Derivation of and Simulations with BiGlobal Stability Equations

MSc Thesis

AE5110

Student : Koen Groot
Studentnumber : 1507060

Supervisor TUD : Dr. B.W. van Oudheusden
Supervisor VKI : Dr. F. Pinna
Technical Advisor TUD : Dr. H.M. Schuttelaars
Technical Advisor TUD : Dr. M. Kotsonis
Abstract

Laminar to turbulent transition has an important role in the aerospace domain in view of its impact on aerodynamic drag and, regarding the high velocity regime, heat transfer. State of the art computational methods, like DNS, LES and RANS are found to be too expensive or rely on case dependent turbulence models to be used for obtaining information regarding the transition phenomenon. Transition is typically initiated by the onset of instability of the laminar flow. Linear stability theory describes the eigenmode growth mechanism. Although this yields a restriction, because additional mechanisms play a role too, the eigenmode growth phase establishes an important base in many practical situations. However, the linearization provides a considerable step in the simplification of the analysis, while the stability theory can be adapted according to the structure of the given mean flow.

At the Von Kármán Institute (VKI), the VKI Extensible Stability and Transition Analysis (VESTA) toolkit has been developed, which mainly involves methods based on the linear stability theory. In the current project, the main goal was to extend the already present tools to incorporate the BiGlobal stability equations, which, together with appropriate boundary conditions, form an eigenvalue problem. This particular problem is solved for perturbations inhomogeneous in two spatial directions and their complex growth rate and frequency. This extension involved a new version of the tool for the derivation of the BiGlobal stability equations, a tool for their automatic implementation in Matlab via the spectral collocation method and a simulation tool to apply boundary conditions and execute the analysis corresponding to a prescribed mean flow.

The derivation of the BiGlobal equations and their verification formed the first part of the project. Both incompressible and compressible versions are derived for different kinds of coordinate systems (e.g. Cartesian and cylindrical) and formulations in the compressible case (e.g. involving temperature and pressure and the energy equation based on static enthalpy). This allowed the verification of the tool with a large number of previously published references. All references, to the knowledge of the current author, that have thus far reported the compressible equations were found to contain errors and had to be cross-verified to yield the ultimate positive outcome. It is hence deemed that the present treatment is the first to report the full compressible BiGlobal stability equations in primitive variable formulation correctly.

The second part of the project involved the verification of the performance of the combination of the derivation, implementation and simulation tools. This was done by
considering three test cases (mean flows). In all cases, the eigenvalue problem was solved using the QZ algorithm. In cases that required high resolution, the Arnoldi algorithm was used in addition, because of its lean performance with respect to required memory. The first test case was the parallel Blasius boundary layer. Because of its one-dimensional nature, this flow has been intensively analysed in the past by means of the classic local stability analysis type (LST). This allowed the BiGlobal analysis of this mean flow to be thoroughly verified in both the incompressible and supersonic regime.

The second case involved the developing incompressible Blasius boundary layer. This flow was chosen because of its better affinity with the actual Blasius boundary layer flow, which has an intrinsic developing nature. The BiGlobal approach involved artificial in- and outflow boundary conditions. Analyses were performed on a domain with a small and large streamwise extent to focus on a flow that is weakly and strongly developing, respectively. The former analyses were again compared to LST simulations to yield an internal verification and consistency check. The results of the analyses on the larger domain could be compared to the literature and were found to agree well in a qualitative sense. The Tollmien-Schlichting branch obtained in this study was found to lie too high with respect to the one reported in the literature. Although the exact reason for this could not yet be established, the most likely cause is a (small) difference in the prescribed mean flow. It is expected that the test case will yield identical results when exactly the same mean flow will be used, as some key differences can be identified in the literature in this regard. It was found that the artificial boundary conditions caused an odd/even effect with respect to the continuous eigenmode branches in the spectrum when the number of points in the streamwise direction was taken to be either odd or even. A similar behaviour was observed when consulting the literature, although the effect was never elaborated on explicitly.

Lastly, the incompressible complex lamellar bidirectional vortex was considered. This mean flow is defined on a cylindrical coordinate system and is highly inhomogeneous in at least two spatial directions. Therefore, this case requires the BiGlobal approach and all power of the newly developed tools could be tested. A test case handled in the literature was very precisely reconstructed. Although it was found that no part of the spectrum was converged, the results were nearly identically retrieved.

The solutions to all three test cases have been obtained successfully and compare reasonably well with the literature. It is therefore concluded that all capabilities of the newly developed tools have been tested successfully and the tools can be considered to be verified.
Acknowledgements

First of all, I like to thank Dr. Fabio Pinna for his excellent guidance, comments and advice during the evolution of my master thesis project at the Technical University of Delft (TU Delft) and my internship at the Von Kármán Institute (VKI). Both the internship and master thesis project were precisely the projects I was hoping to be working on; exhausting all power of the word precisely. Also, I greatly appreciate his efforts for taking the time to visit to the university for the project meetings and his never ceasing availability for frequent contact and discussions via Skype. I hope that our partnership will continue to grow in the future and will prove to be fruitful for future collaborations between the VKI and the university.

I like to thank Dr. Bas van Oudheusden especially for the opportunity to work on a thesis project that was building on the internship executed at the VKI. Furthermore, I like to thank him for his supervision, guidance and feedback given throughout the master project.

I am thankful for the guidance and useful high-level hints of Dr. Henk Schutteleaars with respect to the used numerical methods and the currently used discretization method.

Furthermore, I would like to thank Dr. M. Kotsonis for useful references with respect to mappings for semi-infinite domains.

I thank Dr. Joshua Will Batterson of the University of Tennessee Space Institute for his highly elaborative mail conversations and for the Matlab function that allowed me to precisely reconstruct the mean flow he used for his PhD work. Due to his efforts in these forms and work, I was able to intensively verify the current tools using a highly sophisticated test case.

I like to thank all my student-colleagues at the VKI and those at TU Delft for the wonderful time we had.

I want thank my family for their encouragements and support.

Last, but certainly not least, I want to thank my partner, Jorien Meijer, for her love, support and (mainly) endurance during those times in which the thesis work consumed me fully.
3.3.1 Continuity equations ........................................ 46  
3.3.2 Momentum equations ..................................... 47  
3.3.3 Energy equations ......................................... 47  

4 Derivation Results and Verification 51  
4.1 Derived Equations ........................................... 51  
4.2 General Comments on Verification Procedure .............. 52  
4.3 Incompressible Equation Verifications ...................... 54  
4.3.1 Cartesian Spanwise Equations .......................... 55  
4.3.2 Cartesian Streamwise Equations ......................... 56  
4.3.3 Curvilinear Equations .................................. 56  
4.4 Compressible Equation Verifications ....................... 59  
4.4.1 Robitaillié-Montané [70] ................................. 60  
4.4.2 Theofilis [84] ............................................ 61  
4.4.3 Theofilis & Colonius [86] ............................... 62  
4.4.4 Robinet [69] ............................................. 63  
4.5 Declaration of Verification ................................ 64  

5 Discretization and Implementation 67  
5.1 Chebyshev Polynomials .................................... 67  
5.2 Characteristic Lagrange Polynomials ....................... 69  
5.3 Spectral Derivatives ....................................... 73  
5.4 Mapping proposed by Malik [59] ............................ 76  
5.5 Amplitude-Coefficient Matrices (ACM’s) .................... 77  
5.5.1 (Generalized) Eigenvalue Problem Level ............... 78  
5.5.2 Derivative Level ....................................... 79  
5.5.3 Equation-Variable Level ............................... 79  
5.5.4 Procedure for Quadratic Eigenvalue Problem .......... 81  
5.6 Automatic Equation Implementation Process .............. 82  
5.7 Boundary Conditions ..................................... 84  
5.7.1 Dirichlet conditions .................................... 85  
5.7.2 Neumann conditions .................................... 86  
5.7.3 Robin conditions ....................................... 87  
5.7.4 Periodic conditions .................................... 88  
5.7.5 Compatibility conditions ............................... 90  
5.8 Automatic Boundary Condition Implementation Process ..... 91  

6 Eigensolver Essentials 95  
6.1 The QZ Algorithm: Accuracy of Matlab’s eig ............. 95  
6.2 Accuracy of the Mean Flow ................................. 98  
6.3 Bauer-Fike Theorem & Numerical Modes ................... 100  
6.4 The Arnoldi Algorithm: Essentials of Matlab’s eigs ..... 101  
6.5 Local Solvers & Absolutely Small Eigenfunctions .......... 103  

Table of contents
## Table of contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Parallel Blasius Boundary Layer</td>
<td>107</td>
</tr>
<tr>
<td>7.1</td>
<td>Mean Flow Set-up and Characteristics</td>
<td>107</td>
</tr>
<tr>
<td>7.2</td>
<td>Mean Flow Accuracy</td>
<td>111</td>
</tr>
<tr>
<td>7.3</td>
<td>LST Simulations</td>
<td>111</td>
</tr>
<tr>
<td>7.3.1</td>
<td>Set-up and Results</td>
<td>112</td>
</tr>
<tr>
<td>7.3.2</td>
<td>Convergence of the Most Unstable Modes w.r.t. ( N_\eta )</td>
<td>115</td>
</tr>
<tr>
<td>7.3.3</td>
<td>Convergence of the Spectra w.r.t. ( N_\eta )</td>
<td>117</td>
</tr>
<tr>
<td>7.3.4</td>
<td>Convergence with respect to ( y_{\text{max}} )</td>
<td>119</td>
</tr>
<tr>
<td>7.3.5</td>
<td>Solving the ( M = 10^{-6} ) Case with the Compressible Equations</td>
<td>120</td>
</tr>
<tr>
<td>7.4</td>
<td>BiGlobal Simulations</td>
<td>122</td>
</tr>
<tr>
<td>7.4.1</td>
<td>Set-up &amp; Results</td>
<td>122</td>
</tr>
<tr>
<td>7.4.2</td>
<td>Convergence of the Most Unstable Mode w.r.t. ( N_\eta ) and ( N_\xi )</td>
<td>128</td>
</tr>
<tr>
<td>7.4.3</td>
<td>Convergence of the Spectrum w.r.t. ( N_\eta ) and ( N_\xi )</td>
<td>130</td>
</tr>
<tr>
<td>7.4.4</td>
<td>Spatial Stability Calculation</td>
<td>135</td>
</tr>
<tr>
<td>7.5</td>
<td>Parallel Blasius Flow Conclusions</td>
<td>137</td>
</tr>
<tr>
<td>8</td>
<td>Developing Blasius Boundary Layer</td>
<td>139</td>
</tr>
<tr>
<td>8.1</td>
<td>Mean Flow Set-up and Characteristics</td>
<td>140</td>
</tr>
<tr>
<td>8.2</td>
<td>BiGlobal Simulations</td>
<td>142</td>
</tr>
<tr>
<td>8.2.1</td>
<td>Set-up: Reference Domain used by Alizard [1]</td>
<td>143</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Set-up: Small Domain</td>
<td>145</td>
</tr>
<tr>
<td>8.3</td>
<td>Analysis of the Small Domain (SD)</td>
<td>149</td>
</tr>
<tr>
<td>8.3.1</td>
<td>SD: Spectrum</td>
<td>149</td>
</tr>
<tr>
<td>8.3.2</td>
<td>SD: Convergence Characteristics w.r.t. ( N_\eta ) and ( N_\xi )</td>
<td>153</td>
</tr>
<tr>
<td>8.3.3</td>
<td>SD: Domain of Validity &amp; Converged Mode Analysis</td>
<td>157</td>
</tr>
<tr>
<td>8.3.4</td>
<td>SD: Verification with LST</td>
<td>160</td>
</tr>
<tr>
<td>8.3.5</td>
<td>SD: Utilizing the Secondary Robin Boundary Conditions</td>
<td>165</td>
</tr>
<tr>
<td>8.3.6</td>
<td>SD: Arnoldi Algorithm Assessment</td>
<td>169</td>
</tr>
<tr>
<td>8.3.7</td>
<td>Convergence w.r.t. ( N_\xi ) and Odd/Even Behaviour Revisited</td>
<td>172</td>
</tr>
<tr>
<td>8.4</td>
<td>Intermediate Conclusions</td>
<td>176</td>
</tr>
<tr>
<td>8.5</td>
<td>Analysis of the Large Domain (LD)</td>
<td>177</td>
</tr>
<tr>
<td>8.5.1</td>
<td>LD: Spectrum</td>
<td>177</td>
</tr>
<tr>
<td>8.5.2</td>
<td>LD: Convergence w.r.t. ( N_\eta ) and ( N_\xi )</td>
<td>181</td>
</tr>
<tr>
<td>8.5.3</td>
<td>LD: Verification with Alizard [1] and Alizard &amp; Robinet [2]</td>
<td>182</td>
</tr>
<tr>
<td>8.5.4</td>
<td>LD: Application of Variations of the Robin Conditions</td>
<td>185</td>
</tr>
<tr>
<td>8.6</td>
<td>Developing Blasius Flow Conclusions</td>
<td>187</td>
</tr>
<tr>
<td>9</td>
<td>Complex Lamellar Bidirectional Vortex</td>
<td>195</td>
</tr>
<tr>
<td>9.1</td>
<td>Mean Flow Set-up and Characteristics</td>
<td>195</td>
</tr>
<tr>
<td>9.2</td>
<td>BiGlobal Simulations</td>
<td>200</td>
</tr>
<tr>
<td>9.3</td>
<td>Verification with Batterson’s [10] Results</td>
<td>202</td>
</tr>
<tr>
<td>9.4</td>
<td>Convergence Assessment</td>
<td>203</td>
</tr>
<tr>
<td>9.5</td>
<td>Complex Lamellar Bidirectional Vortex Conclusions</td>
<td>209</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------------------------------------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>10 Conclusions &amp; Recommendations</td>
<td>213</td>
<td></td>
</tr>
<tr>
<td>A Appendix A: Incompressible LST Equations</td>
<td>221</td>
<td></td>
</tr>
<tr>
<td>B Appendix B: Compressible LST Equations</td>
<td>223</td>
<td></td>
</tr>
<tr>
<td>C Appendix C: Incompressible BiGlobal Stability Equations</td>
<td>225</td>
<td></td>
</tr>
<tr>
<td>C.1 Streamwise Cartesian System</td>
<td>225</td>
<td></td>
</tr>
<tr>
<td>C.2 Streamwise Cylindrical System</td>
<td>226</td>
<td></td>
</tr>
<tr>
<td>C.3 Spanwise Cartesian System</td>
<td>227</td>
<td></td>
</tr>
<tr>
<td>D Appendix D: Compressible BiGlobal Stability Equations</td>
<td>229</td>
<td></td>
</tr>
<tr>
<td>D.1 Streamwise Cartesian System</td>
<td>229</td>
<td></td>
</tr>
<tr>
<td>D.2 Streamwise Cylindrical System</td>
<td>232</td>
<td></td>
</tr>
<tr>
<td>D.3 Spanwise Cartesian System</td>
<td>235</td>
<td></td>
</tr>
<tr>
<td>E Appendix E: Compressible Verification Equations</td>
<td>239</td>
<td></td>
</tr>
<tr>
<td>E.1 Verification with Robitaillé-Montané</td>
<td>239</td>
<td></td>
</tr>
<tr>
<td>E.2 Verification with Theofilis</td>
<td>243</td>
<td></td>
</tr>
<tr>
<td>E.3 Verification with Theofilis &amp; Colonius and Robinet</td>
<td>249</td>
<td></td>
</tr>
<tr>
<td>References</td>
<td>255</td>
<td></td>
</tr>
</tbody>
</table>
List of Figures

1.1 Trend of the skin friction coefficient along the SD7003 airfoil, taken from Catalano & Tognaccini [19]. ......................................................... 1
1.2 Heating transfer rate on the surface of a re-entry vehicle, taken from Hamilton et al. [38]. ............................................................... 1
1.3 Illustration of Reynolds’ experiment in his paper published in 1883. .... 3
1.4 Paths to turbulence in wall layers, original by Reshotko [68]. ............ 5
1.5 Illustration of all mechanisms involved in the phenomenon of transition edited from figure 16.23 of Schlichting [74]. It is illustrated which mechanisms are incorporated by which flow predictive methods. ............... 7

2.1 Illustration of the stable and unstable manifolds for the mean solution $\overline{Q}$ and the effect of non-normality of a differential operator, partially inspired by figure 4.1 of Schmid & Henningson [75]. ........................................... 13
2.2 Typical geometry that can be analysed with TriGlobal analysis. ......... 16
2.3 Typical geometry to which streamwise BiGlobal analysis can be applied. 36
2.4 Typical geometry to which spanwise BiGlobal analysis can be applied (parallel flow assumed in the $x$-direction). ................................. 36
2.5 Typical geometry to which LST analysis is applied (parallel flow assumption applied). ................................................................. 37
2.6 Typical geometry to which PSE analysis can be applied (first order non-parallel effects taken into account). ........................................ 37
2.7 Typical geometry to which BiGlobal PSE analysis can be applied (first order non-parallel effects taken into account). ......................... 38
2.8 Illustration of even (dotted) and odd (dashed) variables around the point $x_0$. ................................................................. 39
2.9 Illustration of a vector’s components around symmetric and anti-symmetric boundaries. ......................................................... 39

3.1 Block diagram of the derivation of the Stability Equations. ............... 44

5.1 Illustration of all independent characteristic Lagrange polynomials for $N = 5$ CGL collocation points, see equation (5.7) or (5.10). .......... 70
Table of contents

5.2 Illustration of two-dimensional characteristic Lagrange polynomials $\psi_{32}$, $\psi_{13}$ and $\psi_{44}$, for $N_x \times N_y = 5 \times 5$ CGL collocation points, see equation (5.12) .......................................................... 70

5.3 The points are equally divided around the location $y_i$ (incompressible LST case with $N_y = 60$) .......................................................... 77

5.4 Indication of the rows corresponding to the boundaries of the domain. . . 85

5.5 Indication of the Neumann boundary condition at $\xi = -1$ ................. 87

5.6 Indication of the Neumann boundary condition at $\eta = -1$ .......... 87

5.7 Indication of the periodic boundary condition across $\xi = \pm 1$ .......... 89

5.8 Indication of the periodic boundary condition at $\eta = \pm 1$ .......... 89

5.9 “Clean” matrix $A$ from equation (5.28) corresponding to a compressible framework using the $(p, T)$-formulation, no boundary conditions have been applied. .......................................................... 92

5.10 “Final” matrix $A$ corresponding to figure 5.9, incorporating Neumann conditions on all variables at all boundaries (for clarity). Red dots indicate related coefficients. .......................................................... 92

5.11 “Final” matrix $A$ completely analogous to figure 5.10, except that compatibility conditions are applied to the pressure at all boundaries. Red dots indicate related coefficients. .......................................................... 93

6.1 LST eigenvalue convergence curves for incompressible case. ............ 97

6.2 LST eigenvalue convergence curves for supersonic case. ................. 97

6.3 LST eigenvalue convergence curves that fail to pass algorithm precision in the supersonic case, $\varepsilon_{\text{mean}} = O(10^{-5})$, and those as presented in figure 6.2 ................................................................................. 99

6.4 Location of the numerical mode in the (compressible) LST spectrum. . 101

6.5 Illustration of the eigenfunction $\tilde{\mathbf{v}}$ corresponding to the numerical $P_0$ mode indicated in figure 6.4. ................................................................................. 101

6.6 Demonstration of the diverging behaviour of the numerical $P_0$ mode shown in figure 6.4. ................................................................................. 102

7.1 Horizontal velocity and temperature profile of the compressible Blasius boundary layer ($Re = 3000$ and $M = 2.5$). (Digital version is animated) .. 110

7.2 Zoom on physically interesting LST spectrum for the incompressible ($M = 10^{-6}$) mean flow. .......................................................... 113

7.3 Zoom on physically interesting LST spectrum for the supersonic ($M = 2.5$) mean flow. .......................................................... 113

7.4 Absolute value of the LST perturbation amplitudes relative to the maximum of $|\tilde{u}|$ for the incompressible ($M = 10^{-6}$) mean flow. ............ 115

7.5 Absolute value of the LST perturbation amplitudes relative to the maximum of $|\tilde{u}|$ for the supersonic ($M = 2.5$) mean flow. ............ 115

7.6 Convergence of the LST spectrum for the incompressible ($M = 10^{-6}$) mean flow. “IS” means Indistinguishable Scale, values are indicated in the text. .......................................................... 118
7.7 Convergence of the LST spectrum for the supersonic \((M = 2.5)\) mean flow. 119
7.8 LST eigenvalue convergence curves with respect to \(y_{\text{max}}\) for the incompressible \((M = 10^{-6})\) case. 120
7.9 LST eigenvalue convergence curves with respect to \(y_{\text{max}}\) for the supersonic \((M = 2.5)\) case. 120
7.10 Absolute compressible LST temperature profile in \(M = 10^{-6}\) mean flow calculated with \(e_\text{ig}\) alone and the refined solutions using the Rayleigh and Newton-Raphson local iterative methods. 121
7.11 Horizontal velocity and temperature profile of the compressible Blasius boundary layer augmented together to yield a two-dimensional mean flow \((Re = 3000\) and \(M = 2.5)\) for spanwise BiGlobal analysis. Boundary conditions for the stability problem are indicated. 123
7.12 Zoom on physically interesting BiGlobal spectrum for the incompressible \((M = 10^{-6})\) case. 124
7.13 Zoom on physically interesting BiGlobal spectrum for the supersonic \((M = 2.5)\) case. 124
7.14 Comparison of LST and BiGlobal absolute \(\tilde{u}, \tilde{v}\) and \(\tilde{p}\) profiles in incompressible \((M = 10^{-6})\) mean flow, where \(N_\zeta \times N_\eta = 5 \times 90\). 125
7.15 Comparison of LST and BiGlobal absolute \(\tilde{w}\) profiles in incompressible \((M = 10^{-6})\) mean flow with refinements, where \(N_\zeta \times N_\eta = 5 \times 90\). 125
7.16 Comparison of LST and BiGlobal absolute \(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T}\) and \(\tilde{p}\) profiles in supersonic \((M = 2.5)\) mean flow, where \(N_\zeta \times N_\eta = 12 \times 130\). 126
7.17 LST and BiGlobal convergence for the incompressible \((M = 10^{-6})\) mean flow with \(N_\zeta = 5\) within the convergence region designated in figure 6.1. 128
7.18 LST and BiGlobal convergence for the supersonic \((M = 2.5)\) mean flow with \(N_\zeta = 12\) within the convergence region designated in figure 6.2. 128
7.19 Convergence of the BiG spectrum with respect to \(N_\zeta\) for the incompressible \((M = 10^{-6})\) mean flow for \(N_\zeta = 5\). 131
7.20 Illustration of the scaled real part of the \(\tilde{v}\) amplitude corresponding to the \(\beta \neq 0\) modes forming a group with the most unstable mode. See the magenta box in figure 7.19. 132
7.21 Convergence of the BiG spectrum with respect to \(N_\zeta\) for the incompressible \((M = 10^{-6})\) mean flow for \(N_\eta = 90\). 133
7.22 Convergence of the BiG spectrum with respect to \(N_\zeta\) for the supersonic \((M = 2.5)\) mean flow for \(N_\xi = 12\). 134
7.23 Convergence of the BiG spectrum with respect to \(N_\zeta\) for the supersonic \((M = 2.5)\) mean flow for \(N_\eta = 130\). 135
7.24 Spatial BiGlobal spectrum for the incompressible parallel Blasius mean flow with \(N_\eta \times N_\zeta = 50 \times 5\). Modes with \(\Im\{\alpha\} < 0\) are unstable. 137
7.25 Absolute values of the spatial BiGlobal eigenfunctions \(\tilde{u}, \tilde{v}, \tilde{w}\) and \(\tilde{p}\) for the incompressible parallel Blasius mean flow with \(N_\eta \times N_\zeta = 50 \times 5\). 137
8.1 Developing compressible Blasius boundary layer (Re = 3000 and M = 2.5) set-up with dimensional and non-dimensional coordinate systems. (Digital version is animated) .......................................................... 141
8.2 Indication of the in- and outlet locations on the neutral curve for the reference/large (red line) and small (blue line) domain. .......................................................... 147
8.3 “Bird’s-eye view” of the physically interesting part of the spectrum. Small domain, N$_\xi$ $\times$ N$_\eta$ = 20 $\times$ 90. .......................................................... 150
8.4 Zoom on right half of the physically interesting part of the spectrum. Small domain, N$_\eta$ = 90. .......................................................... 151
8.5 Convergence with respect to even N$_\xi$ of the continuous branches to the exact locations $\Re\{\omega\} = \frac{n\pi}{L}$ for even n, with N$_\eta$ = 90. .......................................................... 154
8.6 Convergence with respect to odd N$_\xi$ of the continuous branches to the exact locations $\Re\{\omega\} = \frac{n\pi}{L}$ for odd n, with N$_\eta$ = 90. .......................................................... 155
8.7 Convergence with respect to N$_\eta$ of the eigenmodes in the domain of interest, with N$_\xi$ = 26. .......................................................... 156
8.8 Convergence with respect to N$_\eta$ of the eigenmodes in the domain of interest, with N$_\xi$ = 25. .......................................................... 157
8.9 Convergence with respect to N$_\eta$ of the eigenmodes in the domain of interest, with N$_\xi$ = 90. .......................................................... 158
8.10 Convergence with respect to N$_\xi$ of the eigenmodes in the domain of interest, with N$_\eta$ = 90. .......................................................... 159
8.11 Absolute value of the perturbation amplitudes of mode I relative to the maximum of $\tilde{u}$ for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 160
8.12 Absolute value of the perturbation amplitudes of mode II relative to the maximum of $\tilde{u}$ for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 160
8.13 Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode I for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 161
8.14 Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 162
8.15 Absolute value of the perturbation amplitudes of mode III relative to the maximum of $\tilde{u}$ for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 163
8.16 Absolute value of the perturbation amplitudes of mode IV relative to the maximum of $\tilde{u}$ for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 163
8.17 Absolute value of the perturbation amplitudes of mode V relative to the maximum of $\tilde{u}$ for N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 163
8.18 Illustration of the spectra with the secondary Robin boundary conditions with N$_\eta$ = 90. .......................................................... 166
8.19 Real and imaginary values of $\alpha_s$ as a function of $\omega$. .......................................................... 167
8.20 Real value of $\alpha_s$ as a function of the $\omega$. .......................................................... 168
8.21 Illustration of the spectra corresponding to the different implementations of the Robin boundary conditions with N$_\eta$ = 90. .......................................................... 169
8.22 Comparison of spectra returned by the QZ and Arnoldi algorithms, N$_\xi$ $\times$ N$_\eta$ = 26 $\times$ 90. .......................................................... 170
8.23 Comparison of eigenfunction returned by the QZ and Arnoldi algorithms, absolute value $|\tilde{u}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 171
8.24 Comparison of eigenfunction returned by the QZ and Arnoldi algorithms, absolute value $|\tilde{v}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 171
8.25 Comparison of eigenfunction returned by the QZ and Arnoldi algorithms, absolute value $|\tilde{w}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 172
8.26 Comparison of eigenfunction returned by the QZ and Arnoldi algorithms, absolute value $|\tilde{p}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 172
8.27 Convergence overview plot of modes I and II, of table 8.4, for increased odd and even values of $N_\xi$, $N_\eta = 90$ .... 173
8.28 Convergence plot of of mode I, for increased odd and even values of $N_\xi$, for $N_\eta = 90$ .... 174
8.29 Convergence plot of of mode II, for increased odd and even values of $N_\xi$, for $N_\eta = 90$ .... 174
8.30 Comparison of eigenfunction profile $|\tilde{u}|/\max|\tilde{u}|$ of mode I for the lowest odd and lowest even value of $N_\xi$, $N_\eta = 90$ .... 175
8.31 Comparison of eigenfunction profile $|\tilde{v}|/\max|\tilde{u}|$ of mode II for the lowest odd and lowest even value of $N_\xi$, $N_\eta = 90$ .... 175
8.32 Overview of the spectrum corresponding to the large domain, $N_\xi \times N_\eta = 153 \times 45$, $\omega_g = 0.045$ and $N_\omega = 1000$ .... 178
8.33 Convergence plot with respect to $N_\eta$ of the spectrum corresponding to the large domain, for $N_\xi = 135$ .... 179
8.34 Convergence plot with respect to $N_\xi$ of the spectrum corresponding to the large domain, for $N_\eta = 38$ .... 179
8.35 Convergence plot with respect to $N_\xi$ of the spectrum corresponding to the large domain, for $N_\eta = 45$ .... 180
8.36 Verification plot for the spectrum obtained using $N_\xi \times N_\eta = 153 \times 45$ (black ● and green ×) with the spectrum reported by Alizard [1] (blue ■ and red △, figure 4.9(a) on page 54) .... 182
8.37 Comparison of spectra corresponding to different Robin conditions to the large domain with slight deviations in the parameters with $N_\xi \times N_\eta = 135 \times 38$ and the primary implementation with $N_\xi \times N_\eta = 136 \times 38$ .... 185
8.38 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{u}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 189
8.39 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{v}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 189
8.40 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{p}|$ relative to the maximum of $|\tilde{u}|$ of mode I .... 189
8.41 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{u}|$ relative to the maximum of $|\tilde{u}|$ of mode II .... 190
8.42 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{v}|$ relative to the maximum of $|\tilde{u}|$ of mode II .... 190
8.43 Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{\rho}|$ relative to the maximum of $|\tilde{u}|$ of mode II. ...................................................... 190
8.44 Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II, for $N_\xi \times N_\eta = 15 \times 90$ .................................................. 191
8.45 Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II, for $N_\xi \times N_\eta = 16 \times 90$ .................................................. 191
8.46 Perturbation amplitude $\Re\{\tilde{u}\}/\max|\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$ ........................................ 192
8.47 Perturbation amplitude $\Re\{\tilde{u}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(b) on page 47). .................................................. 192
8.48 Perturbation amplitude $\Re\{\tilde{v}\}/\max|\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$ ........................................ 192
8.49 Perturbation amplitude $\Re\{\tilde{v}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(c) on page 47). .................................................. 192
8.50 Perturbation amplitude $\Re\{\tilde{p}\}/\max|\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$ ........................................ 193
8.51 Perturbation amplitude $\Re\{\tilde{p}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(d) on page 47). .................................................. 193
8.52 Contour plot of $\Re\{\tilde{u}\}/\max|\tilde{u}|$ corresponding to mode II with $N_\xi \times N_\eta = 153 \times 45$ ........................................ 193
8.53 Perturbation amplitude $\Re\{\tilde{u}\}$ corresponding to mode II reported by Alizard [1] (figure 4.2(a) on page 47). .................................................. 193

9.1 Illustration of the bidirectional vortex flow, flowing in at $r = 1$ at the top, $z = L_a$, moving down and up the chamber and exiting the chamber at the top. (Digital version is animated) .................................................. 196
9.2 Circumferential mean velocity component $\overline{V}$ of the complex lamellar bidirectional vortex. More than 50 nodes are used in the $r$-direction. .......... 198
9.3 Radial mean velocity component $\overline{U}$ of the complex lamellar bidirectional vortex. ............................................................... 199
9.4 Axial mean velocity component $\overline{W}$ of the complex lamellar bidirectional vortex. ............................................................... 199
9.5 Absolute value of the mean velocity field in the eigenfunction plane. ........ 200
9.6 Verification plot for the spectrum obtained using $N_\xi \times N_\eta = 50 \times 50$ (blue $\bullet$) with the spectrum reported by Batterson [10] (black $\square$, figure 6.4 on page 220). ................................................................. 201
9.7 Convergence of the spectrum with respect to $N_\xi$ obtained with the QZ algorithm, $N_\eta = 50$. .................................................. 203
9.8 Convergence of the spectrum with respect to $N_\eta$ obtained with the QZ algorithm, $N_\xi = 50$. .................................................. 204
9.9 Zoom on $\times$-shaped structure in the convergence plot of the spectrum with respect to $N_\xi$ obtained with the QZ algorithm, $N_\eta = 50$. ............... 205
9.10 Zoom on $\times$-shaped structure in the convergence plot of the spectrum with respect to $N_\eta$ obtained with the QZ algorithm, $N_\xi = 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 206
9.11 Zoom on the origin in the convergence plot of the spectrum with respect to $N_\eta$ obtained with the QZ algorithm, $N_\xi = 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 208
9.12 Zoom on the origin in the convergence plot of the spectrum with respect to $N_\eta$ obtained with the QZ algorithm, $N_\xi = 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 208
9.13 Perturbation amplitude $-\Im\{\tilde{u}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 211
9.14 Perturbation amplitude $-\Im\{\tilde{u}\}$ reported by Batterson [10] (Figure 6.7a on page 222). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 211
9.15 Perturbation amplitude $-\Im\{\tilde{v}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 211
9.16 Perturbation amplitude $-\Im\{\tilde{v}\}$ reported by Batterson [10] (Figure 6.7b on page 222). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 211
9.17 Perturbation amplitude $-\Im\{\tilde{w}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 212
9.18 Perturbation amplitude $-\Im\{\tilde{w}\}$ reported by Batterson [10] (Figure 6.7c on page 222). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 212
9.19 Perturbation amplitude $-\Im\{\tilde{p}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 212
9.20 Perturbation amplitude $-\Im\{\tilde{p}\}$ reported by Batterson [10] (Figure 6.7d on page 222). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 212
List of Tables

4.1 Incompressible cross-verification references .......................... 54
4.2 Compressible cross-verification references .......................... 59

5.1 Terms that drop when considering streamwise or spanwise temporal or spatial analysis in *Cartesian coordinate systems*. .......................... 81

7.1 Key parameters for the most unstable modes of the parallel Blasius boundary layer [59, 65]. .............................................. 112
7.2 LST eigenvalue convergence table for incompressible \((M = 10^{-6})\) case. .................. 116
7.3 LST eigenvalue convergence table for supersonic \((M = 2.5)\) case. .................. 116
7.4 LST eigenvalue comparison with Mack [57], Malik [59] and Pinna [65], using \(N_\eta = 90\) and 130 for the \(M = 10^{-6}\) and 2.5 cases, respectively. .................. 117
7.5 BiGlobal eigenvalue \(N_\xi\) convergence table. .......................... 129
7.6 BiGlobal eigenvalue \(N_\eta\) convergence table for incompressible \((M = 10^{-6})\) case. .............................................. 130
7.7 BiGlobal eigenvalue \(N_\eta\) convergence table for supersonic \((M = 2.5)\) case. .................. 130
7.8 Comparison of the spatial BiGlobal eigenvalues with the LST results reported by Danabasoglu & Biringen [21], Jordinson [47] and Pinna [65], using \(N_\eta \times N_\zeta = 50 \times 5\). .............................................. 136

8.1 Parameters for the developing Blasius boundary layer replicating the results of Alizard [1]. .............................................. 143
8.2 Illustration of used procedure to obtain parameters in Gaster-type transformation at the outlet. .............................................. 144
8.3 Parameters for the developing Blasius boundary layer on the small domain. Used length scale: \(l\). .............................................. 147
8.4 Features of the converging modes, maximum converging error is taken over both odd and even \(N_\xi\) simulations. .............................................. 156
8.5 Input parameters, results and comparison of the LST and BiGlobal eigenvalues for modes I and II. .............................................. 161
8.6 Comparison of eigenvalues A to D returned by the QZ and Arnoldi algorithms. .............................................. 172
# List of Symbols

## Acronyms
- ACM: Amplitude Coefficient Matrix
- BiG: BiGlobal
- CAS: Computer Algebra Software
- DNS: Direct Numerical Simulation
- IS: Indistinguishable Scale
- LD: Large Domain
- LES: Large Eddy Simulation
- LST: Linear Stability Theory (and the corresponding equations)
- PSE: Parabolized Stability Equations
- RAM: Random Access Memory
- RANS: Reynolds Averaged Navier-Stokes
- LHS: Left Hand Side
- SD: Small Domain
- VESTA: VKI Extensible Stability and Transition Analysis toolkit
- VKI: Von Kármán Institute
- WKB: Wentzel-Kramers-Brillouin

## Roman symbols - Indicated units correspond to dimensional variables
- $A$: amplitude coefficient matrix or streamwise amplitude function
- $a$: speed of sound or chamber radius
- $B$: amplitude coefficient matrix
- $C$: set of all complex numbers
- $c$: phase or group speed
- $c_p$: specific heat at constant pressure
- $c_v$: specific heat at constant volume
- $e$: internal energy
- $Ec$: Eckert number based on freestream
- $f$: general function or Blasius stream function
- $g$: general function or Blasius enthalpy profile
- $g_{ij}$: (covariant) metric tensor
- $h$: specific enthalpy
- $h_i$: scale factor
- $k$: thermal conductivity

---

Continued on next page
**List of Symbols**

**Roman symbols - Continued from previous page**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{O}$</td>
<td>“Big O”-symbol denoting the order of magnitude</td>
</tr>
<tr>
<td>$P$</td>
<td>mean pressure</td>
</tr>
<tr>
<td>$P_0$</td>
<td>“$P_0$” numerical mode</td>
</tr>
<tr>
<td>$Pr$</td>
<td>Prandtl number based on freestream</td>
</tr>
<tr>
<td>$p$</td>
<td>static pressure</td>
</tr>
<tr>
<td>$Q$</td>
<td>general entity</td>
</tr>
<tr>
<td>$Q^i$</td>
<td>contravariant $i$-component of general vector</td>
</tr>
<tr>
<td>$Q_{,i}$</td>
<td>$i$-component of the covariant derivative of a general entity</td>
</tr>
<tr>
<td>$q$</td>
<td>cylindrical spanwise wave number</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>set of all real numbers</td>
</tr>
<tr>
<td>$R$</td>
<td>gas constant</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$r$</td>
<td>radial direction</td>
</tr>
<tr>
<td>$K$</td>
<td>curvature, see [32]</td>
</tr>
<tr>
<td>$L$</td>
<td>characteristic length in the streamwise direction</td>
</tr>
<tr>
<td>$l$</td>
<td>characteristic length scale</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number based on freestream</td>
</tr>
<tr>
<td>$N$</td>
<td>number of collocation points or $N$-factor of the $e^N$-method</td>
</tr>
<tr>
<td>$S$</td>
<td>Sutherland temperature</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature, Chebyshev polynomial or transformation</td>
</tr>
<tr>
<td>$T^{ij}$</td>
<td>stress tensor</td>
</tr>
<tr>
<td>$\mathbf{T}$</td>
<td>stress tensor in vector notation</td>
</tr>
<tr>
<td>$u'$</td>
<td>perturbation 1-velocity component (streamwise)</td>
</tr>
<tr>
<td>$\overline{U}$</td>
<td>mean 1-velocity component</td>
</tr>
<tr>
<td>$v'$</td>
<td>perturbation 2-velocity component (wall normal)</td>
</tr>
<tr>
<td>$V$</td>
<td>vortex Reynolds number</td>
</tr>
<tr>
<td>$\mathbf{V}$</td>
<td>mean 2-velocity component</td>
</tr>
<tr>
<td>$w'$</td>
<td>perturbation 3-velocity component (spanwise)</td>
</tr>
<tr>
<td>$\overline{W}$</td>
<td>mean 3-velocity component</td>
</tr>
<tr>
<td>$x$</td>
<td>streamwise direction</td>
</tr>
<tr>
<td>$y$</td>
<td>wall-normal direction</td>
</tr>
<tr>
<td>$y_i$</td>
<td>center of collocation point distribution for the mapping introduced by Malik</td>
</tr>
<tr>
<td>$z$</td>
<td>spanwise or axial direction</td>
</tr>
</tbody>
</table>

**Greek symbols - Indicated units correspond to dimensional variables**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1-direction (streamwise) wave number</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>“$\alpha_0$” mode corresponding to $\alpha_s = 0$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>3-direction (spanwise) wave number</td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>displacement thickness</td>
</tr>
</tbody>
</table>

*Continued on next page*
**Greek symbols** - Continued

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$, $\varepsilon$</td>
<td>small parameter</td>
<td>general</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>computational spanwise direction</td>
<td>m</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>ratio of specific heats</td>
<td>—</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Kronecker delta</td>
<td>—</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>small parameter</td>
<td>—</td>
</tr>
<tr>
<td>$\eta$</td>
<td>computational wall-normal direction or Blasius similarity variable</td>
<td>m</td>
</tr>
<tr>
<td>$\eta_j$</td>
<td>Chebyshev Gauss-Lobatto point in $\eta$-direction</td>
<td>m</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>general phase function</td>
<td>—</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle or growth</td>
<td>rad or —</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>eigenfunction condition number or inlet parameter</td>
<td>—</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>second coefficient of viscosity</td>
<td>kgm$^{-1}$s$^{-1}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>first coefficient of viscosity</td>
<td>kgm$^{-1}$s$^{-1}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>kinematic viscosity coefficient</td>
<td>m$^2$s$^{-1}$</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>eigenvector in temporal (linear) problem</td>
<td>general</td>
</tr>
<tr>
<td>$\xi$</td>
<td>computational streamwise coordinate or Blasius similarity variable</td>
<td>m</td>
</tr>
<tr>
<td>$\xi_i$</td>
<td>Chebyshev Gauss-Lobatto point in $\xi$-direction</td>
<td>m</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
<td>kgm$^{-3}$</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>characteristic Lagrange polynomial around point $\xi_j$</td>
<td>—</td>
</tr>
<tr>
<td>$\omega$</td>
<td>frequency of perturbations</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>$\omega_{im}$</td>
<td>maximal $\Im{\omega}$ for a given $Re$</td>
<td>s$^{-1}$</td>
</tr>
</tbody>
</table>

**Sub- and Superscripts**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>freestream value</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}$</td>
<td>mean value of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}^d$</td>
<td>dimensional equivalent of ${\cdot}$ (with physical units)</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}'$</td>
<td>perturbation of ${\cdot}$ or (explicitly) non-dimensional value of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>$\tilde{{\cdot}}$</td>
<td>perturbation amplitude of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}^*$</td>
<td>complex conjugate of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_e$</td>
<td>evaluation of ${\cdot}$ at the edge of the boundary layer</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_{\text{exact}}$</td>
<td>exact version of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}^i$</td>
<td>contravariant i-component of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_i$</td>
<td>covariant i-component, or imaginary value of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>$\Im{\cdot}$</td>
<td>imaginary part of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_{\text{max}}$</td>
<td>maximal value of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_{N \times M}$</td>
<td>discrete version of ${\cdot}$ with $N \times M$ collocation points</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_r$</td>
<td>real value of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>$\Re{\cdot}$</td>
<td>real part of ${\cdot}$</td>
<td>—</td>
</tr>
<tr>
<td>${\cdot}_s$</td>
<td>spatial variant of ${\cdot}$</td>
<td>—</td>
</tr>
</tbody>
</table>

*Continued on next page*
Sub- and Superscripts - *Continued from previous page*

- $\{\cdot\}_t$: temporal variant of $\{\cdot\}$
- $\{\cdot\}_0$: evaluation of $\{\cdot\}$ at the neutral stability curve
- $\{\cdot\}_1$: one-dimensional version of $\{\cdot\}$
Chapter 1

Introduction

Relevance & Background

One of the most important performance characteristics of a commercial airliner is its drag. By reducing the drag coefficient of an airliner by one drag count ($10^{-4}$ unit, see Yechout & Morris [93]) one saves millions of euros in fuel consumption each year.

The laminar to turbulent transition region is the place where a regular laminar flow changes to a chaotic turbulent one. The skin friction associated with a laminar flow is generally significantly less than with the turbulent region. Some trends are shown in figure 1.1, from Catalano & Tognaccini [19]. Transition occurs on the wings of commercial airliners. Therefore, precise knowledge of the location of the transition region is of paramount importance for the determination of the total drag.

![Figure 1.1: Trend of the skin friction coefficient along the SD7003 airfoil, taken from Catalano & Tognaccini [19].](image1)

![Figure 1.2: Heating transfer rate on the surface of a re-entry vehicle, taken from Hamilton et al. [38].](image2)

When a re-entry vehicle returns to earth, it encounters the atmosphere at very high velocities, corresponding to Mach 32 and above. These high velocities give rise to very large forces consisting of mainly drag and to very high heat loads. Drag is, conversely to the case in low speed aerodynamics, embraced, because it helps to decelerate the vehicle to a safe landing speed.
In the high velocity regime, the laminar to turbulent transition region corresponds to a high increase in the heat transfer to the hull of the vehicle. This is illustrated in figure 1.2 from Hamilton et al. [38]. A good knowledge of the location of the transition region therefore provides crucial information concerning heat transfer.

In turn, the latter entities are of indistinguishable importance for the safe progression of the re-entry. A critical discrepancy in the force vector can cause the vehicle to spin out of control. A heat transfer increase at the wrong place can cause the vehicle to incinerate in the atmosphere. In other words: transition is an important phenomenon concerning re-entry aerodynamics as well as for commercial airliners.

Having a tool that can quickly provide accurate information on the transition region with respect to a given flow situation, and therefore function as an engineering tool regarding this problem, is of great value. Generally speaking, one can approach the transition phenomenon in multiple ways. To be able to decide on which kind of approach to focus regarding the former statement, first the essentials of these approaches must be briefly considered. This is done most appropriately by starting at the first introduction of the transition phenomenon in history.

Physics of Classical Transition and Stability

The first experimental analyses of the laminar to turbulent transition phenomenon were performed by Reynolds in 1883. He was the first to properly distinguish the laminar and turbulent flow regimes and to determine the relevant influence to cause the different flow regimes. His experiments involved injecting ink into a water flow through a pipe, using the set-up illustrated in figure 1.3, taken from his paper published in 1883. Due to his contributions, his name is attached to the dimensionless parameter \( \text{Re} \), the ratio of inertial over viscous forces that governs this transition phenomenon. It is interesting to mention that a contemporary of Reynolds, well-known for his own contributions in the field of fluid dynamics: Stokes, was initially against introducing the Reynolds number. He was convinced that “the dimensional properties are already involved in the equations of motion; and there is absolutely nothing to prove that he [Reynolds] has discovered the necessity of an additional constant to define a fluid.” See the review of Jackson & Launder [45].

Considering a specific mean flow, a laminar jet of water, for example. When a disturbance is given to this jet, this disturbance might grow and could ultimately cause the jet to become turbulent. Modes that cause this kind of behaviour are called unstable, whereas modes that decay are referred to as stable. Analysing these kinds of disturbances or perturbations is referred to as performing stability analysis. Stability analysis is one of the classical approaches to transition, although the latter involves far more mechanisms than only unstable perturbation. The will be illustrated more thoroughly later on.

Normally, fluid viscosity is a phenomenon that is associated with the introduction of a damping mechanism. It is responsible for the dissipation of kinetic energy, for example. Hence, the first investigations of fluid stability by Rayleigh, in 1887, were based on inviscid mechanisms only. It was argued that viscosity would only yield a stabilizing
effect on the perturbations. Rayleigh derived the nowadays well-known inflection point criterion. This criterion dictates that, when an inflection point is present in the velocity profile, this is a sufficient condition for the existence of (inviscid) unstable perturbations on the flow. That is to say, perturbations tend to grow and are ultimately leading to the transition of the laminar flow into a turbulent equivalent.

Strong adverse pressure gradients can cause inflection points (in the viscous boundary layer, for example). In relation to the inflection point criterion, those pressure gradients can therefore be responsible for unstable perturbations. Despite of this, examples of flows exist, such as the Poiseuille flow, in which transition occurs while no inflection point is present, see Theofilis [85] for example.

In 1921, Prandtl identified the boundary layer. In combination with stability theory, this brought insight regarding the destabilizing influence of viscosity. Attention was focused in the Orr-Sommerfeld equation, which dates back to 1905. This equation takes into account the latter phenomenon with respect to Rayleigh’s inviscid equation. Because of this, the Orr-Sommerfeld equation contains the Reynolds number as an important parameter. In 1935 and 1933, respectively, Tollmien and Schlichting first showed that this equation has unstable eigenmodes (inherent perturbation shapes) for flows that do not have inflection points, when the Reynolds number is taken to be larger than a certain limit. This limit is referred to as the critical Reynolds number.

Over the years, computer capabilities started to reach a more advanced level and allowed the more thorough investigation of fluid dynamic stability and transition in general. The stability of compressible flows, for example, has been researched intensively by Mack and Malik and their respective co-workers. For a small partition of the work of these men see references [7, 37, 53, 54, 55, 57, 56, 59, 79].

As briefly mentioned before, fluid stability only involves a small part of all mechanisms that lead to transition. Therefore, with the increasing computer capabilities, several other approaches to investigating transition have emerged. When all possible physics are to be included, one involves Direct Numerical Simulation (DNS), see Orszag [64] for the fundamentals. In the case of transition, this can be done by resolving all relevant (Kolmogorov micro-) scales [67, 81]; the full Navier-Stokes equations are solved whilst taking into account the smallest possible structures in a given flow. This automatically requires the use of very fine grids, which are computationally very expensive. Therefore, this method is only applied to flows that correspond to rather basic geometries.
Although counter-intuitive, another approach involves Large Eddy Simulation (LES), see the treatment of Sayadi & Moin [72]. In the conventional LES approach, it is usually assumed that energy “flows” from the large scale structures to the smaller ones, relying on the underlying features of the Richardson energy cascade. When considering transition, the energy cascade could be argued to be reversed. It is generally found that small scales activate large scales to ultimately yield transition. Therefore, applying LES to transition analysis is counter-intuitive; one would logically want to perform “small eddy simulation.”

An important drawback of LES approaches in general is the fact that the solution characteristics are found to depend strongly on the considered Sub-Grid Scale (SGS) models. These models are intrinsic to LES. Usually, these kinds of models have to be tuned for every specific case to yield satisfying results. These simulations are less expensive than DNS and are efficiently applied in regions far from solid walls. When a region near a wall has to be investigated, however, LES requires approximately the same resolution as DNS. Unfortunately, engineering applications often tend to involve these kinds of wall-bounded flows. Moreover, one of the primary requirements of a robust engineering tool is that the involved approach is not case dependent. Therefore, it is argued that LES cannot be implemented as such a tool efficiently.

Another approach to transition involves Reynolds Averaged Navier-Stokes (RANS) simulations. These simulations are cheaper than performing DNS and LES and perform well in regions near a solid wall. As will be explained in chapter 2, stability analysis involves perturbations that do not influence the mean flow, because of their inherent infinitesimal magnitudes. In the RANS approach, finite perturbations are considered. Those kinds of perturbations do have an influence on the mean flow field, through, for example, the Reynolds stresses. See Pope [67], for a thorough consideration. Intrinsic to the RANS approach is the closing problem for the behaviour of the perturbations. Turbulence models have to be introduced to fill in this gap. McKeel [60] handles a number of these models that can be applied to simulate transition. Examples of these models are: a $k$-$\omega$ model (based on the turbulent kinetic energy and frequency\(^1\)) that uses a roughness strip to fix the transition point, a one equation turbulence model that incorporates the first and second mode disturbances in the transition region and intermittency ($\gamma$) based models, which combine the laminar and turbulent flow field. It is generally found, however, as in the case of LES, that the coupled turbulence models all require significant tuning before transition can be predicted in a way that is satisfactory. Therefore, also this approach is dismissed regarding requirements of an engineering tool.

It can be seen that there are a lot of difficulties with respect to the proper numerical analysis of the transition phenomenon. There is no straightforward way in which the problem can be tackled. This is where the stability approach becomes rather interesting, because, although the analysis is fully linear, the approach can be adjusted to a given problem. For all aforementioned approaches, this is usually the other way around; those approaches lend themselves to be applied to a certain class of flows exclusively. Moreover,

\(^1\)In a way, stability analysis considers infinitesimally small eddies.

\(^2\)Here the frequency is related to the turbulent kinetic energy $k$ and the viscous dissipation $\epsilon$ by:

$$\omega = \frac{\epsilon}{k}$$
no tuning is required to yield satisfactory results.

Therefore the current thesis project considers the stability approach. The project is carried out as a collaboration between Delft University of Technology (TUD) and the Von Kármán Institute (VKI). At the VKI, the so called VKI Extensible Stability and Transition Analysis (VESTA) toolkit is under development, see Pinna [65]. This toolkit provides precisely those tools that can perform linear stability analysis on a specific mean flow.

Before the treatment proceeds, the stability approach must be placed into perspective regarding its capabilities. This is done by considering the different transition paths.

The overall complexity of the transition phenomenon is caused by the high number of driving mechanisms that participate in it. The phenomenon is critically altered by the considered geometry and the magnitude of relevant parameters, such as surface roughness, flow Mach number and freestream turbulence level. These different causes can lead to different transition paths. Reshotko [68] gives an overview of these paths for the case of a wall-bounded flow, as sketched in figure 1.4. It can be seen that all paths start with some disturbance to a certain flow. These disturbances are somehow entrained inside the boundary layer. The way in which this occurs is dictated by receptivity mechanisms.

In figure 1.4, it can be seen that from that point on, several paths can be defined which are usually encountered for different kinds of initial disturbance levels. When the initial perturbations are small enough, they can be described by a linear stability theory. This path is indicated by the letter A in figure 1.4 and usually corresponds to the Tollmien-Schlichting mechanism. When the perturbations grow, they can ultimately cause secondary instabilities to emerge, which is followed by breakdown and the ultimate formation of the turbulent flow. It has to be noted that all the latter phenomena (all blocks except the “eigenmode growth” one) are non-linear mechanisms and therefore cannot be coped with by the linear stability analysis. However, it could also occur that perturbations of large magnitude are entrained in the boundary layer. This can be the result of high freestream turbulence levels or surface roughness, for example. In that case, the perturbation amplification can completely by-pass the linear eigenmode growth phase, as illustrated by paths D and E. See Schmid & Henningson [75] and Morkovin [63] for details on the latter mechanisms. For a more thorough elaboration on the different paths illustrated in figure 1.4, see Pinna.
So, it can be seen that the linear stability approach to the transition phenomenon addresses only a part of the total mechanisms that may be involved; only the highlighted mechanism block is considered. A thorough illustration of the exclusion of the transient growth mechanisms will be given in section 2.1. Moreover, while confining the consideration to path A, even then the stability approach alone is insufficient to yield all information regarding the transition process. First of all, the receptivity mechanisms are dropped from consideration (and thus from the scope of the thesis) fully. It can be argued that these mechanisms can be translated into the stability information by reconstructing a finite initial perturbation with the eigenmodes. Such an approach is followed by Balakumar & Malik [7], for example.

Furthermore, all subsequent blocks of path A after the eigenmode growth block are discarded. The so called $e^N$-method, see the treatment of Van Ingen [43] and Smith & Gamberoni [78], proposes to link the prediction of transition to the local flow stability characteristics, in the form of the growth rate. This semi-empirical method relies on the assumption that the linear stability characteristics of imposed perturbations can be extrapolated beyond the subsequent blocks in path A. Accordingly, one can calculate the total relative growth of the perturbations along a certain physical path. Many experiments have demonstrated that the transition location may be correlated with a specific total growth value in this way. Although this method will be touched upon in subsection 2.2.8, its further consideration is outside of the scope of the current project.

To put the assumed scope in a physical perspective, one may consider a representation of the transition phenomenon. The most classical reference in this respect is provided by Schlichting [74]. In one of his figures, he illustrates the different physical mechanisms that come into play regarding the different stages in transition on a flat plate. Figure 1.5 is an edited version, in which all stages are denoted explicitly. As mentioned before, one can perform DNS to take into account all stages illustrated in the figure. The analysis corresponding to these simulations results in probably the most accurate calculations possible. Other predictive methods, such as the LST stability analysis type, can describe only part of the transition process, as indicated in the figure.

**Thesis Scope and Objectives**

When considering stability theory, information about the perturbations is extracted via the so called *stability equations*, which can be solved for the eigenmodes and the corresponding growth rate and frequency or wavenumber. These *amplitude* equations are derived from the instantaneous Navier-Stokes equations linearised for small perturbations, subject to certain assumptions about the to-be-investigated mean flow field. Depending on the specific assumptions made in the stability analysis, different forms of stability equations result, as will be elaborated on in detail in section 2.2. Because the currently considered stability equations are linear, all stages in figure 1.5 that involve non-linear

---

3Often the determination of this path is very non-trivial, this will be referred to in subsection 2.2.8.
mechanisms are not taken into account. It must be noted, however, that it is possible to perform non-linear PSE analysis and this does take non-linearities into account, see the work of Bertolotti [14] and Salinas [71], for example. Furthermore, the receptivity mechanisms, which are dominant near $x = 0$, are discarded. Several stability analysis types are indicated in figure 1.5. It can be seen that the different analysis types could be argued to have some sort of range of validity, as they take into account different features of the flow transition path. The mutual differences will be elaborated on in section 2.2 as well. In this treatment, the main focus will be on the BiGlobal analysis type. This analysis type considers the stability of perturbations that are inhomogeneous in two spatial directions. This is done because of its wide range of application with respect to its equivalents that take less spatial directions into account, as in the case of LST for example. Another important advantage involves its feasibility with respect to memory requirements, see the review of Theofilis [85].

In addition to the aforementioned inherent limitations of the linear stability analysis, there exist several issues concerning the stability approach. First of all, a mean flow field must be supplied (often in discretized form) to the stability equations before these can be solved. This mean flow field must be accurate enough to be able to rely on the corresponding stability results. Gómez et al. [27] report that small errors in the mean flow do not significantly affect the solutions of the eigenvalue problem. However, Drazin & Reid [22] and Theofilis [85] emphasize the contrary. They state that the eigenproblem’s solution is critically dependent on the (second derivative of the) mean flow data. It must be noted that this is particularly the case when the Orr-Sommerfeld equation is considered. Moreover, Barkley et al. [8] have performed mean flow calculations on the
same collocation grid to be used for the eigenvalue problem to avoid data interpolation completely. This seems to form a definite contradiction in the literature; the decisiveness of errors in the discrete mean flow data has been loosened in later references. In this treatment, the errors in the mean flow are made so small that no significant influence is expected to be present, see section 6.2 for more details.

Secondly, it is important to refer to the following. Gómez et al. [26] make the following interesting statement with respect to experimental validation:

“[…] what is largely missing from reported analyses of global flow instability is a culture of using experimental reality both as a sanity check of theory and as guidance for its further development.”

To this end, validating the numerical results with wind tunnel experiments should be performed when possible. However, in the current treatment, either no such validation information is available or the numerical results presented in the literature can be argued to be used as substitutes for experimental validation. The latter involves in particular the analysis of the Blasius boundary layer. The results corresponding to the Orr-Sommerfeld equation and (analogously) the LST analysis type, see subsection 2.2.3, have been experimentally validated in the past. In this regard, no direct experimental validations are deemed to be required and are discarded from consideration.

Thirdly, problems are encountered when artificial boundaries are considered, which is a typical feature for global stability analysis, in contrast to local stability analysis. This case is specifically dealt with when considering the developing Blasius boundary layer in chapter 8, which requires in- and outflow boundary conditions. In subsection 2.3.3 and the aforementioned chapter, all details corresponding to the most appropriate boundary conditions, which are currently deemed to be most appropriate, will be handled.

As a fourth and final point, although not usually considered as an inherent problem of the stability approach, the current treatment points out that the following issue definitely should be taken into account. It involves the fact that the stability equations are rather large when stated in primitive variables, which is required for more involved analyses. The latter fact has the consequence that their derivation and verification is very challenging. Accordingly, many errors are found in the literature, as will be elaborated on thoroughly in chapter 4. The point is that the equations have to be implemented as well. Often, this yields a very tedious work that is apt to result in transcription errors. This last problem forms the heart of the current thesis project. In the following, the goal of the current thesis project is accordingly explained in the form of the objectives and the research question.

The project can be divided into two subsequent main parts. The first part of the project involves the derivation of the BiGlobal stability equations and their verification with the literature. This is done by extending the equation derivation tool of VESTA, such that it can derive these equations. This is carried out in line with the overall VESTA methodology, see the treatment of Pinna [65].

Building on this work, the second part of the project firstly consists of the generation of the implementation tool. This tool allows the user to automatically implement the
stability equations, obtained from the new derivation tool, using the Chebyshev collocation discretization method. Secondly, this part involves the generation of a numerical code with which the stability equations can be solved for specific cases. The solutions obtained are to be verified as thoroughly as possible.

According to these objectives, the research question to be answered is: to what extent and efficiency BiGlobal stability simulations can be set up and executed by the current automatic derivation, implementation and simulation tools in the VESTA toolkit. The “extent” with which the tools can perform simulations is qualitatively determined. First of all, it involves the number of possible forms the derivation tool can output and the implementation tool can implement. Secondly, it involves the kinds of stability problems that can be executed, including the possible boundary conditions, for example.

The efficiency of the set of tools can be measured via their complete execution time. Note that this involves the derivation, implementation and simulation tools altogether. Furthermore, another measure of the efficiency of the tools is the achieved accuracy of the simulations, in terms of the eigenvalue convergence characteristics.

**Thesis Organization**

The remainder of the thesis is set up as follows. First of all, in chapter 2 all theoretical considerations are treated. That is to say, the stability theory and the corresponding equations are explained and the fundamentals of several kinds of (physical) boundaries are explained. Thereafter, the current extension of the automatic derivation tool in VESTA is elaborated on in chapter 3. This is followed by chapter 4, which considers all derived BiGlobal stability equations, using the new version of the derivation tool. These are explicitly stated in appendices C and D. Furthermore, the verification of the derivation tool with a large number of references is performed, with the aid of appendix E. These chapters correspond to the first part of the thesis.

The next part is initiated with chapter 5, in which the discretization with the Chebyshev collocation method is treated, including the treatment of a commonly used transformation. Furthermore, the implementation of the stability equations and boundary conditions is explained and the operation of the automatic implementation tool is briefly illustrated for both cases. Thereafter, the essential information about the used eigensolvers, the accuracy of the mean flow and numerical modes is handled in chapter 6.

After the numerical implementation and accuracy have been explained, the test cases are to be considered. For the current project, it was deemed to be important to start with the analysis of a mean flow that could be thoroughly verified with a simpler analysis type, that itself is verified (e.g. LST). It was therefore chosen to start with the analysis of a parallel Blasius boundary layer in chapter 7. By internal verification with LST simulations, the fact that the latter simulations are experimentally verified is deemed to be also sufficient for the BiGlobal cases. In this case, an incompressible and supersonic mean flow was considered.

For the next test case, it was decided to test the two-dimensional nature of the BiGlobal equations on the developing Blasius boundary layer. This is reported in chapter
8. The reason for selecting this case, is that Alizard & Robinet [2] and Ehrenstein & Gallaire [23] published papers on the flow and the mean flow could be acquired via the parallel flow case. Furthermore, it involved the incorporation of the Robin boundary condition with the Gaster-type relation; one of the artificial boundary conditions. Only the incompressible case was considered due to the already significant complexity of the flow.

In chapter 9, the main objective was to apply the tools to a very demanding three-dimensional test case. The mean flow was chosen to be the complex lamellar bidirectional vortex. An important feature of this mean flow is that it is defined in a cylindrical coordinate system. Testing the tools against this flow meant maximal requirements from the derivation and implementation tools. Moreover, Batterson [10], one of the references that was intensively used for the discretization of the problem, reported in much detail on the stability results. Furthermore, via personal communication, Batterson sent the current author his *Matlab* function for the set-up of the mean flow.

Note that by testing the tools on these three different test cases, a large variety of aspects could be assessed: the difficulties of the compressible regime, the artificial in- and outflow boundary conditions, a mean flow that is highly inhomogeneous in at least two spatial directions and a problem defined in cylindrical coordinates. Therefore, the tool can be argued to be tested in a very large range of application cases.

Separate concluding section are be given for each of the test cases. The main conclusions of the thesis will be discussed in chapter 10.
Chapter 2

Theoretical Foundations

This chapter considers the most important theoretical foundations underlying the work described in this thesis. This is done by first considering the basics in fluid mechanic stability theory in section 2.1. This section is followed by an elaboration on several kinds of stability analyses that can be performed and corresponding closing matters in section 2.2. With respect to the three most commonly used methods (the LST, PSE and BiGlobal analysis types), the link to the $e^N$-method, which is an empirical means to predict transition from stability results, is explained in addition. Lastly, several fundamental considerations on boundary conditions are treated in section 2.3.

2.1 Linear Stability Theory

Basic to all stability equations is to first derive the perturbation equations [17, 22, 74, 75]. This is done by assuming that the (non-dimensional) unsteady flow can be divided into separate components. The first components are $O(1)$ mean or equilibrium quantities (for example the time averaged velocity components, pressure and temperature), denoted by $\bar{Q}$, and corresponding $O(\epsilon)$ perturbation quantities, denoted by $Q'$.\footnote{Here, the big-O order symbol is used as in the conventional sense best described in [48].} Here, $\epsilon$ is assumed to be (infinitesimally) small. This will be commented on in a moment. The instantaneous non-dimensionalized flow field, governed by the Navier-Stokes equations, can then be decomposed as:

$$Q = \bar{Q} + Q'$$  \hspace{1cm} (2.1)

Substituting this decomposition into the Navier-Stokes equations yields the so called mean-plus-perturbation equations. This operation can physically be interpreted as the imposition of perturbations $Q'$ on a mean flow field $\bar{Q}$. Several references, such as Batterson [10] for example, also report on additional kinds of perturbation quantities. Examples are the compressible irrotational wave and the incompressible rotational vortical wave, but these are not considered here. With respect to these fancy names only the so called “fluid-dynamic wave” is regarded.
The assumption is made that the mean flow $\overline{Q}$ is independent of time\(^2\) and satisfies the time-independent (or steady) Navier-Stokes equations. In this context, these equations are also referred to as the *mean equations*. Note that this statement is already based on the fact that the perturbations are infinitesimally small. The mean flow component can thus be truly considered to be an equilibrium solution; it is different with respect to the mean flow considered in RANS approaches. In this case, the mean flow is not influenced by the imposed perturbations.

The instantaneous flow $Q$, composed of the sum of the mean and perturbations, has to satisfy the full (time-dependent) Navier-Stokes equations. Hence, all leading terms of these mean-plus-perturbation equations precisely form the mean equations. This means that when the mean equation is subtracted from the mean-plus-perturbation equation, all $O(1)$ terms cancel. The remaining terms form, what is referred to as, the *perturbation equations*. Because these equations are at least linear in the perturbation quantities, $Q' = 0$ is a trivial solution to these equations. Physically, this means that the mean flow is retrieved without perturbations. This seems trivial, however this leads to an analogy with the theory of fixed points in dynamical systems. This can be done by depicting the mean flow (solution) to be the equivalent of an *equilibrium solution* or a *fixed point*, see for example references [35, 80, 89]. Imposing perturbations on this mean flow, through applying decomposition (2.1), precisely corresponds to the method of determining the stability of the equilibrium solution or fixed point. The analogy can be deduced to go further when considering the small perturbation assumption in more detail.

In the current analysis, the perturbation equations are linearised assuming small perturbations, i.e. taking $\epsilon$ such that $\epsilon^{n+1} \ll \epsilon^n$ for any $n$. Thus a product of more than one perturbation quantity is regarded to be negligibly small with respect to the linear ones. This yields a linearisation of the equations, because only terms linear in the perturbations will remain. Due to this step, linear mechanisms are considered in the current case, exclusively. This means that, for example, vortex breakdown is not incorporated any more. In terms of figure 1.5, this means that the mechanisms beyond the “spanwise vorticity” stage are left out of the consideration. The complete derivation (including the non-dimensionalization) of the linear perturbation equations is illustrated in the case of the equation of state in section 2.2 via equations (2.10) to (2.12).

Returning to the analogy with fixed points in dynamical systems, several theorems can be found that elaborate on the fact that linear stability only provides insight for an infinitesimal small neighbourhood of the mean flow. This is in complete analogy with the fact that *small perturbations* are considered. Technically, one only investigates the “tangent space” of the stable and unstable manifolds at the equilibrium solution (here: the mean flow). Figure 2.1 gives an illustration of these manifolds for a (laminar) solution $\overline{Q}$ in the state space. Unfortunately, the manifolds themselves and new equilibria are not determined through this method, because this critically involves the non-linearity

\(^2\)Note that when the characteristic averaging time of the mean quantities is small with respect to periodic structures in the flow, as encountered in flapping wing configurations for example, the time-averaged flow can be periodic. In these cases, the latter assumption can be attenuated to allow a periodic mean flow, but this involves Floquet theory [9, 40, 85] and is not considered in this treatment.
of the system. Such a new equilibrium, $\mathcal{Q}^*$ in figure 2.1 for example, could be imagined to correspond to another type of mean flow, such as a turbulent equivalent. Hence, determining the precise characteristics of the unstable manifold leading there, can be seen as the core of the laminar-turbulent transition process. However, this is outside the reach of the small perturbation stability theory.

Note that the removal of non-linearities has as consequence that the ultimate effects of the transient growth mechanism are not incorporated here. It is deemed worthwhile to elaborate on this a little further. In figure 2.1, the “mean flow” $\mathcal{Q}^*$ is an example of a flow that can be unstable due to transiently growing perturbations. Consider the blue circle that is drawn around the solutions $\mathcal{Q}$ and $\mathcal{Q}^*$. These circles denote points in the state space that are $\epsilon$-close to these mean solutions. The analysis of the tangent space of the manifolds 1, 2 and 3 (or more straightforward: the calculation of the corresponding eigeninformation), results in the conclusion that they display attracting behaviour toward the solution $\mathcal{Q}$. Accordingly, the mean solution is declared to be linearly stable with respect to these manifolds. This implies that the green non-trivial solutions, that exist between these manifolds, are attracted to $\mathcal{Q}$ as well. Assume that the attracting force of the different manifolds is unequal. Then, these solutions will be approximately tangent to the most attracting manifold at first and end tangentially to the least attracting one.

Figure 2.1: Illustration of the stable and unstable manifolds for the mean solution $\mathcal{Q}$ and the effect of non-normality of a differential operator, partially inspired by figure 4.1 of Schmid & Henningson [75].

It can be seen in figure 2.1 that the green solutions enter the blue circle and stay within it. This means that when the solutions are of $\mathcal{O}(\epsilon)$ initially, they remain of this order for all elapsed time. Now consider the solution $\mathcal{Q}^*$. It has four stable manifolds. Note that the manifold with the label “unstable manifold” is stable with respect to this mean solution $\mathcal{Q}^*$ (note the arrowhead on the line). In this case, the manifolds have to following properties:
• They are nearly tangential to each other, thus significantly non-orthogonal

• The most vertical one has a significantly “stronger attracting force” than the other

It can be seen that the orange solution enters the blue circle at a certain moment. However, due to the strong attracting nature of the most vertical manifold, the solution is hurled out of the circle again. When the manifolds are extremely close to being parallel, this implies that (infinitesimally) small perturbations can become finite before they reach the mean solution. Therefore, this phenomenon is called transient growth.

When the perturbations become large, i.e. no longer of $O(\epsilon)$, non-linearities become dominant and the linear analysis breaks down. This can imply that the solution is unstable, even though the statement of the linear analysis establishes otherwise. Thorough investigation of transient growth involves the construction of a stable manifold itself, not its tangent space only. When this is done, it can be investigated whether initially small stable perturbations become large enough to trigger non-linear mechanisms. Note that the transient growth itself is caused by the linear operator. The fact that large perturbations tend to behave non-linearly, and thereby could become unstable, is not accounted for, however.

After all this said, the aforementioned linearisation seems rather restrictive. However, the eigenmode growth establishes an important instability mechanism in many practical situations, while the linearization provides an important element in the simplification of the stability analysis. Because of the linearity, one can perform a modal decomposition on the linearised perturbation quantities, by writing these as:

$$Q' = \tilde{Q}(x, y, z) e^{i(\Theta(x, y, z) - \omega t)} + \text{c.c.}$$

(2.2)

Here, $\tilde{Q}$ is the complex perturbation amplitude and $\omega$ and $\Theta$ are the generally complex frequency and phase function of the spatial directions, respectively. $\Theta$ is related to the wavenumbers and growth rates of perturbations in space, while $\omega$ is related to the frequency and growth rate of the perturbations in time. The abbreviation c.c. means complex conjugate and is added because the perturbation is a real, not a complex, quantity; recall that adding the complex conjugate is equivalent to taking twice the real part of the considered quantity. The corresponding factor 2 will be avoided as much as possible in the definition of the amplitudes $\tilde{Q}$ throughout the remainder of the treatment. Substituting equation (2.2) into the linearised perturbation equations results in the perturbation amplitude equations or stability equations. Note that this substitution corresponds to taking a general Fourier-Laplace transform.

Different kinds of amplitude equations can be derived by incorporating certain assumptions related to the temporal or spatial variation of the mean flow, which alters the form of $\Theta$ and $\omega$. Note that the imaginary parts of these variables correspond to the attenuation/growth rate of the considered modes and therefore, at least partially, determine the stability of the mean flow. Thus, as noted above, when $\Theta$ and $\omega$ are taken

\footnote{The coordinates $x$, $y$ and $z$ correspond to the streamwise, wall-normal and spanwise directions throughout this chapter.}
to be complex, one allows the modes to grow or decay in both space and time. However, often the analysis is restricted to focus only on either spatial or temporal growth of the modes, not both. This is done by taking Θ complex and ω real-valued or vice versa. This is referred to as considering the spatial or temporal problem, respectively. For now, this distinction is not yet made and Θ and ω in (2.2) can be considered to be complex-valued.

