OPTIMAL AERODYNAMIC ATTITUDE STABILIZATION
OF NEAR EARTH SATELLITES

by

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SUMMARY

A near Earth satellite orbiting in the altitude range of 150 km to 450 km encounters small but non-negligible aerodynamic forces. It is possible to generate sufficient aerodynamic torques by providing 'all-moving' control surfaces of suitable size to achieve active attitude control. This report is a preliminary study of such an active attitude control system.

The satellite configuration considered has four all-moving control surfaces and the dominant gravity gradient torques and aerodynamic torques are considered in the analysis. The resulting equations of motion are linearized and modern optimal control theory concepts are applied to synthesize a feedback control system for controlling the satellite by rotating the control surfaces to obtain the necessary control torques. The numerical studies carried out indicate that it is possible to control a near Earth satellite by using control surfaces of reasonable size. Damping times of the order of from a few orbits to a fraction of an orbit seem reasonable.
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Note: Symbols of limited use are defined where used

\( b^7 \) vector from spacecraft center of mass to the center of pressure of the \( i \)th control surface in \( \mathcal{F}^7 \) coordinate system

\( \text{erf}(x) \) error function of argument \( x \)

\( i \) orbital inclination

\( x_i \) unit vector along the coordinate axis \( x_i \)

\( n^7 \) unit normal to the surface element in \( \mathcal{F}^7 \) coordinate system

\( p^7 \) normal pressure in \( \mathcal{F}^7 \) coordinate system

\( r \) position vector with respect to the satellite center of mass

\( t^7 \) unit vector along the direction in which the shear acts in \( \mathcal{F}^7 \) coordinate system

\( u \) control vector

\( v_R \) unit vector along the relative velocity vector

\( x_i \) vector \( x \) in \( \mathcal{F}^i \) coordinate system

\( x_{ij} \) components of \( x_i \), \( j = 1, 2, 3 \)

\( x \) state vector

\( A \) area of one control surface

\( A_W \) "wetted" surface area

\( A_1, B_1, C_1 \) principal moments of inertia of the satellite about the \( x_1^7, x_2^7, \) and \( x_3^7 \) axes, respectively

\( C_\theta \) \( \cos \theta \)

\( C_D, C_p, C_\tau \) drag, normal force, and shear force coefficients, respectively

\( C_{mA} \) derivative of moment coefficient with respect to angle of attack

\( D \) drag

\( F_A \) aerodynamic force

\( G_G \) gravity gradient torque

\( G_A \) aerodynamic torque
K  matrix of feedback gains
L  characteristic length of center body
Q  positive semidefinite weighting matrix for the state variables in the quadratic performance index
R  orbital radius
R  positive definite weighting matrix for the control variables in the quadratic performance index
Rc  circular orbit radius
S  molecular speed ratio
S  solution of the matrix Riccati equation
SA  cross-sectional area of the center-body
Sθ  sinθ
V  orbital speed of satellite
VR  relative velocity of 'air' with respect to the satellite
X^i_j  coordinate axes in the x^i coordinate system, j = 1,2,3
α  angle between the reversed incoming flow direction and the normal to the surface
η  orbital angle
θ_1,θ_2,θ_3  angles defining orientation of body reference frame with respect to orbital reference frame, correspond to roll, pitch, and yaw respectively
θ^c_i  i th control panel angle
φ^i_j  rotation about X^i_j axis
(θ)_i  transformation matrix for rotation θ around X_i axis
μ  earth's gravitational constant
ρ  density of the atmosphere
σ', σ  normal momentum and tangential momentum accommodation coefficients, respectively
τ^7  shear in ξ^7 coordinate system
ω  argument of perigee
ω  angular velocity vector
\( \omega_E \)  
angular speed of earth

\( \Lambda \)  
angle of attack of center-body

\( (.)_i \)  
pertaining to \( i^{th} \) control surface

\( (.)^i \)  
quantity expressed in \( \mathcal{X}^i \) coordinate system

\( (.)^7 \)  
quantity expressed in \( \mathcal{X}^7 \) coordinate system

(\( \tilde{\cdot} \))  
tilde operator: for arbitrary vectors \( a \) and \( b \) expressed in terms of an arbitrary vector basis \( \{e\} \), \( (a \times b) = \{e\}^T(\tilde{a} \cdot b) \)

\[
\tilde{a} = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\]

(\( \dot{\cdot} \))  
differentiation with respect to time

(\( \dot{\cdot} \))  
differentiation with respect to orbital angle

\( \mathcal{X}^{ij} \)  
transformation matrix from coordinate system \( \mathcal{X}^j \) to \( \mathcal{X}^i \).

\( g \)  
inertia matrix

\( \mathcal{X}^i \)  
coordinate system

\( \mathcal{X}^5 \)  
orbital coordinate system (see Sec. 2.1)

\( \mathcal{X}^7 \)  
body fixed reference coordinate system (see Sec. 2.1)

\( \mathcal{X}^{Hi} \)  
coordinate system fixed in the \( i^{th} \) hinge

\( \mathcal{X}^{si} \)  
principal axes coordinate system fixed in the \( i^{th} \) control surface
I. INTRODUCTION

1.1 General

A large number of satellites have been launched to date and many more will be launched in the future to serve scientific, commercial and military aims. Irrespective of the purpose of the satellite, to be useful and effective, the satellite must be either attitude controllable or its attitude must be measurable. By the former we mean that it should be possible to orient the satellite in a preferred or specified attitude and by the later we mean that if the satellite is uncontrolled it should be possible to obtain a time history of its attitude by suitable instrumentation and telemetry. In general the satellites and the environment in which they operate are far from isotropic. Consequently in the case of scientific satellites, since the measurement of physical parameters made by satellite-borne instruments will be strongly dependent on the orientation of the instruments, it is necessary to either control the attitude of the satellite precisely or provide information regarding the attitude of the satellite to the experimenter to enable him to interpret his results realistically. In the case of communications satellites of active relay type the effective power transmitted from the satellite depends on the satellite transmitter power, transmission efficiency and antenna gain. If the antenna gain is increased the beam width decreases and consequently the satellite will have to be pointed more accurately towards the earth station. The need for the attitude control of military satellites is quite obvious.

Attitude control is one of the most critical areas of space technology. Due to the absence of significant natural sources of damping in the outer space, the slightest internal motion or the smallest external torque is bound to accelerate a satellite into unwanted and undamped rotation. A satellite in orbit may experience disturbing torques due to unbalanced aerodynamic forces, unbalanced solar radiation pressure, gravity gradients, magnetic fields, propulsion units, micrometeoroid impacts, ejected mass, emitted radiation, internal motion etc. The relative magnitude of these torques depends on the orbital radius, the configuration and other aspects of the satellite. Wiggins (Ref.68) studied the environmental torques experienced by a satellite of cylindrical shape, 5 ft. in diameter and 30 ft. in length, in the altitude range of 100 to 1000 nautical miles. Similar studies have been carried out by others on individual satellites. In general the aerodynamic torques are important below 500 km. altitude and become dominant below 300 km. altitude for many satellites. Solar radiation torques become important at very high altitudes. The gravity gradient torques depend upon the radial dependence of the gravitational field, and these are important for near-Earth satellites. Micrometeoroid impacts with a non-zero moment arm to the center of mass can infrequently cause significant angular accelerations but the average effect should be small. The other torques mentioned above are under the control of the designer and their magnitudes can be made negligible by suitable design.

The accuracy requirements of the attitude control system depend on the specific application for which the satellite is intended. Many control concepts have been analyzed for application in satellite attitude control and accuracy, cost, weight, power requirement, availability, reliability, and simplicity are some of the major considerations.
in the selection of a suitable control scheme.

1.2 Attitude Stabilization and Control Techniques

A multitude of fascinating schemes have been devised for attitude control of satellites. These can be broadly classified into the following categories (Refs. 21, 50).

a) Passive control systems: Depend on the environment to generate the necessary control torques and require no onboard sensing, logic or power. In general useful for maintaining an a priori ideal attitude. Spin stabilization, passive gravity gradient systems, stabilization systems which exploit Earth's magnetic field, solar radiation pressure or aerodynamic pressure to generate torques fall under this category.

b) Semipassive Systems: Differ from the above systems to the extent that these require some onboard momentum storage. Gravity gradient stabilization with passive damping by control moment gyros is a typical example in this category.

c) Semiactive Systems: These require partial attitude sensing, minimal onboard logic and momentum control by mass expulsion and/or environmental torque sources. Spin configurations, dual spin configurations and certain types of gravity gradient systems come under this class.

d) Active Systems: Stabilization in an arbitrary ideal attitude is possible. Attitude sensing and control carried out in all degrees of freedom. Momentum management generally by mass expulsion, but environmental torque sources may be used. For internal momentum storage, reaction wheels or control moment gyros are generally used.

e) Hybrid Systems: Involve control of more than three degrees of freedom, the additional degrees of freedom being related to gimbaled auxiliary bodies, which are controlled relative to the main body by internal torquing. These systems allow for stabilization in an arbitrary ideal attitude.

As one moves up the hierarchy from passive to hybrid systems, the accuracy, maneuverability, acquisition capability, power and/or fuel requirements, weight, complexity and cost of the system increases. Sabroff (Ref.50) in his excellent survey article has discussed in considerable detail the classification, historical progress, capabilities and limitations of various types of satellite control techniques.

Several studies* have been carried out on the possible utilization of environmental torques for satellite attitude control. A near Earth satellite experiences forces and torques due to the presence of Earth's atmosphere. The aerodynamic forces form a major source of perturbation on the orbit of the satellite. A number of studies (e.g.

* These are too numerous for individual citation in the text. The list of references in this report and in Sabroff's article provide an exhaustive listing.
Refs. 29, 62) have been carried out on the effect of aerodynamic forces on the orbits of near Earth satellites. The comparison between the measured and calculated orbital parameter changes has provided a means of determining the density of the atmosphere at various orbital altitudes. On the other hand the torque due to the aerodynamic forces forms one of the major perturbations on the rotational motion of the satellite. The problem of stability of satellites influenced by gravitational torques and aerodynamic torques has been studied in the literature (Refs. 16, 32, 42, 57, 58). The possibility of passive stabilization by aerodynamic torques has been discussed by DeBra (Ref. 9), DeBra and Stearns (Ref. 69), Frye and Stearns (Ref. 17) and Wall (Ref. 65). Schrello (Refs. 57, 58) conducted an extensive study of the passive aerodynamic attitude stabilization of near Earth satellites. Sarychev (Ref. 54) studied the stability of a satellite with an aerodynamic stabilization system. The performance of such a satellite (Cosmos-149) equipped with such a system is discussed by Sarychev (Ref. 53).

1.3 Present Study and Contributions:

In the present work, a specific configuration (see Sec. 3.1) which utilizes aerodynamic forces to effectively control a satellite in circular orbits in an earth pointing mode is analyzed. The configuration studied is gravitationally unstable. In the analysis only the gravity gradient and aerodynamic torques are considered, the other torques being assumed to be small. Even though the other torques are not included in the analysis, the general formalism necessary to enable the inclusion of these torques as perturbations is provided. A linearized analysis has been utilized, but the essentials of a general nonlinear formulation necessary for a nonlinear simulation of the system is presented.

The system equations are set up in state variables and the control system synthesis is carried out by means of time domain techniques. The nominal control defined as the set of control angles which minimize the drag due to the control surfaces and maintain the net gravitational and aerodynamic torques acting on the satellite to be zero when the satellite is in zero pointing error attitude are determined. The nominal control angles are in general functions of the orbital angle η. It is found that the nominal control angles are identically zero for equatorial orbits, and for nonequatorial orbits the horizontal control panels are to be set at zero nominal control angle and the vertical control panels have periodic control angles due to the cross-wind arising from the earth's rotation. Further for nonequatorial orbits, the nominal control angles are independent of the local density but are weakly dependent on the orbital altitude through orbital velocity. Numerical results are presented which indicate that the response of the satellite to initial perturbations is unstable with nominal control only.

The well known concepts from modern optimal control theory (Refs. 4, 5, 25, 26, 51) are used to determine the feedback gain matrices which stabilize the satellite in the nominal attitude. The feedback gains are found to be constant for equatorial orbits but are periodic functions of the orbital angle for nonequatorial orbits. Further the feedback gains are functions of altitude. Numerical results are presented showing the response of the satellite to initial perturbations with optimal feedback control at orbital altitudes of 200 km. and 300 km. Further numerical results are presented to show the effect of the vari-
ation in density due to the atmospheric bulge caused by solar heating, and the variation in the surface accommodation coefficients. The effect of using the equatorial orbit feedback gains when the satellite is in nonequatorial orbits and, the effect of using the gain matrix computed for 300 km. altitude orbit when the satellite is in 200 km. altitude orbit is numerically investigated. A second satellite configuration larger than the first is studied numerically. The possibility of stabilizing the satellite in a single axis pointing mode in which one of the satellite axis will point towards the center of earth but with the satellite being allowed to 'weather-vane' around that axis is investigated. An alternative method of synthesizing the feedback control treating the system as a linear inhomogeneous system is suggested.

The effect of having the panels for aerodynamic stabilization of the satellite on the satellite's lifetime is evaluated (Appendix C). It is found that the presence of the control panels reduce the lifetime of the satellite. But the results indicate that it is possible to have reasonable lifetimes in the altitude range of 200 to 400 km. with sufficiently small damping times.

The problem of implementing the aerodynamic control system is discussed. Further it is pointed out that a simple control law can be devised to maintain a similarity in the performance of the satellite for orbits of different altitudes by varying the control panel areas as a function of altitude. If variable area control panels are not utilized, the implementation involves a suitable gain scheduling scheme, the gains being altered at discrete altitudes with sub-optimal performance in the intermediate altitude ranges. In such a case, the performance will be poor at the higher altitudes and good in low altitude orbits.

II. PRELIMINARIES

In general one is interested in various types of stabilization depending on the function of the satellite such as, 1) Inertial stabilization where, the satellite has to maintain a fixed orientation over very long periods of time unless reoriented on purpose (Star pointing satellite). 2) Quasi-inertial stabilization where, the satellite can be considered to have an inertial orientation over short periods of time but, over long periods the orientation is non-inertial (Sun pointing satellite) and 3) Non-inertial stabilization where, the satellite is "driven" at a certain angular rate about a fixed direction in inertial space (Earth pointing satellite). The study of all of the above types of stabilization involve the general steps of proper choice of coordinate systems, derivation of expressions for the disturbing torques and the determination of the equations of motion. In this chapter a general formulation is established which encompasses all the above types under a single framework.

2.1 Coordinate Systems and Transformations:

A judicious choice of coordinate systems is very important and valuable in determining the disturbing torques and equations of motion with ease. Roberson (Ref. 45) has discussed this problem qualitatively at considerable length. In this section several coordinate systems are defined which will be of value in determining the disturbing torques and the equations of motion.
\( \mathbf{\mathbf{X}}^i \) - represents different coordinate systems, \( i = 1, 2, \ldots, A, B, C \ldots \text{etc.} \)

\( X^i_j \) - represent the coordinate axes in the \( \mathbf{X}^i \) coordinate system, \( j = 1, 2, 3 \)

\( \mathbf{L}^i_j \) - represent unit vectors along the coordinate axes \( X^i \)

\( \mathbf{x}^i \) - represent a vector \( \mathbf{x} \) in \( \mathbf{X}^i \) coordinate system

\( x^i_j \) - represents the components of \( x^i \), \( j = 1, 2, 3 \)

\( \theta^i_j \) - represents a rotation about \( X^i \), \( j = 1, 2, 3 \)

\( \mathbf{L}^i_j \) - represents the transformation matrix from coordinate system \( \mathbf{X}^j \) to \( \mathbf{X}^i \)

\[ \mathbf{x}^i = \mathbf{L}^i_j \mathbf{x}^j \]  \hspace{1cm} (2.1)

Note that \( \mathbf{L}^i_j \mathbf{L}^j_i^T = \mathbf{I} \) (Identity matrix) for orthonormality and \( \det[\mathbf{L}^i_j] = +1 \) for pure rotation.

\( (\theta)^i_1 \) - represents transformation matrix for rotation around axis \( X^i_1 \)

\[ C_\theta \triangleq \cos \theta \]

\[ S_\theta \triangleq \sin \theta \]

\( \mathbf{X}^i \rightarrow \mathbf{X}^j \) - implies that the transformation matrix relating the two coordinate systems is the identity matrix.

The fundamental transformation matrices are given by,

\[
(\theta)^i_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & C_\theta & S_\theta \\
0 & -S_\theta & C_\theta \\
\end{bmatrix} ; \quad
(\theta)^i_2 = \begin{bmatrix}
C_\theta & 0 & -S_\theta \\
0 & 1 & 0 \\
S_\theta & 0 & C_\theta \\
\end{bmatrix} ; \quad
(\theta)^i_3 = \begin{bmatrix}
C_\theta & S_\theta & 0 \\
-S_\theta & C_\theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

For a sequence of rotations the resultant transformation matrix is obtained by compounding the individual transformation matrices by multiplication from right to left.

With the above notation the following right handed, orthogonal coordinate systems can be conveniently defined.

\( \mathbf{X}^1 \) - Inertial coordinate system. Earth centered, with \( X^1_2 \) along Earth's axis of rotation. \( X^1_1 \) and \( X^1_3 \) lie in equatorial plane with \( X^1_2 \) pointing in the direction of vernal equinox.

\( \mathbf{X}^2 \) - Inertial coordinate system. Earth centered, with \( X^2_1 \) pointing towards the star of interest. \( \mathbf{X}^2 \) can be obtained from \( \mathbf{X}^1 \) by two rotations. If \( \alpha \) is the right ascension of the star and \( \delta \), its declination (measured
in the usual astronomical sense, $\alpha$ being positive counter clockwise from vernal equinox, $\delta$ being positive towards north pole) the two rotations are,

$$\theta_1^2 = (\alpha - 90) \text{ about } x_2^1$$

$$\theta_2^3 = \delta \text{ about } x_3^2$$

$\mathcal{E}^3$ - Defines orientation of orbit plane in space. Earth centered. Let the nodal longitude of the satellite orbit be $\Omega$ (same as right ascension of nodal line), and the orbital inclination be $i$. $x_3^3$ points towards perigee, $x_1^3$ in the direction of motion at perigee and $x_2^3$ normal to the orbital plane. $\mathcal{E}^3$ can be obtained from $\mathcal{E}^1$ by three rotations, $\theta_1^2 = \Omega$, $\theta_1^3 = i$, and $\theta_2^3 = \omega$ where, $\mathcal{E}^4$ is an intermediate frame of reference obtained by a rotation $\theta_1^2 = \Omega$ and $\omega$ is the argument of perigee.

$\mathcal{E}^5$ - Orbital coordinate system with origin at the vehicle mass center, $x_2^5$ pointing towards Earth's center, $x_2^5$ pointing along the negative direction of the orbital angular momentum vector and $x_1^5$ is such as to form a right handed orthogonal coordinate system. $\mathcal{E}^5$ can be obtained from $\mathcal{E}^1$ by the three rotations $\theta_1^1 = \Omega$, $\theta_2^4 = -(180-i)$ and $\theta_2^3 = 180-(\omega+\eta)$, where $\eta$ is the true anomaly.

$\mathcal{E}^6$ - Body centered coordinate system with $x_1^6$ pointing towards the star. Since the stars are far away in comparison with the orbital radius $\mathcal{E}^6 \Leftrightarrow \mathcal{E}^2$.

$\mathcal{E}^7$ - Body fixed reference coordinate system (for simplicity assumed to be principal axis system). $\mathcal{E}^7$ can be obtained by rotations of $\theta_1^1$, $\theta_2^2$ and $\theta_3^3$ about the $x_1^1$, $x_2^2$ and $x_3^3$ axis respectively from 1) $\mathcal{E}^5$ - if the satellite is to be earth pointing 2) $\mathcal{E}^6 \Leftrightarrow \mathcal{E}^2$ - if the satellite is to be star pointing and 3) $\mathcal{E}^8 \Leftrightarrow \mathcal{E}^3$ - if the satellite is to be sun pointing.

For small angles it can be shown that the equations of motion uncouple such that the order in which the rotations are performed is immaterial.

$\mathcal{E}^8$ - Body centered coordinate system with $x_1^8$ pointing towards the sun. In general $\mathcal{E}^8 \Leftrightarrow \mathcal{E}$.

$\mathcal{E}^S$ - Earth centered solar reference frame, $x_1^S$ points towards the sun. $x_2^S$ lies in the ecliptic plane. At any time $t$, $\mathcal{E}^S$ can be obtained from $\mathcal{E}^1$ by three rotations: $\theta_3^1 = \Lambda^S$. 6
\[ \theta_1 = -90^\circ \text{ and } \theta_3 = (\mu_S - 90^\circ), \text{ where } \mathcal{F}^9 \text{ is an intermediate coordinate system obtained from } \mathcal{F}^1 \text{ by a rotation } \theta_3 = \lambda_S, \lambda_S = 23^\circ.26'.59'', \mu_S = \mu_{s1} + W_s(t-t_o) \text{ and } W_s = \frac{2\pi}{365.2563825} \text{ rad/mean solar day}. \]

\[ \mathcal{F}^E - \text{ Geocentric Earth reference frame, } x^E_2 \text{ along Earth's axis of rotation, } x^E_1 \text{ and } x^E_3 \text{ lying in the equatorial plane with } x^E_3 \text{ passing through the intersection of the equator and } 0^\circ \text{ longitude reference meridian. } \mathcal{F}^E \text{ can be obtained from } \mathcal{F}^1 \text{ by one rotation, } \theta_2 = \mu_E'. \]

where, \( \mu_E = \mu_E \mid_{t=t_0} + W_E(t-t_o) \) and \( W_E = 7.292115 \times 10^{-5} \text{ rad/mean solar sec. is the rate of Earth's rotation.} \)

\[ \mathcal{F}^M - \text{ Geocentric magnetic reference frame; } x^M_2 \text{ points towards the geomagnetic North Pole (south magnetic pole); } x^M_1 \text{ and } x^M_3 \text{ lie on the magnetic equator with } x^M_3 \text{ lying at the intersection of the equatorial plane and the geomagnetic equatorial plane. } \mathcal{F}^M \text{ can be obtained from } \mathcal{F}^1 \text{ by two rotations, } \theta_2 = \beta \text{ and } \theta_3 = \lambda_M, \text{ where } \beta = \beta_M + \mu_E, \beta_M = 19.5^\circ \text{ and } \lambda_M = 17.5^\circ. \]

The coordinate systems defined above are not the only ones necessary and additional coordinate systems will have to be defined as and when necessary to carry out the analysis of any problem on hand. For clarity all the above described coordinate systems are depicted in Fig. 2.1.

The transformation matrix for any pair of coordinate systems can be obtained by suitably compounding transformation matrices of successive pairs of intermediate coordinate systems. Thus one has \( \mathcal{F}^1_{ij} = \mathcal{F}^1_{ik} \mathcal{F}^k_{lj} \) and knowing \( \mathcal{F}^1_{ik}, \mathcal{F}^k_{lj} \), \( \mathcal{F}^1_{ij} \) is easily determined. Thus all the necessary transformation matrices can be derived from the three fundamental transformation matrices, \( (\theta)_1, (\theta)_2 \) and \( (\theta)_3 \) when necessary.

2.2 Disturbing Torques:

The various disturbing torques experienced by a satellite were listed in Section 1.1. The disturbing torques have been the subject of intensive studies by various authors in recent years. In this section the torques acting on a satellite due to its environment are studied using a unified notation and suitable expressions are obtained. This section further provides a demonstration of the elegance of the notation and coordinate systems defined earlier in Section 2.1 for the purposes of the analysis on hand.

2.2.1 Gravity Gradient Torques:

Gravity gradient torques constitute one of the main factors influencing the rotational motion of space vehicles (Refs. 1, 19, 46, 58, 58, etc). The gravitational force and torque acting on a finite
body situated in a simple inverse-square attracting force field are given by

\[ F_G = - \int_{\Omega} \frac{\mu}{R^3} R \rho \, dV \]  
\[ G_G = - \int_{\Omega} \left( r' \times \frac{\mu}{R^3} R \right) \rho \, dV \]

where, \( r' \) - vector from spacecraft center of mass to \( dV \)
\( \rho \) - density of volume element \( dV \)
\( R = R_c + r' \)
\( R_c \) - vector from center of earth to center of mass of spacecraft.

If the spacecraft consists of a set of \( n \) parts connected with each other, let \( r' = \sum_{i=1}^{n} r_i \), \( i = 1, 2, \ldots, n \).

\( c_i \) - vector from spacecraft c.m to the c.m of the \( i \)th part
\( \Gamma_i \) - vector from the c.m of the part to \( dV \).

