Normalized Coprime Factorizations for Systems in Generalized State-Space Form

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Abstract—This note presents a state-space algorithm for the calculation of a normalized coprime factorization of continuous-time generali-
zed dynamical systems. It will be shown that two Riccati equations have to be solved to obtain this normalized coprime factorization.

I. INTRODUCTION

Recent publications have shown the importance of normalized coprime factorization plant descriptions in the fields of control design [6], robustness analysis [13], [15], model reduction [7], and identification for control [12].

In [9], the connection between the state-space realization of a strictly proper plant, and a coprime factorization has been established. The coprime factorization of a generalized dynamical system was presented in [17]. In [8], it has been shown that the resulting structure (2) is obtained by solely interchanging rows and columns containing an $s_j$. Controllability and observability of systems in Jordan form imply nonsingularity of $B_2B_1^T$ and $C_1^TC_2$ [3]. The Jordan form implies $J_{12}J_{21} = 0$ and $J_{12}J_{21}^T = 0$. Now defining

$$A_{11} = \begin{bmatrix} \hat{A} & 0 \\ 0 & J_{11} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ J_{12} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \hat{B} \\ \hat{B}_1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & J_{21} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \hat{C} & \hat{C}_1 \end{bmatrix}, \quad B_2 = : B_2,$n

where the partitions have consistent dimensions, leads to (1), and this proves the proposition.

III. MAIN RESULT

The main result consists of two parts. First, we will show that an NRCF of $P$ is a stable full-rank spectral factor of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (I + P^*P)^{-1} (I - P^*P).$$

(4)

Secondly, we will use this result to obtain a state-space realization of an NRCF of $P$. This will be presented in the form of an algorithm.

Theorem 3.1: Let $P \in \mathcal{F}$ be given. Then the following statements are equivalent:

a) $(N, M)$ is an NRCF of $P$.

b) \[ M \in \mathcal{F}, \text{ where } (N, M) \text{ right coprime is a full-rank spectral factor of} \]

\[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (I + P^*P)^{-1} (I - P^*P). \]
Proof: a) → b). Given \((N, M)\) as an NRCF of \(P\). Then
\[
[M\atop N] \in \mathcal{M} \text{ is full rank and (4) can be written as}
\[
\begin{bmatrix}
I \\
M M + N N
\end{bmatrix}^{-1} = \begin{bmatrix}
I \\
M M + N N
\end{bmatrix}^{-1} = \begin{bmatrix}
M & N
\end{bmatrix} \begin{bmatrix}
M N^{-1} + M N^{-1} N M^{-1} & M N^{-1} N M^{-1}
\end{bmatrix}^{-1} \begin{bmatrix}
M N^{-1} + M N^{-1} N M^{-1}
\end{bmatrix}
\]
which shows b).

b) → a). Let \([M\atop N] \in \mathcal{M}\) be a full-rank spectral factor of (4) with \((N, M)\) right coprime, i.e., (4) equals \([M\atop N] [M^* N^*]\). Premultiplication by \([P - I]\) yields \([P - I] [M\atop N] = 0\), which shows that \((N, M)\) is an RCF of \(P\). Postmultiplication of (4) by \([M\atop N]\) yields
\[
[M\atop N] [M^* N^*]^T [M\atop N] = [I] = [M\atop N] [M^* N^*],
\]
which implies \([M^* N^*] [M\atop N] = I\), and this shows a). □

Based on Theorem 3.1, the following algorithm has been constructed, which will lead to a state-space representation of an NRCF of a system in generalized state-space form. The proof is given in the Appendix.