2.2 Stability Equations

As mentioned before, certain assumptions can be applied to the stability equations to yield simplified versions. Disregarding the aforementioned distinction between the temporal and spatial formulations, one can restrict the analysis to flows which display certain spatial characteristics. The ultimate assumption will lie in the fact that a spatial direction will be treated differentially or spectrally. When a direction is treated in the differential sense, the perturbation amplitude can display all practically interesting behaviour in that direction; the amplitudes are allowed to be differentiable functions or: inhomogeneous. When a direction is treated spectrally, the amplitude is restricted to behave like modal waves, corresponding to the complex exponential function. This function is controlled by the parameters Θ and ω as defined in equation (2.2). This kind of amplitude is also referred to as being homogeneous in this direction. Several kinds of names exist for these different kinds of directions, as will be elaborated on in what follows.

Handling directions differentially or spectrally is analogous to performing global or local analysis in that direction. The latter can be explained best on the basis of an example. Consider the situation in which the stream- and spanwise, x and z, directions are treated spectrally and the wall-normal one, y, differentially. This is typically the type of analysis underlying the Orr-Sommerfeld equation. This situation implies that the characteristics of the amplitudes vary in a general way in the y-direction, but are fixed with respect to Θ and ω in the other spatial directions and in time. Usually, these characteristics are calculated for the mean flow parameter combination (notably, the mean velocity profile and Reynolds number), which generally varies with position, at a given x and z location. Therefore, the analysis can only be regarded to be valid locally for the considered x and z position.

Note that this can be interpreted in another physical way. When regarding local analysis, the effect to mean flow parameter variations are not taken into account in the considered direction. In the global case, this information is taken into account. This implies that when information propagation is carefully considered, one can make a trade-off whether to apply global (expensive) or local (cheap) analysis in a certain direction. This is particularly useful in parabolic systems, in which a single propagation direction can be isolated. Later on in this section, the latter subject will be investigated more thoroughly.

\[^4\text{Note that often growth is considered in only a single spatial direction, yielding that only one wavenumber is taken to be complex valued.}\]
In the remainder of the treatment, a direction that is treated differentially is referred to as a differential or an *eigenfunction direction*. A direction that is treated spectrally, is referred to as an homogeneous or *spectral direction*. In the following subsections, the different kinds of stability equations are discussed, building on the aforementioned considerations. When deemed appropriate, the parabolized analysis type will be introduced.

### 2.2.1 TriGlobal Stability Analysis

When imposing no assumptions regarding information propagation, one considers the case that the modes are *inhomogeneous* in three spatial dimensions and homogeneous in time. To this end, one considers the stability of a fluid volume. In this case, the ansatz can be rewritten as:

$$Q^{\prime}_{\text{TriGlobal}} = \tilde{Q}(x, y, z)e^{-i\omega t} + \text{c.c.}$$  \hspace{1cm} (2.3)

The three-dimensional modes $\tilde{Q}(x, y, z)$ are expected for mean flows that are also inhomogeneous in three directions. For this reason Theofilis [85] has given this analysis type the name TriGlobal and the corresponding equations are subsequently called the *TriGlobal Stability Equations*. An example of a mean flow to which this analysis type can be applied is a three-dimensional roughness element immersed in a boundary layer, as illustrated in figure 2.2. The eigenfunction volume is indicated by the dashed red lines.

The resulting stability equations, together with the corresponding boundary conditions (which will be handled in section 2.3), form a three-dimensional partial derivative eigenvalue problem for $\{\tilde{Q}(x, y, z); \omega\}$. Due to the large computer memory requirements, they are currently still very expensive to solve. Theofilis [85] notes that computations can require in the order of 17.6 [Tb] of RAM.

---

5The latter convention should not be confused with a direction that is discretized with the spectral collocation method; such a direction logically is an eigenfunction direction.
2.2.2 BiGlobal Stability Analysis

A first assumption that one could make, with respect to the spatial characteristics of a mean flow, is that it is two-dimensional. I.e. inhomogeneous in two directions and homogeneous in a third direction and in time. Taking the spanwise coordinate $z$ to be treated spectrally, for example, this implies $\frac{\partial Q}{\partial z} = \frac{\partial \tilde{Q}}{\partial z} = 0$. It should be emphasized that these assumptions do not hold for the perturbations: $\frac{\partial Q'}{\partial z} \neq 0$. Those remain three-dimensional for all types of analysis; their behaviour in the spectral direction is controlled by the parameter that corresponds to it (the spanwise wavenumber $\beta$ in the case of $z$).

Figure 2.3: Typical geometry to which streamwise BiGlobal analysis can be applied.

Figure 4: Typical geometry to which spanwise BiGlobal analysis can be applied (parallel flow assumed in the $x$-direction).

The stability equations corresponding to two-dimensional mean flows are called the BiGlobal Stability Equations (BiG in figure 1.5) [10, 26, 66, 70, 85]. In this case, one considers the stability of an isolated fluid surface, the eigenfunction plane. In figures 2.3 and 2.4, these planes are shown for two different cases, with planes oriented in the streamwise and spanwise directions, respectively. Note that the wall-normal $y$ coordinate is always an eigenfunction direction, because boundary layers are present in this direction in all currently considered cases. Note that influences of neighbouring surfaces in the normal direction are not taken into account. For an eigenfunction plane in the streamwise $x$ and spanwise $z$ directions, the ansatz (2.2) is respectively changed to:

Streamwise: \[ Q'_{\text{BiGlobal}} = \tilde{Q}(x, y)e^{i(\beta z - \omega t)} + c.c. \] (2.4)

Spanwise: \[ Q'_{\text{BiGlobal}} = \tilde{Q}(y, z)e^{i(\alpha x - \omega t)} + c.c. \] (2.5)

Here, in the respective cases, $z$ and $x$ are separately taken to be the spectral direction and $\beta$ and $\alpha$ are the corresponding (again generally complex) wavenumbers in those directions. By considering two-dimensional eigenfunctions, i.e. no spatial restrictions within a specific plane, these equations still take into account the driving mechanisms of recirculations (and thus vortices) in this plane, as illustrated in figure 1.5. In this regard, instabilities in flows around any two-dimensional geometries, also those with a
significant streamwise curvature, such as backward facing steps (see figure 2.3) or airfoils for example, can be analysed.

Together with the appropriate boundary conditions, the BiGlobal stability equations form a two-dimensional partial derivative eigenvalue problem for \( \{ \tilde{Q}(x,y); \beta, \omega \} \). Therefore they are less expensive to solve than the TriGlobal equations. Theofilis notes that they require in the order of 4.3 [Gb] RAM [85]; this (optimistic) indication is already more feasible regarding the required engineering toolkit efficiency/performance. This is one of the reasons the derivation of and numerical simulation with these equations is the main aim of the current project.

2.2.3 Linear Stability Theory (LST)

The next simplification is to assume only one direction to be inhomogeneous, e.g. \( \frac{\partial \tilde{Q}}{\partial x} = \frac{\partial \tilde{Q}}{\partial x} = \frac{\partial \tilde{Q}}{\partial z} = \frac{\partial \tilde{Q}}{\partial z} = 0 \). The resulting equations are the most conventional ones in the literature [22, 74, 75] and are called the linear stability equations, abbreviated by LST in this treatment.\(^6\) The corresponding ansatz for the modal representation is formulated as follows:

\[
Q_{\text{LST}} = \hat{Q}(y)e^{i(\alpha x + \beta z - \omega t)} + c.c. \tag{2.6}
\]

Here, the (streamwise) \( x \)-direction is chosen to be the second homogeneous direction. Thus, the stability is restricted to the mean flow characteristics corresponding to a single \( x \) and \( z \) location. In this case, the stability of an isolated fluid line is considered; influences of neighbouring lines are not taken into account. This means that the streamwise growth rate \( -\Im \{ \alpha \} \) is independent of the flow in the neighbourhood of the considered location. For this reason only the “Tollmien-Schlichting mechanisms stage” is incorporated as is indicated in figure 1.5. In chapters 7 and 8, the error by neglecting the developing nature of a boundary layer will be handled in more detail.

The equations form an ordinary differential eigenvalue problem for \( \{ \tilde{Q}(y); \alpha, \beta, \omega \} \), again together with the appropriate boundary conditions. It can be solved at each \( x \)-location in the domain of interest independently and therefore yields results very quickly. Theofilis states that the required RAM is in the order of 1.0 [Mb]. The inviscid and viscous variants of the equations in the incompressible flow regime are closely related to the Rayleigh and Orr-Sommerfeld equations, respectively, and can be considered as the absolute fundamentals of fluid stability theory [17, 22, 74, 75]. The only difference involves the reformulation of the problem in, for example, the vertical velocity-vorticity formulation.

\(^6\)Note that all equations elaborated on here are linear stability equations. In the context of the current report a name like UniGlobal analysis would be a more consistent name, however “LST” is maintained here for convention.
An important note must be made here about the so called “parallel flow assumption.” Consider the incompressible and compressible mean flow regimes to which one wants to apply LST, taking into account the homogeneity in $x$ and $z$. The compressible mean continuity equation reduces to:

$$
\frac{\partial (\rho U)}{\partial x} + \frac{\partial (\rho V)}{\partial y} + \frac{\partial (\rho W)}{\partial z} = 0
$$

When the incompressible case is considered, the density is constant and therefore can be factored out of the equation. Therefore, the result in that case will be the same. Thus, in both the incompressible and compressible cases, due to the no penetration condition at the wall, the mean wall-normal velocity must be equal to zero. This is explicitly stated by Pinna [65] and Piot [66]. This is equivalent to having streamlines that are parallel to the wall and for this reason mean flows that satisfy the inherent assumptions for LST also satisfy the parallel flow assumption. It must be noted however, that when “extra inhomogeneity” is permitted in the mean flow, the flow is no longer required to be exactly parallel. In the BiGlobal context, for example, the flow is essentially non-parallel, as emphasized/initiated by Theofilis [85]. This implies that complete recirculation regions are allowed in the mean flow for which definitely $V \neq 0$.

Usually, the location of transition is found via the integration of the spatial growth rate $-\Im\{\alpha\}$ over the streamwise direction. Regarding the LST framework, the integrand is only found for one specific $x$ location. Therefore, to evaluate the perturbation amplification, which is the integrated growth rate, the LST calculations need to be performed at several $x$-locations. Note that at different locations the mean flow changes (slightly) and therefore different $\alpha$’s will be found, for a given value of $\omega$. This approach is illustrated in figure 2.5. Because the calculations at different lines are not dependent on each other, they can be performed in parallel.

Due to its simplified framework and hence the relatively fast performance, LST has been performed very intensively in the past. As stated before, the Rayleigh and Orr-Sommerfeld equations belong to the utter fundamentals of the theory.

Strictly, the investigated mean flows should satisfy the simplifications inherently belonging to the kind of stability equations they are analysed with. Although this holds in general, consider the LST equations, for example. Supplying a flow with a recirculation is deemed to be unwise, because in such a flow strong interactions are expected to take place between perturbations at different $x$ locations. However, sometimes “inferior” equations are still used in cases where the mean flow does not strictly correspond to the intrinsic assumptions of those equations. This is done purely due to the fact that dropping assumptions yields a more complex treatment and, accordingly, a higher computational cost. It is argued that certain mechanisms, which correspond to the assumptions, are discarded in those kinds of analysis. When these mechanisms are not dominant, using inferior equations may still yield a reasonable approximation.
On the other hand, it has also been tried to extend the LST equations to incorporate more mechanisms, e.g. non-parallel characteristics of the flow such as mild curvature, while retaining their structure and thus performance characteristics. There are references in which attempts are made to extend the LST equations by including appropriate terms to accommodate certain neglected mechanisms [79]. Through his paper, Itoh [44] brought insight in why these extensions were not consistent within the LST framework. He did this via the appropriate relaxation (though not full alleviation) of the parallel flow assumption.

2.2.4 Parabolized Stability Equations (PSE)

The ultimate consequence was the emergence of the parabolized analysis type, as an extension of the local one. This analysis type is based on Wentzel-Kramers-Brillouin (WKB) expansions, see chapter 4 of Holmes [42] for more details. The corresponding equations are called the Parabolized Stability Equations (PSE). The general assumption intrinsic to these equations is to take into account slow streamwise evolution, which yields an extension of the incorporated mechanisms range in figure 1.5 with respect to the LST approach. In this case the previous assumptions on the mean and amplitude quantities are alleviated. They are allowed to weakly depend on the streamwise direction [14, 26, 41, 66, 71]. This can be characterized by:

$$\frac{\partial}{\partial x} = \mathcal{O}(\epsilon), \quad \frac{\partial}{\partial y} = \mathcal{O}(1), \quad \frac{\partial}{\partial z} = \mathcal{O}(1), \quad \epsilon \ll 1$$

So, the partial derivatives of all mean and amplitude variables with respect to the streamwise direction are assumed to be small; this direction is associated with a large length scale of $\mathcal{O}(\frac{1}{\epsilon})$. It has to be noted that “this $\epsilon$” is a different $\epsilon$ than the one used for the linearisation of the perturbation equations. Physically, with respect to LST, these

---

7 Sometimes, the name Jeffreys is also appended to these names.
Theoretical Foundations

Equations take into account the evolution of the unstable waves downstream (though not upstream) and hence the effects of moderate streamwise curvature of the considered geometry and moderate changes in the mean flow, for example. The ansatz (2.2) is changed in this case to:

$$Q'_{PSB} = \tilde{Q}(x,y)e^{i\left(\int_{x_0}^{x} \alpha(\xi)d\xi + \beta z - \omega t\right)} + c.c. \quad (2.8)$$

Where the amplitudes are subject to the normalization condition $\int_0^{\infty} \tilde{Q}^* \frac{\partial \tilde{Q}}{\partial x} dy = 0$ due to the multiple appearance of $x$, see Herbert [41]; the star denotes complex conjugation. Note that the “history” $\int_{x_0}^{x} \alpha(\xi)d\xi$ is taken into account explicitly, whereas in LST only the local value $\alpha$ was considered. For this reason this kind of analysis is often called non-local.\(^6\) This means that the analysis is no longer “UniGlobal” in $y$, but, say, “Uni+$\epsilon$Global” in the $xy$-plane.

The equations, together with the appropriate initial and boundary conditions, form a two-dimensional parabolized partial differential boundary-initial-value problem for the solutions $\{\tilde{Q}(x,y); \alpha(x), \beta, \omega\}$. Due to the parabolized\(^9\) nature of the ultimate equations, the equations can be solved with a marching algorithm along the streamwise direction. This yields longer solving durations with respect to LST, but overall they are still rather low. A typical set-up is illustrated in figure 2.6.

Here also a note must be given with respect to the equivalent of the parallel flow assumption. The compressible mean continuity equations can be written as:

$$\epsilon \left(\frac{\partial \tilde{p}}{\partial x} \right)_x + \left(\frac{\partial \tilde{p}}{\partial y} \right)_y + \left(\frac{\partial \tilde{p}}{\partial z} \right)_z = 0$$

$$\left(\frac{\partial \tilde{p}}{\partial y} \right)_y = O(\epsilon)$$

$$y = O(1) \Rightarrow \tilde{p} = O(1) \quad \tilde{V} = O(\epsilon)$$

Here, it is found, under the assumption $\tilde{p} = O(1)$, that $\tilde{V} = O(\epsilon)$. Again, in the incompressible case, the density is constant and can be dropped from the continuity equation. The result will therefore be the same. This means that already for the parabolized approach the flow is not strictly parallel any more. Due to this reason, Theofilis [85] and Gómez et al. [26] call the analysis weakly/mildly non-parallel and non-local, respectively. It can be seen that care has to be taken when using the “parallel” terminology. Therefore it is chosen to use the more clear (and exhaustive) aforementioned reasoning referring to “(weak) inhomogeneity” of one or several spatial directions.

---

\(^6\)Note that $\int_{x_0}^{x} \alpha(\xi)d\xi$ reduces to $\alpha(x-x_0)$ when $\alpha \neq \alpha(x)$. The part $-\alpha x_0$ is omitted in the LST ansatz (2.6), because “it does not come down” when differentiating the ansatz; it will be a common factor for every term in the ultimate stability equation.

\(^9\)Note that the equations still have “residual ellipticity,” therefore they are called parabolized instead of parabolic. Due to this feature, care has to be taken when solving the problem on fine meshes, see Li & Malik [53].
2.2.5 BiGlobal PSE

Note that in the current section several kinds of stability equations/analyses were illustrated. However, for completeness it may be stated that a combination can be made between the PSE and BiGlobal approaches, by allowing the homogeneous direction in the BiGlobal analysis to be weakly inhomogeneous. Note that this makes sense when this direction is the streamwise direction. The result is a “Bi+$\epsilon$Global” approach, with the following ansatz:

$$Q'_{\text{BiGPSE}} = \tilde{Q}(x,y,z)e^{i\left(\int_{x_0}^{x}\alpha(\xi)d\xi-\omega t\right)} + \text{c.c.} \quad (2.9)$$

Here, it must be mentioned that $\tilde{Q}$ depends on $x, y$ and $z$. A similar normalization must be given as done for the ansatz (2.8), because $x$ appears in the amplitude and in the exponential function. Piot [66] and Gómez et al. [26] use the name BiGlobal PSE for this kind of analysis. The equations and boundary conditions form a three-dimensional parabolized partial differential initial-boundary-value problem for $\{\tilde{Q}(x,y,z); \alpha(x), \omega\}$ and can be solved by marching along the $x$-direction. Figure 2.7 gives an illustration of a typical set-up. This framework provides an intermediate step between the BiGlobal and TriGlobal analyses, allowing additional weak inhomogeneity of the homogeneous direction with respect to the former analysis.

Figure 2.7: Typical geometry to which BiGlobal PSE analysis can be applied (first order non-parallel effects taken into account).

2.2.6 Mean and Perturbation Equations of State

In the compressible equations one can eliminate one of the thermodynamic variables $p$, $\rho$ or $T$, by substituting the equation of state $p = \rho RT$. Note however, that the equation of state is valid only when it is assumed that a calorically perfect gas is considered. The
non-dimensional mean equation of state can be derived as follows [14, 71, 86]:

\[ p = \rho RT \]

\[ \rho_\infty U_\infty^2 \rho' = \rho_\infty RT_\infty \rho'T' \quad (2.10) \]

\[ \gamma U_\infty^2 \rho' = \gamma RT_\infty \rho'T' = a_\infty^2 \]

\[ \gamma M^2 p = \rho T^\text{Mean} \]

\[ \Rightarrow \gamma M^2 \bar{P} = \bar{p}T \quad (2.11) \]

Here equation (2.10) is obtained by making the equation of state dimensionless using the dynamic pressure, \( \rho_\infty U_\infty^2 \). Note that the non-dimensional variables are temporarily indicated with apostrophes. The form to the left in equation (2.11) is retrieved using the equation \( a_\infty^2 = \gamma RT_\infty \) (which is also only valid for a calorically perfect gas). Furthermore, the definition of the Mach number is used and the apostrophes are omitted. Now deriving the mean-plus-perturbation equation using decomposition (2.1):

\[ \gamma M^2 (\bar{P} + p') = (\bar{\rho} + \rho') (T + T') \]

\[ \gamma M^2 \bar{P} + \gamma M^2 \rho' = \bar{p}T + \rho'T' + \rho'T' \]

Removing all terms that satisfy equation (2.11) and assuming small perturbations of which the second order terms are negligible yields:

\[ \epsilon \gamma M^2 p' = \epsilon \rho' T + \epsilon \bar{p}T' + \epsilon^2 \rho'T' = \underbrace{\mathcal{O}(\epsilon)}_{\mathcal{O}(\epsilon)} + \underbrace{\mathcal{O}(\epsilon^2) = \mathcal{O}(\epsilon)}_{\mathcal{O}(\epsilon)} \]

\[ \gamma M^2 p' = \rho' T + \bar{p}T' \quad (2.12) \]

Equations (2.11) and (2.12) are the mean and perturbation forms of the equation of state, respectively. As stated before, due to the small perturbation assumption, the perturbation equation is linear in its perturbations. Together with the mean equation, one can eliminate of one of the pairs of variables: \( (\bar{\rho}, \rho') \), \( (\bar{P}, \rho') \) or \( (\bar{T}, T') \). In the current treatment, the pair \( (\bar{\rho}, \rho') \) is eliminated always.

### 2.2.7 Transport Coefficients

In the instantaneous compressible momentum and energy equations the transport variables \( \mu, \lambda \) and \( k \) appear. These are conventionally assumed to be functions of temperature, when considering the compressible (calorically perfect) context. Note that this automatically also holds for the mean equivalents. So, when these mean equivalent variables are partially differentiated in some direction, the result will be:

\[ \frac{\partial \bar{\mu}}{\partial x} = \frac{d\bar{\mu}}{dT} \frac{\partial T}{\partial x} \quad (2.13) \]

When setting up the stability equations, the transport coefficients introduce extra perturbation/amplitude variables. This requires explicit relations to be established (e.g.
explicit equations for $\mu'$, $\lambda'$ and $k'$ in terms of temperature). To this extent, it is chosen to perform a Taylor expansion of the perturbation variables with respect to the temperature perturbations around the mean variables. This yields for the first viscosity coefficient, for example [14, 71, 86]:

$$\mu' + \mu|_{T} = \mu|_{T} + \frac{d\mu}{dT}|_{T} (T + T' - T) + \frac{1}{2} \left( \frac{d^{2}\mu}{dT^{2}} \right) |_{T} T'^{2} + O(T'^{3})$$

Equation (2.14) is obtained by neglecting the higher order perturbation terms. In what follows, the explicit indication of the evaluation at $T$, $|T|$, will be omitted from the notation. In this regard, note that $\bar{\mu} = \mu|_{T}$ by definition. It is logical that the transport coefficient perturbations are zero when the temperature perturbation equals zero; the instantaneous coefficients should be equal to the mean coefficients in that case. Equivalent relations hold for the second viscosity coefficient $\lambda$ and thermal conduction coefficient $k$.

By using this formulation, it is clear that at several places in the equations $T$-derivatives will appear (even of second order) through equations (2.13) and (2.14). Note that the problem is now closed. The mean temperature derivatives can be extracted from the mean flow data, which is assumed to be complemented with appropriate constitutive relations for the transport coefficients. Examples of such relations are Sutherland’s law for the first coefficient of viscosity, Stokes’ hypothesis for the second viscosity coefficient and the assumption of constant Prandtl number $Pr$ for the thermal conductivity, see Anderson [4] and White [92].

### 2.2.8 Engineering Application: Relation to the $e^N$-method

As mentioned in subsection 2.2.3, an indication of the location of transition can be found via the integration of the spatial growth rate $-\Im\{\alpha\}$ over, for example, the streamwise direction. By logic, when integrating the growth rate, one receives the total growth over the considered path. In 1956, Van Ingen [43] and Smith & Gamberoni [78] independently proposed, what is nowadays referred to as, the $e^N$-method, which involves an experimental correlation of this total growth and the transition location [75]. The $N$-factor is precisely the aforementioned total growth. This can be shown as follows. Assuming to integrate along the streamwise direction (this will be commented on later), this can be expressed as follows for the LST analysis type:

$$N(x) = -\int_{x_{0}}^{x} \Im\{\alpha\} d\xi = \int_{x_{0}}^{x} \frac{1}{A} \frac{dA}{d\xi} d\xi = \ln\left( \frac{A(x)}{A(x_{0})} \right)$$

or:

$$A(x) = A(x_{0}) e^{N(x)}$$

where: $A \equiv \max_{y} |\tilde{Q}(x, y)|$  \hspace{1cm} (2.15)

Here, $x_{0}$ is the first location for which the perturbations become unstable (pass the neutral curve). The variable $y$ is usually set equal to the wall-normal location for which
the amplitude is maximal. The transition location is the location for which $N$ reaches a certain value. For applications to airfoils, this value is high (8 to 11) when the circumstances are benign (e.g. low freestream turbulence), whereas low values (5 for example) are to be used when adverse external factors come into play. Examples of such factors include high turbulence levels in the freestream and strong roughness effects (induced by the remains of insects or contaminants, for example).

As a method for predicting transition, it has several limitations. The most important one is that it is conventionally based upon linear mechanisms (the current stability theory) only. However, in its defence in this regard, it can also be applied successfully to DNS results, where the full non-linear mechanisms are taken into account. Furthermore, it does not take into account receptivity of the mean flow, see [31] for more details.

It has to be mentioned that in this explanatory section an important simplifying assumption is made regarding the used integration path to obtain the $N$-factor. Currently, it is assumed that this path coincides with the streamwise coordinate direction, which is indeed a sensible approach in two-dimensional flows under incompressible conditions. The point is that when integrating along a different path, one obtains another $N$-factor. In the supersonic regime, for example, the most unstable waves are oblique. Furthermore, in these kinds of flows one can define characteristic lines, related to the Mach waves. In addition to the streamwise direction, one could argue that integrating over the path normal to wave front of the most unstable wave or along the characteristic lines could be logical; the $e^N$-method does not prescribe which direction to use. It is very non-trivial which of these directions to choose. Therefore, the $e^N$-method yields an envelope of envelopes from which the indication of the transition point could ultimately be determined in an optimal way. For more details see the treatment of Arnal [6].

Although equation (2.15) already shows how the $N$-factor is related to the amplitudes, it is deemed to be worthwhile to investigate the relation a little further for the PSE and BiGlobal case. Moreover, in the LST case, the function $A = A(x)$ is not computed; only $\alpha$ is a solution of the analysis. Assuming that the streamwise direction should be integrated over, the following can be derived from the ansatzes of the PSE and streamwise BiGlobal frameworks:

\[
PSE (\alpha \in \mathbb{C}) \quad Q'_{PSE} = \tilde{Q}(x,y)e^{i\left(\int_{x_0}^{x} \alpha(\xi)d\xi + \beta z - \omega t\right)} + \text{c.c.} \\
|Q'_{PSE}| = |\tilde{Q}(x,y)|e^{-\left(\int_{x_0}^{x} \Im\{\alpha(\xi)\}d\xi\right)} \\
\frac{1}{|Q'_{PSE}|} \frac{\partial|Q'_{PSE}|}{\partial x} = \frac{1}{|\tilde{Q}|} \frac{\partial|\tilde{Q}|}{\partial x} - \Im\{\alpha(\xi)\} \\
N_{PSE} \equiv -\int_{x_0}^{x} \Im\{\alpha(\xi)\}d\xi \quad (2.16)
\]

\[
\text{Streamwise BiGlobal } (\omega \in \mathbb{C}) \quad Q'_{BIG} = \tilde{Q}(x,y)e^{i(\beta z - \omega t)} + \text{c.c.} \\
|Q'_{BIG}| = |\tilde{Q}(x,y)|e^{\Im\{\omega\}t} \\
\frac{1}{|Q'_{BIG}|} \frac{\partial|Q'_{BIG}|}{\partial x} = \frac{1}{|\tilde{Q}|} \frac{\partial|\tilde{Q}|}{\partial x} \\
N_{BIG} \equiv \int_{x_0}^{x} \frac{1}{|\tilde{Q}|} \frac{\partial|\tilde{Q}|}{\partial \xi} d\xi = \ln \left|\tilde{Q}(x,y)\right| \\
\left|\frac{\tilde{Q}(x_0,y)}{Q(x_0,y)}\right| \quad (2.17)
\]
Here, in the PSE framework, the partial derivative of $\tilde{Q}$ is nearly zero, because the normalization $\int_0^\infty \tilde{Q}^* \frac{\partial \tilde{Q}}{\partial x} dy = 0$ is applied such that the variation of $\tilde{Q}$ in $x$ is kept minimal \cite{14, 41}. In the case of LST, $\tilde{Q}$ is truly independent of $x$ and hence equation (2.16) exactly reduces to:

$$\text{LST } (\alpha \in \mathbb{C})$$

$$N_{\text{LST}} \equiv - \int_{x_0}^{x} \Im\{\alpha(\xi)\} d\xi \quad (2.18)$$

As noted before, in equations (2.16) to (2.18), the parameter $x_0$ is the first location at which the perturbations become unstable. Thus, the first location for which $-\Im\{\alpha\} = 0$ in the LST and PSE analysis cases and $\frac{\partial \tilde{Q}}{\partial x} = 0$ in the BiGlobal case. It must be noted that in the BiGlobal case, the $N$-factors are dependent on $y$ when representing them in this way. In the case of a developing boundary layer (see Alizard & Robinet \cite{2}), the value of $y$ is chosen for which the considered variable $\tilde{Q}$ achieves its absolute maximum, in line with the definition of the function $A$ in equation (2.15). In general, it could be argued that this choice is very case dependent.

From equation (2.17), it follows that the $N$-factor can be calculated directly from the amplitudes. Note that in that kind of analysis, an equivalent of the function $A$ in equation (2.15) is a part of the solution. This directly illustrates the extra amount of information the BiGlobal analysis type yields with respect to LST and PSE.

The additional information, the BiGlobal framework yields with respect to LST and PSE, can also be illustrated in a different way. It can be seen that the relations for the $N$-factors are nearly the same. Defining $-\Im\{\alpha(\xi)\} = \frac{1}{|\tilde{Q}|} \frac{\partial |\tilde{Q}|}{\partial \xi}$ in the BiGlobal case makes the relations identical (note that in that case, $\alpha$ does not enter the problem, because the $x$-direction is treated differentially). The important point here is that the function $-\Im\{\alpha(\xi)\}$ is different in two ways for the different frameworks.

First of all, the function is determined on the basis of the assumptions for the different frameworks as handled in the previous subsections. One takes into account non-parallel effect, while the other does not, or to a lesser extent. Therefore, the function values will be different in the first place. Secondly, for the different frameworks, more or less information about the function is given per $x$-location. In the LST framework, only the values of $-\Im\{\alpha(\xi)\}$ are calculated per $x$-location, whereas in the PSE framework the first derivatives in $x$ are determined as well. In the BiGlobal framework, due to the fact that the $x$-direction is treated differentially, the function values $\tilde{Q}$, but also the first to $(N_\xi - 1)^{\text{th}}$ derivatives, can be extracted at the collocation points. Here $N_\xi$ is the number of collocation points in the $x$-direction. This theoretically means that the function $-\Im\{\alpha(\xi)\}$ is given up to the $(N_\xi - 2)^{\text{th}}$ derivative at those points.\(^{10}\)

\(^{10}\)The values of the expression $\frac{1}{|\tilde{Q}|} \frac{\partial |\tilde{Q}|}{\partial \xi}$ can be simply determined at each collocation point via the values and first derivative of $|\tilde{Q}|$. Thereafter, these values can be combined with the nodal basis to yield a polynomial function over the considered domain. This polynomial function is of the order $N_\xi - 1$ and can be integrated easily.
Lastly, a number of general statements can be made regarding the theory of the stability equations in this section. Firstly, it must be noted that in this section it is assumed that a wall-bounded flow is considered, with $y$ as the wall-normal coordinate direction. Due to this fact and the existence of the boundary layer near the wall due to the no-slip boundary condition, $y$ is always taken to be an inhomogeneous direction in the flow. In other situations, the combination of inhomogeneous directions may need to be chosen differently. In a cylindrical pipe flow, for example, the radial $r$-direction is the wall-normal direction. In those cases, it is logical to let the eigenfunction depend on (at least) the $r$-direction, as is done by Batterson [10].

Secondly, the literature was carefully studied for reported BiGlobal stability equations to verify the current derivations with. A thorough elaboration of this subject, including very important results for the evolution of the project, can be found in chapter 4.

### 2.3 Boundary Conditions

In the previous section, the underlying theory of the stability equations was elaborated on, however the boundary conditions to make the systems complete were not yet considered; this will be done in this section. Note that for the PSE and BiGlobal PSE analysis types also an initial condition must be supplied to start the marching algorithm. This is not treated here. Details on this can be found in the works of Bertolotti [14] and Herbert [41].

Before proceeding further, it is important to reconsider the form of the stability equations. All currently considered ones form eigenvalue problems in the form $A \Xi = \lambda B \Xi$, where $\Xi$ is the eigenfunction vector. Although, the following statement is rather basic regarding the nature of the problem, it is deemed to be important with respect to the consideration of the theory of the boundary conditions. As any other eigenvalue problem, the system is **homogeneous**. All terms in the system are accompanied by at least a single element of $\Xi$. Otherwise, the problem would not be an eigenvalue problem. Solving the eigenvalue problem corresponds to finding a **homogeneous** solution to the related partial differential equations, which in general might be equipped with non-homogeneous forcing terms or non-homogeneous boundary conditions. This means that all possible boundary conditions for the stability problem need to be homogeneous as well, so that one can only specify homogeneous Dirichlet, homogeneous Neumann conditions, etc. In the remainder of the treatment the prescript homogeneous will hence be dropped occasionally.

Simple examples of inhomogeneous boundary conditions are non-zero Dirichlet conditions, which prescribe non-zero perturbations at the boundaries. By definition, the eigenvalue problem cannot cope with these kinds of conditions; one needs to find a **particular** solution for this purpose.

In what follows, several fundamental boundary types are handled: the solid wall, the symmetric boundary and the in- or outflow boundary. The boundary conditions used for the specific test cases are handled in the corresponding chapters.
2.3.1 Solid Boundaries: No Penetration and No-Slip Conditions

A solid wall is per definition a boundary through which no flow can pass. This means that the velocity component normal to the wall is zero. This condition is referred to as the no penetration condition. In inviscid flows, these kinds of conditions are sufficient to close the simulation of the flow along a solid wall. In viscous flows, however, the wall exerts a shear stress on the fluid. This withholds the fluid from sliding along it. This translates into a condition that is referred to as the no-slip condition; the tangential velocity components along the wall must be zero. These two conditions are often both referred to as the no-slip conditions. All the currently considered flows are viscous. Therefore, a solid wall is modelled by imposing zero normal and tangential velocity as a boundary condition.

These boundary conditions are imposed on the boundary $\Gamma$ of the considered instantaneous and mean flow fields. This means that the decomposition (2.1) and the ansatz (2.2) must be used to obtain the perturbation (and thus amplitude) boundary conditions. For convenience, the variable $t$ is absorbed into the phase function $\Theta$. The decomposition becomes:

$$Q|_{\Gamma} = \overline{Q}|_{\Gamma} + Q'|_{\Gamma}$$  \hspace{1cm} (2.19)

Thus for the velocity perturbations the following no-slip boundary condition can be set as well, taking the $x$-component for example:

$$\overline{U}|_{\Gamma} = \overline{U}|_{\Gamma} + u'|_{\Gamma} \quad \Rightarrow \quad u'|_{\Gamma} = 0$$  \hspace{1cm} (2.20)

For the amplitude $\tilde{u}$, this condition trivially yields $\Re\{\tilde{u}\} = \Im\{\tilde{u}\} = 0$, thus the no-slip condition for the instantaneous flow field translates to homogeneous Dirichlet conditions on the amplitudes. Noting that $\Theta$ is not associated with the variables that involve the boundary conditions, because it only consists of the variables that are treated spectrally. Therefore, the similar reasoning applies to all other kinds of boundary conditions.

From the underlying physics, no direct boundary conditions for the pressure (or density) can be derived in the case of a solid wall. Nevertheless, they are required to complete the system of equations. Therefore compatibility conditions are derived. These conditions are boundary conditions that ensure that the pressure (or density) at the boundaries satisfy the momentum equations, see the treatments of Gómez et al. [26] and Theofilis [85].

These conditions are derived by taking the momentum equations and evaluating them at the corresponding no-slip boundary. For matters of completeness, the derivation process is illustrated using the incompressible Cartesian $y$-momentum stability equation (see section C.1) considering an $y = e^{\pm \beta}$ boundary to be a no-slip boundary. First, all non-derivative velocity amplitudes are zeroed:

$$i \beta \tilde{v} \overline{\nabla} + \tilde{v} \nabla y + \tilde{u} \nabla x + \tilde{v}_y \overline{V} + \tilde{v}_x \overline{U} - i \omega \tilde{v} = -\tilde{p}_y + \frac{\tilde{v}_{yy}}{Re} + \frac{\tilde{v}_{xx}}{Re} - \frac{\beta^2 \tilde{v}}{Re}$$

$$\overline{\tilde{v}} \nabla y + \tilde{v}_x \overline{U} = -\tilde{p}_y + \frac{\tilde{v}_{yy}}{Re} + \frac{\tilde{v}_{xx}}{Re}$$
Due to the fact that along the complete \( y = c^{st} \) boundary all velocity amplitudes are zero, all tangential (in this case \( x \)-) derivatives of these amplitudes vanish as well:

\[
\tilde{v}_y \nabla = -\tilde{p}_y + \frac{\tilde{v}_{yy}}{Re}
\]

Lastly, zero mean velocities are imposed at the solid boundary, this yields the final result:

\[
\tilde{p}_y = \frac{\tilde{v}_{yy}}{Re}
\]  \hspace{1cm} (2.21)

Equation (2.21) forms a compatibility equation for the pressure at \( y = c^{st} \). The typical form of the amplitude coefficient matrices are very case dependent, but very similar to the momentum equations.

The derivation of these compatibility equations for all boundaries (taking \( x \)-momentum for the \( x = c^{st} \) boundary for example) are performed with the derivation tool. It must be noted that in those derivations, the mean velocities are not zeroed; the used mean data sets should provide zero values at the boundaries themselves.

In the compressible case, a condition must be specified for the temperature also. One could specify the temperature, corresponding to a wall that has its temperature controlled, or assume an adiabatic wall. The latter implies that the heat flux through the wall is zero, i.e. the temperature gradient from the wall to the fluid is zero. Hence, a Neumann condition should be applied to model the latter option.

### 2.3.2 Symmetry Boundaries

In some cases, it is useful to divide a full physical situation into parts that are symmetrical or anti-symmetrical. Consider the geometry shown in figure 2.2, for example, subject to a freestream with \( \nabla = \nabla f = 0 \). In that case, the expected mean flow is symmetric with respect to the \( xy \)-plane. Accordingly, the perturbation field is expected to be symmetric with respect to this plane as well. It is very handy to take this into account in the definition of the ultimate problem. One can specify the \( xy \)-plane to be a symmetric boundary and regard only one half of the domain, thereby saving a significant amount of expensive grid points.

![Figure 2.8: Illustration of even (dotted) and odd (dashed) variables around the point \( x_0 \).](image-url)
The to-be-applied boundary condition is simply the Dirichlet or Neumann condition. But which one to use is dependent on the features of the considered variables. To do this properly, the concept of even and odd variables has to be introduced. See section 3.3 of Haberman [36], for example. Consider a one-dimensional situation, where $x = x_b$ corresponds to the boundary. Then the even and odd variables are defined as follows:

$$\text{Even: } f(x - x_b) = f(x_b - x) \quad \text{Odd: } f(x - x_b) = -f(x_b - x)$$

(2.22)

Typical examples of these variables are given in figure 2.8. Note that the odd variables might have a jump at $x = x_b$, but the value $\frac{f(x_b^+) + f(x_b^-)}{2}$ must be equal to zero. The even variables always have a zero derivative at $x = x_b$. Therefore, these variables are always associated with the Dirichlet and Neumann boundary condition, respectively.

The last step to be taken is how to identify whether a variable is odd or even. This depends on the nature of the boundary. Consider a symmetric boundary, which acts as a mirror. All scalar quantities are an even function with respect to this boundary. On the other hand, all scalar quantities are odd with respect to an anti-symmetric boundary. Vector quantities are a little more difficult in this respect. The behaviour of the components around a symmetric and anti-symmetric boundary are shown in figure 2.9. It can be seen that the symmetric boundary truly acts as a mirror, while the anti-symmetric boundary does precisely the opposite. Whether the vector component is odd or even with respect to the boundary can be extracted from the figure by looking whether the sign of the arrow has changed. When it switches, the variable is odd and when it remains the same, the variable is even around the boundary. This is shown in the figure as well.

![Figure 2.9: Illustration of a vector’s components around symmetric and anti-symmetric boundaries.](image)

2.3.3 In-/Outflow Boundaries: Robin Conditions equipped with the Gaster-Type Transformation

Robin boundary conditions involve the specification of the in-/outgoing flux of a quantity proportional to the value of the same quantity evaluated at the considered boundary. This kind of boundary condition is excellent for the simulation of waves that flow into
and out of the domain with a certain wavenumber and amplification rate. The equation for a streamwise velocity wave for a streamwise \((x = c st)\) boundary is:

\[
\frac{\partial \tilde{u}}{\partial x} = i\alpha \tilde{u} \Rightarrow \tilde{u} = e^{i\alpha x} = e^{-\Im\{\alpha\} x + i\Re\{\alpha\} x}
\] (2.23)

So, this condition simulates a right travelling wave with local wavenumber \(\Re\{\alpha\}\) and local growth rate \(-\Im\{\alpha\}\). For fixed \(\alpha\), every eigenmode will have these features at the corresponding boundary. Although better approaches exist, one could use this condition for a parameteric approach to receptivity analysis \([31, 85]\), because one can study the response to a certain incoming wave.

Because in general the features of the incoming and outgoing waves are different for every other eigenmode, using a fixed \(\alpha\) in condition (2.23) is very restrictive. To alleviate this restriction, it is attempted to associate a specific \(\alpha\) to a corresponding mode. In the temporal problem, this can done through relating \(\alpha\) to the eigenvalue \(\omega\), using the local spatial dispersion relation \([2, 23]\):

\[
\alpha_s = \alpha_s(\omega_{s,r})
\] (2.24)

Here, the subscript \(s\) emphasizes that it concerns local spatial, instead of temporal, analysis. In that kind of analysis, \(\omega\) is real-valued; here this is indicated by the extra subscript \(r\) of \(\omega_{s,r}\); imaginary parts will be indicated with an \(i\). This is done regarding the Robin boundary condition only, as the use of \(\Re\{\cdot\}\) and \(\Im\{\cdot\}\) would make the reading very tedious. In the following, the variables corresponding to the temporal and spatial analysis type are distinguished by the first subscript \(s\) or \(t\). This is done only at those places where the distinction is deemed absolutely necessary.

Note that when performing local spatial analysis, one already incorporates an approximation. The local dispersion relation is based on the LST framework, which is restricted to the analysis of parallel mean flows; all non-parallel mechanisms are thrown out locally. No straightforward alternative approach is found in the considered literature in this respect.

Another problem is that the dispersion relation is not available in closed form for \(\alpha\). Otherwise, assuming that the relation exists in polynomial form, one could eliminate \(\alpha\) from equation (2.23) in terms of (a polynomial function of) \(\omega\). This is where the next approximation comes in, which is derived as follows. First of all, the dispersion relation (2.24) is expanded in a Taylor series around the (real) frequency \(\omega_{s,0}\):

\[
\alpha_s = \left.\alpha_{s,r}\right|_{\omega_{s,0}} + i\left.\alpha_{s,i}\right|_{\omega_{s,0}} + \left.\frac{d\alpha_{s,r}}{d\omega_{s,r}}\right|_{\omega_{s,0}} (\omega_{s,r} - \omega_{s,0}) + i \left.\frac{d\alpha_{s,i}}{d\omega_{s,r}}\right|_{\omega_{s,0}} (\omega_{s,r} - \omega_{s,0}) + \frac{d^2\alpha_s}{d\omega_{s,r}^2} (\omega_{s,r} - \omega_{s,0})^2 + \ldots
\] (2.25)

\[\text{Note that the dispersion relation depends on } Re \text{ in the incompressible flow regime—and additionally on } M \text{ and } Pr \text{ in the compressible regime—as well. Later on, during applications of the Robin boundary conditions at different boundaries, these relations become important. For now, the dependency on } \omega_{s,r} \text{ is focused on primarily.}\]
Gaster [24] has derived a transformation between the spatial and temporal parameters. This transformation is based on the assumption that $\alpha$ is an analytic function of $\omega$ and vice versa, without specifying the restriction of spatial or temporal analysis. Furthermore, the following quantities are assumed to be small:

$$\omega_{t,i} = O(\omega_{im}), \quad \alpha_{s,i} = O(\omega_{im}), \quad \frac{\partial \alpha_{s,i}}{\partial \omega_{s,r}} = O(\omega_{im}), \quad \omega_{im} \ll 1 \quad (2.26)$$

Here, the subscript $t$ denotes the variables that are associated with the temporal analysis. The small parameter $\omega_{im}$ is the maximum value of $\omega_i$ for the given Reynolds number. Gaster notes that for the cases of Blasius and Poiseuille mean flows, the value of $\omega_{im} = O(10^{-3})$. Therefore, the use of the ultimate relation in the cases of these mean flows is deemed sufficiently supported. In addition to the assumptions (2.26) made by Gaster, the following assumption is made with respect to the difference factors in the Taylor series:

$$\omega_{s,r} - \omega_{s,0} = O(\varepsilon_T) \quad \omega_{im} < \varepsilon_T \ll 1 \quad (2.27)$$

Here, the subscript $T$ stands for Taylor. The small parameter $\varepsilon_T$ is taken to be larger than $\omega_{im}$ for convenience. Thus, when $z \in \mathbb{C}$, the convention is used that the statement $z = O(\varepsilon)$ implies $z_r = O(\varepsilon)$ and $z_i = O(\varepsilon)$.

At this point one can use Gaster’s relations, which hold when assumptions (2.26) are satisfied. These relations relate the variables corresponding to the temporal and spatial problems to each other:

$$\alpha_{s,r} = \alpha_{t,r} + O(\omega_{im}^2), \quad \omega_{s,r} = \omega_{t,r} + O(\omega_{im}^2), \quad \frac{\omega_{t,i}}{\alpha_{s,i}} = -\frac{\partial \omega_{s,r}}{\partial \alpha_{s,r}} + O(\omega_{im}^2) \quad (2.29)$$
The second and last equation are substituted into equation (2.28). This yields:

\[
\alpha_s = \alpha_{s,r}|_{\omega_s,0} - i \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}|_{\omega_s,0} \omega_{t,i} + \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}|_{\omega_s,0} (\omega_{t,r} - \omega_s,0) + O(\omega_{im}^2) + O(\varepsilon_T^2)
\] (2.30)

\[
\alpha_s = \alpha_{s,r}|_{\omega_s,0} + \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}|_{\omega_s,0} (\omega_{t,r} - i \omega_{t,i} - \omega_s,0) + O(\varepsilon_T^2)
\] (2.31)

Note the very small adjustment the second equation makes in the third term on the right hand side in equation (2.30). In equation (2.31), \(\omega^*_t\) denotes the complex conjugate of the temporal complex frequency \(\omega_t\). Recall that \(\omega_t\) is the eigenvalue in the currently regarded temporal eigenvalue problem.

Equation (2.31) provides a relatively accurate estimate for \(\alpha_s\) in the boundary condition (2.23). Unfortunately, it contains \(\omega^*_t\). This quantity cannot be conveniently linked to the eigenvalue problem \(A\Xi = \omega_t B\Xi\). Indeed, an additional term \(\omega^*_t B'\Xi\) has to be included; \(B'\) denotes the matrix containing the amplitude coefficients corresponding to \(\omega^*_t\). This more general problem cannot be solved with the current tools, because one cannot access the imaginary part of \(\omega_t\). Remember that among others, this is one of the unknowns for which the equations are to be solved.

To get as close as possible to a formulation involving \(\omega_t\) only, equation (2.31) is rewritten to yield:

\[
\alpha_s = \alpha_{s,r}|_{\omega_s,0} + \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}|_{\omega_s,0} (\omega_t - \omega_s,0) - 2 \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}|_{\omega_s,0} \omega_{t,i} + O(\varepsilon_T^2)
\] (2.32)

Finally, equation (2.32) is obtained by recalling that \(\omega_{t,i} = O(\omega_{im})\), as stated amongst assumptions (2.26). This equation can be used without formulation problems, Alizard & Robinet [2] and Ehrenstein & Gallaire [23] refer to this equation as the Gaster-type transformation.

It can be seen that the error in the approximate dispersion relation (2.32) is of first, instead of second, order in \(\omega_{im}\). So, although the system is solvable now, a penalty has been paid. Note however, that the first order error entirely involves the imaginary part, the growth/damping rate, of the simulated wave at the boundary. The wavenumber of this wave is still accurate up to the second order in \(\varepsilon_T\).

Moreover, the first order error is physically interpretable. Of course, the locally simulated waves have incorrect growth behaviour. Instead of growing with the rate \(-\alpha_{s,i}\), they are decaying with the rate \(-\alpha_s\). Fortunately, the most interesting (least stable) modes in the to be analysed flows have small growth rates and assumptions (2.26) and (2.27) hold. Note that the conditions force the system in a certain way. For this particular case, this forcing is rather dominant for the ultimate solution of the system.
Although this forcing is physically erroneous, its effects are not expected to introduce adverse convergence characteristics.

An important point is that next to the least stable eigenmodes, the complete spectrum of the aforementioned mean flows consists of branches that tend to $-i\infty$. For these modes, the assumptions (2.26) and (2.27) definitely do not hold. Only the modes that satisfy these assumptions will be considered in the analysis when use is made of the Robin boundary condition in combination with the Gaster-type transformation. The assumptions (2.26) and (2.27) (the latter in combination with the relations (2.29)) define a rectangular region in the spectrum centered around $\omega_t = \omega_s,0 + i0$ with width $2\varepsilon_T$ and height $2\omega_{im}$. This region will be indicated in the resulting plots of the spectra to be able to easily identify the modes that are modelled reasonably well by the Robin boundary conditions.

There are three parameters in equation (2.32), $\alpha_{s,r}|_{\omega_s,0}$, $\left.\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}\right|_{\omega_s,0}$, and $\omega_{s,0}$. Note that $\left.\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}\right|_{\omega_s,0}$ is the inverse of the local group speed $c_s|_{\omega_s,0}$. It can be seen that $\omega_{s,0}$ has a central role in the condition. It is chosen such that it corresponds to negative $\alpha_{s,i}$ at both boundaries following Ehrenstein & Gallaire [23]. In addition, it is chosen such that $\alpha_{s,i}$ is very near to the neutral curve and satisfies assumptions (2.26). In this respect, it is important to mention that Alizard & Robinet [2] and Ehrenstein & Gallaire [23] both report that the ultimate solution is independent of the assumed value of $\omega_{s,0}$. Nevertheless, the assumption (2.27) must remain valid for the method to be consistent.

When incorporated, the Gaster-type transformation yields the following form of the boundary condition (2.23):

$$\frac{\partial \tilde{u}}{\partial x} = i\alpha_s \tilde{u} = i\left(\alpha_{s,r}|_{\omega_s,0} + \left.\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}\right|_{\omega_s,0} (\omega_t - \omega_{s,0})\right) \tilde{u} + O(\omega_{im}) + O(\varepsilon_T^2)$$

By multiplying with $c_s|_{\omega_s,0}$, one obtains:

$$c_s|_{\omega_s,0} \frac{\partial \tilde{u}}{\partial x} - i\left(c_s|_{\omega_s,0} \alpha_{s,r}|_{\omega_s,0} - \omega_{s,0}\right) \tilde{u} = i\omega_t \tilde{u} + O(\omega_{im}) + O(\varepsilon_T^2) \quad (2.33)$$

where $c_s|_{\omega_s,0} = \left(\left.\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}\right|_{\omega_s,0}\right)^{-1}$.

This relation corresponds directly to equations (8) and (2.9) reported by Alizard & Robinet [2] and Ehrenstein & Gallaire [23], respectively.

\[\text{\footnotesize{13}}\text{It has to be noted that the errors are given in order terms only. This means that the boundaries of the domain of validity in the spectrum are not "as solid" as this statement claims. Nevertheless, the indication is used to give an illustration of the extent of this domain.}\]
Note that equation (2.33) yields a complex element at the left hand side of the generalized eigenvalue problem. When two-dimensional ($\beta = 0$) perturbations are considered, as done in chapter 8, this boundary condition yields the only contribution that makes the left hand side complex-valued. Moreover, it must be revealed that the condition seems to yield different results when an even or odd value of collocation points is used in the normal direction with respect to the wall. This feature is conjectured to be the result of the inherent structure of the pseudo-spectral differentiation matrices corresponding to the Chebyshev collocation method, see the corresponding treatment in section 8.3.1.

A secondary implementation of the Robin boundary condition is therefore considered. This implementation serves a twofold purpose. Firstly, the left hand side of the problem can be kept real-valued and hence requires half the storage capacity with respect to its complex relative. Secondly, it allows the investigation of the influence of the inherent structure of the differentiation matrices on the ultimate results. To derive this secondary form, one needs to consider the second order equivalent of equation (2.23):

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \alpha^2_s \tilde{u} = 0 \quad (2.34)$$

Following the same procedure for the establishment of equation (2.33), keeping the Gaster-type transformation parameters as they were, the following is found:

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = -\alpha^2_s \tilde{u} = - \left( \alpha_{s,r}|_{\omega_{s,0}} + \left. \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}} \right|_{\omega_{s,0}} (\omega_t - \omega_{s,0}) \right)^2 \tilde{u} + \mathcal{O}(\omega_{im}) + \mathcal{O}(\epsilon_T^2)$$

$$= - \left( \alpha^2_{s,r}|_{\omega_{s,0}} + 2 \alpha_{s,r}|_{\omega_{s,0}} \left. \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}} \right|_{\omega_{s,0}} (\omega_t - \omega_{s,0}) + \left( \left. \frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}} \right|_{\omega_{s,0}} \right)^2 (\omega_t - \omega_{s,0})^2 \right) \tilde{u} + \ldots$$

Note that by neglecting the second order term at this stage, it can be argued that a small part of information is thrown away. Therefore, although the condition is asymptotically valid, it is slightly less accurate with respect to the first order equivalent. Note furthermore that, although it is small, it destroys the (near) anti-symmetry of the function $\alpha_{s,r}(\omega_{s,r})$ around $\omega_{s,r} = 0$. This will be thoroughly handled in section 8.3.5.

Note that the second order term could be kept. This yields a quadratic eigenvalue problem that could be solved with the procedure explained in section 5.5.4. However, this procedure doubles the size of the complete problem. Therefore, this method is discarded by virtue of its first purpose.

By multiplying equation (2.3.3) with $c_s|_{\omega_{s,0}}$, one obtains:

$$c_s|_{\omega_{s,0}} \frac{\partial^2 \tilde{u}}{\partial x^2} + \left( c_s|_{\omega_{s,0}} \alpha^2_{s,r}|_{\omega_{s,0}} - 2 \alpha_{s,r}|_{\omega_{s,0}} \omega_{s,0} \right) \tilde{u} = -2 \alpha_{s,r}|_{\omega_{s,0}} \omega_{s,0} \tilde{u} + \mathcal{O}(\omega_{im}) + \mathcal{O}(\epsilon_T^2) \quad (2.35)$$

It can be seen clearly that this equation consists of real coefficients only. Using this slightly altered condition reduces the required storage capacity of the left hand side significantly.
This concludes the treatment of the theoretical foundations. The next chapter will build on the current by illustrating the new implementations in the Automatic Derivation Tool.
Chapter 3

The Automatic Derivation Tool in VESTA

During the author’s internship at the Von Kármán Institute (VKI) in Rhode Saint Genèse, the main goal was to extend the already present stability equation derivation tool, written in Computer Algebra Software (CAS) Maxima [73]. This derivation tool is one of the parts of the VKI Extensible Stability and Transition Analysis (VESTA) toolkit, initiated by Pinna [65]. During the internship the tool was extended to derive the curvilinear compressible PSE equations. In the course of the project, it was found that it is very important to derive the stability equations instead of copying them plainly from the literature; an amount of PSE references was found to contain errors. It is guessed that these errors occur due to the significant effort required to derive the more complicated (for example compressible curvilinear) stability equations. See the corresponding technical report [32] for all details.

The same intention is thus extrapolated to the BiGlobal case. Moreover, it has to be noted in this regard that the analysis of stability with BiGlobal equations is still rather young. The earliest found work in the current context is that of Theofilis [83]. This yields that only several practitioners that have published (and could have cross-checked) their equations, certainly in the compressible case. Furthermore, these compressible equations are very dependent on the form of the system of equations that is started with. One cannot easily compare systems that are derived with the primitive variable set \( \{ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T} \} \) with another that applies \( \{ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{T} \} \), for example. If that is not enough, the energy equation can be formulated in different ways, with respect to pressure work or internal energy, for example. This makes cross-verification very difficult. The derivation tool in the VESTA toolkit is programmed such, however, that an exhaustive number of forms and variable sets can be started with. This yields a program that can very thoroughly verify the equations. The forms and variables sets are from now on referred to as formulations.

In this chapter, the current extensions and adjustments of the derivation tool to perform the derivation of the BiGlobal stability equations are concisely elaborated on. This is done by describing the current extensions of the tool in section 3.1, while building
on the work already present in the internship report, reference [32]. Afterwards, the more involved extensions are handled in sections 3.2 and 3.3, which contribute several practical matters building on the material presented in chapter 2.

3.1 Extensions/Adjustments in Equation Derivation Tool in VESTA

In chapter 2 the derivation process of the stability equations was elaborated on extensively. In figure 3.1 an updated version of the block diagram 3.1 of the VKI report [32] is shown. It summarizes all derivation steps in chronological order focusing on the LST, PSE and BiGlobal stability equations. The orange blocks represent the newly added features to derive the latter ones. For a very thorough treatment of the separate derivation stages, reference [32] should be considered. Here, only the newly added/adjusted blocks are elaborated on. These include:

1. **Set Flow Directions** - With this option one can label the 1-, 2- and 3-directions with any permutation of the streamwise, wall-normal and homogeneous/spanwise flow directions, so to be able to derive the stability equations “in the correct flow direction.” Furthermore, the wavenumbers $\alpha$ and $\beta$ (in the case of a cylindrical system) are fixed to the streamwise and spanwise flow directions to yield a consistent use throughout the derivation tool.

2. **Set (In)Homogeneous Directions** - This option is created such that the eigenfunction plane in the BiGlobal context can be oriented in the appropriate directions. Note that the LST/PSE equations are fixed already by the flow directions, which are set with the previous option.

3. **Form Energy Equation** - By using the equation of state ($p = \rho RT$) the energy equation can be presented in different kinds of forms. In addition to the internal energy and enthalpy based temperature equations, one is also able to use the pressure work equation and the (conservative) total energy equation formulations. These two formulations are added to the standard enthalpy and internal energy ones. The main purpose of this is being able to verify the equations of Theofilis & Colonius [86] and Theofilis [84], respectively.

4. **Representation Formulation** - In the former derivation tool, the representation formulation block, in which the equation of state is used to eliminate one of the thermodynamic variables in compressible equations, was situated in the Stability Equation Derivation stage. It was chosen to move the block to an earlier stage so to be able to use the perturbation form without having to bother with the different stability analysis types. In addition, it has to be noted that the previous code had

\[ These numbers correspond to the coordinate indices of the Maxima \texttt{ctensor} coordinate systems. I.e. for a three-dimensional Cartesian coordinate system these correspond to the $x$-, $y$- and $z$-directions, respectively. \]
to be corrected to appropriately derive the (rather inefficient) \((p,\rho)\)-formulation.\(^2\)

In the previous code, derivatives with respect to the mean temperature accidentally dropped due to the substitution of \(\overline{T} = \gamma M^2 \frac{\bar{\rho}}{\bar{\rho}}\). To this end, a pair of functions is written to evade this problem, resulting in the temperature derivatives to be unaltered.

5. **LST and PSE or BiGlobal, Implement BiGlobal Ansatz & Perturbation Amplitude Equation** - Due to the two different possible orientations of the eigenfunction plane in a wall-bounded flow, the BiGlobal ansatz cannot be rewritten into the LST/PSE ansatz in a general way. Note that this could be done for the PSE case with respect to LST. Therefore an if-statement is created to apply the appropriate ansatz to ultimately derive the required amplitude equation.

6. **BiGlobal Stability Equation** - In BiGlobal analysis one performs local analysis with respect to the homogeneous direction (BiGlobal PSE is not considered). Due to this fact no order of magnitude analysis had to be performed to yield the ultimate stability equations. To this end, the second order of magnitude analysis stage for LST/PSE is evaded by creating corresponding if-statements. A last if-statement was created in the \LaTeX output cell so to output the correct equations.

Items 1, 2, 5 and 6 are argued to be clear from the above description and will not be elaborated on more thoroughly. Moreover, the corresponding code in \textit{wxMaxima}\(^3\) is commented sufficiently to be self-explanatory. The remaining items are handled in the following sections due to their more thorough involvement.

### 3.2 Adjustment in Representation Formulation

The representation formulation block is mainly built on the equation of state. Due to the fact that it is an algebraic equation, it can be substituted into the remaining governing equations rather simply. In this way the number of variables and equations is reduced by one. This section is mainly based on the theory handled in sections 2.2.6 and 2.2.7.

Suppose that the equations are to be presented in the \((p,\rho)\)-formulation. In that case all temperature variables have to be replaced; also those in the derivatives. This causes a non-trivial problem. In \textit{Maxima}, the transport coefficient derivatives will be rewritten in the following incorrect way:

\[
\frac{d\bar{\mu}}{d\overline{T}} = \frac{\partial \bar{\mu}}{\partial \left( \gamma M^2 \frac{\bar{\rho}}{\bar{\rho}} \right)} = \frac{1}{\gamma M^2} \frac{\partial \bar{\mu}}{\partial \left( \frac{\bar{\rho}}{\bar{\rho}} \right)} \quad \bar{\rho} \neq \bar{\mu}(\bar{P},\bar{\rho}) \quad \text{in Maxima}
\]

Accidentally the derivatives are dropped due to the prescribed independence on \(\bar{\rho}\) and \(\bar{P}\) in the derivation tool. This problem can be approached in the following way. One

\(^2\)It has to mentioned that the formulation is not 100% consistent because all \(\overline{T}\)-derivatives in the equations are kept. This is done because alternative representations are (even more) inefficient with respect to the ultimate number of terms in the resulting stability equations.

\(^3\)This is the considered user interface of \textit{Maxima} and is nearly the same as that of CAS \textit{Maple}.
Figure 3.1: Block diagram of the derivation of the Stability Equations.
could allow the transport coefficients to depend on $P$ and $\rho$ through $T$. Using the basic rules of calculus and the mean equation of state, one can rewrite the aforementioned derivative to “solve” this problem:

$$\frac{d\mu(P,\rho)}{dT} = \frac{\partial \mu}{\partial P} \frac{\partial P}{\partial T} + \frac{\partial \mu}{\partial \rho} \frac{\partial \rho}{\partial T} = \frac{\partial \mu}{\partial P} \frac{\rho}{\gamma M^2} - \frac{\partial \mu}{\partial \rho} \frac{\gamma M^2 P}{\rho}$$

$$= \frac{\rho}{\gamma M^2} \left( \frac{\partial \mu}{\partial P} - \frac{\rho^2}{P} \frac{\partial \mu}{\partial \rho} \right)$$

The point is that one derivative is replaced by two terms. In total these terms consist of 2 derivatives and 3 other variables, counting $\rho^2$ as one. When differentiating combination (3.1), one ends up with 5 terms instead of just the one $\partial^2 \mu / \partial T^2$. Hence, rewriting the equation using the transport coefficients as functions of $P$ and $\rho$ is argued to be very inefficient, regarding the length of the equations. Moreover, it has to be noted that this formulation of the transport coefficients is never chosen in the considered literature. Most of the times the $T$-derivatives are just unaltered.

3.2.1 The functions DerVartoName and NametoDerVar

Because of the inefficiency of writing $\mu$ as a function of $P$ and $\rho$, it is decided to leave the derivatives with respect to the mean temperature unaltered. This is a rather non-trivial procedure however, from the point of view of programming. Therefore a pair of functions called DerVartoName and NametoDerVar were created, which can be loaded from the file Derivative_Rewriter_batch.mac. These functions both have three arguments:

1. The considered equation (possibly in vector form), in this case the momentum and energy equations
2. A vector ‘vec’ of variables of which the derivative is taken, in this case the variables $\mu$, $\lambda$ and $k$
3. The variable ‘var’ which is used to differentiate the variables in the previous vector with, in this case the variable $T$

The function DerVartoName substitutes an appropriate name for all first and second (not higher) order ‘var’-derivatives of the elements of the vector ‘vec’. NametoDerVar does precisely the opposite. When calling DerVartoName just before and NametoDerVar just after the substitution of equation (2.11) with the arguments as indicated above, all $T$-derivatives are neatly unaltered.

Note that due to the fact that $T$-derivatives are still present in the equations, the temperature is not fully eliminated. The derived equations are valid, but their formulation is inconsistent. However, as mentioned before, it is (very) inefficient to rewrite the
temperature derivatives into the equivalent $\mathcal{T}$- and $\mathcal{P}$-derivatives. Moreover, the often occurring derivatives of the mean temperature itself (heat conduction terms as $\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right)$ etc.) cause a lot of extra terms as well. Therefore, this inconsistency is allowed as it is for the moment. The $(p, \rho)$-formulation will probably never be implemented in the future due to its inefficiency. Furthermore, recall that in the current context the formulation is implemented only for verification purposes.

### 3.3 Instantaneous Navier-Stokes Equations

When deriving the stability equations, conventionally the non-conservative form of the instantaneous Navier-Stokes equations is used in the derivation tool in VESTA [32, 65]. This is done because the conservative form is simply a multiple of the continuity equation added to the non-conservative form and does not yield extra information. This goes even further when regarding the energy equation. Normally, when deriving the energy equation from first principles, the so called (conservative) total energy formulation is derived. From this formulation not only the continuity equation can be subtracted, but also the inner product of the momentum equations with the velocity vector. The latter expression is also referred to as the theorem of stress means. When subtracting all the corresponding terms, quite a significant number, the resulting system of equations is as lean as possible. This minimizes possible bugs during implementation.

In this section all implemented instantaneous compressible Navier-Stokes equations that are implemented in the derivation tool in Maxima are stated. This is done to have a clear overview of the equations that can be used to derive the stability equation with. This is done in tensor notation, as equally done in Maxima, for two purposes. Firstly, to save space and secondly, to be able to implement the coordinate system later on. Maxima has very handy function packages in this respect that are used to evaluate, for example, the covariant derivatives (which is denoted by $\{\cdot\}_j$). These kind of derivatives take into account curvature of the considered geometry. Note that the Einstein summation convention applies. Since the energy equation is used only in the compressible context, its incompressible equivalent is not considered.

The variables are all made dimensionless with their static freestream equivalents, except for the pressure for which the dynamic pressure $\rho_\infty U_\infty^2$ is used. The non-dimensional parameters $\gamma$, $M$, $Re$, $Pr$ and $Ec$ are all taken as conventional (see section 2.4 of reference [32] for an exhaustive overview), using the static temperature in the Eckert number. Furthermore, it is assumed that $\lambda$ scales as $\mu$.

#### 3.3.1 Continuity equations

The incompressible and compressible continuity equations are stated as equations (3.2) and (3.3), respectively:

<table>
<thead>
<tr>
<th>Incompressible</th>
<th>Compressible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^i_i = 0$</td>
<td>$\frac{\partial \rho}{\partial t} + \rho_\mathcal{v} u^i_i + \rho u_i^i = 0$</td>
</tr>
</tbody>
</table>
The Automatic Derivation Tool in VESTA

In the derivation tool, one can specify several standard options to derive any equation in physical form (with the conventional physical vectors instead of the contravariant equivalents). For the continuity equations, one can specify the choice of either the incompressible or compressible system and set an appropriate coordinate system.

3.3.2 Momentum equations

The incompressible and compressible momentum equations are stated as equations (3.4) and (3.5), respectively:

\[
\begin{align*}
\text{Incompressible:} & \quad \rho \frac{\partial u^i}{\partial t} + \rho u^j u^i_j = -g^{ij} p_j + \frac{\mu}{Re} \left( g^{jk} u^i_{jk} + g^{ik} u^j_{ik} \right) \quad (3.4) \\
\text{Compressible:} & \quad \rho \frac{\partial u^i}{\partial t} = -g^{ij} p_j + \frac{1}{Re} \left( \lambda g^{ij} u^k_{jk} + \mu g^{jk} u^i_{jk} \right) \right. \\
& \left. + \mu_{ij} g^{ik} u^j_{ik} \right) \quad (3.5)
\end{align*}
\]

Next to the choice between the incompressible or compressible versions, one can specify an appropriate coordinate system and set the dependence of the transport coefficients \(\mu, \lambda\) and \(k\) on temperature in the compressible case. In the incompressible case, these coefficients are independent of temperature always. Furthermore, one can apply Stokes’ hypothesis \((\lambda = -\frac{2}{3} \mu)\). Note that these are only the non-conservative versions, no conservative forms are implemented.

3.3.3 Energy equations

The energy equation is represented in many ways in the derivation tool. In the older version, the temperature equations were implemented being derived with respect to specific static enthalpy \((h = c_p T)\) and specific internal energy \((e = c_v T)\) and assuming a calorically perfect gas \((c_p\) and \(c_v\) are constant). These equations are stated in equations (3.6) and (3.7), respectively:

\[
\begin{align*}
\text{Enthalpy:} & \quad \rho c_p \frac{\partial T}{\partial t} + \rho c_p u^j T_j = E_c \left( \frac{\partial p}{\partial t} + u^j p_j \right) + \frac{1}{Re Pr} \left( k g^{ij} T_i \right) + \frac{Ec}{Re} \Phi \\
\text{Internal Energy:} & \quad \rho c_v \frac{\partial T}{\partial t} + \rho c_v u^j T_j = -\gamma E_c p u^j_j + \frac{\gamma}{Re Pr} \left( k g^{ij} T_i \right) + \frac{\gamma Ec}{Re} \Phi \quad (3.7)
\end{align*}
\]

Here, \(\Phi\) is the dissipation function, which can be written as:

\[
\Phi = \lambda u^j_j u^m_m + 2 \mu g^{ik} g^{jm} u^k_m u^j_i \quad (3.8)
\]

Due to various appearances of the pressure work formulation of the energy equation (for example in reference [86]), this formulation was implemented in addition to the
The Automatic Derivation Tool in VESTA

The aforementioned ones. This formulation can be derived by substituting the equation of state for $T$ into either equation (3.6) or (3.7). The result is equation (3.9):

$$M^2 \left( \frac{\partial p}{\partial t} + u^j p_{,j} + \gamma p u_j^j \right) = \frac{1}{RePr} \left( k g^{ij} T_{,i} \right)_j + \frac{E_c}{Re} \Phi \quad (3.9)$$

Equations (3.6), (3.7) and (3.9) are the standard energy equation formulations present in the derivation tool. The several standard options available in the code allow the user to set:

- The specification of an appropriate coordinate system
- The dependence of the transport coefficients $\mu$, $\lambda$ and $k$ on temperature
- The use of the Eckert number number instead of the Mach number (merely to save space in the case of the occurrence of the combination $(\gamma - 1)M^2$)
- The different energy equation formulations
- The use of Stokes’ hypothesis

As for the other equations, this can be done by specifying single options in the code in \textit{wxMaxima}.

In reference [84], Theofilis uses the (conservative) total energy formulation for the energy equation. As stated before, this formulation is the lengthiest one possible and, with respect to implementation for numerical calculations, the most inefficient one. To shed light on this, it is shown explicitly which terms can be dropped in what follows. First start with the conservative total energy formulation with respect to the total energy $E = e + \frac{1}{2} g_{ik} u^i u^k = \frac{1}{(\gamma - 1)} \rho + \frac{1}{2} g_{ik} u^i u^k$:

$$\text{Total Energy:} \quad \frac{\partial (\rho E)}{\partial t} + (\rho E u^j)_j + (p u^j)_j = \frac{1}{PrReEc} \left( k g^{ij} T_{,i} \right)_j + \frac{1}{Re} \left( g_{ij} \mathbb{T}^{ik} u^j \right)_k \quad (3.10)$$

Where $\mathbb{T}^{ik} = \lambda g^{ik} u_j^j + \mu (g^{ij} u_j^k + g^{jk} u_j^i)$ is the viscous stress tensor [5, 17]. This equation

\footnote{Note that when writing the specific heats $c_p$ and $c_v$ within the partial time and covariant derivatives, the calorically perfect gas assumption can be attenuated to the thermally perfect gas assumption for which $c_p$ and $c_v$ are function of $T$. Retaining the variables $h$ and $e$ would be even more general (no implicit assumptions are imposed), however this poses a closing problem. When deriving the pressure work formulation of the energy equation the equation of state is substituted, forcing the assumption of a calorically perfect gas, definitely.}
can be fully expanded to yield:

\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + p_j u_j^i + pu_j^i \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^k u_j^j + \rho g_{ik} u^i u^k \rho_{ij} \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \right) + \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \rho_{ij} \right) \]

\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + p_j u_j^i + pu_j^i \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^k u_j^j + \rho g_{ik} u^i u^k \rho_{ij} \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \right) + \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \rho_{ij} \right) \]

\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + p_j u_j^i + pu_j^i \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^k u_j^j + \rho g_{ik} u^i u^k \rho_{ij} \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \right) + \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \rho_{ij} \right) \]

\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + p_j u_j^i + pu_j^i \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^k u_j^j + \rho g_{ik} u^i u^k \rho_{ij} \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \right) + \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \rho_{ij} \right) \]

\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + p_j u_j^i + pu_j^i \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^k u_j^j + \rho g_{ik} u^i u^k \rho_{ij} \right) + \left( \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \right) + \frac{1}{2} \rho g_{ik} u^i u^j u_k^j \rho_{ij} \right) \]

In equation (3.11) all terms that together satisfy the continuity equation and the inner product of the momentum equation with the velocity vector are shown and can thus be removed from the equation. For details on the theorem of stress means, see Aris [5]. When performing this, it can be seen that precisely the pressure work formulation (3.9) is retrieved (after multiplying with the Eckert number and noting that \( Ec = (\gamma - 1)M^2 \) in this context). Without proof it can be assumed that the number of redundant terms in equation (3.11) is enormous. Theofilis [84] notes 231 terms in his stability equation (not taking into account counting errors and the fact that several terms can be merged). The pressure work stability equation with similar assumptions derived with VESTA has 103 terms. This roughly means 55% less implementation work in which fatal bugs could occur.