Hence,

\[ F_{G_i} = - \int_{\Omega_i} \frac{\mu}{R^3} R \rho \, dV \]

and

\[ G_{G_i} = - \int_{\Omega_i} \left( r' \times \frac{\mu}{R^3} R \right) \rho \, dV \]

Since only the torque expression is of interest here, we have

\[ G_{G_i} = - \int_{\Omega_i} \mu \left\{ (c_i + \Gamma_i) \times (R_c + c_i + \Gamma_i) \right\} R^{-3} \rho \, dV \]

\[ = - \mu \int_{\Omega_i} \left\{ (c_i + \Gamma_i) \times R_c \right\} R^{-3} \rho \, dV \]

\( R^{-3} \) can be approximated as follows:

\[ R^{-3} = \left( \frac{R}{R_c} \right)^{-\frac{3}{2}} = \left\{ \left( \frac{R_c + c_i + \Gamma_i}{R_c} \right) \right\}^{-\frac{3}{2}} \]

\[ = \left\{ \frac{R_c \cdot R_c + 2 \frac{R_c}{R_c} \cdot (c_i + \Gamma_i) + \left( |c_i + \Gamma_i| \right)^2}{R_c \cdot R_c} \right\}^{-\frac{3}{2}} \]

\[ = R_c^{-3} \left\{ \frac{R_c}{R_c} \cdot \frac{R_c}{R_c} + \frac{2}{R_c} \cdot \left( \frac{c_i + \Gamma_i}{R_c} \right) + \left( \frac{|c_i + \Gamma_i|}{R_c} \right)^2 \right\}^{-\frac{3}{2}} \]

\[ = R_c^{-3} \left\{ \left( \frac{R_c + c_i + \Gamma_i}{R_c} \right) \right\}^{-\frac{3}{2}} \]  
\[ \approx \left( \frac{1 + 2 \frac{R_c}{R_c} \cdot \frac{c_i + \Gamma_i}{R_c} }{R_c} \right)^{-\frac{3}{2}} \]

\[ (2.8) \]
In $\mathfrak{X}^5$, $\frac{R_c}{R_c} = - \frac{j_3}{3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The torque expression in $\mathfrak{X}_7$ being of primary interest, in $\mathfrak{X}^7$

$$\frac{R_c}{R_c} \approx L^{75} \frac{j_3}{3}$$

and

$$R^3 \approx R_c \{ 1 - \frac{2}{R_c} L^{75} \frac{j_3}{3} \cdot (\xi_i + r_i) \} \approx R_c \{ 1 + \frac{3}{R_c} L^{75} \frac{j_3}{3} \cdot (\xi_i + r_i) \}$$

Hence $G^7_{G_i} \approx \frac{\mu}{R_c^3} \int_{V_i} \{ - R_c (L^{75} \frac{j_3}{3} \times (\xi_i^7 + r_i^7)) \{ 1 + \frac{3}{R_c} L^{75} \frac{j_3}{3} \cdot (\xi_i^7 + r_i^7) \} \rho d\nu$

$$G^7_{G_i} \approx \frac{\mu}{R_c^3} \left\{ - R_c L^{75} \frac{j_3}{3} \times \left[ \int_{V_i} \xi_i^7 \rho d\nu + \int_{V_i} r_i^7 \rho d\nu \right] 
- 3 \int_{V_i} \left( L^{75} \frac{j_3}{3} \times (\xi_i^7 + r_i^7) \right) \left( L^{75} \frac{j_3}{3} \cdot (\xi_i^7 + r_i^7) \right) \rho d\nu \right\}$$

Noting that $\int_{V_i} r_i^7 \rho d\nu = 0$ and $\int_{V_i} \rho d\nu = m_i$, the mass of part $i$ one obtains

$$G^7_{G_i} \approx - \frac{\mu}{R_c^3} \left\{ \left( + R_c L^{75} \frac{j_3}{3} \times c_i^7 m_i \right) + 3 \int_{V_i} \left( L^{75} \frac{j_3}{3} \times (\xi_i^7 + r_i^7) \right) \left( L^{75} \frac{j_3}{3} \cdot (\xi_i^7 + r_i^7) \right) \rho d\nu \right\}$$

Hence the total gravity gradient torque acting on the satellite is given by

$$G^7_G = \sum_{i=1}^{n} G^7_{G_i} = - \sum_{i=1}^{n} \frac{3 \mu}{R_c^3} \int_{V_i} \left( L^{75} \frac{j_3}{3} \times (\xi_i^7 + r_i^7) \right) \left( L^{75} \frac{j_3}{3} \cdot (\xi_i^7 + r_i^7) \right) \rho d\nu$$

(2.10)

since $\sum_{i=1}^{n} c_i^7 m_i = 0$ by definition of c.m of satellite

For Earth pointing satellites $L^{75} = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1$

For Star pointing satellites $L^{75} = L^{76} L^{64} L^{15}$ where $L^{76} = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1$

For Sun pointing satellites $L^{75} = L^{78} L^{84} L^{15}$ where $L^{78} = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1$

(2.11)

For convenience define $K = L^{75} \frac{j_3}{3}$

Then

$$G^7_G = - \sum_{i=1}^{n} \frac{3 \mu}{R_c^3} \int_{V_i} \left( K \times (\xi_i^7 + r_i^7) \right) \left( K \cdot (\xi_i^7 + r_i^7) \right) \rho d\nu$$

For a satellite made up of a single body
and let \( K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \)

consequently

\[
\mathbf{G} = \frac{3M}{r^2} \int \left( \mathbf{r}_i \times K \right) \left( \mathbf{K} \cdot \mathbf{r}_i \right) \rho \, d\mathbf{r}
\]

\[
\approx \frac{3M}{r_i^3} \int \left[ x_i^7 - x_2^7, x_3^7, x_4^7 \right] \left\{ K_1 x_i^7 + K_2 x_2^7 + K_3 x_3^7 \right\} \rho \, d\mathbf{r} \approx \frac{3M}{r_i^3} \begin{bmatrix} K_2 K_3 (c - b) \\ K_3 K_1 (a - c) \\ K_1 K_2 (b - a) \end{bmatrix}
\]

where

\[
\begin{align*}
\int \rho \, d\mathbf{r} &= 0, \quad i \neq j, \\
\int \left[ (x_i^7)^2 + (x_i^2)^2 \right] \rho \, d\mathbf{r} &= A \\
\int \left[ (x_2^7)^2 + (x_3^7)^2 \right] \rho \, d\mathbf{r} &= B \\
\int \left[ (x_1^7)^2 + (x_i^7)^2 \right] \rho \, d\mathbf{r} &= C
\end{align*}
\]

The above analysis holds for an inverse-square attracting force field only, but it is a simple matter to extend the analysis to include the effects of anomalies in the force field.

2.2.2 Aerodynamic Torques:

For satellites orbiting below 700 km altitude the aerodynamic effects form the major source of perturbation. The orbital dynamics and the attitude dynamics of these satellites are profoundly affected by the aerodynamic effects, (Refs. 1,7,9,16,17,27,29,30,42,53,54,57,58, 62 and 65). Further at these altitudes free molecular flow conditions exist (see Appendix B).

Under free molecular flow conditions the normal, tangential and drag force coefficients are respectively given by

\[
\begin{align*}
C_p &= \left\{ \frac{2 - \sigma'}{2 \sqrt{\pi}} S_{\alpha} + \frac{\sigma'}{2 \sqrt{2}} \cdot \frac{T}{T} \right\} e^{-\left( SC_{\alpha} \right)^2} + \left[ \left( \frac{2 - \sigma'}{2 \sqrt{2}} \right) \left( SC_{\alpha} + \frac{1}{2 \sqrt{2}} \right) + \frac{\sigma'}{2 \sqrt{2} \sqrt{\pi}} \right] e^{-\left( SC_{\alpha} \right)^2} \left[ 1 + \text{erf} \left( SC_{\alpha} \right) \right] \\
C_t &= \frac{\sigma'}{\sqrt{\pi}} S_{\alpha} \left\{ e^{-\left( SC_{\alpha} \right)^2} + \sqrt{\pi} \left( SC_{\alpha} \right) \left[ 1 + \text{erf} \left( SC_{\alpha} \right) \right] \right\} \\
C_d &= C_p C_{\alpha} + C_t S_{\alpha}
\end{align*}
\]

where \( \sigma' \) - Normal momentum accommodation coefficient

\( \sigma \) - Tangential momentum accommodation coefficient

\( \alpha \) - Angle between the reversed incoming flow direction and the normal to the surface.
S - molecular speed ratio, defined as the ratio of the speed of the satellite to the mean thermal speed of the molecules. Under infinite speed ratio assumption, (see Appendix B)

\[ C_p = 2(2 - \sigma^2) \frac{C_a}{|C_a|} \]
\[ C_c = 2\sigma \frac{S_a}{|C_a|} \]  \hspace{1cm} (2.14)
\[ C_d = 2 \left\{ (2 - \sigma^2) \frac{C_a^2}{|C_a|} + \sigma S_a^2 \right\} \frac{1}{|C_a|} = 2(2 - \sigma^2) \frac{C_a^2}{|C_a|} + 2\sigma \frac{C_a}{|C_a|} \]

The above approximation is useful in computing the forces and moments acting on bluff bodies in free molecular flow.

The aerodynamic forces and moments acting on the satellite will have to be determined in \( \mathcal{X} \).

\[ \mathbf{V}_R^5 = -\mathbf{V}_5^5 - k \mathbf{\omega}_E \mathbf{R}_C^5 = -(\mathbf{V}_5^5 - k \mathbf{\omega}_E \mathbf{R}_C^5) = -(\mathbf{V}_5^5 - k \mathbf{\omega}_E \mathbf{R}_C^5) \]  \hspace{1cm} (2.15)

where \( \mathbf{V}_R \) - relative velocity of "air" with respect to satellite

\( k \) - a coefficient which depends on altitude and accounts for any relative rotation of the atmosphere with respect to earth; \( 0 < k < 1 \).

\( \mathbf{\omega}_E \) - angular speed of earth

\( \mathbf{V} \) - orbital speed of satellite

\[ \mathbf{V}_R^5 = \mathbf{V}_5^5 + k \mathbf{\omega}_E \mathbf{R}_C^5 = \mathbf{V} \begin{bmatrix} -C_{\gamma^*} & S_{\gamma^*} \\ 0 & 0 \end{bmatrix} + k \mathbf{\omega}_E \begin{bmatrix} 0 \\ -C_i \\ -S_i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -V \mathbf{C}_i^* \mathbf{R}_C^i \mathbf{C}_i \end{bmatrix} + k \omega_E \mathbf{R}_C^i \mathbf{S}_i \mathbf{C}_i \mathbf{C}_{\omega^*} \]  \hspace{1cm} (2.16)

where \( \gamma^* \) is the inclination of the orbital flight path to horizontal and \( \tan \gamma^* = \frac{\mathbf{\hat{r}}}{\mathbf{\hat{r}}_\eta} = \frac{\epsilon \mathbf{\hat{r}}}{(1 + \epsilon)} \). For near circular orbits \( \gamma^* \) is small since the eccentricity \( \epsilon \to 0 \). Further for low altitude orbits \( k \to 1 \)

Hence

\[ \mathbf{V}_R^5 = \begin{bmatrix} -V + \omega \mathbf{R}_C^i \mathbf{C}_i \\ \omega \mathbf{R}_C^i \mathbf{S}_i \mathbf{C}_{\omega^*} \\ 0 \end{bmatrix} \]  \hspace{1cm} (2.17)

The magnitude of \( \mathbf{V}_R \) can be approximated as follows:

\[ \left( \frac{\mathbf{V}_R}{\mathbf{V}} \right)^2 = 1 - 2 \left( \frac{\omega \mathbf{R}_C}{\mathbf{V}} \right) \mathbf{C}_i + \left( \frac{\omega \mathbf{R}_C}{\mathbf{V}} \right)^2 \left\{ 1 - S_i^2 \right\} \]  \hspace{1cm} (2.18)
For the altitudes of interest \( R \omega_E \approx 0(7 \times 10^2) \text{ m/sec}, V = 0(7 \times 10^3) \text{ m/sec.} \) and hence \( \omega_E R_c / V \) is small compared to unity.

Inverting (2.18), the following expression is obtained

\[
\frac{V}{V_R} = \frac{1}{1 - 2 \left( \frac{\omega_E R_c}{V} \right) C_i + \left( \frac{\omega_E R_c}{V} \right)^2 \left[ 1 - S_{\omega + \gamma}^2 S_{\eta}^2 \right]} \approx 1 + \frac{\omega_E R_c}{V} C_i + 0 \left( \frac{\omega_E R_c}{V} \right)^2 (2.19)
\]

Hence

\[
\frac{V_R}{V} \approx \frac{1}{1 + \frac{\omega_E R_c}{V} C_i} \approx 1 - \frac{\omega_E R_c}{V} C_i \quad (2.20)
\]

From (2.17)

\[
V^5_R \approx V \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\} \left[ \begin{array}{c} -1 \\ \omega_E R_c S_i C_{\omega + \gamma} \\ 0 \end{array} \right] \approx V \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\} \left[ \begin{array}{c} -1 \\ \omega_E R_c S_i C_{\omega + \gamma} \\ 0 \end{array} \right] (2.21)
\]

and

\[
V^5_R \approx V_R V^5_R \quad \text{where} \quad V_R = V \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\}
\]

\[
V^7_R = V \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\} \mathcal{L}^7 \quad \mathcal{L}^7 \approx V_R V^7_R \quad (2.22)
\]

where \( \mathcal{L}^7 \) is given by (2.11)

If \( n^7 \) represents the unit inward normal to the elemental surface area and \( t^7 \) the vector lying at the intersection of the tangent plane at the centre of the elemental area and the plane containing \( n^7 \) and \( v^7_R \), we have

\[
\cos \alpha = n^7 \cdot \frac{v^7_R}{|v^7_R|}
\]

and

\[
t^7 = - \cot \alpha \frac{n^7}{|n^7|} + \cosec \alpha \frac{v^7_R}{|v^7_R|} (2.23)
\]

\( t^7 \) may also be expressed as

\[
t^7 = \frac{(n^7 \times v^7_R) \times n^7}{|n^7 \times v^7_R|} x n^7 \]

The normal pressure and shear acting on the elemental area are given by

\[
\mathcal{P}^7 = \frac{1}{2} \rho V^2 C_p \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\}^2 n^7
\]

\[
\mathcal{Z}^7 = \frac{1}{2} \rho V^2 C_t \left\{ 1 - \frac{\omega_E R_c}{V} C_i \right\}^2 t^7 \quad (2.24)
\]

The net aerodynamic force on the satellite is given by

\[
\mathcal{F}_A^7 = \int_A \left( \mathcal{P}^7 + \mathcal{Z}^7 \right) dA \quad (2.25)
\]

If \( \mathcal{F}_A^7 \) is the position vector of the center of the elemental
surface area, then net aerodynamic torque on the satellite is given by

\[ G_A^T = \int_{A_w} r^7 \times (\hat{r}^T + \hat{r}^T) \, dA \]  

(2.26)

where \( A_w \) is the "wetted" surface area.

2.2.3 Solar Radiation Torques:

For satellites orbiting at altitudes in excess of 1000 km the solar radiation torques form a major source of perturbation on the rotational motion of the satellite (Refs. 1, 6, 19, 23, 37, 40, 44, 59, 60 and 67). In the case of satellites travelling in interplanetary space the solar radiation torques form the major source of perturbation apart from torques due to meteoroid impacts.

The incident energy flux from sun in the vicinity of earth

\[ \Phi = E_s / 4 \pi r^2, \]

where \( E_s = 3.86 \times 10^{26} \text{ joules/sec} \) is the solar constant and \( r = 1.5 \times 10^{11} \text{ meters} \) is the mean solar distance from the earth.

\[ \Phi = mc^2 \]

by virtue of Einstein's mass-energy equivalence relation.

\( c \) = velocity of light in vacuum.

Hence the incident momentum flux \( V = mc = \Phi / c = E_s / 4 \pi r^2 c = 4.64 \times 10^{-7} \text{ kg/m}^2 \). Let \( \alpha \), \( \tau' \) and \( \alpha' \) be the coefficients of reflectivity, transmissivity and absorptivity respectively, characterizing the satellite surface element. For continuity \( \alpha + \tau' + \alpha' = 1 \) and in general \( \tau' = 0 \). Hence \( \alpha' = (1-\alpha) \).

Consider an elemental surface area, the position vector of whose centroid is given by \( \hat{r}^T \). Let \( \hat{n}^T \) be the unit vector along its outward normal. Further let \( \hat{v}^s \) be a unit vector directed from the vehicle center of mass to the sun.

\[ \text{In } \mathbb{X}^S \leftrightarrow \mathbb{X}^S, \hat{v}^S = \hat{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and in } \mathbb{X}^T, \hat{v}^T = \mathbb{I}_{78}^T \hat{v}^S = \mathbb{I}_{78}^T \hat{v} \]

For earth pointing case \( \mathbb{I}_{78}^T = \mathbb{I}_{75}^T \mathbb{I}_{51}^T \).

For star pointing case \( \mathbb{I}_{78}^T = \mathbb{I}_{76}^T \mathbb{I}_{61}^T \).

For sun pointing case \( \mathbb{I}_{78}^T = \mathbb{I}_{78}^T \).

\[ \text{The net force on the satellite due to solar radiation pressure can be easily shown to be given by,} \]

\[ f_{SR}^T = -V \left\{ \int_{A_w} (1+\alpha) |\hat{v}^T \hat{n}^T| (\hat{v}^T \hat{n}^T) \hat{n}^T \, dA + \int_{A_w} (1-\alpha) |\hat{v}^T \hat{n}^T| (\hat{n}^T \times \hat{v}^T) \times \hat{n}^T \, dA \right\} \]  

(2.28)

The net torque on the satellite due to solar radiation pressure is given by

\[ G_{SR}^T = -V \left\{ \int_{A_w} (1+\alpha) |\hat{v}^T \hat{n}^T| \hat{r}^7 \times (\hat{v}^T \hat{n}^T) \hat{n}^T \, dA + \int_{A_w} (1-\alpha) |\hat{v}^T \hat{n}^T| \hat{r}^7 \times [(\hat{n}^T \times \hat{v}^T) \times \hat{n}^T] \, dA \right\} \]  

(2.29)

The value of \( V \) is \( 4.64 \times 10^{-7} \text{ kg/m}^2 \) when the satellite is in direct
sunlight and zero when the satellite is in the umbral region of the earth's shadow. \( V \) varies from \( 4.64 \times 10^{-7} \) kg/m\(^2\) to zero in the penumbral region of the earth's shadow and the nature of this variation is not treated in this study. Further, the earth reflected solar radiation and the earth emitted radiation has not been included in the analysis, their effect being small in comparison with direct solar radiation.

### 2.2.4 Torques Due to Earth's Magnetic Field:

A satellite in orbit experiences disturbing torques due to the interaction of the satellite's magnetic dipole with the earth's magnetic field. The satellite dipole moment is due to either permanent magnets or closed current loops present in the satellite. The magnetic torques acting on earth satellites have been extensively studied in recent years. (e.g. Refs. 1, 37, 59, etc).

The earth's magnetic field may be approximated as the field due to a simple magnetic dipole at the center of the earth with the dipole axis inclined at 17.5° to the spin axis of the earth. The potential function for the magnetic dipole in spherical polar coordinates is given by

\[
\Phi = -\frac{m \sin \phi}{r^2} \tag{2.30}
\]

where

- \( m \) - dipole strength
- \( r \) - magnitude of the radius vector from dipole to the vehicle c.m.
- \( \phi \) - latitudinal position of vehicle c.m. relative to the magnetic equator.

The magnetic field \( \mathbf{B} \) is given by

\[
\mathbf{B}^{\phi} = -\nabla \Phi = -\frac{m}{r^3} \begin{bmatrix} 0 \\ -C_{\phi} \\ 2S_{\phi} \end{bmatrix} \quad \mathbf{x}^{\phi} = \begin{bmatrix} x_{\lambda} \\ x_{\phi} \\ x_r \end{bmatrix} \tag{2.31}
\]

where \( \mathbf{x}^{\phi} \) is a geomagnetic reference spherical polar coordinate system.

\[
\mathbf{x}^{\phi} = \mathbf{L}^{\phi M} \mathbf{x}^M, \quad \text{where} \quad \mathbf{L}^{\phi M} = \begin{bmatrix} C_{\lambda} & 0 & -S_{\lambda} \\ -S_{\phi} S_{\lambda} & C_{\phi} & -S_{\phi} C_{\lambda} \\ C_{\phi} S_{\lambda} & S_{\phi} & C_{\phi} C_{\lambda} \end{bmatrix} \tag{2.32}
\]
Hence,

\[ \frac{B}{M} = -\frac{m}{r^3} \left[ \begin{array}{c} 3S_{\phi} S_{\lambda} \\ 3S_{\phi} - 1 \\ 3S_{\phi} C_{\lambda} C_{\lambda} \end{array} \right] \] (2.33)

\[ X^5 = \mathcal{L}^{51} \mathcal{L}^{1M} X^M = \mathcal{L}^{5M} X^M = \left[ \begin{array}{c} l_{ij} \end{array} \right] X^M \] (2.34)

since \[ \frac{X^5}{3} = \frac{1}{r} \] comparing \[ -\mathcal{L}^M \text{ and } \mathcal{L}^{5M} \], the following are obtained.

\[ \begin{align*}
  l_{31} &= -c_{\phi} S_{\lambda} \\
  l_{32} &= -S_{\phi} \\
  l_{33} &= -c_{\phi} C_{\lambda}
\end{align*} \] (2.35)

Hence

\[ \frac{B}{M} = -\frac{m}{r^3} \left[ \begin{array}{c} 3l_{31} l_{32} \\ 3l_{32} - 1 \\ 3l_{32} l_{33} \end{array} \right] \] (2.36)

\[ \frac{B}{5} = \mathcal{L}^{5M} \frac{B}{M} = -\frac{m}{r^3} \left[ \begin{array}{c} 3l_{32} l_{31} l_{11} + 3l_{12} l_{32} - l_{12} + 3l_{13} l_{32} l_{33} \\ 3l_{32} l_{31} l_{21} + 3l_{22} l_{32} - l_{22} + 3l_{23} l_{32} l_{33} \\ 3l_{32} l_{31} l_{32} + 3l_{32}^2 - l_{32} + 3l_{32} l_{33} \end{array} \right] = \frac{m}{r^3} \left[ \begin{array}{c} l_{12} \\ l_{22} \\ -2l_{32} \end{array} \right] \] (2.37)

since \( l_{ij} l_{ik} = l_{ji} l_{ki} = \delta_{jk} \)

\[ \mathcal{L}^{51} = \left[ \begin{array}{ccc}
  c_{i} c_{\alpha} c_{\omega \eta} - s_{\omega \eta} & s_{i} c_{\omega \eta} & -c_{i} s_{\omega \eta} - c_{\alpha} s_{\omega \eta} \\
  s_{i} c_{\alpha} & -c_{i} & -s_{i} s_{\alpha} \\
  -c_{i} c_{\alpha} s_{\omega \eta} - s_{\omega \eta} & -s_{i} s_{\omega \eta} & c_{i} s_{\alpha} s_{\omega \eta} - c_{\alpha} c_{\omega \eta}
\end{array} \right] \quad \mathcal{L}^{1M} = \left[ \begin{array}{ccc}
  c_{\rho} c_{\lambda} - c_{\beta} s_{\lambda} & s_{\lambda} & c_{\rho} \\
  s_{\lambda} & c_{\lambda} & 0 \\
  -s_{\rho} c_{\lambda} & s_{\rho} s_{\lambda} & c_{\rho}
\end{array} \right] \]

consequently \( \frac{B}{5} = \frac{m}{r^3} \left[ \begin{array}{ccc}
  c_{\omega \eta} (s_{i} c_{\lambda} - s_{\lambda} c_{\rho} - c_{\rho} - c_{\beta} - S_{\rho}) & S_{\rho} - c_{\lambda} c_{\beta} - c_{\lambda} c_{\beta} - S_{\lambda} \\
  -(s_{i} s_{\lambda} c_{\rho} + c_{\lambda} c_{i}) & 2 \{c_{\omega \eta} (s_{i} c_{\lambda} - c_{i} s_{\lambda} c_{\beta} - c_{\lambda} c_{\beta} - S_{\rho}) + c_{\omega \eta} S_{\lambda} S_{\rho} \}
\end{array} \right] \] (2.38)

\[ \frac{B}{7} = \mathcal{L}^{75} \frac{B}{5} \text{, where } \mathcal{L}^{75} \text{ is given by (2.31)} \] (2.39)

In general the torque on a satellite due to the interaction of the earth's magnetic field with magnetic moments fixed or generated within the satellite is given by (expressed in \( \mathcal{L}^{7} \) coordinate system)

\[ G_{M}^{7} = M^{7} \times B^{7} \] (2.40)

where \( M^{7} \) is the magnetic moment of the satellite composed of:
\( M^T_0 \) = due to permanent magnets in instruments and current carrying devices

\( M^T_I \) = due to magnetization of the satellite hull in the geomagnetic field.

In general \( M^T_I = (v/4\pi)g \) where \( v \) - volume of the satellite hull.

\( g \) - is a matrix representing a symmetric tensor.

Hence \( C^T_M = (M^T_0 + M^T_I) \times B^7 \) \hspace{1cm} (2.41)

Torques Due to Eddy Currents:

When a satellite has an angular velocity relative to the magnetic field, eddy currents are induced and the torques produced by the eddy currents (Ref.1) tend to reduce the component of the angular velocity perpendicular to the external magnetic field.

The torque due to the eddy currents is given by

\[ C^T_{EC} = -k_s \left\{ \frac{\vec{v}^7 \times \vec{\omega}^7}{\vec{p}^7} \right\} \times \vec{B}^7 \] \hspace{1cm} (2.42)

where

\( \vec{\omega}^7 \) - angular velocity of the satellite

\( k_s \) - a constant called the dissipation coefficient, which can be approximated for the particular satellite in question by

\( k_s = \alpha J t/R \)

where \( \alpha \) - a non-dimensional coefficient if B and R are expressed in electromagnetic units.

\( J \) - longitudinal or transverse moment of inertia of the satellite hull.

\( t \) - thickness of the satellite hull.

\( R \) - specific volume resistance of hull material.

2.3 System Description and Control Synthesis:

Having established the formalism necessary to determine the torque acting on the satellite, in this section the equations of motion will be obtained and the problem of synthesizing a suitable control will be posed. There are several alternate formulations available for the derivation of the differential equations of rotational motion. Hughes (Ref.22) discusses the relative merits of some of these formulations. In the present study the Euler's equations will be employed to obtain the differential equations for the dynamics of the system.

The rotational dynamics of the satellite can be represented by the following set of Euler's equations in \( X^7 \) coordinate system.
\[
\frac{d}{dt} \left\{ J^7 \omega^7 \right\} + \omega^7 \times (J^7 \omega^7) = G_d^7 + G_c^7 \tag{2.43}
\]

where

\( J^7 \) - Inertia matrix of the satellite

\( \omega^7 \) - angular velocity vector with respect to inertial coordinates.

\( G_d^7 \) - disturbance torque vector

\( G_c^7 \) - control torque vector

The expressions for \( \omega^7 \) for various pointing schemes can be obtained as follows:

For the earth pointing satellite

\[
x^7 = J^7 \dot{x}^7 = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (180 - (\omega \cdot \eta))_2 (-180 + t)_3 (\Delta)_2 \dot{x}^4
\]

consequently

\[
\omega^7 = \dot{j}_3 \dot{\theta}_3 + (\theta_3)_3 j_2 \dot{\theta}_2 + (\theta_3)_3 (\theta_2)_2 j_1 \dot{\theta}_1 - (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 j_2 (\dot{\omega} + \dot{\eta})
\]

\[
- + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (180 - (\omega \cdot \eta))_2 j_3 \dot{j}_3 + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (180 - (\omega \cdot \eta))_2 (-180 + t)_3 \dot{j}_2 \dot{\Delta} \tag{2.44}
\]

For star pointing satellites

\[
x^7 = J^7 \dot{x}^7 = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (\delta)_3 (\alpha - 90)_2 x^4
\]

consequently

\[
\omega^7 = \dot{j}_3 \dot{\theta}_3 + (\theta_3)_3 j_2 \dot{\theta}_2 + (\theta_3)_3 (\theta_2)_2 j_1 \dot{\theta}_1 + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 \dot{j}_3 \dot{\delta} + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (\delta)_3 \dot{j}_2 \dot{\Delta} \tag{2.45}
\]

For sun pointing satellites

\[
x^7 = J^7 \dot{x}^7 = (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (\mu_s - 90)_3 (-90)_1 (\Lambda_s)_3 x^4
\]

consequently

\[
\omega^7 = \dot{j}_3 \dot{\theta}_3 + (\theta_3)_3 j_2 \dot{\theta}_2 + (\theta_3)_3 (\theta_2)_2 j_1 \dot{\theta}_1 + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 \dot{j}_3 \dot{\mu}_s + (\theta_3)_3 (\theta_2)_2 (\theta_1)_1 (\mu_s - 90)_3 (-90)_1 \dot{j}_3 \dot{\Lambda}_s \tag{2.46}
\]

In general the angular rates other than \( \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3 \) and \( \dot{\eta} \) are small and hence can be neglected. Depending on the orientation scheme under study the respective equation in (2.44)-(2.46) can be inverted to obtain \( \delta^7 \) in terms of \( \theta^7 \) and \( \omega^7 \) which being the kinematic differential equation of the system. Alternatively the expressions for \( \omega^7 \) given by (2.44)-(2.46) can be substituted in (2.43) to obtain a set of three second order nonlinear ordinary differential equations.

Thus the following set of first order differential equations describing the kinematics and dynamics of the system are obtained.

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} = f_k(\theta_i, \omega_i), \quad i = 1, 2, 3 \tag{2.47}
\]
\[
\begin{bmatrix}
\omega_1^7 \\
\omega_2^7 \\
\omega_3^7 \\
\end{bmatrix}
= f_p(\omega_1^7, \theta_i, u_j, t) = \begin{bmatrix}
\omega_2^7 w_1^7 \left( \frac{B - C}{A} \right) \\
\omega_3^7 w_1^7 \left( \frac{C - A}{B} \right) \\
w_1^7 w_2^7 \left( \frac{A - B}{C} \right)
\end{bmatrix} [g^T]^{-1} g(\theta_i, \omega_i^7, u_j, t)
\]

(2.48)

where,
\[
g^T = g_p + g_c
\]

\[
u = \begin{bmatrix} u_j \end{bmatrix}
\]
is the control vector.