**Algorithm:** Let \(P(s)\) be a real rational (possibly nonproper) transfer function of McMillan degree \(r\).

1. **Step 1:** Perform the construction of a system in the particular generalized state-space form (1) having the structure defined by (2) yielding (3) as formulated in Proposition 2.2 and its Proof.
2. **Step 2:** Calculate \(W_1\) as the stabilizing solution to the Riccati equation
\[
C_1^2 C_2 + W_2 A_{22} + A_{22}^T W_2 - W_2 B_2 B_2^T W_2 = 0.
\]
3. **Step 3:** Define \(Y, Z, C, B, \tilde{A}\) to be
\[
Y := -(W_2 A_{22} + C_1^2 C_2)^{-1} (A_{12} - W_2 B_2 B_2^T),
\]
\[
Z := -(W_2 A_{22} + C_1^2 C_2)^{-1} (C_1^2 C_1 + W_2 A_{12}),
\]
\[
\tilde{C} := C_1 + C_2 Z,
\]
\[
\tilde{A} := A_{11} + (A_{12} + Y^T C_2 C_1) Z + Y^T C_2 C_1,
\]
\[
\tilde{B} := B_1 - (A_{12} - B_2 B_2^T W_2) (A_{22} - B_2 B_2^T W_2)^{-1} B_2.
\]
4. **Step 4:** Calculate \(W_1\) as the stabilizing solution to the Riccati equation
\[
\tilde{C}^T \tilde{C} + \tilde{A}^T W_1 \tilde{A} + W_1 \tilde{B} \tilde{B}^T W_1 = 0.
\]
5. **Step 5:** A regular state-space realization \((A_n, B_n, C_n, D_n)\) of the NRCF \([M\atop N]\), having an order identical to the McMillan degree of \(P(s)\), is obtained, with
\[
A_n := \tilde{A} - \tilde{B} \tilde{B}^T W_1,
\]
\[
B_n := B_1 (I - B_2 B_2^T) + A_{12} W_2^{-1} B_2^T,
\]
\[
C_n := (B_2 B_2^T - I) B_2^T W_1 - B_2^T A_{21},
\]
\[
D_n := (I - C_2 C_2^T) C_1 - C_2^T A_{12} W_1.
\]

The following corollary enables the construction of both an NRCF and NLCF of a plant using the algorithm presented above.

**Corollary 3.2:** If \((M, N)\) is an NRCF of system \(P\), then \((M^T, N^T)\) is an NLCF of \(P^T\).

**IV. Example**

Assume that our nonproper system is a double differentiator \(P(s) = s^2\). A generalized state-space form of \(P(s)\) is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u
\]
\[
y = \begin{bmatrix}
-1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}.
\]

Using Proposition 2.2, we can bring this system in the form (1), (3)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u
\]
\[
y = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}.
\]

Therefore, \(M(s) = (1/(s^2 + \sqrt{2} s + 1))\), \(N(s) = (s^2/(s^2 + \sqrt{2} s + 1))\); and \(M(s), N(s) \in \mathcal{M}\); \((M(s)N(s)^{-1} = P(s))\), and \(M^*(s)M(s) + N^*(s)N(s) = I\).

**V. Conclusions**

In this note, a state-space algorithm for the calculation of a normalized coprime factorization of continuous-time generalized dynamical systems has been given. It has been shown that two Riccati equations have to be solved in the calculation of this normalized coprime factorization. As shown in the Appendix, these Riccati equations are well defined.

**APPENDIX**

In this Appendix, we prove the existence of an NRCF \((M, N)\) of \(P \in \mathcal{M}\) as constructed in the algorithm.

Let the generalized state-space realization of the system be partitioned according to Proposition 2.2, and apply operations of restricted system equivalence [11] to a generalized state-space
realization of \([I + P^*P]^{-1}[I + P^*] \) as follows:
\[
\begin{bmatrix}
I & -W & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
C^T & -sE^T - A^T & 0 \\
se - A & -BB^T & B^T \\
0 & -C & 0
\end{bmatrix}
\begin{bmatrix}
C \\
0 \\
0
\end{bmatrix}
\]
which equals a generalized state-space realization of the transfer function \(\frac{M}{N}\) with
\[
\begin{bmatrix}
M \\
N
\end{bmatrix} = \begin{bmatrix}
E & A + BF \\
F & B
\end{bmatrix}. 
\]
Now it can be easily checked that \(P(s) = N(s)M^{-1}(s)\). Using operations under restricted system equivalence \([11]\), the constructed generalized state-space realization of \(\frac{M}{N}\) is reduced to the state-space form

\[
\begin{bmatrix}
Q \\
-SE - A + BB^TW \\
B^TW \\
-C
\end{bmatrix}
\begin{bmatrix}
I & -B^T \\
0 & I \\
0 & 0
\end{bmatrix}
\]
with \(B^2 = B_1(B_2B_3)^{-1} \) and \(C = (C_1^T C_2)^{-1} C_2 \). Hence, \(\frac{M}{N}\) is proper and asymptotically stable. This shows that the presented algorithm will lead to a state-space representation of an NRCF of a system in generalized state-space form.

**REFERENCES**