It has to be emphasized that equation (3.11) is the only equation in the derivation tool that is in conservative form and includes the terms according to the stress means theorem. It was included only to verify the energy equation of Theofilis [84] (which is, as illustrated above, extremely impractical to do by hand). The results of the verification are stated in chapter 4.
Chapter 4

Derivation Results and Verification

In this chapter the current derivation results and verifications with external sources are presented. This is done by first elaborating on the standard results in section 4.1. Thereafter, the verification procedure is explained in section 4.2. This section is followed by the actual verification of the incompressible and compressible equations in sections 4.3 and 4.4, separately. The chapter is concluded with the declaration of verification with respect to the derivation of the equations.

4.1 Derived Equations

For the current project it is deemed sufficient to derive the incompressible and compressible BiGlobal stability equations for several cases. Firstly, the equations in Cartesian and cylindrical coordinate systems, where the eigenfunction plane is taken in the streamwise direction, are very useful. Corresponding applications could be boundary layers and flows in cylindrical pipes. Secondly, in the case that the eigenfunction plane is taken normal to the streamwise direction, the Cartesian coordinate system is considered. In this way they could be directly applied to cases involving Görtler vortices on an airfoil and Hagen-Poiseuille flow (see chapters 1 and 17 of Schlichting [74]), respectively.

The incompressible and compressible equations are both derived and stated in appendices C and D, respectively. Several specifications have to be made:

- All derivatives are shown in abbreviated form, e.g. $\overline{U}_x$ is the $x$-derivative of $\overline{U}$.

- The equations are stated in the $(p, T)$-formulation, which means that $\rho$ is eliminated from the equation with the equation of state. This is done because this formulation is more efficient than the full $(p, T, \rho)$- or $(p, \rho)$-formulations, as explained in chapters 2 and 3. The $(T, \rho)$-formulation not used because it is considered in the literature to a lesser extent.
The transport coefficients $\mu$, $\lambda$ and $k$ are taken to be functions of $T$ and Stokes’ hypothesis is not applied. The latter is done so to be general in this respect. In all performed simulations in this treatment, the hypothesis is ultimately applied, though.

The energy equation is presented in the enthalpy formulation as presented in equation (3.6), so to use an efficient formulation. Furthermore, the Eckert number is used instead of the Mach number in this equation, because this gives a more compact representation.

The amplitude variables are always represented by $\{\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T}, \tilde{p}\}$\textsuperscript{1} and the mean variables by $\{U, V, W, T, P\}$. In the incompressible cases the temperature variable and energy equation are left out of the consideration.

The wavenumbers are always taken with respect to a certain direction: $\alpha$ and $\beta$ (and $q$) are the streamwise and rectilinear (and curvilinear) spanwise wavenumbers, respectively.

All instantaneous equations, used to perform the stability equation derivations with, are in non-conservative form, for the reasons explained in chapter 3.

For the equations in Cartesian coordinates, the streamwise, wall-normal and spanwise flow directions are always taken to be the $x$-, $y$- and $z$-directions, respectively. In the cylindrical equations the streamwise flow direction is along the $z$-direction. In the “streamwise” equations, the eigenfunction plane is spanned by the streamwise and wall-normal direction and in the “spanwise” equation, it is spanned by the spanwise and wall-normal direction. Note that the wall-normal direction, due to the presence of the inhomogeneous boundary layer, is always taken as one of the eigenfunction directions.

To yield a minimum of interaction with the equations during the transport from the \textit{wxMaxima} interface to \LaTeX, the derivation tool is programmed such that complete strings are output. In this way no copy-paste errors could occur. Furthermore, the only modification of the \LaTeX code of the equations in appendices C and D is the addition of line breaks, to prevent the equations from going off the page.

In appendices A and B, the LST equations are reported. This is done because these equations will be used for internal verification purposes as the project evolves. These equations are derived with options equivalent to the ones stated above.

\section*{4.2 General Comments on Verification Procedure}

In what follows the derivation tool is verified with the existing literature. For this verification, case specific equations were derived, using the power of the derivation tool

\textsuperscript{1}The velocities in the 1-, 2- and 3-direction are always indicated with a lower or upper case “u”, “v” and “w”.

to its utmost extent. The general idea is that the tool is configured to output precisely the same systems as stated in the considered reference; this will be elaborated on per reference.

The subsequent verification is performed using the terms “errors” and “discrepancies,” which have rather specific definitions in this context. Errors are differences in the terms found when comparing the expanded equation delivered by the derivation tool versus the transcribed version in \textit{wxMaxima} of the original equations. Discrepancies are deviating principles through which these errors emerge; in other words, the collective \textit{causes} of the errors. The latter verification terms are used mostly, because they are invariant to changes in the formulation of the equations. This will be handled more elaborately in section 4.5.

The difference between errors and discrepancies can be illustrated as follows. Say one wants to verify the compressible continuity equation with a found reference \(X\). Firstly, the equation is written so that all intrinsic assumptions of reference \(X\) are incorporated. Then the equations are compared. This can be reported most clearly in the following way:

\[
\begin{align*}
\text{Example by VESTA} & \quad \text{Example by reference X} \\
i \beta \tilde{\rho} \tilde{W} + \tilde{\rho} \tilde{V}_y + \tilde{\rho}_y \tilde{V} + \left[\tilde{\rho} \tilde{U}_x + \tilde{\rho}_x \tilde{U}\right] \\
- i \omega \tilde{\rho} + \tilde{v} \tilde{p}_y + \tilde{u} \tilde{p}_x + i \beta \tilde{\omega} \tilde{p} + \tilde{v} \tilde{p}_y \\
+ \tilde{u}_x \tilde{p} = 0 & \quad \text{(4.1)}
\end{align*}
\]

Clearly, the terms \(\tilde{\rho} \tilde{U}_x\) and \(\tilde{\rho}_x \tilde{U}\) are not present in equation (4.2). For this reason they are highlighted in equation (4.1). Furthermore, a tilde appears where a bar should be present in equation (4.2), which is also highlighted. In the following cases often a comment will be presented around the highlight boxes to directly illustrate the origin of the difference. In this case, assuming that equation (4.1) is correct, there are 3 errors present in equation (4.2).

The incorrectly appearing tilde can be argued to be an individual typo. Therefore, the cause/discrepancy is simply said to be a typo. In the case of the missing terms, however, it can be argued that they are the result of the absence of the term \((\tilde{\rho} \tilde{U})_x\). Thus there might be only a single cause. Therefore, the absence of the terms is counted as one discrepancy. All in all, there are 3 errors, but 2 discrepancies in equation (4.2). The number of discrepancies gives the best indication of the number of mistakes made during the derivation, see section 4.5.

Similarly, when clearly a group of terms is incorrect due to a collective reason, for example when the complete right hand side has the incorrect sign, then this is counted as one discrepancy too. One could have replaced the equal sign without changing the signs accordingly, yielding one action that causes many erroneous terms.

The above gives the standard approach for the verification. With several exceptions, the following is done to finally present the reported equivalents of equations (4.1) and (4.2). Firstly, the reference equation is checked on assumptions, after which the corresponding equation is derived with the derivation tool. Then the reference equation is
carefully transcribed to *wxMaxima*. This is done for two reasons. Firstly, the equations can be altered very easily when a transcription error occurs. Secondly, *Maxima* orders the terms in a similar way as it does with the derived equation, this makes comparing very easy.

After transcribing, the next thing that is done is to output *\LaTeX* strings that can be directly copied into a (separate) comparison document. Note that the equations can be copied with a single click on the string, minimizing the possibility that only part of the equation is selected. Then the formal comparison is done in the corresponding document, without touching the equation *\LaTeX* code. All errors are carefully assessed to determine the discrepancy count. Note that in this way, it is very easy to filter out transcription errors, because the *\LaTeX* code is not touched other than writing it in the comparison document.

When this has been performed, the equation from the derivation tool is directly copied to the report twice. “Enters” are inserted in the equation’s code manually and the second copy is altered such that it precisely resembles the reference equation. This is done, because often the reference equations are in a different form with respect to the equation of VESTA. In this way they are exactly aligned, posing less possibilities for the introduction of transcription errors in the code.

### 4.3 Incompressible Equation Verifications

Relative to the number of compressible references, the number of incompressible references is rather high. Furthermore, the former references need far more attention than the latter ones. Therefore, however in less detail, the incompressible verification will be done separately from the compressible ones.

All considered references are summarized in table 4.1. It can be seen that several coordinate systems and orientations of the eigenfunction plane are used. These cases will be handled separately. Because the equations were derived with a very large variety in assumptions, the verification equations corresponding to this section are not stated. Only the found errors and discrepancies are stated and elaborated on.

<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>Spcd</th>
<th>Author(s)</th>
<th>Ref.</th>
<th>Page(s)</th>
<th>Discr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian (x,y,z)</td>
<td>x</td>
<td>González et al.</td>
<td>28</td>
<td>2798</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>González et al.</td>
<td>30</td>
<td>927</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Jacquin et al.</td>
<td>46</td>
<td>587</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Piot</td>
<td>66</td>
<td>51</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Robitaillie-Montané</td>
<td>70</td>
<td>38</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>z</td>
<td>Brehm &amp; Fasel</td>
<td>15</td>
<td>2-3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>González et al.</td>
<td>29</td>
<td>842</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Merle et al.</td>
<td>61</td>
<td>912</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Merzari et al.</td>
<td>62</td>
<td>2-3</td>
<td>-</td>
</tr>
</tbody>
</table>

*Continued on next page*
Table 4.1 – Continued from previous page

<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>Spcd</th>
<th>Author(s)</th>
<th>Ref.</th>
<th>Page(s)</th>
<th>Discr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Robitaille-Montané</td>
<td>70</td>
<td>51</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Simens et al.</td>
<td>77</td>
<td>90</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Theofilis</td>
<td>83</td>
<td>4</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>De Vincente et al.</td>
<td>90</td>
<td>194</td>
<td>-</td>
</tr>
<tr>
<td>Cylindrical $(r,\theta,z)$</td>
<td>$\theta$</td>
<td>Batterson</td>
<td>10</td>
<td>291</td>
<td>1</td>
</tr>
<tr>
<td>Curvilinear $(\zeta_1,\zeta_2,\zeta_3)$</td>
<td>$\zeta_3$</td>
<td>Kitsios et al.</td>
<td>49</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Kitsios et al.</td>
<td>50</td>
<td>14-16</td>
<td>2</td>
</tr>
</tbody>
</table>

4.3.1 Cartesian Spanwise Equations

In table 4.1, “Spcd” is the in wxMaxima used variable name for the SPeCtral Direction, see section 2.1, here more often referred to as the homogeneous direction. Firstly, the equations are considered in which the BiGlobal eigenfunction plane is oriented in the $yz$-plane. As earlier, the $z$-direction is the spanwise direction. Therefore, the equations are referred to as the spanwise equations.

When comparing the results of González et al. [28] and Piot [66] with equations (C.9) to (C.12), precisely the same systems were found, yielding 100% matching systems.

In the case of Jacquin et al. [46], the same problem formulation was used. However, only amplitude variables are stated in the equations; no mean ones. To be exact, all amplitude variables in equation (25) and all velocity derivatives stated between brackets in equation (24) must be mean variables. Except for this inconvenient discrepancy, the equations match for 100% with equations (C.9) to (C.12).

When considering the case of González et al. [30], a single discrepancy was found. The only difference in their formulation consists of taking $\omega t$ in the ansatz instead of $-i\omega t$. In their equations, the subscript $j$ is summed over the numbers 2 and 3. The discrepancy involves the absence of the terms:

$$i\alpha \bar{U}\bar{u}, \ i\alpha \bar{U}\bar{v}, \ i\alpha \bar{U}\bar{w}, \ \hat{v}\nabla y \text{ and } \hat{w}\nabla z$$  \tag{4.3}

It is highly probable that these three terms are missing due to incorrect summations over $j$ in the terms $\bar{U}_j \frac{\partial \bar{u}}{\partial \zeta_j} + \bar{u}_j \frac{\partial \bar{U}}{\partial \zeta_j}$ in their equation (5). Note that all terms should come from these terms and the $x$-derivative of $\bar{U}$ is equal to zero. Therefore, one discrepancy is counted.

In the work of Robitaille-Montané a case specific derivation was performed for the Poiseuille flow for which $\bar{V} = \bar{W} = 0$. When substituting this into the equations in the appendix, no discrepancies are found.

Finally, Merzari et al. [62] consider a Poiseuille flow in which the streamwise coordinate is denoted by $z$. It has to be emphasized that this is the only exception in which the $x$-direction is not taken to be the streamwise direction. Just as Robitaille-Montané, they set the mean wall-normal velocities equal to zero, in this case $\bar{U} = \bar{V} = 0$. Again, no discrepancies were found.
4.3.2 Cartesian Streamwise Equations

The next references have defined their equations via the eigenfunctions in the $xy$-plane. Where the $x$-direction is the streamwise coordinate. The equations are called the streamwise equations accordingly.

In this case, the equations of Theofilis [83] and De Vincente et al. [90] are the only sets that completely matched with equations (C.1) to (C.4) in the appendix. That is, without adjusting the formulation of the problem.

The most common adjustment of the formulation includes the equation of $W$ with zero. Brehm & Fasel [15], González et al. [29], Merle et al. [61] and Simens et al. [77] have used this assumption. The systems of González et al. [29] and Merle et al. [61] are not based on any further assumptions and match for the full 100% with equations equations(C.1) to (C.4), when substituting $W = 0$ there.

In the work of Brehm & Fasel [15], one discrepancy is found. They state the spanwise wavenumber $β$ without an $i$, which is inconsistent with the reported ansatz. A minor remark must be made that in this reference the mean quantities are the variables with tildes. Simens et al. [77], report the ansatz with an extra minus in the exponent: $e^{-iβz}$. However, they make precisely the same mistake as found in Brehm & Fasel, no $i$ is present in front of the $β$’s. As in the previous reference, this is counted as one discrepancy.

Lastly, in the thesis of Robitaillé-Montané the equations are derived from the application to Hiemenz flow, for which probably it is assumed that $V$ and $W$ do not vary with $x$. This is in line with the compressible elaboration of the same author handled later on. This is guessed to be such, because the terms $\tilde{u}V_x$ and $\tilde{u}W_x$ are dropped. Because the difference is tractable, it is not counted as discrepancy. Furthermore, the term $\bar{V}U_x$ appears which should be $\bar{v}U_x$. This is only a minor discrepancy.

4.3.3 Curvilinear Equations

Batterson [10] and Kitsios et al. [49, 50] have reported several curvilinear stability equations. Batterson’s results can be directly compared to equations (C.5) to (C.8). It is found that the following terms are kept by Batterson:

\[ \frac{\partial}{\partial r} \tilde{v} \bar{U}_\theta, \quad \frac{\tilde{v}}{r} \bar{V}_\theta \quad \text{and} \quad \frac{\tilde{v}}{r} \bar{W}_\theta \] (4.4)

Recall that the $θ$ denotes a $θ$-derivative in this report, in contrast to Batterson’s notation. These terms must be removed, because in the BiGlobal theory the derivatives of the mean flow with respect to the homogeneous ($θ$-) direction have to be zero. This is counted as a discrepancy.

In the cases of Kitsios et al. [49, 50], it is more complicated to verify the equations. In these two references, the equations are given for a general coordinate system that is curvilinear in the $ζ_1ζ_2$-plane; the $ζ_3$-direction is taken homogeneous and rectilinear. Any curvilinear coordinate pair can be chosen for $ζ_1$ and $ζ_2$. It is argued that when supplying the cylindrical system, the equations can be checked with the derivation tool. This done
by specifying:
\[ \zeta_1 = r, \quad \zeta_2 = \theta, \quad \zeta_3 = z \]
\[ h_1 = h_3 = 1, \quad h_2 = r \]
\[ \xi_1 = u, \quad \xi_2 = v, \quad \xi_3 = w \]

In both references, it is stated that the specific case with \( h_3 = 1 \) is used in the treatment, therefore the cylindrical system is chosen, not the spherical one, for example. The statement is rather strange in reference [49], because \( h_3 \) and its derivatives are still present in the equations. In leaving these terms the suspicion is created that the equations are stated for general \( h_3 \). Nevertheless, the cylindrical system is chosen. When substituting all above variables in equations (25) to (29) of Kitsios et al. [49] and (B.2) to (B.14) of Kitsios et al. [50], the following equations are retrieved:

**Continuity**

<table>
<thead>
<tr>
<th>Equation by VESTA</th>
<th>Equation by Kitsios et al. in [50] ((\overline{W} = 0))</th>
<th>Equation by Kitsios et al. in [49]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ i q \ddot{w} + \frac{\ddot{v}_\theta}{r} + \ddot{u} + \frac{\ddot{u}_r}{r} = 0 ] (4.5)</td>
<td>[ i q \ddot{w} + \frac{\ddot{v}_\theta}{r} + \ddot{u} + \frac{\ddot{u}_r}{r} = 0 ] (4.6)</td>
<td>[ i q \ddot{w} + \frac{\ddot{v}_\theta}{r} + \ddot{u} + \frac{\ddot{u}_r}{r} = 0 ] (4.7)</td>
</tr>
</tbody>
</table>

**r-Momentum**

<table>
<thead>
<tr>
<th>Equation by VESTA Missing in (4.9) and (4.10)</th>
<th>Equation by Kitsios et al. in [50] ((\overline{W} = 0)) Missing in (4.9) and (4.10) in (4.10)</th>
<th>Equation by Kitsios et al. in [49]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ i q \ddot{u} \overline{W} - \frac{2 \dot{v}<em>\theta}{r} + \frac{\ddot{u}</em>\theta \overline{V}}{r} ] (4.9)</td>
<td>[ \ddot{u}_\theta \overline{V} ]</td>
<td>[ \ddot{u}_r \overline{V} ]</td>
</tr>
</tbody>
</table>
| \[ + \ddot{u} \overline{U}_r + \ddot{u}_r \overline{U} 
- \frac{\ddot{u}}{r^2 Re} \] (4.8) | \[ + \ddot{u} \overline{U}_\theta + \frac{\ddot{u}_r}{r \ Re} + \frac{\ddot{u}_r}{r} 
- \frac{\ddot{u}}{r^2 Re} \] (4.9) | \[ + \ddot{u}_\theta \overline{U}_r + \ddot{u}_r \overline{U} 
+ \frac{\ddot{u}_\theta}{r^2 Re} + \frac{2 \ddot{u}_r}{r \ Re} + \frac{2 \ddot{u}_r}{r Re} \] (4.10) |
### θ-Momentum

<table>
<thead>
<tr>
<th>Equation by VESTA</th>
<th>Equation by Kitsios et al. in [50] ((\mathbf{W} = 0))</th>
<th>Equation by Kitsios et al. in [49]</th>
</tr>
</thead>
</table>
| \[
i q \hat{v} \hat{W} + \frac{\hat{v} \nabla_{\theta}}{r} + \hat{u} \nabla_{r} + i q \hat{\varpi} \hat{W} + \frac{\hat{\varpi} \hat{\varpi}}{r^2 \Re} + \hat{\varpi}_r \hat{U} - i \omega \hat{\varpi} \] | \[
\hat{v} \nabla_{\theta} + \hat{u} \nabla_{r} + \frac{\hat{\varpi}}{r^2 \Re} + \hat{\varpi}_r \] | \[
\hat{v}_z \hat{W} + \frac{\hat{v} \nabla_{\theta}}{r} + \hat{u} \nabla_{r} + \frac{\hat{\varpi}}{r^2 \Re} + \hat{\varpi}_r \] |
| \(r \) & \(2 \hat{\varpi}_r \Re \) & \( \hat{\varpi}_r \Re \) |
| Missing in \((4.12) \& (4.13)\) | Missing in \((4.13)\) | Missing in \((4.16)\) |

### z-Momentum

<table>
<thead>
<tr>
<th>Equation by VESTA</th>
<th>Equation by Kitsios et al. in [50] ((\mathbf{W} = 0))</th>
<th>Equation by Kitsios et al. in [49]</th>
</tr>
</thead>
</table>
| \[
\frac{\hat{v} \hat{W}_{\theta}}{r} + \hat{u} \hat{W}_r + i q \hat{\varpi} \hat{W} + \frac{\hat{w}_{\theta \theta}}{r^2 \Re} + \frac{\hat{w}_{\theta r}}{r \Re} - \frac{q^2 \hat{w}}{\Re} \] | \[
\frac{\hat{w}_{\theta \theta}}{r^2 \Re} + \frac{\hat{w}_{\theta r}}{r \Re} + \hat{w}_r \hat{U} - i \omega \hat{\varpi} \] | \[
i q \hat{\varpi} \hat{W} + \frac{\hat{u}_{\theta \theta}}{r^2 \Re} + \frac{\hat{u}_{\theta r}}{r \Re} + \hat{u}_r \hat{\varpi} + \frac{2 \hat{u}_r}{r^2 \Re} \] |
| \(r \) & \(2 \hat{w}_r \Re \) & \( \hat{w}_r \Re \) |
| Missing in \((4.16)\) | Missing in \((4.16)\) | Missing in \((4.16)\) |

It must be noted that the equations in the second and third columns are derived by hand, whereas the ones in the first are derived using the derivation tool. Furthermore,
all equations in reference [50] are based on the assumption $W = 0$. It can be seen that all continuity equations give a 100% match. This cannot be said for the momentum equations however.

In reference [49], the signs of the $\omega$ terms are incorrect (1) and the right hand side of each equation are heavily subject to error. This latter fact is probably due to the derivatives of $h_3$ contained in the right hand sides (2). These should not be present because $h_3$ is assumed to be equal to 1. The right hand side terms of the $z$-momentum equation are precisely equal to those of the $r$-momentum equation, being obviously incorrect (3). Furthermore errors must be present in both expressions forming all elements of $\tau'_{ij}$ (4). Several curvature related terms, $-\frac{2 \tilde{v} \vartheta}{r^2 Re}$, $\frac{\tilde{u}}{r^2 Re}$ and $\frac{\tilde{v}}{r^2 Re}$, are not present (5) and lastly, $z$-derivatives are present that should be terms with $i\beta$ instead (6). All discrepancies are numbered with the number between brackets. Thus 6 discrepancies are argued to be present, though it must be stressed that they cannot be well-defined because of the large number of errors.

In reference [50], the $z$-momentum equation is found be matching perfectly with that of VESTA, noting that $W = 0$. In the other equations, the same curvature related terms are missing as in the other reference. Furthermore, in the right hand sides the following terms are missing:

\[-\frac{2 \tilde{v} \vartheta}{r^2 Re}, \quad -\frac{\tilde{u}}{r^2 Re}, \quad -\frac{\tilde{v}}{r^2 Re}, \quad \text{and} \quad \frac{2 \tilde{u} \vartheta}{r^2 Re}\]

(4.17)

Note the apparent symmetry of the terms. As the curvature related terms in the left hand side, these terms are probably due to a curvature related issue in the equations. All in all, it is concluded that this reference has two discrepancies.

### 4.4 Compressible Equation Verifications

In the literature, 4 references were found that reported the compressible stability equations. They are summarized in table 4.2. All references consider a Cartesian coordinate system of which the spanwise $z$-direction is taken to be the homogeneous one. All verification equations are stated in appendix E.

<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>Spcd</th>
<th>Author(s)</th>
<th>Ref.</th>
<th>Pages</th>
<th>Discr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian (x,y,z)</td>
<td>$z$</td>
<td>Robinet</td>
<td>[69]</td>
<td>108-109</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Robitaillé-Montané</td>
<td>[70]</td>
<td>68-69</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Theofilis</td>
<td>[84]</td>
<td>7-11</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Theofilis &amp; Colonius</td>
<td>[86]</td>
<td>11-12</td>
<td>1</td>
</tr>
</tbody>
</table>

#### 4.4.1 Robitaillé-Montané [70]

In her PhD thesis, Robitaillé-Montané notes the compressible BiGlobal stability equations for the application to Hiemenz flow. The $(p,T)$-formulation is used and the en-
thalpy formulation for the energy equation is equipped. Several important, non-standard and even deviating, features have to be reported. Stokes’ hypothesis is applied, for example, yielding only a minor adjustment.

Due to the application to a specific flow case, Robitaillie-Montane prescribes several intrinsic dependencies of the mean flow variables (see pages 70-71 in [70]):

\[
\begin{align*}
U &= U(x, y) \\
V &= V(y) \\
W &= W(y) \text{ Linear in } x \\
\bar{T} &= \bar{T}(y), \quad \bar{\rho} = \bar{\rho}(y) \quad \text{and} \quad \bar{P} = \bar{P}(x) \text{ Boundary layer assumption}
\end{align*}
\]

The linear variation of \( U \) in the \( x \)-direction is logical with respect to the exact Hiemenz flow solution analogue, see page 96 of Schlichting [74]. Furthermore, the boundary layer assumption is made, which results in a constant mean pressure in the \( y \)-direction. Regarding the variation of the mean pressure in \( x \), equation (5.1) in [70] is of the correct form, which corresponds to equation (2.12) of this report. However, after investigation it is found that during the derivation of Robitaillie-Montane the following form of the mean equation of state has been used:

\[
\bar{\rho} = \frac{1}{\bar{T}}
\]  

Equation (4.18) can be derived from the mean equation of state (2.11) by taking the mean pressure \( \bar{P} = \frac{1}{\gamma M^2} \), thus equal to the freestream static pressure, see Bertolotti [14]. However, this implies also that the mean pressure is taken to be completely constant and \( \bar{P} = \bar{P}(x) \). Note that this assumption about the mean flow is not consistent. It is made when performing LST, where one considers parallel mean flows only. The Hiemenz flow case is essentially non-parallel. This forms a rather fatal discrepancy, because nearly all terms are affected by it. The complete basis of the derivation is incorrect. To still be able to compare the resulting equations, specific derivation files were created incorporating equation (4.18) as the mean equation of state.

Some further less major discrepancies are present. The thermal conductivity coefficient is not written as \( k \), but as \( \mu \). This yields 7 erroneous terms. Furthermore, an individual error is present in the energy equation. In the term \( \frac{\bar{T}y}{\bar{T}y + \bar{\rho}} \), the derivative \( \bar{T}_y \) is missing. Lastly, several derivatives of \( \bar{\rho} \) were not expanded using the chain rule (2.13); this is not counted as a discrepancy because it is not incorrect.

Robitaillie-Montane’s has presented all terms at the left hand side of the equations. For ease of comparison in section E.1 all equations have been first copied precisely from the VESTA version and afterwards edited carefully to yield precisely the same equations (though with non-zero right hand sides) as reported in the reference.

4.4.2 Theofilis [84]

In reference [84] Theofilis reports the compressible BiGlobal stability equations for general application. He uses the unadjusted \((p, \rho, T)\)-formulation, so the equation of state
is never inserted, see section 3.2. Furthermore, all equations are derived from the conservative formulation and the energy equation using the total formulation, yielding probably the longest equations possible. Lastly, it must be noted that Stokes’ hypothesis is applied.

To get a good insight in the difference between the stability equations derived from the conservative and non-conservative momentum equations, it is decided to just state the stability equation form of the left hand sides. It has to be emphasized that the z-direction is taken to be homogeneous. Note that $\rho$ is never substituted for in the left hand sides in the considered references, making comparisons straightforward to perform. This yields equations (4.19) to (4.21):

**Conservative x-momentum equation LHS**

$$-i \omega \tilde{u} \tilde{p} + 2 \tilde{u}_x \tilde{p} \tilde{U} + \tilde{v}_y \tilde{p} \tilde{U} + i \beta \tilde{w} \tilde{p} \tilde{U} + 2 \tilde{u} \tilde{p}_x \tilde{U} + \tilde{v}_y \tilde{p} \tilde{U} + i \omega \tilde{p} \tilde{U} + \tilde{p}_x \tilde{U}^2$$

$$+ 2 \tilde{u} \tilde{p} U_x + 2 \hat{p} \tilde{U} U_x + \tilde{v} \tilde{p} U_y + \tilde{u}_y \tilde{p} \tilde{V} + \tilde{u}_y \tilde{p}_y \tilde{V} + \hat{p} \tilde{U} \tilde{V} + \hat{p}_y \tilde{V} + \hat{u} \tilde{p} \tilde{V}_y$$

$$+ \hat{p} \tilde{V} \tilde{V}_y + i \beta \hat{p} \tilde{W} + i \beta \hat{p} \tilde{W} \tilde{W}$$

(4.19)

**Conservative y-momentum equation LHS**

$$-i \omega \tilde{v} \tilde{p} + \tilde{v}_x \tilde{p} \tilde{U} + \tilde{v}_y \tilde{p} \tilde{U} + \tilde{v}_y \tilde{p} \tilde{U} + \tilde{u}_x \tilde{p} \tilde{V} + i \beta \tilde{w} \tilde{p} \tilde{V} + \tilde{u}_y \tilde{p} \tilde{V}$$

$$+ 2 \tilde{v} \tilde{p}_y \tilde{V} - i \omega \hat{v} \tilde{V} + \hat{v}_x \tilde{V} + \hat{v}_y \tilde{V} + \hat{p}_y \tilde{V}^2 + \tilde{u} \tilde{p} \tilde{V}_x + \hat{p} \tilde{U} \tilde{V}_x + 2 \tilde{v} \tilde{p} \tilde{V}_y$$

$$+ 2 \hat{p} \tilde{V} \tilde{V}_y + i \beta \hat{v} \tilde{W} + i \beta \hat{p} \tilde{V} \tilde{W}$$

(4.20)

**Conservative z-momentum equation LHS**

$$-i \omega \tilde{w} \tilde{p} + \tilde{w}_x \tilde{p} \tilde{U} + \tilde{w}_y \tilde{p} \tilde{U} + \tilde{w}_y \tilde{p} \tilde{U} + \tilde{w}_y \tilde{p} \tilde{U} + \tilde{u}_x \tilde{p} \tilde{W} + \tilde{v}_y \tilde{p} \tilde{W} + i \omega \hat{w} \tilde{W} + \hat{p}_x \tilde{W} \tilde{W} + \hat{p}_y \tilde{W} \tilde{W}$$

$$+ \tilde{v}_x \tilde{p} \tilde{W} + 2 i \beta \hat{w} \tilde{p} \tilde{W} + \hat{u}_x \tilde{p} \tilde{W} + \hat{v}_y \tilde{p} \tilde{W} + i \omega \hat{p} \tilde{W} + \hat{p}_x \tilde{W} \tilde{W} + \hat{p}_y \tilde{W} \tilde{W}$$

$$+ \hat{p} \tilde{V} \tilde{W} + i \beta \hat{p} \tilde{W}^2 + \tilde{u} \tilde{p} \tilde{W}_x + \hat{p} \tilde{U} \tilde{W}_x + \tilde{v} \tilde{p} \tilde{W}_y + \hat{p} \tilde{V} \tilde{W}_y$$

(4.21)

Here the blue boxes, as in equation (3.11), contain the terms that can be removed using the continuity equation. In section E.2, where all the verification equations are stated, all blue boxed terms will be found to be extra in the momentum equations of Theofilis [84]. Note that this proves that this is just another valid, though less efficient, representation.

Because Theofilis does not write the transport coefficient amplitudes through temperature amplitudes, it is additionally chosen to create separate derivation files that substitute the amplitude ansatz for $\mu'$ and $k'$ as well. This is done in both the cases.

\[2\text{Theofilis [84] reported the following, after the statement of his equations: "[the perturbation equation of state] [...] may be used to eliminate one of $\hat{p}, \hat{p}, T$ and cast the final system to be solved in terms of five two-dimensional disturbance amplitude functions, the three velocity components and two thermodynamic variables." In his treatment the hats denote the amplitude variables.}\]
of the momentum and energy equations. Furthermore, separate instantaneous equation functions are created that specify the coefficients as a function the coordinates instead of temperature. All in all, this yields the momentum equations (E.13), (E.15) and (E.17).

The total energy formulation of the energy equation is so gigantic that it is chosen to add this formulation in the tool; writing out all additional terms as done for the momentum equations is of no practical use and easily subject to errors. For this reason this formulation is the only conservative equation implemented in the tool. The result is the 175 term equation (E.19).\footnote{An interesting note is that this equation lasted 3 times as long to derive (one minute), with respect to normal energy equation formulations, due to the required divergence of the viscous stress tensor. When incorporating curved coordinate systems it is even worse.}

As done in the case of Robitaillié-Montané, to be able to compare the currently derived equations, Theofilis’ momentum and energy equations are written by first copying the equations from VESTA and then altering them to retrieve precisely the equations reported in reference [84]. Note that the right hand sides are therefore no longer zero.

The continuity and momentum equations give a magnificent 100\% match, after checking all blue boxed terms with those in equations (4.19) to (4.21). However, when considering the energy equation two discrepancies occur. First of all, as with Robitaillié-Montané, the first coefficient of viscosity \( \mu \) appears at the location of the thermal conductivity coefficient \( k \), yielding 9 erroneous terms.

This is not the largest problem however, because the second discrepancy is that nearly all “right hand side” terms have the wrong sign. Exceptions are the thermal conductivity terms and terms that are noted with an extra minus in the nominator of the common factor \( -\frac{\mu \alpha}{4\pi c} \). This extra minus occurs probably accidentally, see the first line on page 11 of reference [84]. During reporting, Theofilis probably moved all the right hand side terms to the left without altering the sign at some stage. Furthermore, all terms involving the frequency \( \omega \) have the wrong sign as well. This is probably caused by the equipment of the wrong sign of \( \omega t \) in the ansatz (2.2). Ultimately, this causes 94 terms with the incorrect sign.

Lastly, Theofilis states a collection of terms without sign three times, see the sixth line of the energy equation on page 8 and the fifth and sixteenth on page 9. To be able to compare the equations it was chosen to take the signs positive, consistent with the previously made sign errors. Taking these negative would yield 16 less erroneous signs. These typos form the third discrepancy.

4.4.3 Theofilis & Colonius [86]

In their paper, Theofilis & Colonius [86] state the BiGlobal stability equations in the \((p,T)\)-formulation. However, they still use \( \rho \) where appropriate, e.g. in the (conventional) left hand sides of the equations. The energy equation is stated in the pressure work formulation and Stokes’ hypothesis is applied again. Lastly, the pressure is made dimensionless using the static, instead of the dynamic, pressure. This is accounted for during the transcription to \textit{wxMaxima} by multiplying all pressure amplitudes with \( \gamma M^2 \).
It has to be noted that, after the equations were transcribed to Maxima, the equation of state is substituted into the equations for $\bar{\rho}$ and $\tilde{\rho}$. In this way the equations can be checked with the standard output from the derivation tool. Furthermore, Theofilis & Colonius state their equations in the generalized eigenvalue problem form, $\mathbf{A} \mathbf{x} = \omega \mathbf{B} \mathbf{x}$. This means that the right hand sides consist only of $\omega$ terms. As done in the earlier verifications, first the equations derived with VESTA are copied and then rewritten carefully to yield precisely the equations from the literature. In this way the equations can be compared very easily. The ultimate equations are stated in section E.3. The blue boxes in this section can be ignored for the moment.

The continuity and energy equations were verified for the full 100%. However, an important note must be made. In the collection of terms $L_{\hat{c}w}$, a single $\tilde{w}$ is still present (using the tilde instead of the hat). Strictly, this would mean that a non-linear $\tilde{w}^2$-term would be present. By looking at the equation by VESTA, it is found that this $\tilde{w}$ should have been removed to correctly create the matrix with $L$’s. This is interpreted as a minor typo here and is not even regarded as a discrepancy.

When considering the momentum equations, a single discrepancy is found. Every term precisely matches except for a repeating collection of 6 terms per equation. By using the mean equation of state, per equation these can be factored to:

$$\tilde{u} \left( (\bar{\rho} U)_x + (\bar{\rho} V)_y \right), \quad \tilde{v} \left( (\bar{\rho} U)_x + (\bar{\rho} V)_y \right) \quad \text{and} \quad \tilde{w} \left( (\bar{\rho} U)_x + (\bar{\rho} V)_y \right) \quad (4.22)$$

When comparing these terms to the ones in equations (4.19), (4.20) and (4.21), it can be seen that they all correspond to a certain blue boxed term. Thus they are “traces” of the fact that a conservative instantaneous equation was used to derive the stability equations.

Regarding this, the following hypothetical explanation for the discrepancy seems feasible. Probably, Theofilis & Colonius derived the momentum equations using the conservative instantaneous form. A hint to this is the occurrence of equation (2) in [86], which is conservative. Thereafter, they probably found out that writing the system in the non-conservative way would yield less terms and thus be more efficient. They did this by subtracting the BiGlobal continuity equation from the conservative equations with the appropriate factor. In doing so, they accidentally used an incorrect continuity equation without the terms $(\bar{\rho} U)_x$ and $(\bar{\rho} V)_y$. In this way, nearly all terms that appear in the conservative equations are removed, except those that are a multiple of these terms.

### 4.4.4 Robinet [69]

Robinet reports precisely the same kind of system as done by Theofilis & Colonius. At a first glance it seems reshuffled and different. However, after transcribing them they nearly exactly match to the ones of Theofilis & Colonius. Even the same extra terms in the momentum equations of Theofilis & Colonius are encountered.

This leads to the idea that Robinet copied the equations of Theofilis & Colonius. It is argued that this is proven in the following way. In the last sentence of the paragraph
that describes the equations, he states (page 99):

“This computational strategy has been successfully used for the first time in a compressible global eigenproblem by Theofilis & Colonius (2004) for a compressible open cavity problem.”

However, in addition to the extra terms in the momentum equations of Theofilis & Colonius, Robinet has made new errors. First of all, Robinet has wrongly corrected the extra $\tilde{w}$ in his $L(\tilde{c})\hat{w}$, as touched upon in the last subsection. Instead of removing it, he has rewritten/interpreted it as the mean variable $\overline{W}$. Furthermore, a $\beta$ is missing in the collection $L(\hat{z})$. Lastly, a few typos have been made. The bars are missing on derivatives of $\rho$ in $L(\hat{c})\hat{T}$ and a minus sign is missing in $L(\hat{z})\hat{T}$. Only this last typo, due to the same policy in the verification with Theofilis [84], is counted as a discrepancy. This sums to a total of 4 discrepancies, including the one present in the work of Theofilis & Colonius.

With the exception of the mentioned extra discrepancies and typos, all equations are precisely identical. For this reason the extra discrepancies are illustrated along with the equations of Theofilis & Colonius in section E.3. To distinguish them clearly, they are presented in blue boxes, which are not to be confused with those used in the verification of Theofilis to distinguish extra conservative terms.

4.5 Declaration of Verification

From tables 4.1 it can be seen that the incompressible stability equations are verified multiple (8) times, through having 100% matched systems. The discrepancies found in the other references are all accordingly reasoned to be the result of mistakes in the corresponding derivations. For this reason the tool is declared to be verified with respect to the incompressible equations.

It is not trivial to state the same result for the compressible equations. In all references at least one discrepancy was found. Luckily, the discrepancies occur at different places in the equations. In both the cases of Theofilis [84] and Theofilis & Colonius [86] the continuity equation is matched precisely. Though a discrepancy occurs in the momentum equations of Theofilis & Colonius, their energy equation delivers a perfect match as well. This happens the other way around in the work of Theofilis, discrepancies occur in the energy equation while the momentum equations are fully matched. Therefore, the derivation tool is declared to be verified through cross-verification of references [84] and [86]. Accordingly, all reported discrepancies are reasoned to be due to mistakes during the derivations of the corresponding treatments.

Note that the systems presented in references [84] and [86] are derived using a different formulation. Therefore, one could argue that only a part of the derivation tool is verified per reference. Verifying one equation in one formulation not necessarily implies the verification of the same equation in another. However, this reasoning would be applicable only when the errors were considered, not the discrepancies. When a formulation is
changed, the terms and thus the errors change accordingly. Indeed, they could even dis-
appear. However, the underlying causes of these errors do not change, *discrepancies are invariant with respect to the choice of formulation.* If an equation contains discrepancies in one formulation, it would contain those in another as well. This is the reason why discrepancies are considered for the verification, not (the number of) isolated erroneous terms.
Chapter 5

Discretization and Implementation

In the course of the current project, the VKI Extensible Stability and Transition Analysis (VESTA) toolkit has been equipped with the automatic implementation tool. In this chapter, all necessary features regarding the discretization and the implementation of the problem is presented using the Chebyshev spectral collocation method. In sections 5.1 to 5.4, the discretization of the problem is handled. Thereafter, the structure of the problem and the principles of the thereupon based automatic equation implementation tool are elaborated on in sections 5.5 and 5.6. The chapter is concluded with a treatment of the discretization of the boundary conditions in section 5.7 and, concisely, the implementation thereof in section 5.8. The information is mainly collected from the work of Batterson [10], Canuto et al. [17, 18], Piot [66], Robitallié-Montané [70] and Trefethen [88].

5.1 Chebyshev Polynomials

Before the discretization method can be elaborated on, several basic steps have to be taken. First of all, the Chebyshev polynomials are concisely touched upon. Note that the Chebyshev spectral collocation method does not directly use the Chebyshev polynomials themselves to represent the to-be-approximated functions. They are used to derive a special form of the characteristic Lagrange polynomials, handled in the next section.

The Chebyshev polynomials are the solution to a specific Sturm-Liouville differential equation [17]:

\[
\frac{d}{d\xi} \left( \sqrt{1-\xi^2} \frac{dT_{N-1}}{d\xi} \right) + \frac{(N-1)^2}{\sqrt{1-\xi^2}} T_{N-1} = 0
\]

or:

\[
(1-\xi^2) \frac{d^2 T_{N-1}}{d\xi^2} - \xi \frac{dT_{N-1}}{d\xi} + (N-1)^2 T_{N-1} = 0 \tag{5.1}
\]

By making the change of variables \( \xi = \cos \theta \), i.e. \( d\xi = -\sin \theta d\theta \), in equation (5.1), the
following will result:
\[
(1 - \cos^2 \theta) \frac{d\theta}{d\xi} \frac{d\xi}{d\theta} \left( \frac{d\theta}{d\xi} \frac{dT_{N-1}}{d\theta} \right) - \cos \theta \frac{d\theta}{d\xi} \frac{dT_{N-1}}{d\theta} + (N - 1)^2 T_{N-1} = 0
\]
\[
- \sin^2 \theta \frac{d}{d\theta} \left( - \frac{1}{\sin \theta} \frac{dT_{N-1}}{d\theta} \right) + \cos \theta \frac{dT_{N-1}}{d\theta} \frac{dT_{N-1}}{d\theta} + (N - 1)^2 T_{N-1} = 0
\]
\[
- \sin \theta \left( \frac{\cos \theta}{\sin^2 \theta} \frac{d^2 T_{N-1}}{d\theta^2} - \frac{1}{\sin \theta} \frac{dT_{N-1}}{d\theta} \right) + \cos \theta \frac{dT_{N-1}}{d\theta} \frac{dT_{N-1}}{d\theta} + (N - 1)^2 T_{N-1} = 0
\]
\[
- \frac{\cos \theta}{\sin \theta} \frac{d^2 T_{N-1}}{d\theta^2} \frac{dT_{N-1}}{d\theta} \frac{dT_{N-1}}{d\theta} + \cos \theta \frac{dT_{N-1}}{d\theta} \frac{dT_{N-1}}{d\theta} + (N - 1)^2 T_{N-1} = 0
\]
\[
\frac{d^2 T_{N-1}}{d\theta^2} + (N - 1)^2 T_{N-1} = 0 \tag{5.2}
\]

The differential equation reduces to the simple oscillator equation (5.2) in \( \theta \). The fundamental solutions to the equation are called the Chebyshev polynomials of first and second kind:
\[
T_{N-1} = \cos((N - 1)\theta), \quad U_{N-1} = \sin((N - 1)\theta) \tag{5.3}
\]

When using the inverse transformation \( \theta = \arccos \xi \), it must be noted that it is not uniquely defined out of the interval \( \xi \in [-1, 1] \). Therefore, the resulting polynomials are considered only within it:
\[
T_{N-1} = \cos((N - 1)\arccos \xi), \quad U_{N-1} = \sin((N - 1)\arccos \xi), \quad \xi \in [-1, 1] \tag{5.4}
\]

Illustrations of these polynomials can be found throughout the literature [10, 65]. These are not reported here, because only the extrema of the Chebyshev polynomial of the first kind are used in the current development. These can be found through solving:
\[
\frac{dT_{N-1}}{d\xi} = - \frac{N - 1}{\sqrt{1 - \xi^2}} \sin((N - 1)\arccos \xi) = - \frac{N - 1}{\sqrt{1 - \xi^2}} U_{N-1} = 0, \quad \xi \in [-1, 1]
\]
\[
\sin((N - 1)\arccos \xi) = 0, \quad \xi \in [-1, 1]
\]
\[
(N - 1)\arccos \xi_i = (i - 1)\pi, \quad \xi \in [-1, 1], \quad i \in \mathbb{Z} \tag{5.5}
\]
\[
\xi_i = \cos \frac{(i - 1)\pi}{N - 1}, \quad i = 1, 2, \ldots, N \tag{5.6}
\]

Here, in equation (5.6) the range of values of \( i \) is determined such to represent a unique set of \( \xi \) values in the \( \xi \)-domain. In equation (5.5), the factor \( i - 1 \) was introduced to let \( i \) start at 1 instead of 0. With similar reasoning, the rather awkward choice of the parameter \( N - 1 \) in equation (5.1) was introduced. This is done to have \( N \) distinct extrema locations in equation (5.6). Using this notation is rather handy with respect to the implementation in Matlab, because zero indices are not allowed in that language. Furthermore, in the following it is more practical to consider the number of discretization points instead of the maximal polynomial order (which would yield a little more neat notation in this section).
The points defined by equation (5.6) are generally referred to as Chebyshev Gauss-Lobatto (CGL) points, see page 86 of Canuto et al. [17]. It must be emphasized that these points are ordered from right to left.

5.2 Characteristic Lagrange Polynomials

Ultimately, one wants to solve for a solution that satisfies a partial derivative eigenvalue problem. To do this, one must be able to represent its derivatives, at least at the points for which the solution is required. Before the derivatives of a function can be considered, one must choose a way to represent the function itself.

In this case, a nodal (Lagrange) basis is considered [17]. This implies that the to-be-found function is described at pre-defined locations by individual basis functions. The perfect example of a set of these functions is a combination of discrete delta functions. For a one-dimensional case, these can be represented as follows:

\[ \psi_j(\xi_k) = \delta_{jk} \begin{cases} = 1 & \text{for } j = k \\ = 0 & \text{for } j \neq k \end{cases} \tag{5.7} \]

Here, \( \psi_j \) are the discrete delta functions and \( \xi_k \) are the collocation points; the points for which the function is approximated. One can express these functions as polynomials in the following way:

\[ \psi_j = \prod_{j \neq k}^{1 \leq j,k \leq N} \frac{\xi - \xi_k}{\xi_j - \xi_k} \tag{5.8} \]

These polynomials are called the characteristic Lagrange polynomials and are illustrated in figure 5.1. The collocation points are highlighted with circles on the \( \xi \)-axis. It can be seen that the first function \( \psi_1 \), which has value 1 at the first collocation point, is the far right one.

A complete function can be represented by multiplying the basis functions with the function values (samples) at the collocation points and adding all resulting polynomials together. The formula is thus as follows:

\[ f_N(\xi) = \sum_{j=1}^{N} f_j \psi_j(\xi), \quad \text{where } f_j = f_{\text{exact}}(\xi_j) \tag{5.9} \]

The distinction between \( f_{\text{exact}} \), \( f_N \) and \( f \) must be emphasized. The discrete version of the exact function \( f_{\text{exact}} \) is denoted with the subscript \( N \), showing that \( N \) collocation points are used. In the two-dimensional case, the number of points in both directions is subscribed. In this case, \( f \) is the coefficient vector whose \( j^{th} \) element is the weight of the \( j^{th} \) basis function \( \psi_j \). In the two-dimensional case \( f \) will be a matrix. The obtained approximation \( f_N \) of the function \( f_{\text{exact}} \) can be used to obtain certain properties, such as derivatives. This is based on the principle of analytic substitution [39].

\[ ^1 \text{Other names are used as well, a collection can be found on page 42 of Trefethen [88].} \]
When CGL collocation points given by equation (5.6) are considered (as already done in figure 5.1) some special features can be used. For example, representation (5.7) can be written more compactly in terms of the Chebyshev polynomials:

\[
\psi_j = \lambda_j = \frac{(-1)^j}{\pi_j(N-1)^2} \frac{1}{\xi - \xi_j} \frac{dT_{N-1}}{d\xi} = \frac{(-1)^{j+1}}{\pi_j(N-1)} \sqrt{1-\xi^2} U_{N-1}
\]

where \( \pi_j = 2 \) for \( j = 1 \lor j = N \) and \( \pi_j = 1 \) for \( 2 \leq j \leq N - 1 \)

Here, equation (5.10) is acquired by using the derivative of \( T_{N-1} \) evaluated in the derivation of equation (5.5). See, for example, Batterson [10], Canuto et al. [17], Piot [66] and Robitaille-Montané [70]. Next to the aforementioned specific description of the characteristic Lagrange polynomials, the use of CGL collocation points is notorious for the resulting accuracy when solving differential equations [10, 17]. According to Burden & Faires [16], the equipment of these nodes provides “an optimal placing of interpolating points to minimize the error in Lagrange interpolation.” See section 8.3 of their treatment for an excellent elaboration on these matters.

In this treatment, one wants to represent two-dimensional functions. This is simply done by extending equation (5.9):

\[
f_{N_\xi \times N_\eta}(\xi, \eta) = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} f_{ji} \psi_{ji}(\xi, \eta), \quad \text{where } f_{ji} = f_{\text{exact}}(\xi_i, \eta_j)
\]

It is logical to consider \( f_{ji} \) and \( \psi_{ji} \) as the components of a matrix \( f \) of sample values and a matrix \( \psi \) of functions of \( \xi \) and \( \eta \), respectively. Thus, \( f \in \mathbb{C}^{N_\xi \times N_\eta} \) and \( \psi \in \mathbb{R}^{N_\xi \times N_\eta} \). Here, \( N_\xi \) and \( N_\eta \) are the numbers of collocation points in the \( \xi \)- and \( \eta \)-coordinates.

The conventional notation as in Golub & Van Loan [25] is used.
direction, respectively. From this point on the CGL collocation points will be denoted by $\xi_i$ and $\eta_j$ in the corresponding direction.

At this moment, it is important to elaborate on the orientation of these matrices. In this treatment, data for constant $\xi$-locations (constant $i$) is stored in the columns and for constant $\eta$-locations (constant $j$) in the rows (therefore the awkward subscripting in equation (5.11)). This is chosen as such because $\eta$ usually is taken as the numerical vertical coordinate direction; by using this convention “vertical information” remains vertical in the matrices.

The components of $\psi$ can be expressed as follows:

$$
\psi_{ji}(\xi, \eta) = \lambda_i(\xi)\mu_j(\eta)
$$

$$
\lambda_i = \frac{(-1)^i}{\bar{c}_i(N-1)^2} \frac{1 - \xi^2}{\xi - \xi_i} \frac{dT_{N-1}(\xi)}{d\xi} = \frac{(-1)^i+1}{\bar{c}_i(N-1)} \sqrt{1 - \xi^2} U_{N-1}(\xi)
$$

$$
\mu_j = \frac{(-1)^j}{\bar{c}_j(N-1)^2} \frac{1 - \eta^2}{\eta - \eta_j} \frac{dT_{N-1}(\eta)}{d\eta} = \frac{(-1)^j+1}{\bar{c}_j(N-1)} \sqrt{1 - \eta^2} U_{N-1}(\eta)
$$

where

$$
\bar{c}_i = 2 \quad \text{for} \quad i = 1 \lor i = N\xi
$$

$$
\bar{c}_i = 1 \quad \text{for} \quad 2 \leq i \leq N\xi - 1
$$

$$
\bar{c}_j = 2 \quad \text{for} \quad j = 1 \lor j = N\eta
$$

$$
\bar{c}_j = 1 \quad \text{for} \quad 2 \leq j \leq N\eta - 1
$$

See Piot [66] or Robitaillié-Montané [70]. Several components of $\psi$ are plotted against $\xi$ and $\eta$ in figure 5.2 for $(i, j) = (2, 3), (3, 1)$ and $(4, 4)$.

When solving the ultimate partial derivative eigenvalue problem, it is practical to use a different storage of the components $f_{ji}$. One wants to solve for the two-dimensional eigenfunctions through linking them to the eigenvectors of the discrete system. Therefore, the components of the matrix $f$ must be organized in a single vector. This is done simply by stacking all columns of $f$. This yields:

$$
f =
$$

$$
\begin{bmatrix}
\xi = 1 & \xi = -1 \\
\uparrow & \downarrow
\end{bmatrix}
\begin{bmatrix}
f_{11} & f_{12} & \cdots & \cdots & f_{1N\xi} \\
f_{21} & f_{22} & \cdots & \cdots & f_{2N\eta} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
f_{N\eta 1} & \cdots & \cdots & f_{N\eta N\xi} \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
\xi = 1 \\
\vdots \\
\xi = -1
\end{bmatrix}
\begin{bmatrix}
f_{11} \\
\vdots \\
f_{N\eta 1} \\
\end{bmatrix}
\begin{bmatrix}
\xi = 1 \\
\vdots \\
\xi = -1
\end{bmatrix}
\begin{bmatrix}
f_{12} \\
\vdots \\
f_{N\eta 2} \\
\end{bmatrix}
\begin{bmatrix}
\xi = 1 \\
\vdots \\
\xi = -1
\end{bmatrix}
\begin{bmatrix}
f_{1N\xi} \\
\vdots \\
f_{N\eta N\xi} \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
\xi = \cos \frac{\pi}{N\xi-1} \\
\vdots \\
\xi = 1 \\
\end{bmatrix}
\begin{bmatrix}
f_{11} \\
\vdots \\
f_{N\eta 1} \\
\end{bmatrix}
\begin{bmatrix}
\xi = \cos \frac{\pi}{N\eta-1} \\
\vdots \\
\xi = 1 \\
\end{bmatrix}
\begin{bmatrix}
f_{12} \\
\vdots \\
f_{N\eta 2} \\
\end{bmatrix}
\begin{bmatrix}
\xi = \cos \frac{\pi}{N\xi-1} \\
\vdots \\
\xi = 1 \\
\end{bmatrix}
\begin{bmatrix}
f_{1N\xi} \\
\vdots \\
f_{N\eta N\xi} \\
\end{bmatrix}
$$

(5.13)
Note the small distinction between the component \( f_{N_N} \) and the represented function \( f_{N_N \times N_N} \), which is first introduced in equation (5.11). The new column-vector is \( N_N \times N_N \) elements high and is from here on referred to as the stacked (column-)vector. This is equivalent to considering a tensor-product grid. Again the subscripting seems rather awkward. This time this has to do with the fact that the collocation points run from higher to lower values as implicitly stated in equation (5.6). This convention is precisely the same as that of Batterson [10]. As he emphasizes, it is very important to recall this structure when applying boundary conditions.

Consider the seemingly trivial case of equating two functions \( \xi_N f(\xi, \eta) \) and \( g(\xi, \eta) \), discretely. First the varying coefficient \( \xi_N \) must be discretized, this is done similarly as handled before. Through using a nodal basis, it is logical to satisfy the equation at the CGL collocation points. Using the stacked notation as in equation (5.13), this can be done rather efficiently. It yields the following equation:

\[
\begin{align*}
(\xi_N)_{N_N \times N_N} f_{N_N \times N_N} &= g_{N_N \times N_N} \\
\sum_{i=1}^{N_N} \sum_{j=1}^{N_N} \xi_N f_{ji}(\xi_i, \eta_j) &= \sum_{i=1}^{N_N} \sum_{j=1}^{N_N} g_{ji}(\xi_i, \eta_j) & \text{at the CGL points} \\
\xi_N f_{ji} &= g_{ji} & \text{for } i = 1, \ldots, N_N & \text{and } j = 1, \ldots, N_N
\end{align*}
\]

(5.14)

(5.15)

It is important to see that, due to the fact that \( \psi_{ji}(\xi_k, \eta_l) = 0 \) whenever \( (\xi_k, \eta_l) \neq (\xi_i, \eta_j) \), equation (5.14) can be evaluated non-trivially at each collocation point pair \((\xi_i, \eta_j)\). Because there are \( N_N \times N_N \) distinct collocation point pairs, this yields the \( N_N \times N_N \) distinct equations (5.15). These equations can be written in a matrix form:

\[
\begin{bmatrix}
(\xi_N)_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (\xi_N)_{N_N1} \\
\end{bmatrix}
\begin{bmatrix}
f_{11} \\
\vdots \\
f_{N_N1} \\
\end{bmatrix}
= 
\begin{bmatrix}
g_{11} \\
\vdots \\
g_{N_N1} \\
\end{bmatrix}
\]

(5.16)

Note that, for example, \( (\xi_N)_{N_N} = \xi_{N_N} \eta_{N_N} = 1 \), \( (\xi_N)_{1N_N} = \xi_1 \eta_{N_N} = -1 \), \( (\xi_N)_{N_N1} = \xi_{N_N} \eta_1 = -1 \), and \( (\xi_N)_{11} = \xi_1 \eta_1 = 1 \), by considering equation (5.6). In the matrix the current form is used to show what index combinations a general coefficient would have.

Due to the fact that \( \psi_{ji}(\xi_k, \eta_l) = 0 \) whenever \( (\xi_k, \eta_l) \neq (\xi_i, \eta_j) \), the matrix in equation (5.16) is diagonal. So when a variable in an equation has a coefficient that involves no derivative, this coefficient will be represented by a diagonal matrix. When derivatives are involved it becomes trickier, this will be handled next.
5.3 Spectral Derivatives

Partial derivatives of a function can be calculated by taking the corresponding derivative of equation (5.11), equipped with equations (5.12):

\[
\frac{\partial f_{N_\xi \times N_\eta}(\xi, \eta)}{\partial \xi} = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} f_{ji} \frac{d\lambda_i(\xi)}{d\xi} \mu_j(\eta), \quad \frac{\partial f_{N_\xi \times N_\eta}(\xi, \eta)}{\partial \eta} = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} f_{ji} \lambda_i(\xi) \frac{d\mu_j(\eta)}{d\eta}
\]

When regarding figure 5.1, it can be seen that \( \frac{d\lambda_i(\xi)}{d\xi} (\xi_k) \neq 0 \) for \( k \neq i \) (recall that \( \lambda_i(\xi_i) = 1 \)). Therefore, when writing out the equation \( \frac{\partial f(\xi,\eta)}{\partial \xi} = g(\xi,\eta) \) in a similar way as equation (5.16), the matrix that multiplies the vector of discrete values of \( f \) is not diagonal. The elements of that so-called pseudo-spectral differentiation matrix, \( D_{\xi,ji} \) in this case, can readily be calculated beforehand. For the one-dimensional case, these are equal to [10, 17, 66, 70, 88]:

\[
1D_\xi = \begin{cases}
1D_{\xi,11} = \frac{2(N_\xi-1)^2+1}{2(N_\xi-1)^2+1} \\
1D_{\xi,1i} = -\frac{6}{2(N_\xi-1)^2+1} \\
1D_{\xi,ii} = \frac{-\xi_i}{2(1-\xi_i^2)} & \text{for } i = 2, \ldots, N_\xi - 1 \\
1D_{\xi,ji} = \frac{\tau_i (-1)^{i+j}}{\xi_i - \xi_j} & \text{for } i \neq j, \ i, j = 2, \ldots, N_\xi - 1 
\end{cases}
\] (5.17)

Where \( \tau_i \) and \( \tau_j \) are defined as in equation (5.12). The elements of the differentiation matrix with respect to the other direction, \( 1D_{\eta,ji} \), can be calculated similarly. The elements of the second order differentiation matrices, \( 1D_{\xi\xi,ji} \) and \( 1D_{\eta\eta,ji} \), can be obtained by squaring the first order matrices. These coefficients of the aforementioned differentiation matrices can be efficiently calculated by the Matlab function chebdif [10, 88]. This function is originally created by Weideman & Reddy [91].

Usually, the considered physical domain does not correspond to the perfect square defined by the domain \( (\xi, \eta) \in [-1, 1] \times [-1, 1] \), from now on referred to as the computational domain. Therefore, most of the time the physical domain is projected onto the computational one, involving a transformation (or: mapping) \( \xi = \xi(x) \), or inversely: \( x = x(\xi) \). Here \( x \) is the (in this case non-dimensional) physical coordinate in the direction of \( \xi \). In section 5.4, the specific transformation used in the current treatment will be handled. With respect to the discretization, this transformation yields only a change in the pseudo-spectral differentiation matrices, which can be illustrated using the chain rule for the exact case:

\[
\frac{\partial f_{\text{exact}}(\xi(x), \eta)}{\partial x} = \frac{d\xi(x)}{dx} \frac{\partial f_{\text{exact}}(\xi, \eta)}{\partial \xi}
\] (5.18)

Here \( \frac{d\xi(x)}{dx} \) is known from the applied transformation; it is the factor that scales the derivatives. In the discrete case precisely the same applies. To get the physical differentiation matrix, the computational one must be multiplied with the appropriate factor:

\[
1D_x = 1T_x 1D_\xi
\]

For the one-dimensional case, these are equal to [10, 17, 66, 70, 88]:
Note that the discrete analogue of the factor $\frac{df(x)}{dx}$ is a diagonal matrix $1_{T_x} \in \mathbb{R}^{N_x \times N_x}$, which consists of the values of $\frac{df(x)}{dx}$ at the collocation points $x_i = x(\xi_i)$, see equation (5.6). The transformation of the $\eta$ coordinate is performed analogously. Note that for second order derivatives, the derivative of equation (5.18) must be taken once again. As a consequence of the product differentiation rule, it is possible that a combination of lower order derivatives is required to evaluate the higher order ones.

In the following, the treatment will make use of the computational derivatives only. The physical counterparts will be obtained by substituting $1_{T_x} \cdot 1_{D_\xi}$ for $1_{D_\xi}$. The same holds for the $\eta$-derivative; substituting $1_{T_y} \cdot 1_{D_\eta}$ for $1_{D_\eta}$ yields the physical equivalent. This will be proved when appropriate.

Hitherto, the differentiation matrices are derived only for a “one-dimensional” case. This can be illustrated in the following way. Note that $1_{D_\xi} \in \mathbb{R}^{N_x \times N_x}$. Consider one row of the matrix form of the two-dimensional $f \in \mathbb{C}^{N_x \times N_x}$, to the left in equation (5.13). I.e. the values of $f$ for constant $\eta$. One can post-multiply this row-vector with $1_{D_\xi}$ to yield the $\xi$-derivative of $f_{N_x \times N_x}$ at constant $\eta$. Similarly, one can take some column of $f$ and pre-multiply it with $1_{D_\eta} \in \mathbb{R}^{N_x \times N_x}$ to yield the $\eta$-derivative of $f_{N_x \times N_x}$ at constant $\xi$. This means that the partial derivatives of $f_{N_x \times N_x}$ can be represented by appropriately multiplying $f$ in matrix form:

$$\frac{\partial f_{N_x \times N_x}(\xi, \eta)}{\partial \xi} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} \left[ \begin{array}{c} f_{j_1, k_1} \\ \vdots \\ f_{j_N, k_N} \end{array} \right] \lambda_i(\xi) \mu_j(\eta), \quad (5.19)$$

$$\frac{\partial f_{N_x \times N_x}(\xi, \eta)}{\partial \eta} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} \left[ \begin{array}{c} f_{j_1, k_1} \\ \vdots \\ f_{j_N, k_N} \end{array} \right] \lambda_i(\xi) \mu_j(\eta), \quad (5.20)$$

Equations (5.19) and (5.20) are really handy to acquire the partial derivatives of the fields. The problem is that one wants to express the derivatives using not the matrix form of $f$, but the stacked column-vector form as presented to the right in equation (5.13). Later on it will be illustrated why. The two-dimensional pseudo-spectral differentiation matrices will be in $\mathbb{R}^{N_x N_x \times N_x N_x}$, just like the matrix in equation (5.16).

This kind of matrix can be efficiently generated by involving the so-called Kronecker product.$^3$ This product is denoted by $A \otimes B = C$, where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ and $C \in \mathbb{C}^{mp \times nq}$. Following Batterson [10], the product manifests itself by:

$$A \otimes B = \begin{bmatrix} A_{11} B & \cdots & A_{1n} B \\ \vdots & \ddots & \vdots \\ A_{m1} B & \cdots & A_{mn} B \end{bmatrix} \quad (5.21)$$

Thus each element of $A$ is multiplied by all elements of $B$. The Kronecker product satisfies the mixed-product property: $(D \otimes E)(F \otimes G) = DF \otimes EG$, when the conventional matrix products $DF$ and $EG$ exist, see Tracy & Singh [87].

$^3$This product is also referred to as the tensor product.
Consider the upperleft $N_\eta \times N_\eta$ part of the matrix in equation (5.16). Note that this part multiplies the column-vector at constant $\xi = 1$. Consider replacing the $N_\eta \times N_\eta$ part of the matrix with the one-dimensional pseudo-spectral differentiation matrix $1D_\eta$. Now, multiplication would yield the discrete $\eta$-derivative of $f_{N_\eta \times N_\xi}$ at $\xi = 1$ in $[g_{11} \cdots g_{N_\eta 1}]^T$, just as arrived upon above. Performing the same for the next $N_\eta \times N_\eta$ block along the diagonal, one will acquire the $\eta$-derivative of $f_{N_\eta \times N_\xi}$ at $\xi = \cos \frac{\pi}{N_\xi - 1}$ in $[g_{12} \cdots g_{N_\eta 2}]^T$.

By induction, the complete $\eta$-derivative at all $\xi$-locations will be stored in the stacked vector $g$ when continuing this process. The resulting matrix with square $N_\eta \times N_\eta$ blocks precisely equals the Kronecker product $I_{N_\xi} \otimes 1D_\eta$. Thus, the equation for the two-dimensional pseudo-spectral differentiation matrix in the $\eta$-direction yields:

$$D_\eta = I_{N_\xi} \otimes 1D_\eta \quad (5.22)$$

It can be imagined that this matrix consists of $N_\eta \times N_\eta$ “diamonds” around the diagonal. In a similar way, gathering all elements in $f$ at constant $\eta$, will yield the following expression for the two-dimensional differentiation matrix in the $\xi$-direction:

$$D_\xi = 1D_\xi \otimes I_{N_\eta} \quad (5.23)$$

This matrix covers the main diagonal and the diagonals in the off-diagonal $N_\eta \times N_\eta$ blocks with respect to the former matrix. Batterson [10] and Trefethen [88] illustrate the structure of the above matrices more clearly at page 99 and 69, respectively.

In a similar way as the derivative matrices, the two-dimensional transformation matrix $T^\xi_x = 1T^\xi_x \otimes I_{N_\eta}$. Through the mixed-product identity, it follows that:

$$D_x = T^\xi_x D_\xi = (1T^\xi_x \otimes I_{N_\eta})(1D_\xi \otimes I_{N_\eta})$$
$$= 1T^\xi_x 1D_\xi \otimes I^2_{N_\eta}$$
$$= 1T^\xi_x 1D_\xi \otimes I_{N_\eta} \quad (5.24)$$

This identity proves the above statement that substituting $1T^\xi_x 1D_\xi$ for $1D_\xi$ yields the corresponding physical result. Analogous results hold for the other derivatives.

Using the above identities, the second order and cross differentiation matrices can be derived:

$$D_{\xi\xi} = (1D_\xi)^2 \otimes I_{N_\eta}, \quad D_{\eta\eta} = I_{N_\xi} \otimes (1D_\eta)^2, \quad D_{\xi\eta} = (1D_\xi \otimes I_{N_\eta})(I_{N_\xi} \otimes 1D_\eta)$$
$$= 1D_\xi I_{N_\xi} \otimes I_{N_\eta} 1D_\eta$$
$$= 1D_\xi \otimes 1D_\eta \quad (5.25)$$

The Kronecker product can be computed with Matlab through the function kron. Note that the definitions of the pseudo-spectral differentiation matrices are dependent on the way that the stacked vector in equation (5.13) is obtained. One could also have linked all the rows head to tail and transpose the result, for example. In that case all Kronecker products in equations (5.22) to (5.25), should be reversed to obtain the correct results. One could compare Batterson [10] and Trefethen [88] to see the two cases independently, as noted before the current convention corresponds to that of Batterson.

On page 99 and 100 Batterson reports that:
“[...] the nature of the equation with its boundary conditions can require one definition [of the order of the Kronecker products] versus another. In some cases a numerical oscillation can be observed in the direction of the derivative defined by $I_N \otimes D_N$. This oscillation can range from minor to significant. For consistency and accuracy we attempt to use the derivatives as defined above whenever possible.”

Batterson refers to the derivative that directly corresponds to equation (5.22), the $\eta$-derivative matrix. In the current treatment these “oscillations” might manifest themselves by the numerical $P_0$ modes, which are defined later on.

This issue can only be investigated thoroughly by “switching” the convention from considering stacked column vectors to stacked row vectors and is therefore omitted in the current treatment. The $P_0$ modes will always be identified and omitted from the consideration.

### 5.4 Mapping proposed by Malik [59]

Now that the main discretization machinery is handled, the specific mapping used in the current treatment can be focused on. It was proposed by Malik [59] and Pinna [65] continued to make use of it.

The transformation is given for the wall-normal $y$-direction as follows:

$$y = \frac{y_i y_{\text{max}} (1 + \eta)}{y_{\text{max}} - \eta (y_{\text{max}} - 2y_i)}$$  \hspace{1cm} (5.26)

This transformation maps $\eta \in [-1, 1]$ into $y \in [0, y_{\text{max}}]$. Here, $y_{\text{max}}$ is the maximal $y$-coordinate of the considered physical domain. Half the number of collocation nodes lies in $[0, y_i]$, while the other half lies in $[y_i, y_{\text{max}}]$. This can be verified by substituting the values $\eta = -1, 0$ and 1 into the transformation (5.26). Malik himself used a slightly different version of the transformation that considered $\eta$ to be in $[0, 1]$. It must be noted that when $y_i$ is set equal to $\frac{y_{\text{max}}}{2}$, the linear transformation $y = \frac{y_{\text{max}}}{2} (1 + \eta)$ is received. This linear form is used by Batterson [10], for example.

As reported by Grosch & Orszag [33], one can use an algebraic transformation to map the upper grid points to infinity. In that way, they report that eigenmodes can be determined very accurately. The algebraic mapping they propose can be derived from equation (5.26) as well. By letting $y_{\text{max}} \to \infty$, one receives:

$$y = y_i \frac{1 + \eta}{1 - \eta}$$  \hspace{1cm} (5.27)

This is precisely the mapping Grosch & Orszag propose.

---

4They also consider an exponential transformation. However, the algebraic one is pointed out to give more accurate results.
The main advantage of the current transformation is that it can be used to map essentially half the collocations points into the boundary layer. Different definitions exist on the location of the edge of the boundary layer, therefore this statement can only be made in a qualitative sense. The general intention is to sufficiently resolve the modes in the neighbourhood of the boundary. This is done by appropriately setting the parameter $y_i$. Also being able to specify $y_{\text{max}} < \infty$ yields an advantage.

One can optimize the combination $(y_i, y_{\text{max}})$ such that a relatively small amount of collocation points is required for a very accurate solution; the linear transformation and transformation (5.27) can be seen as extremes at two ends of the overarching parameter space. In the current treatment, the parameter $y_i$ is fixed to the values found in the literature, which tends to resolve the boundary layer well. Typical values of $y_i$, using the Blasius length scale $l$ defined in section 7.1, are 3 to 10. Accordingly, the parameter $y_{\text{max}}$ can be made relatively large, without “wasting” a large number of collocation points on filling the uninteresting part of the domain that tends toward infinity. By experience, typical values for $y_{\text{max}}$ of $\mathcal{O}(200)$ yield results that are converged up to algorithm precision. See chapter 7 for the related investigation. Figure 5.3 illustrates the distribution of the collocation points for the combination $(y_i, y_{\text{max}}) = (10, 100)$.

From equation (5.26), the vertical equivalent of the derivative $\frac{d\xi}{dx}$ in equation (5.18) can be determined, so the transformation can be incorporated in the pseudo-spectral derivative matrices. Lastly, it must be noted that the horizontal direction is equipped with the transformation as well. However, the horizontal equivalent of $y_i$, the parameter $x_i$, is set equal to $x_{\text{max}}$ always, because in the current treatment there is no need to map collocation points a certain way in this eigenfunction direction.

At this point all the tools have been treated that are required for the discretization of the eigenvalue problem. The next sections will handle the set up of the problem and the incorporation of the boundary conditions.

5.5 Amplitude-Coefficient Matrices (ACM’s)

At this stage one is interested in the implementation of the previously handled tools leading to the complete discrete system. This is done by zooming in at the structure of the eigenvalue problem with an appropriate pace. This is done to provide a base to construct an optimized implementation tool to generate the discrete system automatically in its entirety. At the end of the section, the procedure for handling quadratic
eigenvalue problems will be elaborated on.