Equations (2.47) and (2.48) form a set of six first order nonlinear ordinary differential equations. These can be expressed as
\[
\dot{x} = f(x, u, t)
\]

(2.49)

where
\[
x^T \Delta (\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)
\]
\[
f^T \Delta (f^T_k, f^T_D)
\]

The above system which is to have \(\bar{x} = 0\) as its equilibrium state is to be controlled to remain in the neighbourhood of \(\bar{x} = 0\) in the presence of small perturbations. The control necessary to ensure that \(\dot{x} = 0\) when the satellite is at \(x = 0\) state in the absence of all perturbations is given by \(f(x^0, u^0, t) = 0\). \(u^0\) is called the nominal control and \(x^0 = 0\) is called the nominal state. Various methods of determining \(u^0\) are discussed in Appendix D. In the present study, the aerodynamic forces on the control panels used to generate control torques also contribute to the drag on the satellite which causes orbital decay. Of the several possible \(u^0\), the present study considers the particular \(u^0\) which will also minimize the drag on the satellite in order to ensure that the satellite lifetime does not get reduced drastically because of the aerodynamic control.

Since the perturbing forces and torques on the satellite are small, the excursions in the state of the system from the nominal state due to the external disturbances can be assumed to be small, consequently as an approximation, the system equations given by equation (2.49) can be linearized to obtain,
\[
\dot{X} = A(t) X + B(t) U
\]

(2.50)

where
\[
U = u - u^0
\]
\[
X = x - x^0 = \bar{x}
\]
\[
A(t) = \left. \frac{\partial f}{\partial x} \right|_{x = x^0 = 0, u = u^0}
\]
The problem of control synthesis consists of determining $U$ as a function of $X$ and $t$. Since the system has been linearized a linear feedback control of the form $U = -K(t) X$ can be obtained (see Appendix E). Such a linear feedback control is advantageous from the viewpoint of both synthesis and implementation of the control.

III. EARTH POINTING SATELLITES

Earth pointing satellites are a class of satellites which are intended to have one of their axis pointing towards the center of the earth continuously. Earth pointing satellites find promising applications in present day space programs in such varied areas as communication, navigation, geodesy, meteorology, earth resources surveys and reconnaissance. Several control schemes have been applied for the attitude control of such satellites. In this chapter a specific configuration is studied for possible application of the aerodynamic forces for attitude control.

3.1 Configuration, Coordinate System and Transformation Matrices:

The configuration under study is shown in Fig. 3.1. It essentially consists of a center-body with a long axis of symmetry with cruciform control surfaces mounted as shown. These are 'all-movable' control surfaces which are capable of being rotated about their centroidal axes lying normal to the center-body's axis of symmetry. The control surfaces are to be of light weight construction and of suitable area so as to provide sufficient aerodynamic torque but producing negligible inertial torques due to their angular accelerations.

Apart from the coordinate systems defined in Sec. 2.1 the following coordinate systems will be of use in subsequent analysis.

- $H^i$ - coordinate system fixed in the $i^{th}$ hinge,
- $H^1$ is obtained from $H^7$ by one rotation: $\theta^7_3 = -90^\circ$
- $H^2$ is obtained from $H^7$ by one rotation: $\theta^7_3 = +90^\circ$
- $H^3$ is obtained from $H^7$ by two rotations: $\theta^7_2 = 90^\circ, \theta^7_1 = 90^\circ$
  Alternative choice: $\theta^7_2 = 90^\circ, \theta^7_3 = -90^\circ$

$H^4$ is obtained from $H^7$ by two rotations: $\theta^7_2 = -90^\circ, \theta^7_1 = 90^\circ$
  Alternative choice: $\theta^7_2 = -90^\circ, \theta^7_3 = -90^\circ$

Note that $X^H_2$ and $X^H_3$ point in $X^7_1$ direction whereas $X^H_2$ and $X^H_3$ along the negative $X^7_1$ axis.
$\mathbf{x}_{i}^{si}$ - Principal axis coordinate system fixed in $i^{th}$ control surface, $\mathbf{x}_{i}^{si}$ is obtained from $\mathbf{x}_{i}^{hi}$ by a rotation $\theta_{i}^{hi} = \theta_{i}^{c}$

$b_{i}^{7}$ - Vector from spacecraft c.m to the center of pressure of the $i^{th}$ control surface.

$$
\begin{align*}
\mathbf{b}_{i}^{7} &= \begin{bmatrix} -l_{1}^{7} \\ l_{2}^{7} \\ 0 \end{bmatrix} \\
\mathbf{b}_{i}^{2} &= \begin{bmatrix} -l_{1}^{2} \\ l_{2}^{2} \\ 0 \end{bmatrix} \\
\mathbf{b}_{i}^{3} &= \begin{bmatrix} -l_{1}^{3} \\ 0 \\ l_{2}^{3} \end{bmatrix} \\
\mathbf{b}_{i}^{4} &= \begin{bmatrix} 0 \\ 0 \\ l_{2}^{4} \end{bmatrix}
\end{align*}
$$

(3.1)

Some of the useful transformation matrices are given below:

$$
\mathbf{q}^{51} = \begin{bmatrix}
-c_{i} & 0 & -s_{i} \\
0 & 1 & 0 \\
s_{i} & 0 & c_{i}
\end{bmatrix}
$$

(3.2)

$$
\mathbf{q}^{75} = \begin{bmatrix}
-c_{i} & s_{i} & 0 \\
-s_{i} & c_{i} & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

(3.3)

$$
\mathbf{q}^{si,hi} = \begin{bmatrix} 1 & 0 & 0 \\
0 & c_{i} & s_{i} \\
0 & -s_{i} & c_{i}\end{bmatrix}
$$

$$
\mathbf{q}^{hi,si} = \begin{bmatrix} 1 & 0 & 0 \\
0 & c_{i} & -s_{i} \\
0 & s_{i} & c_{i}\end{bmatrix}
$$

(3.4)

$$
\mathbf{q}^{h1,7} = \begin{bmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1\end{bmatrix}
$$

$$
\mathbf{q}^{7,h1} = \begin{bmatrix} 0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1\end{bmatrix}
$$

(3.5)

$$
\mathbf{q}^{h2,7} = \begin{bmatrix} 0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1\end{bmatrix}
$$

$$
\mathbf{q}^{7,h2} = \begin{bmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1\end{bmatrix}
$$

(3.6)

$$
\mathbf{q}^{h3,7} = \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0\end{bmatrix}
$$

$$
\mathbf{q}^{7,h3} = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & -1 \\
-1 & 0 & 0\end{bmatrix}
$$

(3.7)
3.2 Assumptions:

Generally any assumptions made are constraints on the generality of the analysis and these can be easily relaxed at the cost of added complexity of the analysis. The following assumptions have been made since the present study is in a sense a feasibility study and as such it is advantageous to maintain a certain degree of simplicity.

1) The vehicle is assumed to be in a circular orbit.

2) The rate of change of the orbital parameters $\Omega, i, \omega$ and $R_c$ due to the earth's oblateness and atmospheric effects are small in comparison with the orbital rate of the satellite. The order of magnitude of the ratio of the rate of change in the orbital parameters to the orbital rate of the satellite can be estimated approximately (Refs. 1, 29, 58) as,

$$0\left(\frac{\dot{\Omega}}{\Omega}\right) \approx 0\left(\frac{\dot{i}}{i}\right) \approx 0\left(\frac{\dot{\omega}}{\omega}\right) \approx 10^{-3}$$

$$\frac{\dot{R}_c}{\bar{\eta}} = -\frac{C_D A_e}{m} R_c^2 \rho ,$$

where $C_D$ - drag coefficient of satellite

$A_e$ - effective cross-sectional area of satellite

$m$ - mass of the satellite

$\rho$ - atmospheric density

For a typical satellite $\dot{R}_c/\bar{\eta}$ varies approximately from a value of $10^{-5}$ m./rad. at 700 Km altitude to $10^{-1}$ m./rad. at 150 Km altitude. Hence the orbital parameters may be assumed to remain constant over several orbits.

3) The isodensity contours are geocentric circles and the density at any orbital radius is taken from the 1962 U.S. standard atmosphere. But the effect of the diurnal density variations on a control synthesized with this assumption is investigated numerically by introducing a periodic variation in density at any given altitude (see Sec. 4.2).

4) In the altitude range under study free molecular flow conditions are assumed to exist (see Appendix B).

5) The control panels are assumed to be rigid and consequently the effects due to their flexibility are neglected.

6) The control panels are of light weight construction and hence the moment of inertia of the control panels are assumed to be small to enable one to neglect the variation in the moments of inertia of

\[ \mathbf{d}^{HA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{d}^{HA} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \]
the satellite and the inertia effects arising from the rotation of the control panels.

For order of magnitude purposes the moment of inertia of a spacecraft may be approximated as being $O(mL^2)$ where $m$ is the mass of the spacecraft and $L$ is characteristic dimension. Similarly the largest contribution to the satellite moment of inertia from a control panel of mass $m_c$ may be approximated as $O(m_c l^2)$, where $l$ is the distance of the control panel center of mass from that of the satellite. Hence ratio of the change in the satellite's moment of inertia due to control panel rotation to the moment of inertia of the satellite is approximately $O(m_c/m)$. The present generation solar panels on spacecraft have a mass density of about 5 Kg/m$^2$. Hence solar panels of reasonable size will lead to $O(m_c/m) \approx 0.025$. On the other hand it is possible to construct panels using either metallic foils or Mylar sheets stretched over lightweight frames to obtain mass densities of 0.25 Kg/m$^2$ leading to $O(m_c/m) \approx 0.002$.

7) Only aerodynamic torques and gravity gradient torques are considered in the analysis and the other torques are assumed to be negligible. The expressions for the other disturbing torques obtained in Sec. 2.2 may be utilized to study the effect of these perturbations on the performance of the attitude control system.

8) Since the external disturbances acting on the satellite are small, the excursions in the state of the satellite from the nominal state is small and consequently a linearized analysis is carried out. The essential details necessary for a nonlinear formulation is presented and it may be used to carry out a nonlinear simulation to study the effect of nonlinearities on the system performance.

9) The atmosphere is assumed to be rotating at the same angular velocity as the earth.

10) The shadowing of the control panels from the oncoming flow when the center-body is in an off-nominal attitude is not considered. The effect of such shadowing on the control can be eliminated by having the "all-movable" control panels a certain distance away from the center-body.

3.3 Determination of Drag and Dominant Torque:

The control surfaces can be considered as flat plates of negligible thickness. Let $C_p$, $C_t$, and $C_d$ represent the normal force, tangential force and drag force coefficients for the $i^{th}$ control surface in free molecular flow. Further let $C_m$ denote the moment coefficient of the center-body alone. $C_m$ is a function of $\Lambda$, the angle between the velocity vector of the body relative to the atmosphere and the axis of symmetry of the body. The drag on the satellite due to the center-body leads to orbit perturbation and is treated in Appendix C.

Let $\hat{j}_{3i}$ be the unit normal to the $i^{th}$ control surface. Then,

$$\hat{n}_{i} = \hat{\varphi}_{i}^{Hi} \hat{\varphi}_{i}^{Hi, Si} \hat{j}_{3i} = \hat{\varphi}_{i}^{Hi} \hat{\varphi}_{i}^{Hi, Si} \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$$
Let \( \mathbf{t}_d \) be of unit magnitude and lie at the intersection of the control surface and the plane containing \( \mathbf{n}_d \) and \( \mathbf{v}_R \).

In the sequel the superscript 7 will be dropped and all vectors without any superscript are to be treated as having superscript 7.

\[
(n_d \cdot v_R) = \cos \alpha, \quad t_d = -\cos \alpha n_d + \csc \alpha v_R
\]

(3.9)

\( t_d \) may also be represented as

\[
t_d = \frac{(n_d \times v_R) \times n_d}{\| (n_d \times v_R) \times n_d \|}
\]

Due to the control surfaces alone:

\[
P_{d} = \frac{1}{2} \rho v_{R}^{2} A_{1} C_{p_{1}} p_{d}
\]

(3.10)

\[
\begin{align*}
P_{d} &= \frac{1}{2} \rho v_{R}^{2} A_{1} C_{p_{1}} p_{d} \\
\mathbf{c}_{d} &= \frac{1}{2} \rho v_{R}^{2} A_{1} C_{p_{1}} \mathbf{t}_{d} = \frac{1}{2} \rho v_{R}^{2} A_{1} C_{p_{1}} \left\{ \csc \alpha_{1} v_{R} - \cot \alpha_{1} n_{1} \right\}
\end{align*}
\]

where \( A_{1} \) - area of the \( i \)th control surface.

For the center-body, for small angles \( C_{m} = \frac{2 C_{m} m}{\partial \Lambda} \Lambda = C_{m} \Lambda \).

\( C_{m} \) is a function of body shape and location of the center of mass.

\[
\Lambda = \cos^{-1} \left\{ -v_{R} \cdot \mathbf{j}_{1} \right\}
\]

\[
G_{A_{c.b}} = \frac{1}{2} \rho v_{R}^{2} S_{A} L C_{m} \Lambda (v_{R} \times \mathbf{j}_{1})
\]

where, \( c.b \) - center-body

\( S_{A} \) - characteristic cross-sectional area of the body.

\( L \) - characteristic length of the body.

\[
G_{A_{c.s}} = G_{A_{c.b}} = \frac{1}{2} \rho v_{R}^{2} S_{A} L \left\{ C_{m} \Lambda (v_{R} \times \mathbf{j}_{1}) + \sum A_{i} b_{i} (n_{i} C_{p_{i}} + \csc \alpha_{i} v_{R} C_{t_{i}} - \cot \alpha_{i} n_{i} C_{t_{i}}) \right\}
\]

where \( A_{i} = \frac{A_{1}}{S_{A}}, \quad b_{i} = \frac{b_{1}}{L} \)

(3.11)

The drag due to the control surfaces are given by
\[ D = \frac{1}{2} \rho V_R^2 S_A \sum_{i=1}^{4} A_i C_{D_i} \quad (3.12) \]

The gravity gradient torques acting on the satellite can be approximated by:

\[ G_G \approx \frac{3\mu}{R_c^3} \begin{bmatrix} K_2K_3(C_1 - B_1) \\ K_3K_1(A_1 - C_1) \\ K_1K_2(B_1 - A_1) \end{bmatrix} \quad (3.13) \]

\[ K = \mathcal{L}^{75} K^5 = \mathcal{L}^{75} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} S_{\theta_2}S_{\theta_3} - C_{\theta_2}S_{\theta_3} \\ S_{\theta_2}C_{\theta_3} + C_{\theta_2}S_{\theta_3} \\ C_{\theta_2}C_{\theta_3} \end{bmatrix} \]

Under steady state zero pointing error condition, \( \mathcal{L}^{75} = I \) since \( \theta = 0 \)

\[ K \bigg|_{\theta = 0} = K^5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ V_R \bigg|_{\theta = 0} = V^5_\theta = \begin{bmatrix} \frac{-\Omega + \omega_R C_1}{V_R} \\ \frac{-\Omega - \omega_R S_1 C_1}{V_R} \end{bmatrix} \]

### 3.4 Determination of Nominal Control Angles:

The notion of nominal control enunciated earlier in section 2.3 will be utilized here. The drag acting on the satellite due to the control surfaces, which is a direct consequence of using aerodynamic control leads to a reduction in satellite life. Under steady state zero pointing error condition in the absence of other perturbations it is necessary to maintain the combined aerodynamic and gravitational torques on the satellite at zero in order to maintain the satellite in that attitude. But it is also essential to ensure that in the process of doing so, very little additional drag due to such a nominal control is imposed on the satellite. Thus the nominal control SC (\( \theta_\text{c} \)) has to be determined so as to minimize the total drag on the satellite under steady state zero pointing error condition (\( \theta = 0, \dot{\theta} = 0 \)) subject to the constraint that \( x|_{x=0} = 0 \) where, \( x^T = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \theta_3, \dot{\theta}_3) \).

\[ x \bigg|_{x=0} \quad \text{leads to } \quad (G_G + G_A) \bigg|_{\theta=0} = 0 \]

Hence \( \int_0^{2\pi} D d\eta \bigg|_{\theta=0} \) will have to be minimized subject to the constraints

\[ (G_G + G_A) \bigg|_{\theta=0} = 0. \]

It is easy to show that the above calculus of variations problem degenerates to a ordinary minimum problem for each value of \( \eta \) because of the absence of derivatives of dependent variables in the functionals. Therefore \( D \bigg|_{\theta=0} \) has to be minimized subject to the
constraint \( \{ G_A + G_C \}_{\Theta=0} = 0 \) for various \( \eta \) values from 0 to \( 2\pi \). The problem of function minimization subject to constraints is treated in Appendix D.

The force coefficients are given by (refer to Appendix B)

\[
\begin{align*}
C_{\rho_i} & = \frac{2(2-\sigma')}{\sqrt{\pi}} S \alpha_i \left\{ e^{-(SC\alpha_i)^2} + \left( \frac{2(2-\sigma')}{S^2} \right) C_{\alpha_i} \right\} \text{erf}(SC\alpha_i) \\
C_{\tau_i} & = \frac{2}{\sqrt{\pi}} S \alpha_i \left\{ e^{-(SC\alpha_i)^2} + \sqrt{\pi} S C_{\alpha_i} \left[ \text{erf}(SC\alpha_i) \right] \right\} \\
C_{\delta_i} & = \frac{2}{\sqrt{\pi}} S \alpha_i \left\{ \left( 2-\sigma'\sigma \right) C_{\alpha_i}^2 + \sigma \right\} e^{-(SC\alpha_i)^2} + 2 \left\{ \left( 2-\sigma'\sigma \right) C_{\alpha_i} + \left( \frac{2-\sigma'}{2S^2} + \sigma \right) C_{\alpha_i} \right\} \text{erf}(SC\alpha_i)
\end{align*}
\] (3.14)

Substituting equations (3.14) in (3.11) and (3.12) there results,

\[
G_A = \frac{1}{2} \rho V^2 A \left\{ \sum_{i=1}^{4} C_{\alpha_i} \cos^4 \left( -\psi_R \cdot \mathbf{a}_i \right) \right\}
\]
\[
\begin{align*}
&+ \frac{4}{S \alpha_i} \sum_{i=1}^{4} \mathbf{b}_i \left[ \text{erf}(SC\alpha_i) \left\{ \left( 2-\sigma'\sigma \right) C_{\alpha_i}^2 + \frac{2-\sigma'}{S^2} C_{\alpha_i} \right\} \mathbf{n}_i + 2\sigma C_{\alpha_i} \psi_R \right] \\
&+ e^{-(SC\alpha_i)^2} \left\{ \left( \frac{2(2-\sigma'\sigma)}{\sqrt{\pi}} C_{\alpha_i} \right) \mathbf{n}_i + \frac{2\sigma}{\sqrt{\pi}} \right\}
\end{align*}
\] (3.15)

Drag due to control surfaces alone becomes,

\[
D = \frac{1}{2} \rho V^2 A \sum_{i=1}^{4} \left\{ e^{-(SC\alpha_i)^2} \left[ \frac{2(2-\sigma'\sigma)}{\sqrt{\pi}} C_{\alpha_i}^2 + \frac{2\sigma}{\sqrt{\pi}} \right] \\
+ \text{erf}(SC\alpha_i) \left[ 2(2-\sigma'\sigma) C_{\alpha_i}^3 + \left( \frac{2-\sigma'}{2S^2} + \sigma \right) C_{\alpha_i} \right] \right\}
\] (3.16)

The drag due to the center-body is not included here since once a configuration is selected for the center-body, under nominal attitude condition its drag is fixed and cannot be altered without altering the center-body configuration.

When \( \Theta = 0 \), the following results since \( \mathbf{e}_1^T = 1 \).

\[
C_{\alpha_i} = (n_i \cdot \mathbf{e}_1) \\
n_i = \mathbf{e}_1^T H \mathbf{e}_1^T S \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} = \mathbf{e}_1^T S \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

Hence

\[
\mathbf{n}_1 = \begin{bmatrix}
-S_{\theta z}
0
C_{\theta z}
\end{bmatrix} ; \mathbf{n}_2 = \begin{bmatrix}
S_{\theta z}
0
-C_{\theta z}
\end{bmatrix} ; \mathbf{n}_3 = \begin{bmatrix}
-S_{\theta z}
0
C_{\theta z}
\end{bmatrix} ; \mathbf{n}_4 = \begin{bmatrix}
S_{\theta z}
0
-C_{\theta z}
\end{bmatrix}
\]
\[
\mathbf{v}_R \cdot \mathbf{j}_1 = \begin{bmatrix}
-V + \omega_R \mathbf{e}_1 C_{\theta z}
0
0
\end{bmatrix}
\]
\[
\mathbf{v}_R \cdot \mathbf{j}_1 = \begin{bmatrix}
0
0
\omega_R \mathbf{e}_1 S \mathbf{C}_{\omega + \gamma}
\end{bmatrix}
\]
For earth pointing satellites under nominal attitude con-
dition

\[ \mathbf{K} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

and hence substituting in (3.13), \( G_0 \bigg|_{\theta=0} = 0 \). Hence the constraint rela-
tion reduces to, \( G_0 \bigg|_{\theta=0} = 0 \). 

(3.17)

Referring to Appendix D on function minimization with constraints
it becomes quite clear that most of the methods require the gradient of
the function to be evaluated. The necessary derivatives are evaluated
in Appendix F. The first derivatives are given by equations (F.14) and
(F.15) whereas for minimization methods involving the utilization of
second derivatives the necessary derivatives are given by equations
(F.16), (F.18), (F.19) and (F.20).
Of the several computational schemes discussed in Appendix D, the Penalty Function approach using gradient method for minimization gave good results. A listing of the computer program for the above is given in Appendix F. From trials made using other methods it was found that even though these require fewer iterations, the computation time required was higher because of the increased number of computational steps involved.

For equatorial orbits all control angles are zero for all altitudes. For other orbits the nominal control angles are functions of orbital angle, orbital altitude and inclination. Figures 4.3 and 4.4 show the nominal control angles for a particular satellite configuration (configuration A: see Chap. 4) in polar orbits at 200 Km and 300 Km altitude respectively. It is clear from these figures that the altitude dependence is very slight and \( \theta^c_1(\eta) \) and \( \theta^c_2(\eta) \) are always zero for nominal control.

It should be pointed out here that the method enunciated above for the determination of nominal control is very general and powerful. Thus the above formal approach permits the determination of the nominal control when other deterministic but periodic disturbing torques are included and when the orbit is elliptical. In the case of circular orbits, if only aerodynamic and gravitational torques are considered, recognizing the quadratic nature of the drag, the nominal control can be easily obtained by setting the yaw torque under nominal attitude condition equal to zero and thus obtaining an algebraic equation for the vertical control panel angles which are related by \( \theta^c_3 = -\theta^c_4 \). Since the resulting algebraic equation will be nonlinear in the unknown, it can only be solved by some iterative scheme such as the gradient method. In the case of elliptic orbits both the yaw torque and the pitch torque will have to be set equal to zero (under nominal attitude conditions leading to two algebraic equations for the vertical and horizontal control panel angles respectively with the conditions \( \theta^c_1 = \theta^c_5, \theta^c_3 = -\theta^c_4 \). These equations can again be solved by using any one of the several variations of the gradient method. In the presence of other deterministic but periodic disturbances such an approach may not be useful if those torques were included in determining the nominal control angles. Thus the second method described above is not general, but the computational scheme in cases where it is applicable involves one form or other of the gradient method employed in the approach used in the present study. Obviously the second method requires lesser amount of computations than in the method utilized in this study for the circular orbit case with only aerodynamic and gravitational torques being considered.

3.5 Equations of Motion:

In this section the equations of motion for the configuration under study are derived. These are then linearized around the nominal state and nominal control. Referring to section 2.3, for an earth pointing satellite,

\[
\dot{x}^7 = \mathcal{L}^{75} \mathcal{L}^{51} x^1 = (\theta_3)_3 (\theta_2)_2 (\theta_4)_1 (180 - (\omega + \eta))_2 (-180 + i)_3 (\Omega)_2 x^1 \tag{3.17}
\]
Hence \( \omega^7 \) is given by,

\[
\dot{\omega}^7 = \hat{j}_3 \hat{\theta}_3 + (\hat{\theta}_3)_3 \hat{j}_2 \hat{\theta}_2 + (\hat{\theta}_3)_3 (\theta_3) \hat{j}_1 \hat{\theta}_1 - (\hat{\theta}_3)_3 (\theta_3) \hat{j}_1 \hat{\theta}_1 (\dot{\omega} + \dot{\eta}) \\
+ (\hat{\theta}_3)_3 (\theta_3)_2 (\theta_3)_1 (180 - (\omega + \eta)) \hat{j}_2 \hat{\eta} \\
+ (\hat{\theta}_3)_3 (\theta_3)_2 (\theta_3)_1 (180 - (\omega + \eta)) (-(180 - \hat{\eta})) \hat{j}_2 \hat{\eta}
\]

(3.18)

For Keplerian motion of the c.m of the satellite, \( \omega, i \) and \( \Omega \) are constant. Earth's oblateness, atmospheric drag and solar radiation forces cause secular and/or periodic variations in \( \omega, i \) and \( \Omega \), but in general \( \dot{\omega}, \dot{i} \) and \( \dot{\Omega} \) are small (Assumption 2, Sec. 3.2). Hence neglecting \( \dot{\omega}, \dot{i} \) and \( \dot{\Omega} \) equation (3.18) becomes,

\[
\dot{\omega}^7 = \hat{j}_3 \hat{\theta}_3 + (\hat{\theta}_3)_3 \hat{j}_2 \hat{\theta}_2 + (\hat{\theta}_3)_3 (\theta_3) \hat{j}_1 \hat{\theta}_1 - (\hat{\theta}_3)_3 (\theta_3) \hat{j}_1 \hat{\theta}_1 \dot{\omega}
\]

(3.19)

since \( \hat{j}_3^T = [1, 0, 0] \), \( \hat{j}_2^T = [0, 1, 0] \) and \( \hat{j}_3^T = [0, 0, 1] \)

\( \dot{\omega}^7 \) becomes

\[
\dot{\omega}^7 = \\
\begin{bmatrix}
S_{\theta_3} \hat{\theta}_3 + C_{\theta_3} \hat{\theta}_3 \hat{\theta}_3 + -(S_{\theta_3} S_{\theta_2} C_{\theta_3} + C_{\theta_1} S_{\theta_3}) \hat{\eta} \\
C_{\theta_3} \hat{\theta}_3 - S_{\theta_3} C_{\theta_3} \hat{\theta}_3 + -(C_{\theta_1} C_{\theta_3} - S_{\theta_1} S_{\theta_2} S_{\theta_3}) \hat{\eta} \\
\hat{\theta}_3 + S_{\theta_3} \hat{\theta}_3 + C_{\theta_1} C_{\theta_3} \hat{\eta}
\end{bmatrix}
\]

(3.20)

Inverting (3.20) expressions for \( \dot{\theta}_1, \dot{\theta}_2 \), and \( \dot{\theta}_3 \) can be obtained in terms of \( \dot{\omega}^7, \dot{\eta} \) and \( \hat{\theta} \) and these represent the nonlinear differential equations describing the kinematics of the system.

