching moment of an aerofoil due to a uniform angular velocity of pitch. A.R.C. R. and M. 1216, 1929.


c. Wings in unsteady flow


Zusammenfassender Bericht über instationären Auftrieb von Flügeln. Luftfahrtforschung, Bd. 1936, S 410/424. (in German)

Allgemeine Tragflächentheorie. Luftfahrtforschung, Bd. 17, 1940, S. 370/378. (in German)

8.25. R.T. Jones
The unsteady lift of a wing of finite aspect ratio.
NACA Rep. 681, 1940.

H. Lomax
Two-dimensional unsteady lift problems in supersonic flight.

8.27. H. Lomax
M.A. Heaslet
F.B. Fuller
L. Sluder
Two- and three-dimensional unsteady lift problems in high-speed flight.

8.28. C.E. Watkins
Air forces and moments on triangular and related wings with subsonic leading edges oscillating in supersonic potential flow.

8.29. J.A. Drischler
Approximate indicial lift functions for several wings of finite span in incompressible flow as obtained from oscillatory lift coefficients.

8.30. J.A. Drischler
Calculation and compilation of the unsteady-lift functions for a rigid wing subjected to sinusoidal gusts and to sinusoidal sinking oscillations.

d. Downwash behind a wing

8.31. A. Silverstein
S. Katzoff
Design charts for predicting downwash angles and wake characteristics behind plain and flapped wings.
<table>
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<th>Author(s)</th>
<th>Title</th>
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<tr>
<td>8.34</td>
<td>J.C. Martin</td>
<td>The calculation of downwash behind wings or arbitrary plan form at supersonic speeds. NACA T.N. 2135, 1950.</td>
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**e. Unsteady downwash behind a wing**

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<th>Title</th>
</tr>
</thead>
</table>
8.40. H.S. Ribner
Time-dependent downwash at the tail and the 
pitching moment due to normal acceleration at 
supersonic speeds. 

f. Characteristics of tailplanes and control surfaces

8.41. D.J. Lyons
P.L. Bisgood
An analysis of the lift slope of aerofoils of 
small aspect ratio including fins with design 
charts for aerofoils and control surfaces. 

8.42. J. de Young
Spanwise loading for wings and control sur-
faces of low aspect ratio. 

8.43. J. de Young
Theoretical symmetric span loading due to flap 
deflection for wings of arbitrary plan form at 
subsonic speeds. 

8.44. J.B. Dods Jr.
B.E. Tinling
Summary of results of a windtunnel investiga-
tion of nine related horizontal tails. 

8.45. R.A. Lagerstrom
M.E. Graham
Linearized theory of supersonic control sur-
faces. 

8.46. K.L. Goin
Equations and charts for the rapid estimation 
of hinge moment and effectiveness parameters 
for trailing edge controls having leading and 
trailing edges swept ahead of the Mach lines. 
Airplane moments of inertia


8.49. C.H. v.d. Linden Experimental determination of the three A. Blauw moments of inertia and the product of inertia in the plane of symmetry of the N.L.R. Laboratory aircraft Siebel 204-D-1. N.L.R. V. 1912, 1963. (in Dutch)


Fig. 7.52: Asymmetric stability-and control derivatives