### 5.5.1 (Generalized) Eigenvalue Problem Level

The stability equations considered in this treatment all form a partial differential eigenvalue problem. This means that the equations can be written in the following top "eigenvalue problem level" form of a generalized non-linear eigenvalue problem:

\[ A \Xi = \lambda B \Xi + \lambda^2 B_2 \Xi \]

where \( \lambda \equiv \beta \) for spatial analysis and \( \lambda \equiv \omega \) for temporal analysis \((5.28)\)

Here, \( \lambda \) is a general eigenvalue defined to be equal to \( \omega \) when considering temporal analysis and equal to \( \beta \) in spatial analysis.\(^5\) \( A \) is the left hand side coefficient matrix, \( B \) is the right hand side coefficient matrix associated with a single eigenvalue \( \lambda \) and \( B_2 \) is the right hand side coefficient matrix being the coefficient of the eigenvalue squared. Note that for temporal analysis this matrix is identically zero, because in that case the eigenvalue appears only linearly in the equation. \( \Xi \) is the eigenvector of the system.

The eigenvector consists of four or five physical variables (in discrete form) in the incompressible or compressible case, respectively. For the incompressible case the conventional set is \( \{ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p} \} \). In the compressible case, in addition to the velocity amplitudes one can switch between the pairs \( (\tilde{p}, \tilde{T}) \), \( (\tilde{\rho}, \tilde{T}) \), or \( (\tilde{p}, \tilde{\rho}) \) through the equation of state. In this section, the combination \( \{ \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{T} \} \) is taken. However, it must be noted that the implementation tool allows a switch between the pairs as a simple set-up option.

The discrete amplitudes all have the stacked column-vector form having \( N_\xi N_\eta \) elements each, as introduced in equation (5.13). In turn these vectors are stacked on each other forming \( \Xi \), to yield:

\[
\Xi_{\text{incomp}} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{p} \end{bmatrix} \in \mathbb{R}^{4 N_\xi N_\eta} \quad \Xi_{\text{comp}} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{T} \\ \tilde{p} \end{bmatrix} \in \mathbb{R}^{5 N_\xi N_\eta} \quad (5.29)
\]

From this it can be seen already that the compressible case requires more power to solve, more variables are to be resolved.

The eigenvector of a matrix can be represented in this way only, this is the reason why the rather technical stacked column-vector form of the physical matrices and Kronecker product pseudo-spectral differentiation matrices (5.22) and (5.23) had to be introduced in the previous sections.

---

\(^5\) In this section only the streamwise BiGlobal theory is regarded, with a Cartesian spanwise coordinate direction. In the cases of the spanwise eigenvalue plane or a non-Cartesian spanwise coordinate, the symbol should strictly be an \( \alpha \) or \( q \), respectively.
5.5.2 Derivative Level

Now let’s pay more attention to the operator matrices. When going one layer deeper into equation (5.28), one could argue to arrive at the “derivative level”. The equation can then be decomposed as follows:

\[
\begin{align*}
(A_{\xi\xi} + A_{\xi\eta} + A_{\eta\eta} + A_{\xi} + A_{\eta} + A_c) \Xi &= \lambda (B_{\xi} + B_{\eta} + B_c) \Xi + \lambda^2 B_{2c} \Xi \\
= A &= \Xi = B
\end{align*}
\]  

(5.30)

Note that \( B_{2c} = B_2 \). The top-level \( A \) and \( B \) matrices are divided into sub-matrices that consist of common derivatives; the subscripts indicate the directions of these derivatives. The \( c \) indicates coefficients that do not take a derivative of (a part of) the eigenvector.

The following observations can be made:

- Because only two directions are considered in the BiGlobal context, only two derivative directions are considered. Together with the total absence of the third derivative, this yields the fact that only a single cross-derivative related matrix is present in \( A \).

- The \( B \) matrix either consists of the coefficients of the time derivative terms or single “homogeneous direction derivative terms” in the temporal or spatial analysis case, respectively. Furthermore, recall that the highest order of derivative in the Navier-Stokes equations is of second order. Therefore second order or cross-derivatives are not present in \( B \).

- In the same vein as for the last observation, the matrix \( B_2 \) consists of the coefficients of second order homogeneous direction derivative terms. Again, because the highest order derivatives in the governing equations are of second order, no derivative coefficients are present in \( B_2 \).

These observations were used to optimize the current implementation tool. This is specifically done by omitting to program the extraction of the non-existent derivative coefficients in the right hand side \( B \) and \( B_2 \) matrices. In turn, the tool is currently general enough to transcribe BiGlobal and LST based stability equations.

5.5.3 Equation-Variable Level

The next layer that can be considered is not necessarily a zoom in on the previous level. It is called the “equation-variable level.” It focuses on the amplitudes and equations in the system: the distinction between (non-) derivative contributions is no longer made. The matrices \( A \), \( B \) and \( B_2 \) are now divided into the sub-matrices that multiply the individual variables of \( \Xi \) for the continuity, 1-, 2-, 3-momentum and energy equations, separately. This means that the earlier presented machinery of sections 5.2 and 5.3 is applicable to precisely these sub-matrices.
The resulting decomposition is presented in equation (5.31), being equivalent to the representations reported by Robinet [69] and Theofilis & Colonius [86]:

$$
\begin{bmatrix}
L^u_1 & L^v_1 & L^w_1 & L^T_1 \\
L^u_2 & L^v_2 & L^w_2 & L^T_2 \\
L^u_3 & L^v_3 & L^w_3 & L^T_3 \\
L^u & L^v & L^w & L^T
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w} \\
\tilde{T}
\end{bmatrix}
= \lambda
\begin{bmatrix}
R^u_1 & R^v_1 & R^w_1 & R^T_1 \\
R^u_2 & R^v_2 & R^w_2 & R^T_2 \\
R^u_3 & R^v_3 & R^w_3 & R^T_3 \\
R^u & R^v & R^w & R^T
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w} \\
\tilde{T}
\end{bmatrix}
$$

(5.31)

The subscripts denote the equation and the superscripts the amplitude. Note that $2R^u_c = 2R^v_c = 2R^w_c = 2R^T_c = 0$, due to the fact that no second order derivatives are present in the continuity equation. The $\lambda^2$ related sub-matrices are present only due to second order homogeneous direction derivatives. The remaining zeros are handled in a moment. The blocks of equation (5.31) are from now on referred to as the \textit{amplitude coefficients}. They correspond to the discrete versions of the coefficients of the amplitudes in the stability equations and thus, as mentioned before, correspond to the matrix in equation (5.16). Each coefficient in equation (5.31) is therefore a matrix in $\mathbb{C}^{N_xN_y\times N_xN_y}$. This means that the matrices $A$, $B$ and $B_2$ are in $\mathbb{C}^{5N_xN_y\times 5N_xN_y}$.

However, when the incompressible case is considered, the energy equation decouples from the system and the temperature amplitude becomes superfluous to obtain the velocity field. Thus, for incompressible flow, one can set, or rather remove, the following coefficients:

\text{Incompressible flow :} \quad \begin{cases}
L^u_c = L^v_c = L^w_c = L^T_c = 0 \\
R^u_c = R^v_c = R^w_c = R^T_c = 0 \\
2R^u_c = 2R^v_c = 2R^w_c = 2R^T_c = 0 \quad (5.32)
\end{cases}

This means that only $4 \times 4$ amplitude coefficients are present in $A$, $B$ and $B_2$. Therefore, in the incompressible case these matrices are in the slightly more modest $\mathbb{C}^{4N_xN_y\times 4N_xN_y}$.

Furthermore, when considering several specific systems and making the distinction of considering streamwise or spanwise temporal or spatial analysis, one can zero several additional terms in the right hand side. This can be done, because the eigenvalue simply does not occur in several coefficients, purely due to non-existent derivatives in the Navier-Stokes equations. These coefficients are presented in table 5.1. Note that this table corresponds to Cartesian coordinate systems only; curvilinear coordinates can
introduce extra coefficients. Recall that for the temporal case no squared eigenvalues enter the problem. Therefore, the complete $B_2$ matrix is zero in addition to the terms presented in the second column of table 5.1.

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Temporal</th>
<th>Spatial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenfunction plane</td>
<td>Both ($\omega$)</td>
<td>Streamwise ($\beta$)</td>
</tr>
<tr>
<td>$R_u \hat{v}, R_c \hat{v}, R_w \hat{w}$</td>
<td>$R_u \hat{v}, R_c \hat{v}$</td>
<td>$R_u \hat{v}, R_c \hat{v}$</td>
</tr>
<tr>
<td>$R_1 \hat{v}, R_1 \hat{v}$</td>
<td>$2R_1 \hat{v}$</td>
<td>$2R_1 \hat{v}$</td>
</tr>
<tr>
<td>$R_2 \hat{v}, R_2 \hat{v}$</td>
<td>$2R_2 \hat{v}$</td>
<td>$2R_2 \hat{v}$</td>
</tr>
<tr>
<td>$R_3 \hat{v}, R_3 \hat{v}$</td>
<td>$2R_3 \hat{v}$</td>
<td>$2R_3 \hat{v}$</td>
</tr>
<tr>
<td>$R_1 \hat{v}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_2 \hat{v}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_3 \hat{v}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zero coefficients</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Because the zero coefficients in table 5.1 correspond to specific Cartesian coordinate systems only, not all seemingly ever-present ones are zeroed in equation (5.31). This is done to keep that equation as general as possible. Note in addition that one could argue that the energy equation should have several zero coefficients as well. This depends strongly on what formulation is considered. In the total formulation of the energy equation (nearly) all coefficients are non-zero, for example. However, when considering another formulation of the equation, several coefficients are zero most of the time.\(^6\)

One could zoom in even further on the amplitude coefficients to distinguish each individual term in the stability equations and arrive at the “term-level.” The stability equation in the appendix correspond to this level of detail. The implementation tool writes these individual terms to an .m-file. How this is done is elaborated in section 5.6.

### 5.5.4 Procedure for Quadratic Eigenvalue Problem

In this treatment mainly the temporal problem will be considered. I.e. $\lambda = \omega$ and $B_2 = 0$. This means that the linear generalized eigenvalue problem $A \Xi = \omega B \Xi$ has to be solved, which can be done with standard tools in Matlab. However, some of the parts of the treatment will be built on the spatial theory; some spatial LST simulations will be performed for the determination of parameters for the Gaster-type transformation in chapter 8. It has to be mentioned that this is done with the verified code of Pinna [65]. This kind of problem involves in the current consideration the quadratic generalized eigenvalue problem, which has the general form illustrated in equation (5.28).

This problem can be reduced to a linear eigenvalue problem, such that it can be solved with standard means again. This is done by augmenting the trivial equation set

\(^6\)This is one of the reasons why the total energy equation formulation is rather inefficient.
\( \lambda \Xi = \lambda \Xi \) to the system:

\[
\begin{align*}
- \lambda B \Xi + A \Xi &= \lambda^2 B_2 \Xi \\
\lambda \Xi &= \lambda \Xi
\end{align*}
\]

By orienting the system in this way, one can identify terms that multiply \( \Xi \) alone and terms that multiply \( \lambda \Xi \). By doing this in a smart way, the system can be factored to yield the following system:

\[
\begin{bmatrix}
-B & A \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\Xi
\end{bmatrix}
= \lambda
\begin{bmatrix}
B_2 & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\Xi
\end{bmatrix}
\]

(5.33)

Here, \( I \) and 0 are the identity and zero matrix that have the same dimensions as \( A, B, \) and \( B_2 \). By introducing the new eigenvector \( \tilde{\Xi} \equiv [\lambda \Xi, \Xi]^T \), the problem can be written as follows:

\[
\tilde{A} \tilde{\Xi} = \lambda \tilde{B} \tilde{\Xi}
\]

(5.34)

This system is linear in \( \lambda \), note that the non-linearity of the system is hidden in the eigenvector. This technique is also referred to as the companion matrix technique, see Danabasoglu & Biringen [21]. This system can be given to the standard tools again. Using this same technique, any polynomial order eigenvalue problem can be analysed. The only catch is the fact that the system doubles in size in the quadratic case. A single computation is performed in chapter 7 to just touch upon the corresponding BiGlobal results.

5.6 Automatic Equation Implementation Process

The first part of the implementation tool is programmed in Maxima and derives the stability equations corresponding to a number of adjustable options. Subsequently, it collects all terms to yield the amplitude coefficients, defined above. Then, it rewrites the variables in the terms (e.g. mean or amplitude derivative terms) to a Matlab compatible form and saves each coefficient to an individual .txt-file in string format.

Thereafter, an .m-file loads the strings from the .txt-files and rewrites these to corresponding Matlab code. Finally, this code is written to an .m-file that can be directly used for computations.

How the terms are rewritten can be best illustrated with an example. Assume, for example, that the following term has to be collected into the Cartesian streamwise
Discretization and Implementation 79

coefficient $\mathcal{L}_e^\tilde{u}$:

$$
\frac{2 Ec \tilde{u}_x \bar{\mu} V_x}{Re} \Rightarrow \left[ \begin{array}{ccc}
\frac{2 Ec}{Re} & \bar{\mu} V_x & \frac{\partial}{\partial x} \\
\text{Constants} & \text{Mean variables} & \text{Derivative} \\
\end{array} \right] \tilde{u}
$$

(5.35)

In general, the coefficients (5.35) can be broken up into a constant, mean and derivative part, when each exists. The variables in these three parts are rewritten to Matlab compatible names and saved to string format accordingly:

$$
\frac{2 Ec}{Re} \bar{\mu} V_x \frac{\partial}{\partial x}
$$

Matlab Compatible Names

$Dx \frac{2 Ec}{Re} \mu v x$

Save to String

$Dx*2*Ec/Re*mu*vx;$

This string is loaded by LoadnWriteCoefficient.m. The mean variables and constants are to appear on the diagonal of the ultimate amplitude-coefficient matrix. See equation (5.16). To this end, the string is rewritten as follows:

$$
Dx*2*Ec/Re*mu*vx;
$$

Substitute Dots

$Dx.*2.*Ec./Re.*mu.*vx;$

Write in diag()

$diag(Dx.*2.*Ec./Re.*mu.*vx);$

Note that it is assumed that the provided mean variables are in stacked column-vector form as illustrated in equation (5.13). The next step is to remove the derivative matrix from within $\text{diag}(..)$ and code the derivative appropriately. This is done as follows:

$$
\text{diag}(Dx.*2.*Ec./Re.*mu.*vx); \Rightarrow \text{diag}(2.*Ec./Re.*mu.*vx);
$$

Remove D.-terms

$\text{diag}(2.*Ec./Re.*mu.*vx)*\text{kron}(D1x,I2);$  \hspace{1cm} (5.36)

where

$$
\begin{align*}
I1 &= \text{eye}(N1); \\
I2 &= \text{eye}(N2);
\end{align*}
$$

Here, the expression $\text{kron}(D1x,I2)$ corresponds directly to equation (5.24); the differentiation matrix transformed to the physical domain. Note that $N1 = N_\xi$ and $N2 = N_\eta$.

In the case of a zero coefficient, it is rewritten to:

$$
0; \Rightarrow \text{zer}; \quad \text{where} \quad \text{zer} = \text{zeros}(N1*N2,N1*N2); \quad (5.37)
$$

---

5 When more terms are present this poses no problem, because all coefficients are collected for common amplitude derivatives instead of amplitudes alone. This means that coefficient of $\tilde{u}$ is distinct from that of $\tilde{u}_x$. The coefficients are therefore actually collected corresponding to a mix of the the derivative and equation-variable level presented earlier on.

8 The location of the derivative matrix $Dx$ is changed because of the internal ordering of Maxima.
The code presented in equation (5.36) is appropriately written to the .m-file BuildAandB.m. Of course, a modest for- and if-clause structure is built to perform this operation for every amplitude-coefficient using a partition for the amplitude derivatives.

Note that in the implementation tool no distinction is made between the \((p, T, \rho)\) pairs. It will generate an extra \(\rho\) related amplitude coefficient column in \(A, B\) and \(B_2\); this is done for flexibility. Because the equation of state is applied during the derivation of the stability equations, the implementation tool will thus generate matrices of which one complete amplitude coefficient column contains zeros only. In the code following the generation of these matrices, these zero columns are removed to yield the \(5N_\xi N_\eta \times 5N_\xi N_\eta\) or \(4N_\xi N_\eta \times 4N_\xi N_\eta\) system.

### 5.7 Boundary Conditions

Now the system of stability equations is automatically implemented by constructing the ACM's, the next step is to consider the boundary conditions. These conditions complete the eigenvalue problem.

For the tool to be as general as possible, a high number of different kinds of boundary conditions is implemented. One can set Dirichlet, Neumann, Robin (all homogeneous) and periodic boundary conditions for any variable at any boundary. Furthermore, one can specify compatibility conditions for the pressure and density (via the momentum equations), also at any boundary.

With combinations of these boundary conditions all set-ups found in the literature can be replicated when appropriate mean data sets are available. Moreover, the influence of different kinds of boundary conditions can be studied by applying different conditions to similar mean flows. Later on, this will be treated in more detail.

Boundary conditions replace the function of the stability equations at the boundaries. They “rule” over the boundaries, whereas the stability equations rule over the interior of the considered domain. In this way, the boundary conditions can be interpreted quite effectively as, rather trivial, equations. These equations are quite similarly implemented as the stability equations, by specifying the coefficients of the amplitudes at the correct locations within the ACM’s \(A, B\) and \(B_2\).

Recall equations (5.13) and (5.16). They illustrate in which way each row in the coefficient matrix corresponds to an equation evaluated at a specific location within the domain. To emphasize which rows in the amplitude coefficients correspond to the boundary, the element of the diagonal of the matrix in equation (5.16) are coloured appropriately in figure 5.4 for a \(5 \times 5\) grid. In this figure, the black dots indicate the rows corresponding to the interior locations in the domain, whereas the coloured dots correspond to the indicated boundaries. Doubly coloured dots correspond to corners.

These rows are to be overwritten with the coefficients of the boundary or compatibility conditions. Note that figure 5.4 corresponds to one variable only; only one amplitude coefficient is considered. In the full ACM’s, multiple of these blocks are augmented as shown in equation (5.31). The convention is used to equip the first row of blocks with the boundary condition on \(\tilde{u}\), the second with that on \(\tilde{v}\), etc.
5.7.1 Dirichlet conditions

Dirichlet boundary conditions involve the specification of the value of an amplitude at the boundary. In the homogeneous case considered here, this can be illustrated as follows. Take, for example, the velocity amplitude $\tilde{u}$. The Dirichlet boundary condition at $\xi = -1$ could be written as:

$$
\tilde{u}_{\text{exact}}(\xi = -1, \eta) = 0
$$

Discretize

$$
\tilde{u}_{N_\eta \times N_\xi}(\xi = -1, \eta) = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \tilde{u}_{ji} \psi_{ji}(\xi = -1, \eta) = 0
$$

$$
\sum_{j=1}^{N_\eta} \tilde{u}_{jN_\xi} \psi_{jN_\xi}(\xi = -1, \eta) = 0
$$

$$
\frac{1}{\text{coefficient for Dirichlet condition}} \cdot \tilde{u}_{jN_\xi} = 0 \quad \text{for } j = 1, \ldots, N_\eta
$$

(5.38)

---

9Here the same conventions are used as introduced in section 5.2.
It is clear that the coefficients in the amplitude coefficients have to be set equal to 1 to yield the Dirichlet boundary condition. In figure 5.4, it can be seen that the rows corresponding to the elements \( \tilde{u}_{jN_\xi} \) are indicated by the light blue dots. In this specific case, the blue dots are precisely indicating the locations corresponding to the coefficients, the ones. From equation (5.38), it follows that all other elements in the rows must be zero, also those corresponding to other variables. With the previously mentioned convention, this implies that the corresponding rows in the off-diagonal amplitude coefficient blocks are zero when this condition is applied. Because the condition does not involve the eigenvalues, only zeros will be introduced in the \( B \) and \( B_2 \) matrices.

Because the Dirichlet conditions are simply specifying zero amplitudes at the boundaries, these values do not have any influence on the remainder of the problem. For that reason, the row and column corresponding to the substituted 1 can be and are entirely omitted; they are removed from the ACM’s, thereby reducing their size slightly.

### 5.7.2 Neumann conditions

The Neumann boundary condition involves the specification of the normal derivative of an amplitude with respect to the boundary. This is again illustrated using an example. The Neumann boundary condition at \( \xi = -1 \) is written as:

\[
\left. \frac{\partial \tilde{u}}{\partial \xi} \right|_{\text{exact}} (\xi = -1, \eta) = 0
\]

Discretize \( \Rightarrow \)

\[
\frac{\partial \tilde{u}_{N_\eta \times N_\xi}}{\partial \xi}(\xi = -1, \eta) = \sum_{j=1}^{N_\eta} \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1D_{\xi,ki} \psi_{ji}(\xi = -1, \eta) = 0
\]

\[
\sum_{j=1}^{N_\eta} \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1D_{\xi,kN_\xi} \psi_{jN_\xi}(\xi = -1, \eta) = 0 \quad \text{for } \eta = \eta_j
\]

\[
\sum_{k=1}^{N_\xi} 1D_{\xi,kN_\xi} \cdot \tilde{u}_{jk} = 0 \quad \text{for } j = 1, \ldots, N_\eta \tag{5.39}
\]

In equation (5.39), it can be seen that the \( \tilde{u}_{jk} \) are summed over the index in the \( \xi \)-direction. This means that for a fixed \( \eta \)-position (\( j \) index) in the grid, all values in the \( \xi \)-direction are equipped with the coefficients \( 1D_{\xi,kN_\xi} \), for \( k \in 1, \ldots, N_\xi \). Note that the pseudo-spectral differentiation matrix \( 1D_{\xi} \) is used in its one dimensional form as introduced in equation (5.17), in this way the notation is argued to be most convenient. This will be continued for the remainder of this section.

In figures 5.5 and 5.6, it is illustrated what form a (diagonal\(^{10}\)) amplitude coefficient attains when a Neumann boundary condition is applied to the boundaries \( \xi = -1 \) and \( \xi = 1 \).

\(^{10}\) The off-diagonal amplitude coefficients are all zero again due to the fact that no other variables are involved.
\[ \eta = -1, \text{ respectively. Again, the } 5 \times 5 \text{ grid is considered. Of course, the coefficients corresponding to the } \eta = -1 \text{ boundary are those of the } \eta \text{-derivative, } 1D_{\eta,N,k}, \text{ see equation } (5.20). \]

When comparing figures 5.5 and 5.6, it can be seen that row 25 is occupied in both cases. This row corresponds to the corner \((\xi, \eta) = (-1, -1)\). Therefore, in general this means that at the corners, only one boundary condition can be applied. The one that is firstly implemented is simply overwritten by the second. Only in the case of two Dirichlet conditions this can be justified; in that case this means that the amplitude is single valued at the corner. For all other cases one strictly has to make a choice. In the current implementation, the no-slip boundary conditions are given preference always. This is done because it is argued that these conditions are most critical with respect to physical significance. Fortunately, due to the inherent homogeneous nature of the boundary conditions considered in this treatment, they automatically imply all the conditions with respect to the other boundary.

### 5.7.3 Robin conditions

The implementation of this boundary condition, incorporating the Gaster-type relation (see section 2.3.3) is again illustrated using an example, the \(\xi = -1\) boundary. For convenience, all explicit indications of the evaluations at \(\omega_{s,0}\) and the distinction between the temporal and spatial variables are dropped. Evaluating equation (2.33) at the \(\xi = -1\) boundary yields:

\[
c \frac{\partial \tilde{u}}{\partial \xi} \bigg|_{\text{exact}} (\xi = -1, \eta) - i (c \alpha_r - \omega_0) \tilde{u}_{\text{exact}} (\xi = -1, \eta) = i \omega \tilde{u}_{\text{exact}} (\xi = -1, \eta)
\]
In discrete form, this equation reads:

$$c \sum_{k=1}^{N_\xi} 1 D_{\xi,k,N_\xi} \tilde{u}_{jk} - i (c\alpha_r - \omega_0) \tilde{u}_{jN_\xi} = i\omega \tilde{u}_{jN_\xi} \quad \text{for } j = 1, \ldots, N_\eta$$  \hfill (5.40)

To arrive at equation (5.40), the implementation expressions for the Dirichlet and Neumann conditions were used. It can be seen that through the involvement of the summation of $\tilde{u}_{jk}$ over all possible $k$, the structure of the Robin boundary condition is very similar to that of the Neumann condition as shown in figures 5.5 and 5.6. The primary difference is the inclusion of non-zero elements in the diagonal of right hand side of the eigenvalue problem.

In the case of the investigation of two-dimensional perturbations ($\beta = 0$), equation (2.35) might be used instead of equation (2.33) to minimize the required storage capacity. It is deemed to be sufficient to directly show the equivalent of equation (5.40):

$$c \sum_{k=1}^{N_\xi} 1 D_{\xi,k,N_\xi} \tilde{u}_{jk} + (c\alpha_r^2 - 2\alpha_r \omega_0) \tilde{u}_{jN_\xi} = -2\alpha_r \omega \tilde{u}_{jN_\xi} \quad \text{for } j = 1, \ldots, N_\eta$$  \hfill (5.41)

This equation clearly involves real coefficients only.

It must be noted that the equations are implemented slightly differently than reported here; they are first divided by $c_n |\omega_s,0|$ to avoid factors multiplying the derivative terms, which introduces (only minor) unnecessary complications in the current implementation.

Lastly, it has to be emphasized that one does not have to equip the Robin boundary conditions with the Gaster-type transformation. They can as well be used to simulated incoming perturbations at a fixed (complex) wavenumber. This method might be used to impose forcing perturbations on a given flow field and hence yield a good tool for receptivity analysis.

### 5.7.4 Periodic conditions

In the case of periodic boundary conditions, one equates the values and first normal derivatives of the amplitudes at contrary boundaries. Again, this is best illustrated with an example. The values across the $\xi = \pm 1$ boundaries are equated as follows:

$$\tilde{u}_{\text{exact}}(\xi = -1, \eta) = \tilde{u}_{\text{exact}}(\xi = 1, \eta)$$

In discrete form, this equation reads:

$$\sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \tilde{u}_{ji} \psi_{ji}(\xi = -1, \eta) = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \tilde{u}_{ji} \psi_{ji}(\xi = 1, \eta)$$

$$\sum_{j=1}^{N_\eta} \tilde{u}_{jN_\xi} \psi_{jN_\xi}(\xi = -1, \eta) = \sum_{j=1}^{N_\eta} \tilde{u}_{j1} \psi_{j1}(\xi = 1, \eta)$$

$$\tilde{u}_{jN_\xi} - \tilde{u}_{j1} = 0 \quad \text{for } j = 1, \ldots, N_\eta$$  \hfill (5.42)
Now the values are equated, the normal derivatives have to be equated as well. This is illustrated for the case across the $\xi = \pm 1$ boundaries as follows:

$$\frac{\partial \tilde{u}}{\partial \xi}_{\text{exact}}(\xi = -1, \eta) = \frac{\partial \tilde{u}}{\partial \xi}_{\text{exact}}(\xi = 1, \eta)$$

In discrete form, this equation reads:

$$N_\xi \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \left[ \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1 D_{\xi,ki} \right] \psi_{ji}(\xi = -1, \eta) = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \left[ \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1 D_{\xi,ki} \right] \psi_{ji}(\xi = 1, \eta)$$

$$N_\eta \sum_{j=1}^{N_\eta} \left[ \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1 D_{\xi,kN_\xi} \right] \psi_{ji}(\xi = -1, \eta) = \sum_{j=1}^{N_\eta} \left[ \sum_{k=1}^{N_\xi} \tilde{u}_{jk} 1 D_{\xi,k1} \right] \psi_{jj}(\xi = 1, \eta)$$

$$\sum_{k=1}^{N_\xi} 1 D_{\xi,kN_\xi} \tilde{u}_{jk} - \sum_{k=1}^{N_\xi} 1 D_{\xi,k1} \tilde{u}_{jk} = 0 \quad \text{for } j = 1, \ldots, N_\eta$$

$$(1 D_{\xi,kN_\xi} - 1 D_{\xi,k1}) \tilde{u}_{jk} = 0 \quad \text{for } j = 1, \ldots, N_\eta$$

From equation (5.43), it can be seen that the form of the periodicity in the derivatives results in a similar form of the amplitude coefficient as the Neumann condition in equation (5.39). Together, the relation of the values and derivatives yields the forms as presented in figures 5.7 and 5.8. The latter shows the case across the $\eta = \pm 1$ boundaries.

Figure 5.7: Indication of the periodic boundary condition across $\xi = \pm 1$.

Figure 5.8: Indication of the periodic boundary condition at $\eta = \pm 1$.

Due to the fact that the periodicity conditions involve 2 equations, a choice has to be made in which rows to put them. Because the conditions are mutual, this choice does
not have any influence on the result. In this treatment, it is chosen to relate the values at the rows corresponding to the $\xi = 1$ or $\eta = 1$ boundaries and the derivatives at the rows corresponding to the $\xi = -1$ or $\eta = -1$ boundaries. This is reflected in figures 5.7 and 5.8.

### 5.7.5 Compatibility conditions

In section 2.3.1, the theoretical background and the derivation procedure of the compatibility equations is elaborated on. It is important to recall that these conditions are derived from the momentum equations by evaluating the latter at the boundaries. In doing so, Dirichlet conditions on the velocities, and potentially temperature, have to be incorporated in these equations. The resulting equations can be implemented quite similarly as the stability equations. Therefore this aspect is not further elaborated on. Several statements have to be made, though.

Note that from the first generation of the ACM’s, the momentum equations are implemented in the interior as well as on the boundary of the domain. The application of homogeneous Dirichlet conditions at the boundaries automatically zeros the amplitudes $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ and their tangential derivatives along the boundary. This means that when the Dirichlet conditions have been applied to yield the no-slip boundary condition and, in doing so, the momentum equations on the boundary have not been replaced, the latter are automatically converted into the compatibility equation form. Thus, actually the compatibility equations would not have to be derived and inserted into the system separately. Just applying the Dirichlet conditions implies the formation of the compatibility conditions within the system directly.

Despite of this, the compatibility conditions are derived and implemented individually anyway. This is done because of the earlier mentioned convention on the location of the implementation of the boundary conditions. This convention causes the momentum equations at the boundaries to be overwritten; the velocity boundary conditions are implemented in the amplitude coefficients related to the momentum equations. In the compressible $(p,T)$-formulation, the Dirichlet or Neumann boundary condition on the temperature is implemented alongside the continuity equation and the compatibility condition on the pressure alongside the energy equation. Effectively, this means in this case that the compatibility conditions are removed alongside the momentum equations and substituted back into the system alongside the energy equation. In the incompressible case, they are substituted alongside the continuity equation. This is done only with the argument of straightforward implementation. Later on, this could be optimized, because this unnecessary operation requires more time to execute, logically.

Another reason why this method is unaltered, is due to the fact that it illustrates the power of the combination of the derivation and implementation tool. The compatibility conditions are implemented completely analogously as the stability equations themselves. Performing operations on the gigantic stability equations can be done very easily and implementing the results even more so.
5.8 Automatic Boundary Condition Implementation Process

As an extension of the automatic generation of the ACM’s the boundary conditions are logically implemented automatically as well. In section 5.7, the discretized versions of the conditions are handled all. Here, the implementation is shortly elaborated on so to reveal the method used.

First of all, the conditions are all implemented in a certain order. Firstly, the stability equations are implemented for the complete domain, so including the boundaries.\textsuperscript{11} Thereafter, the Dirichlet, Neumann and periodic boundary conditions are applied, without removing any of the rows or columns of the ACM’s.

This is done by generating the appropriate coefficient blocks corresponding to the equations derived in section 5.7. Recall the convention of the locations of the boundary conditions in the ACM’s: the conditions on the velocities are implemented alongside the corresponding momentum equations and the remaining ones alongside the continuity and, if applicable, energy equations.

In figures 5.9 and 5.10, the non-zero elements of the matrix $A$ from equation (5.28) are shown for a spanwise Cartesian compressible case using the $(p, T)$-formulation (the influence of $\rho$ is eliminated through the application of the equation of state) for a $5 \times 5$ discretization. In figure 5.9, the clean version is shown. This is the result from the implementation of the stability equations. In figure 5.10, the final form of the matrix is shown incorporating Neumann conditions on all variables at all boundaries; the red dots show all the associated coefficients. Note that they correspond to the dots shown in figures 5.5 and 5.6, although the conditions in the corners are overwritten by the ones applied to the $\eta = c_{st}$ boundaries. The Neumann conditions are applied to most clearly indicate the rows that are influenced by the boundary conditions.

In figure 5.10 it is shown which combination of rows corresponds to the boundary condition on which variable. For the order of the equations, the convention illustrated in equation (5.31) is used. From top to bottom first the 1-, 2- and 3-momentum equations are encountered followed by the continuity and energy equations. This causes the nearly symmetrical structure of the matrix.

The next step that is performed is the implementation of the compatibility equations. These equations are implemented quite similarly as the stability equations. This is done after they have been derived as illustrated in subsection 5.7.5 by the automatic derivation tool. Note that this implies that these equations have been first implemented separately for the complete domain. Thereafter, precisely those equations evaluated at the appropriate boundary locations are substituted at the boundary locations in the ACM’s.

The equivalent of figure 5.10, after compatibility conditions are applied to the pressure at all boundaries, is illustrated in figure 5.11. It must be mentioned that the

\textsuperscript{11}Recall the fact that the momentum equations on the boundaries are precisely the compatibility conditions for the pressure when the no-slip boundary conditions are appropriately applied in this system; i.e. when these momentum equations are not overwritten.
numbers at the vertical axis indicates the row indices, it was not intended to let the indices of the left and right matrices match. To the left, the structure of the z- and y-compatibility equations is shown (recall that the system was spanwise), the z-equation is the uppermost one. Note the structural identity when comparing them with the z- and y-momentum equations in figure 5.9. The red elements denote the $z = c_{st}$ boundary locations in the z-compatibility equation and the $y = c_{st}$ boundary locations in the y-compatibility equation.

The arrows indicate how (parts of) these equations are substituted in the ACM $A$. When the eigenvalue is present in one of these equations similar substitutions are performed for the $B$ matrix. It can be seen that the no-slip related $y$-compatibility condition overwrites the condition in the $z$-direction at the corners. This is related to the fact that the $y$-direction is usually taken as the wall-normal direction.

After this is done only the rows and columns corresponding to the (homogeneous) Dirichlet conditions have to be removed. Due to the fact that no distinction has to be made between the thermodynamic variables any longer (via the compatibility equations) or compressibility the superfluous parts of the ACM’s can be removed. Before this operation, the following standard convention for the eigenvector is used:

$$
\Xi_{\text{standard convention}} = \begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w} \\
\tilde{\rho} \\
\tilde{T} \\
\tilde{\rho}
\end{bmatrix} \in \mathbb{C}^{6N_xN_y} \quad (5.44)
$$
In figure 5.9, this convention is clearly visible through the fact that the columns corresponding to \( \tilde{\rho} \) are completely zero. After the compatibility conditions are implemented, these zero columns are removed. In an incompressible case, the energy equation and the temperature variable would be removed in addition. This in all cases yields a square matrix, as shown in figure 5.10, for example. After this operation is executed, the rows and columns corresponding to the homogeneous Dirichlet boundary conditions are removed.

Thereafter, the eigeninformation is extracted from the system and the spectrum is plotted. When homogeneous boundary conditions are present in the physical problem, the eigenvector will be smaller than \( 4N_\xi N_\eta \times 1 \) or \( 5N_\xi N_\eta \times 1 \), for the incompressible or compressible case, respectively. The last boundary layer related code therefore involves the “re-substitution” of the Dirichlet conditions in the eigenvector. The complete discretized eigenfunctions can thus be extracted from the resulting vector and are ready for post-processing. Note that the eigenvector contains the nodal values of the eigenfunctions, \( f_{ji} \) in equation (5.11). The most complete representation of the functions involves the evaluation of equation (5.11) using the characteristic Lagrange polynomials to obtain \( f_{N_\eta \times N_\xi} \).

This concludes the treatment of the discretization and implementation. In the next chapter, the essential features of the used eigensolvers are considered. The main focus will be on the to be expected accuracy.
Chapter 6

Eigensolver Essentials

To be able to thoroughly verify the performance of the currently developed and used tools, several things have to be considered. The ultimate accuracy of the stability calculations has to be based on a chain of several potential sources of error. First of all, the primary solver used to handle the problems is the standard function \texttt{eig} in Matlab, which performs the QZ algorithm to retrieve the complete spectrum. The accuracy characteristics of this function are handled in section 6.1.

All stability calculations are based on a certain mean flow, which is often only represented discretely. Therefore, the requirement on the mean flow is elaborated on in section 6.2. Thereafter, so called numerical modes are concisely handled in relation to the (resulting indications of the) Bauer-Fike theorem in section 6.3. If extra sources of error are introduced via alternative routes,\footnote{Examples of this are the test cases in which the Robin boundary conditions equipped with the Gaster-type relation are incorporated.} this will be assessed in the result chapter accompanying the specific test case.

Because in many cases not the complete spectrum is required while the storage requirements are quite demanding, the Arnoldi algorithm is an appropriate substitute for the QZ algorithm. The former algorithm can be executed through Matlab’s function \texttt{eigs}. The essential features of this algorithm are handled in section 6.4. In certain cases, some of the eigenfunctions are small in absolute sense. In this case, both the QZ and Arnoldi algorithm have problems resolving these functions. In section 6.5, the approach to this issue is elaborated on.

6.1 The QZ Algorithm: Accuracy of Matlab’s \texttt{eig}

Due to the fact that the ACM $A$ is generally complex non-Hermitian and $B$ complex and singular, Matlab’s \texttt{eig} applies the \textit{QZ algorithm} (see section 7.7 of Golub & Van Loan [25]) to obtain the spectrum and the corresponding invariant subspace. To this end, the function calls LAPACK routines [3]. All these routines have a common accuracy statement; they return the solution to a problem that lies close to the to-be-solved
problem. The results are hence accurate up to the following criterion:

\[ \varepsilon_{\text{crit}} = \max (\varepsilon \|A\|_F, \varepsilon \|B\|_F) \]  
\hspace{1cm} (6.1)

Here, \( \| \cdot \|_F \) denotes the Frobenius norm and \( \varepsilon \) is the machine precision \([3, 25]\). In Matlab, the latter number conventionally equals \( 2.2204 \cdot 10^{-16} \). This criterion, here also referred to as the algorithm precision, is very important, as in the remainder of the complete thesis this is a fixed threshold that determines whether the stability calculations are converged. Irrespective of the accuracy of other calculations, one cannot calculate a more accurate solution than one at algorithm precision.

Moreover, the eigenvectors the solver produces are also “to-be-trusted” up to this accuracy. So, when fluctuations or errors of the order \( \mathcal{O}(\varepsilon_{\text{crit}}) \) occur, they can be judged to be the consequence of the finite precision of the eigensolver.

When the results are generated, it is important to know whether the modes are converging. When the threshold (6.1) is reached and passed, the eigensolver is no longer capable of more accurately determining the solution. In those cases, the solution is declared to be “converged to algorithm precision.”

One or several quantities have to be chosen to measure whether the calculation is converged. In this treatment these quantities are chosen to be both the imaginary and real part of the eigenvalue. To be able to check whether they are converging, several computations are done with increasing numbers of the collocation points. For each calculation, the absolute difference between the current and previously calculated eigenvalue is determined. Thus:

\begin{align*}
\text{LST:} & \quad \omega|_{N_{\eta}} - \omega|_{N_{\eta}-N_{\#}} \\
\text{BiG:} & \quad \omega|_{N_{\eta},N_{\zeta}} - \omega|_{N_{\eta}-N_{\#,N_{\zeta}-N_{\dagger}}} 
\end{align*}  
\hspace{1cm} (6.2)

Where \( N_{\#} \) and \( N_{\dagger} \) are, respectively, the increments in the numbers of collocation points in the \( \eta \)- and \( \zeta \)-direction. The latter direction is sometimes exchanged with the \( \xi \)-direction, when appropriate. The real and imaginary parts of this absolute error can be compared to \( \varepsilon_{\text{crit}} \) magnificently well, because the algorithm precision also refers to the absolute error in the eigeninformation.

In figures 6.1 and 6.2, comparisons of the imaginary and real parts of the aforementioned errors with \( \varepsilon_{\text{crit}} \) are shown versus \( N_{\eta} \), for an incompressible and a supersonic LST simulation, respectively. In these figures \( N_{\#} \) is chosen to be equal to 5 to have sufficient resolution. With the exception of some oscillations, it can be seen that the eigenvalues neatly converge initially. At a certain \( N_{\eta} \), the real and imaginary parts of the errors both pass the threshold \( \varepsilon_{\text{crit}} \). As mentioned earlier, this is the \( N_{\eta} \) for which the solution is declared to be converged to algorithm precision and is argued to be the most accurate possible.

Note that in both the cases of figures 6.1 and 6.2, the eigenvalues tend to converge a little further after they have passed the algorithm precision. Thereafter, the influence of the errors induced by the solver become very apparent. One might argue that the most accurate solution might be amongst those with the globally minimal errors. However,
in the worst case, those solutions are still corrupted with an error of $O(\varepsilon_{ \text{crit} })$. Moreover, the threshold increases for a larger number of collocation points. Therefore, it is decided to use the solution for which the errors lie just beyond the threshold.

Because of the appearance of the Frobenius norm in equation (6.1), the algorithm precision will increase when an LST simulation is mimicked by a BiGlobal relative. In those simulations multiple (instead of single) $\zeta$-locations are resolved. It is interesting to note, that when using the familiar 2-norm in these particular simulations, this precision would remain practically equal. Furthermore, recall that via the basic inequality $||A||_2 \leq ||A||_F$, the Frobenius norm causes this precision to be definitely more critical than the 2-norm. Although all norms are equivalent, the Frobenius norm is kept to remain consistent with the LAPACK user manual [3] and the corresponding Numerical Algorithms Group (NAG) Fortran Library Routine Documents (e.g. F08WAF and F08WNF).

To obtain figures 6.1 and 6.2 many time consuming simulations have to be performed, certainly in the case of BiGlobal simulations. Luckily, the algorithm precision tends to increase only slightly when $N_\zeta$ increases moderately; the highest resolved direction most dominantly increases the norm of the matrices. Furthermore, the errors tend to behave similarly as for the LST simulations. Therefore, at least in the case of the BiGlobal simulations that mimic the LST ones, a convergence region is pinned down in each of the aforementioned figures. Those values of $N_\eta$ will be searched so to obtain converged solutions in the BiGlobal framework without having to perform a large number of calculations. Lastly, in this respect, note that the supersonic simulations require a larger $N_\eta$ with respect to the incompressible case. Compressible simulations are notorious for this requirement and this behaviour was expected accordingly.
6.2 Accuracy of the Mean Flow

As mentioned at the beginning of this section, all stability calculations are based on a mean flow dataset. When errors are present in these data sets, one basically solves the incorrect problem even before considering the stability equations. Therefore, it is very important to have an accurate mean flow representation, if practically feasible, converged to machine precision completely.

For some flows, this is a very strong requirement. Indeed, for the compressible mean flows an extremely fine mesh has to be used to obtain such a dataset. Fortunately, in view of section 6.1, a slight alleviation is possible to this requirement. The eigeninformation is accurate up to the threshold $\varepsilon_{\text{crit}}$. Therefore it is argued that the mean data must be accurate up to this threshold as well. Moreover, this holds only for $\varepsilon_{\text{crit}}$ evaluated at the $(N_\eta, N_\zeta)$-combination for which the solution was declared to be converged to algorithm precision. To be absolutely sure that no noticeable mean flow errors are present, the mean datasets are converged a sufficient pace beyond the threshold.

The accuracy of the mean flows is represented by the value $\varepsilon_{\text{mean}}$. The calculation of this value is very case specific. It mainly depends on the way the flow is represented, i.e. on a numerical grid or with a closed form function of the coordinates. In the case of the latter, the values of the variables can be calculated at the to-be-used Chebyshev grid directly to machine precision. In those cases, $\varepsilon_{\text{mean}}$ therefore automatically equals the machine precision.

In the cases corresponding to the Blasius boundary layer, the flow field is represented on a numerical grid. As reported in chapter 7, the primary calculation of $\varepsilon_{\text{mean}}$ has to be based on the grid convergence. However, the flow field has to be projected on the Chebyshev grid via an interpolation routine, yielding additional errors. To handle this in an integral way, the following procedure is followed.

First of all, two sets of Blasius profiles are calculated with $N + 1$ and $N$ points. The solution on the grid with $N + 1$ points is thereafter interpolated on the one with $N$ points using spline interpolation. In this way the absolute errors between the solutions on the different grids can be calculated. Note that the interpolation method introduces an additional error in this method. Ultimately, the mean data is projected on the Chebyshev grid with the same interpolation method. Therefore, it is argued that when the absolute errors are calculated in the aforementioned way, the interpolation error is taken into account. When the datasets are converged, even under the influence of errors introduced by the interpolation mechanism, they must be fully trustworthy.

In figures 6.1 and 6.2 the horizontal magenta lines indicate the values of $\varepsilon_{\text{mean}}$ for the different mean flows. The values are argued to be small enough to negligibly influence the solutions that are converged to algorithm precision with respect to the errors induced by the eigensolver.

In the case of a numerically represented mean flow, it is suggested to follow the following roadmap to perform the stability calculation sufficiently accurately:

1. Start by performing a low resolution mean flow calculation, where $\varepsilon_{\text{mean}}$ is relatively large, but still allows the eigenvalues to converge to and beyond the threshold
\( \varepsilon_{\text{crit}} \) properly.

2. Perform stability calculations with this mean flow. Scan the most dominantly resolved (in this case the \( \eta \)-) direction, with moderate resolution in the other direction, to find the collocation point number for which the calculation is converged to algorithm precision.

3. Define a convergence region as shown in figures 6.1 and 6.2. Accordingly, determine the required value of \( \varepsilon_{\text{mean}} \) and perform a corresponding mean flow calculation.

4. Use the high resolution mean flow data to obtain fully converged stability calculations. To save effort, the calculations can be restricted to the earlier defined convergence region.

All results that are based on numerically represented mean flows are generated using the previous roadmap. In the case of the Blasius boundary layer, the “moderate resolution in the other direction” mentioned in the second step resulted in considering LST simulations to define the convergence region for the BiGlobal ones.

![Figure 6.3: LST eigenvalue convergence curves that fail to pass algorithm precision in the supersonic case, \( \varepsilon_{\text{mean}} = \mathcal{O}(10^{-5}) \), and those as presented in figure 6.2.](image)

Using the low resolution mean data does not work well for finding the eigeninformation. The latter is notorious to be very sensitive to errors in the mean data. Using this data does work quite efficiently with respect to finding the convergence region. Although the convergence is towards the incorrect solution, its characteristics are found to be qualitatively similar with respect to the high resolution mean data.

It is interesting to mention that in the case of the “fully developed” supersonic Blasius boundary layer, the eigenvalues already converged properly for \( \varepsilon_{\text{mean}} \) of \( \mathcal{O}(10^{-9}) \). Figure 6.3 gives the indication of what can happen in this respect; the eigenvalue convergence curve does not reach the threshold properly. This happened for \( \varepsilon_{\text{mean}} = \mathcal{O}(10^{-5}) \).
This issue was resolved merely by increasing the number of points of the mean flow integration scheme. In all cases, the eigenvalues must converge to algorithm precision, otherwise errors might have been introduced via the mean dataset. This error component is made negligible by making $\varepsilon_{\text{mean}}$ smaller than the threshold $\varepsilon_{\text{crit}}$.

Note that in this section nothing particular is said about the spacing of the grid. In several cases, the Chebyshev grid is transformed to more intensively resolve specific areas of the considered domain. This can be done to put an extra emphasis on boundary layers, see section 5.4 for example, which are nearly always the origin of fluid dynamic instabilities. If one does not (or cannot) resolve boundary layers well enough, one might miss very important physical features and hence find incorrect solutions. However, handling these kinds of transformations is very case specific. Therefore, they are handled individually for each considered mean flow case in the corresponding chapters.

### 6.3 Bauer-Fike Theorem & Numerical Modes

An interesting consideration is the *Bauer-Fike theorem* [13, 25]. This theorem considers the *ordinary eigenvalue* $\mu$ of the matrix $A + E$. The theorem (7.2.2) is presented on page 321 by Golub & Van Loan [25]. It is repeated here for completeness:

$$\text{If } \mu \text{ is an eigenvalue of } A + E \in \mathbb{C}^{n \times n} \text{ and } X^{-1}AX = D = \text{diag}(\lambda_1, \ldots, \lambda_n),$$

$$\text{then } \min |\lambda - \mu| \leq \kappa_p(X)||E||_p \quad (6.3)$$

where $|| \cdot ||_p$ denotes any of the $p$-norms.

Here, $\lambda$ is an element of the exact spectrum of $A$, $\kappa_p$ denotes the condition number based on the $p$-norm and $X$ is the matrix containing the exact eigenvectors of $A$. The matrix $E$ can be equated with the (of course unknown) influence of round-off errors. Then this relation gives an upper bound for the errors in the eigenvalues. Although the Bauer-Fike theorem is derived for an ordinary eigenvalue problem, it is assumed that it provides a qualitative indication of the sensitivity of the generalized eigenvalues.

An important point concerning the theorem is that $\lambda$ can be *any* eigenvalue in the spectrum. When one takes a look at a typical calculated spectrum, one will observe a large number of *numerical modes*. These modes are non-physical and purely the result of the fact that the problem is discretized. They can be distinguished by heavy oscillations with very small wavelengths; in the order of the distance between the collocation nodes. Nevertheless, they are elements of the exact invariant subspace of $A$.

It is not easy to filter out these modes. This can be accomplished only by investigating their behaviour when the grid size is varied. When this is done, they are notorious to jump (sometimes violently) through the complex plane. Accordingly, *they do not converge* when the grid is made finer.

The upper bound (6.3) reflects the extreme sensitivity of the numerical modes in addition to the physical modes. Accordingly, $p = 2$ condition numbers are found ranging...
from $\mathcal{O}(10^4)$ to $\mathcal{O}(10^{20})$, for a small to large total number of collocation points, respectively. Assuming the round-off induced error $||E||_2$ to be in the order of the machine precision $\epsilon = \mathcal{O}(10^{-16})$, equation (6.3) often yields a startling sensitivity. However, the physical spectrum, which in the discrete case is bounded by a certain (complex) frequency range, is found to converge very neatly. It is therefore suspected that the high eigenvalue sensitivity is caused by the numerical modes. Therefore in the remainder of the treatment the sensitivity indications obtained via the Bauer-Fike theorem will be discarded.

One of the numerical modes deserves extra attention. In figure 6.4, a typical LST spectrum is shown for the Blasius boundary layer. The most unstable mode seems to be the mode indicated to the right. However, when considering the eigenfunctions, the eigenmode is found to be non-physical; it contains extremely wild fluctuations in the (discrete representation of the) eigenfunction $\tilde{v}$. This is shown in figure 6.5. In accordance with the definition of the “P family” by Mack [57, 75], this mode and its relatives are referred to as the $P_0$ modes.

![Figure 6.4: Location of the numerical mode in the (compressible) LST spectrum.](image)

![Figure 6.5: Illustration of the eigenfunction $\tilde{v}$ corresponding to the numerical $P_0$ mode indicated in figure 6.4.](image)

In nearly all the simulations presented in this chapter these numerical modes are observable (in the BiGlobal case several relatives appear as well). It can be shown however that these modes ultimately diverge linearly when the number of collocation nodes in the wall-normal direction is increased, see figure 6.6. This behaviour demonstrates that they are non-physical. As conjectured at the end of section 5.3, these modes could be the consequence of the “oscillations” Batterson [10] refers to.

### 6.4 The Arnoldi Algorithm: Essentials of Matlab’s `eigs`

The function `eig`, through the inherent set-up of the QZ algorithm, returns the complete spectrum. As explained in the previous section, this yields a fair amount of modes that are not physically interesting, because the current problem is discretized. To be able
to retrieve the complete spectrum, the function \texttt{eig} requires a significant amount of storage capacity and execution time.

Hence, when an increased number of collocation points is required to yield a sufficiently fine grid, the latter algorithm becomes too expensive to use. Another solver that can be used is the \textit{Arnoldi Algorithm}. In contrast to the QZ algorithm, which is a \textit{direct method} to solving the eigenvalue problem, the latter algorithm is an \textit{iterative} one. It is based on the construction of the \textit{Krylov subspace}, which takes “full advantage of the intricate structure of the sequence of vectors naturally produced by the power method,” see chapter 4 of Lehoucq \textit{et al.} [52] (the ARPACK users’ guide) for an excellent elaboration. This construction demands only a relatively small execution time and required storage capacity. Moreover, one is able to specify the number of eigenmodes \(N_\omega\) to be resolved and a \textit{shift} (or eigenvalue guess) \(\omega_g\) so to focus only on a physically interesting area within the spectrum. This reduces the amount of work and required memory even more.

The individual modes returned by the algorithm can be expected to be accurate up to the following criterion [52]:

\[
\varepsilon_{\text{Arnoldi} \text{ crit}} = \epsilon |\omega|
\]

Here, \(\epsilon\) once again denotes the machine precision and \(\omega\) is the eigenvalue returned by the algorithm. In the current treatment, \(|\omega|\) will nearly always be smaller than unity, which means that the results can be expected to be accurate up to machine precision. In the case of exceptions this will be commented on separately.

In the current treatment, the Arnoldi algorithm is used to determine eigeninformation for problems that require very high resolution, see chapters 8 and 9. Usually, the QZ algorithm is used first to identify the overall structure of the spectra. Only when the areas of interest within the spectra are pinpointed or known via the literature, the
Arnoldi algorithm can be deployed to its full power. Otherwise, one could focus on a region that is (physically) rather uninteresting.

A note of caution must be made on the required storage capacity by the function `eigs`. In the first instance, Arnoldi’s algorithm requires relatively little memory. However, when an execution ends, memory problems arise when data (in the form of the L and U matrices of an LU-factorization) has to be transferred from one data structure, corresponding to UMFPACK routines, to the other, corresponding to Matlab. This transfer is very inefficient and can require up to twice the amount of RAM required for the whole remainder of the computation, for a short period of time. Therefore, the use of `eigs` is useful up to a certain memory limit only.

For this reason the computations on the large domain in chapter 8 could not be performed with $N_{\xi} \times N_{\eta} = 180 \times 45$ collocation nodes to replicate the simulations of Alizard & Robinet [2] and Ehrenstein & Gallaire [23].

6.5 Local Solvers & Absolutely Small Eigenfunctions

As will be seen in chapter 7, sometimes eigenfunctions are zero over the complete domain. The function `eig` outputs the eigenvectors such that the eigenfunctions are scaled relative to each other. When the systems become very large, `eig` might run into trouble resolving the smallest modes, outputting wildly oscillating profiles near the wall.

Equation (6.1) is based on the fact that the eigensolver neglects elements that have smaller absolute value than $\varepsilon_{\text{crit}}$. It is conjectured that the following link can be made. The boundary layer involves many scales, also very small ones, which leads to small elements in the ACM’s A and B. It is argued that by neglecting these, the eigeninformation mainly within the boundary layer becomes corrupted with an error of $O(\varepsilon_{\text{crit}})$. Illustrations of this will be shown in chapter 7.

In the currently considered cases, errors of $O(\varepsilon_{\text{crit}})$ are quite small, but in the case of nearly or exactly zero eigenfunctions they can become relatively very apparent. For this reason a warning mechanism is integrated in the current simulation tool, which tracks the magnitude of the individual eigenfunctions. It outputs the “eigenfunction condition number.” It must be emphasized that it is an entirely different number than the eigenvector condition number. Noting that the amplitudes are in $\mathbb{C}^{N_{\eta} \times N_{\xi}}$, it has the following features:

$$\kappa_{\Xi} = \begin{cases} \max(\kappa_{\bar{u}}, \kappa_{\bar{v}}, \kappa_{\bar{w}}, \kappa_{\bar{p}}) & \text{Incompressible mean flow} \\ \max(\kappa_{u}, \kappa_{v}, \kappa_{w}, \kappa_{p}) & \text{Compressible mean flow} \end{cases} \quad \text{where } \kappa_{\tilde{q}} = \begin{cases} 1 & \text{if } \tilde{q}_{ji} = 0 \text{ exactly for } i = 1, \ldots, N_{\xi} \text{ and } j = 1, \ldots, N_{\eta} \\ \max_{i \neq k} \left(1/\diag(\tilde{q}^H \tilde{q})\right) & \text{if } \tilde{q}_{jk} = 0 \text{ exactly for } j = 1, \ldots, N_{\eta} \\ \max_{i} \left(1/\diag(\tilde{q}^H \tilde{q})\right) & \text{otherwise} \end{cases} \quad (6.5)$$

Here, the superscript $H$ denotes the complex conjugate transpose. The division $1/\diag(\tilde{q}^H \tilde{q})$
indicates element-wise division. Note that $\kappa_{\tilde{q}} \geq 1$, because the entire eigenvector is normalized to have its 2-norm equal to 1. For clarity, the three statements to the right of the big accolade will be explained; they are handled from bottom to top.

Basically, $\kappa_{\tilde{q}}$ is very large when the eigenfunction $\tilde{q}$ has a very small, but not identically zero, maximal absolute value at a certain position along the wall. In equation (6.5) the index $j$ is always taken to be running over the elements that are oriented in the wall-normal direction. Therefore, wild oscillations of $O(\varepsilon_{\text{crit}})$ in the boundary layer will always be captured. This is the primary purpose of the “otherwise” statement.

The other statements are defined to rule out the possibility of infinite $\kappa_{\tilde{q}}$. One would obtain this result when an eigenfunction is completely zero at a specific location along the wall. In those cases, it is argued that no error is present at all. Therefore, $\kappa_{\tilde{q}}$ is defined to be equal to 1 locally in those cases. Because 1 is the smallest possible value for $\kappa_{\tilde{q}}$, the local values of $\kappa_{\tilde{q}}$ at the other locations will overrule this local value via the maximum operation. This is handled by the second statement. When the eigenfunction is zero everywhere, $\kappa_{\tilde{q}} = 1$ globally. This is returned by the first statement.

Ultimately, $\kappa_{\Xi}$ will be printed when analysing a specific eigenmode. If this value is large, the functions $\text{eig}$ and $\text{eigs}$ might have problems resolving a certain eigenfunction. One can use this as a validity check of the returned information.

When this occurs, the results have to be “refined.” This is where local eigensolvers come in. A key difference when using these solvers in this case, is that they do not discard small elements in the matrices. Furthermore, these solvers consider, per definition, only a single eigenvalue and -vector, based on an initial guess of the eigeninformation [25]. Therefore, these solvers are very cheap to execute, while leaving the initial problem unchanged. These local solvers can therefore provide an efficient refinement service. Recall that the results from the QZ and Arnoldi algorithm are very accurate already; only the eigenfunctions have to be improved. The results obtained from those methods can thus be given as input to the local solvers as initial guesses.

Currently, the two solvers implemented by Pinna are used, hence the following specifications are taken from his treatment, reference [65]. The first is the Rayleigh algorithm, which is an inverse power method extended with the Rayleigh quotient to compute the eigenvalue [20, 51]. This algorithm converges rapidly (cubically), but makes use of the inverse of $A - \omega g B$ to iteratively determine the ultimate eigenvalue more precisely. Here, $\omega$ denotes the guess for the eigenvalue. Per definition of the eigenvalue, the pencil $A - \omega B$ tends to become singular when $\omega$ approaches the exact eigenvalue. Therefore, the results of this algorithm can be argued to be questionable for more and more accurate values of $\omega$.

The second local solver is based on a Newton-Raphson iterative method, which looks for the root of the following system:

$$ F(\Xi, \omega) = \begin{bmatrix} (A - \omega B)\Xi \\ \Xi^H \Xi - 1 \end{bmatrix} $$

(6.6)

Here, $\Xi^H \Xi - 1 = 0$ is the normalization condition on the (right) eigenvector $\Xi$. Due to the fact that the Jacobian matrix can be determined analytically, the only numerical
operation involves finding the solution of the system $J(\Xi, \omega)[\Delta\Xi \Delta\omega]^T = F$ for the correction $[\Delta\Xi \Delta\omega]^T$. This algorithm does not become ill-conditioned when $\omega$ approaches the exact value. Ultimately, this method converges quadratically, thus a penalty has been paid with respect to the Rayleigh algorithm.

Several results of these local methods will be shown in chapter 7 for small but finite and truly zero eigenfunctions. In the finite case both solvers return extremely accurate results. In the zero eigenfunction case, the Newton-Raphson method is found to be more accurate. This is expected due to its aforementioned advantage with respect to the Rayleigh algorithm.
Chapter 7

Parallel Blasius Boundary Layer

To be able to verify the built tools, initially a canonical flow is chosen: the Blasius self-similar boundary layer. This flow has been analysed throughout the complete history of fluid stability theory and its characteristics are therefore well established and documented in the literature; an excellent feature for verification purposes. Furthermore, the analysis of this flow using the parallel flow assumption can be regarded as a basis for the following chapter, in which this assumption is alleviated.

The current chapter will start with an elaboration on the mean flow’s set-up and characteristics in section 7.1. This is followed by section 7.2, in which the mean flow’s accuracy is considered. In this chapter two kinds of analysis will be performed, LST and spanwise BiGlobal. This is done to be able to naturally deduce what features appear and why. Each of these analysis types is separately elaborated on in sections 7.3 and 7.4.

Section 7.3 handles the set-up of the stability problem and subsequently the corresponding results. These matters are followed by several convergence studies. These included the study of the behaviour of the most unstable modes and the spectra with respect to $N_\eta$. Furthermore, the convergence of the most unstable modes with respect to $y_{max}$ is handled. The section is concluded with an illustration of a compressible result with $M = 10^{-6}$, while involving the results of the local Rayleigh and Newton-Raphson algorithms.

Section 7.4 is structured similarly as the previous. After the treatment of the set-up of the stability problem and the excepted results, the convergence of the unstable modes and the spectra will be considered with respect to $N_\eta$ and $N_\xi$ only. This section is concluded by performing a spatial BiGlobal stability calculation to perform a (shallow) verification with the corresponding LST based literature accordingly.

In section 7.5 the conclusions of the current chapter are stated.

7.1 Mean Flow Set-up and Characteristics

The Blasius boundary layer corresponds to viscous fluid flow along a flat plate at zero incidence to the incoming flow. The velocity at the edge of the boundary layer is hence
constant in the streamwise direction. The compressible Blasius flow is described by the following set of equations and boundary conditions, see White [92]:

\[
\begin{align*}
\text{Momentum:} & \quad (c f''(\eta))' + f g' = 0 \\
\text{Energy:} & \quad \left( \frac{\rho c_p}{\mu} g' \right)' + f g' + \left( \gamma - 1 \right) M^2 c f'' = 0
\end{align*}
\]

where

\[
\begin{align*}
\text{Boundary conditions:} & \quad \left\{ \begin{array}{l}
f'(0) = 0, \quad f(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = 1 \\
g'(0) = 0, \quad \lim_{\eta \to \infty} g(\eta) = 1
\end{array} \right.
\end{align*}
\]

Here, the prime denotes differentiation with respect to \( \eta \), in this case the non-dimensional wall-normal coordinate. It must be emphasized that this \( \eta \) is different from the one used in chapter 5. As done before, all quantities corresponding to the mean flow are indicated with an overbar. The superscript \( d \) denotes the dimensional variables, i.e. those with physical units. Note the rather unusual unit of \( \xi \), \( [(\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1})^2] \). Although the notation seems messy, it is very important to make the distinction; the stability equations are made dimensionless differently with respect to the conventions used for problem (7.1).

In the remainder of the text, the mean quantities that are made dimensionless with the conventions of the stability equations are noted without superscripts, e.g. \( U \), \( V \), \( W \), \( T \) and \( \overline{P} \). The non-dimensional coordinates for the stability equations are denoted by \( x \), \( y \) and \( z \).

The variable \( h^d \) is the mean specific enthalpy related to the static temperature through \( h^d = c_p^d T^d \); the gas is assumed to be calorically perfect. Furthermore, \( \overline{p}^d \) and \( \overline{\mu}^d \) are the fluid density and dynamic viscosity, respectively. The subscripts \( e \) and \( w \) denote the evaluation at the edge of the boundary layer and the wall, respectively.

Lastly, it must be noted that the profiles \( U^d(\eta) \) and \( T^d(\eta) \) are not constant in the streamwise direction, they are self-similar. They develop in the streamwise direction, but retain their relative shape in the wall-normal direction. The \( \eta \) coordinate is therefore a function of \( y^d \) and \( x^d \).

The used non-dimensional coordinates and Reynolds number are based on the Blasius length scale \( l \) (which is related to the boundary layer thickness \( \delta^* = 1.7208l \)):

\[
\begin{align*}
l & = \sqrt{\frac{T_e^d x^d}{U_e}} \quad \text{where} \quad \nu^d = \frac{\overline{\mu}^d}{\overline{\rho}^d} \\
Re & = \frac{U_e^d x^d}{\nu^d} = \sqrt{\frac{U_e^d x^d}{\nu_e}} = \sqrt{Re_{xd}}
\end{align*}
\]

Here, \( \nu^d \) is the kinematic viscosity. All other dimensionless parameters are determined trivially using the variables evaluated at the boundary layer edge.
Means to integrate the system were already incorporated in VESTA, it involves a shooting method in combination with a standard fourth order Runge-Kutta method, see Pinna [65] for more details. In this chapter, an incompressible ($Re = 580$, $M = 10^{-6}$ and $T_e = 300$ [K]) and supersonic ($Re = 3000$, $M = 2.5$ and $T_e = 148.1$ [K]) case are considered, just as considered by Malik [59] and Pinna [65]. In figure 7.1, the profiles for the supersonic case are shown.

![Figure 7.1: Horizontal velocity and temperature profile of the compressible Blasius boundary layer ($Re = 3000$ and $M = 2.5$). (Digital version is animated)](image)

It is worthwhile to mention that all other mean velocity components are zero and the mean pressure is constant and equal to its static value at the boundary layer edge:

$$\bar{V} = W = 0 \quad \bar{P} = \frac{1}{\gamma M^2}$$  \hspace{1cm} (7.4)

In the incompressible case, the mean pressure is set equal to the pressure at the edge of the boundary layer. Note however, that due to the fact that the pressure is involved in the incompressible instantaneous equations in an isolated and linear way only, the mean pressure is not present in the incompressible stability equations.

The mean dynamic viscosity is related to the mean temperature through a Sutherland-like law. Moreover, it is assumed that the Prandtl number $Pr$ is constant and equal to 0.7, together with the mean viscosity this yields a relation for the thermal conductivity. Lastly, Stokes’ hypothesis is assumed to be valid, which related the second coefficient of viscosity $\lambda$ to $\mu$. All together this yields:

$$\bar{\mu}^d = \mu_S \left(\frac{T_e}{T^d}\right)^{\frac{3}{2}} \quad \bar{k} = \frac{c_p Pr}{Pr} \quad \bar{\lambda} = -\frac{2}{3} \bar{\mu}$$  \hspace{1cm} (7.5)

where

$$\begin{align*}
\mu_S &= 1.458 \cdot 10^{-6} \left[\frac{kg}{ms\sqrt{K}}\right] \\
S &= 110.4 \ [K]
\end{align*}$$
All other constants are fixed to their usual values. This yields all required information so to be able to obtain the mean flow.

The dependency of $Re$ on $x^d$ in its definition (7.3) reflects the developing nature of the Blasius boundary layer. In this chapter, however, the mean flow is assumed to be parallel, i.e. all variations in the streamwise direction are negligibly small. Although this assumption is physically inconsistent, it is well-established in the field of fluid stability. The LST analysis—which is inherently confined to parallel/non-developing flows—of the Blasius boundary layer is a classic example in this regard. A counterargument of this assumption is that the streamwise variations are small, making this assumption a valid approximation for boundary layer flows. This counterargument will be commented on in chapter 8. The first superset analysis type that takes these small variations into account specifically is the PSE framework.

In what follows, the calculated profiles will be used as if they represent a parallel flow, i.e. invariant of $z$ and $x$ location. To verify the stability of this flow thoroughly, two different approaches are adopted. First of all, the “conventional” LST analysis will be performed. Thereafter, a BiGlobal analysis with a spanwise oriented eigenfunction plane is considered, see figure 7.11. The latter orientation is used, so to mimic the LST simulations, because both types of analysis consider modal waves in the streamwise direction.

### 7.2 Mean Flow Accuracy

Due to the aforementioned independence of the mean flow on the $x$- and $z$-direction, the profiles are simply augmented together to yield the two-dimensional mean flow fields. All accuracy therefore rests upon the calculation of several of the variables presented in problem (7.1):

$$f', f'', f''', g, g', \bar{\rho}^d, (\bar{\rho}^d)', \text{ and } \bar{\mu}^d$$

(7.6)

It is important to note that the variables $f$ and $(\bar{\mu}^d)'$ are not directly involved in the generation of the mean data that is used as input for the stability equations. The Runge-Kutta integration method is programmed such that, per grid, it converges in absolute sense to $10^{-15}$ or beyond. Therefore, only the grid convergence has to be considered.

This is done by performing two calculations. One with $N + 1$ and another with $N$ points. The solution on the grid with $N + 1$ points is thereafter interpolated on the one with $N$ points using spline interpolation. In this way the absolute errors between the solutions on the different grids can be calculated. Subsequently, the maximum of the errors in the variables stated in equation (7.6) is determined. This maximal error is denoted by $\varepsilon_{\text{mean}}$ and is to be compared to the threshold $\varepsilon_{\text{crit}}$.

It is very important to note that the interpolation method introduces an additional error. Still, this interpolation is incorporated, because ultimately the data has to be projected on the Chebyshev grid. The key point is that this is done with the same interpolation method. When the datasets are converged, even under the influence of the superimposed errors introduced by the interpolation, they are argued to be fully trustworthy.
In figures 6.1 and 6.2 the (magenta) horizontal lines indicate the values of $\varepsilon_{\text{mean}}$ for the different mean flows. Clearly, they are small enough to negligibly influence the solutions that are converged to algorithm precision with respect to the errors induced by the eigensolver. Where in the incompressible case this required $N = 10^4$ points, in the compressible case this required the extreme number of $N = 3.2 \cdot 10^6$ points. In the former case, the profile of $f'''$ had the largest error equal to $3.9 \cdot 10^{-14}$, while in the latter case the variable $g$ was found to have the largest error, which was equal to $4.3 \cdot 10^{-13}$.

7.3 LST Simulations

Before the BiGlobal framework was considered, LST simulations were executed for several purposes. First of all, reference calculations had to be made to compare the BiGlobal results with. Secondly, the convergence characteristics were studied. The latter is done with respect to the number of collocation nodes $N_\eta$ and the truncation parameter $y_{\text{max}}$. These matters are very important for the investigation of the ultimate BiGlobal results. Furthermore, the compressible analysis of a $M = 10^{-6}$ case is considered. The case is interesting, because it involves the local algorithms to obtain a qualitatively accurate solution. This section will elaborate on all these matters subsequently.

7.3.1 Set-up and Results

As mentioned before, two flow cases are handled. One approaching the incompressible limit ($M = 10^{-6}$) and the other moderately supersonic ($M = 2.5$). To analyse the stability of these flows, at least the compressible LST equations are required. This system and its incompressible relative are stated in appendices B and A, respectively.

The considered non-dimensional domain is taken from $y = 0$ to $y = y_{\text{max}}$. In dimensional coordinates this means from 0 to $y_{\text{max}}$ times the Blasius length scale $l$ as defined in equation (7.2). At these boundaries, boundary conditions are to be applied. At $y = 0$, the conventional no-slip conditions are applied. This is done by applying homogeneous Dirichlet conditions on the velocity and temperature amplitudes. Furthermore, the (momentum equation based) compatibility condition is applied on the pressure.

The boundary $y = y_{\text{max}}$ is artificial. Strictly, $y_{\text{max}}$ should tend to $\infty$ to yield a physical boundary. To do this efficiently, without wasting too much collocation points on uninteresting regions of the domain. The transformation elaborated on in section 5.4 is used to map about half the collocations points into the boundary layer. The values of $y_i$ for the incompressible and supersonic simulations were set equal to 10 and 6, respectively. The influence of the value of $y_{\text{max}}$, in terms of the corresponding convergence characteristics, will be considered in subsection 7.3.4.

In the conventional case, at infinity the perturbation amplitudes should decay. In the Blasius case, this decay can be proven to be according to $e^{-\sqrt{\alpha^2 + \beta^2}}$, see Schmid & Henningson [75]. Therefore, it is argued that the velocity and temperature amplitudes can set equal to zero at $y_{\text{max}}$ by applying homogeneous Dirichlet conditions, when $y_{\text{max}}$ is sufficiently large. Whether $y_{\text{max}}$ is currently taken to be large enough is handled in
subsection 7.3.4. This could also be done for the pressure amplitude, as done by Piot [66] for example. It is argued, however, that applying the compatibility conditions is better, because through applying these conditions the momentum equations are satisfied at the boundary. Note that this would be only the case if actually $y_{\text{max}} \to \infty$. In any other case, the additional terms in the compatibility equations are still non-zero and ought to have an influence on the pressure amplitude.

Table 7.1: Key parameters for the most unstable modes of the parallel Blasius boundary layer [59, 65].