On linearization (3.20) leads to

\[
\dot{\omega}^7 \approx \\
\begin{bmatrix}
\dot{\theta}_1 - \hat{\theta}_3 \dot{\eta} \\
\dot{\theta}_2 - \hat{\theta}_1 \dot{\eta} \\
\dot{\theta}_3 + \hat{\theta}_1 \dot{\eta}
\end{bmatrix}
\]

(3.21)

For circular orbits \( \dot{\eta} = 0 \). Hence differentiating (3.20),

\[
\dot{\omega}^7 = \\
\begin{bmatrix}
\hat{\theta}_2 S_{\theta_3} + C_{\theta_3} \hat{\theta}_3 S_{\theta_2} C_{\theta_3} S_{\theta_1} \hat{\theta}_1 + \hat{\theta}_2 C_{\theta_3} S_{\theta_2} C_{\theta_3} \hat{\theta}_1 + (C_{\theta_3} S_{\theta_2} C_{\theta_3} \hat{\theta}_1 + S_{\theta_3} S_{\theta_1} S_{\theta_2} S_{\theta_3} C_{\theta_1} C_{\theta_3}) \hat{\eta} \\
-S_{\theta_3} \hat{\theta}_3 - C_{\theta_3} \hat{\theta}_3 S_{\theta_2} C_{\theta_3} \hat{\theta}_1 - (S_{\theta_3} C_{\theta_2} \hat{\theta}_1 - C_{\theta_3} S_{\theta_2} \hat{\theta}_1) S_{\theta_2} S_{\theta_3} \hat{\theta}_1 + S_{\theta_3} C_{\theta_2} \hat{\theta}_1 + (C_{\theta_3} C_{\theta_2} \hat{\theta}_1 - S_{\theta_2} S_{\theta_3} \hat{\theta}_1) \hat{\eta}
\end{bmatrix}
\]

(3.22)

Linearizing (3.22),

\[
\dot{\omega}^7 = \\
\begin{bmatrix}
\dot{\theta}_1 - \hat{\theta}_2 \dot{\eta} \\
\dot{\theta}_2 - \hat{\theta}_1 \dot{\eta} \\
\dot{\theta}_3 + \hat{\theta}_1 \dot{\eta}
\end{bmatrix}
\]

(3.23)

Note that (3.23) is same as the expression resulting from (3.21) on differentiation. The equations of motion given by (2.43) are,
Hence the nonlinear differential equations representing the dynamics of the system can be written as,

\[
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} + \begin{bmatrix}
\omega_1 \omega_2 (C_1 - B_1) \\
\omega_2 \omega_3 (A_1 - C_1) \\
\omega_3 \omega_2 (B_1 - A_1)
\end{bmatrix} = G^7 + \gamma^7
\]

(3.24)

As before the superscript 7 will be dropped for convenience in the subsequent analysis. Only the linearized form of the equations of motion will be utilized in the present study. The nonlinear form of equations are provided in order that they may be used for nonlinear simulation if needed.

For notational convenience define \( \{ \cdot \}^{\gamma = 0} = \{ \cdot \}^{*} \)

Then in linearized form (3.24) can be written as,

\[
\begin{bmatrix}
A_1 \dot{\theta}_1 \\
B_1 \dot{\theta}_2 \\
C_1 \dot{\theta}_3
\end{bmatrix} = \begin{bmatrix}
\dot{\theta}_1 (A_1 + C_1 - B_1) + \dot{\theta}_2 (C_1 - B_1) \\
0 \\
\dot{\theta}_3 (B_1 - A_1) - \dot{\theta}_2 (B_1 - A_1)
\end{bmatrix} + \{G_6 + G_A\}^{*} + \left\{ \frac{\partial}{\partial \theta_j} (G_6 + G_A) \right\}^{*} \theta_j \\
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix}
\]

(3.25)

\( \{G_6 + G_A\}^{*} = 0 \) since this was the constraint used for determining \( \theta^c \).

\( \frac{\partial G_6}{\partial \theta_j} = 0 \) and referring to Appendix F, \( \frac{\partial}{\partial \theta_j} \begin{bmatrix} G_A \end{bmatrix} \bigg|_{\theta = 0} \) is given by equation (F.15). Hence \( \left\{ \frac{\partial}{\partial \theta_j} (G_A) \right\}^{*} \) is obtained by substituting \( \theta^c = \theta^c \) in (F.15). \( \left\{ \frac{\partial}{\partial \theta_j} (G_6) \right\}^{*} \) is given by equation (F.24). \( \left\{ \frac{\partial}{\partial \theta_j} (G_A) \right\}^{*} \) can be obtained by substituting \( \theta^c = \theta^c \) in (F.29). Define \( x^T = [\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3] \)

\( u = \delta \theta^c = \theta^c - \theta^c \) and \( u_\theta = \theta^c \). Further at this point it is convenient to change the independent variable from \( t \) to \( \eta \).

\[
\frac{d}{dt} = \sqrt{\frac{\mu}{R^3_c}} \frac{d}{d\eta}
\]

Hence \( \dot{x} = x \dot{\eta} \)

\[
\ddot{x} = \dot{x} \dot{\eta} + \ddot{\eta} = x \dot{\eta}^2 \text{ since for a circular orbit } \dot{\eta} = 0,
\]

\( \dot{\eta} = \sqrt{\frac{\mu}{R^3_c}} \) The dot denotes differentiation with respect to \( t \) and the prime denotes differentiation with respect to \( \eta \).
Then the linearized form of the equations of motion can be expressed as,

\[ \dot{x} = A(u_0(\eta), \eta) x + B(u_0(\eta), \eta) u \]  

(3.26)

where,

\[ A = \begin{bmatrix} 0 (3 \times 3) & I (3 \times 3) \\ C (3 \times 3) & D (3 \times 3) \end{bmatrix} \]

\[ C = \begin{bmatrix} 4 \left( \frac{C_4 - B_4}{A_4} \right) & 0 & 0 \\ 0 & 3 \left( \frac{C_4 - A_4}{B_4} \right) & 0 \\ 0 & 0 & \frac{A_4 - B_4}{C_4} \end{bmatrix} + \frac{1}{\eta^2} \begin{bmatrix} \frac{1}{A_4} & 0 & 0 \\ 0 & \frac{1}{B_4} & 0 \\ 0 & 0 & \frac{1}{C_4} \end{bmatrix} \{ \frac{3}{\partial \theta_k} G_A \}^* \]

\[ D = \begin{bmatrix} 0 & 0 & \left( \frac{A_4 + C_4 - B_4}{A_4} \right) \\ 0 & 0 & 0 \\ \left( \frac{B_4 - A_4 - C_4}{C_4} \right) & 0 & 0 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 (3 \times 4) \\ \frac{E (3 \times 4)}{E (3 \times 4)} \end{bmatrix} \]

\[ E = \frac{1}{\eta^2} \begin{bmatrix} \frac{4}{A_4} & 0 & 0 \\ 0 & \frac{1}{B_4} & 0 \\ 0 & 0 & \frac{1}{C_4} \end{bmatrix} \{ \frac{2}{\partial \theta_j} G_A \}^* \]

0 - null matrix, I - unit diagonal matrix.

Note that for equatorial orbits \( u_0 = \theta_c = 0 \) and consequently the matrices A and B become constant matrices and Eqn (3.26) represents a stationary linear differential system. On the contrary, for non-equatorial orbits \( u_0 = \theta_c \) is periodic in \( \eta \) with period \( 2\pi \) and hence the matrices A and B are periodic matrices with period \( 2\pi \) and Eqn(3.26) represents a periodic linear differential system.

3.6 Synthesis of Feedback Control:

Referring to Eqn(3.26), if \( u \) is set to zero the system under the nominal control \( u_0 \) takes the form,

\[ \dot{x} = A(u_0(\eta), \eta) x \]  

(3.27)

In general the system represented by (3.27) is not asymptotically stable as can be verified by studying the eigen-values of the system when \( A(u_0(\eta), \eta) \) is constant or its characteristic multipliers (Ref.73) when \( A(u_0(\eta), \eta) \) is periodic, or by studying the solution system \( \Phi(\eta) \) of the differential equation (3.27) (obtained by solving equation(3.27) with the initial condition \( \Phi(0) = I \) where I is the unit diagonal matrix) if \( A(u_0(\eta), \eta) \) happens to be an arbitrary function of \( \eta \). In this section the well known theory of optimal control (see Appendix E) is utilized to obtain a feedback control which will provide asymptotic
stability.

The system can be represented by

\[ \dot{x} = A(\eta) x + B(\eta) u \quad , \quad x(0) = x_0. \quad (3.28) \]

The aim is to determine a suitable control \( u(\eta) \) as a function of \( x \) given by \( u(\eta) = f(x) \) which is optimal in some sense and assures asymptotic stability \( (\dot{x}(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty) \). Minimizing a quadratic performance index of the form,

\[ J = \int_0^\infty \left[ x^T(\eta) Q(\eta) x(\eta) + u^T(\eta) R(\eta) u(\eta) \right] d\eta \quad (3.29) \]

where, \( Q(\eta) \) and \( R(\eta) \) are positive semi-definite and positive definite respectively, leads to an optimal control (see Appendix E) which is a linear function of the state and is given by,

\[ u(\eta) = -R^{-1}(\eta) B^T(\eta) S(\eta) x(\eta) \quad (3.30) \]

where, \( S(\eta) \) is the solution of the matrix Riccati equation given by,

\[ S = -S(\eta) A(\eta) - A^T(\eta) S(\eta) + S(\eta) B(\eta) R^{-1}(\eta) B^T(\eta) S(\eta) - Q(\eta), S(\eta \rightarrow \infty) = 0 \quad (3.31) \]

\( Q(\eta) \) and \( R(\eta) \) may be chosen as unit diagonal matrices for convenience. Such a choice also implies that all the control and state variables are equally weighted in the cost function. For alternative choices see Ref.4.

\( u(\eta) \) can be represented as \( u(\eta) = -K(\eta) x(\eta) \quad (3.32) \)

where, \( K(\eta) = R^{-1}(\eta) B^T(\eta) S(\eta) \) are the feedback gains.

Hence with the optimal feedback control (3.32), Eqn.(3.28) leads to

\[ \dot{x} = [A(\eta) - B(\eta) K(\eta)] x \quad , \quad x(0) = x_0 \quad (3.33) \]

Equation (3.33) represents the feedback controlled system and its stability characteristics can be studied by studying its solutions. For equatorial orbits the system is stationary and consequently the \( S \) matrix and \( K \) matrix are constant. But when the satellite is in nonequatorial orbits the system is periodic and consequently the \( S \) matrix and \( K \) matrix are also periodic.

For implementing the control system it is necessary to store the gain matrix \( K \) and utilize it to set the control angles according to the control law (3.32). But such an operation requires a perfect knowledge of the state \( x(\eta) \) of the system at all values of \( \eta \) and the problem of measuring and estimating the state from the measurements is a complex problem and is not treated in this study.

3.7 Single Axis Pointing: A Special Case*

The feedback control synthesis of Sec.(3.5) was carried out

* The author wishes to thank Prof. B. Etkin for pointing out this possibility.
with the aim of maintaining the body fixed reference coordinate system \( \mathcal{X} \) in coincidence with the orbital coordinate system \( \mathcal{X}^5 \). For non-equatorial orbits there exists a cross-wind due to the rotation of the earth and the atmoosphere and consequently there is a periodic disturbance torque acting on the satellite and this is taken care of by the nominal control angles which are periodic functions of \( \eta \). The question arises as to whether the satellite can be stabilize in a single axis pointing mode. That is, the \( X^1 \) axis is to be maintained in coincidence with \( X^3 \) allowing the other two axes to move around by virtue of rotation around \( X^3 \) (In other words, the satellite is to earth point as before but is allowed to freely "weather-vane with the wind" in yaw). Further it is desired to eliminate the vertical control surfaces (3 and 4) or maintain them fixed and consequently making them useless for control purposes. Assuming that it is desired to maintain control surfaces 3 and 4 fixed, the following equations are obtained.

\[
\begin{align*}
\dot{x} &= g(Q, \eta) + A(Q, \eta) x + B(Q, \eta) u \\
\end{align*}
\]  

where \( A \) and \( B \) are defined in equation (3.26) and \( g(Q, \eta) = \left\{ \frac{\partial g}{\partial x} + \frac{\partial g}{\partial \eta} \right\} \). Note that \( g(Q, \eta) \) is periodic in \( \eta \) and equals zero for equatorial orbits.

\[
J = \int_0^\infty \left[ x^T(\eta) Q(\eta) x(\eta) + u^T(\eta) R(\eta) u(\eta) \right] d\eta
\]  

where \( Q(\eta) \) and \( R(\eta) \) are chosen as,

\[
Q(\eta) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
R(\eta) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R_0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Note that \( Q(\eta) \) is positive semidefinite and \( R(\eta) \) is positive definite.

With the above choice of the performance index, a feedback control can be synthesized by solving the Riccati equation (3.31). Even though \( u \) has been set equal to zero the \( A \) and \( B \) matrices are still periodic functions of \( \eta \) since the relative wind velocity vector is changing its direction as a periodic function of \( \eta \).

The feedback control obtained by solving the Riccati equation applies to the system,

\[
\dot{x} = A(Q, \eta) x + B(Q, \eta) u,
\]

with control panels 3 and 4 being kept fixed and \( \theta_3 \) and \( \dot{\theta}_3 \) being allowed to vary freely. Comparing (3.34) and (3.36) it can be concluded that
equation (3.34) represents the system described by (3.36) with a periodic disturbance \(g(0, \eta)\) acting on it.

Consequently, corresponding to equation (3.34) we have,

Uncontrolled system: \[
\dot{x} = A(\eta) x + g(\eta) \tag{3.37}
\]

Feedback controlled system: \[
\dot{x} = [A(\eta) - B(\eta) R^{-1}(\eta) B^T(\eta) S(\eta)] x + \hat{g}(\eta) \tag{3.38}
\]

Note that due to the periodic disturbance \(g(\eta)\), the system represented by (3.38) may have pointing errors in elements of the state vector which are feedback controlled in addition to the pointing errors in the uncontrolled elements \(\theta_3\) and \(\dot{\theta}_3\) of the state vector.

3.8 Feedback Synthesis as a Linear Inhomogeneous System:

In this section an alternative form of feedback control will be discussed. In section 3.4 the problem of the determination of nominal control angles was discussed. The results indicate that the nominal control angles for equatorial orbits are zero and for non-equatorial orbits \(\Theta^e_1\) and \(\Theta^e_2\) are zero, whereas \(\Theta^e_3\) and \(\Theta^e_4\) are periodic with maximum amplitudes which are less than \(7^\circ\) for the range of satellite and orbital parameters considered. In the process of obtaining the nominal control angles, the constraint \(\{S_A + S_B\}_x = 0\) was utilized. Suppose the nominal control angles were set equal to zero irrespective of the orbital inclination then, \(\{S_A + S_B\}_x = 0\) for all non-equatorial orbits. Under these circumstances the equations of motion are given by Eqn(3.34) and can be represented by,

\[
\dot{x} = A(\eta) x + B(\eta) u + g(\eta), \quad x(0) = x_0 \tag{3.39}
\]

The above is a linear inhomogeneous system of differential equations. Defining the performance index to be minimized as,

\[
J = \int_0^{\infty} \left\{ x^T(\eta) Q(\eta) x(\eta) + u^T(\eta) R(\eta) u(\eta) \right\} d\eta
\]

the feedback control can be obtained as (see Appendix E),

\[
u(\eta) = -R^{-1}(\eta) B^T(\eta) S(\eta) x - R^{-1}(\eta) B^T(\eta) U(\eta) \tag{3.40}
\]

where \(S(\eta)\) is obtained from the matrix Riccati equation

\[
\dot{S} = -S(\eta) A(\eta) - A^T(\eta) S(\eta) + S(\eta) B(\eta) R^{-1}(\eta) B^T(\eta) S(\eta) - Q(\eta), \quad S(\eta \to \infty) = 0
\]

and \(U(\eta)\) is obtained from,

\[
\dot{U} = -[A(\eta) - S(\eta) B(\eta) R^{-1}(\eta) B^T(\eta)] U(\eta) - S(\eta) \hat{g}(\eta), \quad U(\eta \to \infty) = 0 \tag{3.41}
\]

The feedback controlled system is represented by,

\[
\dot{x} = [A(\eta) - B(\eta) R^{-1}(\eta) B^T(\eta) S(\eta)] x - B(\eta) R^{-1}(\eta) B^T(\eta) U(\eta) + \hat{g}(\eta), \quad x(0) = x_0 \tag{3.42}
\]
Defining \( K(\eta) = R^{-1}(\eta) B^T(\eta) S(\eta) \)

and \( u_{inh}(\eta) = R^{-1}(\eta) B^T(\eta) U(\eta) \) where subscript \( inh. \) represents the inhomogeneous part, Equations (3.40) and (3.42) become,

\[
\dot{u}(\eta) = -K(\eta) x - u_{inh}
\]

where

\[
\dot{\mathbf{x}} = [A(\eta) - B(\eta) K(\eta)] \mathbf{x} - B(\eta) u_{inh} + \mathbf{g}(\eta) , \quad \mathbf{x}(0) = x_0 \quad (3.44)
\]

In the above analysis, no attempt was made to minimize the drag on the satellite due to the control panels and it was not brought into consideration in any part of the analysis. \( U(\eta) \) is a matrix of dimension \((6 \times 1)\). Thus the function minimization problem that had to be solved for the feedback control of section 3.5 (to determine the nominal control angles) has been replaced now by a set of six differential equations (3.41) which have to be solved in conjunction with the matrix Riccati equation (3.31) and the additional computations involved for solving (3.41) and (3.31) as against solving (3.31) alone are negligible in comparison with the computations involved in determining the nominal control. Further in addition to storing \( K(\eta) \) for \( 0 < \eta < 2\pi \), we have to store \( u_o = \theta^c \) as a function of \( \eta \) for \( 0 < \eta < 2\pi \) in the first case whereas in the present case \( u_{inh}(\eta) = R^{-1}(\eta) B(\eta) U(\eta) \) will have to be stored instead. On the other hand in spite of its relative merits, the analysis of this section does not help one to "get a handle" on the problem of minimizing the drag due to the control surfaces.

IV. RESULTS

The analysis carried out in earlier sections was applied for two specific satellite configurations. Configuration \( \mathbb{A} \) whose parameters are shown in Table 4.1 is of moderate size whereas configuration \( \mathbb{B} \) whose parameters are shown in Table 4.2 is comparable in size with the space bases that are being currently proposed. The configuration \( \mathbb{A} \) was also studied for the possibility of implementing the single axis pointing scheme analyzed in section 3.6. The present chapter consists of two sections. In the first section the computational considerations pertaining to the numerical study carried out are discussed, whereas in the second section the results obtained are discussed.

4.1 Computational Considerations:

In this section the computational considerations involved in the synthesis and the performance evaluation of the feedback control system are discussed. First certain general remarks are in order. Most of the computations were done on the IBM 360/65 computer at the Institute of Computer Sciences of the University of Toronto. All the programs were written in single precision FORTRAN IV. The various computations are discussed below and the listings of most of the computer programs used are presented in Appendix G. A schematic of the computation is shown in Fig. 4.1.

4.1.1. Determination of Nominal Control:

Of the several methods discussed in Appendix D on function minimization with constraints, the penalty function approach utilizing
the gradient method was used for computing the nominal control angles. The penalty function

\[ F = D + K \left\{ G_G + G_A \right\} \left[ g_0 \right] \]

with a K value of 10^{20} was used. The initial step size for the gradient method was taken as 0.1 and the steps were halved whenever there was an increase in the value of F. The gradient method was terminated when the step size reached a value of 10^{-6} or when the change in the value of F became less than 10^{-15}. The starting values for the control angles were always set as \( \phi_c = 0 \) and the nominal control angles were determined at intervals of 10° in orbital angle for one complete orbit. A Fourier cosine series containing 21 terms was fitted to each of nominal control angle as a function of \( \eta \) when they were nonzero and periodic. The programs used in the computations are listed in Appendix G. The total computation time on the IBM 360/65 was approximately 15 mins.

4.1.2 Response of Satellite to Initial Disturbances with Nominal Control:

After having determined the nominal control angles, the homogeneous linear differential equation \( \dot{x} = A(\eta)x \) was solved with a set of linearly independent initial state vectors to test whether the satellite is stable with nominal control alone. The computations were carried out in matrix form by solving the set of equations given by \( X = A(\eta)X \) with the initial condition

\[
X(0) = \begin{bmatrix}
5.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.4
\end{bmatrix}
\]

whose columns are six linearly independent set of initial state vectors. The integration was carried out using a modified fourth order Runge-Kutta formula due to Gill. The initial increment in \( \eta \) was set at \( 2^0 \). The increment was adjusted to satisfy the accuracy requirements during the process of integration. The upper error bound was set equal to 10^{-6}. The integration was carried out for two complete orbits. The total computation time was approximately 5 mins. on the IBM 360/65 computer. A listing of the computer program is given in Appendix G.

4.1.3 Solution of the Matrix Riccati Equation:

The matrix Riccati equation given by eqn.(3.31) was integrated backwards with the initial condition \( S(\eta_f) = 0 \). The integration was carried out using a modified fourth order Runge-Kutta formula due to Gill. \( \eta_f \) was set at 3600°. The initial decrement in \( \eta \) was set at \( 2^0 \). The upper error bound was set equal to 10^{-6}. The integration was carried out over four orbits and \( K(\eta) \) was computed from the solution \( S(\eta) \) continuously. In cases where \( S(\eta) \) reached a constant value the corresponding \( K(\eta) \) value was read out. On the other hand if \( S(\eta) \) reached a steady periodic value 21 term Fourier series were fitted to the elements.
of \(K(\eta)\) as functions of \(\eta\) over one orbit. The coefficients of the Fourier series were read out along with the norm \(\sum \|S_{ij}(\eta)\|\) of \(S(\eta)\). In cases where \(S(\eta)\) did not reach either a constant or periodic steady state value the configuration was abandoned. The computation time on the IBM 360/65 computer was approximately 15 mins. A listing of the computer program is given in Appendix G.

4.1.4 Response of Satellite to Initial Disturbances with Feedback Control:

After having computed the feedback gain matrix \(K(\eta)\), to study the response of the satellite to initial conditions, the differential equation \(\dot{X} = [A(\eta)-B(\eta)K(\eta)] X\), representing the satellite with feedback control was solved with a set of linearly independent initial state vectors. The computations were carried out in matrix form solving the set of equations given by \(\dot{X} = [A(\eta)-B(\eta)K(\eta)] X(\eta)\) with the initial condition given by Equation \((4.1)\). The integration was carried out using a modified fourth order Runge-Kutta formula due to Gill. The initial increment in \(\eta\) was set at 2° and the increment was suitably adjusted to satisfy the accuracy requirements during the process of integration. The upper error bound was set equal to \(10^{-6}\). The integration was carried out for two complete orbits. The total computation time was approximately 8 mins. on the IBM 360/65 computer. A listing of the computer program is given in Appendix G.

4.2 Results and Discussion:

In the present section the results obtained are briefly described and their implications are discussed. Both configuration \(A\) and \(B\) in the absence of aerodynamic torques and in the presence of gravitational torques alone are unstable. The gravitational stability of satellites can be conveniently represented using the non-dimensional variables

\[
\frac{A_1}{\text{Trace } 9}, \quad \frac{B_1}{\text{Trace } 9}, \quad \text{and} \quad \frac{C_1}{\text{Trace } 9}.
\]

It is clear from the above Magnus variable representation (Ref. 74) of configuration \(A\) and \(B\) shown in Fig. 4.2 that both these configurations are gravitationally similar since they plot on to the same point in the Magnus plane and are unstable. Of the two configurations most of the computations were carried out for configuration \(A\).

4.2.1 Nominal Control Angles:

Plots of nominal control angles for the satellite configuration \(A\) in polar orbit at 200 Km altitude and 300 Km altitude are shown in Figs. 4.3 and 4.4 respectively. It is seen that the horizontal control panel angles remain zero and the vertical control panel angles vary periodically. Physically, the cross-wind present due to the rotation of the atmosphere introduces a periodic yawing moment on the satellite due to the center-body and the above moment is balanced by creating a yawing moment in an opposite sense by suitably rotating the vertical control panels. The nominal control angles are all zero for equatorial orbits since in equatorial orbit the cross-wind is zero. Since the density appears as a multiplier both in the expression for the drag and the constraints in the present case, the nominal control angles are not affected by local density variations. From Figs. 4.3 and 4.4 it is clear that
the altitude dependence of the nominal control angles is very slight. Consequently in the case of non-equatorial orbits the nominal control need be altered only at discrete altitudes which may be separated by several tens of kilometers.

4.2.2 Response of Satellite with Nominal and Feedback Controls:

The results of the response calculations made for satellite configurations A, B and for configuration A in a single axis pointing mode are shown in Figs. 4.6-4.20 and Fig. 4.22. All these figures are nothing but the plots of the transition matrices of the linearized system with nominal control and with feedback control for different altitudes, orbital inclinations, various parameter changes etc., computed with the initial conditions given by Eqn. (4.1). In each plot with its own scale, the elements in a column of the transition matrix are shown (only those elements which are significant are plotted) with the corresponding initial condition. Generally transition matrices are computed with a unit diagonal matrix as the initial condition, but in the present situation all nonzero initial angle variables are set at 5° and all nonzero initial angle rates are set at 0.4 rad./rad. instead. It should be pointed out that these plots may not correspond to the performance of the actual satellite subjected to an initial disturbance given by Eqn. (4.1) due to the following reasons: 1) The effect of nonlinearities. 2) The above initial conditions have been chosen with no regard to the fact that at higher altitudes where the gains are large, these may lead to very large control panel angles which lie outside the permissible range of operation. For sufficiently small initial conditions they are proportional to the satellite's response.

Of the aerodynamic forces acting on the control panels only the lift forces are effective in creating rolling moment on the satellite whereas both the lift and drag forces contribute to the yawing and pitching moments acting on the satellite. For small deviations in the attitude of the satellite from the nominal, the moment arm for the drag forces and lift forces are proportional to the sine and cosine of the attitude angle respectively and hence the lift forces contribute the major portion of the control torques since its moment arm is large in comparison with that for the drag forces. Figure 4.5 shows a plot of the lift coefficient as a function of the control panel angle of attack \( \theta \) which is approximately equal to the control angle \( \theta^c \) for small deviations in the attitude of the satellite from the nominal. The linear approximation is also shown in the figure. It can be seen that the linear approximation holds if \( \theta^c \) lies in the range of \( \pm 10^\circ \). But it is possible to have control angles in the range of \( \pm 50^\circ \) since in this range the nonlinearity is of the hard spring type and the linear approximation underestimates the actual control torque. But under no circumstances should the control panel angle lie outside the range of \( \pm 50^\circ \) since the linear approximation is grossly inadequate in such a case.

Thus the results presented in the plots and the corresponding initial conditions must be suitably scaled to maintain the control angles in the above operational range since if the control panel angles were to lie outside this range, the plot will not represent even qualitatively the response of the satellite. That is, in the feedback controlled case, if the control panel angles lie outside the operational range during part of the control history, the plots may show that the satellite stabilizes in or acquires the nominal attitude but in actuality the satellite may be unstable.
The results obtained for satellite configuration A with nominal control, with feedback control, and the effect of varying certain parameters are shown in Figs. 4.6 - 4.18. Figure 4.6 shows the response of the satellite to initial disturbances with nominal control and with feedback control, in 200 Km equatorial orbit. The norm of the S-matrix is also shown plotted. From the plots it can be deduced that the satellite is unstable with nominal control alone and with feedback control all the modes of the satellite's rotational motion are damped out in approximately 1/2 orbit. Figure 4.7 shows similar results for 200 Km polar orbit. The plot of the norm of the S-matrix clearly shows the periodic nature of the S-matrix. The response of the satellite in polar orbit with feedback control seems to be almost the same as in equatorial orbit. Figures 4.8 and 4.9 show the response of the satellite configuration A in 300 Km equatorial orbit and polar orbit respectively. The norm of the S-matrix for the polar orbit shown in Fig. 4.9 again shows the periodic character of S-matrix. The responses of the feedback controlled satellite are similar for equatorial and polar orbits but at 300 Km orbits the satellites rotational motion is damped out in about one orbit for disturbances in certain modes but takes almost 2-1/2 orbits for disturbances in others. From the results shown in Figs. 4.6 - 4.9 it can be concluded that the satellite is unstable with nominal control alone but is stable when feedback control is utilized. Further for a given altitude the results obtained for polar orbit do not seem to differ appreciably from the results obtained for the equatorial orbit and hence more attention will be paid to the response of the satellite in equatorial orbits. The above results also indicate that the performance of the satellite attitude control system becomes less effective with increasing altitude. This is to be expected since with $\gamma$ as the independent variable, the disturbing gravity gradient terms remain constant with altitude while as the aerodynamic control terms which are proportional to the density decrease with altitude.