<table>
<thead>
<tr>
<th>Test case</th>
<th>$M$</th>
<th>$Re$</th>
<th>$T^d_s$ [K]</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10^{-6}$</td>
<td>580</td>
<td>300</td>
<td>0.179</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>3000</td>
<td>148.1</td>
<td>0.06</td>
<td>0.1</td>
<td>6</td>
</tr>
</tbody>
</table>

As done by Malik [59] and Pinna [65], the temporal analysis is considered to be able to verify the results. This means that $\omega$ is unknown and the (real) wavenumbers $\alpha$ and $\beta$ are to be supplied. The aforementioned references are consulted in this respect. They note values for $\alpha$ and $\beta$ for which the modes are most unstable keeping the mean flow field fixed. In this case, a verification is deemed to be performed successfully when these specific most unstable modes are reconstructed correctly. Table 7.1 provides a summary of all information that differs for the different test cases. Note that this information is the same for the BiGlobal simulations, except for the fact that several variables are omitted.

In figures 7.2 and 7.3, physically interesting regions of the resulting spectra are shown for the cases 1 and 2 in table 7.1, respectively. The first figure is obtained using $N_\eta = 90$ and the second with $N_\eta = 130$. In both cases $y_{\text{max}}$ was taken to be equal to 283. Later on, the figures of amplitudes will be cut off at $y = 15$ to have a good view of the shapes of the eigenfunctions near the wall. The convergence characteristics of the eigeninformation with respect to $N_\eta$ and $y_{\text{max}}$ will be elaborated on further in sections 7.3.2 and 7.3.4.

First of all, note the asymmetrical structure of the spectra with respect to the $\Re\{\omega\} = 0$ axis. This structure can be understood by looking at the relation for the real phase speed:

$$c_{\text{ph}} = \frac{\Re\{\omega\}}{\Re\{\alpha\}} \quad (7.7)$$

For both cases $\alpha$ is fixed and positive. This means that when the eigenvalues are located to the right of the $\Re\{\omega\} = 0$ axis, they correspond to waves with a positive phase velocity. This means that the waves propagate in the streamwise direction. If the spectra would be symmetric around this axis, every wave would have a brother that has the same amplification characteristics in time, but propagates in the opposite, upstream, direction. Clearly, the asymmetric structure of the spectrum indicates that no such brothers exist in these cases.

\footnote{This \textit{phase} speed is not to be confused with the \textit{group} speed as introduced in equation (2.33).}
This behaviour can be extracted from the equations in the following way. Consider the transformation $\omega \mapsto -\omega^*$, where the star denotes complex conjugation. Applying this transformation to the system of stability equations does not leave the form (of its real and imaginary parts) invariant.\(^2\)

It can be seen that the spectra have common characteristics. In both cases, a branch of modes enters the figure from below and roughly splits in two at a certain moment. The branch that splits toward the left consists of modes that are discrete. Even in the exact spectrum they are discrete. One is specifically interested in this kind of modes, because they correspond to the modal waves. The branch that splits toward the right, together with the branch it originates from, correspond to the discretized version of a continuous branch, which is a characteristic feature of the (truncated) semi-infinite domain. In the incompressible case, this branch is related to viscosity, whereas in the compressible case an extra continuous branch corresponding to entropy is found, see the treatment of Balakumar & Malik \[7\].

It is very interesting to note that the phase speed (7.7) can be approximately related to the mean flow velocity via the relation:

$$U|_{y = y_c} = c_{ph}$$  \hspace{1cm} (7.8)

Here, $y_c$ is a good estimate of the location where the mode is most dominant, see Schmid & Henningson \[75\]. In the case of the most unstable incompressible mode, the frequency $\Re\{\omega\} \approx 0.0652$, while $\alpha = 0.179$, thus $c_{ph} = 0.364$. This means that this mode has its maximum amplitude close to the wall, as will become apparent later.

The stable modes at the top of the right branch approach $\Re\{\omega\} = 0.179$, which means that $c_{ph} \to 1$. Hence these modes correspond to modes that are dominant in the outer flow. This is reinforced by the conclusions of Balakumar & Malik \[7\]. The

\(^2\)Obviously, the same applies for the transformation $\omega \mapsto \omega^*$.\n
---

Figure 7.2: Zoom on physically interesting LST spectrum for the incompressible ($M = 10^{-6}$) mean flow.

Figure 7.3: Zoom on physically interesting LST spectrum for the supersonic ($M = 2.5$) mean flow.
latter authors note that the vertical continuous branches correspond to vorticity waves, which are usually convected with the characteristic velocity $U_e$. In the exact spectrum, derived by Grosch & Salwen [34], all modes on the continuous branch indeed have phase speed $U_e$, also see figure 3.4(b) and equation (3.84) of Schmid & Henningson [75]. In the discretized case, the phase speed decreases for more negative imaginary parts of $\omega$. This discrepancy decays when $N_\eta$ increases. This will be demonstrated in section 7.3.2.

Note that the spectrum corresponding to the supersonic mean flow has quite similar characteristics. In addition, it contains two horizontally oriented branches along the real axis of which small parts are shown in figure 7.3. These branches are argued to be features of the compressible nature of the flow. Balakumar & Malik [7] note that the horizontal branches consist of acoustic waves. This is reinforced by the fact that the modes have very dominant temperature eigenfunctions, as is already reflected in figure 7.5. The latter authors note in addition that the compressible spectra have two vertical continuous branches; one corresponding to vorticity and the other to entropy waves. For high Reynolds numbers, they report that these branches collide and therefore are indistinguishable.

Above the red lines in figures 7.2 and 7.3 one can clearly distinguish two modes. As elaborated on in section 6.3, the mode above the branch with $c_{ph} \approx 1$ is numerical, it is the $P_0$ mode. The highly oscillatory shape of the eigenfunction $\tilde{v}$ is illustrated in figure 6.5. This function is obtained for the supersonic mean flow as well, having very similar divergence characteristics as its incompressible relative. The other unstable modes are the physically interesting ones.

In figures 7.4 and 7.5, the resulting absolute values of the amplitudes corresponding to these most unstable physical modes are shown for the cases 1 and 2 in table 7.1, respectively. In this treatment, all absolute amplitude profiles are scaled with the maximal absolute value of the streamwise velocity component. In figure 7.4, the signature shape of the Tollmien-Schlichting waves can be observed [66, 74, 75]. It has to be noted that $\tilde{w}$ is identically zero for all $y$ values. This is a consequence of $\beta$ being 0 for this case. Later on, it will be observed that numerical issues are encountered with these kinds of small amplitudes.

In figure 7.5, the structure of the compressible Tollmien-Schlichting wave is shown. Note that from the spectrum, the group velocity $c$ approximately equals: $\frac{0.0367}{0.06} = 0.612$. Which illustrates that the eigenfunctions should be dominant at a location further away from the wall with respect to the incompressible most unstable modes. This is clearly reflected in figure 7.5. The eigenfunction $\tilde{w}$ is far from zero, which is a consequence of the fact that $\beta = 0.1$ in this case.

### 7.3.2 Convergence of the Most Unstable Modes w.r.t. $N_\eta$

The convergence of the profiles with respect to $N_\eta$ was focused on thoroughly in section 6.1. There, it was shown through figures 6.1 and 6.2 that using $N_\eta = 90$ and $N_\eta = 130$, respectively, yields absolutely converged eigeninformation up to algorithm precision. It has to be emphasized however, that those convergence plots only apply to the most unstable modes, whose characteristics were tracked.
Figure 7.4: Absolute value of the LST perturbation amplitudes relative to the maximum of $|\tilde{u}|$ for the incompressible ($M = 10^{-6}$) mean flow.

Figure 7.5: Absolute value of the LST perturbation amplitudes relative to the maximum of $|\tilde{u}|$ for the supersonic ($M = 2.5$) mean flow.

An important sample of the tracked data is reported in tables 7.2 and 7.3, in the form of the eigenvalues. They are shown with the error $\varepsilon_{\text{crit}}$ introduced by the eigensolver. The decimals of real and imaginary parts of the eigenvalues that are corrupted with this error are highlighted with the red boxes. Note that the information is said to be converged when the absolute error between the subsequent cases is smaller than the value of $\varepsilon_{\text{crit}}$, which is not directly apparent from these tables. This information could already be seen in figures 6.1 and 6.2. Moreover, it has to be emphasized that there the spacing $N_\# = 5$ was used, see equation (6.2).

Table 7.2: LST eigenvalue convergence table for incompressible ($M = 10^{-6}$) case.

<table>
<thead>
<tr>
<th>$N_#$</th>
<th>$\Im{\omega}$</th>
<th>$\Re{\omega}$</th>
<th>$\varepsilon_{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>0.00142478995</td>
<td>70921</td>
<td>0.065177993307</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5757</td>
</tr>
<tr>
<td>80</td>
<td>0.00142478994</td>
<td>49376</td>
<td>0.06517799329</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>57733</td>
</tr>
<tr>
<td>90</td>
<td>0.00142478994</td>
<td>53745</td>
<td>0.06517799329</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>24719</td>
</tr>
<tr>
<td>100</td>
<td>0.00142478994</td>
<td>54451</td>
<td>0.06517799329</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>27671</td>
</tr>
<tr>
<td>110</td>
<td>0.00142478994</td>
<td>54894</td>
<td>0.06517799329</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>23365</td>
</tr>
</tbody>
</table>

Table 7.3: LST eigenvalue convergence table for supersonic ($M = 2.5$) case.

<table>
<thead>
<tr>
<th>$N_#$</th>
<th>$\Im{\omega}$</th>
<th>$\Re{\omega}$</th>
<th>$\varepsilon_{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>0.00057214234</td>
<td>08711</td>
<td>0.03669107124</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>89327</td>
</tr>
<tr>
<td>120</td>
<td>0.00057214231</td>
<td>40594</td>
<td>0.03669107124</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>68370</td>
</tr>
<tr>
<td>130</td>
<td>0.00057214231</td>
<td>73720</td>
<td>0.03669107124</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>43963</td>
</tr>
<tr>
<td>140</td>
<td>0.00057214231</td>
<td>67778</td>
<td>0.03669107124</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>44786</td>
</tr>
<tr>
<td>150</td>
<td>0.00057214231</td>
<td>71544</td>
<td>0.03669107124</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>49375</td>
</tr>
</tbody>
</table>
Table 7.4: LST eigenvalue comparison with Mack [57], Malik [59] and Pinna [65], using $N_\eta = 90$ and 130 for the $M = 10^{-6}$ and 2.5 cases, respectively.

<table>
<thead>
<tr>
<th>Reference</th>
<th>$M = 10^{-6}$ ($\alpha = 0.179$)</th>
<th>$M = 2.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>$c = \frac{\omega}{\alpha} = 0.364123 + 0.007960i$</td>
<td>$\omega = 0.0366911 + 0.0005721i$</td>
</tr>
<tr>
<td>Mack [57]</td>
<td>$0.3641 + 0.0080i$</td>
<td>$0.0367339 + 0.0005840i$</td>
</tr>
<tr>
<td>Malik [59]</td>
<td>$0.3641 + 0.0079i$</td>
<td>-</td>
</tr>
<tr>
<td>Pinna [65]</td>
<td>$0.36411 + 0.00796i$</td>
<td>$0.0366795 + 0.0005728i$</td>
</tr>
</tbody>
</table>

Mack [57], Malik [59] and Pinna [65] report the values of these eigenvalues as well. In table 7.4, the comparison of the currently obtained values with those of these references is shown; the same comparison is done as reported by Pinna [65]. It can be seen that in the incompressible case the results are nearly identical. In the supersonic case, the values slightly differ. It is argued that this could be the consequence of the extremely high precision mean dataset used for the current calculations; the results were found to be quite dependent on its accuracy. Nevertheless, the values differ only slightly. Therefore, the simulations are declared to be verified.

### 7.3.3 Convergence of the Spectra w.r.t. $N_\eta$

It is important to have a good look at the convergence characteristics of the complete spectra. In figure 7.6, the spectrum of the incompressible case is shown for multiple values of $N_\eta$. It can be seen very clearly that the discrete part of the spectrum is quite well resolved at low values of $N_\eta$. However, the modes that correspond to the continuous branch are more problematic. In this case, they seem to be approaching the final solution faster when they are closer to the real axis. It seems that the discrete spectrum, when “unfolded” from the continuous spectrum is quite accurate for small values of $N_\eta$ already.

To investigate this more deeply, several zooms are plotted over figure 7.6. First of all, it is demonstrated that the $P_0$ mode is “wandering” quite significantly. As explained in section 6.1, this mode is not physical.

Furthermore, zooms are made on two of the discrete modes. The first is the most unstable mode, which has the approximate value $6.518 \cdot 10^{-2} + i1.425 \cdot 10^{-3}$. This mode is converged so far, that the zoom could not represent the difference in the coordinates. Therefore, the abbreviation “IS” for Indistinguishable Scale is inserted. The averages of the coordinates of the dotted lines are: $\Re\{\omega\} = 0.065177993300$ and $\Im\{\omega\} = 0.0014247899515$. The differences between the coordinates are: $\Delta \Re\{\omega\} = 2.2 \cdot 10^{-11}$ and $\Delta \Im\{\omega\} = 1.7 \cdot 10^{-11}$. These differences are extremely small with respect to the other discrete mode. This mode just emerged from the continuous branch and has the differences: $\Delta \Re\{\omega\} = 2.1 \cdot 10^{-4}$ and $\Delta \Im\{\omega\} = 1.5 \cdot 10^{-3}$.

In figure 7.7, a similar spectrum convergence plot is shown for the supersonic mean flow. In this case, the $P_0$ mode does not need a zoom to show that it moves significantly.
It can be seen that its growth is again approximately linear, as shown in figure 6.6 as well.

In this case, the spectrum consists of three distinguishable continuous branches. Again, it is very apparent that the vertical one is hard to resolve. When zooming in on the horizontal branches, it is observed that they are well resolved at their ends, but, as the vertical one, diverge quite rapidly when moving to infinity. When taking a close look at this, one might argue that one can draw an ellipse (note the different scaling) in which the continuous branches are converged to the same precision. This ellipse has a horizontal major and a vertical minor axis. It seems that the center of this ellipse would be the point $\omega = \Re\{\alpha\} + i0 = 0.06$. Several of these ellipses are drawn in figure 7.7. From inner to outer, they correspond to errors of $5 \cdot 10^{-4}$, $6 \cdot 10^{-3}$ and $7 \cdot 10^{-3}$. A note of caution must be given, because it is very hard to compare which modes belong to each other far away from $\omega = 0.06$.

It is very interesting that the discrete branch of modes does not comply with this heuristic conjecture. In a zoom on the “intersection” of the discrete and continuous modes, it is illustrated that the discrete modes converge very fast with respect to their close neighbours in the continuous branch. The modes that lie to the left of the magenta line are nearly indistinguishable, while those to the right of it are converging very slowly.
It is very clear that the full spectrum requires a significant amount of collocation points before it converges to its ultimate state. Note that in this case still only a one-dimensional profile is analysed. To obtain a fully converged BiGlobal spectrum, more points are required. Fortunately, the above analysis shows that the discrete part of the spectrum converges rather rapidly, when it is unwrapped from the continuous branch. In this particular case, the most unstable mode is therefore relatively easy to obtain at high precision.

### 7.3.4 Convergence with respect to $y_{\text{max}}$

The parameter $y_{\text{max}}$ is a truncation parameter. Therefore, the convergence characteristics with respect to it are of paramount importance for the error in the ultimate results. From the theory, it follows that the eigenfunctions decay as $e^{-y\sqrt{\alpha^2+\beta^2}}$ in the $y$-direction. Therefore, estimates of $y_{\text{max}}$ can be made for the two cases so that the Dirichlet conditions at the boundary $y = y_{\text{max}}$ are justified. In other words, such that these conditions imply the correct result up to algorithm precision, $\varepsilon_{\text{crit}}$. It is assumed that the functions are of $O(1)$ near the solid boundary, then the following estimates can be made:
\[ M = 10^{-6}, \alpha = 0.179, \beta = 0 \]
\[ \varepsilon_{y_{\text{max}}} = 10^{-13} = e^{-0.179 y_{\text{max}}} \]
\[ y_{\text{max}} = \frac{-13 \ln 10}{-0.179} \approx 167 \]
\[ M = 2.5, \alpha = 0.06, \beta = 0.1 \]
\[ \varepsilon_{y_{\text{max}}} = 10^{-12} = e^{-y_{\text{max}} \sqrt{0.06^2 + 0.1^2}} \]
\[ y_{\text{max}} = \frac{-12 \ln 10}{-\sqrt{0.06^2 + 0.1^2}} \approx 237 \]

With respect to the above values, it can be seen that taking \( y_{\text{max}} = 283 \) is very appropriate. However, it can be shown that these estimates are rather pessimistic. This can be done by tracking the convergence of the computed eigenvalues with respect to \( y_{\text{max}} \).

It is deemed to be sufficient to show the convergence plots corresponding to a single pair of computation sequences, involving the incompressible and supersonic mean flows individually. They were performed using \( N_{\eta} = 90 \) and \( N_{\eta} = 130 \) for the respective cases, using all conventional settings otherwise.

In figures 7.8 and 7.9, the convergence plots are shown with respect to \( y_{\text{max}} \); the errors on the vertical axes are to be identified with \( \varepsilon_{y_{\text{max}}} \). It can be clearly seen that in both cases, the errors introduced by the currently used value of \( y_{\text{max}} = 283 \) are in the order of \( 10^{-14} \).

It follows from the figures that \( y_{\text{max}} = 120 \) and \( y_{\text{max}} = 160 \) would have yielded a sufficiently small error in the incompressible and supersonic case, respectively. Nevertheless, the value of \( y_{\text{max}} = 283 \) was used to be absolutely sure that no errors were present for these particular simulations.

### 7.3.5 Solving the \( M = 10^{-6} \) Case with the Compressible Equations

In addition to the supersonic mean flow, the performance of the compressible equations was assessed with the (nearly) incompressible mean flow. Note that the Mach number was not taken to be equal to zero exactly in the mean flow that was formerly addressed as the “incompressible case.”
All convergence characteristics are nearly identical with those for the true incompressible case. Therefore, this section handles only a particular issue with these kinds of calculations. The temperature amplitudes are extremely small in these cases, not identically zero. In figure 7.10, it can be seen that the absolute temperature profile’s overall magnitude is approximately $O(10^{-13})$. In this particular computation the algorithm precision $\varepsilon_{\text{crit}}$ was determined to be equal to $7.9 \cdot 10^{-13}$. Recall that this is an upper bound of the error. Therefore, when $\text{eig}$ would output such a profile, one would expect relatively large errors. The eigenfunction condition number $\kappa_\Xi$ defined in section (6.5) acted to this expectation accordingly. For this computation its value was equal to $5.8 \cdot 10^{23}$.

![Figure 7.10: Absolute compressible LST temperature profile in $M = 10^{-6}$ mean flow calculated with $\text{eig}$ alone and the refined solutions using the Rayleigh and Newton-Raphson local iterative methods.](image)

As can be seen in figure 7.10, the black line which corresponds to the output of $\text{eig}$ is indeed very oscillatory. Because of this behaviour, the eigenvalue and -vector were refined using the algorithms discussed in subsection 6.5. The results are plotted alongside the original in figure 7.10. It is observed that the refinement methods yield a very smooth result. Moreover, despite the ill-conditioning of the Rayleigh algorithm, the two refinement results are indistinguishable. It has to be emphasized that the results are indistinguishable on a $O(10^{-13})$ scale, so it can be said that they are (in absolute sense) identical to machine precision.

The refinement tools are programmed such that they output the total correction they applied to the eigenvalue. In this case, these corrections were:

- Rayleigh algorithm: $\Delta \omega = -6.2 \cdot 10^{-15} + i3.2 \cdot 10^{-15}$
- Newton-Raphson algorithm: $\Delta \omega = -6.9 \cdot 10^{-15} + i3.1 \cdot 10^{-15}$

This illustrates that the calculations performed by $\text{eig}$ are already very accurate. The
upper bound $\varepsilon_{\text{crit}}$ is large with respect to these corrections. The Rayleigh and Newton-Raphson algorithms are therefore only used as refinement tools.

### 7.4 BiGlobal Simulations

Now all the characteristics of the LST simulations have been handled in detail, they can be compared to BiGlobal results. It has to be emphasized that the latter simulations are purely meant to reproduce the LST results. Together with their set-up these results are handled first. Thereafter, the convergence characteristics of the BiGlobal results are handled. This section is concluded with a spatial spanwise BiGlobal verification.

#### 7.4.1 Set-up & Results

The primary BiGlobal verifications on the Blasius boundary layer are performed in a spanwise plane. This means that the $z$-direction is treated as a dependent variable for the eigenfunction and the $x$-direction is a spectral direction accompanied with the wavenumber $\alpha$. The corresponding BiGlobal equations are stated in sections C.3 and D.3 for the incompressible and compressible case, respectively. The two-dimensional mean dataset was generated by repeating the Blasius profiles used for the LST simulations in the $z$-direction. The collocation points are mapped into the boundary layer using the same transformation as used in the LST cases; the parameters $y_i$ are left unchanged. This verification method was performed following the work of Piot [66] very closely. In figure 7.11, the resulting set-up is illustrated.

Due to the primarily one-dimensional nature of the resulting mean flow field, it suffices to use relatively small $N_\zeta$ with respect to $N_\eta$. The main purpose of these simulations is to replicate the LST results in the form of the most unstable modes. In the case of incompressible LST, the most unstable mode had the spanwise wavenumber $\beta = 0$, see table 7.1. This means that in the incompressible BiGlobal calculations an eigenfunction constant in the $z$-direction must be represented. Therefore, using $N_\zeta = 5$ in this particular case is already more than sufficient. Nevertheless, $N_\zeta$ is increased to 10 and 15 to investigate the convergence characteristics in that direction. Because the profile is constant, the (non-dimensional) domain length in the $z$-direction can be chosen arbitrarily. In this case, $L_z = 24$ is used, i.e. the domain is $24l [m]$ wide.

For the compressible LST case, $\beta = 0.1$, see table 7.1. Therefore, firstly, the domain has to be adjusted to accommodate the wave with this wavenumber. Otherwise, in line with the symmetry boundary conditions, a wave with a distinct wavenumber would be preferred. Note that this is related to the used boundary conditions. To do this, a domain length $L_z$ of $\frac{2\pi}{\beta} \approx 62.8$ is used. Secondly, this particular wave must be represented. Having $N_\zeta > 5$ is definitely required to achieve the latter. Therefore, the values of 10 and 12 are investigated in addition. Due to the relatively high resolution required for the $y$-direction, the highest value for $N_\zeta$ cannot be larger than 12. In the compressible regime, a total collocation point number $N_\eta N_\zeta$ exceeding 1800 is already apt to yield an out of
Figure 7.11: Horizontal velocity and temperature profile of the compressible Blasius boundary layer augmented together to yield a two-dimensional mean flow ($Re = 3000$ and $M = 2.5$) for spanwise BiGlobal analysis. Boundary conditions for the stability problem are indicated.
memory error on a computer with 4 [Gb] of RAM.\textsuperscript{3} It is therefore expected that these BiGlobal simulations will not converge fully up to and beyond algorithm precision. The convergence characteristics will be handled in further detail in the following subsections.

\textbf{Figure 7.12: Zoom on physically interesting BiGlobal spectrum for the incompressible ($M = 10^{-6}$) case.}

\textbf{Figure 7.13: Zoom on physically interesting BiGlobal spectrum for the supersonic ($M = 2.5$) case.}

It has to be emphasized very strongly, that only the most unstable LST modes are aimed to be reproduced. In the spanwise BiGlobal framework general behaviour is allowed in the spanwise direction. Therefore, modes can be found that do not have a wave like character. Conversely, the modes that do behave like waves in the $z$-direction might have a different wave number than those imposed in the LST simulations. Moreover, the domains are created such that the most unstable LST modes fit precisely. Waves with smaller wavenumbers might be present in the continuum. A simple example is the incompressible case, in which $\beta = 0$. Next to the constant solution, waves that fit into the domain and satisfy the boundary conditions will be found as well. It is therefore expected that the BiGlobal spectra do not contain all physically interesting modes due to the limitations of the current set-up. For this reason, the BiGlobal spectra will be considered superficially.

Note that when the wavenumber of the most unstable mode is unknown \textit{a priori}, it is most effective to change the boundary conditions. Using Robin conditions with a fixed wavenumber is more effective, because the obtained perturbations will no longer be dependent on the size of the domain. One can perform a wavenumber sweep to see whether a most unstable mode can be identified. Note that in all cases, the domain must be able to represent half a wave (corresponding to the fixed wavenumber). In the latter case, the length of the domain can be fixed corresponding to the smallest wavenumber in the to-be-performed sweep.

Once again, boundary conditions are to be applied at all the boundaries of the

\textsuperscript{3}It has to be mentioned that only for this mean flow this computer was used. For all subsequent chapters a computer with approximately 64 [Gb] of RAM was used to be able to execute high resolution calculations.
domain. For the $y = 0$ and $y = y_{max}$ boundaries the same conditions are applied as done in the LST case. It has to be noted that this yields a very slight departure from Piot’s approach, as she applies Dirichlet conditions on the pressure amplitudes at $y = y_{max}$. In this treatment the compatibility conditions are applied. This is argued to be more consistent with respect to the truncated character of this boundary, although the differences are expected to be negligible regarding the large value of $y_{max}$ (again 283 is used).

On the spanwise boundaries, $z = 0$ and $z = L_z$, Piot applies symmetry conditions. This is related to the fact that the ultimate application concerns a roughness element that is set up in a spanwise array. This means that all scalar variables, $\tilde{p}$ and $\tilde{T}$, and symmetric velocity components, $\tilde{u}$ and $\tilde{v}$, are to be treated as even variables in the spanwise direction around these boundaries [36]. See section 2.3.2 for a detailed explanation. The velocity component $\tilde{w}$ is anti-symmetric with respect to the boundaries and must therefore be treated as an odd variable. This yields a combination of Dirichlet and Neumann conditions as illustrated in figure 7.11.

Because in the current tool periodic boundary conditions are implemented, next to the case with Piot’s symmetry conditions, a case with periodic spanwise boundary conditions applied to all variables was considered. With the exception of the influence of the random phase shift, which is allowed by the periodic conditions, all results were identical.

In figures 7.12 and 7.13, the physically interesting parts of the BiGlobal spectra are
illustrated for both mean flows. In the incompressible case $N_\zeta \times N_\eta = 5 \times 90$ nodes were used, whereas in the supersonic case $N_\zeta \times N_\eta = 12 \times 130$. In section 7.4.2, it will be shown that these values correspond to the most converged solutions. In the following, the BiGlobal results are mainly compared to the LST ones. It is deemed to be sufficient to omit a thorough discussion of the results, because they match very well with the LST results that are already discussed.

As in the LST spectra, the both BiGlobal ones are found to be asymmetric around the imaginary axis and the continuous branches can be clearly distinguished. There are some qualitative differences with the LST spectra, though. First of all, the discrete branch in the two cases consists of many more modes. Secondly, there appear to be, instead of a single, $N_\zeta - 2$ numerical $P_0$ modes. So, one for every internal grid point in the $z$-direction. From now on, the incompressible and supersonic case will be handled subsequently.

Despite these qualitative differences, the locations of the most unstable physical modes in the spectra are very close to corresponding locations in the LST spectra. In figures 7.14 and 7.15, the eigenfunctions corresponding to the most unstable mode are shown for the incompressible case. Considering the current mean flow, it is argued to be sufficient to consider only a vertical profile of the BiGlobal modes (the same will be done in the supersonic case). Note that due to the fact that $\beta = 0$ for the incompressible simulations, the choice of the specific $z$-location does not matter. In figure 7.14, the BiGlobal profiles are plot over the LST ones. It can be seen that the red
crosses corresponding to the BiGlobal profiles are located in the very center of the black circles, which correspond to the LST simulations.

As noted in subsection 7.3.5, \texttt{eig} has problems resolving small eigenfunctions when large systems are involved.\footnote{The error threshold is linearly linked to the Frobenius norm, which per definition increases when the system becomes larger. This in contrast to the 2-norm, which might remain equal, as experienced when testing this particular flow case.} This fact is reflected by figure 7.15, which displays the absolute \( \tilde{w} \) profile. The green data corresponds to the output of \texttt{eig}, which has rather strong oscillations, although of very small magnitude. The eigenfunction condition number had the value \( 6.8 \cdot 10^{21} \), implying the need for the application of a refinement method to circumvent the small-element-deletion feature of \texttt{eig}. In this case, the application of the refinement methods is increasingly interesting, because the “exact” eigenfunction is identically zero; truly testing the capabilities of the local eigensolvers. After applying both the Rayleigh and Newton-Raphson algorithms, the blue and red eigenfunctions were obtained, respectively. It can be seen that in this case the Newton-Raphson algorithm performs better than the Rayleigh algorithm, which is expected regarding the ill-conditioning of the latter. Note that the results of the Newton-Raphson algorithm are still moderately oscillatory. Nevertheless, its results are nearly accurate up to machine precision (\( \epsilon = 2.2 \cdot 10^{-16} \)).

In the supersonic case, the branches close to the real axis have closed in on the point \( \omega = \alpha + 0i \) nearly symmetrically. Regarding the LST framework, Balakumar & Malik\footnote{The error threshold is linearly linked to the Frobenius norm, which per definition increases when the system becomes larger. This in contrast to the 2-norm, which might remain equal, as experienced when testing this particular flow case.} report that the locations of the branch points of these horizontal branches are related to acoustic waves that travel with the phase speeds \( c_{ph} = \frac{1}{U_e} (U_e \pm a_e) = 1 \pm \frac{1}{M} \); as is typical for these kinds of waves. Hence, the fact that the branch points have closed in on the point \( \omega = \alpha + 0i \) is a strange observation. The latter authors note that their analysis is only valid for the LST framework. Therefore no specific conclusion is drawn in this regard. Lastly, a “diagonal” branch of modes appears in this spectrum when \( N_\zeta \) is increased. In subsection 7.4.3 all these qualitative difference will be considered in more detail.

A very important point has to be made regarding the eigenmodes in the supersonic case. As noted, \( \beta \neq 0 \) in this specific case, which means that the absolute profiles differ when considering a different \( z \)-location. If all the eigenfunctions were in-phase, no problem would occur, because the oscillating behaviour would be taken into account and “scaled away.” As explained before, in this case all variables except \( \tilde{w} \) are even with respect to the boundaries. This implies that \( \tilde{w} \), being an odd variable in this sense, is out-of-phase with respect to the other variables. In turn, this implies that the \( \tilde{w} \) profile cannot be directly compared to that of the LST simulations. To be able to do so anyway, the absolute \( \tilde{w} \) profile will be scaled such that the maximum of the LST and BiGlobal profiles are identical. All the eigenfunctions are shown in figure 7.16. As in the incompressible case, all eigenfunctions overlap the LST ones very precisely.

In addition to the convergence characteristics which are considered in a second, the presented results yield the verification of the BiGlobal simulations with respect to the LST ones.
7.4.2 Convergence of the Most Unstable Mode w.r.t. $N_\eta$ and $N_\xi$

In section 6.1, the LST convergence plots were given in which the “convergence regions” were designated. These regions were used to set $N_\eta$ in the BiGlobal simulations. To show the very close resemblance of the convergence characteristics of the LST and BiGlobal simulations in $N_\eta$, figures 7.17 and 7.18 show the convergence characteristics for the two cases alongside each other. It can be seen that the curves are very close to each other whilst above the criterion $\varepsilon_{crit}$. This shows that only when eigensolver errors are introduced, the errors become significantly different. Hence, the assumption that the convergence characteristics of the BiGlobal simulations in $N_\eta$ are similar as those for LST is verified.

![Figure 7.17: LST and BiGlobal convergence for the incompressible ($M = 10^{-6}$) mean flow with $N_\xi = 5$ within the convergence region designated in figure 6.1.](image1)

![Figure 7.18: LST and BiGlobal convergence for the supersonic ($M = 2.5$) mean flow with $N_\xi = 12$ within the convergence region designated in figure 6.2.](image2)

This is a very handy characteristic, because this means that, for example, a boundary layer must be represented in the vertical direction with a significant number of collocation nodes irrespective of the streamwise number of points. Consider using the transformation parameter $y_i$ as a measure of the boundary layer thickness. Then it is demonstrated that using at least 45 and 65 collocation points in the current incompressible and supersonic cases, respectively, is sufficient to resolve the boundary layer to algorithm precision.

Because of the aforementioned equivalence, it is assumed that the convergence characteristics with respect to $y_{max}$ of the BiGlobal simulations coincide with those observed for the LST simulations. Hence it is decided that having $y_{max} = 283$ is more than sufficient and that the treatment of the convergence with respect to $y_{max}$ can be dropped.

From figures 7.17 and 7.18, it can be clearly seen that the threshold $\varepsilon_{crit}$ has moved upward with respect to those corresponding to the LST simulations. As noted in section 6.1, the use of the Frobenius norm causes this. In figure 7.18, it can be seen that this definition is consistent, because the errors at $N_\eta = 120$ are suddenly off relatively far with respect to the errors at $N_\eta = 115$ and $N_\eta = 125$. When using the 2-norm, the
BiGlobal and LST threshold would practically overlap. Therefore, the aforementioned distinction in the errors could only be explained using the appropriate definition of the norm in equation (6.1).

Before considering the values of the BiGlobal eigenvalues, a global view is given of the convergence characteristics encountered when increasing the number of nodes in the \( z \)-direction. This is done via table 7.5. The table shows the maximal\(^5\) error between the eigenvalues corresponding to the two indicated values of \( N_\xi \). It can be seen that in the incompressible case, the maximal error is already smaller than the threshold when going from \( N_\xi = 5 \) to \( N_\xi = 10 \). So, it can be safely decided that this case is converged up to algorithm precision completely. It is interesting to note that the smallest individual error is in the order of machine precision.

Unfortunately, in the case of the supersonic mean flow this cannot be said, because the smallest error indicated for that flow in table 7.5 is of \( O(10^{-7}) \). This error is much larger than the algorithm precision. This is the consequence of the fact that a complete wave must be represented in the \( z \)-direction in this case, as noted in the previous subsection.

The reason why the \( z \)-direction cannot be resolved any further concerns memory problems. As mentioned earlier, they cause 12 to be the maximum value for \( N_\zeta \), while requiring full convergence in the \( y \)-direction. Simple ways around this problem involve changes in the set-up. When using a combination of symmetric and anti-symmetric boundary conditions, one only has to resolve a quarter of the complete wave. This would ultimately require a domain width \( L_z = \frac{12\pi}{4f} \approx 15.7 \). Furthermore, one could decrease the values of \( y_i \) and \( y_{\text{max}} \) on the basis of figure 7.9, yielding a smaller required value for \( N_\eta \). It is expected that when using one or a combination of these set-up changes, the computations will converge completely to algorithm precision.

Performing the corresponding computations is however not attempted due to the time constraint; the fact that the method converges is considered to be sufficient proof of the fact that the code is consistent. An absolute error of \( O(10^{-7}) \) is judged to be already relatively accurate for 12 nodes in the \( z \)-direction.

Table 7.5: BiGlobal eigenvalue \( N_\xi \) convergence table.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( N_\xi )</th>
<th>( N_\eta (\Delta \Re {\omega}, \Delta \Im {\omega}) )</th>
<th>( \max N_\eta (\varepsilon_{\text{crit}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10(^{-6})</td>
<td>5(\rightarrow)10</td>
<td>1.3e-13</td>
<td>5.1e-13</td>
</tr>
<tr>
<td></td>
<td>10(\rightarrow)15</td>
<td>1.2e-13</td>
<td>7.6e-13</td>
</tr>
<tr>
<td>2.5</td>
<td>5(\rightarrow)10</td>
<td>2.3e-4</td>
<td>1.1e-12</td>
</tr>
<tr>
<td></td>
<td>10(\rightarrow)12</td>
<td>2.0e-7</td>
<td>1.1e-12</td>
</tr>
</tbody>
</table>

The values of the eigenvalues are shown in tables 7.6 and 7.7, as in the LST related section. Again, a column showing the error threshold \( \varepsilon_{\text{crit}} \) is augmented. It is deemed to be sufficient to show the convergence results corresponding to the \( N_\xi \) value for which the results are converged farthest in the \( z \)-direction. That is to say, \( N_\xi = 5 \) and \( N_\xi = 12 \)

---

\(^5\)The maximum is taken over all values of \( N_\eta \) within the convergence region.
in the incompressible and supersonic case, respectively. The orange boxes in table 7.7 indicate the decimals that are corrupted with the maximal error reported in table 7.5.

Table 7.6: BiGlobal eigenvalue \( N_\eta \) convergence table for incompressible \((M = 10^{-6})\) case.

<table>
<thead>
<tr>
<th>( N_\eta )</th>
<th>( \Im {\omega} )</th>
<th>( \Re {\omega} )</th>
<th>( \varepsilon_{\text{crit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>0.00142478995</td>
<td>71093</td>
<td>0.065177993307</td>
</tr>
<tr>
<td>80</td>
<td>0.00142478994</td>
<td>48669</td>
<td>0.06517799329</td>
</tr>
<tr>
<td>90</td>
<td>0.00142478994</td>
<td>54890</td>
<td>0.06517799329</td>
</tr>
<tr>
<td>100</td>
<td>0.00142478994</td>
<td>54857</td>
<td>0.06517799329</td>
</tr>
</tbody>
</table>

When comparing the lowest row in table 7.6 with the one corresponding to the incompressible LST simulations, it can be seen that all to-be-trusted decimals of the eigenvalues are matching. Even the first 2-3 decimals in the red boxes match. This result is considered to be a quantitative verification of the incompressible BiGlobal simulations.

Table 7.7: BiGlobal eigenvalue \( N_\eta \) convergence table for supersonic \((M = 2.5)\) case.

<table>
<thead>
<tr>
<th>( N_\eta )</th>
<th>( \Im {\omega} )</th>
<th>( \Re {\omega} )</th>
<th>( \varepsilon_{\text{crit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>0.0005724134</td>
<td>86926</td>
<td>0.03669107547</td>
</tr>
<tr>
<td>120</td>
<td>0.0005724132</td>
<td>19931</td>
<td>0.03669107546</td>
</tr>
<tr>
<td>130</td>
<td>0.0005724132</td>
<td>54040</td>
<td>0.03669107546</td>
</tr>
<tr>
<td>140</td>
<td>0.0005724132</td>
<td>52430</td>
<td>0.03669107546</td>
</tr>
</tbody>
</table>

When doing the same for the supersonic eigenvalues, of course the convergence error in the \( z \)-direction must be taken into account primarily. To that end, all values are identical. When considering the decimals within the orange boxes, it can be seen however, that the next 3 decimals are identical as well. Therefore, it is also in the supersonic case concluded that the BiGlobal simulations are quantitatively verified.

### 7.4.3 Convergence of the Spectrum w.r.t. \( N_\eta \) and \( N_\xi \)

As stated in the previous subsection, these BiGlobal simulations are set-up so to mimic the LST simulations, in the form of the most unstable eigenmodes. This means that not all converging modes in the spectrum are expected to have physical significance. Nevertheless, the convergence of the spectra will be studied to get a feel for their behaviour. This is thought to be of paramount importance for the study of numerically determined spectra that correspond to more complicated mean flows.

The convergence of the spectra is considered in both directions. The convergence of the incompressible spectrum with respect to \( N_\eta \) is considered first in figure 7.19, for
$N_\xi = 5$. It can be directly seen that the continuous branch moves to the right when $N_\eta$ increases, just as observed in the LST counterparts. A zoom on the non-physical $P_0$ modes has been superimposed on the figure. It can be clearly seen that 3 such modes are returned in each simulation. Furthermore, they still display the linear “wandering” behaviour with respect to $N_\eta$.

Secondly, a zoom is given for a group of three modes that is “unfolding” from the continuous branch as it moves to the right. When zooming in even further, it will be observed that all three modes converge at approximately the same rate. This is completely in line with the behaviour observed in the LST convergence studies. All discrete modes tend to converge very rapidly once they are unfolded from the continuous branch.

It can be observed that all over the spectrum clusters of modes are formed. Each of these clusters consist of three ($N_\xi - 2$) modes that are related to each other. When taking a close look at these modes, the reason for this becomes clear. In this specific case, all groups have a “zeroth” mode that is constant in $z$, a “first” mode that approximately has wavenumber $\frac{2\pi}{2L_x} = 0.038$ and a “second” mode with approximate wavenumber $\frac{2\pi}{L_z} = 0.076$.

As mentioned before, due to the one-dimensional nature of the mean flow, the spec-
trum contains all possible modes that can be represented in the domain while satisfying the boundary conditions. Amongst these modes are those that behave like waves with $\beta \neq 0$. In figure 7.20, an indication is given of the shape of the modes shown in the magenta box in figure 7.19. Because of the small $N_\xi$, the first and second mode are not as resolved as the leading one. Therefore, these modes are expected to wander significantly when $N_\xi$ is increased. Precisely this behaviour can be observed in figure 7.21, the convergence plot with respect to $N_\xi$.

In figure 7.21, another zoom is giving on the numerical $P_0$ modes. In that particular zoom, it is clearly shown that for a simulation with $N_\xi$ nodes in the $z$-direction, one obtains clusters that contain $N_\xi - 2$ modes that are related to each other as explained above. This illustrates that, in this particular case, the single modes observed in the LST simulations have a one-to-one correspondence with a cluster of modes in the BiGlobal case. In figure 7.21, all discrete clusters are indicated with the labels $C_1$ to $C_{10}$. In this way, they can be compared to the 10 discrete LST modes shown in figure 7.2. Two of the clusters have been tracked, by checking the shapes of the eigenmodes. This yielded the magenta and blue boxes. Note the apparent coherent structure of the clusters on, above and below the diagonal $C_2$-$C_4$-$C_6$-$C_8$-$C_{10}$. Clearly, the modes with $\beta \neq 0$ behave
Figure 7.21: Convergence of the BiG spectrum with respect to \( N_\xi \) for the incompressible \((M = 10^{-6})\) mean flow for \( N_\eta = 90 \).

qualitatively similarly on those parts of the discrete branch.

Now the spectra of the incompressible flow are handled, those corresponding to the supersonic mean flow are next. In figure 7.22, the convergence of the spectrum with respect to \( N_\eta \) is shown, with \( N_\xi = 12 \). Again, the vertical continuous branch displays the usual convergence behaviour. Moreover, it seems that one can again draw the elliptical convergence areas around the point \( \omega = \alpha + 0i \), indicating similar levels of convergence for the continuous branches (only). This follows from the fact that the zooms on the branches near the real axis (which have equal scales in the real direction) display divergence at approximately the same distance from this point.

In the zoom on the \( P_0 \) cluster, it can be seen that there are indeed \( 12 - 2 = 10 \) such modes and again they are diverging linearly with \( N_\eta \). Again, this gives an illustration of the fact that these modes diverge and are not physical. When zooming in on the intersection of the discrete branch and the vertical continuous one, it can be clearly seen that the discrete modes are quite converged with respect to the continuous ones, again in line with the LST results.

In the previous subsection it was noted that an “odd” diagonal branch seems to emerge when \( N_\xi \) is taken equal to 10 and 12. In figure 7.23, a zoom on this “branch” is given. It can be clearly seen that actually, it is a mode cluster corresponding to the
most unstable mode with $\beta = 0.1$. This can argued, because for the simulations with $N_\xi$ nodes, $N_\xi - 2$ modes are counted within the zoom. Furthermore, the mode clusters are designated with the labels $C_1$ to $C_{14}$, as in the incompressible case. These clusters have a one-to-one correspondence to the 14 discrete modes in the LST spectrum in figure 7.3. Again, the modes that have $\beta \neq 0$ seem to display a coherent structure with respect to their particular location in the discrete branch.

In this chapter, it has become very clear that the structure of the BiGlobal spectra can become very complex. Note that the maximal value for $N_\xi$ is $\mathcal{O}(10)$ in figures 7.21 and 7.23. Moreover, note that the one dimensional nature of the current mean flow simplifies the behaviour as well. Yet, the resulting spectrum is already rather complicated. When traversing to more complicated mean flows, which require increased amounts of collocation points in both spatial directions, the analysis of the spectra will become extremely complex.

Nevertheless, the results, which are based on a parallel boundary layer, are in full agreement with the one-dimensional LST results. In the sense of the eigenvalues corresponding to the incompressible mean flow, this agreement is up to (and even beyond) algorithm precision. It is conjectured that this is true for the supersonic mean flow as

Figure 7.22: Convergence of the BiG spectrum with respect to $N_\eta$ for the supersonic ($M = 2.5$) mean flow for $N_\xi = 12$. 
Figure 7.23: Convergence of the BiG spectrum with respect to $N_\xi$ for the supersonic ($M = 2.5$) mean flow for $N_\eta = 130$.

well. The current supersonic results converged and are equal to the LST results to the highest acquired accuracy. Unfortunately, this accuracy is not fully up to the algorithm precision, which would prove the conjecture.

7.4.4 Spatial Stability Calculation

The main focus of the thesis is on the temporal stability analysis type. This is reflected in the current chapter already. The spatial application of the BiGlobal framework is at the current moment to a certain extent less interesting than the temporal one. The main reason for this is that the extra inhomogeneous direction provides the corresponding information as well, when the eigenfunction plane is oriented in the streamwise plane. The first application that is expected to become interesting regards the spanwise eigenfunction plane. In that case, one might be interested in the spatial stability characteristics of two-dimensional spanwise modes, examples are the Görtler vortices [82].

To show that the current code can be deployed for the spatial stability type as well, a shallow verification case is set up for the incompressible parallel Blasius case. This is done by replicating the results of Danabasoglu & Biringen [21], Jordinson [47] and Pinna [65]. This is done with the spanwise BiGlobal analogue of their LST computations as already
used in this chapter. To this end, the equations stated in section C.3 are implemented where now $\alpha$ is the eigenvalue and the quadratic eigenvalue problem is linearised using the companion matrix method illustrated in section 5.5.4.

The references use the displacement thickness $\delta^* = 1.7208 l$ as length scale. Transforming the parameters to the current scaling yields:

$$Re \equiv Re_l = Re_{\delta^*} \frac{l}{\delta^*} = \frac{336}{1.7208}, \quad \omega \equiv \omega_l = \omega_{\delta^*} \frac{l}{\delta^*} = \frac{0.1297}{1.7208} \quad \text{and} \quad \beta = 0 \quad (7.9)$$

It has to be noted that the final values of $\alpha_l$ had to be rescaled again, to $\alpha_{\delta^*}$, to yield the appropriate parameter to compare to the literature. Because $\beta = 0$, it suffices to use a small number of nodes in the $z$-direction as demonstrated in the previous sections. The limit is exhausted and $N_\zeta$ is set equal to 5. By Danabasoglu & Biringen [21] and Pinna [65] it is specifically reported that the value for $N_\eta$ is taken equal to 50, so this is done in this case as well. Furthermore, Danabasoglu & Biringen [21] note that they use a “uniform” grid with $y_{\max} = 6 \delta^*$. This is translated to the use of the parameters $(y_i, y_{\max}) = \left(6 \cdot \frac{1.7208}{2}, 283\right) \approx (5, 283)$. Note that this yields a relatively small total amount of collocation points and accordingly a cheap calculation.

### Table 7.8: Comparison of the spatial BiGlobal eigenvalues with the LST results reported by Danabasoglu & Biringen [21], Jordinson [47] and Pinna [65], using $N_\eta \times N_\zeta = 50 \times 5$.

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\Im{\alpha_{\delta^*}}$</th>
<th>$\Re{\alpha_{\delta^*}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>0.007940</td>
<td>0.308349</td>
</tr>
<tr>
<td>Danabasoglu &amp; Biringen [21]</td>
<td>0.00799</td>
<td>0.30864</td>
</tr>
<tr>
<td>Jordinson [47]</td>
<td>0.0079</td>
<td>0.3084</td>
</tr>
<tr>
<td>Pinna [65]</td>
<td>0.008004</td>
<td>0.308357</td>
</tr>
</tbody>
</table>

Because the current analysis is BiGlobal, the higher order modes are found again, as elaborated on in section 7.4. The mode corresponding to $\beta = 0$, the mode that is constant in the $z$-direction, is selected to perform the verification (again the higher modes are consistently found to be more stable than the former). It is argued to be sufficient to only consider the first case (to which the parameter in equation (7.9) correspond) handled in the references, because no further spatial analyses are undertaken.

In table 7.8, the found eigenvalue $\alpha_{\delta^*}$ is shown. It can be seen that the values are nearly identical; the largest error is of $O(10^{-4})$. It is argued to be a bit strange why a different value is found with respect to the value reported by Pinna [65]. Because the code of the latter author is in possession of the current, an LST simulation was performed with precisely the same parameters as given as input to the BiGlobal calculation. When this was done precisely the same values were found up to the digits reported in table 7.8. Probably, Pinna had used slightly different input parameters to obtain the value reported in his treatment. This cannot be checked, because he does not report all used
parameters in reference [65]. The agreement of the values in table 7.8 is argued to yield the verification of this test case.

To be complete, the physically interesting part of the spectrum and the eigenfunctions corresponding to the mode reported in table 7.8 are shown in figures 7.24 and 7.25. Note that in both figures variables are shown that scale with $\delta^*$ and therefore have this subscript.

Note that in this type of analysis, $\Im\{\alpha_{\delta^*}\} < 0$ corresponds to the unstable domain of the spectrum. First of all, when carefully looking at the modes around the value illustrated in table 7.8, it can be seen that 3 ($= \text{ceil}(\frac{N^2}{2})$) modes are very closely packed. They form the cluster corresponding to the unstable mode and its higher order relatives. It can be seen that the $P_0$ modes are also present when performing the spatial analysis type. They are located slightly below the continuous spectrum in the middle of the domain. A continuous branch is observed along the imaginary axis in the spectrum (it is very poorly resolved with these scales of the figure). As reported by Balakumar & Malik [7], this branch corresponds to acoustic waves. The vertical branch with its branch point at the real axis is associated with vorticity waves. The asymptotically determined value of the branch point, derived by Balakumar & Malik, should be $\alpha_{\delta^*} = \omega_{\delta^*} + 0i = 0.1297$. The value of the mode closest to the branch point is given by the label. It can be seen that this value is indeed obtained approximately.

Because the modes are constant over in the $z$-direction, only the profiles at the second node in this direction are shown in figure 7.25. The Tollmien-Schlichting wave shape is obtained as expected.
7.5 Parallel Blasius Flow Conclusions

This chapter handles the stability of the parallel Blasius boundary layer. This is done, because its stability is well established and documented in the literature. Furthermore, the currently obtained features can be used as a basis for the following chapter, in which the developing flow is considered.

The analysis is performed for an incompressible and supersonic flow case. The corresponding mean flows are obtained using a shooting method in combination with a standard fourth order Runge-Kutta method, as provided by Pinna [65] in VESTA. The incompressible and supersonic mean datasets are fully converged in “time” and up to $3.9 \cdot 10^{-14}$ and $4.3 \cdot 10^{-13}$ in space, respectively.

Both LST and BiGlobal analyses were performed to be able to compare the results corresponding to the latter analysis type to the former. The LST simulations were found to be converged up to algorithm precision for $N_n$, the number of collocation points in the vertical direction, equal to 90 and 130 in the incompressible and supersonic case, respectively. The results, in the form of the Tollmien-Schlichting mode’s eigenvalue, were compared to the literature and were found to be nearly identical.

Furthermore, the convergence characteristics with respect to $y_{\text{max}}$ were determined. It was found that using $y_{\text{max}} = 120$ and 160 in the incompressible and supersonic case is sufficient to obtain solutions that are converged up to algorithm precision. Lastly, the $M = 10^{-6}$ case was solved with the compressible equations, which yielded a very small temperature amplitude. Because eig has problems resolving this kind of functions, the local Rayleigh and Newton-Raphson algorithms were used to refine the solution. Due to the finite nature of the function, both algorithms delivered the same function up to machine precision.

The BiGlobal simulations were handled next. The eigenfunction plane was chosen to be in the spanwise direction, so to simulate modal waves in the streamwise direction. In the spectra, $N_\zeta - 3$ additional discrete modes are encountered that correspond to the higher harmonics of the LST results. The combined $N_\zeta - 2$ modes form a cluster within the spectra. For the verification, the modes corresponding to their LST relatives were compared. Both the eigenvalues and -functions were found to be matching up to the highest acquired accuracy for the BiGlobal simulations. In the incompressible case, this meant algorithm precision, while in the supersonic case the convergence error with respect to $N_\zeta$ provided the largest error, of $O(10^{-7})$. Because the BiGlobal results can be argued to converge toward the LST ones, they are declared to be verified.

Lastly, a spatial spanwise BiGlobal verification was performed using a single test case. This is done to show that the current tools can be deployed for those kinds of problems as well. The eigenvalues were found to match nearly identically to the results in the literature. Slight differences are expected to have arisen due to differences in the used mean flow and the specific parameters for the stability problem.
Chapter 8

Developing Blasius Boundary Layer

Now the parallel Blasius boundary layer flow has been analysed thoroughly, a more physical relative can be considered: the developing Blasius boundary layer. This flow is interesting, because it’s better affinity with the actual Blasius boundary layer flow. The “extent of development” can be scaled by considering a short or a long domain. Therefore, this flow is excellent for verification purposes.

Hence, first, a small domain is considered, which is expected to involve a weakly developing mean flow. Thereafter, a large domain is considered which corresponds to a strongly developing mean flow. The flow corresponding to this large domain has been considered in a BiGlobal context by Alizard [1], Alizard & Robinet [2] and Ehrenstein & Gallaire [23]. Due to the inherent complexity of the flow only the incompressible version is analysed. Regarding the involvement of forcing Robin boundary conditions in the streamwise direction, the results are expected to display slight differences with respect to the parallel LST simulations.

This chapter is started with an elaboration on the used mean flow and its characteristics in section 8.1. The modes on the small domain require much less resolution and can therefore be analysed already very thoroughly with the QZ algorithm only. The large domain is set up so to be able to compare the results to those in the aforementioned references. The set-up of the stability problem for each of these domains is handled in section 8.2.

After this is done, the analysis of the small domain is handled in section 8.3. Along with the elaboration on the results and the verifications with LST simulations, the results corresponding to the secondary Robin conditions are handled and the performance of the Arnoldi algorithm is assessed. When this is done, a revision of the observed odd/even behaviour is given by involving the latter algorithm. All the findings up to that point are concluded in the intermediate conclusion in section 8.4.

---

1It has to be mentioned that the treatments of Alizard [1] (PhD thesis) and Alizard & Robinet [2] (paper) are intimately related. It is assumed that they are based upon the same project. Therefore, the latter reference is mostly referred too.
Thereafter, the results on the large domain are considered in section 8.5. These are mainly calculated using the Arnoldi algorithm as well. The results are mainly compared with those of Alizard [1]. The overall findings are subsequently concluded in section 8.6.

8.1 Mean Flow Set-up and Characteristics

As stated in chapter 7, the parallel Blasius analysis case involves a physically restrictive assumption of no streamwise changes. Accordingly, that direction is treated as a spectral direction. Nevertheless, problem (7.1) describes a developing flow in the $x^d$ direction. In this case, this developing flow is considered.

To acquire the two-dimensional mean flow field, the previously used profiles in $\eta$ have to be scaled in the streamwise direction. This can be done by using the set-up shown in figure 8.1. Note that the $M = 2.5$ profiles are shown. This is done only for illustration purposes; only the incompressible case is considered in this chapter. First of all, the Blasius length scale $l$, as introduced in equation (7.2), is clearly a function of $x^d$; the square root dependence is clearly visible in figure 8.1. In the former computations, the used length scale was based on the streamwise distance $x^d = x^d l$. From now on the symbol $l$ will denote the Blasius length scale at $x^d = x^d l$, the symbol $l_x$ will be
the substitute for the varying scale designated by equation (7.2). For convenience, the non-dimensional coordinates are defined to span the following domain: $x \in [0, L]$ and $y \in [0, y_{\text{max}}]$. With respect to the dimensional coordinates, they are defined as follows:

$$y \equiv \frac{y^d}{l^d \sqrt{2}}, \quad x \equiv \frac{x^d - x^d_1}{l}$$ (8.1)

These coordinates are represented by the blue axes in the overview figure.

The Blasius length scale $l_x$ can be expressed in $x$ in the following way:

$$l_x = \sqrt{\nu^d_ e x^d} = \sqrt{\frac{\nu^d_ e (x^d_1 + x)}{U^d_ e}} = \sqrt{\frac{\nu^d_ e x^d_1}{U^d_ e} + \frac{\nu^d_ e x l^2}{U^d_ e l^2}} = \sqrt{l^2 + l^2 \frac{x}{Re}} = l \sqrt{1 + \frac{x}{Re}}$$ (8.2)

Here, $Re$ is the Reynolds number based on the constant length scale $l$. This means that the definition is the same as displayed in equation (7.3). However, with the exception that here $l$ is constant and evaluated at $x^d = x^d_1$ or equivalently: $x = \frac{x^d - x^d_1}{l} = 0$. Accordingly, the Reynolds number based on $l_x$ yields:

$$Re_{l_x} \equiv \frac{U^d_ e l_x}{\nu^d_ e} = \frac{U^d_ e l}{\nu^d_ e} \sqrt{1 + \frac{x}{Re}} = Re \sqrt{1 + \frac{x}{Re}}$$ (8.3)

Note that at $x^d = x^d_1$ and $x = \frac{x^d - x^d_1}{l} = \frac{l}{l} = L$. It must be mentioned that the above Reynolds number is slightly different with respect to the number $Re_{x^d}$, defined in chapter 7. Evaluating the two previous equations at $x = L$, one finds:

$$l_L = l \sqrt{1 + \frac{L}{Re}}, \quad Re_L \equiv Re_{l_l} = Re \sqrt{1 + \frac{L}{Re}}$$ (8.4)

The non-dimensional coordinate $\eta$, for which the incompressible Blasius boundary layer is self-similar, is:

$$\eta = y^d \sqrt{\frac{U^d_ e}{2 \nu^d_ e x^d}} = \frac{y^d}{l_x \sqrt{2}} \frac{1}{\text{eq. (8.2)}} \frac{1}{l \sqrt{2} \sqrt{1 + \frac{x}{Re}}} = \frac{y}{\sqrt{1 + \frac{x}{Re}}}$$ (8.5)

See White [92], for example. This relation shows that to get from the self-similar coordinate to the variable non-dimensional one, one has to multiply the former with the factor $\frac{Re}{Re_{l_x}} = \sqrt{1 + \frac{x}{Re}}$, which is growing monotonically in $x$. This implies that for larger values of $x$, equal profile values will lie at larger values of $y$; the profiles are developing.

This is an excellent moment to show that the counterargument, in favour of the parallel flow assumption, presented at the end of section 7.1 is justified up to a certain threshold. The dividing factor in equation (8.5) is very close to 1 when $Re \gg x$. Indeed, in practice $Re$ is usually rather large with respect to the streamwise length scale. Moreover, this factor is also large when a short domain is considered in $x$. Therefore, in those situations the parallel flow assumption is vindicated.
Except for the change of the location of the values of the streamwise velocity and temperature, their derivatives change as well. This change can be found via the chain rule:

\[
\frac{\partial Q}{\partial y} = \frac{n \partial Q}{\partial y} d\eta = \frac{1}{\sqrt{1 + \frac{x}{Re}}} \frac{dQ}{d\eta} \frac{\partial^2 Q}{\partial y^2} = \frac{n \partial Q}{\partial y} \frac{d\eta}{\partial y} = \left( \frac{\partial \eta}{\partial y} \right)^2 \frac{d^2 Q}{d\eta^2} = \frac{1}{1 + \frac{x}{Re}} \frac{d^2 Q}{d\eta^2}
\] (8.6)

In addition to the required parameters presented in chapter 7, the parameter \( L \) and the aforementioned equations are all tools required to set up the developing mean flow field. This implies that the accuracy of the mean flow still rests upon the calculation of the profile \( \overline{U} \) as handled in section 7.2.

### 8.2 BiGlobal Simulations

Now the mean flow is elaborated on, the BiGlobal simulations are to be discussed. This is done by handling the used domains and boundary conditions first.

In figure 8.1, it can be seen that in this case a streamwise eigenfunction plane is chosen. Therefore, the streamwise BiGlobal stability equations are to be solved in the interior of the domain. These equations are presented in section C.1. In this chapter the main focus will be on two-dimensional perturbations in the \((x,y)\)-plane; the spanwise wavenumber \( \beta \) is set equal to 0 for all cases. The next thing that has to be treated are the boundary conditions.

As can be seen in figure 8.1, the boundaries in the wall-normal direction are again a solid wall, at \( y = 0 \), and a truncation boundary at \( y = y_{\text{max}} \). The corresponding boundary conditions are taken precisely the same as in the parallel Blasius case. The value(s) of \( y_{\text{max}} \) will be treated later on.

A problem is encountered in the \( x \)-direction. Strictly, one wants to use infinitely far boundaries in these directions. Because this is practically infeasible, the domain is truncated. Unfortunately, there are no boundary conditions that close this truncated problem in a thoroughly physically sound manner. Intrinsically, perturbations are not confined to the truncated domain. It is expected that perturbations will enter and exit the domain at \( x = 0 \) and \( x = L \), respectively. Thus, one cannot simply apply Dirichlet conditions at the in- or outlet.

To be able to close this problem as “tightly” as currently possible, Robin boundary conditions equipped with the Gaster-type transformation are applied. Thereby the same approach is intended as used by Alizard & Robinet [2] and Ehrenstein & Gallaire [23]. To do this properly, several considerations have to be made; the domain size is the first on the list. In order to thoroughly verify the results associated with this mean flow, two domains are set up. One is relatively small and the second rather large. The latter will be elaborated on first.
8.2.1 Set-up: Reference Domain used by Alizard [1]

This domain corresponds to the results presented mainly by Alizard [1] (the treatment of Alizard & Robinet [2] is equivalent and Ehrenstein & Gallaire [23] use a different domain size). First of all, it has to be noted that all references use the displacement thickness $\delta^*$ as the characteristic length scale; the used (incompressible) Blasius length scale $l$ in this treatment is related to it by: $1.7208l = \delta^*$, see Malik [59] for example. This means that all non-dimensional lengths are 1.7208 times larger than the ones reported in their work. Conversely, the Reynolds numbers are 1.7208 times smaller. This yields the parameters as stated in table 8.1.

It can be seen that the references use a relatively long domain. It has to be noted that Ehrenstein & Gallaire [23] use the value $y_{d,max} = 15\delta^*$ for the analysis, whereas Alizard & Robinet [2] use the value $y_{d,max} = 20\delta^*$. The latter is chosen in this case, because this one is argued to be the more accurate option.

Table 8.1: Parameters for the developing Blasius boundary layer replicating the results of Alizard [1].

<table>
<thead>
<tr>
<th>Reference case</th>
<th>Replication case</th>
<th>Gaster relation parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta^* = 1.7208l$</td>
<td>$l$</td>
<td>$\alpha_s</td>
</tr>
<tr>
<td>$Re_{\delta^*} = 610$</td>
<td>$Re = Re_l = 354.49$</td>
<td>$\frac{\partial \omega_s, r}{\partial x, r}</td>
</tr>
<tr>
<td>$L^d = 400\delta^*$</td>
<td>$L^d = 688.32l$</td>
<td>$\frac{Re}{Re_L} \alpha_s</td>
</tr>
<tr>
<td>$y_{d,max} = 20\delta^*$</td>
<td>$y_{d,max} = 34.416l$</td>
<td>$Re_{L,\delta^*} = 1046.2$</td>
</tr>
<tr>
<td>$Re_{L,\delta^*} = 1046.2$</td>
<td>$Re_L = Re_{L,l} = 608.00$</td>
<td>$\omega_0 = 0.049490 \frac{\tau^d}{T}$</td>
</tr>
<tr>
<td>$\omega_0 = 0.08 \frac{\tau^d}{T}$</td>
<td></td>
<td>$\frac{\partial \omega_s, r}{\partial x, r}</td>
</tr>
</tbody>
</table>

Recall that in figure 7.8, a $y_{max}$ convergence plot is shown for the parallel Blasius case. Note that in the simulations corresponding to that figure, the mapping explained in section 5.4 was incorporated. This is not done in this particular case, because the domain is already confined to a relatively thin region along the wall. Despite this slight discrepancy and the fact that the current mean flow is developing, it is assumed that the convergence curve is nearly identical for this case. When slightly extrapolating the information in the figure, it can be seen that using $y_{d,max}^d = 34.416l$ corresponds to an approximate error of $O(10^{-4})$. Therefore, by using this particular value for the truncation parameter $y_{d,max}^d$, it is argued that it does not make any sense to let the eigenvalues converge further than just below $10^{-4}$. Lastly, in this regard, it must be noted that 400 uniformly spaced points will be used to represent the Blasius flow in the streamwise direction.

Now the size of this domain is handled, an important theoretical consideration has to be mentioned regarding the scaling of the frequency $\omega$ and, for example, the wavenumber $\alpha$ in relation to the ultimately used Robin boundary conditions. In the BiGlobal problem,
both are taken to scale with the constant length scale $l$:

$$\omega^d = \omega \frac{U_e^d}{T}, \quad \alpha^d = \frac{\alpha}{l} \tag{8.7}$$

Note that $U_e^d$ is constant over the complete domain. This means that, just like the Reynolds number $Re$, both the dimensional and non-dimensional relatives are global variables that are invariant with respect to arbitrary position changes.

For the evaluation of the parameters in the Robin boundary conditions, the results of local LST simulations at the in- and outlet must be used. These two sets of simulations both obey a different characteristic length scale: $l$ and $l_L$, respectively. This results in a difficulty at the outlet, because $l_L \neq l$. Effectively, the LST results have to be scaled back to yield values that obey the BiGlobal scaling. For the outlet of this particular domain, this approach is best illustrated using table 8.2.

The parameters $Re$ and $\omega_0$ are chosen to correspond to a certain behaviour in the domain. Alizard & Robinet report that $\omega_0$ “is chosen at inflow as well as outflow really near the neutral curve.” This is very reasonable, because of the first order error in equation (2.32); simulating waves that are locally close to the neutral curve corresponds to a more accurate set of boundary conditions. Nevertheless, this criterion should be checked per eigenvector $a posteriori$.

<table>
<thead>
<tr>
<th>BiGlobal Scaling</th>
<th>LST Scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$, $Re$</td>
<td>$\frac{Re_L}{Re} \omega_0$, $Re_L$</td>
</tr>
<tr>
<td>$\frac{Re_L}{Re} \omega_0$</td>
<td>$\frac{Re_L}{Re} \omega_0$, $Re_L$</td>
</tr>
<tr>
<td>$\alpha_{s,r}$</td>
<td>$\frac{Re_L}{Re} \omega_0$, $Re_L$</td>
</tr>
</tbody>
</table>

In table 8.2, it is shown that the parameter $\omega_0$ is unchanged at the end of the process; this value is effectively not influenced by the difference in scaling at the outlet. Unfortunately, this is not the case for the parameters $\alpha_{s,r}$ and $\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}$. These parameters are obtained from the complex dispersion relation through the LST calculations, not by merely scaling the respective variables at the inflow. It has to be noted that the inverse group speed $\frac{\partial \alpha_{s,r}}{\partial \omega_{s,r}}$ is calculated (at constant Reynolds$^2$ as dictated by Gaster’s relation) using fourth order finite differences with a $\omega_{s,r}$ spacing of 0.005 units. This quantity

$^2$It is interesting to note that a function is approximated, whose geometrical representation in fig-
is invariant to changes in the used length scale; the length scales cancel out. Despite this fact, this parameter is still evaluated at \( \left( \frac{R_e L}{R_e L}, \omega_0, R_e L \right) \), not at \( (\omega_0, R_e) \), which yields a slight difference in the ultimate parameters. The resulting parameters are shown in table 8.1. This table completes all required information for the stability calculations on the domain handled by Alizard [1].

Consider the average of the streamwise wavenumbers at the in- and outlet: \( \alpha \approx 0.126 \). The corresponding non-dimensional wavelength is \( \lambda = \frac{2\pi}{0.126} = 49.8 \). Noting that \( L = 400 \), the number of to-be-represented waves in the streamwise direction can be expected to be equal to \( \frac{400}{49.8} \approx 8 \). In the treatment of the parallel supersonic case in chapter 7, it was found that a decent amount of collocation points is required to represent a simple sinusoid; at least 20 to 30. Therefore, to resolve all interesting modes, the corresponding stability calculations require a significant amount of points and, accordingly, memory. In all references, 180 points are used in the streamwise direction, which very roughly means \( \frac{180}{8} = 22.5 \) collocation points per wave. The overall grid consists of 180 \times 45 collocation points, which is far too many to be able to solve the problem using \texttt{eig}. Therefore, it is decided that the Arnoldi algorithm (\texttt{eigs} in \textit{Matlab}) has to be used to obtain solutions that approximate this resolution.

### 8.2.2 Set-up: Small Domain

To carefully analyse the intrinsic features of the current flow, it is decided to set up a smaller domain. Primarily, this domain is set up such that the stability problem can already be thoroughly analysed with the QZ algorithm. Moreover, it will be made such that it complies with all previous findings regarding the parallel mean flow as much as possible. Note that this contrasts with the reference domain. By following this twofold method, the stability of this mean flow is analysed from the point of view of both the literature as well as the already established knowledge.

To achieve all these interests, a small domain has to be designed. This is done in the following way. To attain an as accurate solution as possible, a list of corrupting influences is made, starting with the most demanding feature:

1. Robin boundary conditions with Gaster-type relation
2. Truncation of the domain at \( y_{\text{max}} \)
3. Grid convergence
4. Mean flow accuracy
5. Algorithm precision

It must be noted that it is not claimed that this list is complete, it is merely used as an indication of the chosen path. It is clear that all contributions should be minimized individually to obtain the most accurate solution. Points 4 and 5 are intensively covered in previous sections and are argued to be practically minimal already; their maximal characteristic error is of \( O(10^{-12}) \) for this mean flow.\(^3\) These are therefore discarded from the following discussion.

\(^3\)The accuracy of the Arnoldi algorithm is assessed later on in this chapter.
The largest errors are associated with the Robin boundary conditions, because of the first (in $\omega_{im}$) and second (in $\varepsilon_T$) order errors introduced in equation (2.32). The equation illustrates that these errors are mode dependent. In the considered case, these errors are a posteriori found to be minimally of $O(10^{-5})$ and $O(10^{-7})$, respectively. The reason for this is explained as follows.

As thoroughly elaborated on during its derivation, equation (2.32) fails to satisfy Gaster’s third relation in the set (2.29). The imaginary part of $\alpha_s$ has the wrong sign. The only way to minimize this (qualitative) error, is to simulate waves that have $\alpha_{s,i}$ as close to zero as possible at the boundaries. This implies that the boundaries must be located such that they coincide with the neutral ($\alpha_i = 0$) curve in the $(\omega, Re)$-plane. The references report the same statement; it is expected that they use the same argumentation. When doing this, one has to fix the parameter fraction $\frac{\omega_0}{Re}$, which is equivalent to specifying the dimensional frequency $\omega_d$ in equation (8.7) at the inlet of the domain. This enables one to derive the domain size and the parameters of the Gaster-type transformation. When this is done, one would find zero error in the Robin boundary conditions when the mode’s eigenvalue $\omega$ precisely equals $\omega_0 + 0i$. Unfortunately, the eigenvalues are always located relatively far from this point. This causes the related errors to be most significant in the given list.

One of several ways to minimize this problem with respect to the real direction, is to iterate on the definition of $\omega_0$. One could set it equal to the frequency $\Re\{\omega\}$ of a potentially interesting mode, derive the domain size and Gaster parameters and perform the simulations again. It is conjectured that $\Re\{\omega\} \rightarrow \omega_0$ in this way. Big disadvantages of this approach are that only a single mode is focused on and that the domain is resized at every iteration. Therefore, the formerly explained method is followed. The subsequent design process is described next.

Figure 8.2: Indication of the in- and outlet locations on the neutral curve for the reference/large (red line) and small (blue line) domain.
Figure 8.2 gives a qualitative illustration of the neutral curve in the \((\omega, Re)\)-plane. The red line indicates the path of a perturbation that is about to enter the reference domain. As it moves in the streamwise direction, the Reynolds number increases. At a certain moment, at \(x_d = x_d^L\), the perturbation has zero spatial amplification rate. As explained before, this position is taken to be the inlet of the domain. When travelling further, one again crosses the neutral curve at \(x_d^L = x_d^L\). Analogously, this location corresponds to the outlet.

In all cases, the new domain has to be smaller than the one defined by the references. The right equation in the set (8.4) can be rewritten to yield:

\[
L = Re \left( \left( \frac{Re_L}{Re} \right)^2 - 1 \right) \tag{8.8}
\]

This means that when the circles in figure 8.2 come closer together \((\frac{Re}{Re_L} \rightarrow 1)\), the domain length \(L\) decreases. This means that the slope of the line corresponding to the small domain must be larger than that of the red line.

A few paragraphs back, it was shown that the non-dimensional wavelength is approximately 50 units. It is argued that when the small domain length is taken smaller than this, the streamwise direction should be resolved rather well with a small number of collocation points. To this end, the target value of \(L\) was taken to be 30 units. Note that equation (8.8) cannot be solved for the ratio \(\frac{Re}{Re_L}\). Therefore, a trial and error procedure involving a large number of spatial LST simulations was used to find the corresponding locations on the neutral curve. The result is the blue line in figure 8.2, all parameters that are acquired accordingly are stated in table 8.3.