Focussing attention to 200 Km and 300 Km equatorial orbits some further observations can be made. Table 4.3 and 4.4 show the feedback gain matrices (K-matrices) of configuration A for equatorial orbits at 200 Km and 300 Km respectively. An inspection of the elements of the 200 Km K-matrix shows that at 200 Km altitude it is possible to acquire the nominal attitude from large initial angles commensurate with the linearization carried out in the analysis and initial rates of nearly 1.2 rad./rad., maintaining the control angles in the range of $\pm 50^\circ$ during the process. On the other hand an examination of the elements of the 300 Km equatorial orbit K-matrix shows that at 300 Km altitude it is possible to acquire the nominal attitude from large initial roll and yaw angles and roll and yaw rates of nearly 0.5 rad./rad., but it is not possible to acquire the nominal attitude from initial pitch angles larger than a few degrees and pitch rates greater than approximately 0.1 rad./rad. since in such cases the control angles may lie outside of $\pm 50^\circ$ range. Thus in the case of initial pitch and pitch rate disturbances the response plots for 300 Km orbits must be suitably scaled to meet the above limitations.

Effect of Variations in Certain Parameters:

The results shown in Figs. 4.6 - 4.9 were computed for certain fixed values of parameters some of which are left to the choice of the designer ($C_m A$, $A$, $S_\alpha$, $L_1$, $A_1$, $B_1$ and $C_1$) and some others
which are not known to the designer apriori. Further no consideration was given to the possibility of operating at off-design values of altitudes and orbital inclinations. In this section the effect of operating the satellite at off-design values of orbital inclination and orbital altitude, the effect of variable density due to the diurnal bulge, and the effect of off-design values of $\sigma$ and $\sigma'$ on the satellite response will be discussed.

The periodic nature of the feedback gain matrices in the case of nonequatorial orbits makes it complex for implementation. Earlier it was pointed out that the response of the satellite in an equatorial orbit and a polar orbit at a fixed altitude are very similar. Figures 4.10 and 4.11 show the effect of using the 200 Km and 300 Km equatorial orbit K-matrices (which are constant matrices) for polar orbits at the respective altitudes. The plots do not differ much from the plots (Figs. 4.7 and 4.9) obtained using the proper polar orbit K-matrices. Thus it can be concluded that the constant K-matrix computed for the equatorial orbit at a given altitude may be used for nonequatorial orbits at the same altitude but the nominal control has to be the one computed for the particular orbital inclination in question.

Due to the orbital decay of the satellite, the orbital altitude continually changes. Hence theoretically the feedback gains must be computed as a function of altitude and implemented. Such an approach is impractical since it introduces complications in the implementation of the control and requires a phenomenal amount of computation. Alternatively a form of gain scheduling scheme (see Sec. 4.2.4 on implementation) in which the feedback gains are changed at discrete altitudes separated by several tens of kilometers may be adopted. For such a scheme to be practical the system should perform reasonably well though not optimally at off-design altitudes.

Figures 4.12 and 4.13 show the effect of using the 300 Km equatorial orbit K-matrix at 200 Km equatorial orbit and the 300 Km polar orbit K-matrix at 200 Km polar orbit respectively. The plots indicate that the performance in the roll mode is not as good as would obtain with the proper 200 Km K-matrix. In other modes the performance is about the same. On the other hand since the 300 Km gain values are rather high, even though the damping times remain reasonably good, the permissible initial disturbances from which acquisition is possible is reduced sharply from the values possible with the use of 200 Km gain matrix. On the contrary these permissible initial disturbances are certainly as large as or even larger than the ones allowable at 300 Km. Since the acquisition problem is important only at the initial altitude, once having acquired the nominal attitude, the need for reacquiring seldom occurs because large disturbances which can throw the system off nominal by a large amount are rarely encountered. Consequently the feedback control system can be allowed to operate at a lower altitude with the gains computed for some higher altitude.

In computing most of the response results presented in this study the density at any given altitude has been assumed to be constant. But the atmospheric densities are highly variable as a function of time at all altitudes in excess of 150 Km (see Appendix A). The satellite experiences a periodic density variation with a period approximately equal to the orbital period due to the diurnal variation in density. Other regular variations in density experienced by the satellite have large time constants. The nature and magnitude of the diurnal density variations at the various altitudes are discussed in Appendix A. To study the effect of the above periodic density variations on
the performance of the feedback controlled satellite, a density variation of the form $\rho = \rho_0 (1 + k \cos \eta)$, where $\rho_0$ is the density given by the 1962 U.S. standard atmosphere, and $k$ is the diurnal density variation factor at the given altitude (see Appendix A) was introduced along with the K-matrix computed with a constant density of $\rho = \rho_0$.

The results obtained for the 200 Km and 300 Km equatorial orbits are shown in Fig. 4.14 and 4.15 respectively. A comparison with Figs. 4.6 and 4.8 obtained with constant density values indicate that the variable densities do not produce any appreciable change in the performance of the system at the above altitudes.

In computing most of the results, the accommodation coefficients were set at $\sigma = \sigma' = 0.5$. Since there is a lack of experimental results which provide the accommodation coefficients for typical gases in the atmosphere interacting with various material surfaces at satellite velocities, with the theoretical estimates of the accommodation coefficients for clean surfaces (see Ref. 76) as a guide, the above values seem to be reasonable. Computations were made to study the effect of the accommodation coefficients being different from the values given above. Figure 4.17 shows the response of the satellite at 200 Km equatorial orbit with $\sigma = \sigma' = 0.75$. The results indicate that due to the higher value of the accommodation coefficients, the damping times have increased and so are the peak amplitudes. This is to be expected since an increase in the accommodation coefficients lead to a decrease in the lift forces and consequently the control torques generated due to a given control panel angle. The response of the satellite at 200 Km equatorial orbit with $\sigma = \sigma' = 0.75$ using the K-matrix computed for $\sigma = \sigma' = 0.5$ is displayed in Fig. 4.18. The results show an increase in damping times and peak amplitudes of the transients in comparison with the results shown in Fig. 4.6. On the other hand Fig. 4.19 shows that the response of the satellite in a 200 Km equatorial orbit with $\sigma = \sigma' = 0.5$ using the K-matrix computed for $\sigma = \sigma' = 0.75$, is not appreciably altered from results shown in Fig. 4.6. Hence it can be concluded that as long as the K-matrix is computed with values of $\sigma$ and $\sigma'$ higher than those obtaining in actual operating conditions, the performance of the satellite will be close to the design performance. The physically unrealistic values $\sigma = \sigma' = 0.0$ would result in the best performance.

The results of the response studies carried out on configuration $\text{B}$ are shown in Figs. 4.19 and 4.20. Configuration $\text{B}$ is comparable in size to the space bases currently being proposed by NASA. As pointed out earlier configuration $\text{B}$ is gravitationally similar to configuration $\text{A}$ but on the other hand it has a more favourable set of aerodynamic parameters (see Sec. 4.2.4). Figure 4.19 shows the response of configuration $\text{B}$ in a 200 Km equatorial orbit with the nominal control alone and with feedback control. From the plots it can be deduced that the configuration is unstable with nominal control alone and with feedback control all the modes of the satellite's rotational motion are damped out in approximately 1/3 orbit. Figure 4.20 shows similar results for the 300 Km equatorial orbit. With feedback control all the modes are damped out in approximately 1-1/2 orbits. The K-matrices computed for the 200 Km, and 300 Km, equatorial orbits are shown in Tables 4.5 and 4.6 respectively. An examination of the elements of these K-matrices show that at 200 Km altitude it is possible to acquire the nominal attitude from large initial angles and initial rates of 1.5 to 2.0 rad./rad. maintaining the control angles within the range of $\pm 50^\circ$ during the process. At 300 Km altitude acquisition from large initial angles seem possible but initial rates are limited to approximately 1.2 rad./rad. in yaw and roll and 0.4 rad./rad. in pitch.
4.2.3 Single Axis Pointing:

The computations carried out on configuration A for the single axis scheme described in Sec. 3.6 are shown in Figs. 4.21 - 4.23. Figure 4.21 shows the norm of the S-matrix obtained by solving the Riccati equation with the weighting matrices Q and R given in eqn.(3.35) for a 200 Km polar orbit. It is clear that the solution has not reached either a periodic or constant steady state behaviour even after 3 orbits. Consequently the solution had to be abandoned as per the stipulation made earlier (Sec.4.1.3). Physically the cross wind forces the satellite to oscillate in yaw and because of the coupling between yaw and roll, the roll mode is continuously forced with a periodic input and consequently it is not possible to control roll completely without controlling yaw as well. The coupling between yaw and roll occurs because of the two off-diagonal nonzero elements in the matrix D given in equation 3.26. These elements may be set equal to zero and a feedback controller can be synthesized to control roll and pitch alone and treat the forcing of the roll mode due to the coupling between roll and yaw as an external disturbance experienced by the feedback controlled system. Figure 4.22 shows the results of the response studies carried out on the satellite utilizing a single axis controller synthesized for the decoupled system of equations. The pitch mode damps out within 1/2 orbit. Yaw and roll modes seem to asymptotically tend to steady periodic motion. The amplitude of oscillations in yaw and roll tend to their asymptotic value at a very slow rate. Consequently to determine the steady state behaviour of the system, the response of the system with zero initial condition was computed and is shown plotted in Fig. 4.23. The influence of the initial conditions seem to die out fairly fast and from the plot it can be concluded that in the absence of other perturbations the steady state error in yaw and roll have maximum amplitudes of approximately 0.5 deg. and 0.2 deg. respectively due to the cross-wind experienced by the satellite.

4.2.4 Implementation:

It was pointed out in earlier sections that the nominal control has a very weak dependence on the altitude. Further the use of the feedback gain matrix computed for a higher altitude at a lower altitude orbit did not lead to any serious problems in the performance of the system. Consequently to avoid computing the nominal and feedback controls continuously as a function of altitude for implementation, it is possible to compute and update these at discrete altitudes. This is a form of gain scheduling. A schematic of the gain scheduling is shown in Fig. 4.24. It should be noted that in the altitude band between the two altitude values at which the gains are updated the system operates in a suboptimal fashion. The computations to determine the nominal and feedback control Fourier coefficients may be done on-board if a computer with sufficient capacity is carried onboard. The controller itself requires certain amount of storage and computational capability to store the Fourier coefficients and to compute the control angles using the Fourier coefficients, state variables and the orbital angle.

Since the density varies considerably with altitude, the
above scheme provides good performance at low altitudes and relatively poor performance at higher altitudes, as a consequence of the control torque that could be generated decreasing with altitude. But if the control panel area could be suitably changed with altitude it may be possible to obtain similar performances for a wide range of altitudes. The equations of motion with \( \eta \) as the independent variable are given by Eqn. (3.26). A closer examination of these equations show that in the absence of aerodynamic effects the motion of the satellite under the action of the gravitational torques alone can be represented by a single set of response curves for all altitudes with \( \eta \) as the independent variable. Or in other words the transformation used to change the independent variable from \( t \) to \( \eta \) happens to be a similarity transformation in the above case, and which eliminates the orbital radius from the equations. But when the aerodynamic effects are present, the orbital radius \( R_c \) is not eliminated from the equation (since \( \dot{\eta} \), \( \rho \) and \( V_R \) are functions of the orbital radius) and hence the response of the satellite is a function of \( R_c \). The following nondimensional parameters may be defined:

For the center-body:

\[
\alpha_{c,b} = \frac{1}{\eta^2} \left\{ \frac{1}{\text{trace } \mathcal{G}} \right\} \left( \frac{1}{2} \rho V_R^2 \right) S L \mathcal{C}_{m_A}
\]

For the control panels:

\[
\alpha_{c,s} = \frac{1}{\eta^2} \left\{ \frac{1}{\text{trace } \mathcal{G}} \right\} \left( \frac{1}{2} \rho V_R^2 \right) A \| \textbf{b} \|
\]

If it were possible to maintain both these parameters constant for all altitudes, then the equations of motion will be independent of \( R_c \). Of the two parameters \( \alpha_{c,b} \) cannot be maintained constant. But it is possible to make \( \alpha_{c,b} \) to remain at zero by suitably positioning the center of mass along the satellite’s axis of symmetry (i.e., at the aerodynamic center of the main body) thus making \( \mathcal{C}_{m_A} \) zero. Such an approach on the other hand eliminates all the stiffness provided to the system by the moments created by the aerodynamic forces acting on the center body. In contrast to \( \alpha_{c,b} \), \( \alpha_{c,s} \) can be easily maintained a constant by varying the control panel areas. The control panel area for any altitude will be given by,

\[
A = \frac{\alpha_{c,s} \eta^2 \text{trace } \mathcal{G}}{(\frac{1}{2} \rho V_R^2) \| \textbf{b} \|}
\]

Note that in the above expression \( \rho \) varies by an order of magnitude over the altitude range of interest. But with the current available technology in deployable and retractable structures it should not be too difficult to vary the control panel areas by orders of magnitude. Such variable area control panels will lead to better performance at higher altitudes in comparison with fixed area control panels.

4.2.5 Choice of Weighting Matrices Q and R:

In this section some general remarks will be made regarding the rationale used for choosing \( Q \) and \( R \). In general \( Q \) and \( R \) should be chosen so as to make the resulting feedback control system meet certain performance requirements. The relationship between the chosen values of \( Q \) and \( R \) and the performance of the system is not known very well. But certain general remarks can be made all the same. If the weights on the elements of the state vector (elements of \( Q \)) are made larger in comparison with the weights on the elements of the control vector (elements of \( R \)), the resulting system will be stiff and the natural frequencies
will be higher than in the case when the elements of $Q$ and $R$ are of comparable value. Such a stiff system will have high gains and small damping times but if the controls were to be limited to a certain region, the initial states from which acquisition is possible becomes very small. Of the elements of the $Q$ matrix, the larger the weights on the rate variables the higher the damping. Further it is possible to selectively give more weight to some state and/or control variables to improve the performance in certain modes alone.

V. COMPARISON WITH OTHER SYSTEMS AND CONCLUDING REMARKS

The performance characteristics and capabilities of various satellite attitude control systems have been compared in Sabroff's paper (Ref. 50). Comparison of the present system with other systems on the basis of the various characteristics enunciated in the above paper is possible only if a fairly complex simulation of the overall system is carried out. Consequently only the general range of performance of the present system will be indicated. In the present case stabilization is possible only about the nominal attitude whereas in mass expulsion type (with or without momentum storage devices) systems, stabilization about arbitrary attitude is possible. Initial attitude angles from which acquisition is possible are fairly large, but the allowable initial rates are relatively small. This compares favourably with most systems except momentum storage-mass expulsion systems. Acquisition times can be made as small as $1/3$ orbit which compares favourably with all other systems except mass expulsion systems. The present system requires attitude sensing and attitude rate information about all three axes. Because of the feedback control scheme utilized, the range of orientation accuracy is expected to be reasonably good. Control cost involves only the power required to rotate the control panels and does not involve any fuel expenditure.

The results of this study indicate that it is possible to exploit the aerodynamic forces acting on a satellite orbiting at certain altitudes to actively control the attitude of the satellite in an earth pointing mode. The density variations due to the diurnal bulge have been shown to be of no serious consequence in devising a suitable feedback control system. Further it has been shown that the extreme variations in density with altitude can be handled by either a gain scheduling scheme or by providing a set of control panels whose areas are varied suitably with altitude. Realistic values of surface accommodation coefficients have been used in the computations. Numerical results indicate that the effect of off-design values of surface accommodation coefficients do not cause any serious problems. The need for having variable gain feedback matrices for nonequatorial orbits has been shown to be unnecessary. The evaluation of the effect of having the panels for aerodynamic stabilization on the lifetime of the satellite has been carried out and the results indicate that lifetimes of the order of two to three years are possible.

A great deal of work needs to be done before the present system can be implemented in practice. To assess the capabilities of the system it is essential to carry out a complete simulation of the system. Such a simulation will be very valuable in assessing the following.
1) The effect of the flexibility of the control panels.

2) The effect of disturbing torques due to other environmental sources.

3) The effect of the ellipticity of the orbit.

4) The effect of nonlinearities.

Synthesis of a nonlinear controller which takes into account the nonlinearities in the lift and drag forces merits investigation since such a controller can be expected to give improved performance. A hybrid system, involving momentum storage devices, which utilizes the aerodynamic torques for momentum management might prove to be very attractive. Such a system will have the high pointing accuracy capabilities of momentum storage-mass expulsion type systems without the necessity of fuel for momentum management.
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The utilization of aerodynamic torques for satellite attitude control depends to a large extent on the proper characterization of the properties of the atmosphere and the interaction of the control surfaces of the satellite with the atmosphere. The properties of the atmosphere are highly variable and in this appendix the problems involved in characterizing the atmosphere will be discussed. A satellite in an eccentric orbit loses a certain amount of energy per orbit when it passes through the denser layers of the atmosphere at perigee. This leads to a contraction in orbit and consequently the apogee height decreases till the eccentricity becomes essentially zero. For a satellite inserted into an elliptic orbit the life time is defined as the time taken to reach $e = 0$ condition. For circular orbits the life time is defined as the time taken for the satellite to have its period $T$ reduced to 87 minutes (which corresponds to 130 Km altitude) since most satellites make very few resolutions after this time. The initial altitude is governed by the required life time (Appendix C) and in the present instance also by the control torques that can be generated by utilizing the aerodynamic forces. The attention will be focussed to the properties of the atmosphere which are relevant to our study, for the altitude range of 100 Km - 700 Km.

The atmosphere is composed of gases which are far from being perfect at these altitude ranges. A true characterization of the atmosphere should account for all the dynamic physical and chemical processes taking place in the atmosphere at all points inside the gaseous envelope of earth (which we choose to term as atmosphere) for all times. A quick look at some of the dynamic processes taking place in the earth's atmosphere and its causes should convince that it may not be possible in the foreseeable future to have a complete description of the atmosphere. The variations in the physical properties of the atmosphere with altitude, latitude, longitude, and time has been subjected to intensive study in recent years (Refs. 24, 28, 35, 38, 64, 70). For our purpose we are interested only in those properties of the atmosphere, namely, density, temperature (kinetic), mean molecular weight and particle speed (mean molecular velocity: function of temperature and molecular weight), which affect the aerodynamic torque and drag experienced by the satellite. Of these properties only the density governs the forces and torques acting on the satellite directly.

The atmospheric densities are highly variable in the altitude range of interest. The principal variations can be classified as,

1) Diurnal variations: Regular day and nighttime variations over a 24 hour period caused by ultra-violet, X-ray and corpuscular radiation.

2) Regular semi-annual density variations caused by solar wind effect.

3) Regular atmospheric density variations that occur over an eleven year period due to the solar-cycle effect.

4) Irregular geomagnetic activity effects, due to magnetic storms arising from corpuscular radiation.
For our present study the diurnal variation is the most important one. All the above variations depend on altitude, being more pronounced at higher altitudes and almost negligible below 200 Km. The periods of all regular variations in density except the diurnal variation is of the order of several hundreds of orbital periods. On the other hand a satellite in orbit will experience the diurnal density variation as a variation in density with a period approximately equal to the orbital period. Equivalently, an observer located at the center of the sun would observe the earth orbiting around the sun with the atmosphere of the earth always bulging towards the sun. Consequently, for circular orbits of a given altitude a satellite in an equatorial orbit will experience maximum density variations and a satellite in a sun-synchronous orbit will experience minimum density variations. For circular orbits below 200 Km the density variations due to the atmospheric bulge will be negligible. Figure A.1 shows the plot of altitude versus the diurnal density variation factor

\[ k = \frac{\rho_{\text{Day}} - \rho_{\text{Night}}}{\rho_{\text{mean}}} \]

which is a measure of the maximum possible density variation experienced by a satellite in a circular orbit at the given altitude, the minimum value of \( k \) being nearly zero for orbits normal to the earth-sun line. Figure A.2 shows the diurnal density variations at 350 Km during medium solar activity.

The variation of atmospheric properties with altitude has been studied by scientific organizations in several countries and several standard atmospheres have been proposed. These models do not account for the periodic variations in the atmospheric properties but provide only an approximation to the variation of the mean values with altitude. These standard atmosphere models get updated periodically as more measurements from satellites become available. Figure A.3 shows the atmospheric properties of interest obtained from the U.S. standard atmospheres of 1956, 1959 and 1962. Figure A.4 shows the speed ratio \( S \) which is the ratio of the satellite orbital speed to particle speed as function of altitude based on the 1962 U.S. standard atmosphere. For altitudes of interest \( S \) lies in the range of 5.0 - 10.0.

From Fig. A.1 it is obvious that for altitudes in excess of 300 Km the diurnal density variations in any orbit other than near sun-synchronous orbits will be too large. Figure A.3 indicates that the infinite speed ratio assumption does not correspond to reality.
APPENDIX B: AERODYNAMIC ANALYSIS

The aerodynamic forces acting on satellites orbiting below 700 Km warrant careful study since the drag force acting on the satellite leads to orbital decay and the aerodynamic moments acting on the satellite affect the attitude of the satellite. In the altitude range of 150-700 Km the Knudsen Number for satellites which are several meters in size is greater than 10 and consequently the continuum hypothesis breaks down and the molecular nature of the atmosphere must be recognized. The forces and moments acting on satellites under such free molecular flow conditions can be evaluated with the aid of suitably defined surface interaction parameters called accommodation coefficients.

The energy, tangential momentum and normal momentum accommodation coefficients are defined as,

\[
\bar{a} = \frac{E_i - E_r}{E_i - E_w}
\]

\[
\sigma = \frac{\tau_i - \tau_r}{\tau_i - \tau_w}
\]

\[
\sigma' = \frac{p_i - p_r}{p_i - p_w}
\]

where \(E\) represents energy flux, \(\tau\) and \(p\) represent the tangential and normal momenta respectively. The subscripts \(i\) and \(r\) represent the incident and reflected quantities respectively whereas subscript \(w\) represents the reflected quantities when the incident molecules are being reemitted in Maxwellian equilibrium with the surface. \(\bar{a}\), \(\sigma\) and \(\sigma'\) vary from 0 to 1. Diffuse reflection occurs when \(\bar{a} = \sigma = \sigma' = 1\) and specular reflection corresponds to \(\bar{a} = \sigma = \sigma' = 0\). It is to be noted that these coefficients are useful engineering parameters, but they provide an incomplete description of the boundary conditions for kinetic theory and are inadequate for concave geometries. The basic assumption for theoretical free molecular flow calculations is that the flow of incident molecules is undisturbed by the presence of the body. This allows one to calculate the aerodynamic heating and force characteristics by considering separately the incident and reflected flows of the molecules since the above assumption also implies that the incident and reflected molecules do not interact.

With the conventional approximations for a molecular stream in Maxwellian equilibrium, the normal and tangential components of the momentum flux per unit area may be written as (Ref. 55 and 56, e.g.)

\[
\tau_i = \frac{\rho v^2}{2\sqrt{\pi}} \left\{ e^{-(S \sin \theta)^2} + \sqrt{\pi} (S \sin \theta) \right\} \left[ 1 + \text{erf} (S \sin \theta) \right]
\]

\[
\tau_r = \frac{\rho v^2}{2\sqrt{\pi}} \left\{ e^{-(S \sin \theta)^2} + \sqrt{\pi} (S \sin \theta) \right\} \left[ 1 + \text{erf} (S \sin \theta) \right]
\]
where $S$ is the molecular speed ratio and $\theta$ is the angle of attack of the surface element (see Fig. B.1a).

Using B.1 one obtains,

$$P = P_i + P_r = (2-\sigma') P_i + \sigma' P_w$$

$$\tau = \tau_i - \tau_r = \sigma \tau_i$$

$P_w$ is given by (Ref. 55 and 56, e.g.)

$$P_w = \frac{P}{2} \sqrt{2\pi RT_w} \sqrt{\frac{RT}{2\pi}} \left\{ e^{-\left(S\sin\theta\right)^2} + \sqrt{\pi} \left(S\sin\theta\right) \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

(B.3)

By combining the above equations, the following expressions for the total pressure and shear coefficients can be easily obtained.

$$C_p = \frac{P}{2\rho V^2} = \frac{1}{S^2} \left\{ \left(S - \sigma' S\sin\theta + \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} \right) e^{-\left(S\sin\theta\right)^2} + \left[(2-\sigma')(S^2\sin^2\theta + \frac{1}{2}) + \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} (S\sin\theta)\right] \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

$$C_\tau = \frac{\tau}{2\rho V^2} = \frac{\sigma \cos \theta}{S} \left\{ e^{-\left(S\sin\theta\right)^2} + \sqrt{\pi} \left(S\sin\theta\right) \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

(B.4)

The pressure coefficient depends on the temperature ratio $T_w/T$ which is small for orbital altitudes under consideration.

For a flat plate at angle of attack $\theta$ with both surfaces exposed to the flow (Fig. B.1b),

for the front side,

$$P_F = \frac{\rho V^2}{2S^2} \left\{ \left(S - \sigma' S\sin\theta + \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} \right) e^{-\left(S\sin\theta\right)^2} + \left[(2-\sigma')(S^2\sin^2\theta + \frac{1}{2}) + \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} (S\sin\theta)\right] \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

$$\tau_F = \frac{\sigma \rho V^2 \cos \theta}{2S \sqrt{\pi}} \left\{ e^{-\left(S\sin\theta\right)^2} + \sqrt{\pi} \left(S\sin\theta\right) \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

By replacing $\theta$ by $(180 + \theta)$ in the above expressions, the resulting expressions for the rear side are,

$$P_R = \frac{\rho V^2}{2S^2} \left\{ \left(S - \sigma' S\sin\theta + \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} \right) e^{-\left(S\sin\theta\right)^2} + \left[(2-\sigma')(S^2\sin^2\theta + \frac{1}{2}) - \frac{\sigma'}{2} \sqrt{\frac{RT}{T}} (S\sin\theta)\right] \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

$$\tau_R = -\frac{\sigma \rho V^2 \cos \theta}{2S \sqrt{\pi}} \left\{ e^{-\left(S\sin\theta\right)^2} - \sqrt{\pi} \left(S\sin\theta\right) \left[1 + \text{erf} \left(S\sin\theta\right)\right]\right\}$$

Hence for a flat plate exposed to flow on both sides, the pressure
coefficient \( C_p \), the shear coefficient \( C_t \), and the drag coefficient \( C_d \) (based on area of one side of the flat plate being the reference area) are given by,

\[
C_p = \frac{P_T - P_a}{\frac{1}{2} \rho V^2} = \frac{2(2-\sigma')}{\sqrt{\pi} S} \sin \theta e^{-\left(\sin \theta\right)^2} + \left\{(2-\sigma') \sin^2 \theta + \frac{2-\sigma'}{S^2}\right\} \text{erf} \left(\frac{S \sin \theta}{\sqrt{T}}\right) + \frac{\sigma'}{S} \sqrt{\frac{\pi T}{T}} \sin \theta
\]

\[
C_t = \frac{\tau_T - \tau_a}{\frac{1}{2} \rho V^2} = \frac{2\sigma}{\sqrt{\pi} S} \cos \theta \left\{ e^{-\left(\sin \theta\right)^2} + \frac{\sqrt{T}}{S} \sin \theta \text{erf} \left(\frac{S \sin \theta}{\sqrt{T}}\right) \right\}
\]

\[
C_d = C_p \sin \theta + C_t \cos \theta = \left\{(2-\sigma') \frac{\sin^2 \theta}{\sqrt{T}} + \frac{2\sigma}{\sqrt{\pi} S} \cos^2 \theta\right\} e^{-\left(\sin \theta\right)^2} + \frac{\sigma'}{S} \sqrt{\frac{\pi T}{T}} \sin^2 \theta
\]

\[+ 2 \sin \theta \left\{(2-\sigma') \left(\sin^2 \theta + \frac{1}{2S^2}\right) + \sigma \cos^2 \theta\right\} \text{erf} \left(\frac{S \sin \theta}{\sqrt{T}}\right)\]

\( (B.5) \)

B.1 Control Panels:

The control panels are essentially flat plates with both surfaces exposed to the flow. In the body fixed coordinate system let \( v_x^i, n_x^i, \) and \( t_x^i \) denote the unit vector along the velocity vector, unit normal to the \( i \)th control surface and a unit vector lying at the intersection of the plane containing \( n_x^i \) and \( v_x^i \) with the control surface respectively. If \( \alpha_i \) denotes the angle between \( v_x^i \) and \( n_x^i \) (Fig. B.1.c), the following results can be easily deduced (note \( \alpha_i \) is not the angle of attack).

For the \( i \)th control panel of area \( A_i \), deleting the superscript 7,

\[
B_i \cdot v_x^i = \cos \alpha_i, \quad t_x^i = -\cot \alpha_i n_x^i + \csc \alpha_i v_x^i = \frac{(B_i \cdot v_x^i) \times n_x^i}{(B_i \cdot v_x^i) \times n_x^i}, \quad P_i = \frac{1}{2} \rho v_x^i A_i C_p n_x^i
\]

\[
T_i = \frac{1}{2} \rho v_x^i \cos \alpha_i C_t \quad B_i = \frac{1}{2} \rho v_x^i \cos \alpha_i C_d
\]

\[
C_p = \frac{2(2-\sigma')}{\sqrt{\pi} S} \cos \alpha_i \left(\cos a_i \right)^2 + \left\{ 2(2-\sigma') \cos^2 a_i + \frac{2-\sigma'}{S^2}\right\} e^{-\left(\cos \alpha_i\right)^2} + \frac{\sigma'}{S} \sqrt{\frac{\pi T}{T}} \cos^2 \alpha_i
\]

\[+ 2 \cos \alpha_i \left\{(2-\sigma') \left(\cos^2 a_i + \frac{1}{2S^2}\right) + \sigma \cos^2 a_i\right\} \text{erf} \left(\frac{S \cos \alpha_i}{\sqrt{T}}\right)\]

\( (B.6) \)

For the orbital altitudes under consideration the factor \( \frac{1}{2} \sqrt{\frac{T}{T}} \) is very small and hence can be neglected. Therefore

\[
C_p = \frac{2(2-\sigma')}{\sqrt{\pi} S} \cos \alpha_i \left(\cos a_i \right)^2 + \left\{ 2(2-\sigma') \cos^2 a_i + \frac{2-\sigma'}{S^2}\right\} e^{-\left(\cos \alpha_i\right)^2}
\]

\[
C_t = \frac{2\sigma}{\sqrt{\pi} S} \sin \alpha_i \left\{ e^{-\left(\cos \alpha_i\right)^2} + \frac{\sqrt{T}}{S} \cos \alpha_i \left[\text{erf} \left(\frac{S \cos \alpha_i}{\sqrt{T}}\right)\right]\right\}
\]

\[
C_d = \left\{ 2(2-\sigma') \cos^2 a_i + \frac{2\sigma}{\sqrt{\pi} S} \sin^2 a_i\right\} e^{-\left(\cos \alpha_i\right)^2} + 2 \cos \alpha_i \left\{(2-\sigma') \left(\cos^2 a_i + \frac{1}{2S^2}\right) + \sigma \sin^2 a_i\right\} \text{erf} \left(\frac{S \cos \alpha_i}{\sqrt{T}}\right)\]

\( (B.7) \)

If the infinite speed ratio approximation is made \( (B.7) \) simplifies to,

B3
\[ C_{p_i} = 2 \left( 2 - \sigma' \right) \cos \alpha_i \left| \cos \alpha_i \right| \]
\[ C_{\tau_i} = 2 \sigma \sin \alpha_i \left| \cos \alpha_i \right| \]
\[ C_{D_i} = 2 \left( 2 - \sigma' - \sigma \right) \cos^2 \alpha_i \left| \cos \alpha_i \right| + 2 \sigma \left| \cos \alpha_i \right| \tag{B.8} \]

(B.8) is a good approximation only when \( S \cos \alpha_i \) is large. The speed ratio \( S \) lies in the range of 5 to 10. Consequently (B.8) is useful only when \( \alpha_i \) is small, which corresponds to the control surface being normal to the incident stream. Since one is interested in keeping the control surface at low angles of incidence to reduce the drag due to the control panels (B.8) is not useful. Further these expressions lead to discontinuities in the derivatives of the torque and drag expressions and thus lead to numerical problems. Consequently it is advisable to use the expressions given by (B.7) and these are shown plotted as a function of \( \alpha_i \) for various values of accommodation coefficients and speed ratios of interest in Fig. B.2.

**B.2 Center Body:**

Considering an elemental surface area \( dA \), the position vector of its center being \( \mathbf{r} \) in \( \mathbb{R}^7 \) coordinate system one can easily obtain the following:

\[ \mathbf{r} = \frac{1}{2} \rho V_k^2 C_p \mathbf{n} \]
\[ \mathbf{z} = \frac{1}{2} \rho V_k^2 C_{\tau} \mathbf{t} \]
\[ C_p = \frac{2 \left( 2 - \sigma' \right) \cos \alpha }{ \sqrt{\pi} S } \left( \cos \alpha \right)^2 \left\{ \left( \frac{2 \left( 2 - \sigma' \right) \cos \alpha \left( \cos \alpha \right)^2 + \frac{2 - \sigma'}{S^2} \right) \right\} \operatorname{erf} \left( S \cos \alpha \right) \]
\[ C_{\tau} = \frac{2 \sigma }{ \sqrt{\pi} S } \sin \alpha \left\{ e^{-\left( \cos \alpha \right)^2} + \sqrt{\pi} S \cos \alpha \left[ \operatorname{erf} \left( S \cos \alpha \right) \right] \right\} \]
\[ \cos \alpha = \mathbf{n} \cdot \mathbf{v} \]
\[ \mathbf{t} = \frac{ \left( \mathbf{n} \times \mathbf{v} \right) \times \mathbf{n} } { \left| \left( \mathbf{n} \times \mathbf{v} \right) \times \mathbf{n} \right| } \]

where \( \mathbf{n} \) is the unit inward normal of the elemental area in \( \mathbb{R}^7 \) system. Then the net aerodynamic force on the center body is given by

\[ \mathbf{F}_A = \int_{A_w} \left( \mathbf{F} + \mathbf{z} \right) dA \]

where \( A_w \) is the "wetted" surface area.

\[ \mathbf{G}_A = \int_{A_w} \mathbf{F} \times \left( \mathbf{F} + \mathbf{z} \right) dA \]

These results can be applied to any convex bodies. Davison (Ref.7) made a detailed study of the drag forces and moments on body shapes such as
sphere, cone, cylinder, hemisphere-cylinder etc. From his results we can conclude that for bodies with one axis of symmetry (the symmetry axis being nearly along the direction of motion) whose \( l/D > 1 \) the drag coefficient is approximately 2.2. The pitching moment coefficient is a linear function of angle of attack for angles less than around 20\(^\circ\). Further the slope of the pitching moment curve can be arbitrarily fixed within a fairly wide range by suitably fixing the location of the c.m. The rolling moment is zero because of symmetry. Because of the above conclusions the moments acting on the center body can be treated for our purpose as follows.

For the center body for small angles the moment coefficient \( C_m \) is a function of body shape and the location of c.m.

\[
\Lambda = \cos^4(-\frac{\theta}{2}) \frac{j_1^7}{|j_1^7|} \quad \text{where } j_1^7 \text{ is the unit vector along the symmetry axis.}
\]

Then the aerodynamic moment on the center body is given by

\[
G_{A_{CB}} = \frac{1}{2} \rho V_R^2 S_A L C_{m_\Lambda} \cos \left\{ \frac{-\theta}{2} \frac{j_1^7}{|j_1^7|} \right\} (Y_{R} \times j_1^7)
\]

(B.11)

where
- \( C.B. \) - center body,
- \( S_A \) - characteristic cross-sectional area of the body
- \( L \) - characteristic length of the body.
APPENDIX C: SATELLITE LIFETIMES

In this appendix a simplified analysis of the lifetime of an aerodynamically stabilized satellite will be carried out. A longer useful life of a satellite increases the cost effectiveness of the satellite. For near-earth satellites the major perturbation leading to the orbital decay of the satellites is the atmospheric drag. The estimation of the orbital lifetimes of near-earth satellites has been a subject of intense study in recent years (e.g. Refs. 28, 29, 62, 71). The aim of the analysis in this section will be to obtain gross estimates of the lifetimes of aerodynamically stabilized satellites and to demonstrate that it is possible to obtain a reasonable useful lifetime for such satellites.

In the analysis, the following simplifying assumptions are made:

1) Since the satellite is attitude stabilized, the effective cross-sectional area of the center-body is assumed to be the cross-sectional area normal to the axis of symmetry.

2) The drag coefficient of the center-body is assumed to be 2.2.

3) The oblateness of the atmosphere and the rotation of the atmosphere are not considered.

In view of assumption (3), only the lifetimes of a satellite in equatorial orbits for which the nominal control angles are all zero will be estimated. For nonequatorial orbits, not all the nominal control angles are zero due to the rotation of the atmosphere (see Sec. 3.3). Consequently for nonequatorial orbits the lifetimes are bound to be lower than that for an equatorial orbit. A qualitative study of the effect of nonzero nominal control angles on the lifetime of the satellite will be made in the present study by setting the vertical control panel angles at some fixed value and observing its effect on the lifetime.

For near circular orbits, from Lagrange's planetary equations, the following equation for the time rate of change of the orbital radius can be easily obtained (see Ref. 29).

\[ \frac{dR}{dt} = - \frac{C_D A_e}{m} \rho \sqrt{\mu R} \]  

(C.1)

where

- \( R \) = radius of the satellite,
- \( C_D \) = total satellite drag coefficient,
- \( A_e \) = effective cross-sectional area of the satellite
- \( m \) = mass of the satellite
- \( \rho \) = density of the atmosphere.

For the configuration under study, using the notation previously introduced.
\[ C_D A_e = C_D A_S + A \sum_{i \leq 4} \left\{ e^{-\left(\frac{2(2-\sigma'-\sigma)}{\sqrt{\mu}}\right)} \cos^2 \alpha_i + \frac{z^2}{\sqrt{\mu}} \right\} + \text{arctan}\left(\frac{2(2-\sigma'-\sigma)}{\sqrt{\mu}}\right) \cos \alpha_i \]  

\[(C.2)\]

where, as per assumption (2) above \(C_D\) - drag coefficient of the center-body is taken as 2.2.

Inverting (C.1),

\[ \frac{dt}{dR} = \frac{m}{C_D A_e \rho} \cdot \frac{1}{\sqrt{\mu R}} \]  

\[(C.3)\]

For a satellite placed in a circular orbit of \(R_0\), the lifetime \(T_L\) is given by

\[ T_L = \int_{R_e}^{R_0} \frac{m}{C_D A_e \rho} \cdot \frac{1}{\sqrt{\mu R}} \ dR \]  

\[(C.4)\]

where \(R_e\) is the mean radius of the earth. The above lifetime corresponds to the total time elapsed before the satellite hits the earth's surface. Introducing altitude as the independent variable, (C.4) becomes,

\[ T_L = \int_{0}^{h_0} \frac{m}{C_D A_e \rho} \cdot \frac{1}{\sqrt{\mu (R_e + h)}} \ dh \]  

\[(C.5)\]

where \(h_0\) is the initial orbital altitude.

The above integral was evaluated using density values from 1962 U.S. Standard Atmosphere. Figure C.1 shows a plot of lifetime as a function of orbital altitude for configuration \(A\) with the control surface area as a parameter. The \(A = 0\) curve corresponds to the satellite with a non-aerodynamic attitude mechanism and no control panels. \(A = 5.0 \text{ m}^2\) curve corresponds to the configuration for which numerical results of the response of the attitude control system are presented earlier in this report. Figure C.2 shows a plot of lifetime as a function of orbital altitude for configuration \(A\) with \(A = 5 \text{ m}^2\) and for various values of constant vertical control panel angles. From the results presented in Fig. C.1 it can be concluded that for a given initial altitude the lifetime decreases as the control panel area is increased. On the other hand the increased control panel area leads to an increase in the upper bound of the altitude range in which acceptable performance can be achieved thus leading to an overall increase in the lifetime of the satellite. Consequently the results of the above elementary analysis presented in Figs. C.1 and C.2 indicate that it is possible to attain lifetimes of the order of 2 to 3 years for a satellite with an aerodynamic attitude control system of the type synthesized in this report with reasonably sized control panels.
APPENDIX D: FUNCTION MINIMIZATION WITH CONSTRAINTS

In this section some of the available methods for minimizing a function of several variables subject to a set of constraints will be reviewed. Constrained minimization of multivariable, nonlinear functions has been extensively studied in recent years (Refs. 4, 15, 39, 48, 61, e.g). The problem on hand can be stated as follows.

To minimize \( f(x) \) subject to a set of constraints
\[ g(x) = 0 \]
where
\[ x = [x_i] \quad i = 1, 2, \ldots n \]
\[ g = [g_j] \quad j = 1, 2, \ldots m, \quad m < n. \]  
(D.1)

In general, except in the case of very simple problems computerized numerical methods are necessary to determine the optimal value of \( x \) denoted by \( x^* \) which satisfies the following relations:

\[ f(x^*) \leq f(x) \text{ for all } x \text{ such that } g(x) = 0 \]
\[ g(x^*) = 0 \]  
(D.2)

Among the several methods available, the penalty function approach, the first-order gradient method and the second-order gradient method will be discussed here.

D.1: Penalty Function Approach: In this method the constrained minimization problem is replaced by an approximately equivalent unconstrained minimization problem. Thus instead of solving the given problem (D.1) we minimize a "penalty function" defined by

\[ F = f(x) + \sum_{j=1}^{m} K_j g_j^2(x) \]  
(D.3)

Without loss of generality we can choose \( K_j = K \) for \( j = 1, \ldots m \).

Hence
\[ F = f(x) + K \| g(x) \|^2 \]  
(D.4)

If \( x^* \) represents the value of \( x \) for which \( F(x^*) \leq F(x) \), for all \( x \), \( K \) being large, then \( f(x^*) \approx f(x^0) \) and \( g(x^*) \approx 0 \).

Further
\[ \lim_{K \to \infty} x^* = x^0 \quad \text{and} \quad \lim_{K \to \infty} f(x^*) = f(x^0) \]

The determination of \( x^* \) forms an unconstrained function minimization problem and this can be solved by one of the several well known methods such as the gradient method, the simplex method (Ref. 39) etc.

For example, use of the gradient method is presented below.

Define \( \Delta = \left[ \frac{\partial F}{\partial x} \right]^T \), guess an initial value \( x = x^{(0)} \).
Then \( x^{(i+1)} = x^{(i)} - h \frac{A^{(i)}}{A^{(i)}} \) where \( 0 < h < 1 \) is the step size.

The stopping criterion can be one of the following:

\[ |F^{(i+1)} - F^{(i)}| < \delta \]

or \( |x^{(i+1)} - x^{(i)}| < \varepsilon \) for suitably chosen (small) values of \( \delta \), and \( \varepsilon \).

The penalty function approach is computationally appealing and involves simple computations. On the other hand it has its own disadvantages. It creates steep narrow valleys where search techniques fail and further may create artificial minima not present in the original problem.

D.2: First-Order Gradient Method: This method is also known as the method of steepest descent. It is characterized by iterative algorithms for improving the estimates of \( x^0 \). The method is outlined below.

Partition \( x \) as \( x = [u, v] \) where \( u = [x_i], i = 1, 2, \ldots, m \)

\( v = [x_i], i = m+1, \ldots, n \)

Hence \( u = [u_i], i = 1, 2, \ldots, m \)

\( v = [v_i], i = 1, 2, \ldots, (n-m) \)

The steps involved are,

1) guess a \( y^{(0)} \)

2) satisfy the constraints:

Algorithm: \( u^{(i+1)} = u^{(i)} - h \frac{A^{(i)}}{A^{(i)}} \) \( f^{(i)} \)

\( = u^{(i)} - k J^{(i)} g^{(i)} \) (saves computation time).

Stopping criterion: \( \|f^{(i)}\| < \varepsilon \) or \( \|u^{(i+1)} - u^{(i)}\| < \delta \)

where \( J = \frac{\partial g^{(i)}}{\partial u} \)

\( h \) is the step size.

3) Define \( F = f(u, v) + \lambda^{T} g(u, v) \)

Determine \( \lambda \) from \( \lambda^{T} = - \frac{\partial f^{(i)}}{\partial y^{(i)}} [J^{(i)}]^{-1} \)

4) Compute \( \frac{\partial F^{(i)}}{\partial u^{(i)}} = \frac{\partial f^{(i)}}{\partial u^{(i)}} + \lambda^{T} \left( \frac{\partial g^{(i)}}{\partial u^{(i)}} \right) \)

5) Determine an improved estimate of \( y^0 \) by

\( y^{(i+1)} = y^{(i)} - k \left[ \frac{\partial F^{(i)}}{\partial y^{(i)}} \right] \)

where \( k \) is the step size.
6) Repeat steps (2) through (5) until one of the following stopping criteria is satisfied.

\[
\begin{aligned}
\left\| x^{(t+1)} - x^{(t)} \right\| &< \delta \\
\left\| p^{(t+1)} - p^{(t)} \right\| &< \epsilon
\end{aligned}
\]

where, \( P = f + g \frac{T}{T} g \)

The convergence is very rapid in the first few iterations but the convergence is poor as the optimum is approached. The initial values have to be chosen properly to ensure convergence to the optimum but in general the first-order gradient methods do not pose any major problems in this regard.

D.3: Second-Order Gradient Method: In this method the first and second derivatives in the \( v \)-space are utilized. The method is outlined below.

The necessary conditions for the stationarity of \( F = f(u,v) + \lambda^T \theta(u,v) \) 
(\( u, v \) are defined in Sec. D.2) are given by

\[
\begin{aligned}
F_u = \frac{\partial F}{\partial u} &= 0 \\
F_v = \frac{\partial F}{\partial v} &= 0 \\
g = 0
\end{aligned}
\]

1) Guess \( u^{(0)}, v^{(0)}, \lambda^{(0)} \)

2) Determine 
\[
\begin{aligned}
F_u (u^{(0)}, v^{(0)}, \lambda^{(0)}) &= F_u^{(0)} \\
F_v (u^{(0)}, v^{(0)}, \lambda^{(0)}) &= F_v^{(0)} \\
g(u^{(0)}, v^{(0)}) &= g^{(0)}
\end{aligned}
\]

3) Linearizing the necessary conditions about \( u^{(0)}, v^{(0)}, \lambda^{(0)} \)

\[
\begin{aligned}
F_u^{(0)} + F_u^{(0)} du + F_v^{(0)} dv + g_u^{(0)} d\lambda &= 0 \\
F_v^{(0)} + F_u^{(0)} du + F_v^{(0)} dv + g_v^{(0)} d\lambda &= 0 \\
g^{(0)} + g(u^{(0)}, v^{(0)}) du + g_v^{(0)} dv &= 0
\end{aligned}
\]

4) Improved values of \( u, v \) and \( \lambda \) are obtained from

\[
\begin{bmatrix}
u^{(i+1)} \\
v^{(i+1)} \\
\lambda^{(i+1)}
\end{bmatrix}
= \begin{bmatrix}
u^{(i)} \\
v^{(i)} \\
\lambda^{(i)}
\end{bmatrix}
- \Gamma^{-1} \begin{bmatrix}
F_u^{(i)} \\
F_v^{(i)} \\
g^{(i)}
\end{bmatrix}
\]

where, \( \Gamma = \begin{bmatrix}
F_{uu} & F_{uv} & g_u^{T} \\
F_{vu} & F_{vv} & g_v^{T} \\
g_u & g_v & 0
\end{bmatrix}\)

Define \( \omega = \begin{bmatrix}\omega_u \\ \omega_v \end{bmatrix} \), \( x = [x_i], i = 1, 2, ..., n \); \( \lambda = [\lambda_j], j = 1, 2, ..., m \)

\( \omega = [\omega_i], i = 1, 2, ..., m + n \)

\( \Delta = \begin{bmatrix}
F_u^{T} \\
F_v^{T} \\
g_v
\end{bmatrix} \), \( g_x \)
Then the above algorithm may be restated as
\[ \omega^{(i+1)} = \omega^{(i)} - f^{-1}(t) \Delta^{(i)} \]

5) The stopping criterion can be one of the following.
\[ \| \omega^{(i+1)} - \omega^{(i)} \| < \delta \]
\[ \| p^{(i+1)} - p^{(i)} \| < \epsilon \]
where \( P = f + G^T g \)

Since in the second-order gradient method the information contained in the first and second derivative are utilized it has very good convergence characteristics as the optimal solution is approached. But the proper choice of a set of starting values is important for the successful application of this method. A combined first and second order gradient method where the first-order gradient method is used during the initial phases of the iterative process and the second-order gradient method is used during the terminal phase (in the neighbourhood of the optimum) is very appealing. In any case the first-order and second-order gradient methods are computationally more complex than the penalty function approach.
APPENDIX E: OPTIMAL REGULATORS

Optimal linear feedback control is one of the most widely studied problems in the field of optimal control in recent years (Refs. 4, 5, 25, 26, 51, 72). Optimal regulators form a sub-class of linear feedback systems. In this appendix a brief review of optimal linear feedback systems will be presented and results applicable to optimal regulators will be deduced.

Consider a linear dynamical system represented by,

\[ \dot{x} = A(t) x + B(t) u \] (E.1)

where

- \( x \) = n-dimensional state vector
- \( u \) = r-dimensional control vector

The initial condition is given by,

\[ x(0) = x_0 \] (E.2)

It is desired to take the system to a terminal state \( x(t_f) = 0 \), where \( t_f \) is the terminal time, by applying a suitable control \( u(t) = f(x, t) \) which is optimal in some sense. Consequently a performance index has to be defined and minimized. Define a quadratic performance index of the form,

\[ J = \frac{1}{2} x^T S_f x + \frac{1}{2} \int_{t_0}^{t_f} [x^T Q(t) x + u^T R(t) u] \, dt \] (E.3)

Without loss of generality, \( Q(t) \), \( R(t) \) and \( S_f \) can be assumed to be symmetric matrices. Further \( S_f \), \( Q(t) \) and \( R(t) \) are respectively positive semidefinite, positive definite matrices with \( Q(t) \) and \( R(t) \) being continuous in \( t \). An appropriate choice of \( S_f \), \( Q(t) \) and \( R(t) \) must be made for obtaining acceptable performance of the system (Ref. 4). Note that to obtain linear feedback law it is not sufficient to have a linear system but it is also necessary to have a cost functional of the form given in (E.3) and further such a cost functional keeps the problem mathematically tractable (see Ref. 72).

The Hamiltonian is given by

\[ H[x(t), u(t), \lambda(t), t] = \frac{1}{2} x^T Q(t) x + \frac{1}{2} u^T R(t) u + \lambda^T [A(t) x + B(t) u] \] (E.4)

By the maximum principle

\[ \frac{\partial H}{\partial u} = 0 = R(t) u(t) + B^T(t) \lambda(t) \] (E.5)

\[ \frac{\partial H}{\partial x} = -\dot{\lambda} = Q(t) x(t) + A^T(t) \lambda(t) \] (E.6)
Terminal condition:

\[ A(t_f) = S_f \mathbf{x}(t_f) \]  \hspace{1cm} (E.7)

Hence from (E.5),

\[ \mathbf{u}(t) = -R^{-1}(t)B^T(t)A(t) \]  \hspace{1cm} (E.8)

Assuming that the solution to \( A(t) \) has the same structure as (E.7), \( A(t) \) can be represented by,

\[ A(t) = S(t) \mathbf{x}(t) \]  \hspace{1cm} (E.9)

Hence,

\[ \dot{\mathbf{x}} = A(t) \mathbf{x}(t) - B(t) R^{-1}(t) B^T(t) S(t) \mathbf{x}(t) \]  \hspace{1cm} (E.10)

\[ \dot{A} = \dot{S} \mathbf{x}(t) + S(t) \dot{\mathbf{x}} = Q(t) \mathbf{x}(t) - A^T(t) S(t) \mathbf{x}(t) \]  \hspace{1cm} (E.11)

Combining (E.10) and (E.11), the following equation can be obtained.

\[ \left\{ \dot{S} + S(t) A(t) + A^T(t) S(t) - S(t) B(t) R^{-1}(t) B^T(t) S(t) - Q(t) \right\} \mathbf{x}(t) = 0 \]

The above equation must hold for all nonzero \( x(t) \) and hence \( S(t) \) is an \((n \times n)\) symmetric matrix satisfying the matrix Riccati equation given by

\[ \dot{S} = -S(t) A(t) - A^T(t) S(t) + S(t) B(t) R^{-1}(t) B^T(t) S(t) - Q(t) \]  \hspace{1cm} (E.12)

with the terminal condition \( S(t_f) = S_f \)  \hspace{1cm} (E.13)

Hence the optimal control law is given by,

\[ \mathbf{u}(t) = -K(t) \mathbf{x}(t) \]  \hspace{1cm} (E.14)

where \( K(t) \) is the \((n \times n)\) feedback gain matrix given by,

\[ K(t) = R^{-1}(t) B^T(t) S(t) \]  \hspace{1cm} (E.15)

The corresponding optimal cost is given by \( J^* = \frac{1}{2} \mathbf{x}(t_f) S_f \mathbf{x}(t_f) \)  \hspace{1cm} (E.16)

For finite values of \( t_f - t_o \) the solution to the Riccati equation will in general be time-varying even in the case of constant coefficient or stationary systems.

E.1: Methods of Solution of the Terminal Controller Problem:

For finite values of \( t_f - t_o \), the problem above is called a terminal controller problem. The various methods of determining the feedback gains for such a terminal controller is given below:
Consider the canonical (Hamiltonian) differential equations given by

\[ \dot{x} = \frac{\partial H}{\partial \lambda} - A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t) \]  
(E.17)

\[ \dot{\lambda} = -\frac{\partial H}{\partial x} = -Q(t)x(t) - A^T(t)\lambda(t) \]  
(E.18)

Initial condition \( x(t_0) = x_0 \)  
(E.19)

Terminal condition \( \lambda(t_f) = S(t_f)x(t_f) = S_fx(t_f) \)  
(E.20)

\( \Lambda(t) = S(t)x(t) \) is to be determined. Since the above differential system is linear, a set of linearly independent solutions can be generated and by linear superposition a solution satisfying the boundary conditions can be obtained.