<table>
<thead>
<tr>
<th>Domain parameters</th>
<th>Gaster relation parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Re = 305.93)</td>
<td>(Re_L)</td>
</tr>
<tr>
<td>(L = 30.689)</td>
<td>(\alpha_s</td>
</tr>
<tr>
<td>(y_{\text{max}} = 104.90)</td>
<td>(\frac{\partial \omega_s}{\partial \alpha_s}</td>
</tr>
<tr>
<td>(y_i = 10)</td>
<td>(\omega_0 = 0.074359)</td>
</tr>
<tr>
<td>(Re_L = 320.90)</td>
<td>(\frac{\partial \omega_s}{\partial \alpha_s}</td>
</tr>
</tbody>
</table>

Several of the parameters stated in the table have to be elaborated on. First of all, it can be seen that the imaginary parts of the \(\alpha\)'s are very small. Furthermore, \(y_{\text{max}}\) is taken such that it corresponds to an error of \(O(10^{-10})\) in figure 7.8, while regarding the length scale \(l_L\). In this case, the mapping elaborated on in section 5.4 is used. The parameter \(y_i\) is set equal to 10 to place half the collocation points under \(y = 10\) as done in chapter 7. It must be noted that about 150 uniformly spaced points were used to
represent the Blasius flow in the streamwise direction. Lastly, the fourth order finite difference method to obtain the inverse of the group speed is again used with an \( \omega_{s,r} \) spacing of 0.005 units, which yields an estimated error of \( O(0.005^4) = O(6.25 \cdot 10^{-10}) \). It is clear that an error of \( O(10^{-10}) \) was allowed to calculate all parameters. This is done to isolate the error due to the Gaster-type transformation as thoroughly as possible. In line with this approach, all parameters are implemented up to the eleventh significant digit in the code.

When comparing tables 8.1 and 8.3, it can be seen that in the case of the reference domain the fraction \( \frac{Re_L}{Re} \approx 1.715 \), while in the case of the small domain \( \frac{Re_L}{Re} \approx 1.049 \). As shown in section 8.1, this parameter indicates “how much” the considered mean flow develops. Logically, the mean flow in the small domain develops rather moderately, while the flow in the reference domain develops significantly. This implies that the results corresponding to the small domain are expected to compare well to LST results. An LST verification is hence performed as a part of the analysis of the results on the small domain.

Going back to the list of possible errors, only point 3 has to be treated. As noted, the Gaster-type transformation introduces a minimal overall error of \( O(10^{-3}) \). This means that one could be satisfied when the calculations are converged up to this level. However, it is tried too assess the converge characteristics irrespective of the errors induced by this transformation. Therefore, it is attempted to let the calculations converge to the next threshold is reached, which is that of \( O(10^{-10}) \).

The number of collocation points in the wall-normal direction is taken to be equal to 90, 100 and 110. In chapter 7, it was shown that \( N_\eta = 90 \) is already enough to yield fully converged solutions. Additional calculations are done with \( N_\eta = 100 \) and 110 to verify this feature in this case. In the streamwise direction, the values 15, 16, 20, 21, 25 and 26 are used, this will be elaborated on later.

### 8.3 Analysis of the Small Domain (SD)

In this section the complete analysis of the small domain is elaborated on. This is done in a logical sequence that is the reverse of the one followed in chapter 7. In that chapter, the expected final result was known \textit{a priori} and had to be reconstructed. Therefore, the results could be extracted in the beginning and the convergence treated afterwards. In this particular section, the results are not known beforehand. Therefore, to analyse every detail optimally, the following approach is used.

First of all, the qualitative structure of the spectrum is investigated in subsection 8.3.1. It is attempted to elaborate on all features and to confine the qualitative domain of interest accordingly. This domain is further restricted quantitatively by considering the convergence of the spectrum and several specific modes in subsection 8.3.2. The modes that are found to be relevant in this case are handled in the subsequent subsection. Thereafter, in subsection 8.3.4, these results are compared to LST simulations for verification purposes. Thereafter, in subsection 8.3.5, the results corresponding to the secondary implementation of the Robin boundary conditions are handled. In sub-
section 8.3.6, the performance of the Arnoldi algorithm is assessed so to be able to use this tool consistently in the remainder of the work. Finally, in subsection 8.3.7, the observed even/odd features are focused on intensively to demonstrate the validity of the to-be-handled results.

8.3.1 SD: Spectrum

A characteristic overview of the physically interesting spectrum is shown in figure 8.3. It can be seen that the spectrum is significantly different than its spanwise relative handled in chapter 7. Regarding the large structures, it can be seen that the spectrum is nearly symmetric around the imaginary axis. This can be explained to be the consequence of the forcing nature of the Robin boundary conditions. Note that in this section, all computations are performed using the primary implementation of these conditions. The secondary implementation will be specifically referred to in subsection 8.3.5.

![Figure 8.3: "Bird’s-eye view" of the physically interesting part of the spectrum. Small domain, $N_x \times N_y = 20 \times 90$.](image-url)
The sign of $\alpha_{s,r}$ is practically equal to the sign of $\mathcal{R}\{\omega\}$; it can be shown that the real part of the Gaster-type transformation \textit{approximately} behaves as an odd function\textsuperscript{4} with respect to the origin. This means that in this particular case for every $\omega$ with large absolute value, there exists an eigenvalue $-\omega^*$ representing the same mode as $\omega$. This similarity is expressed through the fact that it has the same (average) phase speed and equal amplification characteristics.

Furthermore, in figure 8.3, it can be seen that in the lower half-plane a structure of adjacent vertical branches exists. These branches are the equivalents of the vertical branch in the parallel framework. The one nearest to the imaginary axis coincides with the one observed in chapter 7. In this case, higher harmonics are forced via the Robin boundary conditions, which yields the adjacent branches.

As stated by Alizard & Robinet [2], the locations of these branches can be predicted by assuming the parallel flow assumption. In that case, the BiGlobal equations can be reduced to the traditional Orr-Sommerfeld equation. The continuous branch of that equation can be shown to obey the following relation:

$$c_{ph} = \frac{\mathcal{R}\{\omega\}}{\mathcal{R}\{\alpha\}} = \bar{U}_e = 1$$

(8.9)

For more details, see the treatment of Grosch & Salwen [34]. In this particular case, Alizard & Robinet argue that the largest wave that can be represented has wavenumber $\mathcal{R}\{\alpha\} = \frac{2\pi}{L} \approx 0.205$. However, this involves a \textit{complete} wave. Actually, as supported by the ultimate results, the minimal wavenumber corresponds to a wave that has the wavelength $2L$ and $\mathcal{R}\{\alpha\} = \frac{2\pi}{2L} \approx 0.102$ accordingly. All higher harmonics (the waves with wavenumber $\mathcal{R}\{\alpha\} = n \frac{2\pi}{L}$ where $n \in \mathbb{Z}$) and the constant solution in $x$ will be present in the spectrum as well. Equation (8.9) dictates that the corresponding branches will lie approximately at $\mathcal{R}\{\omega\} = n \frac{2\pi}{L}$. These locations are given by the dashed black lines in figure 8.4.

Discarding the color difference for now, it can be seen that the continuous branches lie very close to these lines near the imaginary axis. This reflects the fact that the higher wavenumber modes require more collocation points to converge. At the right of figure 8.4, it can be observed that the branches start to “wander” through the domain. It is clear from figure 8.3 that the branches wander more violently farther from the imaginary axis. This is the consequence of the fact that waves that correspond to those branches are increasingly under-resolved; this will be shown in subsection 8.3.2. For this reason, the eigenmodes of these branches will be excluded from consideration later on. More continuous branches are expected to align themselves when $N_{\xi}$ is increased. It is found that, for sufficiently large $N_{\xi}$, the branch corresponding to $n = \pm \text{round} \left( \frac{N_{\xi}}{2} \right)$ starts to wander, where \text{round} rounds the argument to the nearest even or odd integer when $N_{\eta}$ is even or odd, respectively. This can be explained on the basis of the Nyquist-Shannon sampling theorem, see Shannon [76].

Next to the continuous branches, clear discrete clusters of modes are observed. In figure 8.4, the main diagonal of these modes, commented on in section 7.4.3, is highlighted

\textsuperscript{4}See section 2.3.2 for the definition of an odd function.
with a green line. Also the numerical $P_0$ modes have passed over from the parallel computations; several are pointed out in the figure. It must be noted that the continuous branch that lies in the upper half-plane in the far left part of the figure has a $P_0$ mode as well, this time it is located below the real axis.

The spectrum in figure 8.4 consists of black and red eigenvalues, which correspond to an even and odd value of $N_\xi$, respectively. This behaviour is observed for any parameter combination.\footnote{Even the mapping that maps half the collocation points to the left of the $x$-location $x_i$ was applied to try to destroy a potential symmetry in the grid. This did not affect the current observation.} This is very peculiar, because it seems that only one half of the spectrum is found for a single value of $N_\xi$.

An unproven explanation for this behaviour is based upon the implementation of the Robin boundary conditions, which involves the inherent structure of the pseudo-spectral differentiation matrices. The values of the elements in these matrices are oscillatory in the $x$-direction. This means that the function values at different locations have different (oscillatory) weights. In the case of even and odd $N_\xi$, an even or odd number of half-
Developing Blasius Boundary Layer waves is present, respectively. It is assumed that this causes the even and odd parts of the spectrum to be preferred independently.

If this assumption is correct, this behaviour is inherent to the current discretization method. Therefore, currently the only way to cope with this problem is to perform two simulations with \( N_\xi \) and \( N_\xi + 1 \) and regard the superposition of the two spectra as one solution. It can be seen that the discrete modes appear in both the even and odd spectrum. An important point is that they are near, but significantly distinct for small values of \( N_\xi \). Due to this behaviour, the assessment of the convergence characteristics has to be performed by comparing eigenmodes that correspond to the same “kind” of \( N_\xi \), for the moment. Otherwise, close but significantly distinct solutions are compared. To perform the latter with the QZ algorithm, \( N_\xi \) is set equal to the combinations \{15, 21, 25\} and \{16, 20, 26\}. Later on, in subsection 8.3.7, it will be shown that both even and odd solutions appear to be converging toward the same limit as \( N_\xi \to \infty \).

In the treatment of Alizard & Robinet [2], only the continuous branches are shown that correspond to even \( n \). This is in line with the fact that they use an even number (180) of nodes in the streamwise direction. Ehrenstein & Gallaire [23] do not report the continuous branches at all. It is therefore assumed that the current result is in line with what is reported in the literature.

The last qualitative features in figure 8.4 involve the branches of modes in a very close proximity to the imaginary axis. The nearest is highlighted blue. This branch of discrete modes is referred to as the \( \alpha_0 \) branch from now on, as it corresponds to the real value of \( \omega \) for which the Gaster-type transformation yields \( \alpha_{s,r} = 0 \). Using the parameters from table 8.3 and substituting those into equation (2.32), the corresponding \( \Re\{\omega\} \) can be calculated to be equal to:

\[
\alpha_{s,r} = 0 = \alpha_{s,r}\omega_0 + \frac{1}{\partial \omega_0 / \partial \omega} (\Re\{\omega\} - \omega_0)
\]

\[
0 = 0.18605 + \frac{1}{0.47054} (\Re\{\omega\} - 0.074359)
\]

\[\Rightarrow \Re\{\omega\} = -0.18605 \cdot 0.47054 + 0.074359 \approx -0.013185 \quad \text{(8.10)}\]

Thus the linear function \( \alpha_s = \alpha_s(\omega) \) is zero for \( \Re\{\omega\} = -0.013185 \); the transformation is only approximately, not exactly, odd around the origin. The existence of this branch is related to the fact that the Robin boundary conditions trigger eigenmodes that are constant in the streamwise direction. These modes display oscillatory behaviour in the wall-normal direction; they do not decay as \( y \) increases. Although the modes on this branch have large imaginary parts, the corresponding eigenfunctions will not be considered in further detail.

The other branch is a continuous one and also corresponds to highly oscillatory behaviour in the wall-normal direction, outside of the boundary layer. Some of the modes are oscillating so violently that the question is raised whether this branch consists of numerical modes only. Therefore, the modes of this branch are not considered for a more specific analysis. These structures are inherent to the primary implementation of
the Robin boundary condition. In subsection 8.3.5, the structures corresponding to the secondary condition handled.

Now all qualitative features of the spectrum are handled, the domain of interest in the spectrum can be pinpointed. It is argued that the modes close to the imaginary axis are most converged with respect to the used value of $N_\xi$. It is therefore decided to confine the domain of interest to the interval spanned by the first two continuous branches in either side of the real direction. That is to say, $\Re\{\omega\} \in [-\frac{2\pi}{L}, \frac{2\pi}{L}] = [-0.205, 0.205]$. Due to the aforementioned symmetry, it is decided to focus on the right hand side of this particular domain.

### 8.3.2 SD: Convergence Characteristics w.r.t. $N_\eta$ and $N_\xi$

The next step is to restrict the aforementioned domain of interest using quantitative means. In the first place, this involves the convergence characteristics and, in the second, the domain of validity corresponding to the Gaster-type transformation. By investigating which modes converge properly, the domain of validity can be set and the accompanying error fixed.

![Figure 8.5: Convergence with respect to even $N_\xi$ of the continuous branches to the exact locations $\Re\{\omega\} = \frac{n\pi}{L}$ for even $n$, with $N_\eta = 90$.](image)

Before this is done, the convergence of the continuous branches with respect to $N_\xi$ are shown in figures 8.5 and 8.6, for even and odd values of $N_\xi$, respectively. It can be seen very clearly that the branches converge toward the exact positions as $N_\xi$ increases.
Figure 8.6: Convergence with respect to odd $N_\xi$ of the continuous branches to the exact locations $\Re\{\omega\} = \frac{n\pi}{L}$ for odd $n$, with $N_\eta = 90$.

The branches at the left hand side of the imaginary axis behave similarly. In these figures the relation for the value of $n$ for which the branches are starting to wander, $n = \pm \text{round} \left( \frac{N_\xi}{2} \right)$, is very apparent. In the cases of $N_\xi = 15, 21$ and $25$, for example, it can be seen in the figure that the branches start to diverge when $n = \pm 7, \pm 11$ and $\pm 13$, respectively. This is precisely in line with the relation.

Now, the convergence characteristics can be assessed in the domain of interest. First of all, the convergence with respect to $N_\eta$ is considered. Corresponding convergence plots are shown in figures 8.7 and 8.8 for $N_\xi = 26$ and $25$, respectively. Recalling the results in chapter 7, an important qualitative consideration with respect to these figures involves the identification of the $P_0$ modes. For the respective cases, 4 and 3 such modes are found and indicated in the figures; each corresponds to one of the continuous branches. It can be seen that the mode corresponding to the branch in the upper half-plane diverges towards $-\infty$.

Another important conclusion that can be drawn from the figures is that the discrete spectrum is once again converged very well. When checking the convergence in $N_\eta$ of the ultimately investigated modes (see table 8.4), the convergence error is at most $4.3 \cdot 10^{-8}$. For the used numbering of the modes, see figures 8.7 to 8.10. The precise reason for the numbering will be handled in a moment. Mode III is the only mode that has errors of the latter order. Later on, this mode will be discarded for other reasons. Nearly all other modes have convergence errors of and below $O(10^{-10})$; only in a single case,
mode II displays the error of $\mathcal{O}(10^{-9})$. All in all, the conjecture that the convergence characteristics with respect to $N_\eta$ are analogous to the parallel equivalents, is observed to be reasonably valid. The results are generally not fully converged up to algorithm precision, which is of $\mathcal{O}(10^{-12})$. However, they certainly do comply with the $\mathcal{O}(10^{-10})$ estimate given in subsection 8.2.2, which is based on the error threshold used for the calculation of the parameters for the Gaster-type transformation.\footnote{Note that this involves a different notion than the relatively large non-physical errors introduced by the use of the transformation itself.} Accordingly, it is decided to regard the $N_\eta = 90$ case as sufficiently converged.

The next thing that has to be done is to consider the convergence plots with respect to $N_\xi$. These are shown in figures 8.9 and 8.10. As in the previous figures, all $P_0$ modes and the modes corresponding to the $\alpha_0$ branch are indicated. It can be seen that the figures contain numerous modes that are not converged. It is clear that a higher value of $N_\xi$ than 26 is required to let all discrete modes converge sufficiently. Nevertheless, for the currently used values of $N_\xi$ a few eigenmodes can be found that appear to be converged reasonably. These modes are determined via figures 8.9 and 8.10 and are indicated as mentioned earlier. The features of these “converged” modes are shown in table 8.4. An important point is that, although the differences in the eigenvalues of these modes are relatively small, it is hard to establish whether the modes converge to a certain point in the spectrum. This will be more thoroughly handled in subsection 8.3.7.
Figure 8.8: Convergence with respect to $N_\eta$ of the eigenmodes in the domain of interest, with $N_\xi = 25$.

Table 8.4: Features of the converging modes, maximum converging error is taken over both odd and even $N_\xi$ simulations.

<table>
<thead>
<tr>
<th>Mode label</th>
<th>$\omega_{\text{even} N_\xi}$</th>
<th>$\omega_{\text{odd} N_\xi}$</th>
<th>$\max \Delta \omega_{N_\eta}$</th>
<th>$\max \Delta \omega_{N_\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$0.088 + 0.018i$</td>
<td>$0.087 + 0.016i$</td>
<td>$4.2 \cdot 10^{-11}$</td>
<td>$1.1 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>II</td>
<td>$0.043 - 0.0057i$</td>
<td>$0.046 - 0.0027i$</td>
<td>$1.4 \cdot 10^{-9}$</td>
<td>$3.0 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>III</td>
<td>$0.0057 - 0.0197i$</td>
<td>$0.0054 - 0.0202i$</td>
<td>$4.3 \cdot 10^{-8}$</td>
<td>$5.1 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>IV</td>
<td>$0.0161 - 0.0229i$</td>
<td>$0.0151 - 0.0232i$</td>
<td>$9.2 \cdot 10^{-13}$</td>
<td>$6.6 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>V</td>
<td>$-0.1564 + 0.0557i$</td>
<td>$-0.1566 + 0.0559i$</td>
<td>$2.7 \cdot 10^{-11}$</td>
<td>$2.0 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

It can be seen that these 5 modes are found at qualitatively identical locations within the spectrum in both results corresponding to the simulations with odd and even values of $N_\xi$. Also the eigenfunctions are observed to be qualitatively equal. Unfortunately, the eigenvalues are off relatively far for the currently considered values of $N_\xi$, in the extreme case up to $O(10^{-3})$. It is therefore decided to compare only the modes of one kind of simulations to extract the respective convergence characteristics at the moment. Again, a thorough investigation of the convergence characteristics will be revisited in subsection 8.3.7.

As presented in table 8.4, the convergence errors with respect to $N_\xi$ are relatively large with respect to those with respect to $N_\eta$; the minimal error was found to be of $O(10^{-7})$. Although the errors are not of $O(10^{-10})$, it will be shown later on that the modes are converging. This is the main subject of subsection 8.3.7. For now, as done in the supersonic case handled in chapter 7, the currently found errors are judged to be quite small already.
Therefore, the current results are further analysed in the following subsections. It is assumed that considering the results corresponding to even values of $N_\zeta$ is sufficient. The differences between the even and odd computations are handled in subsection 8.3.7.

### 8.3.3 SD: Domain of Validity & Converged Mode Analysis

All the aforementioned modes can be analysed with respect to the convergence characteristics. However, at this moment the errors induced by the Gaster-type relation have not been considered. Recall that this relation is expanded around the value $\omega = \omega_0 + 0i \approx 0.074$, this point is highlighted with a blue × in the convergence figures 8.7 and 8.10. Any change in the vertical position yields a first order error, whereas a change in the horizontal one corresponds to a second order error. As can be seen, several of the considered modes lie far from the point $\omega_0 + 0i$. Therefore a compromise was made between the number of modes that is considered further and the error introduced by the Gaster-type transformation, to keep the latter reasonable.

It is argued that mode I is most interesting, because it seems to be qualitatively most equivalent to the unstable mode analysed in chapter 7. It is very important to note that mode V is more unstable than mode I. The latter mode lies so far from $\omega_0 + 0i$, however, that it is argued that the Gaster-type transformation is no longer valid for this mode. The eigenmode is converged and does not contain heavy oscillations, but is argued to be polluted with large non-physical errors due to the Gaster-type transformation.
The imaginary part of mode I is rather large.\(^7\) Therefore, it is decided to set the small parameter \(\omega_{im}\) using mode I as a reference. In the odd and even case, it is set equal to 0.017 and 0.018, respectively, slightly larger than the imaginary part of mode I. To determine the parameter \(\varepsilon_T\), the error it causes is equated to that induced by \(\omega_{im}\). Due to its second order influence, see equation (2.32), it yields the same error when \(\varepsilon_T^2 = \omega_{im}\). Thus, \(\varepsilon_T = \sqrt{0.017} \approx 0.130\) in the odd case and \(\varepsilon_T = \sqrt{0.018} \approx 0.134\) in the even case. These parameters define the dimensions of the blue boxes in figures 8.7 to 8.10, with respect to the blue \(\times\). So, in those boxes, the errors introduced by the Gaster-type transformation are of \(\mathcal{O}(\omega_{im})\). It can be seen that in both cases, only modes I and II are included within the box. These will be analysed in some detail in section 8.3.3. The others will be just commented on briefly.

Both modes I and II are Tollmien-Schlichting-like. The wall-normal profiles for even \(N_\xi\) are shown in figures 8.11 and 8.12. This kind of relative profiles collapse onto each other with respect to different \(x\) locations.\(^8\) As noted before, the differences of the eigenfunctions with their odd relatives are considered in subsection 8.3.7. One of the mutual differences of the profiles, corresponding to mode I and II, involves the maximum values of the absolute amplitudes \(\tilde{v}\) and \(\tilde{p}\) relative to \(\tilde{u}\). These are smaller for mode II. Another difference is the location of these maxima. For mode II, these maxima lie at

---

\(^7\)Gaster [24] himself points out that an error of \(\mathcal{O}(10^{-3})\) is usually the case.

\(^8\)The flow is developing so little that the effect is barely distinguishable.
Figure 8.11: Absolute value of the perturbation amplitudes of mode I relative to the maximum of $|\tilde{u}|$ for $N_\xi \times N_\eta = 26 \times 90$. The maximum of $\tilde{u}$ for $N_\xi \times N_\eta = 26 \times 90$.

Figure 8.12: Absolute value of the perturbation amplitudes of mode II relative to the maximum of $|\tilde{u}|$ for $N_\xi \times N_\eta = 26 \times 90$. The maximum of $\tilde{u}$ for $N_\xi \times N_\eta = 26 \times 90$.

A larger value of $y$. The mode II can hence to argued to be more “diffused” out of the boundary layer with respect to mode I.

Using the real part of the Gaster-type transformation at the inlet on $\omega$, the inlet wavenumbers of modes I and II are found to be approximately equal to 0.22 and 0.12, respectively. This means that the modes are expected to approximately have a non-dimensional wave length equal to 28.6 and 52.4 units, respectively. Recalling that the domain length approximately equals 31 units, it can be clearly observed to the left in figures 8.13 and 8.14, that the resulting modes approximately have these wave lengths in the $x$-direction. At the right of these figures, the absolute value of $|\tilde{u}|$ is plot. It can be deduced using these plots, that $\Im\{\omega\}$ and $\Im\{\alpha\}$ have the same sign, which is enforced by the Gaster-type transformation. This means that temporally unstable modes decay in the streamwise direction and vice versa, precisely the opposite with respect to the statement of the third equation of the set (2.29). As explained before, there is no way around this non-physical discrepancy, due to implementation related problems. Every analysed mode displays this kind of behaviour.

The same kind of discrepancy is found in the work of Alizard & Robinet [2] and Ehrenstein & Gallaire [23]. All eigenmodes they consider have negative $\Im\{\omega\}$, while the perturbations grow in the streamwise direction (at the inlet).

The modes III and V are Tollmien-Schlichting-like as well. The corresponding profiles are shown in figures 8.15 and 8.17. Mode III has eigenfunctions that decay relatively slowly when $y \to y_{\text{max}}$. This probably explains the moderate convergence with respect to $N_\eta$. Mode V corresponds to eigenfunctions that decay very strongly in the $x$-direction.

The most characteristic features of this mode are the strange indentation near the peak of the $\tilde{u}$ function and the fact that the absolute value does not go to zero around $y = 3$

---

9The wavenumber of the outlet is not precisely the same, but using the parameter at the inlet yields a good qualitative agreement.
Figure 8.13: Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode I for $N_\xi \times N_\eta = 26 \times 90$.

Mode IV is of a totally different kind than the others. It corresponds to a pure $\tilde{w}$ eigenfunction; all other functions are zero. Note that for this reason, figure 8.16 plots the profiles relative to max $|\tilde{w}|$. It is observed that the corresponding wave moves toward the wall in time.

### 8.3.4 SD: Verification with LST

To be able to check the current results on internal consistency, it is decided to verify them against temporal LST simulations. These simulations are considered at the inlet for several reasons:

1. The (complex) wavenumber $\alpha$ imposed on the BiGlobal simulations is known via the Gaster-type transformation and the BiGlobal eigenvalues.

2. The scaling for the LST problem precisely coincides with that of the BiGlobal problem at the inlet only.

It is argued that this is a very efficient and precise way to reproduce the BiGlobal profiles. A problem is, though, that the imposed wavenumbers are not real, opposed to the standard in the conventional temporal LST approach. Therefore, to check potential differences, both the complex $\alpha$’s and their real parts are imposed separately. All required parameters can be constructed using tables 8.3 and 8.4. The resulting values for $\alpha$ are stated in table 8.5.

The values of $\alpha$ in table 8.5 are supplied to the LST equations stated in appendix A, together with the parameters $\beta = 0$, $Re = 305.93$, $y_{max} = 283$ (note the deviation with
Figure 8.14: Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II for $N_\xi \times N_\eta = 26 \times 90$.

Table 8.5: Input parameters, results and comparison of the LST and BiGlobal eigenvalues for modes I and II.

<table>
<thead>
<tr>
<th>Mode I</th>
<th>Mode II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.21550 + 0.037241i$</td>
<td>$\alpha = 0.12015 - 0.012087i$</td>
</tr>
<tr>
<td>$\omega_{LST} = 0.090098 + 0.017645i$</td>
<td>$\omega_{LST} = 0.044490 - 0.006858i$</td>
</tr>
<tr>
<td>$\omega_{LST} - \omega_{BIG} = 1.8835e-3 + i1.2173e-4$</td>
<td>$\omega_{LST} - \omega_{BIG} = 1.1391e-3 - i1.1707e-3$</td>
</tr>
<tr>
<td>$\omega_{LST} = 0.21550 + 0i$</td>
<td>$\omega_{LST} = 0.12015 + 0i$</td>
</tr>
<tr>
<td>$\omega_{LST} - \omega_{BIG} = 8.5776e-6 + i1.8410e-2$</td>
<td>$\omega_{LST} - \omega_{BIG} = 5.8283e-4 + i4.2179e-2$</td>
</tr>
</tbody>
</table>

$y_{max}$ for the BiGlobal problem) and $y_i = 10$. This yields the values $\omega_{LST}$, which are compared to the BiGlobal values in table 8.4. It is deemed to be sufficient to compare only to the case $N_\xi \times N_\eta = 26 \times 90$. The resulting values and differences are stated in table 8.5. From these results it can already be seen that quite different eigenvalues are obtained for different input parameters. It is quite interesting that when the imaginary part of $\alpha$ is supplied, the imaginary part of $\omega_{LST}$ corresponds rather well to that of the BiGlobal simulations. When this part is dropped, the real part is captured well instead. From the table it is observed that supplying the LST simulations with complex instead of real wavenumbers, i.e. regarding spatially growing or decaying waves instead of just neutral waves, yields significant differences in the temporal growth rates. It can be seen that, for these parameter combinations, approximately half $\Re\{\alpha\}$ is effectively added to $\Im\{\omega\}$ (note that $\partial \omega / \partial \alpha \approx 0.47$). This means that spatially decaying waves are more unstable and...
Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.

Recall that in the LST cases the parallel flow assumption holds. This means that the factor $\frac{Re_L}{Re_c}$ has been assumed to be equal, or very close, to 1. In this particular case, $\frac{Re_L}{Re_c} \approx 1.049$, which is indeed reasonably close to 1. In the BiGlobal framework this assumption is totally alleviated. The errors between the LST and BiGlobal simulations are expected to be of this approximately order as well in this specific case. Note that this spatially growing ones more stable.\textsuperscript{10} Due to the fact that the imaginary parts of the BiGlobal eigenvalues are approximated reasonably, this means that those equations also produce this result. Therefore, the complex nature of the parameter $\alpha$ is argued to be the reason for the large value of $\Im\{\omega\}$ in the BiGlobal spectrum corresponding to the developing mean flow.
Developing Blasius Boundary Layer 159

is not true in general, because when $\frac{Re}{Re}$ becomes large, the errors between the LST and BiGlobal case should remain small locally. Nevertheless, the errors are assumed to be of $O(10^{-2})$ in this case. The errors shown in table 8.5 reflect this argument. Therefore, the LST eigenvalues are judged to be close enough to the BiGlobal ones to proceed with the verification.

Now that the eigenvalues are compared, the eigenfunctions will be considered. Because $\tilde{w} = 0$ for all $y$, this function is discarded from the treatment. Furthermore, although the relative BiGlobal profiles do not depend on $x$ distinguishably, the profiles at the inlet are used. This yields figures 8.38 to 8.43. In these figures zooms are given on interesting spots within the figures. All figures are cut-off at the value of $y$ for which the profiles become indistinguishable. For both cases the profiles overlap very well. It is determined that the maximal point-wise error in the profiles for mode I is at most $3.4 \cdot 10^{-2}$ in the $\Im\{\alpha\} \neq 0$ case and $7.4 \cdot 10^{-2}$ in the $\Im\{\alpha\} = 0$ case. In a moment, it will be explained that the latter case can be dropped. For mode II, for both cases the error is at most $4.6 \cdot 10^{-2}$. All in all, it can be concluded that the results are perfectly in line with the expected $O(10^{-2})$ error.

For mode II, it can be observed that the BiGlobal profiles have a slightly larger value than the LST ones. For this mode, it is not very clear which of the LST cases matches best. After a quantitative investigation, the approximation of the $\Im\{\alpha\} = 0$ case is determined to be best. However, the difference with the $\Im\{\alpha\} \neq 0$ case is off only slightly.

For mode I, the opposite holds. In that case it can be clearly observed that the $\Im\{\alpha\} \neq 0$ case overlaps the BiGlobal profile for a significant part of the domain. The LST profile corresponding to $\Im\{\alpha\} = 0$ seems to continuously undershoot the profile. This yields the large maximal error $7.4 \cdot 10^{-2}$ in this case. It is argued that this is caused by the relatively large value of $\Im\{\alpha\}$ for this mode. It is therefore decided that the $\Im\{\alpha\} = 0$ case for this mode can be ruled out. As a last qualitative statement, it is observed that the BiGlobal profile undershoots the $\Im\{\alpha\} \neq 0$ profile where the latter achieves local extrema.

The excellent agreement with the LST simulations leads to the declaration of internal verification and consistency of the BiGlobal simulations. Due to this successful verification, comparison of the results with those produced by Alizard & Robinet [2] and (equivalently) Ehrenstein & Gallaire [23] are the next step to full (internal and external) verification. Before this is done, several detailed analyses were performed.

First of all, the mise en place of the computations corresponding to the reference domain must be handled. This is done by handling the secondary implementation of the Robin conditions in subsection 8.3.5 and the assessment of the Arnoldi algorithm in subsection 8.3.6. Thereafter, the even and odd computations performed with the Arnoldi algorithm are reported so to intensively investigate the convergence characteristics of the even and odd results corresponding to the small domain. This is mainly done to see whether the corresponding results converge to the same ultimate solution.
It is clear from the currently handled results that the Robin boundary conditions have a major impact on the characteristics of the eigensolutions, due to their forcing nature. From the verification with the LST results, it became clear that the imaginary part of $\alpha_s$, that the condition prescribes, heavily affects the temporal growth rate. Contrary to all other kinds of boundary conditions, the Robin conditions truly force the solution. A specific spectrum seems to be preferred in one case (even $N_\xi$) opposed to another (odd $N_\xi$). Therefore, the suspicion arises whether the Robin conditions are truly physical, despite the fact that the imaginary part of $\alpha_s$ is physically incorrect by the very statement of the Gaster-type transformation (2.32).

Note that the LST verification does not test whether the conditions are physical, because those tests were all subjected to the same input prescribed by the Robin conditions. Another way had to be found to check the hypothesis. Currently, the only other candidate for this involves the incorporation of the secondary form of the Robin boundary conditions as derived at the end of section 2.3.3. In section 8.3.1, the conjecture is posed that the inherent structure of the pseudo-spectral differentiation matrices are locking the solutions in a certain way. By using second, instead of first, order “Robin conditions,” it is expected that the inherent structure is changed.\(^{11}\)

To investigate the general characteristics of these secondary conditions, two small domain test cases were executed. One is performed with $N_\xi \times N_\eta = 15 \times 90$ and the other with $N_\xi \times N_\eta = 16 \times 90$, so to focus on the odd/even effect. The resulting spectra are shown in figure 8.18.

The main result from the investigation is immediately clear. Although not the first but the second order derivatives are involved, the spectra for even and odd $N_\xi$ are still qualitatively distinct. The main difference in this regard, though, is that the preferences have switched. Whereas for the primary implementation the branches corresponding to odd $N_\xi$ (the black continuous branches in figure 8.18) were obtained with odd $N_\xi$, now they are obtained with even $N_\xi$. The same holds the other way around.

Next to this fact, another very distinctive feature has appeared. This feature involves a set of continuous branches along the negative real axis. It starts in a point on the positive real axis, just beyond the origin. A zoom on this point is given in figure 8.18. It can be seen that from the end of the horizontal branches a vertical branch starts off upwards and ultimately bends downwards. These branches are argued to be presented solely due to the negligence of the second order term in equation (2.3.3), in section 2.3.3. There, it was stated that dropping the quadratic term destroyed the near anti-symmetry of the Gaster-type transformation. The new features in the spectrum are argued to be the consequence of this. This can be explained by looking at the “secondary Gaster-type transformation” more closely.

In figure 8.19, the real and imaginary parts of $\alpha_s$ are shown versus the real and imaginary parts of $\omega$. In figure 8.20, a three-dimensional plot of $\alpha_{sr}$ versus the combination

\(^{11}\)Note that, if the conjecture is correct, the best thing one could try to achieve is to break the inherent structure. However, without leaving the Chebyshev collocation discretization technique, this is not possible.
Developing Blasius Boundary Layer

Figure 8.18: Illustration of the spectra with the secondary Robin boundary conditions with $N_\eta = 90$.

of real and imaginary parts of $\omega$ is shown, in addition. In this figure, the red surface corresponds to $\alpha_{s,r}$ calculated via the secondary method (via its squared value), while the blue surface denotes the primary (linear) relation. When investigating figure 8.20 closely, it can be seen that the red and blue surfaces are intersecting and tangent at the point $\omega = \omega_0$. Furthermore, it can be seen that the blue surface nearly crosses the line $(\Re\{\omega\}, \alpha_{s,r}) = (0,0)$, this is more clearly illustrated in the upper left subplot of figure 8.19 (the value is slightly off due to the resolution of the figure). This caused the “anti-symmetric character” of the condition with respect to the imaginary $\omega$ axis. Because in the secondary case, the square of $\alpha_s$ is approximated by its tangent at $\omega = \omega_0$, $\alpha_s^2$ can become negative for certain real values of $\omega$. The corresponding square root behaviour can be clearly seen in the plot to the right in figure 8.19. Unfortunately, the square root’s cusp is located rather near the point $\omega = \omega_0$ in the current case. It is argued that all the modes that have $\Re\{\omega\}$ smaller than the real location of the cusp (and small $\Im\{\omega\}$), can be declared to be non-physical, because the Gaster-type transformation displays incorrect behaviour beyond this point.
The cusp location can be calculated to be equal to:

$$\Re\{\omega\}|_{\alpha_{s,r}} = 0 = \omega_0 - \frac{1}{2} \alpha_{s,r}|_{\omega_0} c|_{\omega_0} \approx 0.03059$$

This point is indicated in figure 8.20 (the value in the figure is again slightly off due to the used resolution to plot the surface). Going back to figure 8.18, it can be seen in the zoom, that the new sets of continuous branches originate very close to the point $\omega = 0.0306$ (slightly to the right of it). An interesting observation is that the horizontal continuous branches have imaginary parts that vary as the square root close to the point $\omega = 0.0306$. I.e., they have identical behaviour as $\Im\{\alpha_s\}$, which is shown to the lower right in figure 8.19. Accordingly, these complete branches are argued to be corrupted with the non-physical features of the secondary relation.

Lastly, it might be argued that the “conventional” vertical continuous branches in the lower half-plane seem to be still rather symmetrical with respect to the imaginary axis. This is not the case however. This can be seen most directly by noticing that the branches that diverge do not do so in a symmetric way any more.

Hitherto, two major conclusions can be drawn with respect to the Robin conditions. Firstly, the odd/even locking behaviour remains present and permutes when the primary or secondary implementation is used. Per definition, this means that the solutions on the continuous branches are dependent on the implementation and hence could be argued to be influenced by non-physical behaviour. Secondly, the secondary form of the Gaster-type transformation behaves correctly in only a very restricted domain of the eigenvalue space around $\omega_0$. The current value is indicated by the blue $\times$ in figure 8.18.
Only solutions that correspond to an eigenvalue with real part larger than the parameter $\Re\{\omega\}_{\alpha^2_{s,r}=0}$ are argued to be physically interesting. Recall that these two strong disadvantages of the conditions have to be added to the non-physical effect introduced by the error in the imaginary part of $\omega$ introduced when deriving equation (2.32). It is clear that the equipment of Robin conditions with the Gaster-type transformation is very tricky and can only be trusted up to a limited extent.

To really assess the differences the secondary implementation of the Robin condition introduces with respect to the primary one, two spectra corresponding to different implementations are superimposed in figure 8.21. Although the secondary condition introduces many non-physical errors, it can be seen that the overall structure of the physically interesting spectrum is preserved. It can be seen that all discrete modes are shifted downward. This is argued to be the result of the rather complicated $\alpha_{s,r}$ distribution shown to the lower right in figure 8.19. Although the differences in the eigenvalues are of $\mathcal{O}(10^{-3})$, the eigenfunctions are observed to be qualitatively the same.

Furthermore, it can be seen that the black continuous branches are converged further toward the dashed lines than the red ones are. This is purely the consequence of the fact that the results corresponding to the secondary implementation have a single collocation point more than the other (recall the odd/even switch).

It is decided that the secondary implementation of the Robin boundary conditions delivers results that are qualitatively matching to those corresponding to the primary one in the domain of interest. Even though the conditions are based upon a form of the Gaster-type transformation that is (even) less consistent with the original statement of the proper Gaster relations. In combination with the significant required memory decrease it yields, it is argued that the conditions can be used to obtain stability information in the region of interest in the subsequent work. Considering modes around the non-physical branches is to be avoided.
8.3.6 SD: Arnoldi Algorithm Assessment

In this section, the performance of the Arnoldi algorithm is assessed by comparing the results with respect to those obtained with the QZ algorithm. This is done by supplying precisely the same system to the algorithm as before. It is assumed that considering one case, \( N_\xi \times N_\eta = 26 \times 90 \), is enough to extract all relevant information.

Initially, several choices have to be made regarding the settings of the algorithm. First of all, one needs to set the amount of to-be-resolved eigenvalues. The algorithm will stop if it has calculated \( N_\omega \) converged eigenvalues. Furthermore, one can specify to search around a specific eigenvalue guess, \( \omega_g \). In this particular case, the interesting eigenvalues are all quite close to the origin. Which means that setting this guess to zero is the proper choice. When the algorithm is provided with a non-zero guess, it requires more time to execute. Therefore, setting a non-zero guess is only useful when continuous branches are clouding the physically interesting eigenvalues; these branches “eat” a significant amount of modes.

Two cases were executed, one resolving 500 and another with 1000 eigenmodes.
Figure 8.22: Comparison of spectra returned by the QZ and Arnoldi algorithms, $N_\xi \times N_\eta = 26 \times 90$.

Note that for the particular test case, the QZ algorithm scales with $(N_\xi N_\eta)^3 = 12.8 \cdot 10^9$, whereas the Arnoldi algorithm scales with $N_\xi N_\eta N_\omega + N_\omega^2 = 1.4 \cdot 10^6$ and $3.3 \cdot 10^6$, respectively, as reported by Lehoucq et al. in the ARPACK users’ guide [52]. This yields an alleviation factor of $10^4$ to $10^5$. The convergence flag returned by the algorithm was 0 in both cases (and in all other performed simulations). This means that the results are converged up to machine precision. The resulting spectra are superimposed on each other in figure 8.22.

The algorithm with $\omega_g$ searches for the eigenvalues of smallest magnitude. Thus for both cases a circle can be drawn that consists of all resolved eigenvalues, these circles can be seen in figure 8.22. Several samples are chosen within these circles to compare the results. The first of these samples corresponds to mode I in table 8.4. A zoom on this mode is given within the figure. The scales of this sub figure are indistinguishable and thus denoted by the abbreviation IS, the averages of the coordinates of the dotted lines are: $\mathcal{R}\{\omega\} = 0.08821452250$ and $\mathcal{I}\{\omega\} = 0.017523366835$. The spacings between the dotted lines in the zoom on figure 8.22 are: $\Delta \mathcal{R}\{\omega\} = 9.0 \cdot 10^{-11}$ and $\Delta \mathcal{I}\{\omega\} = 5.0 \cdot 10^{-11}$. These differences are well within the expected convergence level of $O(10^{-10})$. Therefore, for both cases mode I is represented to the same accuracy as done via the QZ algorithm.

Next to the eigenvalue, the profiles of the eigenfunctions of mode I are checked. Comparisons of all profiles are shown in figures 8.23 to 8.26. In the cases of the eigenfunctions $\tilde{u}$, $\tilde{v}$ and $\tilde{p}$, the profiles are overlapping so closely that they are nearly fully indistinguishable. In a zoom on figure 8.23, the scales are indistinguishable once again.
The averages of the coordinates of the dotted lines are: $\frac{|\tilde{u}|}{\max |\tilde{u}|} = 0.30383712991$ and $y = 4.932073173098725$. The spacings between the dotted lines are: $\Delta(\frac{|\tilde{u}|}{\max |\tilde{u}|}) = 1.2 \cdot 10^{-10}$ and $\Delta y = 5.0 \cdot 10^{-14}$. The maximal point-wise error found between all eigenfunctions (also including $\tilde{w}$) is equal to $3.6 \cdot 10^{-10}$.

In the case of $\tilde{w}$, the profiles corresponding to the Arnoldi algorithm both differ from that obtained with the QZ algorithm. It can be seen that the Arnoldi algorithm has troubles representing the small eigenfunctions, probably even more so with respect to the QZ algorithm. It must be noted, however, that the horizontal axis of the figure scales with $10^{-14}$. Because the profile corresponding to QZ is identically zero, this plot directly gives the error. This is 4 orders less than the maximal encountered error and also far below the algorithm precision of the QZ algorithm. These profiles can therefore be regarded as practically zero. To be very neat, the eigenmode could be refined using the local Newton-Raphson method or local Rayleigh algorithm; the eigenfunction condition number in the $N_\omega = 500$ case is equal to $7.7 \cdot 10^{28}$. In this case this is disregarded, because the obtained results are expected to be very similar to those shown in figure 7.15. The current results are argued to be accurate enough to yield a solid verification of the performance of the algorithm.

The rest of the considered modes in figure 8.22 are denoted by A to D. These modes are chosen such that they lie as close to the boundary of the resolved eigenvalue space as possible. It is deemed to be sufficient to only consider the difference in the eigenvalues. These are reported in table 8.6. It can be seen that the eigenvalues are very close to each other. In this case the difference is even smaller than the expected error of $O(10^{-10})$. The number $N_\omega$ and the location of the eigenvalues with respect to the outer boundary seems to have a negligible influence on the accuracy of the results.
Because the results agree extremely well, at least up to the expected convergence criterion for the QZ algorithm, the Arnoldi algorithm is verified to be at least as precise as QZ algorithm. In the next section this algorithm will therefore be applied to the reference domain, for which an extreme number of collocation points is required.

Table 8.6: Comparison of eigenvalues A to D returned by the QZ and Arnoldi algorithms.

<table>
<thead>
<tr>
<th>Mode label</th>
<th>$\Delta \omega_{N_\omega=500}$</th>
<th>$\Delta \omega_{N_\omega=1000}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$4.666e-12 - i1.831e-12$</td>
<td>$1.868e-12 + i2.62e-13$</td>
</tr>
<tr>
<td>B</td>
<td>$-4.012e-13 + i4.303e-13$</td>
<td>$-1.200e-12 + i5.07e-13$</td>
</tr>
<tr>
<td>C</td>
<td>-</td>
<td>$-5.511e-12 - i1.579e-12$</td>
</tr>
<tr>
<td>D</td>
<td>-</td>
<td>$8.58e-13 + i6.211e-12$</td>
</tr>
</tbody>
</table>

8.3.7 Convergence w.r.t. $N_\xi$ and Odd/Even Behaviour Revisited

In subsection 8.3.2, it was mentioned that using $N_\xi \in \{16, 20, 26\}$ yields small convergence errors, but does not demonstrate that the modes converge to a specific limit. Moreover, the solutions corresponding to the odd values of $N_\xi$ should converge to the same limit. Therefore, it was decided to use the Arnoldi algorithm to be able to increase the number of collocation points in the streamwise direction. The main goal of the current subsection is to show that the mode I and II, see table 8.4, tend toward the same limit. Thereby demonstrating that the discrete modes corresponding to the current mean flow are present in both the computations with odd and even values of $N_\xi$.

The number $N_\xi$ is taken to be an element of the sets $\{41, 55, 71\}$ and $\{40, 50, 70\}$. In section 8.3.2, it was shown that using $N_\eta = 90$ yields sufficiently converged solutions in the $y$-direction. So, this number is kept as it was. The combination $71 \times 90$ is the highest possible resolution before memory issues are encountered.
Figure 8.27: Convergence overview plot of modes I and II, of table 8.4, for increased odd and even values of $N_\xi$, $N_\eta = 90$.

The overall results can be observed in figure 8.27. In this plot, the locations of the modes I and II are shown for odd and even values of $N_\xi$. Ultimately, limiting eigenvalues are determined, which will be handled in a moment. Around these limiting eigenvalues, the $O(\omega_{im})$ error threshold is drawn. It is very important to mention that all solutions, also the ones corresponding to smaller values of $N_\xi$, lie within this threshold. Therefore, the errors of the solutions are less than the expected non-physical error introduced by the Gaster-type transformation. By this statement, it can already be argued that the solutions are converged with respect to the non-physical error the Gaster-type transformation introduces.

Nevertheless, it is important to consider the separate modes more closely, because when $N_\xi \to \infty$ the solutions should converge to the same solution. This is done with figures 8.28 and 8.29. The value of $N_\xi$ is shown along each of the plotted modes. In both cases, it can be seen that the errors for initial values of $N_\xi$ are rather small. However, when $N_\xi$ is increased, in both cases the modes are observed to wander significantly through the spectrum. Moreover, it can be clearly seen that modes corresponding to
odd and even values of $N_\xi$ tend to approach each other. Although the final limits are far from reached, estimates of these limits can be determined. This is currently done by using a \textit{linear extrapolation} of the real and imaginary parts of the last two solutions as parametric functions of the odd and even values of $N_\xi$. The resulting estimates are shown as the green dots in the figures 8.27, 8.28 and 8.29, in the latter ones together with their numerical values.

It is expected that the solutions corresponding to odd and even values of $N_\xi$ ultimately converge towards a region close to these green dots. Several circles are drawn in figures 8.28 and 8.29 to illustrate the converging behaviour; the radii are indicated. Note that these circles appear as ellipses because of the unequal scales of the axes. An interesting point involves the fact that in both figures one “kind” of $N_\xi$ values seems to be converging faster than the other. The actual cause is that the current estimates are quite inaccurate and will definitely move when more computations are performed. Currently, both estimates seem to be biased toward the left side of the domain. It is expected that the solution corresponding to the different kinds of $N_\xi$ converge as fast toward the ultimate limit.

Because the estimates are based on the real and imaginary parts of the eigenvalues as parametric functions of $N_\xi$, estimates for those numbers are retrieved as well. For mode I and II, the worst estimates were 120 and 175, respectively. Again, it must be emphasized that the estimates are based on the linear extrapolation of the last two eigenmodes and therefore are expected to be off from the true values significantly.

At this moment, it is demonstrated that the eigenvalues corresponding to even and odd values of $N_\xi$ approach each other. It is therefore expected that the even and odd discrete modes represent one and the same solution of the problem. It is argued that the last thing that has to be done, so to reinforce the latter statement, is to regard
Figure 8.30: Comparison of eigenfunction profile $|\tilde{u}|/\max|\tilde{u}|$ of mode I for the lowest odd and lowest even value of $N_\xi$, $N_\eta = 90$.

Figure 8.31: Comparison of eigenfunction profile $|\tilde{u}|/\max|\tilde{u}|$ of mode II for the lowest odd and lowest even value of $N_\xi$, $N_\eta = 90$.

The eigenfunctions. First of all, absolute profiles of the even and odd functions in the $y$-direction are shown in figures 8.30 and 8.31; these profiles are located at $x = 0$. The $\tilde{w}$ profiles are omitted, because they are identically zero in both cases.

It is decided to use the smallest values of $N_\xi$, 15 and 16, to show that even for a small number of collocation points the eigenfunctions are very similar. In the case of mode I, it can be seen that the profiles nearly coincide. The red profiles undershoot the extrema of the black profiles only very slightly. When comparing the black and red profiles, the maximal point-wise error is found to be equal to 0.0087. This error is found in the profile corresponding to $|\tilde{u}|$. It is argued to be still of $O(\omega_{im}) = O(0.005)$.

In the case of mode II, the maximal error is found to be equal to 0.065. Again, this error is found in the profile corresponding to $|\tilde{u}|$. This error is no longer of $O(0.005)$. It has to be mentioned that the second largest error is found in the $|\tilde{v}|$ profile and is equal to 0.0090. The latter error is of comparable magnitude of the largest error in mode I. Why the large error in the $|\tilde{u}|$ profile occurs is not clear. It is argued that this is related to the fact that in the case of mode II, the eigenvalues display a relatively large error as well, see figure 8.27.

Figures 8.44 and 8.45 show the similarity for the full two-dimensional eigenfunctions for mode II. An important point is that the left sub figures seem to display a different eigenfunction. The point is that the real part of the eigenfunction is considered, which is invariant with respect to multiplication with -1 (taking into account the standard eigenvector scaling $||\Xi||_2 = 1$). However, when the eigenfunction is reflected in the $\Re\{\tilde{u}\} = 0$ plane, the eigenfunctions remain distinct. It can be seen that this mostly involves the growth of the modes instead of the wavenumber in the streamwise direction.

In figure 8.27, it can be seen that the errors in the eigenvalues is most pronounced in the imaginary parts, which can differ by a factor 2. This factor is directly visible to the
right in figures 8.44 and 8.45. In the case of $N\xi = 15$, corresponding to the eigenvalue with the smallest imaginary part, the eigenfunction has a maximal value at $x = 0$ of 0.86, while in the case of $N\xi = 16$ this value is 0.69. This illustrates that the latter mode grows approximately twice as much as the former one. This is precisely in line with the factor in the imaginary parts of the eigenvalues.

Similar behaviour is observed in the eigenfunctions that correspond to mode I. In that case, the differences are less distinguishable and hence are omitted from consideration. It has to be repeated that this is directly in line with the relatively small errors in the eigenvalues corresponding to mode I, see figure 8.27.

In all cases, it is observed that when $N\xi \to \infty$, the eigenfunctions overlap more and more. Therefore, this reinforces the conclusion that the modes corresponding to odd and even values of $N\xi$ converge toward the same solution.

8.4 Intermediate Conclusions

At this point, the solution corresponding to the small domain are completely analysed. The most important findings involve the fact that the structure of the spectrum is very similar to that of the parallel Blasius case. Adjacent continuous branches are obtained, because these are exited by the Robin boundary conditions. It is found that these branches display an even/odd behaviour with respect to the parameter $N\xi$. When even values of $N\xi$ are set, the even continuous branches are retrieved and vice versa. It is argued that this is a feature of the current discretization technique. It is found that the solutions corresponding to even and odd values of $N\xi$ tend to converge to the same limit as $N\xi \to \infty$.

According to the convergence of the modes with respect to $N\eta$ and $N\xi$ in combination with the domain of validity determined via the errors introduced by the Gaster-type transformation, two Tollmien-Schlichting-like modes are analysed in detail. They are found to compare very well with corresponding LST simulations, that were provided with both the real and complex variants of $\alpha_s$ prescribed by the Gaster-type transformation. The found errors could be related to the error related to the non-parallel nature of the developing mean flow (which is equal to $\frac{ReL}{Re} - 1$).

Thereafter, the results corresponding to the secondary implementation of the Robin boundary conditions are handled. It is observed that these conditions introduce a new kind of non-physical structure in the spectrum. This is the consequence of the fact that the latter conditions are based on a variant of the Gaster-type transformation that is less consistent with respect to the primary implementation. It is therefore advised to exclude the modes that are close to the non-physical structure from the consideration. Despite of this, the results are found to be qualitatively the same. Using these conditions saves a significant amount of required memory. It is decided that these conditions will be used in the analysis of the reference domain.

Because the large domain requires a high resolution, the performance of the Arnoldi algorithm is assessed. As expected, it is found that this algorithm returns results that are as accurate as the results obtained with the QZ algorithm. Moreover, it is found that
the precision is not related to the location of the modes with respect to the considered domain of resolved modes.

## 8.5 Analysis of the Large Domain (LD)

Contrary to the analysis of the small domain, the reference domain has been analysed in the literature. Therefore, the main objective of this section is to retrieve the results of Alizard [1], Alizard & Robinet [2] and Ehrenstein & Gallaire [23] (indirectly). It is argued that this can be done most optimally by mimicking their simulations as closely as possible. The parameters in table 8.1 are derived precisely for this purpose. From now on, the reference domain will also be referred to as the large domain, due to its large streamwise extent.

In all literature, the combination $N_\xi \times N_\eta = 180 \times 45$ is used. This amount of collocation nodes is so large that while executing both the QZ and the Arnoldi algorithms memory problems were encountered. Therefore, it was decided to perform calculations on slightly coarser meshes to see whether those results would converge toward the solutions presented in the literature. In view of the findings in section 8.3, it is decided to consider odd values of $N_\xi$ in combination with the secondary Robin conditions. The former is done to retrieve the even continuous branches so to be able to compare the results to those of Alizard & Robinet. The latter conditions are incorporated to minimize the required memory capacity. The following collocation point combinations were considered:

$$
N_\xi \times N_\eta = \begin{array}{cccc}
117 \times 45 \\
135 \times 38 & 135 \times 45 & 135 \times 51 \\
153 \times 38 & 153 \times 45 \\
181 \times 38 
\end{array}
$$

It can be seen that the maximal total number of collocation points is about 6900. It can be seen that the second column represents calculations with $N_\eta$ equal to that considered in the literature. The first column of computations is included to see the significance of being able to use $N_\xi = 181$. Note that in this case one extra node is taken to yield an odd value again. One extra $135 \times 51$ computation is added in the second row to be able to perform a convergence study with respect to $N_\eta$.

### 8.5.1 LD: Spectrum

To perform as well resolved computations as possible, the Arnoldi algorithm was used. This means that an eigenvalue guess and a number of to-be-resolved eigenvalues have to be specified. First of all, the eigenvalue guess was taken such that the spectrum reported in the literature could be retrieved. The average of the real part of the eigenvalues analysed in the literature was 0.045, which is very close the value of $\omega_0$ reported in table 8.1. Next was the choice of $N_\omega$. In figure 8.32, an overview of a typical spectrum is shown. The part of the spectrum reported in the literature is indicated with the green
Note that in section 8.3.3, a domain of validity was introduced to indicate the domain in which the physical errors induced by the Gaster-type transformation were of reasonable magnitude. Recall that it was set up by arguing that mode I in figure 8.9 was most interesting. From that point on, other modes were analysed that were contained in the resulting domain of validity. In this case, the most interesting modes are those reported in the literature. Therefore, the domain that reported there is considered in this case as well. The largest imaginary part considered (corresponding to mode II, see further on) is equal to $-3.8 \times 10^{-3}$. This means that the modes removed $\sqrt{3.8 \times 10^{-3}} \approx 6.2 \times 10^{-2}$ units in the horizontal direction from $\omega_0$ are still within the domain of validity, i.e. $\Re\{\omega_0\} \in [0.02, 0.14]$. This interval overlaps the domain considered in the references almost perfectly.

In figure 8.32, the overall characteristics of the current solution can be observed. First of all, note that the eigenvalues are shown while scaled with the length scale $\delta^*$. Therefore, the subscript in the figure. This is done so to be able to compare the results to those in the references directly. Several nearly vertical narrow mode branches represent the (even) continuous branches. It can be seen that when taking $N_\eta = 45$, the green domain is chosen such that the part of the branches that is not converged is discarded. Furthermore, the main diagonal discrete mode branch is clearly visible once more.

The same non-physical structure as considered in section 8.3.5 is—unfortunately—
Figure 8.33: Convergence plot with respect to $N_\eta$ of the spectrum corresponding to the large domain, for $N_\xi = 135$.

Figure 8.34: Convergence plot with respect to $N_\xi$ of the spectrum corresponding to the large domain, for $N_\eta = 38$. 
again located in the domain of interest. In this case the cusp of the square root is located at \( \Re\{\omega^d\} = 0.01786 - 0.03073 \frac{\delta^d}{\delta^s} \). This can be calculated as done earlier, now using the parameters reported in table 8.1. As mentioned before, the modes close to and to the left of this structure are argued to be effected by the qualitative errors introduced by the secondary form of the Gaster-type transformation. Therefore, these modes are not considered in the subsequent analysis.

Before considering the spectrum reported in the literature more thoroughly, the seemingly most unstable temporal mode must be commented on. This mode, having the eigenvalue \( \omega = 0.07276 + 0.02342 \), has the most positive imaginary part in the considered area. It is a Tollmien-Schlichting-like mode, that is damped very heavily in the streamwise direction. This is the consequence of the Gaster-type transformation in combination with the large imaginary part of the eigenvalue. The Gaster-type transformation involves a first order physical error with respect to the imaginary part of \( \omega \). This mode is therefore argued to be highly corrupted with this kind of error. Furthermore, this mode is not considered in the references. Therefore, it is currently decided to omit it from the consideration.

At this point, the part of the spectrum handled in the literature can be considered. In the literature, this spectrum is fully confined to the green box illustrated in figure 8.32. It consists of the continuous branches and, next to the main discrete diagonal, of a discrete branch of Tollmien-Schlichting modes. The main focus will be on the latter branch.
When comparing the Tollmien-Schlichting branch with that reported in the literature, one can immediately spot a qualitative difference in figure 8.32. The Tollmien-Schlichting branch lies far higher in the currently obtained spectra. Before this difference will be analysed, it will be first shown that the spectrum shown in figure 8.32 can be argued to be converged. On the way, several other features of the spectrum will be touched upon.

8.5.2 LD: Convergence w.r.t. $N_\xi$ and $N_\eta$

In figure 8.33, the convergence of the spectrum is shown with respect to $N_\eta$. As in the previous cases, the continuous branches are better resolved for larger $N_\eta$. A zoom is given on the Tollmien-Schlichting branch. It can be seen that the modes corresponding to this branch are indistinguishable. The convergence is quantitatively verified for the modes that are ultimately considered. These modes were taken to correspond as closely as possible to those analysed by Alizard [1] and Alizard & Robinet [2] (for the numbering see figure 8.34). The maximal difference between the $N_\eta = 38$ and $N_\eta = 45$ computations is $1.3 \cdot 10^{-4}$, while that between the $N_\eta = 45$ and $N_\eta = 52$ computations is $2.3 \cdot 10^{-5}$ (both differences correspond to mode II). With respect to the aforementioned error introduced by the truncation of the domain at $y_{max}$, of $O(10^{-4})$, the above differences demonstrate that the computations can be declared to be converged when $N_\eta$ is taken to be equal to 45.

In the zoom, the $P_0$ modes can be identified clearly; they are indicated with the red circles. An important statement must be made about the $P_0$ modes close to the imaginary axis. It can be seen that the modes located to the left of the non-physical structure lie rather high with respect to those that lie to the right of it. In figure 8.4, corresponding to the small domain with the primary Robin condition, it was observed that the $P_0$ modes lie on a nearly horizontal line (the real axis in that particular case). In the current case, it can therefore be clearly seen that the spectrum is affected in the region to the left of the non-physical structure. This is therefore an illustration of the fact that caution must be taken in considering this part of the spectrum.

Next, the convergence characteristics with respect to $N_\xi$ are handled. In figures 8.34 and 8.35, convergence plots are shown of the spectra. Note that these figures correspond to the first and second column of the combinations of $N_\xi \times N_\eta$ shown at the beginning of the current section, respectively. Several of the $P_0$ modes are indicated, so to be able to distinguish the Tollmien-Schlichting branch easily. In both figures it seems that the modes wander through the domain rather significantly. However, when quantifying this, the maximal difference in the eigenvalues is found to be $1.2 \cdot 10^{-4}$ and $1.5 \cdot 10^{-4}$ in figures 8.34 and 8.35, respectively. This illustrates that in both cases, the results could be argued to be converged up to the expected precision of $O(10^{-4})$.

In the case of figure 8.34, caution must be assumed regarding the convergence declaration, because the increment of collocation points is not uniform. When this is taken into account, it is in both cases found that mode II converges, but I does not. In defence of the validity of the latter modes, it is argued that the diverging behaviour is still within the limits of the error induced by the truncation parameter $y_{max}$. Therefore, the found Tollmien-Schlichting modes are declared to be converged far enough. Moreover, it is
expected that, ultimately, these modes converge to a certain limit when $N_\xi \to \infty$, as shown in the case of the Tollmien-Schlichting modes in subsection 8.3.7. Accordingly, the currently obtained solutions will be compared with those presented in the literature.

Before this is done, a last statement must be given about figures 8.34 and 8.35. The exact locations of the continuous branches are again indicated by dashed vertical lines. It can be seen, with the exception of the third branch from the origin, all continuous branches seem to be converged very close up to these lines. The third branch is argued to be corrupted by the influences of the secondary form of the Gaster-type transformation.

### 8.5.3 LD: Verification with Alizard [1] and Alizard & Robinet [2]

In his thesis, Alizard [1] reports the spectrum in an isolated and magnified manner with respect to those reported in the paper by Alizard & Robinet [2]. Therefore, the former reference is taken to extract the verification figure: figure 4.9(a) on page 54. Figure 1(a) reported by Alizard & Robinet [2] reports qualitatively the same result, only with slightly different simulation parameters. On top of this figure, the current result, the case $N_\xi \times N_\eta = 153 \times 45$, is superimposed to be able to compare the results. This yields figure 8.36.

![Figure 8.36: Verification plot for the spectrum obtained using $N_\xi \times N_\eta = 153 \times 45$ (black and green x) with the spectrum reported by Alizard [1] (blue ■ and red △, figure 4.9(a) on page 54).](image)

In this figure, the modes I and II are indicated. In the current case, they have the following values:

$$\omega_I = 0.0702 + i0.0004 \quad \omega_{II} = 0.1135 - i0.0038 \quad (8.12)$$

About these modes, Alizard only notes the real values: 0.068 and 0.113, respectively. Modes I and II were chosen to match these values as closely as possible. Note that these modes are located at the qualitatively correct position along the branch.

---

12Mainly the simulation parameters for the mean flow are slightly different. Such as the number of collocation point in the streamwise direction and the convergence criterion in time.
The thing that stands out most, is the difference between the continuous branches. In the current case, neat slightly slanted branches are obtained. In the references by Alizard [1] and Alizard & Robinet [2], two significantly distinct branches are observed. Note, however, that only the even branches are present in the figures reported by these references. It must be mentioned that having two collapsed continuous branches is in line with all previously obtained results. Note, in addition, that the corresponding branch points approximate their exact locations, indicated by the dashed lines, very precisely. The approximation gets worse when one considers parts of the spectrum that are located far away from the point $\omega_0$. This is argued to be logical regarding the physical error introduced by the Gaster-type transformation for distant points in the spectrum.

Furthermore, as mentioned before, it is immediately clear that the currently obtained Tollmien-Schlichting branch is located too high. On top of that, the horizontal locations of the modes are not correct as well; several seem to be shifted toward the right. Lastly, the main discrete diagonal seems to be absent in the references. When consulting figure 4.1(a) on page 45 of Alizard [1], the main diagonal is indicated and it can be seen that several modes that lie on the diagonal are present in the bottom left part of figure 8.36. This demonstrates once more that the currently obtained modes lie higher than those in the literature.

It must be noted that the spectra reported by Ehrenstein & Gallaire [23] are very similar to the references considered up to this point, with the exception of the fact that the continuous branches are not reported in the latter reference. Therefore, all current considerations are applicable to that reference as well. The current results will not be directly compared with those of Ehrenstein & Gallaire, however, because they consider a slightly larger domain ($L = 460$ instead of 400).

The precise reason for all the aforementioned differences is unclear. Explanations could involve slight differences in the parameters for the Gaster-type transformation and the mean flow used in the references. The former parameters are not reported in the references, so no direct conclusion can be drawn in their regard. A comparison of results with different parameters and the primary implementation of the Robin boundary conditions is reported in subsection 8.5.4. The differences in the mean flows are summarized in what follows.

First of all, the references have obtained the mean flow in a stream function-vorticity formulation with a second order semi-implicit Crank-Nicholson discretization in time and a second order finite difference discretization in both spatial directions. Furthermore, specific outflow boundary conditions are used. Alizard [1] and Alizard & Robinet [2] report that the convergence criterion in time is set equal to $10^{-13}$ and $10^{-8}$, respectively. No information is reported with respect to the grid convergence.

In the current case, the Blasius profile is obtained with a fourth order accurate method, see section 7.1, which is subsequently transformed to obtain the profile at the other horizontal locations. The shooting method in combination with the Runge-Kutta method converges to machine precision and grid convergence errors are decreased below $3.9 \cdot 10^{-14}$. Currently, the non-dimensional height $y_{max}^d = 283l$ was used to obtain the mean flow with 10001 integration points, while Alizard [1] and Alizard & Robinet [2] have
used 150 points and \( y_{\text{max}}^d = 25\delta^* \). Therefore, many more points are used to obtain the required \textit{grid convergence}, currently. It might be possible that this causes an important difference.

Furthermore, it can be argued that using \( y_{\text{max}}^d = 25\delta^* \) yields a rather low upper bound of the domain. The continuous branches can be analysed by considering the equations as \( y_{\text{max}} \to \infty \), see the treatment of Balakumar & Malik [7]. The previous considerations with respect to \( y_{\text{max}} \) are therefore argued to be very important features with respect to the reported structure of the continuous branches in the literature. In the treatment of Alizard & Robinet, a convergence plot of the spectrum is shown with \( y_{\text{max}}^d = 20\delta^* \) and \( 25\delta^* \) (figure 14). There no significant difference can spot in the branches.

It must be noted that currently 400 points are used to represent the mean flow in the streamwise direction versus 450 and 500 points used by Alizard & Robinet [2] and Alizard [1], respectively. The corresponding error is argued to be negligible.

When disregarding the non-physical structure induced by the secondary Robin boundary conditions, the most important difference in the current spectrum is the location of the Tollmien-Schlichting branch. Note that the structure of this branch is very similar. Therefore, it is argued that the corresponding eigenfunctions can still be compared with those reported in the literature.

Alizard [1] and Alizard & Robinet [2] both report the streamwise velocity amplitude corresponding to modes I and II as indicated in figure 8.36. Again the figures are taken from the work of Alizard, because those figures are presented in an isolated and magnified manner. Moreover, Alizard reports also the wall-normal and pressure amplitudes corresponding to mode I; all plots are shown in figure 4.2 on page 47 of his treatment. These plots are repeated here in figures 8.47, 8.49, 8.51 and 8.53. Before each of these figures, the corresponding perturbation amplitudes are presented as obtained in the current project. In the latter contour plots, the scaled amplitudes with respect to the maximal absolute value of \( \tilde{u} \) are given. Note that in these figures the coordinates are (again) provided with the subscript \( \delta^* \). This indicates the fact that the coordinates scale with that length scale instead of \( l \). Alizard [1] uses this scale by standard and therefore does not include this subscript. Note furthermore that the complete domain is shown; no part of the domain is trimmed off.

It was found in subsection 8.5.1, that the currently obtained Tollmien-Schlichting branch lies higher than that reported in the literature. Therefore, the current boundary conditions impose less amplifying incoming and outgoing perturbations. This is very clear when figures 8.46, 8.48, 8.50 and 8.52 are compared to those corresponding to the work of Alizard [1]. It can be seen that all amplitudes reported by Alizard enter the domain with nearly zero amplitude. When considering mode I, the amplitudes increase significantly around \( x_{\delta^*} = 300 \). This expected difference is thus evident.

However, when considering the shape of the contours, they are found to be very similar. Of course, arbitrary phase shifts can be allowed in the \( x \)-direction due to the Robin boundary conditions. When allowing this, it can be seen that the shapes of the modes are nearly identical in both cases. On top of this, by simply counting the number of waves, the wavenumber in the \( x \)-direction is found to be approximately identical for
While accounting for the high location of the Tollmien-Schlichting branch, the current flow case is argued to be verified in a “weak” qualitative sense. I.e., regarding the mode shapes and the wavenumbers. It is argued that minor changes in the parameters of the Robin conditions could give insight in what could be causing these differences.

### 8.5.4 LD: Application of Variations of the Robin Conditions

To the former end, it was decided to execute several small, $135 \times 38$, cases to identify the individual influences of specific changes to the input parameters. The changes were chosen as follows:

1. Use the complex values of $\alpha_{s_r|\omega_0}$ instead of $\alpha_{s_r|\omega_0}$, see table 8.1, in the secondary implementation of the Robin conditions
2. Apply the primary implementation of the Robin conditions with the nominal parameters and using \(N_\xi \times N_\eta = 136 \times 38\)

3. Use the values \(2c|_{\omega_0}\) instead of \(c|_{\omega_0}\) in the secondary implementation of the Robin conditions

Here, the first case was argued to be interesting because the Gaster-type transformation uses the complex-valued \(\omega\). If complex values of \(\alpha_s\) are supplied, one could argue that more information is taken into account. Logically, when \(\omega = \omega_0\), \(\alpha_s = \alpha_s|_{\omega_0}\) which is generally complex-valued.

The second case involved the logical consideration of the primary implementation of the Robin conditions. It is important to see whether the secondary implementation does not change the spectrum significantly. Note that \(N_\xi = 138\) is used to obtain the even continuous branches again.

In case number 3, a last attempt was made to lower the Tollmien-Schlichting branch. This was done by noting that \(\alpha_{s,i} = (c|_{\omega_0})^{-1}\omega_{t,i}\). When assuming the nominal spectrum is correct, then when doubling \(c|_{\omega_0}\), \(\omega_{t,i}\) must attain double values as well. In this way, it was attempted to lower at least the parts of the spectrum that are below the real axis. Moreover, when regarding equation (8.11), it can be seen that this case yields a significantly smaller value of \(\Re\{\omega\}|_{\alpha_s^2 = 0}\). Therefore the non-physical structure moves toward the left (in this case out of the domain of interest).

The resulting spectra, together with the nominal case, are stated in figure 8.37. Again \(\omega_0\) is indicated by the blue \(\times\). First of all, it can be seen that case number 1 causes a deviation in the Tollmien-Schlichting branch only in the vicinity of \(\Re\{\omega\} \approx \omega_0\). It is interesting to see that, with a handful of exceptions, the red branch has lowered slightly with respect to the black one. Furthermore, it can be seen that the modes do not suddenly jump upwards for \(\Re\{\omega\}\) slightly larger than \(\omega_0\). The branch seems to be "more natural" in this way. This is argued to be logical, because of the aforementioned reasoning that \(\alpha_s = \alpha_s|_{\omega_0}(\in \mathbb{C})\) when \(\omega = \omega_0\). Despite the latter fact, the red modes do not lie significantly lower than the nominal branch.

In the case number 2, the green modes lie lower when \(\omega < \omega_0\) and higher when \(\omega > \omega_0\). Note that the non-physical structure has disappeared in this case. It lies far to the left of considered part of the spectrum. It can be seen that the secondary implementation of the Robin conditions attenuates the growth rates to the left of \(\omega_0\). Despite of this, the decrease is not significant to reach the branch reported in the literature. It is interesting to notice, though, that the green modes lie very close to the black ones in the vicinity of \(\omega_0\). This shows that the primary and secondary implementations of the conditions compare rather well in this neighbourhood.

In case number 3, the observation is the other way around with respect to case number 2. Note that this is against the expectations; all imaginary parts were expected to be doubled. In this case, the non-physical structure can be seen at the left side of the figure. Although the blue modes lie below the nominal modes around and to the right of \(\omega_0\), this case also does not lower the branch significantly.
Note that all changes made in the three test cases were major. No small perturbations of the conditions were considered. This illustrates that the currently obtained spectra are not different from those in the literature purely due to small errors in the parameters of the Robin boundary conditions. As mentioned in subsection 8.5.3, differences in the mean flow are expected to have caused the currently observed discrepancies. The current results are therefore only “verified” in a weak qualitative sense.