Let \( X(t) \), \( \Lambda(t) \) be a pair of matrix solutions of the above equations satisfying the terminal conditions \( x(t_f) = I \), \( \Lambda(t_f) = S(t_f) \). Then \( \Lambda(t) = S(t)x(t) \). Let \( \Theta(t,t_f) \) is the transition matrix of the canonical system (E.17) and (E.18), satisfying the terminal conditions,

\[ x(t_f) = I \]
\[ \Lambda(t_f) = I \]  
(E.21)

That is if \( \Theta(t,t_f) = \begin{bmatrix} \theta_{11}(t,t_f) & \theta_{12}(t,t_f) \\ \theta_{21}(t,t_f) & \theta_{22}(t,t_f) \end{bmatrix} \) is the transition matrix of the equation

\[ \dot{\Theta} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & A^T(t) \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \]  
(E.22)

with the terminal condition \( \Theta(t_f) = I \)

then \( S(t) = \left[ \theta_{21}(t,t_f) + \theta_{22}(t,t_f)S(t_f) \right] \left[ \theta_{11}(t,t_f) + \theta_{12}(t,t_f)S(t_f) \right]^{-1} \)  
(E.23)

Hence \( S(t) \) can be computed for \( t_0 \leq t \leq t_f \) and \( K(t) = R^{-1}(t)B^T(t)S(t) \) can be computed and stored for \( t_0 \leq t \leq t_f \). The stored values of \( K(t) \) can be retrieved and used as the gains in the feedback loop when the control is being implemented.

E.1.2: Direct Solution Method: In this method the matrix Riccati equation (E.12) is numerically solved backward from the terminal time \( t = t_f \) with the condition \( S(t_f) = S_f \). From the solution \( S(t) \), \( K(t) \) can be computed.
and stored for \( t_0 \leq t \leq t_f \) and used when required.

E.2: The Regulator Problem:

Definition: A regulator is a feedback control system in which the reference input is constant for long duration, and often for the entire duration of system operation. Succinctly, a regulator corresponds to a terminal controller with \( (t_f-t_o) \rightarrow \infty \).

For general time varying systems it is impractical to solve for, and store \( K(t) \) for \( 0 \leq (t-t_o) \leq \infty \) by either of the above methods. For stationary systems and periodic systems the regulator problem can be solved by suitable methods discussed below.

E.2.1: Time Invariant Systems: In this case \( A \) and \( B \) are constant matrices. If \( R \) and \( Q \) are chosen as constant matrices satisfying all the conditions stated earlier, then the matrix Riccati equation becomes,

\[
\dot{S} = -S(t)A - A^T S(t) + S(t)BR^{-1}B^T S(t) - Q \tag{E.24}
\]

As \( (t_f-t_o) \rightarrow \infty, \dot{S} = 0 = -SA^T S + SB^{-1}B^T S - Q \Rightarrow S(t) \rightarrow S^0 \), where \( S^0 \) is a constant matrix. Hence \( S^0 \) satisfies the equation

\[
SA + A^T S - SBR^{-1}B^T S + Q = 0 \tag{E.25}
\]

The above equation is called the steady state or the algebraic matrix Riccati equation. It can be solved iteratively to determine \( S^0 \) and hence the corresponding constant gain matrix \( K_0 \) (see Ref.3).

Alternatively Kalman (Ref.25) has shown that given any symmetric, nonnegative definite matrix \( C \), the matrix Riccati equation (E.12) has a unique solution \( \pi(t; c, t_f) \) which takes on the value \( C \) at \( t = t_f \), and \( \pi(t; c, t_f) \) exists for all \( t \leq t_f \). Further if the system is completely controllable then \( \lim_{t \rightarrow \infty} \pi(t; 0, t_f) = \dot{S}(t) \) exists for all \( t \) and is a solution of (E.12) and if \( A, B, R \) and \( Q \) are constants \( \lim_{t \rightarrow \infty} \pi(t; 0, t_f) = S^0 \).

Consequently \( S^0 \) can be determined by solving equation (E.24) backwards with the condition \( S(t_f) = 0 \) starting from a large value of \( t = t_f \) until \( \dot{S} \approx 0 \).

E.2.2: Periodic Systems: In this case \( A(t + T) = A(t) \) and \( B(t + T) = B(t) \) where \( T \) is the period. Then as \( (t_f-t_o) \rightarrow \infty, S \neq 0 \). Consequently the following alternative proposition can be made use of.

Proposition: If 1) \( A(t + T) \) and \( B(t + T) = B(t) \), 2) \( R \) and \( Q \) are constant matrices, and 3) the system is completely controllable, then \( \lim_{t \rightarrow \infty} \pi(t; 0, t_f) = \dot{S}(t) \) exists for all \( t \), \( \dot{S}(t+T) = \dot{S}(t) \), and is a solution of (E.12).

Hence for periodic systems equation (E.12) can be solved backwards with the initial conditions \( S(t_f) = 0 \) starting from a large value of \( t = t_f \) until the condition \( \dot{S}(t+T) = \dot{S}(t) \) is satisfied. \( K(t) \)
satisfies the condition $K(t + T) = K(t)$ and is given by $K(t) = R^{-1}B(t)\delta(t)$. $K(t)$ can either be stored for one orbit or a series in orthogonal polynomials can be fitted to $K(t)$ over one period and the coefficients stored. Attempts to prove the above proposition were unsuccessful. But in all the numerical results obtained in the present study the above proposition was found to be true.

E.3: Optimal Control for Linear Inhomogeneous Systems:

In this section an optimal feedback control is obtained for a linear inhomogeneous system based on the quadratic performance criteria. The derivation follows closely that given by Garber (Ref. 75).

Consider the linear inhomogeneous dynamical system represented by

$$\dot{x} = A(t)x + B(t)u + \delta(t) \tag{E.26}$$

where $\delta(t)$ is an $n$-vector

$x$ is an $n$-vector

$u$ is an $r$-vector

The initial condition is given by $x(0) = x_0$.

Define a quadratic performance index of the form

$$J = \frac{1}{2} x^T S_f x + \frac{1}{2} \int_{t_0}^{t_f} [x^T Q(t)x + u^T R(t)u] \text{dt} \tag{E.27}$$

The Hamiltonian is given by

$$H[x(t), u(t); A(t), t] = \frac{1}{2} x^T Q(t)x + \frac{1}{2} u^T R(t)u + A^T [A(t)x + B(t)u + \delta(t)] \tag{E.28}$$

By the maximum principle

$$\frac{\partial H}{\partial y} = 0 = R(t)u(t) + B^T(t) \lambda(t) \tag{E.29}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda} = Q(t)x(t) + A^T(t)\lambda(t) \tag{E.30}$$

Terminal condition:

$$\lambda(t_f) = S_f x(t_f) \tag{E.31}$$

Hence from (E.29),

$$u(t) = R^{-1}(t)B^T(t)\lambda(t) \tag{E.32}$$

Equations (E.26), (E.30) and (E.32) can be combined to obtain,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix} \tag{E.33}$$
The above linear inhomogeneous equation suggests a solution of the form

\[ \Lambda(t) = S(t) x(t) + u(t) \quad , \quad u(t) = -R(t) B^T(t) [S(t) x(t) + u(t)] \]  \hspace{1cm} (E.34)

Since \[ \Lambda(t_f) = S_f x(t_f) \]
\[ u(t_f) = 0 \quad \text{and} \quad S(t_f) = S_f \]

Differentiating (E.34) and substituting (E.26),
\[ \dot{\Lambda} = \dot{x} + S(t) [A(t) x + B(t) u + g(t)] + \dot{u} \]  \hspace{1cm} (E.35)

Substituting (E.34) in (E.30),
\[ \dot{A} = -Q(t) x - A^T(t) [S(t) x + u(t)] \]  \hspace{1cm} (E.36)

Equating (E.35) and (E.36) and collecting terms,
\[ \begin{bmatrix} \dot{x} + S(t) A(t) + A^T(t) S(t) - S(t) B(t) R^{-1}(t) B^T(t) S(t) + Q(t) \end{bmatrix} x + \begin{bmatrix} \dot{u} + A^T(t) u - S(t) B(t) R^{-1}(t) B^T(t) u(t) + S(t) g(t) \end{bmatrix} = 0 \]  \hspace{1cm} (E.37)

Hence \( S(t) \) is the solution of
\[ \dot{s} = -S(t) A(t) - A^T(t) S(t) + S(t) B(t) R^{-1}(t) B^T(t) S(t) - Q(t), \quad (S(t_f) = S_f) \]  \hspace{1cm} (E.38)

and \( u(t) \) is the solution of
\[ \dot{u} = -[A(t) - S(t) B(t) R^{-1}(t) B^T(t)] u(t) - S(t) g(t) \quad , \quad u(t_f) = 0 \]  \hspace{1cm} (E.39)

As in section E.2 if \( t_f \to \infty \) the above problem becomes a regulator problem and method of solution is described below for time-invariant systems and periodic systems.

**E.3.1: Time-Invariant Systems:** \( A, B, R \) and \( Q \) are constant matrices and \( g \) is a constant vector.

Then as \( (t_f - t_o) \to \infty \) \quad \( S(t) \to S^o \) and \( u(t) \to U^o \)
The constant matrix \( S^o \) satisfies the equation
\[ SA + A^T S - S B R^{-1} B^T S + Q = 0 \]

and \( U^o \) satisfies the equation
\[ AU - S B R^{-1} B^T U + S g = 0 \]

Alternatively \( S^o \) and \( U^o \) can be determined by solving equation (E.38) and (E.39) backwards with the conditions \( S(t_f) = 0, \quad U(t_f) = 0 \) starting from a large value of \( t = t_f \) until \( \dot{S} = 0 \) and \( \dot{U} = 0 \).
E. 3.2: Periodic Systems: In this case $A(t + T) = A(t)$, $B(t + T) = B(t)$ and $g(t + T) = g(t)$, where $T$ is the period. Then as $(t_f - t_o) \to \infty$, $S \neq 0$ and $U \neq 0$. The solutions $S(t)$ and $U(t)$ of equations (E.38) and (E.39) respectively can be obtained by solving the above equations backwards with the initial conditions $S(t_f) = 0$, $U(t_f) = 0$, starting from a large value of $t = t_f$ until the conditions $S(t_f + T) = S(t)$ and $U(t + T) = U(t)$ are satisfied.

E. 4: Controllability of Linear Systems: The concept of complete controllability of the linear system given by (E.1) is of central importance and it was tacitly assumed in obtaining the results of the previous sections. In this section some of the results obtained by Kalman (Ref. 25) are presented without proof and it is pointed out that it is not necessary to investigate controllability before attempting to synthesize the feedback control.

Let $\Phi(t; t_o, x_o)$ denote the value of the solution of (E.1) at time $t$, for control vector $u$, and initial state $x$ given at $t_o$. $\phi(t, T)$ will denote a fundamental solution of the homogeneous equation $\dot{x} = A(t) x$ and $\phi(t, t) = I$. With the above notation,

Definition (Kalman): The system (E.1) is completely controllable at $t_o$ if for every initial state $x$, there is a control $u(t)$ depending on $t_o$ and $x$ such that $\Phi(t_f; t_o, x_o) = 0$ for some finite $t_f$.

Theorem (Kalman): The system (E.1) is completely controllable at time $t_o$ if and only if the matrix

$$W(t_o, t_f) = \int_{t_o}^{t_f} \phi(t_o, \tau) B(\tau) B^T(\tau) \phi^T(t_o, \tau) d\tau$$  \hspace{1cm} (E.40)

is nonsingular for some $t_f > t_o$.

The following remarks can be made regarding the above theorem:

1) $W$ is a symmetric positive semi-definite matrix.

2) The condition that $W(t_o, t_f)$ be nonsingular yields a stronger result than required by the definition, that is, any point $x$ can be brought to the origin at time $t_f$, where $t_f$ is independent of $x_o$.

3) If the system (E.1) is completely controllable at $t_o$, then it is completely controllable at any time $t < t_o$ and not necessarily completely controllable at a time $t > t_o$.

For time-invariant systems the following result can be applied.

Corollary (Kalman): If the matrices $A$ and $B$ are constant the system (E.1) is completely controllable if and only if the rank of the matrix $[B, AB, \ldots, A^{n-1}B]$ equals $n$. In this case any point can be controlled to the origin in an arbitrarily small positive interval of time.

It is possible to convert the integral for $W(t_o, t_f)$ given by (E.40) into a differential equation. Thus differentiating (E.40) with respect to $t_o$. 

E7
\[ \frac{\partial W(t_0, t_f)}{\partial t_0} = -\Phi(t_0, t_0) B(t_0) B^T(t_0) \Phi^T(t_0, t_0) + \int_{t_0}^{t_f} \frac{\partial \Phi(t_0, t)}{\partial t_0} B(t) B^T(t) \Phi^T(t_0, t) \, dt \]

\[ + \int_{t_0}^{t_f} \Phi(t_0, t) B(t) B^T(t) \frac{\partial \Phi^T(t_0, t)}{\partial t_0} \, dt \]

But \[ \frac{\partial \Phi(t_0, t)}{\partial t_0} = A(t_0) \Phi(t_0, t) \] and \[ \Phi(t_0, t_0) = I \]

Hence \[ \frac{\partial W(t_0, t_f)}{\partial t_0} = -B(t_0) B^T(t_0) + A(t_0) \int_{t_0}^{t_f} \Phi(t_0, t) B(t) B^T(t) \Phi^T(t_0, t) \, dt \]

\[ + \left\{ \int_{t_0}^{t_f} \Phi(t_0, t) B(t) B^T(t) \Phi^T(t_0, t) \, dt \right\} A^T(t_0) \]

Therefore \[ \frac{\partial W(t_0, t_f)}{\partial t_0} = -B(t_0) B^T(t_0) + A(t_0) W(t_0, t_f) + W(t_0, t_f) A^T(t_0) \quad \text{(E.41)} \]

The terminal condition \( W(t_0, t_0) = 0 \) since if \( W(t_0, t_f) \neq 0 \) it will be possible to transfer any state \( x_0 \) to the origin in zero time.

In the case of regulators \((t_f - t) \to \infty\). Hence to test controllability equation (E.41) can be solved backwards and \( W(t_0, t_f) \) tested for non-singularity. On the other hand equation (E.12) can be solved backwards for \( S(t) \) without bothering to test for controllability. The time taken by the solution \( S(t) \) to reach a limiting value \( (S^0 \text{ in the case of time-invariant systems and } S(t+T) = S(t) \text{ in the case of periodic systems}) \) is a measure of the damping time for the resulting feedback system. Hence if \( S(t) \) does not reach the limiting value within a reasonable value of \((t_f - t)\) it can be concluded that the system will not have useful performance characteristics even though it may be controllable and hence the attempt to synthesize the control may be abandoned.
APPENDIX F: EVALUATION OF DERIVATIVES

In this appendix some of the derivatives necessary for the analysis are obtained. Referring to equations (3.15) and (3.16) of Sec. 3.4, the expressions for the aerodynamic torque on the satellite and the drag due to control surfaces are given by, respectively.

\[ G_\Lambda = \frac{1}{2} \rho v^2 S_A L \left\{ C_{m_\Lambda} \cos^4(-\nu_R \cdot \hat{\nu}_R) \left( \hat{\nu}_R \cdot \hat{\nu}_R \right) \right\} \]

\[ + \frac{A}{S_A L} \sum_{i=1}^{4} \bar{\nu}_i \left[ \text{erf} \left( S \cos \alpha_i \right) \left\{ \frac{2(2-\sigma' - \sigma)}{S^2} \cos^2 \alpha_i + \frac{2-\sigma'}{S^2} \right\} \bar{\nu}_i + 2\sigma \cos \alpha_i \nu_R \right] \]

\[ + \epsilon^{-\left( S \cos \alpha_i \right)^2} \left\{ \left( \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \cos \alpha_i \right) \bar{\nu}_i + \frac{2\sigma}{\sqrt{\pi} S} \nu_R \right\} \right] \]

\[ D = \frac{1}{2} \rho v^2 A \sum_{i=1}^{4} \epsilon^{-\left( S \cos \alpha_i \right)^2} \left\{ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \cos^2 \alpha_i + \frac{2\sigma}{\sqrt{\pi} S} \right\} \]

\[ + \text{erf} \left( S \cos \alpha_i \right) \left\{ 2(2-\sigma' - \sigma) \cos \alpha_i + 2 \left( \frac{2-\sigma'}{2S^2} + \sigma \right) \cos \alpha_i \right\} \]

where \( \cos \alpha_i = \left( \frac{\bar{\nu}_i \cdot \nu_R}{|\bar{\nu}_i|} \right) \)

When \( \theta = \theta_j = 0 \), \( \epsilon = 1 \) and hence \( \nu_R = \nu_R^5 \)

\[ \frac{d}{d\theta^c} \nu_R^5 = 0 \]

Let \( \hat{\nu}_i = \frac{d}{d\theta^i} \nu_i \), \( \beta_i \triangleq (\nu_i \cdot \nu_R^5) \) and \( \gamma_i \triangleq (\hat{\nu}_i \cdot \nu_R^5) \)

\[ G_\Lambda \bigg|_{\theta = 0} = \frac{1}{2} \rho v^2 S_A L \left\{ C_{m_\Lambda} \cos^4(-\nu_R \cdot \hat{\nu}_R) \left( \hat{\nu}_R \cdot \hat{\nu}_R \right) \right\} + \frac{A}{S_A L} \sum_{i=1}^{4} \bar{\nu}_i \left[ \text{erf} \left( S \beta_i \right) \left\{ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i^2 + \frac{2-\sigma'}{S^2} \right\} \right] \]

\[ + 2\sigma \beta_i \nu_R \right] + \epsilon^{-\left( S \beta_i \right)^2} \left\{ \left( \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i \right) \bar{\nu}_i + \frac{2\sigma}{\sqrt{\pi} S} \nu_R \right\} \right] \]

\[ D \bigg|_{\theta = 0} = \frac{1}{2} \rho v^2 S_A L \sum_{i=1}^{4} \epsilon^{-\left( S \beta_i \right)^2} \left\{ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i^2 + \frac{2\sigma}{\sqrt{\pi} S} \right\} \]

\[ + \text{erf} \left( S \beta_i \right) \left\{ 2(2-\sigma' - \sigma) \beta_i^3 + 2 \left( \frac{2-\sigma'}{2S^2} + \sigma \right) \beta_i \right\} \]

\[ \frac{\partial}{\partial \beta^i} \text{erf} \left( S \beta_i \right) = \frac{25}{\sqrt{\pi}} \gamma_i \exp \left( -S^2 \beta_i^2 \right) \]

\[ \frac{\partial}{\partial \beta^i} \exp \left( -S^2 \beta_i^2 \right) = -2S^2 \beta_i \gamma_i \exp \left( -S^2 \beta_i^2 \right) \]

\[ \frac{\partial}{\partial \beta^i} \left\{ \text{erf} \left( S \beta_i \right) \left\{ 2(2-\sigma' - \sigma) \beta_i^2 + \frac{2-\sigma'}{S^2} \right\} \right\} = \]

\[ \frac{25}{\sqrt{\pi}} \gamma_i \exp \left( -S^2 \beta_i^2 \right) \left( 2(2-\sigma' - \sigma) \beta_i^2 + \frac{2-\sigma'}{S^2} \right) + \text{erf} \left( S \beta_i \right) \left\{ 4(2-\sigma' - \sigma) \gamma_i \beta_i \right\} \]
Utilizing (F.5) to (F.13), the following expressions are obtained:

\[
\frac{\partial}{\partial \theta_i} \left\{ \text{erf} (s \beta_i) \cdot 2 \sigma \beta_i \right\} = \frac{\mu \sigma S}{\sqrt{\pi}} Y_i \exp \left( -s^2 \beta_i^2 \right) + \text{erf} (s \beta_i) \cdot 2 \sigma Y_i \\
\frac{\partial}{\partial \theta_i} \left\{ \frac{2 \sigma \exp \left( -s^2 \beta_i^2 \right)}{\sqrt{\pi} S} \right\} = - \frac{\mu \sigma S}{\sqrt{\pi}} Y_i \beta_i \exp \left( -s^2 \beta_i^2 \right) \\
\frac{\partial}{\partial \theta_i} \left\{ \exp \left( -s^2 \beta_i^2 \right) \cdot \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i \right\} = - \frac{2S}{\sqrt{\pi}} Y_i \exp \left( -s^2 \beta_i^2 \right) \left[ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i \right] \\
+ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} Y_i \beta_i \exp \left( -s^2 \beta_i^2 \right) \\
\frac{\partial}{\partial \theta_i} \left\{ \text{erf} (s \beta_i) \cdot \left[ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right] \right\} = \frac{2S}{\sqrt{\pi}} Y_i \exp \left( -s^2 \beta_i^2 \right) \\
\left[ 2(2- \sigma' - \sigma) \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right] + \text{erf} (s \beta_i) \left[ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right] \\
\frac{2}{\partial \theta_i} \left\{ c_{\text{m}_{\lambda}} \cos^{-1} \left( -\frac{\nu_i^5}{\nu_R^5} \right) \left( \nu_i^5 \nu_R^5 \right) \right\} = 0
\]

For minimization methods involving the utilization of second derivatives, the necessary expressions are obtained below:

\[
\left. \frac{\partial^2 D}{\partial \theta_i^2} \right|_{\theta = 0} = \frac{1}{2} \nu_R^2 A \left\{ \left( \theta (2-\sigma' - \sigma) \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right) \exp \left( -s^2 \beta_i^2 \right) \right\} \\
+ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} Y_i \beta_i \exp \left( -s^2 \beta_i^2 \right) \\
+ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} Y_i \beta_i \exp \left( -s^2 \beta_i^2 \right) \\
\frac{\partial^2 G_A}{\partial \theta_i^2} \right|_{\theta = 0} = \frac{1}{2} \rho \nu_R^2 A b_i \left\{ \tilde{n}_i \left[ (2(2-\sigma' - \sigma) \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right] \exp \left( -s^2 \beta_i^2 \right) \right\} \\
+ \frac{2(2-\sigma' - \sigma)}{\sqrt{\pi} S} Y_i \beta_i \exp \left( -s^2 \beta_i^2 \right) \\
+ \left[ \frac{4(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i \right] \exp \left( -s^2 \beta_i^2 \right) \\
\left. \frac{\partial^2 G_A}{\partial \theta_i \theta_j} \right|_{\theta = 0} = \frac{1}{2} \rho \nu_R^2 A \tilde{n}_i \left\{ \tilde{n}_i \left[ (2(2-\sigma' - \sigma) \beta_i + 2 \left( \frac{2-\sigma' + \sigma}{2 \beta_i^2} \right) \beta_i \right] + \tilde{n}_i \left[ \frac{4(2-\sigma' - \sigma)}{\sqrt{\pi} S} \beta_i \right] \exp \left( -s^2 \beta_i^2 \right) \right\}
\]

For minimization methods involving the utilization of second derivatives, the necessary expressions are obtained below:

\[
\left. \frac{\partial^2 D}{\partial \theta_i \partial \theta_j} \right|_{\theta = 0} = 0 \quad \text{for } i \neq j \\
\neq 0 \quad \text{for } i = j \\
\left( \tilde{n}_i \cdot \nu_R^5 \right) = - \left( \tilde{n}_i \cdot \nu_R^5 \right) = - \beta_i
\]

\[\text{P2}\]
\[ \frac{2^2 D}{2 \theta_t^2} \bigg|_{\theta = 0} = \frac{1}{2} \rho V_R^2 A \left\{ \begin{array}{l} 12 (2 - \sigma' - \sigma) Y_t^2 \beta_t - 6 (2 - \sigma' - \sigma) \beta_t^3 - 2 \left( \frac{2 - \sigma'}{2 \sigma^2} + \sigma \right) \beta_t \end{array} \right\} \text{erf} \left( S \beta_t \right) \\
+ \left[ \frac{12 S}{\sqrt{V}} \left( 2 - \sigma' - \sigma \right) Y_t \beta_t^2 + \frac{4S}{\sqrt{V}} \left( \frac{2 - \sigma'}{2 \sigma^2} + \sigma \right) Y_t \right] \exp \left( -S^2 \beta_t^2 \right) \\
+ \left[ \frac{2}{\sqrt{V} S} (6 - 3 \sigma' - 2 \sigma) \left( Y_t^2 - \beta_t^2 \right) \right] \exp \left( -S^2 \beta_t^2 \right) - \frac{3S}{\sqrt{V}} \left( 6 - 3 \sigma' - \sigma \right) \beta_t Y_t^2 \exp \left( -S^2 \beta_t^2 \right) \right\} \]

Rearranging terms, the following expression is obtained.

\[ \frac{2^2 D}{2 \theta_t^2} \bigg|_{\theta = 0} = \frac{1}{2} \rho V_R^2 A \left\{ \begin{array}{l} 12 (2 - \sigma' - \sigma) Y_t^2 \beta_t - 6 (2 - \sigma' - \sigma) \beta_t^3 - 2 \left( \frac{2 - \sigma'}{2 \sigma^2} + \sigma \right) \beta_t \end{array} \right\} \text{erf} \left( S \beta_t \right) \\
+ \left[ \frac{12 S}{\sqrt{V}} (6 - 3 \sigma' - 2 \sigma) \left( Y_t^2 - \beta_t^2 \right) \right] \exp \left( -S^2 \beta_t^2 \right) - \frac{3S}{\sqrt{V}} \left( 6 - 3 \sigma' - \sigma \right) \beta_t Y_t^2 \exp \left( -S^2 \beta_t^2 \right) \right\} \] (F.18)

\[ \frac{2^2 G_A}{\partial^2 \theta_t^2} \bigg|_{\theta = 0} = \begin{cases} 0 & \text{for } i \neq j \\
\theta & \text{for } i = j \end{cases} \] (F.19)

\[ \frac{\partial^2 G_A}{\partial \theta_t^2} \bigg|_{\theta = 0} = \frac{1}{2} \rho V_R^2 A \left\{ \begin{array}{l} -n_i \left\{ \left( 2 - \sigma' - \sigma \right) \beta_t^2 + \frac{3S}{\sqrt{V}} \beta_t \right\} \text{erf} \left( \beta_t \right) + \frac{2}{\sqrt{V} S} \left( N_1 \right) \exp \left( -S^2 \beta_t^2 \right) \right\} \]

Rearranging terms,

\[ \frac{\partial^3 G_A}{\partial \theta_t^3} \bigg|_{\theta = 0} = \frac{1}{2} \rho V_R^2 A \left\{ \begin{array}{l} n_i \left\{ \left( 2 - \sigma' - \sigma \right) \beta_t^2 + \frac{3S}{\sqrt{V}} \beta_t \right\} \text{erf} \left( \beta_t \right) - \left( 12 - 6 \sigma' - 4 \sigma \right) \beta_t + \frac{4S}{\sqrt{V}} \beta_t \beta_t \exp \left( -S^2 \beta_t^2 \right) \right\} \] (F.20)

The linearized form of the equations of motion derived in Sec. 3.4 involve the expression

\[ \frac{\partial}{\partial \theta_t} (G_G + G_A) \bigg|_{\theta = 0} \]

where

\[ \theta^c = \left[ \theta_j^c \right], \quad j = 1, 2, 3, 4 \]

and \( \theta_0^c \) is the nominal value of \( \theta_j^c \) under minimum drag and zero net torque on the satellite in equilibrium state. The expressions necessary to determine \( \frac{\partial}{\partial \theta_t} (G_G + G_A) \bigg|_{\theta = 0} \) are obtained below.