8.6 Developing Blasius Flow Conclusions

In this chapter, the stability of the developing Blasius boundary layer is considered. This is mainly done, because this flow is the natural superior with respect to the parallel Blasius flow. Furthermore, the extra incorporated physics can be scaled, because a short and long domain (in the $x$-direction) corresponds to a weakly and strongly developing case, respectively. Moreover, Alizard [1], Alizard & Robinet [2] and Ehrenstein & Gallaire [23] report on the stability of this flow as well, so that these references could be used to compare the currently obtained results against.

The streamwise boundary conditions form a problem in the current case. No physically closed conditions are available for those boundaries. Therefore, the Robin conditions with the Gaster-type relation are incorporated, which introduce physical errors into the problem. This results in non-physical branches in the spectrum and the need for a domain of validity in which the physical error is argued to be reasonable.

A small domain was designed to be able to calculate solutions that require a moderate resolution. Accordingly, it could be analysed thoroughly and features of different analysis procedures could be tested. Such as the primary and secondary implementations of the Robin conditions and the performance of the Arnoldi algorithm.

The spectrum was found to be qualitatively different for even and odd values of $N_\xi$. Therefore, the Arnoldi algorithm had to be used to generate results with sufficient resolution. These results were used to show that the discrete modes of the distinct solutions converged toward the same limit when $N_\xi \rightarrow \infty$. Hence, it is demonstrated that they represent the same solution.

Furthermore, it was demonstrated that the Arnoldi algorithm returns results that are at least as accurate as those obtained with the QZ algorithm and the primary and secondary implementations of the Robin conditions yielded qualitatively similar results. For a thorough overview of the conclusions with respect to the small domain, see section 8.4.

When the work was performed for the small domain, the considered approaches could be applied to the large domain. The main goal was to retrieve the solutions of Alizard [1]. The secondary Robin conditions were used to minimize the required storage capacity.

When the results were obtained, firstly it was found that both the QZ and Arnoldi algorithms could not handle the $180 \times 45$ resolution used in the literature; memory problems could not be avoided. Therefore, it was tried to see whether solutions on coarser meshes converged to the high resolution result. Indeed, the solutions were found
Developing Blasius Boundary Layer 183
to be converged up to the $O(10^{-4})$ threshold determined by the small value of $y_{\text{max}}$ ($=20$). Therefore, the $153 \times 45$ case was compared against the reference.

Neglecting the presence of the non-physical structure introduced by the secondary Robin boundary conditions, several differences between the currently obtained spectrum and that reported by Alizard [1] were found. First of all, the continuous branches of Alizard are splitting in two, while the current ones form single neat slightly slanted branches (which correspond to two merged branches). It is conjectured that differences in the used mean flows are the cause for this difference; in the parallel mean flow and small domain cases single (merged) branches are obtained in all cases as well.

The second and most important difference is the elevated location of a branch with Tollmien-Schlichting modes. The structure of this branch and its modes are quite similar in the current and reference case. The mode shapes and wavenumbers in the $x$-direction match very well. The significant difference in the imaginary parts of the modes causes a large qualitative difference, however. See figures 8.46 to 8.53. Therefore, the current results on the large domain could only be declared to be verified in a weak qualitative sense.

Using Robin conditions with different parameter combinations and implementations, it was determined that large changes in the parameters of these conditions are not the cause for the location of the current Tollmien-Schlichting branch. It is therefore expected that the differences in the mean flow are responsible for the mismatch. Details of these differences are stated in subsection 8.5.3.
Figure 8.38: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{u}|$ relative to the maximum of $|\tilde{u}|$ of mode I.

Figure 8.39: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{v}|$ relative to the maximum of $|\tilde{u}|$ of mode I.

Figure 8.40: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{p}|$ relative to the maximum of $|\tilde{u}|$ of mode I.
Figure 8.41: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{u}|$ relative to the maximum of $|\tilde{u}|$ of mode II.

Figure 8.42: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{v}|$ relative to the maximum of $|\tilde{u}|$ of mode II.

Figure 8.43: Comparison of BiGlobal with LST eigenfunctions, absolute value $|\tilde{p}|$ relative to the maximum of $|\tilde{u}|$ of mode II.
Figure 8.44: Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II, for $N_\xi \times N_\eta = 15 \times 90$.

Figure 8.45: Real and absolute value of the perturbation amplitude $\tilde{u}$ of mode II, for $N_\xi \times N_\eta = 16 \times 90$. 
Figure 8.46: Perturbation amplitude $\Re\{\tilde{u}\}/\max|\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$.

Figure 8.47: Perturbation amplitude $\Re\{\tilde{u}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(b) on page 47).

Figure 8.48: Perturbation amplitude $\Re\{\tilde{v}\}/\max|\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$.

Figure 8.49: Perturbation amplitude $\Re\{\tilde{v}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(c) on page 47).
Developing Blasius Boundary Layer

Figure 8.50: Perturbation amplitude $\Re\{\tilde{p}\} / \max |\tilde{u}|$ corresponding to mode I with $N_\xi \times N_\eta = 153 \times 45$.

Figure 8.51: Perturbation amplitude $\Re\{\tilde{p}\}$ corresponding to mode I reported by Alizard [1] (figure 4.2(d) on page 47).

Figure 8.52: Contour plot of $\Re\{\tilde{u}\} / \max |\tilde{u}|$ corresponding to mode II with $N_\xi \times N_\eta = 153 \times 45$.

Figure 8.53: Perturbation amplitude $\Re\{\tilde{u}\}$ corresponding to mode II reported by Alizard [1] (figure 4.2(a) on page 47).
Chapter 9

Complex Lamellar Bidirectional Vortex

This chapter involves the BiGlobal stability analysis of the incompressible bidirectional vortex. Several analytical model solutions for this mean flow have been derived by Batterson [10].\(^1\) One of these solutions involves a complex lamellar flow field (see section 3.41 of Aris [5] for more details) to which viscous corrections are applied. An important feature of the bidirectional vortex is that it involves cylindrical coordinates. Verification of the current tools with respect to a curvilinear coordinate system is regarded to be very interesting, because in this way all power of the derivation and implementation tools is used.

The current mean flow is extremely complicated with respect to the flows handled in chapters 7 and 8. Therefore, the main goal of the current analysis is to retrieve the results presented in the literature. Batterson reports simulations that use the combination \(N_\xi \times N_\eta = 50 \times 50\) and that “the spectrum appears to lack convergence,” see page 215 of his work. In the current case, this observation is retrieved. Therefore, next to the main goal, an attempt is made to find a solution that is converged.

In the first two sections, the mean flow and the set-up of the stability problem are elaborated on. This is followed by section 9.3, in which the results are compared with those of Batterson. Thereafter, the convergence characteristics of the spectrum obtained with the QZ algorithm are handled in section 9.4. The chapter is concluded in section 9.5.

9.1 Mean Flow Set-up and Characteristics

The mean flow considered in this chapter is derived by Batterson [10]. The flow is defined in a cylindrical coordinate system \((r, \theta, z)\) of which the \(z\)-axis coincides with the centerline of a rocket’s combustion chamber. The mean velocity vector corresponding to the coordinate system is denoted, in line with the remainder of the treatment, by

\(^1\)The accompanying conference papers are references [11, 12].
Figure 9.1: Illustration of the bidirectional vortex flow, flowing in at \( r = 1 \) at the top, \( z = \frac{L}{a} \), moving down and up the chamber and exiting the chamber at the top. (Digital version is animated)
(\(U, V, W\)). First, Batterson derives the inviscid version of the mean flow solution, by solving the steady Euler equations on the non-dimensional domain \((r, z) \in [0, 1] \times [0, L/a]\), with the boundary conditions:

\[
\begin{align*}
U\big|_{r=0} &= 0, \quad U\big|_{r=1} = 0, \quad W\big|_{z=0} = 0, \quad (9.1) \\
V\big|_{r=1} &= 1 \quad \text{and} \quad 2\pi \int_0^{r_{exit}} W\big|_{z=L/a} rdr = Q_i \quad (9.2)
\end{align*}
\]

Here, \(L\) is the chamber length in the \(z\)-direction, \(a\) is the radius of the chamber, \(r_{exit}\) is the exit port radius and \(Q_i\) is the non-dimensional total volume flow that is led through the chamber. The length scale \(a\) is used to make the variables dimensionless. The conditions imply that at \(r = 0, 1\) and \(z = 0\) (also referred to as the headwall) the no penetration condition is enforced, while at \(z = L/a\) the flow exiting the chamber is prescribed. The flow resulting from this set-up is shown in figure 9.1.

Batterson solves the Euler equations with these conditions and reports the inviscid solution as follows:

\[
\begin{align*}
U &= -\kappa \frac{\sin(\pi r^2)}{r} \\
V &= \frac{1}{r} \\
W &= 2\pi \kappa z \cos(\pi r^2)
\end{align*}
\]

where:\n
\[
\kappa = \frac{Q_i a}{2\pi L} \quad (9.4)
\]

Here, \(\kappa\) is referred to as the inlet parameter. Note that through setting \(W\big|_{r_{exit}} = 0\), one obtains \(r_{exit} = \sqrt{2}\).

In the current context, one is specifically interested in the viscous case, because the resulting boundary layers are the primary source of disturbances. To this end, Batterson derives viscous corrections for the boundaries \(r = 0\) and \(r = 1\), assuming \(V\) to be large. With the resulting viscous corrections, the velocities become:

\[
\begin{align*}
U &= -\kappa \frac{\sin(\pi r^2)}{r} \left( 1 - e^{-\frac{V\alpha}{4} (1-r^2)} \right) \\
V &= \frac{1}{r} \left( 1 - e^{-\frac{Vz}{4} - e^{-\frac{V\alpha}{4} (1-r^2)}} \right) \quad (9.5) \\
W &= 2\pi \kappa z \cos(\pi r^2) \left( 1 - e^{-\frac{V\alpha}{4} (1-r^2)} \right)
\end{align*}
\]

where:\n
\[
\begin{align*}
V &\equiv 2\pi \kappa Re \\
Re &\equiv \frac{U^d d}{\nu d} \\
\alpha &\equiv \left( 1 + \frac{\pi^2}{6} \right)
\end{align*}
\]

Here, \(V\) is the so called vortex Reynolds number and \(\nu^d\) is the kinematic viscosity. The characteristic velocity \(U^d_e\), with which the velocities are made dimensionless, is defined
via the dimensional passed volume flow $Q_i^d$ and the corresponding inflow injection area $A_i^d$, see Majdalani & Chiaverini [58]:

$$\overline{U}_c^d \equiv \frac{Q_i^d}{A_i^d}$$

(9.6)

Note that no viscous corrections are given for the headwall. Via personal communication, Batterson comments that no closed-form correction exists that is not oversimplified. Therefore, the boundary layer at the headwall was discarded in his treatment. The main goal of the current work is to replicate Batterson’s results. Therefore this boundary is treated the same way. In addition to the velocities noted in equation (9.5), Batterson reports a relation for the pressure (difference). Because the incompressible BiGlobal stability equations do not require the mean pressure as input, it is not considered here.

It is clear that the solution is in this case given in closed form. This implies that at any location in the domain the mean flow variables are accurate up to machine precision. Therefore, no intensive grid refinement strategy has to be followed to assure the latter in this case.

There exist some practical problems with the second relation stated in equation (9.5). Basically, when constructing the composite expression, the contributions of the expansions for the wall and core of the vortex conflict with each other. Note that:

$$\lim_{r \to 0} \overline{V} = \lim_{r \to 0} -\frac{e^{-\frac{V_{th}}{r}}}{r} \quad \lim_{r \to 1} \overline{V} = e^{-\frac{V}{r}} \neq 0$$
Theoretically, noting that $V$ is a large parameter in this problem, these terms are transcendentally small, see Kevorkian & Cole [48] or Holmes [42]. Therefore, these terms are negligibly small when $V$ is assumed to be large. In Matlab, however, these kinds of terms have to be forced to zero manually. When doing so, this yields figures 9.2 to 9.5.

Note that this flow has several very interesting features. It contains a solid wall boundary layer and a boundary layer corresponding to the vortex core. Furthermore, it contains a moderate shear layer around $r = r_{exit}$. All these phenomena interact and are expected to have very complex stability characteristics.

Furthermore, recall from chapters 7 and 8 that approximately 45 collocation nodes must be placed within the boundary layer to yield converged eigenvalues. In figure 9.2, 120 nodes were used to plot the mean velocity component on the mapped Chebyshev grid. Approximately 10 to 14 nodes can be argued to lie inside the boundary layer at $r = 1$. About 20 to 30 are placed within the vortex core boundary layer. Currently implemented mapping techniques can map collocation points into the boundary layer at $r = 1$, however this will reduce the resolution of the layer at $r = 0$. Ideally, one requires resolution around both boundary layers and the shear layer. It is highly probable that multi-domain spectral collocation yields the most feasible solution for this purpose in contrast to convoluted mapping techniques.

Batterson uses no refinement mapping at all, just a linear transformation is used to switch from the computational $[-1,1] \times [-1,1]$ space to the non-dimensional physical $[0,1] \times [0, \frac{L_a}{a}]$. Therefore, it is decided to stick with this set-up, although convergence problems are expected/dreaded.

\[2\]This depends on the definition and is thus only an approximate statement.
9.2 BiGlobal Simulations

Now that the mean flow set-up is handled, the set-up of the BiGlobal simulations are next in line. First of all, the considered domain is precisely the same as for the mean flow. Due to the axisymmetric nature of the latter, the eigenfunction plane is chosen to lie in the \((r, z)\)-plane. This is illustrated in figure 9.1. In this case, the coordinate system is cylindrical. Therefore, the corresponding incompressible BiGlobal stability equations, reported in section C.2, are used.

The boundary conditions for the stability problem are copied from Batterson’s work so to replicate his results as well as possible. At the sidewall, \(r = 1\), the no-slip conditions are applied:
\[
\tilde{u} = \tilde{v} = \tilde{w} = 0 \quad (9.7)
\]
Furthermore, it is assumed that no velocity perturbations enter through the headwall, entrance or exit, thus at \(z = 0\) and \(\frac{L}{a}\) the conditions (9.7) are applied as well. Batterson refers to these kinds of boundaries as being “acoustically closed.” With the same reasoning, he therefore defines the pressure boundary conditions at \(r = 1\), \(z = 0\) and \(\frac{L}{a}\) as:
\[
\frac{\partial \tilde{p}}{\partial r} = 0 \quad \text{at } r = 1 \quad (9.8)
\]
\[
\frac{\partial \tilde{p}}{\partial z} = 0 \quad \text{at } z = 0 \text{ and } \frac{L}{a} \quad (9.9)
\]
Regarding the approach followed in the current treatment, these conditions (9.8) and (9.9) deviate. Conventionally, one should apply the compatibility conditions for the pressure at these boundaries.
Lastly, the conditions at $r = 0$ have to be described. As referred to by Batterson, this boundary is “asymmetric,” or more precisely: anti-symmetric. This is required for cases in which the circumferential wavenumber $q \neq 0$. According to Batterson, this means that all variables around this axis have to be equipped with Dirichlet boundary conditions. Thus:

$$\tilde{u} = \tilde{v} = \tilde{w} = \tilde{p} = 0 \quad \text{at } r = 0 \quad (9.10)$$

However, in figure 2.9 it can be seen that the $\tilde{v}$ component should actually be even around $r = 0$ and thus a Neumann condition should be applied to appropriately simulate the anti-symmetric boundary. See section 2.3.2 for a discussion on this. Despite of this, the Dirichlet condition is still incorporated so to be able to replicate Batterson’s results. All boundary conditions are summarized in figure 9.1. Note that they are accompanied with dashed lines that point to the corresponding boundary. The other labels correspond to the boundary conditions on the mean flow.

![Verification plot for the spectrum obtained using $N_{\xi} \times N_{\eta} = 50 \times 50$ (blue •) with the spectrum reported by Batterson [10] (black □, figure 6.4 on page 220).](image)

The current flow is the only flow in this treatment that is completely bounded; the Blasius cases are all defined on a domain that is at least semi-infinite in one direction. One could argue that all the physics are captured when the domain is well resolved at the appropriate places. As mentioned before, this is not very easy, because multiple sources of potential instabilities exist. To that end, Batterson’s approach is again copied; the domain is covered with the Chebyshev grid without any refinement transformation.
Lastly, the used values of the aforementioned parameters are:

\[
\frac{L}{a} = 2, \quad \kappa = 0.1, \quad Re = 10000 \quad \text{and} \quad q = 1 \quad (9.11)
\]

These parameters are set such that corresponding results can be compared to those obtained by Batterson; for this particular combination he reports spectra and the eigenfunctions corresponding to a single mode.

This yields all information to execute computations using the QZ algorithm. First of all, the parameter combination \((N_\xi, N_\eta) = (50, 50)\) is executed to yield results that are directly comparable with those of Batterson. In addition, the ordered pairs \((N_\xi, N_\eta) = (38, 50), (62, 50), (50, 62)\) and \((50, 75)\) are executed to assess the global convergence characteristics in the \(r\) and \(z\) directions. Note that \(N_\xi\) is the number of collocations points in the \(z\)-direction, whereas \(N_\eta\) denotes this number in the wall-normal \(r\)-direction.

### 9.3 Verification with Batterson’s [10] Results

On page 220, Batterson reports the spectrum which corresponds to the currently used parameters. The spectrum obtained from the \(50 \times 50\) replication simulation compares very well to this figure. The current result is superimposed on Batterson’s plot (see the caption of the figure for the full reference) in figure 9.6. Note that all blue dots are within the black squares. Therefore, the spectrum is deemed to be verified.

The next level of the verification involves an eigenmode. Batterson reports that he has considered \(\omega = 0.2178 + 0.2940i\). In the current spectrum, the following approximate (rounding the fifth decimal) value is found:

\[
\omega = 0.21777 + 0.29403i \quad (9.12)
\]

This value directly corresponds to the value reported by Batterson. Lastly, the eigenfunctions are considered individually.

Batterson reports that the following is plotted in figure 6.7 on page 222: “the magnitude of \(u_r, u_\theta, u_z,\) and \(p\) for the first unstable eigenvalue.” Translating to the current case, these variables correspond to \(\tilde{u}, \tilde{v}, \tilde{w}\) and \(\tilde{p}\), respectively. With “magnitude” usually the absolute value is meant. However, Batterson plots negative values in figure 6.7. Therefore, the current results were compared until the correct quantity was found that matched. This quantity turned out to involve the negative of the imaginary part of the amplitudes: \(-\Im\{\tilde{q}\}\). The contours of these quantities computed currently are shown in figures 9.13, 9.15, 9.17 and 9.19. Each of these figures is accompanied with the one presented by Batterson in figure 6.7.

It can be seen that the eigenfunctions match very closely. It is not reported how the functions are scaled. Therefore, the only thing that could be done in this respect was to set the colorbar so to correspond to the values Batterson uses. In the resulting figures this yields small differences with respect to the references. These can be argued to be the consequence of slight shifts of the contours. Therefore, these small changes
are argued to be negligible. Accordingly, the current test case can be declared to be thoroughly verified. An important statement must be made regarding the shape of the amplitude $\tilde{v}$ around the corner $(r, z) = (1, 0)$. It can be seen that there the mode is very fluctuant, the structure has a wavelength very close to the collocation node separation. Usually, modes that display this kind of behaviour are apt to be labelled spurious. For that reason caution is advised when interpreting this mode as physical.

9.4 Convergence Assessment

The spectra corresponding to the aforementioned combinations of $N_\xi$ and $N_\eta$ have been calculated and superimposed upon each other to most conveniently indicate the global overview of the convergence problem with respect to the current test case. The results are shown in figures 9.7 and 9.8.
The first thing that can be seen is that the spectra are immensely complicated. Several contingent large structures can be identified, but when zooming in to distinguish individual modes, the “micro-structure” is never consistently converging. This makes pinpointing “the most important” individual eigenmode impossible. In private communication, Batterson explained that his argumentation to pick the mode that is considered in section 9.3, was that this mode had the smallest eigenvalue in absolute sense. He reasoned that this kind of mode should be converged most of all, because he expected that the absolutely smallest eigenvalue would correspond to the simplest mode structure in space; i.e. the one with the smallest wavenumber. Note that in the case of the developing Blasius boundary layer, a dispersion relation exists that reinforces this statement; the Gaster relations provide a linear relation between $\Re\{\omega\}$ and $\Re\{\alpha_s\}$. Also, the $N_\xi$ convergence characteristics for that case reinforce it.

However, the Gaster relations are only valid for a very restricted part of the eigenvalue space. Furthermore, in the parallel Blasius boundary layer case, higher modes were found to be clustered in a region close to their lower mode relatives. In turn, these clusters
are found at multiple locations within the spectrum. Therefore, in general it cannot be said that absolutely small temporal eigenvalues correspond to spatially mild structures, corresponding to low wavenumbers.

Furthermore, it can be clearly seen in the figures that the spectra are asymmetric with respect to the imaginary axis. Unlike the case of the developing Blasius boundary layer, no forcing mechanisms are currently present that literally enforce the (near) symmetry of the spectrum.

When considering the figures for true convergence characteristics, one can spot a lot of overlapping structures in both of them. In figure 9.8, for example, the outer branches of the main “×” structure around the origin seems to be converged rather well. In figure 9.7, however, these structures move outward rather far. Conversely, it seems that figure 9.7 contains converged branches to the right of the main structure, also notice the small “T” structure at the bottom of the figure. It can be seen that these structures change vigorously when looking them up in figure 9.8 (the red and black T-shaped structures have moved far out of the considered domain). The convergence
characteristics seem to be interchanging when considering internal structures, so the main ×-shaped structure cannot be argued to be converged with \( N_\eta \) in its entirety. So, contrary to all previously considered cases, it is very hard to globally pinpoint structures that seem to be converged already. It is expected that many more collocation points are required to acquire a qualitatively globally converged spectrum. Batterson reports the same conclusion:

“[The spectrum convergence figure (6.1 in this treatment)] does not show the precise convergence achieved with the one-dimensional approach. Due to computational and run-time limitations, a complete parametric study with significantly more collocation points is unfeasible at the present. Nonetheless, even though the spectrum appears to lack convergence, the waveforms predicted by the eigenfunctions are convergent.”

As a reaction to this statement, it is currently argued that it is very hard to see which modes are convergent in the reported spectra, because it is very hard to track them
properly. It is clear that certain structures converge for a constant number of nodes in a certain direction, but not when the number of nodes in both directions is increased. Moreover, it is interesting to exemplify that the T-shaped structure seems to converge when $N_\xi$ is increased, but the complete structure diverges when $N_\eta$ is increased. A similar convergence behaviour is observed for the notorious $P_0$ modes in the Blasius cases.

At this moment a choice has to be made regarding the mean flow. The current test case has a very high Reynolds number, which yields thin boundary layers. It is expected that currently their influence on the stability characteristics might be largely absent, because they are rather badly resolved. Furthermore, several of the boundary conditions have to be changed to be consistent with the theory and the remainder of the cases presented in the thesis. It must be noted that incorrect boundary conditions often lead to spurious modes. Requiring full convergence of the complete spectrum is argued to be far too ambitious. A significant, if not extreme, amount of additional collocation nodes is required to accomplish this; at least the boundary layers must be resolved more thoroughly. In the treatment of the Blasius cases, it was found that about 45 collocation points within the layers are needed to accurately take into account their behaviour. Batterson reports that with the current number of collocation points about 3 nodes are inside the solid wall boundary layer.

When changing the set-up (by using a much smaller Reynolds number, for example), one might achieve reasonable convergence with the currently used number of collocation points and possible non-physical features might be filtered out. This is deemed to be unwise, however, because this will cause the analysis to completely divert from Batterson’s reported solutions. It is decided to stick with the current mean flow and to try to obtain the converged relatives of Batterson’s solutions, which are expected to be located in the neighbourhood of the origin.

Before analysing the local convergence characteristics more intensively in the direct neighbourhood of the origin, several features can be extracted from the global overviews already. In figure 9.7, it can be seen that the branches of the $\times$-shaped structure move outwards with increasing $N_\xi$. It can be seen that the pace in which they move outwards is slowing down. This can be seen as a hint to the fact that the overall structure ultimately converges, although many more better resolved simulations should be done to really demonstrate this.

Zooms on the first internal branches of the $\times$-shaped structure are shown in figures 9.9 and 9.10. As mentioned before, although the outer branches are not very well converged with respect to $N_\xi$, some internal structures do seem to be converged. Several examples of these structures are indicated in figure 9.9. In turn, the menacing thing is that most of these particular structures are shifting with respect to $N_\eta$, as can be observed in figure 9.10. One qualitative exception involves the structures indicated with the magenta ellipse; these structures seem to stay put at their approximate locations in both figures. In this regard, it must be mentioned that the scales in figure 9.9 are deceptive, the structure still moves about 10 units in the real direction. Nevertheless, it can be argued that this is one of the structures that is most converged in both $N_\eta$. 
and $N_\xi$ in the neighbourhood of the origin. Lastly, note that nothing can be said about the convergence characteristics of the modes near the origin, because they are so closely spaced that one cannot distinguish them properly.

The next zoom on the origin is shown in figure 9.11 and 9.12. This is where the lack of convergence of the spectrum really becomes visible. In both plots, a cloud of modes of different colors can be clearly seen. There seems to be no inherent structure (yet); no specific areas exist where the modes overlap. Therefore, it can be concluded safely that Batterson’s solution is far from converged. Moreover, his cited statement about the convergent nature of the modes is doubted seriously. In figure 6.1 of his treatment, Batterson varies $N_\xi$ and $N_\eta$ simultaneously with 10 units, similarly as done here. It can be seen from figures 9.11 and 9.12 that it is impossible to track the path of any mode. Therefore it is equally impossible to say whether a certain mode is convergent or divergent.

A way to find converged modes is proposed as follows. First of all, very high resolution calculations should be performed, possibly with the Arnoldi algorithm. A possible start could be made in the direction neighbourhood around the origin. It is recommended to vary the number of collocation points very gradually, just 2 to 3 more or less. In that way one might be able to trace the eigenmodes through the eigenvalue space and determine whether they are convergent. For now, the fact that Batterson’s solution is retrieved precisely is deemed as an extra insurance that the implementation of the current code is identical to that of Batterson.
9.5 Complex Lamellar Bidirectional Vortex Conclusions

In this chapter, the stability of the complex lamellar bidirectional vortex is considered. This mean flow is considered because it is highly two-dimensional and involves cylindrical coordinates. This is very interesting, because this mean flow requires all power of the derivation, implementation and simulation tools. Verifying with respect to this case therefore truly tests the capabilities of the tools. Because of the complexity of the flow, the main objective is to retrieve the results presented by Batterson [10].

When considering the mean flow set-up, boundary layers are considered at the sidewall and the vortex core, but not at the headwall. Via personal communication, Batterson explains that for this boundary no closed-form viscous correction exists that is not oversimplified. Therefore, the boundary layer at this wall is discarded. No transformations are used to map the collocation points into the boundary layer. With Batterson’s highest resolution, 3 nodes are located in the layer therefore. As apparent from chapters 7 and 8 about 45 collocation nodes are required to resolve the layers appropriately. Batterson assumes “acoustically closed” boundaries at the headwall, entrance and exit, i.e. no perturbations can come in or go out. In addition, Neumann conditions are specified for the pressure instead of the compatibility equations. Lastly, it is assumed that the vortex core is an “asymmetric” boundary, allowing the circumferential wavenumber $q$ to be non-zero. Batterson argued that all amplitudes have to be set equal to zero. However, in section 2.3.2, it is illustrated that one should use a Neumann condition in the amplitude $\tilde{v}$, to correctly set an anti-symmetric boundary. Although a lot of assumptions are made and inconsistent settings are used with respect to the other simulations in the current treatment, Batterson’s approach is fully replicated. This is done, so to be able to retrieve his solution as accurately as possible.

When the replicating solution is obtained, it compares very well with Batterson’s results. The spectrum overlaps very precisely, the reported eigenvalue is obtained to all digits correctly and even the eigenfunctions can be argued to be found nearly perfectly. See figures 9.13 to 9.20. The only difference in the contour plots of the eigenfunctions is related to the fact that Batterson does not scale the functions; just the raw functions are plot. Therefore, it is hard to compare them up to high precision. Because slight shifts in the contours prohibit this. Regarding the upcoming statements, it is argued, though, that the eigenmode is retrieved up to sufficiently high precision.

Because Batterson reports that the spectrum is not converged yet, it is attempted to find a solution that is converged. Several combinations of collocation points were used and convergence plots of the spectrum are considered. It is found that specific structures on the spectrum converge with respect to $N_ξ$ or $N_η$, but not both. It is therefore very hard to pinpoint a region that is about to converge. When a zoom is given on the verification mode in the spectrum, no coherent structure can be found at all. This implies that the spectrum is far from converged, certainly near the location of the verification mode. The attempt to find a converged solution is therefore discontinued.

Batterson notes that the spectrum is not converged, but that the considered modes are convergent. However, it can be seen that no mode can be properly tracked in fig-
ures 9.11 and 9.12. Therefore, the validity of statement that the mode he selected is convergent is questioned.

It is very important to have a converged solution. However, the fact that the currently retrieved spectrum is not converged, boosts the meaning of the currently performed verification. It is argued to be exceptionally hard to precisely retrieve a spectrum that is not converged, because a slight perturbation in one of the parameters should change the overall picture significantly. Only when the computation is replicated very precisely, this is case. Therefore, it is argued that currently the same computation is performed as done by Batterson.
Figure 9.13: Perturbation amplitude $-\Im\{\tilde{u}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$.

Figure 9.14: Perturbation amplitude $-\Im\{\tilde{u}\}$ reported by Batterson [10] (Figure 6.7a on page 222).

Figure 9.15: Perturbation amplitude $-\Im\{\tilde{v}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$.

Figure 9.16: Perturbation amplitude $-\Im\{\tilde{v}\}$ reported by Batterson [10] (Figure 6.7b on page 222).
Figure 9.17: Perturbation amplitude $-\Im\{\tilde{w}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$.

Figure 9.18: Perturbation amplitude $-\Im\{\tilde{w}\}$ reported by Batterson [10] (Figure 6.7c on page 222).

Figure 9.19: Perturbation amplitude $-\Im\{\tilde{p}\}$ obtained using $N_\xi \times N_\eta = 50 \times 50$.

Figure 9.20: Perturbation amplitude $-\Im\{\tilde{p}\}$ reported by Batterson [10] (Figure 6.7d on page 222).
Chapter 10

Conclusions & Recommendations

Verification of Stability Equation Derivation

In the first part of the thesis project, the main objective was to derive the BiGlobal stability equations and to verify these with the literature. At the VKI, the VKI Extensible Stability and Transition Analysis (VESTA) toolkit is set up, see Pinna [65]. During the current author’s internship, the main goal was to create a derivation tool to automatically derive LST and PSE stability equations. Currently, this derivation tool is extended to exhaustively cover also the BiGlobal analysis type.

The verification of the equations revealed several errors in the reported literature in the incompressible case. Because these errors were found in only a small part of the literature, sufficiently many references could be used to cross-verify the equations. Therefore, the errors in the incompressible equations did not directly cause any problems with respect to the current verification.

In the verification of the compressible case, however, errors were found in all the considered references and the verification critically relied on cross-verification to yield the ultimate positive outcome. See section 4.5 for the thorough consideration.

All references that report the compressible equations were considered, at least to the knowledge of the current author. Therefore, it is concluded that the present treatment is the first to report the full compressible BiGlobal stability equations in primitive variable formulation correctly.

Discretization, Implementation and Eigensolvers

In the second part of the thesis project, the automatic implementation tool was set up to implement the equations with the Chebyshev collocation method. The latter discretization method is used, because it has been proven to yield good results and convergence characteristics with respect to the current application, see [10, 16, 66, 65, 70, 75, 88]. The working principle is based on writing the equations derived in Computer Algebra Software (CAS) Maxima to .txt-files and to read these with Matlab.
In the current treatment, the QZ and Arnoldi algorithms, both global algorithms, have mainly been used to solve the eigenvalue problems via Matlab’s functions `eig` and `eigs`. The Arnoldi algorithm does not remove small elements with respect to the QZ algorithm. Therefore, it can be expected to yield (slightly) more precise results than the QZ algorithm. In chapter 8, it is demonstrated that indeed the results corresponding to the two algorithms are accurate up to the algorithm precision corresponding to the QZ algorithm. Moreover, all eigenvalues in the part of the spectrum returned by the Arnoldi algorithm comply with the previous statement; eigenvalues in the core of the resolved part of the spectrum are equally accurately determined as those close to the outer boundary thereof.

It is argued that, when the required memory capacity allows this and the to-be-retrieved spectrum unknown, the application of the QZ algorithm is best, because the complete spectrum is retrieved; no a priori knowledge about the spectrum is required to use this algorithm. However, when the physically interesting part of the eigenvalue space is known, the Arnoldi algorithm might be more practical; in those cases, it can be “tuned” to retrieve only the significant part of the spectrum, requires less storage capacity and requires a less execution time with respect to the QZ algorithm. This means that computations with (slightly) higher resolution can be performed with the Arnoldi algorithm.

In addition to the former algorithms, the Rayleigh algorithm and an eigensolver based on a Newton-Raphson iterative method, both local methods implemented by Pinna [65], were tested when small eigenfunctions were encountered. These solvers are not considered for the retrieval of the complete spectrum, because this would mean that the complete eigenvalue space should be scanned carefully. This is much less efficient with respect to the global algorithms. It was found that the QZ and Arnoldi algorithms have troubles representing small eigenfunctions. Due to the fact that the other algorithms are local solvers, they can be applied after this kind of small eigenfunctions are encountered in the spectra returned by one of the global algorithms.

It is observed that the Newton-Raphson method yields more accurate results when the exact eigenfunction is identically zero. It is argued that this is caused by the fact that the Rayleigh algorithm is based on solving systems that become singular when the exact eigensolution is approached. A warning for small eigenfunctions is incorporated in the current simulation tool via an eigenfunction condition number, which is very different from the conventional eigenvector condition number. This number becomes very large (of \(O(10^{13})\) and larger) when an eigenfunction is nearly, but not exactly, zero. The application of the local algorithms on a small but finite and an identically zero eigenfunction can be found in subsections 7.3.5 and 7.4.1, respectively.

Another important subject involved the accuracy of the used mean flow data. When the data is given in a discrete (interpolated) form, the spatial convergence error (taking into account the interpolation) of the data is currently required to be smaller than the algorithm precision corresponding to the QZ algorithm. In that way, it is argued that one can be sure that the contribution of the errors in the mean data is negligible with respect to errors in this particular eigensolver. In section 6.2, it is shown that errors in
the mean flow can yield the eigensolvers to be unable to converge to algorithm precision properly.

Test Cases

It was attempted to apply the tools to a variety of cases, such that all capabilities could be tested. These capabilities involved the compressibility of the flow, the application of artificial in- and outflow boundary conditions, the use of a curvilinear coordinate system and the analysis of a mean flow that is strongly two- (or at least multi-) dimensional.

In chapter 7, the parallel Blasius boundary layer is handled. Due to the one-dimensional nature of this flow, it has been analysed thoroughly with the LST analysis type in the past. Therefore, all simulations performed in this chapter could be verified with those results. Moreover, due to its relative simplicity, both incompressible and compressible simulations were performed.

Both LST and spanwise BiGlobal simulations were performed to verify the latter with the former and the corresponding literature. The spanwise orientation of the eigenfunction plane was chosen following the treatment of Piot [66]. The structure of the spectrum of the BiGlobal simulations is analysed with respect to that corresponding to LST. This is done for both an incompressible \((M = 10^{-6})\) and a supersonic \((M = 2.5)\) case, so to be able to compare the results with those of Mack [57], Malik [59] and Pinna [65].

All results are found to be convergent and agree very well with the literature. The main difference in the BiGlobal spectra with respect to those corresponding to LST involves the appearance of higher harmonics of the LST modes. Together with the LST modes, the higher harmonics form “clusters” of modes. For the current parameter combinations, these clusters could easily distinguished. For the case of the most unstable mode, these higher harmonics are all found to be stable for the currently used parameter combinations.

The next test case that is considered is the developing Blasius boundary layer. The main features of this flow is the involvement of artificial Robin boundary conditions on the in- and outflow boundaries, see chapter 8 and the fact that it is developing. The flow is the logical successor of the parallel equivalent, because, depending on the streamwise extent, non-parallel effects become dominant. Therefore, the results from the parallel case could be build upon, incorporating the influence of non-parallel effects. This is regarded to be one of the most important capabilities of the BiGlobal framework.

First, a domain of a small streamwise extent, with a length of 30.689l \([\text{m}]\), is considered so to be able to perform all the simulations using the QZ algorithm. This small domain corresponds to only a weakly non-parallel influence. The results were hence compared to LST simulations and were found to agree up to the error expected due to the non-parallel nature of the BiGlobal simulations.

Another set of simulations was performed on a domain with a large streamwise extent, with a length of 400\(\delta^* \approx 688l\) \([\text{m}]\), so to be able to compare the results mainly with the
The results qualitatively agreed well with the literature. However, the discrete branch of Tollmien-Schlichting modes was found to lie too high with respect to the one reported in the literature. The shapes and wavenumbers of the corresponding eigenfunctions were found to agree with those reported in the literature. Therefore, the test case is verified only in a weak qualitative sense. Changes were made to the in- and outflow boundary conditions, which will be treated in a moment, to see whether this would yield the branch to assume a lower position in the eigenvalue space. All attempts did not yield this to a satisfactory extent. Therefore, it is argued that only differences in the currently used mean flow could make up for the difference; the references report that their mean flows are converged at least up to the threshold $10^{-8}$ in time, but do not report the grid convergence error. Moreover, another qualitative discrepancy is found regarding the continuous branches. In the literature, these split in two quite close to the real axis. This is argued to be the consequence of the small height that is used in the references, which in the case of Alizard [1] is equal to $206^* \approx 34\ell$ [m].

An important feature of the simulations involves the artificial boundary conditions, that force the resulting eigensolutions. The corresponding Robin conditions are equipped with the Gaster-type transformation. This is done so to be able to relate the in- and outflowing perturbations to the eigenvalue using parameters obtained via spatial LST results. When an even or odd number of collocation points $N_\xi$ is used in the streamwise direction, qualitatively different spectra are obtained. Using the primary implementation of the condition yields the “odd” continuous branches to be preferred when $N_\xi$ is odd and the even branches when $N_\xi$ is even. When the secondary form of the Robin conditions are considered, based on $\frac{\partial^2 \tilde{u}}{\partial x^2} = -\alpha_s^2 \tilde{u}$ instead of $\frac{\partial \tilde{u}}{\partial x} = i\alpha_s \tilde{u}$, the latter correspondence switches; odd continuous branches suddenly correspond to even values of $N_\xi$. For the domain with a small and large streamwise extent, the primary and secondary Robin conditions are used, respectively. The secondary conditions are implemented to minimize the required storage capacity.

Accordingly, it is deemed to be proven that the Robin boundary conditions yield solutions that are dependent qualitatively on the implementation. This illustrates that one should be hesitant of the physical soundness of the conditions. It must be noted that the verification of the simulations on the small domain with the LST simulations cannot be considered as the remedy for this issue; the LST simulations are subjected to the same forcing the Robin boundary conditions imply in the BiGlobal case. A more thorough investigation has to be performed in this regard to see what can be done to incorporate artificial boundary conditions that are physically sound.

It has to be emphasized that the continuous branches change qualitatively when $N_\xi$ is taken to be odd or even. The similar behaviour can be observed in the results of Alizard [1], although the effect is never elaborated on explicitly. The discrete spectrum seems to be present in both the cases, although relatively large differences are found when comparing the eigenvalues of computations with even and odd $N_\xi$. A thorough investigation on the small domain is performed to see whether the discrete modes converge towards the same limit as $N_\xi$. Although the ultimate limit is far from reached,
both kinds of computations are found to indeed converge to the same limit. Therefore, it is argued that the discrete spectrum can be safely analysed, regarding the soundness warning given above.

Lastly, in chapter 9, the complex lamellar bidirectional vortex is considered. This mean flow is defined on cylindrical coordinates and is highly inhomogeneous in two coordinate directions. Therefore, the corresponding stability analysis requires the BiGlobal approach with all power of the combination of the derivation, implementation and simulation tools. The analysis of this mean flow is reported by Batterson [10] and the work presented in chapter 9 was primarily focused to replicate his results, as verification. This meant that several inconsistencies, that were present in the original work, were also incorporated in the present investigation. These inconsistencies mainly involved the applied kinds of boundary conditions.

The results were found to match identically with those of Batterson, up to the measurable extent. A convergence study was performed, so as to be able to judge on the convergence characteristics of Batterson’s solution. In this study, it was found that nearly no part of the spectrum is converged. Several structures seem to converge when keeping $\eta_N$ or $\zeta_N$ constant, but these disappear or move vigorously when both parameters are varied. Moreover, it is argued that many more collocation points are required to resolve the boundary layers at either side of the domain sufficiently. Currently, Batterson reports that 3 nodes are located in the boundary layer, whereas the results for the Blasius boundary layer in chapters 7 and 8 show that 45 are required to attain a fully converged solution. Hence, the influence of the boundary layers is argued to be largely absent in the spectrum.

It can be seen that solutions to all three test cases have been obtained successfully and compare well with the literature. In the case of the developing Blasius case on the domain with the large streamwise extent, the results are found to match qualitatively with those in the literature, but are different quantitatively. It is expected that this test case will yield identical results when exactly the same mean flow will be used, though. It is therefore concluded that all capabilities of the current combination of tools have been tested successfully and the tools can be declared to be verified.

**Overall Conclusions**

The research question stated in the introduction can now be readdressed. Firstly, the extent to which the current tools can execute BiGlobal simulations is determined by the possible formulations that can be provided by the derivation tool. As can be seen in figure 3.1, a large variety of options is present to derive BiGlobal stability equations up to the moderate supersonic regime. Important capabilities are that the tool can output the equations in any coordinate system, for the incompressible or compressible regime and using different formulations in the compressible case (including the energy equation). Furthermore the stability problem can be set up for any kind of eigenfunction
plane orientation and flow directions. In this regard, a further extension of the tool may involve the chemical equilibrium and chemical non-equilibrium hypersonic regimes (although tools that derive the chemical equilibrium LST equations are present in the toolkit already). Another possible extension would be to include the derivation of the BiGlobal PSE (Bi+Global) and TriGlobal stability equations.

Secondly, the current simulation tool is mainly set up for a single domain, while using Chebyshev spectral collocation techniques to discretize the problem. Due to the large arsenal of implemented boundary conditions (all standard ones to the knowledge of the current author), all flow simulations that are to be analysed in a single domain can be performed. Mapping techniques are used to be able to resolve certain parts of the domain better, which allows the efficient investigation of truncated semi-infinite domains. Problems were encountered when in- and outflow boundaries are considered. In that case, Robin boundary conditions equipped with the Gaster-type transformation are proposed, but these are shown to yield qualitatively different results when the implementation changed or an even or odd number of collocation nodes is used in the in-/outflow direction. It must be noted that the discrete spectrum can be found in both the “odd” and “even” solutions and is observed to converge toward the same limit when $N_ξ \to \infty$.

The efficiency with which the BiGlobal simulations can be set up is extremely high. By setting options, the combination of the derivation and implementation tool can implement any kind of system in mere minutes. Normally weeks or even months are required to derive, implement and verify the equations. Moreover, the boundary conditions can also be simply specified in the simulation tool. Therefore, only the mean flow has to be incorporated to be able to execute the analysis. For computations that involve high resolution ($N_ξ \times N_η = 75 \times 50$) and are solved with the QZ algorithm, the execution time of the simulation tool alone can reach 12 hours on a computer that has 32 [Gb] RAM and 35 [Gb] swap memory. When the Arnoldi algorithm is used, the largest computation performed had $N_ξ \times N_η = 71 \times 90$ nodes and resolved 100 eigenmodes. This computation took about 6 hours on the same computer. This is found to be in the order of what is normally expected ($\texttt{eig}$ or $\texttt{eigs}$ are responsible for the largest elapsed times), see Theofilis [85].

It was found that, in the least requiring case (which is the incompressible parallel boundary layer), the tool can be applied such that it can yield fully converged eigenvalues. That is to say, to algorithm precision: which is usually of the order $O(10^{-13})$. However, in cases that require significant resolution in two spatial directions, this threshold can no longer be attained. This is mainly caused by limits on the required RAM. In those cases, it is argued that sufficient resolution of the boundary layer is of paramount importance. Accordingly, enough collocation points were used in the wall-normal direction to yield full convergence in that direction always. In the worst situation amongst the developing Blasius case, the maximal convergence error with respect to the other direction was of $O(10^{-4})$. This error magnitude in combination with the convergent character of the solutions is still argued to be sufficient for verification. To solve the required RAM problem, it is advised to incorporate a parallel eigensolver.
Conclusions & Recommendations

Recommendations

As mentioned already, future work on the derivation tool may be its extension to include the chemical equilibrium and chemical non-equilibrium hypersonic regimes. Furthermore, the BiGlobal PSE and TriGlobal stability equations can be added to the possible analysis types.

Regarding the simulation tool, the next step is to incorporate a multi-domain (spectral element) method. In this way, more complicated mean flows can be analysed. Furthermore, the incorporation of a parallel eigensolver is deemed to be mandatory, because the test cases in this thesis already define the outer border of what analyses are currently possible when using Matlab. Changing from platform is advised regarding the performance required of the (parallel) eigensolver and to avoid inefficient infrastructures between the corresponding routines.

In the current work, several problems were encountered regarding the artificial boundary conditions in the developing Blasius case. It is argued that this problem has to be investigated more thoroughly to appropriately simulate in- and outflowing perturbations. Boundary conditions that simulate these in- or outflowing perturbations in a physically sound way are deemed to be of very high importance. Examples may involve any kind of flow situation. In the case of the bidirectional vortex, for example, one could ultimately also want to investigate the effect of incoming perturbations from the in- or outlet. Potential solutions could involve the incorporation of a better representation of the (local) spatial dispersion relation $\alpha_s = \alpha_s(\omega_t)$, although it is expected that this does not change the peculiar aspects of the current results. In the literature other kinds of boundary conditions are handled as well, examples are simple extrapolations, see Alizard & Robinet [2].

Furthermore, it is found that the stability results for the complex lamellar bidirectional vortex require much more resolution before the results can be interpreted consistently. To achieve this, many more collocation nodes are required (when scaling linearly, at least a factor 15 in the wall-normal direction). It is argued that the incorporation of a parallel solver is required to be able to handle the large proportions of the resulting problem.
Appendix A

Incompressible LST Equations

This appendix contains the incompressible Linear Stability Theory (LST) equations in Cartesian, cylindrical, spherical and a specific curvilinear coordinate system.

Noting that $g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ where $x^i$ denote the (possibly) curvilinear coordinates and $y^j$ are the Cartesian coordinates, the following conventions apply in the case of this system:

- **Streamwise**: $x^1 = x$ spectral ($\alpha$)
- **Wall-normal**: $x^2 = y$ differential
- **Spanwise**: $x^3 = z$ spectral ($\beta$)

Metric tensor: \[
g = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Note that in this case global analysis is performed in the $y$-direction. This yields the following set of equations:

**Continuity equation**
\[
i \beta \tilde{w} + \tilde{v}_y + i \alpha \tilde{u} = 0 \quad (A.1)
\]

**x-Momentum equation**
\[
i \beta \tilde{u} \tilde{W} + \tilde{v} \tilde{U}_y + i \alpha \tilde{u} \tilde{U} - i \omega \tilde{u} = -i \alpha \tilde{p} + \frac{\tilde{u}_{yy}}{Re} - \frac{\beta^2 \tilde{u}}{Re} - \frac{\alpha^2 \tilde{u}}{Re} \quad (A.2)
\]

**y-Momentum equation**
\[
i \beta \tilde{v} \tilde{W} + i \alpha \tilde{v} \tilde{U} - i \omega \tilde{v} = -\tilde{p}_y + \frac{\tilde{v}_{yy}}{Re} - \frac{\beta^2 \tilde{v}}{Re} - \frac{\alpha^2 \tilde{v}}{Re} \quad (A.3)
\]

**z-Momentum equation**
\[
\tilde{v} \tilde{W}_y + i \beta \tilde{w} \tilde{W} + i \alpha \tilde{w} \tilde{U} - i \omega \tilde{w} = -i \beta \tilde{p} + \frac{\tilde{w}_{yy}}{Re} - \frac{\beta^2 \tilde{w}}{Re} - \frac{\alpha^2 \tilde{w}}{Re} \quad (A.4)
\]
Appendix B

Compressible LST Equations

This appendix contains the compressible Linear Stability Theory (LST) equations in Cartesian, cylindrical, spherical and a specific curvilinear coordinate system.

Noting that $g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ where $x^i$ denote the (possibly) curvilinear coordinates and $y^j$ are the Cartesian coordinates, the following conventions apply in the case of this system:

- **Streamwise**: $x^1 = x$ spectral ($\alpha$)
- **Wall-normal**: $x^2 = y$ differential
- **Spanwise**: $x^3 = z$ spectral ($\beta$)

Metric tensor: $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note that in this case global analysis is performed in the $y$-direction. This yields the following set of equations:

**Continuity equation**

$$-i \beta \frac{T \overline{P} \overline{W}}{T} + i \beta \tilde{p} \overline{W} - i \alpha \frac{T \overline{P} \overline{U}}{T} + i \alpha \tilde{p} \overline{U} - \tilde{v} \overline{P} \overline{T}_y + \frac{i \omega \overline{P}}{T} - i \omega \tilde{p} + \tilde{v} \overline{P}_y + i \beta \tilde{w} \overline{P} + i \alpha \tilde{u} \overline{P} = 0 \tag{B.1}$$

**x-Momentum equation**

$$i \beta \frac{\gamma \tilde{u} M^2 \overline{P} \overline{W}}{T} + \frac{\gamma \tilde{v} M^2 \overline{P} \overline{U}_y}{T} + i \alpha \frac{\tilde{v} M^2 \overline{P} \overline{U}}{T} - i \gamma \omega \frac{\tilde{u} M^2 \overline{P}}{T} = \frac{T \mu_{\tau} \overline{U}_{yy}}{Re}$$

$$+ i \beta \frac{\dot{T} \mu_{\tau} \overline{T}_y \overline{U}_y}{Re} + \frac{\dot{T}_{yy} \mu_{\tau} \overline{U}_y}{Re} + i \alpha \frac{\dot{\tilde{v}} \mu_{\tau} \overline{T}_y}{Re} + \frac{\ddot{u}_y \mu_{\tau} \overline{T}_y}{Re} - i \alpha \frac{\hat{\tilde{p}}}{Re} - \frac{\alpha \beta \tilde{v} \mu}{Re} + \frac{i \alpha \tilde{v}_y \mu}{Re}$$

$$+ \frac{\dot{u}_{yy} \mu}{Re} - \frac{\alpha \beta \tilde{u} \mu}{Re} - \frac{2 \alpha^2 \tilde{u} \mu}{Re} - \frac{\alpha \beta \tilde{w} \mu}{Re} + \frac{i \alpha \tilde{v}_y \mu}{Re} - \frac{\alpha^2 \tilde{u} \mu}{Re} \tag{B.2}$$
y-Momentum equation
\[
\frac{i \beta \gamma \hat{v} M^2 \hat{P} W}{T} + \frac{i \alpha \gamma \hat{v} M^2 \hat{P} U}{T} - \frac{i \gamma \omega \hat{v} M^2 \hat{P}}{T} = \frac{i \beta \hat{T} \hat{P} \hat{W}_y}{Re} + \frac{i \alpha \hat{T} \hat{P} \hat{U}_y}{Re}
\]
\[
+ \frac{2 \hat{v}_y \hat{P} \hat{T}_y}{Re} + \frac{i \beta \hat{w} \hat{X} \hat{T}_y}{Re} + \frac{i \alpha \hat{u} \hat{X} \hat{T}_y}{Re} - \hat{p}_y + \frac{i \beta \hat{w}_y \hat{P}}{Re} + 2 \frac{\hat{v}_{yy} \hat{P}}{Re} - \frac{\beta^2 \hat{v} \hat{P}}{Re} - \frac{\alpha^2 \hat{v} \hat{P}}{Re} + \frac{i \beta \hat{w} \hat{P}}{Re} + \frac{i \alpha \hat{u} \hat{P}}{Re} + \frac{\hat{v}_{yy} \hat{P}}{Re} + \frac{i \alpha \hat{u}_{yy} \hat{P}}{Re}
\]
(B.3)

z-Momentum equation
\[
\frac{\gamma \hat{v} M^2 \hat{P} \hat{W}_y}{T} + \frac{i \beta \gamma \hat{v} M^2 \hat{P} \hat{W}_y}{T} + \frac{i \alpha \gamma \hat{v} M^2 \hat{P} \hat{U}_y}{T} - \frac{i \gamma \omega \hat{v} M^2 \hat{P}}{T} = \frac{\hat{T} \hat{P} \hat{W}_{yy}}{Re}
\]
\[
+ \frac{\hat{T} \hat{P} \hat{T}_y \hat{W}_y}{Re} + \frac{\hat{T}_y \hat{P} \hat{W}_y}{Re} + \frac{\hat{w}_y \hat{P} \hat{T}_y}{Re} + \frac{i \beta \hat{v} \hat{X} \hat{T}_y}{Re} - \frac{i \beta \hat{v} \hat{P}}{Re} - \frac{2 \beta^2 \hat{w} \hat{P}}{Re}
\]
\[
- \frac{\alpha^2 \hat{w} \hat{P}}{Re} - \frac{i \beta \hat{w}_{yy} \hat{P}}{Re} - \frac{\alpha \beta \hat{u} \hat{P}}{Re} - \frac{\beta^2 \hat{w} \hat{X}}{Re} + \frac{i \beta \hat{v} \hat{X}}{Re} - \frac{\alpha \beta \hat{u} \hat{X}}{Re}
\]
(B.4)

Energy equation
\[
\frac{i \beta \gamma \hat{T} M^2 \hat{P} W}{T} + \frac{i \alpha \gamma \hat{T} M^2 \hat{P} U}{T} + \frac{\gamma \hat{v} M^2 \hat{P} \hat{T}_y}{T} - \frac{i \gamma \omega \hat{T} M^2 \hat{P}}{T} = \frac{(\gamma - 1) \hat{T} M^2 \hat{P} \hat{(W_y)}^2}{Re}
\]
\[
+ \frac{2 \gamma (\gamma - 1) \hat{v}_y M^2 \hat{P} \hat{W}_y}{Re} + \frac{2 i \beta (\gamma - 1) \hat{v} M^2 \hat{P} \hat{W}_y}{Re}
\]
\[
+ \frac{2 \gamma (\gamma - 1) \hat{u} M^2 \hat{P} \hat{U}_y}{Re} + \frac{i \alpha (\gamma - 1) M^2 \hat{P} \hat{U}_y}{Re} + \frac{\hat{T} \hat{K} \hat{T}_y \hat{U}_y}{Pr Re} + \frac{\hat{T} \hat{K} \hat{T}_y \hat{(U_y)}^2}{Pr Re}
\]
\[
+ \frac{2 \hat{T}_y \hat{K} \hat{T}_y}{Pr Re} + \frac{\hat{T}_{yy} \hat{K}}{Pr Re} - \frac{\beta^2 \hat{K} \hat{K}}{Pr Re} - \frac{\alpha^2 \hat{K} \hat{K}}{Pr Re}
\]
\[
- \frac{i \gamma (\gamma - 1) M^2 \hat{P} \hat{U} + (\gamma - 1) \hat{v} M^2 \hat{P} \hat{T}_y}{Pr Re}
\]
(B.5)
Appendix C

Incompressible BiGlobal Stability Equations

This appendix states the incompressible BiGlobal stability equations in a Cartesian and cylindrical coordinate system with a streamwise eigenfunction plane in sections C.1 and C.2, respectively. The system in a Cartesian coordinates with a spanwise eigenfunction plane is stated in section C.3.

C.1 Streamwise Cartesian System

Noting that $g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ where $x^i$ denote the (possibly) curvilinear coordinates and $y^i$ are the Cartesian coordinates, the following conventions apply in the case of this system:

- **Streamwise:** $x^1 = x$ differential
- **Wall-normal:** $x^2 = y$ differential
- **Spanwise:** $x^3 = z$ spectral ($\beta$)

Metric tensor:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

Note that in this case the $(x, y)$-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

- **Continuity equation**
  \[
i \beta \tilde{w} + \tilde{v}_y + \tilde{u}_x = 0 \quad \text{(C.1)}
\]

- **x-Momentum equation**
  \[
i \beta \tilde{u} \tilde{W} + \tilde{u}_y \tilde{V} + \tilde{v} \tilde{U}_y + \tilde{u} \tilde{U}_x + \tilde{u}_x \tilde{U} - i \omega \tilde{u} = -\tilde{p}_x + \frac{\tilde{u}_{yy}}{Re} + \frac{\tilde{u}_{xx}}{Re} - \frac{\beta^2 \tilde{u}}{Re} \quad \text{(C.2)}
\]

- **y-Momentum equation**
  \[
i \beta \tilde{v} \tilde{W} + \tilde{v}_y \tilde{V} + \tilde{u} \tilde{V}_x + \tilde{v}_y \tilde{V} + \tilde{v}_x \tilde{U} - i \omega \tilde{v} = -\tilde{p}_y + \frac{\tilde{v}_{yy}}{Re} + \frac{\tilde{v}_{xx}}{Re} - \frac{\beta^2 \tilde{v}}{Re} \quad \text{(C.3)}
\]

- **z-Momentum equation**
  \[
\tilde{v} \tilde{W}_y + \tilde{u} \tilde{W}_x + i \beta \tilde{w} \tilde{W} + \tilde{w}_y \tilde{V} + \tilde{w}_x \tilde{U} - i \omega \tilde{w} = -i \beta \tilde{p} + \frac{\tilde{w}_{yy}}{Re} + \frac{\tilde{w}_{xx}}{Re} - \frac{\beta^2 \tilde{w}}{Re} \quad \text{(C.4)}
\]
C.2 Streamwise Cylindrical System

Noting that \( g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \) where \( x^i \) denote the (possibly) curvilinear coordinates and \( y^i \) are the Cartesian coordinates, the following conventions apply in the case of this system:

**Wall-normal:** \( x^1 = r \) differential

**Spanwise:** \( x^2 = \theta \) spectral \((q)\)

**Streamwise:** \( x^3 = z \) differential

The **Metric tensor** is given by:

\[
g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Note that in this case the \((r, z)\)-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

**Continuity equation**

\[
\tilde{w}_z + \frac{i q \tilde{v}}{r} + \tilde{u}_r + \frac{\tilde{u}}{r} = 0 
\]

**r-Momentum equation**

\[
\tilde{u}_z \tilde{W} - \frac{2 \tilde{v} \tilde{V}}{r} + \frac{i q \tilde{u} \tilde{V}}{r} + \tilde{u} \tilde{U}_z + \tilde{u} \tilde{U}_r + \tilde{u}_r \tilde{U} - i \omega \tilde{u} = -\tilde{p}_r - \frac{2 i q \tilde{v}}{r^2 \text{Re}} + \frac{\tilde{u}_{zz}}{\text{Re}} \\
+ \frac{\tilde{u}_{rr}}{r \text{Re}} - \frac{q^2 \tilde{u}}{r^2 \text{Re}} - \frac{\tilde{u}}{r^2 \text{Re}} 
\]

**\(\theta\)-Momentum equation**

\[
\tilde{v}_z \tilde{W} + \tilde{w} \tilde{V}_z + \tilde{u} \tilde{V}_r + \frac{i q \tilde{v} \tilde{V}}{r} + \frac{\tilde{u} \tilde{V}}{r} + \tilde{v} \tilde{U} + \frac{\tilde{v} \tilde{U}_r}{r} - i \omega \tilde{v} = -\frac{i q \tilde{p}}{r} + \frac{\tilde{v}_{zz}}{\text{Re}} + \frac{\tilde{v}_{rr}}{r \text{Re}} \\
+ \frac{\tilde{v}_r}{r \text{Re}} - \frac{q^2 \tilde{v}}{r^2 \text{Re}} - \frac{\tilde{v}}{r^2 \text{Re}} + \frac{2 i q \tilde{u}}{r^2 \text{Re}} 
\]

**z-Momentum equation**

\[
\tilde{w} \tilde{W}_z + \tilde{u} \tilde{W}_r + \tilde{w}_z \tilde{W} + \frac{i q \tilde{w} \tilde{V}}{r} + \tilde{w}_r \tilde{U} - i \omega \tilde{w} = -\tilde{p}_z + \frac{\tilde{w}_{zz}}{\text{Re}} + \frac{\tilde{w}_{rr}}{r \text{Re}} \\
+ \frac{\tilde{w}_r}{r \text{Re}} - \frac{q^2 \tilde{w}}{r^2 \text{Re}} 
\]
C.3 Spanwise Cartesian System

Noting that \( g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \) where \( x^i \) denote the (possibly) curvilinear coordinates and \( y^i \) are the Cartesian coordinates, the following conventions apply in the case of this system:

- **Wall-normal**: \( x^1 = x \) spectral \((\alpha)\)
- **Spanwise**: \( x^2 = y \) differential
- **Streamwise**: \( x^3 = z \) differential

Metric tensor:

\[
g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Note that in this case the \((y,z)\)-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

Continuity equation

\[
\tilde{w}_z + \tilde{v}_y + i \alpha \tilde{u} = 0 \quad (C.9)
\]

x-Momentum equation

\[
\tilde{u}_z \tilde{W} + \tilde{u}_y \tilde{V} + \tilde{w}_z \tilde{U} + \tilde{v}_y \tilde{U}_y + i \alpha \tilde{u} \tilde{U} - i \omega \tilde{u} = -i \alpha \tilde{p} + \frac{\tilde{u}_{zz}}{Re} + \frac{\tilde{v}_{yy}}{Re} - \frac{\alpha^2 \tilde{u}}{Re} \quad (C.10)
\]

y-Momentum equation

\[
\tilde{v}_z \tilde{W} + \tilde{w}_z \tilde{V} + \tilde{v}_y \tilde{V}_y + i \alpha \tilde{v} \tilde{U} - i \omega \tilde{v} = -\tilde{p}_y + \frac{\tilde{v}_{zz}}{Re} + \frac{\tilde{v}_{yy}}{Re} - \frac{\alpha^2 \tilde{v}}{Re} \quad (C.11)
\]

z-Momentum equation

\[
\tilde{w}_z \tilde{W} + \tilde{v}_y \tilde{W}_y + \tilde{w}_z \tilde{W} + i \alpha \tilde{w} \tilde{U} - i \omega \tilde{w} = -\tilde{p}_z + \frac{\tilde{w}_{zz}}{Re} + \frac{\tilde{w}_{yy}}{Re} - \frac{\alpha^2 \tilde{w}}{Re} \quad (C.12)
\]
Appendix D

Compressible BiGlobal Stability Equations

This appendix states the compressible BiGlobal stability equations in a Cartesian and cylindrical coordinate system with a streamwise eigenfunction plane in sections D.1 and D.2, respectively. The system in Cartesian coordinates with a spanwise eigenfunction plane is stated in section D.3. All equations are shown in the \((p, T)\)-formulation.

D.1 Streamwise Cartesian System

Noting that \(g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}\) where \(x^i\) denote the (possibly) curvilinear coordinates and \(y^i\) are the Cartesian coordinates, the following conventions apply in the case of this system:

Streamwise: \(x_1 = x\) differential  
Wall-normal: \(x_2 = y\) differential  
Spanwise: \(x_3 = z\) spectral \((\beta)\)

Metric tensor: \(g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\)

Note that in this case the \((x,y)\)-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

\[
\begin{align*}
\text{Continuity equation} & \quad -i \beta \frac{\hat{T} \hat{P} W}{T} + i \beta \frac{\hat{T} \hat{P} V y}{T} + \frac{\hat{p} V y}{T} - \frac{\hat{p} \hat{T} y V}{T} + \frac{2 \hat{T} \hat{P} \hat{T} y V}{T^2} - \frac{\hat{T} \hat{P} y V}{T} - \frac{\hat{T} y P V}{T} \\
& + \hat{p}_y V - \frac{\hat{T} \hat{P} U_x}{T} + \hat{p} U_x - \frac{\hat{p} T_x U}{T} + \frac{2 \hat{T} \hat{P} T_x U}{T^2} - \frac{\hat{T} \hat{P} x U}{T} - \frac{\hat{T} x P U}{T} + \hat{p}_x U - \frac{\hat{v} \hat{P} \hat{T} y}{T} \\
& - \frac{\hat{u} \hat{P} \hat{T} x}{T} + i \omega \frac{\hat{T} \hat{P} y}{T} - \frac{i \omega \hat{p} + \hat{v} \hat{P} y + \hat{u} \hat{P} x + i \beta \hat{w} \hat{P} + \hat{v}_y \hat{P} + \hat{u}_x \hat{P}}{T} = 0 \quad (D.1)
\end{align*}
\]
\[ i \beta \gamma \bar{u} M^2 \frac{P W}{T} + \frac{\gamma M^2 \ddot{P} U_y V}{T} - \frac{\gamma \ddot{T} M^2 \ddot{P} U_y V}{T^2} + \frac{\gamma \ddot{u} y M^2 \ddot{P} V}{T} + \frac{\gamma \ddot{v} M^2 \ddot{P} U_y}{T} \\
+ \frac{\gamma M^2 \ddot{P} U_x}{T} - \frac{\gamma \ddot{T} M^2 \ddot{P} U_x}{T^2} + \frac{\gamma \ddot{u} x M^2 \ddot{P} U}{T} - i \gamma \omega \ddot{u} M^2 \ddot{P} \]

\[ = i \beta \ddot{T} \frac{\pi_T W_x}{Re} + \frac{T_y \pi_T \dddot{V}_x}{Re} + \frac{T_x \dddot{V}_x}{Re} + \frac{\dddot{T}_x \dddot{V}_x}{Re} + \frac{\dddot{T} \pi_T \dddot{T}_y \dddot{V}_x}{Re} + \frac{\dddot{T} \pi_T \dddot{T}_y \dddot{V}_x}{Re} \\
+ \frac{2 \dddot{\pi}_x \dddot{\pi}_T \dddot{T}_x}{Re} - \frac{i \beta \dddot{\bar{u} x} \dddot{\bar{u}}} + \frac{i \beta \dddot{\bar{v} x} \dddot{\bar{v}} + \dddot{\bar{u} y} \dddot{\bar{v}}}{Re} + \frac{i \beta \dddot{\bar{v} y} \dddot{\bar{v}} + \dddot{\bar{u} y} \dddot{\bar{u}}}{Re} \]  

(D.2)

x-Momentum equation

\[ \frac{i \beta \gamma \ddot{u} M^2 \frac{P W}{T} + \frac{\gamma M^2 \ddot{P} V}{T} + \frac{\gamma \ddot{v} M^2 \ddot{P} V}{T} + \frac{\gamma \ddot{u} y M^2 \ddot{P} U_y}{T} + \frac{\gamma \ddot{v} y M^2 \ddot{P} U_y}{T} \\
- \frac{\gamma \ddot{T} M^2 \ddot{P} V}{T^2} + \frac{\gamma \ddot{u} y M^2 \ddot{P} V}{T} + \frac{\gamma \ddot{u} x M^2 \ddot{P} U}{T} - i \gamma \omega \ddot{u} M^2 \ddot{P} \]

\[ = i \beta \dddot{T} \frac{\pi_T W_y}{Re} + \frac{2 \dddot{T} \pi_T \dddot{V}_y}{Re} + \frac{T_y \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{\dddot{T}_y \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{T_x \dddot{\pi}_T \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{\dddot{T}_x \dddot{\pi}_T \dddot{\pi}_T \dddot{V}_y}{Re} \\
+ \frac{2 \dddot{\pi}_y \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{T_y \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{T_x \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{\dddot{T}_x \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{\dddot{T}_y \dddot{\pi}_T \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{2 \dddot{\pi}_y \dddot{\pi}_T \dddot{V}_y}{Re} + \frac{i \beta \dddot{\bar{v} y} \dddot{\bar{v}}}{Re} + \frac{i \beta \dddot{\bar{u} y} \dddot{\bar{u}}}{Re} \]  

(D.3)

y-Momentum equation
z-Momentum equation

\[
\begin{align*}
\frac{\gamma M^2 \tilde{p} \nabla W_y}{T} &+ \frac{\gamma \tilde{M}^2 \tilde{P} \nabla W_y}{T^2} + \frac{\gamma \tilde{v} M^2 \tilde{P} W_y}{T} + \frac{\gamma M^2 \tilde{P} W_x}{T} - \frac{\gamma \tilde{M}^2 PU \tilde{W}_x}{T^2} \\
&+ \frac{\gamma \tilde{u} M^2 \tilde{P} W_x}{T} + \frac{i \beta \gamma \tilde{w} M^2 \tilde{P} W}{T} + \frac{\gamma \tilde{w}_y M^2 \tilde{P} V}{T} + \frac{\gamma \tilde{w}_x M^2 \tilde{P} U}{T} - i \gamma \omega \tilde{w} M^2 \tilde{P} \\
&= \frac{\tilde{T} \tilde{P} \tilde{W}_{yy}}{Re} + \frac{\tilde{T} \tilde{P} \tilde{T}_y \tilde{W}_y}{Re} + \frac{\tilde{T}_y \tilde{P} \tilde{W}_y}{Re} + \frac{\tilde{T} \tilde{P} \tilde{T}_x \tilde{W}_x}{Re} + \frac{\tilde{T}_x \tilde{P} \tilde{W}_x}{Re} + \frac{i \beta \tilde{T} \tilde{X}_T \tilde{V}_y}{Re} + \frac{i \beta \tilde{T} \tilde{X}_T \tilde{U}_x}{Re} + \frac{\tilde{w}_y \tilde{P} \tilde{T}_y}{Re} + \frac{\tilde{w}_x \tilde{P} \tilde{T}_x}{Re} + \frac{i \beta \tilde{u} \tilde{P} \tilde{T}_y}{Re} \\
&- i \beta \tilde{p} + \frac{\tilde{w}_y \tilde{u}}{Re} + \frac{\tilde{w}_x \tilde{u}}{Re} - \frac{2 \beta^2 \tilde{w} \tilde{u}}{Re} + \frac{i \beta \tilde{v}_y \tilde{x}}{Re} + \frac{\beta^2 \tilde{w} \tilde{X}}{Re} + i \beta \tilde{v}_y \tilde{X} + \frac{i \beta \tilde{u}_y \tilde{X}}{Re} \\
\end{align*}
\]

Energy equation

\[
\begin{align*}
&\frac{i \beta \tilde{T} M^2 \tilde{P} \tilde{W}}{T} + \frac{\gamma M^2 \tilde{P} \tilde{T}_y \tilde{V}}{T^2} - \frac{\gamma \tilde{T} M^2 \tilde{P} \tilde{T}_y \tilde{V}}{T^2} + \frac{\gamma \tilde{T}_y M^2 \tilde{P} \tilde{V}}{T} + \frac{\gamma M^2 \tilde{P} \tilde{T}_y \tilde{U}}{T} \\
&- \frac{\gamma T M^2 \tilde{P} \tilde{T}_x \tilde{U}}{T^2} + \frac{\gamma \tilde{T}_x M^2 \tilde{P} \tilde{T}_x \tilde{U}}{T^2} + \frac{\gamma \tilde{u} M^2 \tilde{P} \tilde{T}_x}{T} - \frac{i \gamma \omega \tilde{T} M^2 \tilde{P}}{T} \\
&= \frac{\tilde{E} \tilde{T} \tilde{P} (\tilde{W}_y)^2}{Re} + \frac{2 \tilde{E} \tilde{w}_y \tilde{P} \tilde{W}_y}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{W}_y}{Re} + \frac{\tilde{E} \tilde{T} \tilde{P} (\tilde{W}_x)^2}{Re} + \frac{2 \tilde{E} \tilde{w}_x \tilde{P} \tilde{W}_x}{Re} + \frac{i \beta \tilde{E} \tilde{u} \tilde{P} \tilde{W}_x}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{V}_y}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{V}_y}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{V}_y)^2}{Re} + \frac{2 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{V}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_y}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} \\
&+ \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{u}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} \\
&+ \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{u}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} \\
&+ \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \beta \tilde{E} \tilde{u}_x \tilde{P} \tilde{U}_x}{Re} + \frac{2 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{4 \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} + \frac{\tilde{E} \tilde{T} \tilde{X}_T (\tilde{U}_x)^2}{Re} + \frac{2 \beta \tilde{E} \tilde{v}_y \tilde{P} \tilde{U}_x}{Re} \\
&+ \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Pr Re} + \frac{2 \beta \tilde{E} \tilde{w}_x \tilde{P} \tilde{U}_x}{Pr Re} + \frac{2 \beta \tilde{E} \tilde{u}_x \tilde{P} \tilde{U}_x}{Pr Re} + \frac{2 \tilde{E} \tilde{T} \tilde{X}_T (\tilde{T}_x)^2}{Pr Re} + \frac{\tilde{T}_y \tilde{y} \tilde{K}}{Pr Re} + \frac{\tilde{T}_x \tilde{x} \tilde{K}}{Pr Re} + \frac{\beta^2 \tilde{T} \tilde{K}}{Pr Re} \\
&- i \tilde{E} \omega \tilde{p} + \tilde{E} \tilde{v} \tilde{P} y + \tilde{E} \tilde{u} \tilde{P} x \\
&= \text{(D.4)}
\end{align*}
\]
D.2 Streamwise Cylindrical System

Noting that $g_{ij} = \sum_{k=1}^{3} \frac{\partial y^{k}}{\partial x^{i}} \frac{\partial y^{k}}{\partial x^{j}}$ where $x^{i}$ denote the (possibly) curvilinear coordinates and $y^{i}$ are the Cartesian coordinates, the following conventions apply in the case of this system:

Wall-normal:  $x^{1} = r$  differential  
Spanwise:  $x^{2} = \theta$  spectral ($q$)  
Streamwise:  $x^{3} = z$  differential  

Metric tensor:  $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note that in this case the $(r, z)$-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

Continuity equation

\[
-\frac{T \bar{P} \bar{W}_{z}}{T} + \frac{\bar{p} W_{z}}{T} - \frac{\bar{p} T_{z} W}{T} + \frac{2 \bar{T} \bar{P} T_{z} W}{T^{2}} - \frac{\bar{T} P_{z} W}{T} - \frac{\bar{T}_{z} P W}{T} + \bar{p}_{z} W - i q \frac{T \bar{P} V}{r T} \\
+ \frac{i q \bar{p} V}{r} - \frac{T \bar{P} \bar{U}_{r}}{T} + \bar{p} U_{r} - \frac{\bar{p} T_{r} U}{T} + \frac{2 \bar{T} \bar{P} T_{r} U}{T^{2}} - \frac{\bar{T} P_{r} U}{T} - \frac{\bar{T}_{r} P U}{T} - \frac{T \bar{P} \bar{U}}{T^{2}} \\
+ \bar{p}_{r} U + \frac{\bar{p} U}{r} - \frac{\bar{w} \bar{P} T_{z}}{T} - \frac{\bar{u} \bar{P} T_{r}}{T} + i \omega \frac{T \bar{P}}{T} - i \omega \bar{p} + \bar{w} P_{z} + \bar{u} P_{r} + \bar{w} \bar{P}_{r} \\
+ \frac{i q \bar{v} \bar{P}}{r} + \bar{u}_{r} \bar{P} + \frac{\bar{u} \bar{P}}{r} = 0
\]

r-Momentum equation

\[
\gamma M^2 \frac{\bar{p} U_{z} W}{T} - \gamma \frac{T M^2 \bar{P} U_{z} W}{T^{2}} + \gamma \bar{u}_{z} M^2 \frac{\bar{P} W}{T} - \gamma M^2 \frac{\bar{p} V^2}{r T} + \gamma \frac{T M^2 \bar{P} V^2}{r T^{2}} \\
- \frac{2 \gamma \bar{v} M^2 \frac{\bar{P} V}{r T}}{T} + \frac{i q \bar{u} M^2 \frac{\bar{P} V}{r T}}{T} + \gamma \bar{w} M^2 \frac{\bar{P} U_{z}}{T} + \gamma M^2 \frac{\bar{p} U_{r} U_{z}}{T} - \gamma \frac{T M^2 \bar{P} U_{r} U_{z}}{T^{2}} \\
+ \gamma \bar{u} M^2 \frac{\bar{P} U_{r}}{T} + \frac{i \gamma \omega \bar{u} M^2 \bar{P}}{T} = \frac{T \bar{\lambda}_{T T} T_{r} W_{z}}{T} + \frac{T \bar{\lambda}_{T r} W_{z}}{T} \\
+ \frac{T \bar{\mu}_{T} W_{r} z}{T} + \frac{\bar{T} \bar{\lambda}_{T} W_{r} z}{T} + \frac{T \bar{\mu}_{r} T_{z} W_{r}}{T} + \frac{T \bar{\lambda}_{r} W_{r} z}{T} + \frac{i q \bar{T} \bar{\mu}_{T} V_{r}}{T} - \frac{i q \bar{T} \bar{\mu} T V}{T} \\
+ \frac{T \bar{\mu}_{T} T_{z} Z_{z}}{T} + \frac{T \bar{\mu}_{r} T_{z} Z_{z}}{T} + \frac{T \bar{\mu}_{T} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{T} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{r} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{T} \bar{U}_{r} r}{T} \\
+ \frac{T \bar{\mu}_{r} T_{z} U_{r}}{T} + \frac{T \bar{\mu}_{T} T_{z} U_{r}}{T} + \frac{T \bar{\mu}_{T} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{T} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{r} \bar{U}_{r} r}{T} + \frac{T \bar{\lambda}_{T} \bar{U}_{r} r}{T} \\
+ \frac{2 \bar{T} \bar{\mu}_{T} U}{r T} + \frac{T \bar{\lambda}_{T} U}{r T} + \frac{T \bar{\lambda}_{r} T_{z} U}{r T} + \frac{T \bar{\lambda}_{T} T_{z} U}{r T} + \frac{T \bar{\lambda}_{r} \bar{U}_{r} r}{r T} + \frac{2 \bar{u}_{r} \bar{r} T_{r}}{T} + \frac{T \bar{\lambda}_{T} T_{r}}{T} \\
+ \frac{i q \bar{v} \bar{\lambda}_{T} T_{r}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{3 i q \bar{v} \bar{\lambda}_{T} T_{r}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} \\
+ \frac{2 \bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{2 \bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{2 \bar{u}_{r} \bar{r} \bar{\lambda}_{T} T_{r}}{r T} + \frac{i q \bar{v} \bar{\lambda}_{T} \bar{T}}{r T} + \frac{i q \bar{v} \bar{\lambda}_{T} \bar{T}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} \bar{T}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} \bar{T}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} \bar{T}}{r T} + \frac{\bar{u}_{r} \bar{r} \bar{\lambda}_{T} \bar{T}}{r T}
\]

(D.7)
\[ \theta \text{-Momentum equation} \]
\[
\gamma \frac{M^2 \bar{p} V_z W}{T} - \gamma \frac{\hat{T} M^2 \bar{P} V_z W}{T^2} + \gamma \frac{\bar{v}_z M^2 \bar{P} W}{T} + \gamma \frac{\bar{w} M^2 \bar{P} V_z}{T} + \gamma \frac{M^2 \bar{p} \bar{U} V z}{T} - \gamma \frac{\hat{T} M^2 \bar{P} U V z}{r^2 T} - \gamma \frac{\bar{v} M^2 \bar{P} U}{r T} - \gamma \frac{\bar{w} M^2 \bar{P} V}{r T} = i \gamma q \bar{v} M^2 \bar{P} V
\]
\[
+ \gamma \frac{\bar{u} M^2 \bar{P} V}{T} + \gamma \frac{\bar{v}_r M^2 \bar{P} U}{r T} + \gamma \frac{\bar{w}_r M^2 \bar{P} W}{r T} - i \gamma \frac{\omega \bar{v} M^2 \bar{P}}{r T} = i q \bar{T} \lambda T W_z + \bar{T} \bar{p} T V z z
\]

\[ z \text{-Momentum equation} \]
\[
\gamma \frac{M^2 \bar{p} W W z}{T} - \gamma \frac{\hat{T} M^2 \bar{P} W W z}{T^2} + \gamma \frac{\bar{v} \bar{w} M^2 \bar{P} V z}{T} + \gamma \frac{\bar{w} M^2 \bar{P} U W r}{T} - \gamma \frac{\hat{T} M^2 \bar{P} U W r}{T^2}
\]
\[
+ \gamma \frac{\bar{u} M^2 \bar{P} W W r}{T} + \gamma \frac{\bar{v}_r \bar{w} M^2 \bar{P} V z}{T} + \gamma \frac{\bar{w}_r \bar{w} M^2 \bar{P} W z}{T} - i \gamma \frac{\omega \bar{v} M^2 \bar{P}}{T} = i q \bar{T} \lambda T W z + \bar{T} \bar{p} T W z z
\]
\[
= 2 \frac{\bar{T} \bar{p} T W z z}{2 \bar{T} \bar{p} T W z} + \frac{\bar{T} \lambda T W z z}{2 \bar{T} \bar{p} T W z} + 2 \frac{\bar{T} \bar{p} T T z W z}{2 \bar{T} \bar{p} T W z} + \frac{\bar{T} \lambda T T z W z}{2 \bar{T} \bar{p} T W z} + 2 \frac{\bar{T} \bar{p} T W z}{2 \bar{T} \bar{p} T W z} + \bar{T} \lambda T W z
\]
\[
+ 2 \frac{\bar{T} \bar{p} T W z}{2 \bar{T} \bar{p} T W z} - r \frac{\bar{T} \lambda T W z}{2 \bar{T} \bar{p} T W z} + \frac{\bar{T} \bar{p} T T z W z}{2 \bar{T} \bar{p} T W z} + \frac{\bar{T} \bar{p} T T z W z}{2 \bar{T} \bar{p} T W z} + \frac{\bar{T} \bar{p} T T z W z}{2 \bar{T} \bar{p} T W z} + \bar{T} \lambda T W z
\]
Energy equation

\[
\begin{align*}
\frac{\gamma M^2 \tilde{p} T_z W}{T} & - \frac{\gamma \hat{T} M^2 \hat{P} T_z W}{T^2} + \frac{\gamma \hat{T} z M^2 \hat{P} V}{T} + i \gamma q \hat{T} M^2 \hat{P} V + \frac{\gamma M^2 \tilde{p} T_z U}{T} \\
- \frac{\gamma \hat{T} M^2 \hat{P} T_z U}{T^2} & + \frac{\gamma \tilde{w} M^2 \hat{P} U}{T} + \frac{\gamma \tilde{u} M^2 \hat{P} z}{T} - i \gamma \tilde{w} M^2 \hat{P} \\
= & \frac{2 Ec \hat{T} \tilde{p}_T (W_z)^2}{r Re} + \frac{Ec \hat{T} \tilde{w}_T (W_z)^2}{r Re} + \frac{2 Ec \hat{T} \tilde{w}_r U, W_z}{r Re} + \frac{2 Ec \hat{T} \tilde{w}_z U W_z}{r Re} \\
+ & \frac{4 Ec \tilde{w}_z \tilde{p} W_z}{r Re} + \frac{2 Ec \tilde{w}_z \tilde{w} W_z}{r Re} + \frac{2 i Ec \tilde{w} \tilde{w} W_z}{r Re} + \frac{2 i Ec \tilde{w} \tilde{w} W_z}{r Re} + \frac{2 Ec \tilde{u}_r \tilde{w} W_z}{r Re} + \frac{2 Ec \tilde{u}_r \tilde{w} W_z}{r Re} + \frac{Ec \tilde{p}_z W}{r Re} \\
+ & \frac{Ec \hat{T} \tilde{p}_T (V_r)^2}{r Re} + \frac{2 i Ec \tilde{w} \tilde{w} \tilde{V} V}{r Re} + \frac{2 Ec \tilde{w} \tilde{w} \tilde{V} V}{r Re} + \frac{Ec \hat{T} \tilde{w}_r \tilde{V} V}{r Re} + \frac{2 Ec \hat{T} \tilde{w}_z \tilde{V} V}{r Re} \\
+ & \frac{2 Ec \tilde{v}_r \tilde{V} V}{r Re} + \frac{2 Ec \tilde{v}_r \tilde{V} V}{r Re} + \frac{2 i Ec \tilde{v} \tilde{V} V}{r Re} + \frac{Ec \hat{T} \tilde{w}_r \tilde{V} V}{r Re} + \frac{2 i Ec \tilde{w} \tilde{V} V}{r Re} + \frac{2 i Ec \tilde{w} \tilde{V} V}{r Re} \\
+ & \frac{2 Ec \tilde{w} \tilde{V} V}{r^2 Re} + \frac{Ec \hat{T} \tilde{w} \tilde{V} V}{r^2 Re} + \frac{2 i Ec \tilde{w} \tilde{V} V}{r^2 Re} + \frac{2 Ec \tilde{w} \tilde{V} V}{r^2 Re} + \frac{2 Ec \tilde{w} \tilde{V} V}{r^2 Re} \\
+ & \frac{2 i Ec \tilde{v} \tilde{U}_r}{r Re} + \frac{2 Ec \tilde{v} \tilde{U}_r}{r Re} + \frac{2 Ec \tilde{v} \tilde{U}_r}{r Re} + \frac{2 i Ec \tilde{v} \tilde{U}_r}{r Re} + \frac{Ec \hat{T} \tilde{w}_r \tilde{U}_r}{r Re} + \frac{Ec \hat{T} \tilde{w}_r \tilde{U}_r}{r Re} \\
+ & \frac{4 i Ec \tilde{v} \tilde{U}_r}{r^2 Re} + \frac{4 Ec \tilde{v} \tilde{U}_r}{r^2 Re} + \frac{2 Ec \tilde{v} \tilde{U}_r}{r^2 Re} + \frac{2 i Ec \tilde{v} \tilde{U}_r}{r^2 Re} + \frac{2 Ec \tilde{v} \tilde{U}_r}{r^2 Re} + \frac{2 Ec \tilde{v} \tilde{U}_r}{r^2 Re} \\
+ & \frac{\tilde{T}_k \tilde{T}_z T_z}{Pr Re} + \frac{\tilde{T}_k \tilde{T}_z T_z}{Pr Re} + \frac{2 \tilde{T}_z \tilde{T}_k \tilde{T}_z}{Pr Re} + \frac{\tilde{T}_k \tilde{T}_z T_z}{Pr Re} + \frac{\tilde{T}_k \tilde{T}_z T_z}{Pr Re} + \frac{\tilde{T}_k \tilde{T}_z T_z}{Pr Re} \\
+ & \frac{\tilde{T}_z \tilde{T}_k \tilde{T}_r}{Pr Re} + \frac{\tilde{T}_z \tilde{T}_k \tilde{T}_r}{Pr Re} + \frac{\tilde{T}_z \tilde{T}_k \tilde{T}_r}{Pr Re} + \frac{\tilde{T}_z \tilde{T}_k \tilde{T}_r}{Pr Re} + \frac{\tilde{T}_z \tilde{T}_k \tilde{T}_r}{Pr Re} - i Ec \omega \tilde{p} + Ec \tilde{w} \tilde{P}_z + Ec \tilde{u} \tilde{P}_r \\
= & D.10
\end{align*}
\]
D.3 Spanwise Cartesian System

Noting that
\[ g_{ij} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \]
where \( x^i \) denote the (possibly) curvilinear coordinates and \( y^i \) are the Cartesian coordinates, the following conventions apply in the case of this system:

- Wall-normal: \( x^1 = x \) spectral (\( \alpha \))
- Spanwise: \( x^2 = y \) differential
- Streamwise: \( x^3 = z \) differential

Metric tensor: \( g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

Note that in this case the \((y, z)\)-plane is the BiGlobal eigenfunction plane. This yields the following set of equations:

**Continuity equation**
\[
- \frac{\hat{T} \hat{P} \hat{W}_z}{T} + \hat{p} \hat{W}_z - \frac{\hat{p} \hat{T}_z \hat{W}}{T} + \frac{2 \hat{T} \hat{P} \hat{T}_z \hat{W}}{T^2} - \frac{\hat{T} \hat{P}_z \hat{W}}{T} - \hat{T}_z \hat{P} \hat{W} - \frac{\hat{T} \hat{P} \hat{V}_y}{T} + \hat{p} \hat{V}_y \\
- \frac{\hat{p} \hat{T}_y \hat{V}}{T} + \frac{2 \hat{T} \hat{P} \hat{T}_y \hat{V}}{T^2} - \frac{\hat{T} \hat{P}_y \hat{V}}{T} - \hat{T}_y \hat{P} \hat{V} - \hat{p} \hat{V} - i \alpha \frac{\hat{T} \hat{P} \hat{U}}{T} + i \alpha \frac{\hat{p} \hat{U}}{T} - \frac{\hat{T} \hat{p} \hat{P}}{T} + \frac{\hat{p} \hat{V}_p}{T} + i \alpha \hat{u} \hat{P} = 0 \quad (D.11)
\]

**\(x\)-Momentum equation**
\[
\gamma M^2 \frac{\hat{p} \hat{U}_z \hat{W}}{T} - \gamma \frac{\hat{T} \hat{M}^2 \hat{P} \hat{U}_z \hat{W}}{T^2} + \gamma \frac{\hat{u}_z M^2 \hat{P} \hat{W}}{T} + \gamma M^2 \frac{\hat{p} \hat{U}_y \hat{V}}{T} - \gamma \frac{\hat{T} \hat{M}^2 \hat{P} \hat{U}_y \hat{V}}{T^2} \\
+ \frac{\gamma \hat{T} \hat{U}_z \hat{T}}{T} + \gamma \frac{\hat{T} \hat{U}_y \hat{T}}{T} + \gamma \frac{\hat{u}_z \hat{U}_z \hat{T}}{T} + \gamma \frac{\hat{u}_y \hat{U}_y \hat{T}}{T} + \frac{i \alpha \hat{T} \hat{p} \hat{U}_z \hat{W}}{T} + \frac{\hat{T} \hat{p} \hat{U}_z \hat{W}}{T} + \frac{\hat{T}_z \hat{p} \hat{U}_z \hat{W}}{T} + \frac{\hat{T} \hat{p} \hat{U}_z \hat{W}}{T} \\
\gamma \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \gamma \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \gamma \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \gamma \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \frac{i \alpha \hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} + \frac{\hat{T}_z \hat{p} \hat{U}_z \hat{T}}{T} + \frac{\hat{T} \hat{p} \hat{U}_z \hat{T}}{T} \\
- \frac{i \alpha \hat{p} + \frac{i \alpha \hat{u}_z \hat{P}}{Re} + \frac{i \alpha \hat{u}_y \hat{P}}{Re} + \frac{i \alpha \hat{u}_z \hat{X}}{Re} + \frac{i \alpha \hat{u}_y \hat{X}}{Re} + \frac{2 \alpha^2 \hat{u} \hat{X}}{Re} + \frac{i \alpha \hat{u}_z \hat{X}}{Re} + \frac{i \alpha \hat{u}_y \hat{X}}{Re} + \frac{\alpha^2 \hat{u} \hat{X}}{Re}}{Re} = 0 \quad (D.12)
\]
Compressible BiGlobal Stability Equations

\begin{align*}
\gamma M^2 \frac{\ddot{p}}{T} \frac{V_z}{W} & - \frac{\dot{T}}{T^2} M^2 \frac{\ddot{P}}{P} \frac{V_z}{W} + \frac{\ddot{V}_z}{T} M^2 \frac{\ddot{P}}{P} - \frac{\gamma M^2 \ddot{p}}{T} \frac{V_y}{V} + \frac{\dot{T}}{T} \frac{\ddot{P}}{P} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& = \frac{\dot{T}}{T} \frac{\ddot{X}_T T_y W_z}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T W_z}{Re} + \frac{\dot{T} \ddot{p} T_y W_{yz}}{Re} + \frac{\ddot{T} \ddot{X}_T W_{yz}}{Re} + \frac{\ddot{T}_{\mu \nu} T_z W_y}{Re} + \frac{\ddot{T} \ddot{X}_T W_y}{Re} + \frac{\ddot{V}_y M^2 \ddot{P}}{T} + i \alpha \gamma \ddot{\dot{p}} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& + \frac{\dot{T}}{T} \frac{\ddot{X}_T T_y V_y}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T V_y}{Re} + \frac{\dot{T} \ddot{p} T_y V_{yz}}{Re} + \frac{\ddot{T} \ddot{X}_T V_{yz}}{Re} + \frac{\ddot{T}_{\mu \nu} T_z V_y}{Re} + \frac{\ddot{T} \ddot{X}_T V_y}{Re} + \frac{\ddot{V}_y M^2 \ddot{P}}{T} + i \alpha \gamma \ddot{\dot{p}} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& + \frac{2 \ddot{V}_y \ddot{X}_T T_y}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T T_y}{Re} - \frac{\alpha^2 \ddot{\dot{p}}}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{w}_y \ddot{X}_T}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re}
\end{align*}

y-Momentum equation

\begin{align*}
\gamma M^2 \frac{\ddot{p}}{T} \frac{W_z}{W} & - \frac{\dot{T}}{T^2} M^2 \frac{\ddot{P}}{P} \frac{W_z}{W} + \frac{\ddot{V}_z}{T} M^2 \frac{\ddot{P}}{P} - \frac{\gamma M^2 \ddot{p}}{T} \frac{V_y}{V} + \frac{\dot{T}}{T} \frac{\ddot{P}}{P} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& = \frac{\dot{T}}{T} \frac{\ddot{X}_T T_y W_z}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T W_z}{Re} + \frac{\dot{T} \ddot{p} T_y W_{yz}}{Re} + \frac{\ddot{T} \ddot{X}_T W_{yz}}{Re} + \frac{\ddot{T}_{\mu \nu} T_z W_y}{Re} + \frac{\ddot{T} \ddot{X}_T W_y}{Re} + \frac{\ddot{V}_y M^2 \ddot{P}}{T} + i \alpha \gamma \ddot{\dot{p}} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& + \frac{\dot{T}}{T} \frac{\ddot{X}_T T_y V_y}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T V_y}{Re} + \frac{\dot{T} \ddot{p} T_y V_{yz}}{Re} + \frac{\ddot{T} \ddot{X}_T V_{yz}}{Re} + \frac{\ddot{T}_{\mu \nu} T_z V_y}{Re} + \frac{\ddot{T} \ddot{X}_T V_y}{Re} + \frac{\ddot{V}_y M^2 \ddot{P}}{T} + i \alpha \gamma \ddot{\dot{p}} + \frac{\ddot{V}_y}{T} M^2 \frac{\ddot{P}}{P} - i \gamma \omega \ddot{\dot{P}} \frac{M^2}{T} \\
& + \frac{\ddot{T}}{T} \frac{\ddot{X}_T T_y T_z}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T T_z}{Re} - \frac{\alpha^2 \ddot{\dot{p}}}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{w}_y \ddot{X}_T}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re} + \frac{\alpha \ddot{\dot{p}}}{Re} + \frac{\ddot{V}_y}{T} \frac{\ddot{X}_T}{Re}
\end{align*}

z-Momentum equation
Energy equation

\[
\begin{align*}
\frac{\gamma M^2 \tilde{p} T_z W}{T} &- \frac{\gamma \tilde{T} M^2 \tilde{P} T_z W}{T^2} + \gamma \tilde{T}_z M^2 \tilde{P} W - \frac{\gamma M^2 \tilde{p} T_y V}{T} - \frac{\gamma \tilde{T} M^2 \tilde{P} T_y V}{T^2} \\
+ \frac{\gamma \tilde{T}_y M^2 \tilde{P} V}{T} &+ i \alpha \gamma \tilde{T} M^2 \tilde{P} U + \gamma \tilde{w} M^2 \tilde{P} T_z + \gamma \tilde{v} M^2 \tilde{P} T_y - i \gamma \omega \tilde{T} M^2 \tilde{P} \\
= 2 Ec \tilde{T} \tilde{p}_T (W_z)^2 &+ \frac{Ec \tilde{T} \tilde{X}_T (W_z)^2}{Re} + \frac{2 Ec \tilde{T} \tilde{X}_T V_y W_z}{Re} + \frac{4 Ec \tilde{w}_z \tilde{p} W_z}{Re} + \frac{2 Ec \tilde{w}_z \tilde{X} W_z}{Re} \\
+ \frac{2 i \alpha Ec \tilde{u} \tilde{X} W_z}{Re} + Ec \tilde{T} \tilde{p}_T (W_y)^2 &+ \frac{2 Ec \tilde{T} \tilde{p}_T V_z W_y}{Re} + \frac{2 Ec \tilde{w}_z \tilde{p} W_y}{Re} \\
+ \frac{2 Ec \tilde{V} \tilde{p}_T (V_z)^2}{Re} &+ \frac{4 Ec \tilde{v}_y \tilde{p} V_y}{Re} + \frac{2 Ec \tilde{w}_z \tilde{X} V_y}{Re} + \frac{2 Ec \tilde{v}_y \tilde{X} V_y}{Re} \\
+ \frac{2 i \alpha Ec \tilde{u} \tilde{X} V_y}{Re} &+ Ec \tilde{T} \tilde{p}_T (U_z)^2 + \frac{2 i \alpha Ec \tilde{w}_z \tilde{p} U_z}{Re} + \frac{2 Ec \tilde{u}_z \tilde{p} U_z}{Re} \\
+ \frac{Ec \tilde{T} \tilde{p}_T (U_y)^2}{Re} &+ \frac{2 i \alpha Ec \tilde{v} \tilde{p} U_y}{Re} + \frac{2 Ec \tilde{w}_z \tilde{p} U_y}{Re} + i \alpha Ec \tilde{p} U + \frac{\tilde{T} \tilde{k}_T T_{zz}}{Pr Re} \\
+ \frac{\tilde{T} \tilde{k}_T T_{zz}}{Pr Re} &+ \frac{2 \tilde{T}_z \tilde{k}_T T_z}{Pr Re} + \frac{\tilde{T} \tilde{k}_T T_{yy}}{Pr Re} + \frac{\tilde{T} \tilde{k}_T T_{yy}}{Pr Re} + \frac{2 \tilde{T}_y \tilde{k}_T T_y}{Pr Re} + \frac{\tilde{T}_z \tilde{k}}{Pr Re} \\
+ \frac{\tilde{T}_y \tilde{k}}{Pr Re} &- \alpha^2 \frac{\tilde{T} \tilde{k}}{Pr Re} - i \gamma \omega \tilde{T} M^2 \tilde{P} + Ec \tilde{w} \tilde{p}_z + Ec \tilde{v} \tilde{p}_y
\end{align*}
\]  

(D.15)
Appendix E

Compressible Verification Equations

In this appendix all compressible verification equations are reported, performed with Robitaillie-Montane [70], Theofilis [84], Theofilis & Colonius [86] and Robinet [69].

E.1 Verification with Robitaillie-Montane

Continuity equation

Equation by VESTA

\[
- \frac{i \beta T W}{T} + i \beta \gamma M^2 \frac{p W}{T} - \frac{T V_y}{T} + \gamma M^2 \frac{p V_y}{T} \\
- \frac{\gamma M^2 p T_y V}{T} + 2 \frac{T T_y V}{T^2} - \frac{T_y V}{T} + \gamma M^2 \frac{p V_y}{T} \\
- \frac{T U_x}{T} + \gamma M^2 \frac{p U_x}{T} - \frac{T_x U}{T} + \gamma M^2 \frac{p_x U}{T} \\
- \frac{\tilde{v} T_y}{T} + \frac{i \omega T}{T} - i \gamma M^2 \frac{p}{T} + i \beta \tilde{w} \\
+ \tilde{v}_y + \tilde{u}_x = 0 
\] (E.1)

Equation by Robitaillie-Montane in [70]

\[
- \frac{i \beta T W}{T} + i \beta \gamma M^2 \frac{p W}{T} - \frac{T V_y}{T} + \gamma M^2 \frac{p V_y}{T} \\
- \frac{\gamma M^2 p T_y V}{T} + 2 \frac{T T_y V}{T^2} - \frac{T_y V}{T} + \gamma M^2 \frac{p V_y}{T} \\
- \frac{T U_x}{T} + \gamma M^2 \frac{p U_x}{T} - \frac{T_x U}{T} + \gamma M^2 \frac{p_x U}{T} \\
- \frac{\tilde{v} T_y}{T} + \frac{i \omega T}{T} - i \gamma M^2 \frac{p}{T} + i \beta \tilde{w} \\
+ \tilde{v}_y + \tilde{u}_x = 0 
\] (E.2)
x-Momentum equation

Equation by VESTA
\[
\frac{i \beta \dot{u}}{T} + \frac{\gamma M^2 \dot{p} \dot{U}_y}{T} - \ddot{T} \dot{U}_y + \ddot{u} = \frac{2 \ddot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}} + \frac{\dot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}}}{T} + \frac{\dot{p}_x + i \beta \ddot{w}}{3 \text{Re}} + \frac{4 \ddot{u}_x \frac{\dot{p}}{3 \text{Re}} - \beta^2 \ddot{u}}{T} \quad (E.3)
\]

Equation by Robitaille-Montané in [70]
\[
\frac{i \beta \dot{u}}{T} + \frac{\gamma M^2 \dot{p} \dot{U}_y}{T} - \ddot{T} \dot{U}_y + \ddot{u} = \frac{2 \ddot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}} + \frac{\dot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}}}{T} + \frac{\dot{p}_x + i \beta \ddot{w}}{3 \text{Re}} + \frac{4 \ddot{u}_x \frac{\dot{p}}{3 \text{Re}} - \beta^2 \ddot{u}}{T} \quad (E.4)
\]

y-Momentum equation

Equation by VESTA
\[
\frac{i \beta \dot{v}}{T} + \frac{\gamma M^2 \dot{p} \dot{V}_y}{T} - \ddot{T} \dot{V}_y + \ddot{v} = \frac{4 \ddot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}} + \frac{\dot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}}}{T} + \frac{\dot{p}_y + i \beta \ddot{w}}{3 \text{Re}} + \frac{4 \ddot{u}_y \frac{\dot{p}}{3 \text{Re}}}{T} \quad (E.5)
\]

Equation by Robitaille-Montané in [70]
\[
\frac{i \beta \dot{v}}{T} + \frac{\gamma M^2 \dot{p} \dot{V}_y}{T} - \ddot{T} \dot{V}_y + \ddot{v} = \frac{4 \ddot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}} + \frac{\dot{T} \frac{\dot{p} \dot{T}}{3 \text{Re}}}{T} + \frac{\dot{p}_y + i \beta \ddot{w}}{3 \text{Re}} + \frac{4 \ddot{u}_y \frac{\dot{p}}{3 \text{Re}}}{T} \quad (E.6)
\]
z-Momentum equation

Equation by VESTA
\[
\frac{\gamma M^2 \tilde{p} \nabla W_y}{T} - \frac{\dot{T} \nabla W_y}{T^2} + \frac{\dot{\varepsilon} W_y}{T} = \frac{\dot{T} \pi_T W_{yy}}{Re} + \frac{\dot{T} \pi_T \pi_T W_{yy}}{Re} + \frac{\dot{T} \pi_T W_{yy}}{Re} - \frac{2 i \beta \pi_T \nabla y}{3 Re} - \frac{2 i \beta \pi_T \nabla x}{3 Re} + \frac{\dot{\varepsilon} \pi_T \pi_T W_{yy}}{Re} + \frac{\dot{\varepsilon} \pi_T \pi_T W_{yy}}{Re} - \frac{4 \beta^2 \dot{\varepsilon} \pi_T \pi_T}{3 Re} + \frac{i \beta \pi_T \pi_T}{3 Re} + \frac{i \beta \pi_T \pi_T}{3 Re} \quad (E.7)
\]

Equation by Robitaille-Montané in [70]
\[
\frac{\gamma M^2 \tilde{p} \nabla W_y}{T} - \frac{\dot{T} \nabla W_y}{T^2} + \frac{\dot{\varepsilon} W_y}{T} = \frac{\dot{T} \pi_T W_{yy}}{Re} + \frac{\dot{T} \pi_T \pi_T W_{yy}}{Re} + \frac{\dot{T} \pi_T W_{yy}}{Re} - \frac{2 i \beta \pi_T \nabla y}{3 Re} - \frac{2 i \beta \pi_T \nabla x}{3 Re} + \frac{\dot{\varepsilon} \pi_T \pi_T W_{yy}}{Re} + \frac{\dot{\varepsilon} \pi_T \pi_T W_{yy}}{Re} - \frac{4 \beta^2 \dot{\varepsilon} \pi_T \pi_T}{3 Re} + \frac{i \beta \pi_T \pi_T}{3 Re} + \frac{i \beta \pi_T \pi_T}{3 Re} \quad (E.8)
\]
Energy equation

Equation by VESTA

\[
\frac{i \beta \hat{T} \hat{W}}{T} + \frac{\gamma M^2 \hat{p} T_y V}{T} - \frac{\hat{T} T_y V}{T^2} + \frac{\hat{T}_y V}{T} \\
+ \frac{\hat{T}_x U}{T} + \frac{\hat{v}_T T_y}{T} - \frac{i \omega \hat{T}}{T} \\
= \frac{E c \hat{T} \hat{v}_T (W_y)^2}{Re} + \frac{2 E c \hat{w}_y p W_y}{Re} \\
+ \frac{2 i \beta E c \hat{v}_y \hat{p} W}{Re} + i \beta E c \hat{p} W \\
+ \frac{4 E c \hat{T} \hat{v}_y (V_y)^2}{3 Re} - \frac{4 E c \hat{T} \hat{u}_x \hat{U}_y}{3 Re} \\
- \frac{4 E c \hat{u}_x \hat{p} V_y}{3 Re} + E c \hat{p}_y V + \frac{E c \hat{T} \hat{v}_T (U_y)^2}{Re} \\
+ \frac{2 E c \hat{v}_x \hat{p} \hat{U}_y}{3 Re} + \frac{2 E c \hat{u}_y \hat{p} \hat{U}_y}{3 Re} \\
+ \frac{4 E c \hat{v}_x \hat{p} \hat{U}_x}{3 Re} - \frac{4 i \beta E c \hat{w}_x \hat{U}_x}{3 Re} \\
- \frac{4 E c \hat{v}_y \hat{p} \hat{U}_x}{3 Re} + \frac{8 E c \hat{u}_x \hat{p} \hat{U}_x}{3 Re} + E c \hat{p}_x \hat{U} \\
+ \frac{\hat{T} T_y T_{yy}}{Pr Re} + \frac{\hat{T} \hat{v}_T (T_y)^2}{Pr Re} \\
+ \frac{2 T_y \hat{p}_T \hat{T}_y}{Pr Re} + \frac{T_{yy} \hat{E}}{Pr Re} \\
+ \frac{\hat{T}_x \hat{E}}{Pr Re} - i E c \hat{p} + E c \hat{u} \hat{P}_x
\]

(E.9)

Equation by Robitaille-Montané in [70]

\[
\frac{i \beta \hat{T} \hat{W}}{T} + \frac{\gamma M^2 \hat{p} T_y V}{T} - \frac{\hat{T} T_y V}{T^2} + \frac{\hat{T}_y V}{T} \\
+ \frac{\hat{T}_x U}{T} + \frac{\hat{v}_T T_y}{T} - \frac{i \omega \hat{T}}{T} \\
= \frac{E c \hat{T} \hat{v}_T (W_y)^2}{Re} + \frac{2 E c \hat{w}_y p W_y}{Re} \\
+ \frac{2 i \beta E c \hat{v}_y \hat{p} W}{Re} + i \beta E c \hat{p} W \\
+ \frac{4 E c \hat{T} \hat{v}_y (V_y)^2}{3 Re} - \frac{4 E c \hat{T} \hat{u}_x \hat{U}_y}{3 Re} \\
- \frac{4 E c \hat{u}_x \hat{p} V_y}{3 Re} + E c \hat{p}_y V + \frac{E c \hat{T} \hat{v}_T (U_y)^2}{Re} \\
+ \frac{2 E c \hat{v}_x \hat{p} \hat{U}_y}{3 Re} + \frac{2 E c \hat{u}_y \hat{p} \hat{U}_y}{3 Re} \\
+ \frac{4 E c \hat{v}_x \hat{p} \hat{U}_x}{3 Re} - \frac{4 i \beta E c \hat{w}_x \hat{U}_x}{3 Re} \\
- \frac{4 E c \hat{v}_y \hat{p} \hat{U}_x}{3 Re} + \frac{8 E c \hat{u}_x \hat{p} \hat{U}_x}{3 Re} + E c \hat{p}_x \hat{U} \\
+ \frac{T \hat{p}_T T_{yy}}{Pr Re} + \frac{\hat{T} \hat{v}_T (T_y)^2}{Pr Re} \\
+ \frac{T \hat{v}_T \hat{T}_y}{Pr Re} + \frac{T_{yy} \hat{E}}{Pr Re} \\
+ \frac{T \hat{v}_x \hat{P}}{Pr Re} - \frac{\beta^2 \hat{T} \hat{p}}{Pr Re} - i E c \omega \hat{p} + E c \hat{u} \hat{P}_x
\]

(E.10)
E.2 Verification with Theofilis

Continuity equation

\[
\begin{align*}
\text{Equation by VESTA} & \quad i \beta \rho W + \rho V_y + \rho V + \rho U_x + \rho x U - i \omega \rho \\
+ \bar{v}_y \rho_x + \bar{u}_x \rho + i \beta \bar{w} \rho + \bar{v} \rho + \bar{u}_x \rho &= 0 \\
\text{Equation by Theofilis in [84]} & \quad i \beta \rho W + \rho V_y + \rho V + \rho U_x + \rho x U - i \omega \rho \\
+ \bar{v}_y \rho_x + \bar{u}_x \rho + i \beta \bar{w} \rho + \bar{v} \rho + \bar{u}_x \rho &= 0 \\
\text{(E.11)}
\end{align*}
\]

x-Momentum equation

\[
\begin{align*}
\text{Equation by VESTA} & \quad i \beta \bar{u} \rho W + \bar{\rho} U_y V + \bar{\rho} U \rho y V + \bar{\rho} U \rho y V + \bar{\rho} U \rho y V \\
+ \bar{u} \rho U x + \bar{u} x \rho U & \\

-i \omega \bar{u} = & \quad \frac{i \beta \bar{u} W_x}{Re} \\
\text{Equation by Theofilis in [84]} & \quad i \beta \bar{u} \rho W + \bar{\rho} U_y V + \bar{\rho} U \rho y V + \bar{\rho} U \rho y V + \bar{\rho} U \rho y V \\
+ \bar{u} \rho U x + \bar{u} x \rho U & \\

-i \omega \bar{u} = & \quad \frac{i \beta \bar{u} W_x}{Re} \\
\text{(E.13)}
\end{align*}
\]
y-Momentum equation

Equation by VESTA

\[
i \beta \tilde{v} \rho W + \hat{\rho} V V_y + \tilde{v} \rho V_y + \hat{\rho} U V_x \
+ \hat{u} \rho V_x + \tilde{v}_y \rho V
\]

\[
+ \tilde{v}_x \rho U - i \omega \tilde{v} \rho = \frac{i \beta \tilde{\mu} W_y}{Re}
\]

\[
+ \frac{4 \tilde{\mu} V_{yy}}{3 Re} + \frac{4 \tilde{\mu}_y V_y}{3 Re} + \frac{\tilde{\mu} V_{xx}}{Re} + \frac{\tilde{\mu}_x V_x}{Re}
\]

\[
+ \frac{\tilde{\mu}_x U}{Re} + \frac{\tilde{\mu}_y U_y}{3 Re} - \frac{2 \tilde{\mu}_y U_x}{3 Re} - \hat{p}_y - \frac{2 i \beta \tilde{w} \tilde{\mu}_y}{3 Re}
\]

\[
+ \frac{4 \tilde{v}_y \tilde{\mu}_y}{3 Re} - \frac{2 \tilde{u}_x \tilde{\mu}_y}{3 Re} + \frac{\tilde{v}_x \tilde{\mu}_x}{Re} + \frac{\tilde{u}_y \tilde{\mu}_x}{Re} + \frac{3 Re}{3 Re}
\]

\[
+ \frac{4 \tilde{v}_y \tilde{\mu}_y}{3 Re} + \frac{\tilde{v}_x \tilde{\mu}_x}{Re} - \frac{\beta^2 \tilde{v} \tilde{\mu}}{Re} + \frac{\tilde{u}_x \tilde{y} \tilde{\mu}}{3 Re}
\]

(E.15)

Equation by Theofilis in [84]

\[
i \beta \tilde{v} \rho W + \frac{2 \tilde{\mu} V V_y}{Re} + \frac{2 \tilde{v} \rho V_y}{Re} + \frac{\tilde{\mu} U V_x}{Re}
\]

\[
+ \hat{u} \rho V_x + \frac{2 \tilde{v}_y \rho V_y}{Re} + \hat{\nu}_x \rho V_x + \tilde{v} \rho U_x
\]

\[
+ \hat{u}_x \rho V + \frac{i \beta \tilde{w} \rho V}{Re} + \hat{\nu}_x \rho V + \frac{2 \tilde{v} \rho V}{Re}
\]

\[
- \frac{i \omega \tilde{\rho} V}{Re} + \frac{\hat{\rho} U V}{Re} + \frac{\tilde{\rho} U_x V}{Re} + \frac{\tilde{\rho} V}{Re}
\]

\[
+ \frac{i \beta \tilde{\rho} V}{Re} + \tilde{v}_x \rho U - i \omega \tilde{v} \rho = \frac{i \beta \tilde{\mu} W_y}{Re}
\]

\[
+ \frac{4 \tilde{\mu} V_{yy}}{3 Re} + \frac{4 \tilde{\mu}_y V_y}{3 Re} + \frac{\tilde{\mu} V_{xx}}{Re} + \frac{\tilde{\mu}_x V_x}{Re}
\]

\[
+ \frac{\tilde{\mu}_x U}{Re} + \frac{\tilde{\mu}_y U_y}{3 Re} - \frac{2 \tilde{\mu}_y U_x}{3 Re} - \hat{p}_y - \frac{2 i \beta \tilde{w} \tilde{\mu}_y}{3 Re}
\]

\[
+ \frac{4 \tilde{v}_y \tilde{\mu}_y}{3 Re} - \frac{2 \tilde{u}_x \tilde{\mu}_y}{3 Re} + \frac{\tilde{v}_x \tilde{\mu}_x}{Re} + \frac{\tilde{u}_y \tilde{\mu}_x}{Re} + \frac{i \beta \tilde{w} \tilde{\mu}}{3 Re}
\]

(E.16)
**z-Momentum equation**

**Equation by VESTA**

\[ \dot{\rho} V W_y + \dot{\nu} \rho W_y + \dot{\rho} U W_x + \dot{\mu} p W_x + i \beta \dot{w} p W + \dot{w}_y p V \]

\[ + \dot{w}_x p U - i \omega \dot{w} p \]

\[ = \frac{\mu W_{yy}}{Re} + \frac{\dot{\mu} W_y}{Re} + \frac{\mu W_{xx}}{Re} + \frac{\dot{\mu} W_x}{Re} \
- \frac{2i \beta \dot{\mu} V_y}{3 Re} - \frac{2i \beta \dot{\mu} U_x}{3 Re} - i \beta \dot{p} + \frac{\dot{w}_y p_y}{Re} \
+ \frac{i \beta \dot{v} p_y}{Re} + \frac{\dot{w}_x p_x}{Re} + \frac{i \beta \dot{u} p_x}{Re} + \frac{w_{yy} p}{Re} + \frac{w_{xx} p}{Re} \]

\[ - \frac{4 \beta^2 \dot{w} p}{3 Re} + \frac{i \beta \dot{v} p_y}{3 Re} + \frac{i \beta \dot{u} p_x}{3 Re} \]  

(E.17)

**Equation by Theoilis in [84]**

\[ \dot{\rho} V W_y + \dot{\nu} \rho W_y + \dot{\rho} U W_x + \dot{\mu} p W_x + 2i \beta \dot{w} p W + \dot{w}_y p V + \dot{w}_x p U + \dot{w}_y p U_x \]

\[ + \dot{w}_y p V + \dot{w}_x p W + \dot{v}_y p W \]

\[ + \dot{u}_y p W + \dot{v}_y p W - i \omega \dot{p} W + \dot{\rho}_x U W \]

\[ + \dot{\rho}_y V W + \dot{\rho}_y W W + \dot{\rho}_y U W + i \beta \dot{\rho} W^2 \]

\[ + \dot{w}_x p U - i \omega \dot{w} p \]

\[ = \frac{\mu W_{yy}}{Re} + \frac{\dot{\mu} W_y}{Re} + \frac{\mu W_{xx}}{Re} + \frac{\dot{\mu} W_x}{Re} \]

\[ - \frac{2i \beta \dot{\mu} V_y}{3 Re} - \frac{2i \beta \dot{\mu} U_x}{3 Re} - i \beta \dot{p} + \frac{\dot{w}_y p_y}{Re} \]

\[ + \frac{i \beta \dot{v} p_y}{Re} + \frac{\dot{w}_x p_x}{Re} + \frac{i \beta \dot{u} p_x}{Re} + \frac{w_{yy} p}{Re} + \frac{w_{xx} p}{Re} \]

\[ - \frac{4 \beta^2 \dot{w} p}{3 Re} + \frac{i \beta \dot{v} p_y}{3 Re} + \frac{i \beta \dot{u} p_x}{3 Re} \]  

(E.18)
Energy equation

Equation by VESTA
\[
\begin{align*}
\dot{\rho} \nabla \rho \nabla W &= \tilde{v} p \nabla W_y + \tilde{w} p \nabla W_y \\
+ \dot{\rho} U \nabla W_x &= \tilde{u} p \nabla W_x + \tilde{w} p \nabla W_x \\
+ \tilde{v} p \nabla W_y &= \tilde{u} p \nabla W_y + \tilde{w} p \nabla W_y \\
+ \frac{i \beta \dot{\rho} V^2}{2} &= \frac{\dot{\rho} V_y W^2}{2} + \frac{\dot{\rho} V W^2}{2} \\
+ \frac{\dot{\rho} U_x W^2}{2} &= \frac{\dot{\rho} U W^2}{2} - \frac{i \omega \dot{\rho} W^2}{2} \\
+ \frac{\tilde{v} p_y W^2}{2} &= \frac{\tilde{u} p_y W^2}{2} + \frac{3 i \beta \tilde{w} p W^2}{2} \\
+ \frac{\tilde{v} y p W^2}{2} &= \frac{\tilde{u} x p W^2}{2} + \tilde{w} p y W_x \\
+ \frac{i \beta \dot{\rho} V^2}{2} &= \tilde{w} p_y V W + \tilde{y} p y V W \\
+ i \beta \tilde{v} p V W &= \tilde{w} p U W + \frac{i \beta \dot{\rho} U^2 W}{2} \\
+ \frac{i \beta \tilde{v} y p W}{2} &= \frac{i \beta \gamma \dot{p} W}{(\gamma - 1)} + \frac{3 \tilde{p} V^2 y}{2} \\
+ 3 \tilde{v} p \nabla p W_y &= \frac{\tilde{p} U^2 V^2}{2} + \tilde{u} p U V_y \\
+ \frac{\gamma \tilde{p} V y}{(\gamma - 1)} &= \frac{\dot{\rho} V W_x}{2} + \tilde{u} p V V_x \\
+ \tilde{v} p \nabla U_x &= \frac{\dot{\rho} U^3}{2} + \frac{\tilde{p} U x V^2}{2} \\
+ \frac{\tilde{p} x U^2 V^2}{2} &= \frac{3 \tilde{p}_y V^2}{2} \\
+ \frac{\tilde{u} p_x V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u}_x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u} p y V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u} x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u}_x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
- i \omega \tilde{v} p V &= \frac{\gamma \tilde{p} V y}{(\gamma - 1)} + \tilde{v} p U U_y \\
+ \frac{\tilde{v} p U x V}{2} &= \frac{\dot{\rho} U^2 V}{2} + \tilde{u} p U V_x \\
+ \frac{\tilde{p} x U V^2}{2} &= \frac{3 \tilde{p}_y V^2}{2} \\
+ \frac{\tilde{u} p_x V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u}_x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u} p y V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u} x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
+ \frac{\tilde{u}_x p V^2}{2} &= \frac{i \beta \tilde{w} p V^2}{2} + \frac{3 \tilde{v} y p V^2}{2} \\
- i \omega \tilde{v} p V &= \frac{\gamma \tilde{p} V y}{(\gamma - 1)} + \tilde{v} p U U_y
\end{align*}
\]
\[
\begin{align*}
+ \frac{3 \hat{\rho} U^2 U_x}{2} &+ 3 \hat{u} p U_x + \gamma \hat{p} U_x \\
+ \frac{\hat{\rho}_x U^3}{2} &- i \omega \hat{\rho} U^2 + \hat{v}_y \hat{p}_x U^2 \\
+ \frac{3 \hat{u}_x p_x U^2}{2} &+ i \beta \hat{w}_x p U^2 + \hat{v}_y \hat{p}_x U^2 \\
+ \frac{3 \hat{u}_x p U^2}{2} &- i \omega \hat{u}_x U + \gamma \hat{\rho}_x U \\
- \frac{i \omega \hat{\rho}_y}{(\gamma - 1)} &+ \gamma \hat{v}_P y + \gamma \hat{u}_x P \\
+ i \beta \gamma \hat{w}_P &+ \gamma \hat{v}_y P + \gamma \hat{u}_x P \\
+ \frac{3 \hat{u}_x p_y U^2}{2} &+ i \beta \hat{w}_x p U^2 + \hat{v}_y \hat{p}_x U^2 \\
+ \frac{3 \hat{u}_x p U^2}{2} &+ i \omega \hat{u}_x U + \gamma \hat{\rho}_x U \\
+ \frac{\hat{\rho}_y U^3}{2} &+ i \omega \hat{\rho} U^2 + \hat{v}_y \hat{p}_y U^2 \\
+ \frac{3 \hat{u}_y p_y U^2}{2} &+ i \beta \hat{w}_y p U^2 + \hat{v}_y \hat{p}_y U^2 \\
+ \frac{3 \hat{u}_y p U^2}{2} &+ i \omega \hat{u}_y U + \gamma \hat{\rho}_y U \\
+ \frac{\hat{\rho}_x U^3}{2} &+ i \omega \hat{\rho} U^2 + \hat{v}_y \hat{p}_x U^2 \\
+ \frac{3 \hat{u}_x p_x U^2}{2} &+ i \beta \hat{w}_x p U^2 + \hat{v}_y \hat{p}_x U^2 \\
+ \frac{3 \hat{u}_x p U^2}{2} &+ i \omega \hat{u}_x U + \gamma \hat{\rho}_x U \\
= \frac{\hat{\mu} W W_{y y}}{Re} + \frac{\hat{w} p W_{y y}}{Re} + \frac{\hat{\mu} (W_y)^2}{Re} \\
+ \frac{\hat{\mu}_y W W_{y y}}{Re} + i \beta \hat{\mu} V W_{y y} + \hat{w}_y \hat{p}_y W_{y y} \\
+ 2 \hat{w}_y p W_{y y} + i \beta \hat{v}_y p W_{y y} + \frac{\hat{\mu} W W_{y y}}{Re} \\
+ \frac{\hat{w}_y W W_{x x}}{Re} + \frac{\hat{w} \mu W W_{x x}}{Re} + \frac{2 \hat{w}_x \hat{\mu} W}{Re} \\
+ 2 i \beta \hat{u}_y p W_{x x} + i \beta \hat{v}_y p W_{x x} \\
+ \frac{\hat{w}_y \hat{p}_y W}{Re} + i \beta \hat{v}_y \hat{p}_y W + \frac{\hat{w}_y \hat{p}_y W_{x x}}{Re} \\
+ 4 \beta^2 \hat{w}_y W + \frac{3 \hat{\mu} V V_{y y}}{3 Re} + 4 \beta \hat{w}_y p W + \frac{\hat{w}_y \hat{p}_y W_{x x}}{Re} \\
+ \frac{4 \hat{\mu}_y V V_{y y}}{3 Re} + 4 \hat{\mu} \hat{V}_y V_{y y} + \frac{4 \hat{\mu} (V_y)^2}{3 Re} \\
+ \frac{4 \hat{\mu} V V_{x x}}{3 Re} + 4 \hat{\mu} \hat{V}_x V_{x x} + \frac{2 \hat{\mu} \hat{V}_x V_{x x}}{3 Re} \\
+ \frac{4 \hat{v}_y \hat{p}_y V}{3 Re} + \frac{2 \hat{u}_y \hat{p}_y V}{3 Re} + 4 i \beta \hat{w}_y \hat{p}_y V \\
+ \frac{8 \hat{v}_y \hat{p}_y V}{3 Re} - 4 \hat{u}_y \hat{p}_y V + \frac{\hat{\mu} \hat{V} V_{x x}}{3 Re} \\
\frac{\hat{v}_y \hat{p}_y V_{x x}}{Re} + \frac{\hat{\mu} \hat{V} V_{x x}}{3 Re} + \frac{\hat{\mu} \hat{V} V_{x y}}{3 Re}
\end{align*}
\]
\[ + \frac{\tilde{\mu}}{Re} (\nabla_x)^2 + \frac{\tilde{\mu}_x}{Re} \nabla \nabla_{\cdot} + 2 \frac{\tilde{\mu}_y}{Re} \nabla_{\cdot} \]
\[ + \tilde{\mu} \frac{U_{xy}}{Re} + \frac{\tilde{\mu}_y}{Re} \nabla_U \]
\[ + \tilde{v}_x \tilde{p}_y \]
E.3 Verification with Theofilis & Colonius and Robinet

The equations presented in this section are those of Theofilis & Colonius [86]. It is found however that Robinet [69] has taken the same system and reported extra errors based on, among others, wrong interpretations of reference [86]. These changes are presented using blue boxes when applicable.

Continuity equation

Equation by VESTA

\[
-i \beta \frac{T \bar{P} W}{T} + i \beta \bar{p} W - \frac{T \bar{P} V_y}{T} + \bar{p} V_y
\]

\[
-\frac{\bar{p} T_y V}{T} + \frac{2 \bar{T} P T_y V}{T^2} - \frac{\bar{T} P_y V}{T} - \frac{\bar{T}_y P V}{T}
\]

\[
+\bar{p}_y V - \frac{\bar{T} \bar{P} U_x}{T} + \bar{p} U_x - \frac{\bar{p} \bar{T}_x U}{T}
\]

\[
+\frac{2 \bar{T} P T_x U}{T^2} - \frac{\bar{T}_x P U}{T} + \bar{p} \bar{T}_x U
\]

\[
-\frac{\bar{v} \bar{P} T_y}{T} - \frac{\bar{u} \bar{P} T_x}{T} + i \omega \frac{\bar{T} \bar{P}}{T} - i \omega \bar{p} + \bar{v} \bar{P}_y
\]

\[
+\bar{u} \bar{P}_x + i \beta \bar{w} \bar{P} + \bar{v}_y \bar{P} + \bar{u}_x \bar{P} = 0 \quad (E.21)
\]

Equation by Theofilis & Colonius in [86] and Robinet in [69]

\[
-\frac{i \beta \bar{T} P W}{T} + i \beta \bar{p} W - \frac{\bar{T} P V_y}{T} + \bar{p} V_y
\]

\[
-\frac{\bar{p} T_y V}{T} + \frac{2 \bar{T} P T_y V}{T^2} - \frac{\bar{T} P_y V}{T} - \frac{\bar{T}_y P V}{T}
\]

\[
+\bar{p}_y V - \frac{\bar{T} \bar{P} U_x}{T} + \bar{p} U_x - \frac{\bar{p} \bar{T}_x U}{T}
\]

\[
+\frac{2 \bar{T} P T_x U}{T^2} - \frac{\bar{T}_x P U}{T} + \bar{p} \bar{T}_x U
\]

\[
-\frac{\bar{v} \bar{P} T_y}{T} - \frac{\bar{u} \bar{P} T_x}{T} + i \omega \frac{\bar{T} \bar{P}}{T} - i \omega \bar{p} + \bar{v} \bar{P}_y
\]

\[
+\bar{u} \bar{P}_x + i \beta \bar{w} \bar{P} + \bar{v}_y \bar{P} + \bar{u}_x \bar{P} = 0
\]

Wrongly corrected typo (extra \( \bar{w} \)) in Robinet [69] reference [86]

\[
+\bar{u} \bar{P}_x + i \beta \bar{w} \bar{P} + \bar{v}_y \bar{P} + \bar{u}_x \bar{P} = 0 \quad (E.22)
\]
Compressible Verification Equations

x-Momentum equation

Equation by VESTA

\[ \begin{align*}
    \frac{i \beta \gamma \tilde{u} M^2 \overline{P} \overline{W}}{T} &+ \frac{\gamma M^2 \tilde{p} \overline{U}_y \overline{V}}{T} - \frac{\tilde{T} M^2 \overline{P} \overline{U}_y \overline{V}}{T^2} \\
    &+ \frac{\gamma \tilde{u}_y M^2 \overline{P} \overline{V}}{T} - \frac{\tilde{T} \gamma M^2 \overline{P} \overline{U}_y \overline{V}}{T} \\
    &+ \frac{\gamma M^2 \tilde{p} \overline{U}_x}{T} - \frac{\tilde{T} M^2 \overline{P} \overline{U}_x}{T^2} \\
    &+ \frac{\gamma \tilde{u}_x M^2 \overline{P} \overline{U}_x}{T} \\
    &- \frac{i \gamma \omega \tilde{u} M^2 \overline{P}}{T} = \frac{i \beta \tilde{T} \overline{P} \overline{W}_x}{Re} \\
    &- \frac{2 \tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} - \frac{2 \tilde{T}_x \overline{p} \overline{T}_y \overline{V}}{3 Re} + \frac{\tilde{T}_x \overline{p} \overline{T}_y \overline{V}_y}{3 Re} \\
    &+ \frac{\tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} + \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{V}_x}{3 Re} + \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{U}_y}{3 Re} \\
    &+ \frac{4 \tilde{T} \overline{p} \overline{T}_x \overline{U}_y}{3 Re} + \frac{4 \tilde{T}_x \overline{p} \overline{T}_x \overline{U}_x}{3 Re} + \frac{\tilde{T}_x \overline{p} \overline{T}_y \overline{U}_y}{3 Re} \\
    &+ \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{U}_x}{3 Re} + \frac{2 \tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} \\
    &- \frac{2 i \beta \tilde{w} \overline{P} \overline{T}_x}{3 Re} - \frac{2 \tilde{v}_y \overline{p} \overline{T}_x}{3 Re} \\
    &+ \frac{2 \tilde{w}_x \overline{p} \overline{T}_x}{3 Re} - \frac{\tilde{v}_x \overline{p} \overline{T}_x}{3 Re} \\
    &+ \frac{\tilde{u}_y \overline{p} \overline{T}_x}{3 Re} + \frac{\tilde{u}_x \overline{w}_x \overline{p}}{3 Re} - \frac{\tilde{u}_x \overline{v}_y \overline{p}}{3 Re} \\
    &+ \frac{\tilde{u}_y \overline{w}_x \overline{p}}{3 Re} - \frac{\beta^2 \tilde{u} \overline{p}}{3 Re} \quad (E.23)
\end{align*} \]

Equation by Theofilis & Colonius in [86] and Robinet in [69]

\[ \begin{align*}
    \frac{i \beta \gamma \tilde{u} M^2 \overline{P} \overline{W}}{T} &+ \frac{\gamma M^2 \tilde{p} \overline{U}_y \overline{V}}{T} - \frac{\tilde{T} M^2 \overline{P} \overline{U}_y \overline{V}}{T^2} \\
    &+ \gamma \tilde{u}_y M^2 \overline{P} \overline{V} - \gamma \tilde{T} \gamma M^2 \overline{P} \overline{U}_y \overline{V} \\
    &+ \gamma M^2 \tilde{p} \overline{U}_x - \gamma \tilde{T} M^2 \overline{P} \overline{U}_x \\
    &+ \gamma \tilde{u}_x M^2 \overline{P} \overline{U}_x \\
    &- \frac{i \gamma \omega \tilde{u} M^2 \overline{P}}{T} = \frac{i \beta \tilde{T} \overline{P} \overline{W}_x}{Re} \\
    &+ \frac{2 \tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} - \frac{2 \tilde{v}_y \overline{p} \overline{T}_x}{3 Re} + \frac{\tilde{v}_y \overline{p} \overline{T}_x}{3 Re} \\
    &+ \frac{\tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} + \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{V}_x}{3 Re} + \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{U}_y}{3 Re} \\
    &+ \frac{4 \tilde{T} \overline{p} \overline{T}_x \overline{U}_y}{3 Re} + \frac{4 \tilde{T}_x \overline{p} \overline{T}_x \overline{U}_x}{3 Re} + \frac{\tilde{T}_x \overline{p} \overline{T}_y \overline{U}_y}{3 Re} \\
    &+ \frac{\tilde{T}_y \overline{p} \overline{T}_y \overline{U}_x}{3 Re} + \frac{2 \tilde{T} \overline{p} \overline{T}_x \overline{V}_y}{3 Re} \\
    &- \frac{2 i \beta \tilde{w} \overline{P} \overline{T}_x}{3 Re} - \frac{2 \tilde{v}_y \overline{p} \overline{T}_x}{3 Re} \\
    &+ \frac{2 \tilde{w}_x \overline{p} \overline{T}_x}{3 Re} - \frac{\tilde{v}_x \overline{p} \overline{T}_x}{3 Re} \\
    &+ \frac{\tilde{u}_y \overline{p} \overline{T}_x}{3 Re} + \frac{\tilde{u}_x \overline{w}_x \overline{p}}{3 Re} - \frac{\tilde{u}_x \overline{v}_y \overline{p}}{3 Re} \\
    &+ \frac{\tilde{u}_y \overline{w}_x \overline{p}}{3 Re} - \frac{\beta^2 \tilde{u} \overline{p}}{3 Re} \quad (E.24)
\end{align*} \]
y-Momentum equation

Equation by VESTA

\[
\begin{align*}
\frac{i \beta \gamma \bar{v} M^2 \mathcal{P} \mathcal{W}}{T} + \frac{\gamma M^2 \bar{p} \mathcal{V} \mathcal{V}_y}{T} &= \frac{i \beta \gamma \bar{v} M^2 \mathcal{P} \mathcal{W}}{T} + \frac{\gamma M^2 \bar{p} \mathcal{V} \mathcal{V}_y}{T} \\
- \frac{\gamma \dot{T} M^2 \mathcal{P} \mathcal{V} \mathcal{V}_y}{T^2} + \frac{\gamma \bar{v} M^2 \mathcal{P} \mathcal{V} \mathcal{V}_y}{T} &= \frac{-\gamma \dot{T} M^2 \mathcal{P} \mathcal{V} \mathcal{V}_y}{T^2} + \frac{2 \gamma \bar{v} M^2 \mathcal{P} \mathcal{V} \mathcal{V}_y}{T} \\
+ \frac{\gamma \bar{u} M^2 \mathcal{P} \mathcal{V}_x}{T} &= \frac{\gamma \bar{u} M^2 \mathcal{P} \mathcal{V}_x}{T} \\
+ \frac{\gamma \bar{v}_y M^2 \mathcal{P} \mathcal{V}}{T} &= \frac{\gamma \bar{v}_y M^2 \mathcal{P} \mathcal{V}}{T} \\
+ \frac{\gamma \bar{v}_x M^2 \mathcal{P} \mathcal{U}}{T} &= \frac{\gamma \bar{v}_x M^2 \mathcal{P} \mathcal{U}}{T} \\
- \frac{i \gamma \omega \bar{v} M^2 \mathcal{P}}{T} &= \frac{-i \gamma \omega \bar{v} M^2 \mathcal{P}}{T} \\
\end{align*}
\]

From \( \varphi \mathcal{V}_y \)

From \( \varphi \mathcal{U} \)

From \( \varphi \mathcal{V} \)

From \( \varphi \mathcal{U}_x \)

From \( \varphi \mathcal{V}_x \)

From \( \varphi \mathcal{V}_y \)

From \( \varphi \mathcal{V} \)

From \( \varphi \mathcal{U} \)

From \( \varphi \mathcal{U} \)

\[(E.25)\]
z-Momentum equation

Equation by VESTA

\[
\frac{\gamma M^2 \hat{p} \nabla W_y}{T} - \frac{\gamma T M^2 \nabla W_y}{T^2} + \frac{\gamma \hat{v} M^2 \hat{p} W_y}{T} + \frac{\gamma M^2 \hat{p} U W_x}{T} - \frac{\gamma \bar{T} M^2 \bar{p} \bar{U} \bar{W}_x}{T^2} + \frac{\gamma \hat{u} M^2 \hat{p} W_x}{T} + \frac{i \beta \gamma \hat{w} M^2 \bar{p} W}{T} + \frac{\gamma \bar{w}_y M^2 \bar{p} V}{T} + \frac{\gamma \bar{w}_x M^2 \bar{p} U}{T} - i \gamma \omega \bar{w} M^2 \bar{p}
\]

\[
= \frac{T \bar{p}_T W_{yy}}{Re} + \frac{T \bar{p}_T T_y W_y}{Re} + \frac{T \bar{y}_p T W_y}{Re} + \frac{T \bar{p}_T \bar{W}_{xx}}{Re} + \frac{T \bar{p}_T \bar{T}_x \bar{W}_x}{Re} + \frac{T \bar{y}_p \bar{T}_x \bar{W}_x}{Re} - \frac{2 i \beta T \bar{p}_T V_y}{3 Re} - \frac{2 i \beta \bar{p}_T U_x}{3 Re} + \frac{\bar{w}_y \bar{p}_T T_y}{Re} + \frac{\bar{w}_x \bar{p}_T T_x}{Re} + i \beta \bar{u}_x \bar{p}_T T_x - i \beta \bar{p} + \frac{\bar{w}_y y \bar{p}}{Re} + \frac{\bar{w}_x x \bar{p}}{Re} - \frac{4 \beta^2 \bar{w} \bar{p}}{3 Re} + \frac{i \beta \bar{v}_y \bar{p}}{3 Re} + \frac{i \beta \bar{u}_x \bar{p}}{3 Re} \tag{E.27}
\]

Equation by Theofilis & Colonius in [86] and Robinet in [69]

\[
\frac{\gamma M^2 \hat{p} \nabla W_y}{T} - \frac{\gamma T M^2 \nabla W_y}{T^2} + \frac{\gamma \hat{v} M^2 \hat{p} U W_x}{T} - \frac{\gamma \bar{T} M^2 \bar{p} \bar{U} \bar{W}_x}{T^2} + \frac{\gamma \hat{u} M^2 \hat{p} W_x}{T} + \frac{i \beta \gamma \hat{w} M^2 \bar{p} W}{T} + \frac{\gamma \bar{w}_y M^2 \bar{p} V}{T} - i \gamma \omega \bar{w} M^2 \bar{p}
\]

\[
= \frac{T \bar{p}_T W_{yy}}{Re} + \frac{T \bar{p}_T T_y W_y}{Re} + \frac{T \bar{y}_p T W_y}{Re} + \frac{T \bar{p}_T \bar{W}_{xx}}{Re} + \frac{T \bar{p}_T \bar{T}_x \bar{W}_x}{Re} + \frac{T \bar{y}_p \bar{T}_x \bar{W}_x}{Re} - \frac{2 i \beta T \bar{p}_T V_y}{3 Re} - \frac{2 i \beta \bar{p}_T U_x}{3 Re} + \frac{\bar{w}_y \bar{p}_T T_y}{Re} + \frac{\bar{w}_x \bar{p}_T T_x}{Re} + i \beta \bar{u}_x \bar{p}_T T_x - i \beta \bar{p} + \frac{\bar{w}_y y \bar{p}}{Re} + \frac{\bar{w}_x x \bar{p}}{Re} - \frac{4 \beta^2 \bar{w} \bar{p}}{3 Re} + \frac{i \beta \bar{v}_y \bar{p}}{3 Re} + \frac{i \beta \bar{u}_x \bar{p}}{3 Re} \tag{E.28}
\]
Energy equation

Equation by VESTA

\[
\begin{align*}
&\quad i \beta M^2 \tilde{p} \tilde{W} + \gamma M^2 \tilde{p} \tilde{V}_y + M^2 \tilde{p}_y \tilde{V} \\
&+ \gamma M^2 \tilde{p} \tilde{U}_x + M^2 \tilde{p}_x \tilde{U} - i \omega M^2 \tilde{p} + \tilde{u} M^2 \tilde{P} \\
&+ \tilde{u} M^2 \tilde{P}_x + i \beta \gamma \tilde{w} M^2 \tilde{P} + \gamma \tilde{v}_y M^2 \tilde{P} \\
&+ \gamma \tilde{u}_x M^2 \tilde{P} = \frac{Ec \tilde{T} \tilde{p}_T (\tilde{W}_y)^2}{Re} + 2 Ec \tilde{w}_y \tilde{p} \tilde{W}_y \\
&+ \frac{2 i \beta Ec \tilde{v} \tilde{p} \tilde{W}_y}{Re} + \frac{Ec \tilde{T} \tilde{p}_T (\tilde{W}_x)^2}{Re} + 2 Ec \tilde{w}_x \tilde{p} \tilde{W}_x \\
&+ \frac{4 i \beta Ec \tilde{u} \tilde{p} \tilde{V}_y}{3 Re} + \frac{8 Ec \tilde{v}_y \tilde{V}_y}{3 Re} \\
&+ \frac{4 i \beta Ec \tilde{u}_x \tilde{V}_y}{3 Re} + \frac{Ec \tilde{T} \tilde{p}_T (\tilde{V}_x)^2}{Re} + \frac{2 Ec \tilde{u}_x \tilde{V}_x}{Re} \\
&+ \frac{2 Ec \tilde{u}_y \tilde{V}_x}{Re} + \frac{2 Ec \tilde{v}_x \tilde{V}_x}{Re} \\
&+ \frac{4 i \beta Ec \tilde{w} \tilde{U}_x}{3 Re} + \frac{8 Ec \tilde{v}_x \tilde{U}_x}{3 Re} - \frac{4 \tilde{V}_x}{3 Re} \\
&+ \frac{4 i \beta Ec \tilde{w}_x \tilde{U}_x}{3 Re} + \frac{8 Ec \tilde{v}_x \tilde{U}_x}{3 Re} - \frac{4 \tilde{W}_y}{3 Re} \\
&+ \frac{\tilde{T}_y \tilde{T}_{yy}}{Pr Re} + \frac{\tilde{T}_x \tilde{T}_{yx}}{Pr Re} + 2 \tilde{T}_y \tilde{T}_x \tilde{T}_{yy} \\
&+ \frac{\tilde{T}_x \tilde{T}_{xx}}{Pr Re} + \frac{\tilde{T}_y \tilde{T}_{yy}}{Pr Re} + 2 \tilde{T}_x \tilde{T}_y \tilde{T}_{xx} \\
&+ \frac{\tilde{T}_{yy} \tilde{K}}{Pr Re} + \frac{\tilde{T}_{xx} \tilde{K}}{Pr Re} + \beta^2 \tilde{T} \tilde{K} (E.29)
\end{align*}
\]

Equation by Theofilis & Colonius in [86] and Robinet in [69]

\[
\begin{align*}
&\quad i \beta M^2 \tilde{p} \tilde{W} + \gamma M^2 \tilde{p} \tilde{V}_y + M^2 \tilde{p}_y \tilde{V} \\
&+ \gamma M^2 \tilde{p} \tilde{U}_x + M^2 \tilde{p}_x \tilde{U} - i \omega M^2 \tilde{p} + \tilde{u} M^2 \tilde{P} \\
&+ \tilde{u} M^2 \tilde{P}_x + i \beta \gamma \tilde{w} M^2 \tilde{P} + \gamma \tilde{v}_y M^2 \tilde{P} \\
&+ \gamma \tilde{u}_x M^2 \tilde{P} = \frac{Ec \tilde{T} \tilde{p}_T (\tilde{W}_y)^2}{Re} + 2 Ec \tilde{w}_y \tilde{p} \tilde{W}_y \\
&+ \frac{2 i \beta Ec \tilde{v} \tilde{p} \tilde{W}_y}{Re} + \frac{Ec \tilde{T} \tilde{p}_T (\tilde{W}_x)^2}{Re} + 2 Ec \tilde{w}_x \tilde{p} \tilde{W}_x \\
&+ \frac{4 i \beta Ec \tilde{u} \tilde{p} \tilde{V}_y}{3 Re} + \frac{8 Ec \tilde{v}_y \tilde{V}_y}{3 Re} \\
&+ \frac{4 i \beta Ec \tilde{u}_x \tilde{V}_y}{3 Re} + \frac{Ec \tilde{T} \tilde{p}_T (\tilde{V}_x)^2}{Re} + \frac{2 Ec \tilde{u}_x \tilde{V}_x}{Re} \\
&+ \frac{2 Ec \tilde{u}_y \tilde{V}_x}{Re} + \frac{2 Ec \tilde{v}_x \tilde{V}_x}{Re} \\
&+ \frac{4 i \beta Ec \tilde{w} \tilde{U}_x}{3 Re} + \frac{8 Ec \tilde{v}_x \tilde{U}_x}{3 Re} - \frac{4 \tilde{V}_x}{3 Re} \\
&+ \frac{4 i \beta Ec \tilde{w}_x \tilde{U}_x}{3 Re} + \frac{8 Ec \tilde{v}_x \tilde{U}_x}{3 Re} - \frac{4 \tilde{W}_y}{3 Re} \\
&+ \frac{\tilde{T}_y \tilde{T}_{yy}}{Pr Re} + \frac{\tilde{T}_x \tilde{T}_{yx}}{Pr Re} + 2 \tilde{T}_y \tilde{T}_x \tilde{T}_{yy} \\
&+ \frac{\tilde{T}_x \tilde{T}_{xx}}{Pr Re} + \frac{\tilde{T}_y \tilde{T}_{yy}}{Pr Re} + 2 \tilde{T}_x \tilde{T}_y \tilde{T}_{xx} \\
&+ \frac{\tilde{T}_{yy} \tilde{K}}{Pr Re} + \frac{\tilde{T}_{xx} \tilde{K}}{Pr Re} + \beta^2 \tilde{T} \tilde{K} (E.30)
\end{align*}
\]
References