F3
From equation (3.13), Sec. 3.3,

\[ G_G \approx \frac{3\mu}{R_c^3} \begin{bmatrix} K_2 & K_3(C_1 - B_1) \\ K_3 & K_1(A_1 - C_1) \\ K_1 & K_2(B_1 - A_1) \end{bmatrix} \quad (F.21) \]

\[ K = l_7^{75} \frac{j_3}{j} = \begin{bmatrix} S_{e_1}S_{e_2} - C_{e_1}S_{e_2}S_{e_3} \\ S_{e_2}C_{e_1} + C_{e_2}S_{e_2}S_{e_3} \\ C_{e_1}C_{e_2} \end{bmatrix} \]

Linearizing, \( K \approx \begin{bmatrix} -\theta_z \\ \theta_1 \\ 1 \end{bmatrix} \quad (F.22) \)

For circular orbits \( \eta^2 = \mu/R_c^3 \).

\[ G_G \approx 3\eta^2 \begin{bmatrix} \theta_1(C_1 - B_1) \\ -\theta_2(A_1 - C_1) \\ 0 \end{bmatrix} \quad (F.23) \]

Differentiating (F.23) with respect to \( \theta_1 \) and setting \( \theta = 0 \) and \( \theta^c = \theta^c_0 \),

\[ \text{det} \begin{bmatrix} (C_1 - B_1) & 0 & 0 \\ 0 & -(A_1 - C_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3\eta^2 \quad (F.24) \]

\[ G_A \] is given by equation (F.4)

Define

\[ \mathbf{v}_R^{s} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} \]

\[ \mathbf{v}_R = L_7^{75} \mathbf{v}_R^{s} \]

where \( L_7^{75} = [l_{ij}] \) and is given by equation (3.3), Sec. 3.1.

\[ \frac{\partial l_{ij}^{75}}{\partial \theta_1} \bigg|_{\theta = 0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \frac{\partial l_{ij}^{75}}{\partial \theta_2} \bigg|_{\theta = 0} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \frac{\partial l_{ij}^{75}}{\partial \theta_3} \bigg|_{\theta = 0} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (F.25) \]
Define \( \delta_k = \frac{\partial u_k}{\partial \theta_k} \bigg|_{\theta=0} \), \( k = 1, 2, 3 \)

Then \( \delta_1 = u_3^0 \), \( \delta_2 = 0 \), \( \delta_3 = -u_1^0 \) \hspace{1cm} (F.26)

\[
\frac{\partial}{\partial \theta_k} \left\{ \text{erf} \left( S(n_i \cdot u_k) \right) \right\} \bigg|_{\theta=0} = \frac{2S}{\sqrt{\pi}} (n_i \cdot \delta_k) \exp \left( -S^2 \beta_i^2 \right) \hspace{1cm} (F.27)
\]

\[
\frac{\partial}{\partial \theta_k} \left\{ \exp \left( -S^2(n_i \cdot u_k)^2 \right) \right\} \bigg|_{\theta=0} = -2S^2 \beta_i (n_i \cdot \delta_k) \exp \left( -S^2 \beta_i^2 \right) \hspace{1cm} (F.28)
\]

Utilizing (F.27) and (F.28) and differentiating \( G_A \), there results,

\[
\frac{\partial G_A}{\partial \theta_k} \bigg|_{\theta=0} = \frac{1}{2} p_s \vec{v}_A \cdot \vec{v}_L \left\{ C_{\alpha} \left[ \frac{1}{\left( 1 - (u_k^0 \cdot \vec{i}_k) \right) ^2} (\delta_k \cdot \vec{i}_k) (\vec{\omega}_k \cdot \vec{i}_k) + \cos^{-1} (u_k^0 \cdot \vec{i}_k) (\vec{\omega}_k \cdot \vec{i}_k) \right] \right.
\]

\[
+ \frac{A}{S^2 A} \sum_{i=1}^{4} \vec{b}_i \left\{ \frac{1}{\sqrt{\pi}} \left[ \text{erf} \left( S \beta_i \right) 4 (2 - \sigma' \cdot \sigma) \beta_i (n_i \cdot \delta_k) + \frac{2S}{\sqrt{\pi}} (n_i \cdot \delta_k) \exp \left( -S^2 \beta_i^2 \right) \right] \right.
\]

\[
+ \left[ \frac{4 \left( 2 - \sigma' \cdot \sigma \right) S^2 \beta_i^2 + \frac{2\sigma}{S^2} \beta_i \right] \frac{1}{2} (2 - \sigma' \cdot \sigma) \beta_i (n_i \cdot \delta_k) \left. \right\} \}
\]

Rearranging the various terms,

\[
\frac{\partial G_A}{\partial \theta_k} \bigg|_{\theta=0} = \frac{1}{2} p_s \vec{v}_A \cdot \vec{v}_L \left\{ C_{\alpha} \left[ \frac{1}{\left( 1 - (u_k^0 \cdot \vec{i}_k) \right) ^2} (\delta_k \cdot \vec{i}_k) (\vec{\omega}_k \cdot \vec{i}_k) + \cos^{-1} (u_k^0 \cdot \vec{i}_k) (\vec{\omega}_k \cdot \vec{i}_k) \right] \right.
\]

\[
+ \frac{A}{S^2 A} \sum_{i=1}^{4} \vec{b}_i \left\{ \frac{1}{\sqrt{\pi}} \left[ \text{erf} \left( S \beta_i \right) 4 (2 - \sigma' \cdot \sigma) \beta_i (n_i \cdot \delta_k) + \frac{2S}{\sqrt{\pi}} (n_i \cdot \delta_k) \exp \left( -S^2 \beta_i^2 \right) \right] \right.
\]

\[
+ \left[ \frac{4 \left( 2 - \sigma' \cdot \sigma \right) S^2 \beta_i^2 + \frac{2\sigma}{S^2} \beta_i \right] \frac{1}{2} (2 - \sigma' \cdot \sigma) \beta_i (n_i \cdot \delta_k) \left. \right\} \}
\]

\[
\frac{\partial G_A}{\partial \theta_k} \bigg|_{\theta=0} \bigg|_{\theta^c = \theta^c_0} \hspace{1cm} (F.29)
\]

can be obtained from (F.29) by substituting \( \theta^c = \theta^c_0 \).
### TABLE 4.1
Parameters of Configuration A

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Satellite</td>
<td>$m = 10,000$ kg.</td>
</tr>
<tr>
<td>Moments of Inertia</td>
<td>$A_1 = 30,000$ kg.m$^2$, $B_1 = C_1 = 60,000$ kg.m$^2$.</td>
</tr>
<tr>
<td>Characteristic Length of Center-Body</td>
<td>$L = 10$ m</td>
</tr>
<tr>
<td>Cross-Sectional Area of Center-Body</td>
<td>$S_A = 10$ m$^2$.</td>
</tr>
<tr>
<td>Slope of Aerodynamic Moment Curve for the Center-Body</td>
<td>$C_{m_A} = 1.0$</td>
</tr>
<tr>
<td>Area of Each Control Panel</td>
<td>$A = 5$ m$^2$</td>
</tr>
<tr>
<td>Aerodynamic Moment Arms for the Control Panels:</td>
<td>$l_1 = 5$m, $l_2 = 2$m.</td>
</tr>
</tbody>
</table>

### TABLE 4.2
Parameters of Configuration B

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Satellite</td>
<td>$m = 50,000$ kg.</td>
</tr>
<tr>
<td>Moments of Inertia</td>
<td>$A_1 = 1.5 \times 10^6$ kg.m$^2$, $B_1 = C_1 = 3.0 \times 10^6$ kg.m$^2$.</td>
</tr>
<tr>
<td>Characteristic Length of Center-Body</td>
<td>$L = 20$ m</td>
</tr>
<tr>
<td>Cross-Sectional Area of Center-Body</td>
<td>$S_A = 50$ m$^2$.</td>
</tr>
<tr>
<td>Slope of Aerodynamic Moment Curve for the Center-Body</td>
<td>$C_{m_A} = 1.0$</td>
</tr>
<tr>
<td>Area of Each Control Panel</td>
<td>$A = 250$ m$^2$.</td>
</tr>
<tr>
<td>Aerodynamic Moment Arms for the Control Panels:</td>
<td>$l_1 = 20$ m, $l_2 = 10$m.</td>
</tr>
</tbody>
</table>
### TABLE 4.3
Configuration A in 200 km equatorial orbit: K-Matrix.

\[
\begin{bmatrix}
0.49563 & 0.24555 & -0.15934 & 0.66824 & 0.81561 & 0.04807 \\
0.49563 & -0.24555 & -0.15934 & 0.66824 & -0.81561 & 0.04807 \\
0.58831 & 0.0 & -0.22122 & 0.72833 & 0.0 & 0.84062 \\
0.40306 & 0.0 & -0.29656 & 0.60816 & 0.0 & -0.74449
\end{bmatrix}
\]

### TABLE 4.4
Configuration A in 300 km equatorial orbit: K-Matrix.

\[
\begin{bmatrix}
0.46755 & 6.36972 & -0.74864 & 1.25194 & 6.25344 & 0.21895 \\
0.46755 & -6.36972 & -0.74864 & 1.25194 & -6.25344 & 0.21895 \\
0.71714 & 0.0 & -0.85773 & 1.52562 & 0.0 & 1.40118 \\
0.21795 & 0.0 & -0.63954 & 0.97825 & 0.0 & -0.96329
\end{bmatrix}
\]
| TABLE 4.5 |
| Configuration in 200 km equatorial orbit: K-Matrix. |
| | 0.49976 | 0.20034 | -0.03533 | 0.52119 | 0.75326 | -0.03118 |
| | 0.49976 | -0.20034 | -0.03533 | 0.52119 | -0.75326 | -0.03118 |
| | 0.52125 | 0.0 | 0.12375 | 0.51340 | 0.0 | 0.70974 |
| | 0.47827 | 0.0 | -0.19441 | 0.52899 | 0.0 | -0.77209 |

| TABLE 4.6 |
| Configuration in 300 km. equatorial orbit: K-Matrix. |
| | 0.49626 | 1.99616 | -0.19565 | 0.68543 | 2.55915 | -0.17302 |
| | 0.49626 | -1.99616 | -0.19565 | 0.68543 | -2.55915 | -0.17302 |
| | 0.58243 | 0.0 | -0.09966 | 0.64218 | 0.0 | 0.60150 |
| | 0.41009 | 0.0 | -0.29163 | 0.72869 | 0.0 | -0.94754 |
coordinate frames

geocentric inertial reference frame - $\mathbf{X}$

geocentric solar reference frame - $\mathbf{X}$

geocentric earth reference frame - $\mathbf{X}$

geocentric magnetic reference frame - $\mathbf{X}$

figure 2.1 some useful coordinate frames
FUNCTION MINIMIZATION

COMPUTE $u_0(\eta)$

REPRESENT $u_0(\eta) = c_{ij} \theta_j(\eta)$

$\theta_j(\eta)$ - ORTHOGONAL SEQUENCE

in $0 < \eta < 2\pi$.

DETERMINE $A(\eta)$ & $B(\eta)$

SOLVE RICCATI EQN. WITH

$Q = I, R = I, S(\eta_{p}) = 0, \eta_{f}$

Large, Backwards until

$S(\eta + 2\pi) = S(\eta)$.

IF $S(\eta_{p} + 2\pi) = S(\eta_{p})$ determine

$S(\eta)$ for one orbit from $\eta_{p}$ on.

COMPUTE $K(\eta)$ for one orbit,

REPRESENT

$K_{ij}(\eta) = \Lambda_{i,j,k} \theta_k(\eta)$

$\theta_k(\eta)$ - Orthogonal Sequence

in $0 < \eta < 2\pi$

TRANSITION MATRIX OF SYSTEM WITH NOMINAL CONTROL FOR ONE ORBIT OR MORE

TRANSITION MATRIX OF SYSTEM WITH FEEDBACK CONTROL FOR ONE OR MORE ORBITS.

FIGURE 4.1 SCHEMATIC OF COMPUTATIONS
FIGURE 4.2 MAGNUS PLANE REPRESENTATION OF THE GRAVITATIONAL STABILITY OF THE SATELLITE
FIGURE 4.3 NOMINAL CONTROL ANGLES OF SATELLITE CONFIGURATION IN 200 Km. ALTITUDE POLAR ORBIT
FIGURE 4.4 NOMINAL CONTROL ANGLES OF SATELLITE CONFIGURATION
A IN 300 Km. ALTITUDE POLAR ORBIT
FIGURE 4.5 LINEAR APPROXIMATION AND PERMISSIBLE RANGE OF CONTROL PANEL ANGLE OF ATTACK
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km} \]
\[ \rho = 0.3318 \times 10^{-9} \text{ kg m}^{-3} \]
\[ \sigma = 0.5 \]
\[ \sigma' = 0.5 \]
\[ C_m = 1.0 \]
\[ L = 100 \text{ m} \]

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

\[ \theta_1, \theta_2, \theta_3 \]

\( \psi(\theta) = 5.0 \text{ deg} \)

\( \phi(\theta) = 5.0 \text{ deg} \)

FIGURE 4.6 RESPONSE OF CONFIGURATION \( \Delta \) IN 200 Km.
EQUATORIAL ORBIT
FIGURE 4.6  CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES

- $h = 200\text{km}$
- $\sigma = 0.3318 \times 10^{-3} \text{kg.m}^3$
- $\sigma = 0.5$
- $\sigma = 0.5$
- $C_m = 10$
- $A = 5.0\text{m}^2$
- $S_n = 10.0\text{m}^2$
- $A = 5.0\text{m}^2$
- $B = C = 60,000\text{kg.m}^2$
- $m_f = 30,000\text{kg}$
- $m_f = 30,000\text{kg}$
- $L = 10.0\text{m}$
- $L = 10.0\text{m}$

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.7 RESPONSE OF CONFIGURATION A IN 200 Km. POLAR ORBIT
FIGURE 4.7 CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 300 \text{ km} \]
\[ \rho = 3585 \times 10^{-10} \text{kg.m}^{-3} \]
\[ \sigma = 0.05 \]
\[ \sigma' = 0.05 \]
\[ C_m = 4.0 \]
\[ L = 10.0 \text{m.} \]
\[ i = 0.0 \text{ deg.} \]
\[ w = 0.0 \text{ deg.} \]
\[ A = 5.0 \text{m}^2 \]
\[ A_\perp = 30,000 \text{kg.m}^2 \]
\[ B = 60,000 \text{kg.m}^2 \]
\[ A_\perp = 30,000 \text{kg.m}^2 \]

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.8 RESPONSE OF CONFIGURATION \(A\) IN 300 Km.
EQUATORIAL ORBIT
FIGURE 4.8 CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 300 \text{ km,} \]
\[ \rho = 0.3565 \times 10^{-4} \text{ kg m}^{-3} \]
\[ \sigma = 0.05 \]
\[ \sigma' = 0.05 \]
\[ C_{n,40} = 40 \]
\[ L = 400 \text{ m} \]
\[ a = 5.0 \text{ m}^2 \]
\[ S = 100 \text{ m}^2 \]
\[ A = 30,000 \text{ kg m}^{-2} \]
\[ B_1 = C_1 = 60,000 \text{ kg m}^{-2} \]

WITH NOMINAL CONTROL

\[ \theta_1(0) = 5.0 \text{ DEG.} \]

WITH FEEDBACK CONTROL

\[ \theta_1(0) = 5.0 \text{ DEG.} \]

FIGURE 4.9 RESPONSE OF CONFIGURATION A IN 300 Km.
POLAR ORBIT (CONTINUED)
WITH NOMINAL CONTROL

\[ \theta(0) = 0.4 \text{ RAD/RAD.} \]

WITH FEEDBACK CONTROL

\[ \theta(0) = 0.4 \text{ RAD/RAD.} \]

FIGURE 4.9 CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km} \]
\[ \rho = 0.3318 \times 10^{-9} \text{ kg m}^{-3} \]
\[ \sigma = 0.5 \]
\[ \sigma = 0.5 \]
\[ C_{n_h} = 1.0 \]
\[ L = 100 \text{ m} \]
\[ \theta = 90.0 \text{ deg} \]
\[ \omega = 0.0 \text{ deg} \]
\[ A = 50 \text{ m}^2 \]
\[ S = 100 \text{ m}^2 \]
\[ \alpha = 3.000 \text{ kg m}^2 \]
\[ B = 60,000 \text{ kg m}^2 \]

WITH K-MATRIX COMPUTED FOR
\[ \omega = 0.0 \text{ deg} \] ORBIT

FIGURE 4.10 EFFECT ON THE RESPONSE OF USING EQUATORIAL ORBIT K-MATRIX FOR POLAR ORBIT: 200 Km ALTITUDE: CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 300 \text{ km} \quad \omega = 90.0 \text{ deg.} \]

\[ \rho = 0.3585 \times 10^{-10} \text{ kg} \cdot \text{m}^3 \quad \sigma = 0.5 \]

\[ \sigma' = 0.6 \quad S_2 = 10.0 \text{ m}^2 \]

\[ C_{m0} = 1.0 \quad A_1 = 30,000 \text{ kg} \cdot \text{m}^2 \]

\[ L = 10.0 \text{ m.} \quad B_1 = C_2 = 60,000 \text{ kg} \cdot \text{m}^2 \]

\[ A = 5.0 \text{ m}^2 \quad S = 10.0 \text{ m}^2 \]

\[ \epsilon = 30,000 \text{ kg} \cdot \text{m}^2 \]

\[ B_1 = 60,000 \text{ kg} \cdot \text{m}^2 \]

WITH K-MATRIX COMPUTED FOR

\[ \omega = 0.0 \text{ deg.} \]

ORBIT

FIGURE 4.11 EFFECT ON THE RESPONSE OF USING EQUATORIAL ORBIT K-MATRIX FOR POLAR ORBIT: 300 Km ALTITUDE: CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

h = 200 km.
ρ = 0.3318 x 10⁻⁹ kg/m³
ω = 0° deg.
σ = 0.5
σ = 0.5
C₃₄ = 0.001 kg·m⁻²
L = 100 m²

WITH K-MATRIX COMPUTED FOR
h = 300 km. ORBIT:

A = 50 m²
S₄ = 30,000 kg·m²
B₃ = 60,000 kg·m²

FIGURE 4.12 EFFECT ON THE RESPONSE OF USING 300 Km. K-MATRIX AT 200 Km. : EQUATORIAL ORBIT:
CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km.} \quad \iota = 90.0 \text{ deg.} \]
\[ \rho = 0.3318 \times 10^{-9} \text{ kg.m}^3 \quad \omega = 0.0 \text{ deg.} \]
\[ \sigma = 0.5 \quad A = 5.0 \text{ m}^2 \]
\[ \sigma = 0.5 \quad S_t = 10.0 \text{ m}^2 \]
\[ C_{\text{Ma}} = 1.0 \quad A_t = 30,000 \text{ kg.m}^2 \]
\[ L = 1.0 \text{ m.} \]

WITH K-MATRIX COMPUTED FOR:
\[ h = 300 \text{ km.} \quad \iota = 90.0 \text{ deg.} \text{ ORBIT.} \]

FIGURE 4.13 EFFECT ON THE RESPONSE OF USING 300 Km. K-MATRIX AT 200 Km.: POLAR ORBIT: CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km.} \]
\[ e = 0.3318 \times 10^{-9} \text{ kg.m}^2 \]
\[ \sigma = 0.5 \]
\[ \sigma' = 0.5 \]
\[ i = 0.0 \text{ deg.} \]
\[ \omega = 0.0 \text{ deg.} \]
\[ A = 50 \text{ m}^2 \]
\[ \Delta = 5.0 \text{ m}^2 \]
\[ \Delta A = 30,000 \text{ kg.m}^2 \]
\[ B_{\Delta} = 5.60 \times 10^5 \text{ kg.m}^2 \]

\[ p = p'(1 + 0.0501 \cos \lambda) \]

FIGURE 4.14 EFFECT OF VARIABLE DENSITY ON THE RESPONSE:
200 Km. ALTITUDE EQUATORIAL ORBIT:
CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ \rho = 300 \text{ km} \]

\[ \omega = 0 \text{ deg} \]

\[ \sigma = 0.5 \]

\[ \sigma' = 0.5 \]

\[ C_m = 10 \]

\[ L = 400 \text{ m} \]

\[ A = 5 \text{ m}^2 \]

\[ S_n = 100 \text{ m}^2 \]

\[ B(C, = 60,000 \text{ kg} \cdot \text{m}^2 \]

\[ \theta_1 \]

\[ \theta_2 \]

\[ \theta_3 \]

WITH VARIABLE DENSITY

\[ \rho \eta^2(1 + 0.5 \eta \cos \eta) \]

FIGURE 4.15 EFFECT OF VARIABLE DENSITY ON THE RESPONSE:
300 Km. ALTITUDE EQUATORIAL ORBIT:
CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

h = 200 km.
\[ \rho = 0.3318 \times 10^{-9} \text{ kg m}^{-3} \]
\[ \omega = 0 \text{ deg.} \]
\[ \sigma = 0.75 \]
\[ A = 5.0 \text{ m}^2 \]
\[ S_A = 10.0 \text{ m}^2 \]
\[ L = 10.0 \text{ m} \]

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.16 RESPONSE OF CONFIGURATION A IN 200 Km. ALTITUDE
EQUATORIAL ORBIT WITH \( \sigma = \sigma' = 0.75 \)
FIGURE 4.16 CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km} \]
\[ \rho = 0.3318 \times 10^{-9} \text{ kg m}^{-3} \]
\[ \alpha = 0.5 \]
\[ \sigma = 0.5 \]
\[ C_m = 10 \]
\[ L = 10.0 \text{ m} \]
\[ 8 + 0 \text{ deg} \]
\[ \beta_1, \beta_2 \]

With K-matrix computed for \( \sigma = \sigma' = 0.75 \)

FIGURE 4.17 EFFECT ON THE RESPONSE OF USING K-MATRIX
COMPUTED WITH \( \sigma = \sigma' = 0.75 \) WHEN \( \sigma = \sigma' = 0.5 \):
CONFIGURATION (A)
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\begin{align*}
\text{h} & = 200 \text{ km} \\
\rho & = 0.3318 \times 10^{-9} \text{ kg m}^{-3} \\
\sigma & = 0.75 \\
\sigma' & = 0.75 \\
C^H & = 1.0 \\
L & = 100 \text{ m} \\
A & = 50 \text{ m}^2 \\
S & = 10.0 \text{ m}^2 \\
\theta(0) & = 5.0 \text{ deg} \\
\omega & = 0.0 \text{ deg} \\
A_1 & = 30,000 \text{ kg m}^2 \\
B_1 & = 60,000 \text{ kg m}^2 \\
\sigma & = 0.05
\end{align*}

FIGURE 4.18 EFFECT ON THE RESPONSE OF USING K-MATRIX COMPUTED FOR CONFIGURATION A WHEN $\sigma = \sigma' = 0.5$
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.19 RESPONSE OF CONFIGURATION (2) IN 200 Km.
EQUATORIAL ORBIT.
WITH NOMINAL CONTROL

\[ \epsilon_{90} = 0.4 \text{ RAD/RAD.} \]

WITH FEEDBACK CONTROL

\[ \epsilon_{90} = 0.4 \text{ RAD/RAD.} \]

FIGURE 4.19 CONTINUED
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\( h = 300 \text{ km} \)
\( \rho = 0.3585 \times 10^{-10} \text{ kg m}^3 \)
\( \omega = 0.0 \text{ deg} \)
\( \sigma = 0.5 \)
\( A = 250 \text{ m}^2 \)
\( \sigma' = 0.5 \)
\( S_\text{A} = 50 \text{ m}^2 \)
\( C_\text{m} = 10 \)
\( A_1 = 15 \times 10^6 \text{ kg m}^2 \)
\( C_2 = 30 \times 10^6 \text{ kg m}^2 \)
\( L = 20 \text{ m} \)

\( \theta_1 \) \( \theta_2 \) \( \theta_3 \)

\( \Theta(0) = 5.0 \text{ deg} \)

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.20 RESPONSE OF CONFIGURATION \( \textcircled{2} \) IN 300 Km.
EQUATORIAL ORBIT.
WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

FIGURE 4.20 CONTINUED
FIGURE 4.21 PLOT OF THE NORM OF S-MATRIX FOR SINGLE AXIS POINTING: COUPLED CASE: CONFIGURATION A
RESPONSE OF SATELLITE TO INITIAL DISTURBANCES.

\[ h = 200 \text{ km} \]
\[ \rho = 0.03316 \times 10^{-9} \text{kg.m}^{-3} \]
\[ \omega = 0.0 \text{ deg.} \]
\[ \sigma = 0.5 \]

\[ A = 5.0 \text{ m}^2 \]
\[ \sigma = 0.5 \]
\[ S = 40.0 \text{ m}^2 \]
\[ C_n = 10 \]
\[ L = 100 \text{ m} \]
\[ B = C_1 = 60.000 \text{ kg.m}^2 \]
\[ A_1 = 30.000 \text{ kg.m}^2 \]
\[ \theta_1 \]
\[ \theta_2 \]
\[ \theta_3 \]

WITH NOMINAL CONTROL

WITH FEEDBACK CONTROL

\[ \theta(0) = 5.0 \text{ deg.} \]

\[ \theta(0) = 5.0 \text{ deg.} \]

\[ \theta(0) = 5.0 \text{ deg.} \]

\[ \theta(0) = 5.0 \text{ deg.} \]

FIGURE 4.22 RESPONSE OF CONFIGURATION A IN THE SINGLE AXIS POINTING MODE: DECOUPLED CASE: 200 Km. POLAR ORBIT.
FIGURE 4.22 CONTINUED
FIGURE 4.23 CONFIGURATION A IN SINGLE AXIS POINTING MODE: RESPONSE TO ZERO INITIAL CONDITIONS.
Figure 4.24: Schematic of Gain Scheduling.

ATTITUDE CONTROL SYSTEM & MEMORY

GROUNDED STATION & COMPUTER

TELEMETRY OF SYSTEM PERFORMANCE, DENSITY, ORBITAL INCLINATION & OTHER PARAMETERS.

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TELEMETRY OF $c_{ij}$ & $A_{ijk}$
FIGURE A.1 DIURNAL DENSITY VARIATION FACTOR AS A FUNCTION OF ALTITUDE

Data: Ref. 70
Figure A.2: Diurnal Density Variations at 350 Km. During Medium Solar Activity

Data: Ref. 24
FIGURE A.4 SPEED RATIO AS A FUNCTION OF ALTITUDE.
FIGURE B.1 NOMENCLATURE FOR SURFACE ELEMENTS.
FIGURE B.2 FORCE COEFFICIENTS FOR THE CONTROL SURFACES.
FIGURE C.1 LIFETIME AS A FUNCTION OF ALTITUDE: EFFECT OF VARYING CONTROL PANEL AREAS ON LIFETIME.
FIGURE C.2 LIFETIME AS A FUNCTION OF ALTITUDE: EFFECT OF NONZERO VERTICAL CONTROL PANEL ANGLES ON THE SATELLITE LIFETIME.
A near-Earth satellite orbiting in the altitude range of 150 km to 450 km encounters small but non-negligible aerodynamic forces. It is possible to generate sufficient aerodynamic torques by providing "all-moving" control surfaces of suitable size to achieve active attitude control. This report is a preliminary study of such an active attitude control system. The satellite configuration considered has four all-moving control surfaces and the dominant gravity gradient torques and aerodynamic torques are considered in the analysis. The resulting equations of motion are linearized and modern optimal control theory concepts are applied to synthesize a feedback control system for controlling the satellite by rotating the control surfaces to obtain the necessary control torques. The numerical studies carried out indicate that it is possible to control a near-Earth satellite by using control surfaces of reasonable size. Damping times of the order of from a few orbits to a fraction of an orbit seem reasonable.